

Concentration of small Willmore spheres in Riemannian 3-manifolds

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ABSTRACT. Given a 3-dimensional Riemannian manifold (M, g) , we prove that if (Φ_k) is a sequence of Willmore spheres (or more generally area-constrained Willmore spheres), having Willmore energy bounded above uniformly strictly by 8π , and Hausdorff converging to a point $\bar{p} \in M$, then $\text{Scal}(\bar{p}) = 0$ and $\nabla \text{Scal}(\bar{p}) = 0$ (resp. $\nabla \text{Scal}(\bar{p}) = 0$). Moreover, a suitably rescaled sequence smoothly converges, up to subsequences and reparametrizations, to a round sphere in the euclidean 3-dimensional space.

This generalizes previous results of Lamm and Metzger contained in [14]-[15].

An application to the Hawking mass is also established.

Key Words: Willmore functional, Hawking mass, blow up technique, concentration phenomena, fourth order nonlinear Elliptic PDEs.

AMS subject classification:

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1 Introduction

Let Σ be a closed two dimensional surface and (M, g) a 3-dimensional Riemannian manifold. Given a smooth immersion $\Phi : \Sigma \hookrightarrow M$, $W(\Phi)$ denotes the Willmore energy of Φ defined by

$$(1) \quad W(\Phi) := \int_{\Sigma} H^2 dvol_{\bar{g}},$$

where $\bar{g} := \Phi^*(g)$ is the pullback metric on Σ (i.e. the metric induced by the immersion), $dvol_{\bar{g}}$ is the associated volume form, and H is the mean curvature of the immersion Φ (we adopt the convention that $H = \frac{1}{2} \bar{g}^{ij} A_{ij}$ where A_{ij} is the second fundamental form; or, in other words, H is the arithmetic mean of the two principal curvatures).

In case the ambient manifold is the euclidean 3-dimensional space, the topic is classical and goes back to the works of Blaschke and Thomsen in 1920-'30 who were looking for a conformal invariant theory which included minimal surfaces; the functional was later rediscovered by Willmore [36] in the 60'ies and from that moment there have been a flourishing of results (let us mention the fundamental paper of Simon [33], the work of Kuwert-Schätzle [11]-[12]-[13], the more recent approach by Rivière [29]-[30]-[31], etc.) culminated with the recent proof of the Willmore Conjecture by Marques and Neves [19] by min-max techniques (let us mention that partial results towards the Willmore conjecture were previously obtained by Li and Yau [18], Montiel and Ros [26], Ros [32], Topping [34], etc., and that a crucial role in the proof of the conjecture is played by a result of Urbano [35]).

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On the other hand, the investigation of the Willmore functional in non constantly curved Riemannian manifolds is a much more recent topic started in [20] (see also [21] and the more recent joint work with Carlotto [4]) where the second author studied existence and non existence of Willmore surfaces in a perturbative setting.

Smooth minimizers of the L^2 norm of the second fundamental form among spheres in compact Riemannian three manifolds were obtained in collaboration with Kuwert and Schygulla in [10] where the full regularity theory for minimizers was settled taking inspiration from the approach of Simon [33] (see also [25] for minimization in non compact Riemannian manifolds).

Let us finally mention the work in collaboration with Rivière [23]-[24] where, using a “parametric approach” inspired by the Euclidean theory of [29]-[30]-[31], the necessary tools for studying the calculus of variations of the Willmore functional in Riemannian manifolds (i.e. the definition of the weak objects and related compactness and regularity issues) are settled together with applications; in particular the existence and regularity of Willmore spheres in homotopy classes is established.

Since -as usual in the calculus of variations- the existence results are obtained by quite general techniques and do not describe the minimizing object, the purpose of the present paper is to investigate the geometric properties of the critical points of W .

More precisely we investigate the following natural questions: Let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of smooth critical points of the Willmore functional W (or more generally we will also consider critical points under area constraint) converging to a point $\bar{p} \in M$ in Hausdorff distance sense; what can we say about Φ_k ? are they becoming more and more round? Has the limit point \bar{p} some special geometric property?

These questions have already been addressed in recent articles -below the main known results are recalled by the reader’s convenience-, but in the present paper we are going to obtain the sharp answers. Before passing to describe the known and the new results in this direction, let us recall that a critical point of the Willmore functional is called *Willmore surface* and it satisfies:

$$(2) \quad \Delta_{\bar{g}}H + H|A^\circ|^2 + H \operatorname{Ric}(\bar{n}, \bar{n}) = 0,$$

where $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator corresponding to the metric \bar{g} , $(A^\circ)_{ij} := A_{ij} - H\bar{g}_{ij}$ is the trace-free second fundamental form, \bar{n} is a normal unit vector to Φ , and Ric is the Ricci tensor of the ambient manifold (M, g) . Notice that (2) is a fourth-order nonlinear elliptic PDE in the parametrization map Φ .

Throughout the paper we will consider more generally *area-constrained Willmore surfaces*, i.e. critical points of the Willmore functional under area constraint; the immersion Φ is an area-constrained Willmore surface if and only if it satisfies

$$(3) \quad \Delta_{\bar{g}}H + H|A^\circ|^2 + H \operatorname{Ric}(\bar{n}, \bar{n}) = \lambda H,$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier.

The first result in the direction of the above questions was achieved in the master degree thesis of the second author [20] where it was proved that if (Φ_k) is a sequence of Willmore surfaces obtained as normal graphs over shrinking geodesic spheres centered at a point \bar{p} , then the scalar curvature at \bar{p} must vanish: $\operatorname{Scal}(\bar{p}) = 0$.

In the subsequent papers [14]-[15], Lamm and Metzger proved that if $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ is a sequence of area-constrained Willmore surfaces converging to a point \bar{p} in Hausdorff distance sense and such that ¹

$$(4) \quad W(\Phi_k) \leq 4\pi + \varepsilon \quad \text{for some } \varepsilon > 0 \text{ small enough,}$$

then $\nabla \operatorname{Scal}(\bar{p}) = 0$ and, up to subsequences, Φ_k is $W^{2,2}$ -asymptotic to a geodesic sphere centered at \bar{p} . Moreover in [15], using the regularity theory developed in [10], they showed that if (M, g) is any compact Riemannian 3-manifold and a_k is any sequence of positive real numbers such that $a_k \downarrow 0$ then there

¹notice that the normalization of the Willmore functional used in [14]-[15] differ from our convention by a factor 2

exists a smooth minimizer Φ_k of W under the area-constraint $\text{Area}(\Phi_k) = a_k$; moreover such sequence (Φ_k) satisfies (4) and therefore it $W^{2,2}$ -converges to a round critical point of the scalar curvature. Let us mention that the existence of area-constrained Willmore spheres was generalized in [24] to any value of the area.

The goal of this paper is multiple. The main achievement is the improvement of the perturbative bound (4) above to the global bound

$$(5) \quad \limsup_k W(\Phi_k) < 8\pi.$$

Secondly we improve the $W^{2,2}$ -convergence above to *smooth* convergence towards a *round* critical point of the scalar curvature, i.e. we show that if we rescale (M, g) around \bar{p} in such a way that the sequence of surfaces has fixed area equal to one (for more details see Section 2), then the sequence converges smoothly, up to subsequences, to a round sphere centered at \bar{p} , and \bar{p} is a critical point of the scalar curvature of (M, g) .

Finally we give an application of these results to the Hawking mass.

We believe that the bound (5) is sharp in order to have smooth convergence to a *round* point (in the sense specified above); indeed, if (5) is violated then the sequence (Φ_k) may degenerate to a couple of bubbles, each one costing almost 4π in terms of Willmore energy.

Now let us state the main results of the present article. The first theorem below concerns the case of a sequence of Willmore immersions and it is a consequence of the second more general theorem about area-constrained Willmore immersions.

Theorem 1.1. *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of Willmore surfaces satisfying the energy bound (5) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\text{Scal}(\bar{p}) = 0$ and $\nabla \text{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around \bar{p} in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the 3-dimensional euclidean space.

Actually we prove the following more general result about sequences of area-constrained Willmore immersions.

Theorem 1.2. *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of area-constrained Willmore surfaces satisfying the energy bound (5) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \text{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around \bar{p} in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the 3-dimensional euclidean space.

Of course Theorem 1.2 implies Theorem 1.1 except the property $\text{Scal}(\bar{p}) = 0$. This fact follows by the aforementioned [20, Theorem 1.3] holding for Willmore graphs over geodesic spheres, together with the smooth convergence to a round point ensured by Theorem 1.2.

Now we pass to discuss an application to the Hawking mass m_H , defined for an immersed sphere $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ by

$$(6) \quad m_H(\Phi) = \frac{\text{Area}_g(\Phi)}{16\pi^{3/2}} (4\pi - W(\Phi)).$$

Of course, the critical points of the Hawking mass under area constraint are exactly the area-constrained Willmore spheres (see [16] and the references therein for more material about the Hawking mass); moreover it is clear that the inequality $m_H(\Phi) \geq 0$ implies that $W(\Phi) \leq 4\pi$.

Therefore, combining this easy observations with Theorem 1.2, we obtain the following corollary.

Corollary 1.3. *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of critical points of m_H under area constraint having non negative Hawking mass and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \text{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around \bar{p} in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the 3-dimensional euclidean space.

Let us briefly comment on the relevance of Corollary 1.3 despite the triviality of its proof. Recall that, from the note of Christodoulou and Yau [5], if (M, g) has non negative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; on the other hand it is known (see for instance [6] or [28]) that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0 (see also [22] for the non-compact case). Therefore a link between regions with positive Hawking mass and critical points of the scalar curvature was already present in literature, but Corollary 1.3 expresses this link precisely.

We end the introduction by outlining the structure of the paper and the main ideas of the proof. First of all, as already noticed, it is enough to prove Theorem 1.2 in order to get all the stated results. To prove it, we adopt the blow up technique taking inspiration from [17] where the first author analyzed the corresponding questions in the context of CMC-surfaces; such technique was introduced in the analysis of the Yamabe problem which is a second order scalar problem (for an detailed overview of the method including applications see [7]), the technical novelty of [17] was that that a second order *vectorial* problem was considered; the technical originality of the present paper from the point of view of the blow up method is that we study a *fourth order vectorial problem*.

More precisely, in Section 2 we consider normal coordinated centered at the limit point \bar{p} and we rescale appropriately the metric g such that the rescaled surfaces have all diameter one (or, thanks to the monotonicity formula, it is equivalent to fix the area of the rescaled surfaces equal to one); notice that the rescaled ambient metrics g_k are becoming more and more euclidean. In Subsection 2.1, by exploiting the divergence form of the Willmore equation established in [24], we give a decay estimate on the Lagrange multipliers as k goes to infinity.

Section 3 is devoted to the proof of Theorem 1.2; we start in Subsection 3.1 by establishing a fundamental technical result telling that, under the above working assumptions, the sequence (Φ_k) converges smoothly to a round sphere, up to subsequences and reparametrizations. Let us remark that in the proof we exploit in a crucial way the assumption (5), otherwise it may be possible for the sequence to degenerate to a couple of bubbles. Once we have smooth convergence to a round sphere ω , we study the remainder given by the difference between Φ_k and ω : in Subsection 3.3 we use the linearized Willmore operator (recalled in Appendix A) in order to give precise asymptotics of such remainder term and in the final Subsection 3.4 we refine these estimates and conclude the proof.

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2 Notation and preliminaries

Throughout the paper (M, g) is a Riemannian 3-manifold and \mathbb{S}^2 is the round 2-sphere of unit radius in \mathbb{R}^3 . The greek indexes $\alpha, \beta, \gamma, \mu, \nu$ will run from 1 to 3 and will denote quantities in M , latin indexes will run from 1 to 2 and will denote quantities on $\Phi_k(\mathbb{S}^2)$, we will always use Einstein notation on summation over indexes. Given a smooth immersion $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ we call $\bar{g} = \Phi^*(g)$ the pullback metric, $dvol_{\bar{g}}$

the induced area form, $H_{g,\Phi}$ the mean curvature and

$$W_g(\Phi) := \int_{\mathbb{S}^2} |H_{g,\Phi}|^2 dvol_{\bar{g}}$$

is the Willmore functional.

Now let (Φ_k) be a sequence of smooth immersions from \mathbb{S}^2 into M . Under our working assumptions, called $\text{diam}_g(\Omega)$ the diameter of the subset Ω of M with respect to the metric g , we will always have

$$(7) \quad \varepsilon_k := \text{diam}_g(\Phi_k(\mathbb{S}^2)) \rightarrow 0,$$

$$(8) \quad W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g,\Phi_k}|^2 dvol_{\bar{g}_k} \leq 8\pi - 2\delta, \quad \text{for some } \delta > 0 \text{ independent of } k$$

where $dvol_{\bar{g}_k}$ is the area form on \mathbb{S}^2 associated to the pullback metric $\bar{g}_k = \Phi_k^*(g)$ and H_{g,Φ_k} is the mean curvature of Φ_k .

Notice that in case M is compact then (7) is sufficient to ensure that, up to subsequences, $\Phi_k(\mathbb{S}^2)$ converges to a point $\bar{p} \in M$ in Hausdorff distance sense; but since there is no further reason to restrict to a compact ambient manifold we assume the convergence to \bar{p} in the hypothesis of our main results instead of a compactness assumption on M .

In order to efficiently handle the geometric quantities we need good coordinates; let us now introduce them. Take coordinates (x^μ) , $\mu = 1, 2, 3$ around \bar{p} and let $p_k = (p_k^1, p_k^2, p_k^3)$ be the center of mass of $\Phi_k(\mathbb{S}^2)$:

$$p_k^\mu = \frac{1}{\text{Area}_g(\Phi_k)} \int_{\mathbb{S}^2} \Phi_k^\mu dvol_{\bar{g}_k}, \quad \mu = 1, 2, 3,$$

where $\text{Area}_g(\Phi_k) = \int_{\mathbb{S}^2} dvol_{\bar{g}_k}$ is the area of $\Phi_k(\mathbb{S}^2)$. Clearly, up to subsequences, $p_k \rightarrow \bar{p}$.

For every $k \in \mathbb{N}$ consider the exponential normal coordinates centered in p_k and rescale this chart by a factor $\frac{1}{\varepsilon_k}$ with respect to the center of these coordinates. Hence we get a new sequence of immersions $\tilde{\Phi}_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$, in the sequel simply denoted by Φ_k , where the metric g_{ε_k} is defined by

$$(9) \quad g_{\varepsilon_k}(y)(u, v) := g(\varepsilon_k y)(\varepsilon_k^{-1} u, \varepsilon_k^{-1} v).$$

Notice that now we have

$$(10) \quad W_{g_{\varepsilon_k}}(\Phi_k) \leq 8\pi - 2\delta, \quad \text{diam}_{g_{\varepsilon_k}}(\Phi_k(\mathbb{S}^2)) = 1 \quad \text{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}(0, 3/2),$$

where the first inequality is a consequence of the invariance under rescaling of the Willmore functional, and $B_{g_{\varepsilon_k}}(0, 3/2)$ is the metric ball in $(\mathbb{R}^3, g_{\varepsilon_k})$ of center 0 and radius 3/2. By the classical expression of the metric in normal coordinates, we get that (see Appendix B in [17])

$$(11) \quad (g_{\varepsilon_k})_{\mu\nu}(y) = \delta_{\mu\nu} + \frac{\varepsilon_k^2}{3} R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta + \frac{\varepsilon_k^3}{6} R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3),$$

the inverse metric is

$$(12) \quad (g_{\varepsilon_k})^{\mu\nu}(y) = \delta_{\mu\nu} - \frac{\varepsilon_k^2}{3} R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta - \frac{\varepsilon_k^3}{6} R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3),$$

the volume form of g_{ε_k} on can be written as

$$(13) \quad \sqrt{|g_{\varepsilon_k}|}(y) = 1 - \frac{\varepsilon_k^2}{6} \text{Ric}_{\alpha\beta}(p_k) y^\alpha y^\beta - \frac{\varepsilon_k^3}{12} \text{Ric}_{\alpha\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3),$$

and the Christoffel symbols of g_{ε_k} can be expanded as

$$(14) \quad (\Gamma_{\varepsilon_k})_{\alpha\beta}^\gamma(y) = A_{\alpha\beta\gamma\mu}(p_k) y^\mu \varepsilon_k^2 + B_{\alpha\beta\gamma\mu\nu} y^\mu y^\nu \varepsilon_k^3 + o(\varepsilon_k^3)$$

where $A_{\alpha\beta\gamma\mu}(p_k) = \frac{1}{3}(R_{\beta\mu\alpha\gamma}(p_k) + R_{\alpha\mu\beta\gamma}(p_k))$ and $B_{\alpha\beta\gamma\mu\nu}(p_k) = \frac{1}{12}(2R_{\beta\mu\alpha\gamma,\nu}(p_k) + 2R_{\alpha\mu\beta\gamma,\nu}(p_k) + R_{\beta\mu\nu\gamma,\alpha} + R_{\alpha\mu\nu\gamma,\beta}(p_k) - R_{\alpha\mu\nu\beta,\gamma}(p_k))$.

Since by (11) the metric g_{ε_k} is close to the euclidean metric in C^∞ norm on $B_{g_0}(0, 2)$, where $B_{g_0}(0, 2)$ is the euclidean ball in \mathbb{R}^3 of center 0 and radius 2, recalling (10) we get the following lemma.

Lemma 2.1. *Let g_{ε_k} be the metric defined in (9) having the form (11); let $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ be smooth immersions with $\Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}(0, 2)$ satisfying*

$$W_{g_{\varepsilon_k}}(\Phi_k) \leq 8\pi - 2\delta, \quad \text{for some } \delta > 0.$$

Then, for k large enough, we have

$$(15) \quad W_{g_0}(\Phi_k) \leq 8\pi - \delta, \quad \frac{1}{2} \leq \text{diam}_{g_0}(\Phi_k(\mathbb{S}^2)) \leq 2 \quad \text{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_0}(0, 2),$$

where g_0 is the euclidean metric on \mathbb{R}^3 , W_{g_0} is the euclidean Willmore functional and $B_{g_0}(0, 2)$ is the euclidean ball of center 0 and radius 2 in \mathbb{R}^3 . It follows that, for large k , $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ is a smooth embedding and that there exist constants $C_1, C_2 > 0$ such that

$$(16) \quad 0 < \frac{1}{C_1} \leq \frac{1}{C_2} \text{Area}_{g_0}(\Phi_k) \leq \text{Area}_{g_{\varepsilon_k}}(\Phi_k) \leq C_2 \text{Area}_{g_0}(\Phi_k) \leq C_1 < \infty.$$

Proof. The properties expressed in (15) follow from (10) by a direct estimate of the remainders given by the curvature terms of the metric g_{ε_k} ; for such estimates we refer to Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4 in [25].

It is classically known that if the Willmore functional of an immersed closed surface in (\mathbb{R}^3, g_0) is strictly below 8π then the immersion is actually an embedding (see [18] or [33]), so our second statement follows. In order to prove (16) let us recall Lemma 1.1 in [33] stating that

$$\sqrt{\frac{\text{Area}_{g_0}(\Phi_k)}{W_{g_0}(\Phi_k)}} \leq \text{diam}_{g_0} \Phi_k(\mathbb{S}^2) \leq C \sqrt{\text{Area}_{g_0}(\Phi_k) W_{g_0}(\Phi_k)} \quad \text{for some universal } C > 0,$$

which, combined with the bound on $\text{diam}_{g_0}(\Phi_k(\mathbb{S}^2))$ and $W_{g_0}(\Phi_k)$ expressed in (15), gives that there exists a constant $C_0 > 0$ such that

$$0 < \frac{1}{C_0} \leq \text{Area}_{g_0}(\Phi_k) \leq C_0 < \infty;$$

the desired chain of inequalities (16) follows then by estimating the remainders as in Lemma 2.2 in [25]. \square

2.1 The area-constrained Willmore equation and an estimate of the Lagrange multiplier

In the rest of the paper we will work with area-constrained Willmore immersions, i.e. critical points of the Willmore functional under the constraint that the area is fixed. If $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ is a smooth area-constrained Willmore immersion, then it satisfies the following PDE (see for instance Section 3 in [16] for the derivation of the equation)

$$(17) \quad \Delta_{\bar{g}} H_{g, \Phi} + H_{g, \Phi} |A_{g, \Phi}^\circ|_{\bar{g}}^2 + H_{g, \Phi} \text{Ric}_g(\vec{n}_{g, \Phi}, \vec{n}_{g, \Phi}) = \lambda H_{g, \Phi}$$

for some $\lambda \in \mathbb{R}$, where $\vec{n}_{g, \Phi}$ is a normal unit vector to $\Phi(\mathbb{S}^2) \subset (M, g)$, $(A_{g, \Phi}^\circ)_{ij}$ is the traceless second fundamental form $(A_{g, \Phi}^\circ)_{ij} = (A_{g, \Phi})_{ij} - \bar{g}_{ij} H_{g, \Phi}$ (of course $(A_{g, \Phi})_{ij}$ is the second fundamental form of Φ in (M, g)) and $|A_{g, \Phi}^\circ|_{\bar{g}}^2 = \bar{g}^{ik} \bar{g}^{jl} (A_{g, \Phi}^\circ)_{ij} (A_{g, \Phi}^\circ)_{kl}$ is its norm with respect to the metric $\bar{g} = \Phi^*g$.

Now let (Φ_k) be a sequence of smooth area-constrained Willmore immersions of \mathbb{S}^2 into (M, g) satisfying (7)-(8); perform the rescaling procedure described above and obtain the immersions $(\tilde{\Phi}_k)$ of \mathbb{S}^2 into $(\mathbb{R}^3, g_{\varepsilon_k})$ (for simplicity denoted again with Φ_k from now on), where g_{ε_k} is defined in (9), satisfying (10). Since the Willmore functional is scale invariant, the rescaled surfaces are still area-constrained Willmore surfaces so they satisfy the following equation

$$(18) \quad \Delta_{\bar{g}_{\varepsilon_k}} H_{g_{\varepsilon_k}, \Phi_k} + H_{g_{\varepsilon_k}, \Phi_k} |A_{g_{\varepsilon_k}, \Phi_k}^\circ|_{\bar{g}_{\varepsilon_k}}^2 + H_{g_{\varepsilon_k}, \Phi_k} \text{Ric}_{g_{\varepsilon_k}}(\vec{n}_{g_{\varepsilon_k}, \Phi_k}, \vec{n}_{g_{\varepsilon_k}, \Phi_k}) = \lambda_k H_{g_{\varepsilon_k}, \Phi_k}.$$

The first step in our arguments is to show that the Lagrange multipliers λ_k are controlled by ε_k^2 . The idea for obtaining informations on the Lagrange multipliers, as in [15], is to use the invariance under rescaling of the Willmore functional.

Lemma 2.2. *Let (Φ_k) be a sequence of smooth area-constrained Willmore immersions of \mathbb{S}^2 into $(\mathbb{R}^3, g_{\varepsilon_k})$ where g_{ε_k} has the form (11) with $\varepsilon_k \rightarrow 0$, and $\Phi_k(\mathbb{S}^2) \subset B_{g_0}(0, 2)$, the euclidean ball of center 0 and radius 2.*

Then the Lagrange multipliers λ_k appearing in (18) satisfies:

$$(19) \quad \sup_{k \in \mathbb{N}} \frac{|\lambda_k|}{\varepsilon_k^2} < \infty.$$

Proof. Since (Φ_k) are area-constrained Willmore immersions, for every variation vector field \vec{X} on \mathbb{R}^3 we have that

$$(20) \quad \delta_{\vec{X}} W_{g_{\varepsilon_k}}(\Phi_k) = \lambda_k \delta_{\vec{X}} \text{Area}_{g_{\varepsilon_k}}(\Phi_k),$$

where $\delta_{\vec{X}} W$ and $\delta_{\vec{X}} \text{Area}$ are the first variations of the Willmore and the Area functionals corresponding to the vector field \vec{X} . Observe that the vector field corresponding to the dilations in \mathbb{R}^3 is the position vector field \vec{x} , so the first variation of the euclidean Willmore functional in \mathbb{R}^3 with respect to \vec{x} is null: $\delta_{\vec{x}} W_{g_0} = 0$; on the other hand the first variation of euclidean area with respect to the \vec{x} variation is easy to compute using the tangential divergence formula:

$$\delta_{\vec{x}} \text{Area}_{g_0}(\Phi) = -2 \int_{\mathbb{S}^2} \langle \vec{H}, \vec{x} \rangle_{g_0} d\text{vol}_{\bar{g}_0} = \int_{\mathbb{S}^2} \text{div}_{\Phi, g_0} \vec{x} d\text{vol}_{\bar{g}_0} = 2 \text{Area}_{g_0}(\Phi),$$

where div_{Φ, g_0} is the tangential divergence on $\Phi(\mathbb{S}^2)$ with respect to the euclidean metric. The two euclidean formulas give the well known fact that every area-constraint Willmore surface is actually a Willmore surface.

In the present framework, the ambient metric g_{ε_k} is a perturbation of order ε_k^2 of the euclidean metric g_0 , so it is natural to expect that the Lagrange multiplier maybe does not vanish but at least is of order ε_k^2 . Let us prove it. First of all, by the expansion of the Christoffel symbols (14) it follows that the covariant derivative in metric g_{ε_k} of the position vector field \vec{x} has the form

$$(21) \quad \nabla^{g_{\varepsilon_k}} \vec{x} = \text{Id} + O(\varepsilon_k^2).$$

It follows that the tangential divergence of \vec{x} on $\Phi_k(\mathbb{S}^2)$ with respect of the metric \bar{g}_k is $\text{div}_{\Phi, g_{\varepsilon_k}} \vec{x} = 2 + O(\varepsilon_k^2)$ and by the tangential divergence formula we obtain as before

$$\delta_{\vec{x}} \text{Area}_{g_{\varepsilon_k}}(\Phi) = -2 \int_{\mathbb{S}^2} \langle \vec{H}_{\Phi_k, g_{\varepsilon_k}}, \vec{x} \rangle_{g_{\varepsilon_k}} d\text{vol}_{\bar{g}_{\varepsilon_k}} = \int_{\mathbb{S}^2} \text{div}_{\Phi_k, g_{\varepsilon_k}} \vec{x} d\text{vol}_{\bar{g}_{\varepsilon_k}} = [2 + O(\varepsilon_k^2)] \text{Area}_{g_{\varepsilon_k}}(\Phi_k);$$

recalling the uniform area bound given in (16) we get that there exists $C > 0$ such that

$$(22) \quad 0 \leq \frac{1}{C} \leq \delta_{\vec{x}} \text{Area}_{g_{\varepsilon_k}}(\Phi) \leq C < \infty.$$

Now let us compute the variation of the Willmore functional with respect to the variation \vec{x} :

$$(23) \quad \delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = \int_{\mathbb{S}^2} \langle \vec{x}, \vec{n} \rangle_{g_{\varepsilon_k}} \left(\Delta_{\bar{g}_{\varepsilon_k}} H + H|A^\circ|^2 + H \text{Ric}(\vec{n}, \vec{n}) \right) d\text{vol}_{\bar{g}_{\varepsilon_k}}$$

where, of course, all the quantities are computed on Φ_k and with respect to the metric g_{ε_k} . In order to continue the computations, it is useful to rewrite the first variation of W in divergence form. Up to a reparametrization we can assume that Φ_k are conformal, so that the following identity holds (see Theorem 2.1 in [24])

$$(24) \quad \left[\Delta_{\bar{g}_{\varepsilon_k}} H \vec{n} + \vec{H}|A^\circ|^2 - R_{\Phi}^\perp(T\Phi) \right] d\text{vol}_{\bar{g}_{\varepsilon_k}} = D^* \left[\nabla H \vec{n} - \frac{H}{2} D\vec{n} + \frac{H}{2} \star_{g_{\varepsilon_k}} (\vec{n} \wedge D^\perp \vec{n}) \right]$$

where $\vec{H} = H\vec{n}$ is the mean curvature vector of the immersion Φ_k , $\star_{g_{\varepsilon_k}}$ is the Hodge operator associated to metric g_{ε_k} , $D \cdot := (\nabla_{\partial_{x_1}\Phi_k} \cdot, \nabla_{\partial_{x_2}\Phi_k} \cdot)$ and $D^\perp \cdot := (-\nabla_{\partial_{x_2}\Phi_k} \cdot, \nabla_{\partial_{x_1}\Phi_k} \cdot)$ and D^* is an operator acting on couples of vector fields (\vec{V}_1, \vec{V}_2) along $(\Phi_k)_*(T\mathbb{S}^2)$ defined as

$$D^*(\vec{V}_1, \vec{V}_2) := \nabla_{\partial_{x_1}\Phi_k} \vec{V}_1 + \nabla_{\partial_{x_2}\Phi_k} \vec{V}_2.$$

Finally $R_{\Phi_k}^\perp(T\Phi_k) := (\text{Riem}(\vec{e}_1, \vec{e}_2)\vec{H})^\perp = \star_{g_{\varepsilon_k}}(\vec{n} \wedge \text{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H})$, where $\vec{e}_i = \frac{\partial_{x_i}\Phi}{|\partial_{x_i}\Phi|}$ for $i = 1, 2$. Plugging (24) into (23) and integrating by parts we obtain

$$(25) \quad \begin{aligned} \delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) &= \int_{\mathbb{S}^2} \langle -D\vec{x}, \nabla H\vec{n} - \frac{H}{2}D\vec{n} + \frac{H}{2}\star_{g_{\varepsilon_k}}(\vec{n} \wedge D^\perp\vec{n}) \rangle_{g_{\varepsilon_k}} d\text{vol}_{\mathbb{S}^2} \\ &+ \int_{\mathbb{S}^2} \langle \vec{x}, R_{\Phi_k}^\perp(T\Phi_k) + \vec{H} \text{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} d\text{vol}_{\bar{g}_{\varepsilon_k}}. \end{aligned}$$

Since the Riemannian curvature tensor of the metric g_{ε_k} is of order $O(\varepsilon_k^2)$ and both the curvature terms are linear in H , using Schwartz inequality the integral in the second line can be estimated as

$$(26) \quad \int_{\mathbb{S}^2} \langle \vec{x}, R_{\Phi_k}^\perp(T\Phi_k) + \vec{H} \text{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} d\text{vol}_{\bar{g}_{\varepsilon_k}} = O(\varepsilon_k^2) \left(W_{g_{\varepsilon_k}}(\Phi_k) \text{Area}_{g_{\varepsilon_k}}(\Phi_k) \right)^{1/2} = O(\varepsilon_k^2).$$

The first line of the right hand side of (23) can be written explicitly as

$$(27) \quad \begin{aligned} &\int_{\mathbb{S}^2} \langle -\partial_{x^1}\Phi_k - \bar{\Gamma}_{\alpha\beta}^{g_{\varepsilon_k}}(\partial_{x^1}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^1}H)\vec{n} + \frac{H}{2}A_1^j(\partial_{x^j}\Phi_k) + \frac{H}{2}A_2^j\star_{g_{\varepsilon_k}}(\vec{n} \wedge \partial_{x^j}\Phi_k) \rangle_{g_{\varepsilon_k}} d\text{vol}_{\mathbb{S}^2} \\ &+ \int_{\mathbb{S}^2} \langle -\partial_{x^2}\Phi_k - \bar{\Gamma}_{\alpha\beta}^{g_{\varepsilon_k}}(\partial_{x^2}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^2}H)\vec{n} + \frac{H}{2}A_2^j(\partial_{x^j}\Phi_k) - \frac{H}{2}A_1^j\star_{g_{\varepsilon_k}}(\vec{n} \wedge \partial_{x^j}\Phi_k) \rangle_{g_{\varepsilon_k}} d\text{vol}_{\mathbb{S}^2}. \end{aligned}$$

Recalling that $\star_{g_{\varepsilon_k}}(\vec{n} \wedge \partial_{x^1}\Phi_k) = \partial_{x^2}\Phi_k$ and $\star_{g_{\varepsilon_k}}(\vec{n} \wedge \partial_{x^2}\Phi_k) = -\partial_{x^1}\Phi_k$ we obtain that all terms obtained doing the scalar product with $-\partial_{x^1}\Phi_k$ in the first line, and with $-\partial_{x^2}\Phi_k$ in the second line simplify and just the terms containing the Christoffel symbols remain; since $\Phi_k \subset B_{\gamma_{\varepsilon_k}}(0, 2)$ and the Christoffel symbols are of order $O(\varepsilon_k^2)$ by (14), (27) can be written as

$$(28) \quad \int_{\mathbb{S}^2} -\sum_{i=1}^2 \langle \bar{\Gamma}_{\alpha\beta}^{g_{\varepsilon_k}}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^i}H)\vec{n} \rangle d\text{vol}_{\mathbb{S}^2} + O(\varepsilon_k^2) \int_{\mathbb{S}^2} |H_{\Phi_k, g_{\varepsilon_k}}| |A_{\Phi_k, g_{\varepsilon_k}}| d\text{vol}_{\bar{g}_{\varepsilon_k}};$$

using Schwartz inequality of course the second summand can be bounded by

$$(29) \quad O(\varepsilon_k^2) \left(\int_{\mathbb{S}^2} |H_{\Phi_k, g_{\varepsilon_k}}|^2 d\text{vol}_{\bar{g}_{\varepsilon_k}} \right)^{1/2} \left(\int_{\mathbb{S}^2} |A_{\Phi_k, g_{\varepsilon_k}}|^2 d\text{vol}_{\bar{g}_{\varepsilon_k}} \right)^{1/2} = O(\varepsilon_k^2),$$

where we used the Gauss equations, Gauss-Bonnet Theorem and the area bound (16) to infer that

$$\int_{\mathbb{S}^2} |A_{\Phi_k, g_{\varepsilon_k}}|^2 d\text{vol}_{\bar{g}_{\varepsilon_k}} \leq C(W_{g_{\varepsilon_k}}(\Phi_k) + 1) \leq C_1.$$

In order to estimate the first integral of (28) we integrate by parts the derivative on H and we recall (14), obtaining

$$(30) \quad \begin{aligned} &\int_{\mathbb{S}^2} -\sum_{i=1}^2 \langle \bar{\Gamma}_{\alpha\beta}^{g_{\varepsilon_k}}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^i}H)\vec{n} \rangle d\text{vol}_{\mathbb{S}^2} = O(\varepsilon_k^2) \int_{\mathbb{S}^2} (|H_{\Phi_k, g_{\varepsilon_k}}| + |H_{\Phi_k, g_{\varepsilon_k}}| |A_{\Phi_k, g_{\varepsilon_k}}|) d\text{vol}_{\bar{g}_{\varepsilon_k}} \\ &= O(\varepsilon_k^2) \left(W_{g_{\varepsilon_k}}(\Phi_k) \right)^{1/2} \left[\left(\text{Area}_{g_{\varepsilon_k}}(\Phi_k) \right)^{1/2} + \left(\int_{\mathbb{S}^2} |A_{\Phi_k, g_{\varepsilon_k}}|^2 d\text{vol}_{\bar{g}_{\varepsilon_k}} \right)^{1/2} \right] = O(\varepsilon_k^2). \end{aligned}$$

Collecting (25), (26), (27), (28), (29) and (30) we obtain that

$$\delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = O(\varepsilon_k^2).$$

Combining the last equation with (22) and (20) we obtain that $\lambda_k = O(\varepsilon_k^2)$ as desired. \square

3 The blow up analysis and the proof of the main theorem

3.1 Existence of just one bubble and convergence

Lemma 3.1. *Let g_{ε_k} be the metrics on \mathbb{R}^3 defined in (9) having the expression (11) and let (Φ_k) be area-constrained Willmore immersions of \mathbb{S}^2 into $(\mathbb{R}^3, g_{\varepsilon_k})$ satisfying (10); without loss of generality we can assume Φ_k to be conformal with respect to the euclidean metric g_0 . Up to a rotation in the domain we can also assume that, for every $k \in \mathbb{N}$, the north pole $N \in \mathbb{S}^2$ is the maximum point of the quantity $|\nabla\Phi_k|^2 + |\nabla^2\Phi_k|$:*

$$\mu_k := |\nabla\Phi_k|_h^2(N) + |\nabla^2\Phi_k|_h(N) = \max_{\mathbb{S}^2} |\nabla\Phi_k|_h^2 + |\nabla^2\Phi_k|_h,$$

where h is the standard round metric of \mathbb{S}^2 of constant Gauss curvature equal to one and $|\nabla\Phi_k|_h, |\nabla^2\Phi_k|_h$ are the norms evaluated in the h metric.

Called $S \in \mathbb{S}^2$ the south pole and $P : \mathbb{S}^2 \setminus \{S\} \rightarrow \mathbb{R}^2$ the stereographic projection, consider the new parametrizations $\tilde{\Phi}_k$, in the sequel simply denoted with Φ_k , defined by

$$\tilde{\Phi}_k(P^{-1}(z)) := \Phi_k\left(P^{-1}\left(\frac{z}{\mu_k^{1/2}}\right)\right), \quad \forall z \in \mathbb{R}^2.$$

Then $\tilde{\Phi}_k$, a priori just defined on $\mathbb{S}^2 \setminus \{S\}$, extend to smooth conformal immersions of \mathbb{S}^2 into (\mathbb{R}^3, g_0) and converge to a conformal parametrization of a round sphere in $C^l(\mathbb{S}^2, h)$ -norm, for every $l \in \mathbb{N}$.

Proof. Step a: there exists a smooth conformal parametrization $\Phi_\infty : \mathbb{S}^2 \rightarrow (\mathbb{R}^3, g_0)$ of a round sphere in \mathbb{R}^3 endowed with the euclidean metric g_0 such that, up to subsequences, $\tilde{\Phi}_k \rightarrow \Phi_\infty$ in $C_{loc}^l(\mathbb{S}^2 \setminus \{S\})$ -norm, for every $l \in \mathbb{N}$.

Denote by u_k the conformal factor associated to $\tilde{\Phi}_k$, i.e.

$$\tilde{\Phi}_k^*(g_0) = e^{2u_k} h \quad ,$$

where g_0 is the euclidean metric in \mathbb{R}^3 . Observe that, by construction, for any compact subset of the form

$$K := \mathbb{S}^2 \setminus B_\delta^h(S) \quad \text{for some } \delta > 0 \quad ,$$

there holds

$$(31) \quad \sup_{k \in \mathbb{N}} \sup_K \left(|\nabla\tilde{\Phi}_k|_h^2 + |\nabla^2\tilde{\Phi}_k|_h \right) < \infty \quad .$$

Then, for every compact there exists a constant C_K depending just on K such that for every $x_0 \in K$ and every $\rho \in \left(0, \frac{\text{dist}(K, S)}{2}\right)$ it holds

$$\sup_{k \in \mathbb{N}} \sup_{B_\rho^h(x_0)} |\nabla^2\tilde{\Phi}_k|_h^2 \leq C_K \quad ,$$

where $B_\rho^h(x_0)$ is the ball of center x_0 and radius ρ in the metric h . By the conformal invariance of the Dirichlet energy, called $\pi_{\tilde{n}_k}$ the projection on the normal space to $\tilde{\Phi}_k$, we infer that for every $\varepsilon_0 > 0$ there exists $\rho_{\varepsilon_0, K} > 0$ (small enough) depending just on K and on ε_0 but not on $k \in \mathbb{N}$ such that for every $\rho \in (0, \rho_{\varepsilon_0, K})$ and $x_0 \in K$ it holds

$$(32) \quad \begin{aligned} \int_{B_\rho^h(x_0)} |\nabla\tilde{n}_k|_{\tilde{\Phi}_k^*(g_0)}^2 d\text{vol}_{\tilde{\Phi}_k^*(g_0)} &= \int_{B_\rho^h(x_0)} |\nabla\tilde{n}_k|_h^2 d\text{vol}_h = \int_{B_\rho^h(x_0)} |\pi_{\tilde{n}_k}(\nabla^2\tilde{\Phi}_k)|_h^2 d\text{vol}_h \\ &\leq \int_{B_\rho^h(x_0)} |\nabla^2\tilde{\Phi}_k|_h^2 d\text{vol}_h \leq C_K \rho^2 \leq \varepsilon_0 \quad . \end{aligned}$$

Taking $\varepsilon_0 \leq \frac{8\pi}{3}$, for any $x_0 \in K$ and $\rho < \rho_{\varepsilon_0, K}$ we can apply the Hélein moving frame method based on Chern construction of conformal coordinates (for more details see [31], Section 3) and infer that, up to a reparametrization of $\tilde{\Phi}_k$ on $B_\rho(x_0)$, called \bar{u}_k the mean value of u_k on $B_\rho^h(x_0)$, it holds

$$\|u_k - \bar{u}_k\|_{L^\infty(B_\rho^h(x_0))} \leq \tilde{C},$$

for some $\tilde{C} > 0$ independent of $k \in \mathbb{N}$. Covering K by finitely many balls as above, the connectedness of K implies that any two balls of the finite covering are connected by a chain of balls of the same covering and therefore there exists constants $c_{k,K} \in \mathbb{R}, k \in \mathbb{N}$, such that

$$(33) \quad \sup_{k \in \mathbb{N}} \|u_k - c_{k,K}\|_{L^\infty(K)} < \infty.$$

Observe that $\sup_{k \in \mathbb{N}} c_{k,K} < +\infty$; indeed, if $\limsup_k c_{k,K} = +\infty$ then $\limsup_k \text{Area}(\tilde{\Phi}_k(K)) = +\infty$ contradicting the area bound (16) (here we use that K has positive h -volume). Now let us consider separately the case $\sup_k |c_{k,K}| < \infty$ and $\liminf_k c_{k,K} = -\infty$ starting from the former.

Case 1: $\sup_k |c_{k,K}| < \infty$. Estimate (33) yields a uniform bound on the conformal factors u_k on the subset K . Since by assumption the immersions $\tilde{\Phi}_k$ are area-constrained Willmore immersions satisfying (32), then by ε -regularity (ε -regularity for Willmore immersions was first proved by Kuwert and Schätzle in [11]. Here we use the ε -regularity theorem proved by Rivière (see Theorem I.5 in [29]; see also Theorem I.1 in [1]); to this aim observe that the ε -regularity theorem was stated for *Willmore immersions*, but the proof can be repeated verbatim to *area-constrained Willmore immersions in metric g_{ε_k}* : indeed the Lagrange multiplier $\lambda \tilde{H}$ and the Riemannian terms are lower order terms that can be absorbed in the already present error terms \tilde{g}_1, \tilde{g}_2 in the proof of Theorem I.5 at pp. 24-26 in [29]. Of course ε -regularity is a consequence of the ellipticity of the equation.) we infer that for every $l \in \mathbb{N}$ there exists C_l such that

$$|e^{-l u_k} \nabla^l \tilde{\Phi}_k|_{L^\infty(B_{\rho/2}^h(x_0))} \leq C_l \left(\int_{B_\rho^h(x_0)} |\nabla \tilde{n}_k|_h^2 d\text{vol}_h + 1 \right)^{\frac{1}{2}} \leq \hat{C}_l$$

and therefore, by the assumed uniform bound on $|u_k|$ and by covering K by finitely many balls we get that

$$(34) \quad \sup_{k \in \mathbb{N}} |\nabla^l \tilde{\Phi}_k|_{L^\infty(K)} < \infty \quad \forall l \in \mathbb{N} \quad .$$

By Arzelá-Ascoli Theorem and by the estimate on the Lagrange multipliers given in Lemma 2.2, up to subsequences, the maps $\tilde{\Phi}_k$ converge in $C^l(K)$ norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\tilde{\Phi}_\infty$ of K into (\mathbb{R}^3, g_0) ; repeating the above argument to $K = \mathbb{S}^2 \setminus B_\delta^h(S)$, for every $\delta > 0$, we get that, up to subsequences, the maps $\tilde{\Phi}_k$ converge in $C_{loc}^l(\mathbb{S}^2 \setminus \{S\})$ norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\Phi_\infty : \mathbb{S}^2 \setminus \{S\} \rightarrow \mathbb{R}^3$ is a smooth Willmore conformal immersion with finite area and L^2 -bounded second fundamental form, therefore by Lemma A.5 in [30] (let us mention that this result was already present in [27]; see also [9]) the map Φ_∞ can be extended up to the south pole S to a possibly branched immersion; i.e. the south pole S is a possible branch point for Φ_∞ and the following expansion around S holds

$$(35) \quad (C - o(1))|z|^{n-1} \leq \left| \frac{\partial \Phi_\infty}{\partial z} \right| \leq (C + o(1))|z|^{n-1}$$

where z is a complex coordinate around the south pole and $n - 1$ is the branching order. We claim that the branching order is 0, or in other words that Φ_∞ is unbranched; indeed, by the strong convergence of $\tilde{\Phi}_k$ to Φ_∞ and the smooth convergence of g_{ε_k} to the euclidean metric g_0 we have that

$$(36) \quad W_{g_0}(\Phi_\infty) \leq \liminf_k W_{g_{\varepsilon_k}}(\tilde{\Phi}_k) < 8\pi;$$

therefore, by the Li-Yau inequality [18], we get that $n - 1 = 0$, i.e. Φ_∞ is an immersion also at the south pole S . Since Φ_∞ is a smooth Willmore immersion of \mathbb{S}^2 into \mathbb{R}^3 with energy less than 8π , by the classification of Willmore spheres by Bryant [3], Φ_∞ is a smooth conformal parametrization of a round sphere in \mathbb{R}^3 .

Case 2: $\liminf_k c_{k,K} = -\infty$, can not happen. In this case, up to subsequences, we have that $\tilde{\Phi}_k(K) \rightarrow \bar{x} \in M$ in Hausdorff distance sense. Consider then the rescaled immersions

$$(37) \quad \hat{\Phi}_k := e^{-c_{k,K}} \tilde{\Phi}_k$$

of K and observe that by construction $\sup_k |\hat{u}_{k,K}| < \infty$, where $\hat{u}_{k,K}$ is the conformal factor of $\hat{\Phi}_k$. Moreover, since the integrals appearing in (32) are invariant under rescaling, estimate (32) holds for $\hat{\Phi}_k$ as well. Therefore, up to a diagonal extraction, $\hat{\Phi}_k \rightarrow \Phi_\infty$ in $C^l_{loc}(\mathbb{S}^2 \setminus \{S\})$ -norm. In particular $\tilde{\Phi}_k \rightarrow 0$ in $C^2_{loc}(\mathbb{S}^2 \setminus \{S\})$ -norm, which contradicts the fact that

$$|\nabla \tilde{\Phi}_k|_h^2(N) + |\nabla^2 \tilde{\Phi}_k|_h(N) = 1.$$

Step b: $\tilde{\Phi}_k \rightarrow \Phi_\infty$ in $C^l(\mathbb{S}^2)$, for every $l \in \mathbb{N}$; namely the convergence of *Step a* is *on the whole* \mathbb{S}^2 . Observe that if there exists $\bar{\rho} > 0$ such that $\sup_k \sup_{B_{\bar{\rho}}^h(S)} |\nabla \tilde{\Phi}_k|^2 + |\nabla^2 \tilde{\Phi}_k| < \infty$, then in *Step a* we can choose as compact subset K the whole \mathbb{S}^2 and the claim of *Step b* follows by the same arguments of *Step a*. So assume by contradiction that there exists a sequence $\rho_k \downarrow 0$ such that, called

$$\bar{\mu}_k := \sup_{B_{\rho_k}^h(\bar{x})} |\nabla \tilde{\Phi}_k|^2 + |\nabla^2 \tilde{\Phi}_k|,$$

one has

$$\limsup_k \bar{\mu}_k = +\infty.$$

By a small rotation in the domain \mathbb{S}^2 we can assume that, for every $k \in \mathbb{N}$, the maximum of $|\nabla \tilde{\Phi}_k|^2 + |\nabla^2 \tilde{\Phi}_k|$ on $B_{\rho_k}^h(S)$ is attained at the south pole S and that, up to subsequences in k , it holds

$$(38) \quad \lim_k \bar{\mu}_k := \lim_k (|\nabla \tilde{\Phi}_k|^2(S) + |\nabla^2 \tilde{\Phi}_k|(S)) = +\infty.$$

Analogously to above, called $P_N : \mathbb{S}^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ the stereographic projection centered at the north pole N , we consider the reparametrized immersions

$$\bar{\Phi}_k(P_N^{-1}(z)) := \tilde{\Phi}_k \left(P_N^{-1} \left(\frac{z}{\bar{\mu}_k^{1/2}} \right) \right).$$

Observe that, in this way, the compact subsets K 's considered above are shrinking towards the north pole N and, by the arguments above, their $\bar{\Phi}_k$ -images are converging to a round sphere; repeating the arguments above to compact subsets this time containing the south pole S and avoiding the north pole N we infer that, up to subsequences, $\bar{\Phi}_k$ (or a further rescaled of it) converges smoothly, away the north pole N , to a round sphere; namely a second bubble. Combining the bubble formed in *Step a* and this second bubble, since each bubble contributes 4π of Willmore energy, we infer that

$$(39) \quad \limsup_k W_{g_{\epsilon_k}}(\Phi_k) \geq 8\pi \quad ,$$

contradicting the assumption (10). This concludes the proof of the *Step b* and of the lemma .

□

3.2 Expansion of the equation

Recalling that $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ is a smooth immersion satisfying the area-constrained Willmore equation in metric g_{ε_k} , and that g_{ε_k} smoothly converge to the euclidean metric g_0 , in the present section we expand this differential equation with respect to ε_k . Without loss of generality we can assume that Φ_k is conformal with respect to the metric g_{ε_k} . We will see that curvature terms appear at ε_k^2 order while the derivatives of the curvature appear at ε_k^3 order.

From now on, in order to make the notation a bit lighter, we replace ε_k by ε .

Recall that the area-constrained Willmore equation in metric g_ε has the following form

$$(40) \quad \Delta_{\bar{g}_\varepsilon} H_\varepsilon + H_\varepsilon |A_\varepsilon^\circ|_{\bar{g}_\varepsilon}^2 + \text{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) H_\varepsilon = \lambda_\varepsilon H_\varepsilon \quad .$$

Since $\Delta_{\bar{g}_\varepsilon} = \frac{2}{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2} \Delta$, where Δ is the flat laplacian in \mathbb{R}^2 , multiplying equation (40) by $\frac{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2}{2}$, we get

$$(41) \quad \Delta H_\varepsilon + \frac{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon |A_\varepsilon^\circ|_{\bar{g}_\varepsilon}^2 + \frac{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon \text{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) = \lambda_\varepsilon \frac{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon \quad .$$

First of all, recalling that $H_\varepsilon = \frac{g_\varepsilon(\Delta_{\bar{g}_\varepsilon} \Phi_\varepsilon, \vec{n}_\varepsilon)}{2}$, we expand H_ε as

$$(42) \quad H_\varepsilon = \frac{1}{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2} (g_\varepsilon)_{\alpha\beta} \Delta \Phi_\varepsilon^\alpha \sqrt{|g_\varepsilon|} g_\varepsilon^{\beta\gamma} (\vec{\nu}_\varepsilon)_\gamma = \frac{\sqrt{|g_\varepsilon|}}{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2} \Delta \Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}$$

where $\vec{\nu}_\varepsilon$ is the inward pointing unit normal with respect to g_0 . Using (11) and (13), we get

$$\begin{aligned} |\nabla \Phi_\varepsilon|_{g_\varepsilon}^2 &= |\nabla \Phi_\varepsilon|^2 + \frac{\varepsilon^2}{3} R_{\alpha\beta\gamma\eta}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle \\ &\quad + \frac{\varepsilon^3}{6} R_{\alpha\beta\gamma\eta,\mu}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \Phi_\varepsilon^\mu \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle + O(\varepsilon^4) \quad , \end{aligned}$$

so that

$$(43) \quad \begin{aligned} \frac{1}{|\nabla \Phi_\varepsilon|_{g_\varepsilon}^2} &= \frac{1}{|\nabla \Phi_\varepsilon|^2} \left(1 - \frac{\varepsilon^2}{3|\nabla \Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle \right. \\ &\quad \left. - \frac{\varepsilon^3}{6|\nabla \Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \Phi_\varepsilon^\mu \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle + O(\varepsilon^4) \right) \quad , \end{aligned}$$

moreover

$$(44) \quad \sqrt{|g_\varepsilon|} = 1 - \frac{\varepsilon^2}{6} \text{Ric}_{\alpha\beta}(p_k) \Phi_\varepsilon^\alpha \Phi_\varepsilon^\beta - \frac{\varepsilon^3}{6} \text{Ric}_{\alpha\beta,\gamma}(p_k) \Phi_\varepsilon^\alpha \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma + O(\varepsilon^4).$$

Combining (42) with (43) and (44) we can write

$$(45) \quad H_\varepsilon = \frac{\Delta \Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla \Phi_\varepsilon|^2} (1 + \varepsilon^2 S_\varepsilon + \varepsilon^3 T_\varepsilon + O(\varepsilon^4)) \quad ,$$

where

$$S_\varepsilon := -\frac{1}{3|\nabla \Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle - \frac{1}{6} \text{Ric}_{\alpha\beta}(p_k) \Phi_\varepsilon^\alpha \Phi_\varepsilon^\beta$$

and

$$T_\varepsilon := -\frac{1}{6|\nabla \Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k) \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma \Phi_\varepsilon^\mu \langle \nabla \Phi_\varepsilon^\alpha, \nabla \Phi_\varepsilon^\eta \rangle - \frac{1}{6} \text{Ric}_{\alpha\beta,\gamma}(p_k) \Phi_\varepsilon^\alpha \Phi_\varepsilon^\beta \Phi_\varepsilon^\gamma.$$

The combination of (44) and (45) gives

$$(46) \quad \text{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) H_\varepsilon = \varepsilon^2 \frac{\Delta \Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla \Phi_\varepsilon|^2} \text{Ric}_g(p_k)(\vec{\nu}_\varepsilon, \vec{\nu}_\varepsilon) + O(\varepsilon^4).$$

Finally, using (45), (46) and (19), we expand (41) up to ε^2 -order (the term $H_\varepsilon|A_\varepsilon^\circ|_{g_\varepsilon}^2$ will be expanded in the next subsection) as follows

$$(47) \quad \begin{aligned} & \Delta H_\varepsilon + \frac{|\nabla\Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon |A_\varepsilon^\circ|_{g_\varepsilon}^2 + \frac{|\nabla\Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon \text{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) - \lambda_\varepsilon H_\varepsilon \frac{|\nabla\Phi_\varepsilon|_{g_\varepsilon}^2}{2} = \\ & \Delta \left(\frac{\Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2} \right) + \varepsilon^2 \left(\Delta \left(\frac{\Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2} \right) S_\varepsilon + 2 \left\langle \nabla \left(\frac{\Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2} \right), \nabla S_\varepsilon \right\rangle + \frac{\Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2} \Delta S_\varepsilon \right) \\ & + \frac{|\nabla\Phi_\varepsilon|_{g_\varepsilon}^2}{2} H_\varepsilon |A_\varepsilon^\circ|_{g_\varepsilon}^2 + \frac{\varepsilon^2}{2} \Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha} \text{Ric}_g(p)(\vec{\nu}_\varepsilon, \vec{\nu}_\varepsilon) - \frac{\lambda_\varepsilon}{2} \Delta\Phi_\varepsilon^\alpha \vec{\nu}_{\varepsilon\alpha} + o(\varepsilon^2). \end{aligned}$$

3.3 Approximated solutions to the area-constrained Willmore equation

In this section we solve (47) up to the ε^2 order. For this let ω be the inverse of the stereographic projection with respect to the north pole and notice that ω is a solution of the equation when $\varepsilon = 0$. We make the ansatz of looking for a solution up to the order ε^2 of the form $\omega + \varepsilon^2\rho$, for some function ρ . Since $|A^\circ|^2 = 0$ for ω , it is clear that

$$(48) \quad H_\varepsilon |A_\varepsilon^\circ|_{g_\varepsilon}^2 = O(\varepsilon^4);$$

in particular, since for our arguments it is enough to expand the equation up to ε^3 order, this term will never play a role and therefore it will be neglected.

Observing that $\frac{\Delta\omega^\alpha\omega_\alpha}{|\nabla\omega|^2} \equiv -1$, equation (47) implies that ρ must solve

$$(49) \quad \begin{aligned} L_\omega(\rho) = & \Delta \left(\frac{1}{3|\nabla\omega|^2} R_{\alpha\beta\gamma\mu}(p_k)\omega^\beta\omega^\gamma \langle \nabla\omega^\alpha, \nabla\omega^\mu \rangle + \frac{1}{6} Ric_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta \right) \\ & - \frac{|\nabla\omega|^2}{2} Ric_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \frac{\lambda_\varepsilon}{2\varepsilon^2} |\nabla\omega|^2, \end{aligned}$$

where L_ω is the linearized Willmore operator at ω , see Appendix A for more details. Using the identity

$$(50) \quad \langle \nabla\omega^\alpha, \nabla\omega^\beta \rangle = (\delta_{\alpha\beta} - \omega^\alpha\omega^\beta) \frac{|\nabla\omega|^2}{2},$$

equation (49) reduces to

$$(51) \quad \begin{aligned} L_\omega(\rho) = & \frac{1}{3} \Delta (Ric_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta) - \frac{|\nabla\omega|^2}{2} Ric_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \frac{\lambda_\varepsilon}{2\varepsilon^2} |\nabla\omega|^2 \\ = & \left(-Ric_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \left(\frac{\lambda_\varepsilon}{2\varepsilon^2} + \frac{Scal(p_k)}{3} \right) \right) |\nabla\omega|^2. \end{aligned}$$

Hence, we easily check that

$$(52) \quad \rho_\varepsilon = \frac{1}{3} Ric_{\alpha\beta}(p_k)\omega^\beta + \frac{\lambda_\varepsilon}{\varepsilon^2} f(r)\omega$$

with

$$f(r) = \frac{r^2 \ln\left(\frac{r^2}{1+r^2}\right) - 1 - \ln(1+r^2)}{1+r^2}.$$

where $r^2 = x^2 + y^2$, is the desired function. Moreover it is not difficult to check that this perturbed ω satisfies the conformal conditions up to ε^2 order, that is to say

$$(53) \quad \begin{cases} g_\varepsilon((\omega + \varepsilon^2\rho_\varepsilon)_x, (\omega + \varepsilon^2\rho_\varepsilon)_x) - g_\varepsilon((\omega + \varepsilon^2\rho_\varepsilon)_y, (\omega + \varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3) \\ g_\varepsilon((\omega + \varepsilon^2\rho_\varepsilon)_x, (\omega + \varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3) \end{cases};$$

a way to prove it is to use the expansion of the metric with the fact that in dimension 3 one has

$$R_{\alpha\beta\gamma\mu} = (g_{\alpha\gamma} Ric_{\beta\mu} - g_{\alpha\mu} Ric_{\beta\gamma} + g_{\beta\mu} Ric_{\alpha\gamma} - g_{\beta\gamma} Ric_{\alpha\mu}) + \frac{Scal}{2} (g_{\alpha\mu} g_{\beta\gamma} - g_{\alpha\gamma} g_{\beta\mu}).$$

3.4 Proof of Theorem 1.2

Let us briefly recall the setting. Let $\Phi_k : \mathbb{S}^2 \hookrightarrow (M, g)$ be conformal Willmore immersions satisfying

$$(54) \quad \varepsilon := \text{diam}_g(\Phi_k(\mathbb{S}^2)) \rightarrow 0,$$

$$(55) \quad W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g, \Phi_k}|^2 d\text{vol}_{\bar{g}_k} \leq 8\pi - 2\delta, \quad \text{for some } \delta > 0 \text{ independent of } k.$$

Thanks to Lemma 2.2, we associate to Φ_k the new immersion $\Phi^\varepsilon : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_\varepsilon)$, where $g_\varepsilon(y)(u, v) := g(\varepsilon y)(\varepsilon^{-1}u, \varepsilon^{-1}v)$, which satisfies the area-constrained Willmore equation

$$(56) \quad \Delta_{\bar{g}_\varepsilon} H_{g_\varepsilon, \Phi^\varepsilon} + H_{g_\varepsilon, \Phi^\varepsilon} |A_{g_\varepsilon, \Phi^\varepsilon}^\circ|_{\bar{g}_\varepsilon}^2 + H_{g_\varepsilon, \Phi^\varepsilon} \text{Ric}_{g_\varepsilon}(\vec{n}_{g_\varepsilon, \Phi^\varepsilon}, \vec{n}_{g_\varepsilon, \Phi^\varepsilon}) = \lambda_\varepsilon H_{g_\varepsilon, \Phi^\varepsilon}$$

with $\lambda_\varepsilon = O(\varepsilon^2)$. Moreover by Lemma 3.1 we know that, up to conformal reparametrizations and up to subsequences, we have

$$\Phi^\varepsilon \rightarrow \Phi \text{ in } C^2(\mathbb{S}^2)$$

where Φ is a conformal diffeomorphism of \mathbb{S}^2 . Clearly, up to reparametrizing our sequence, we can assume that $\Phi = Id$. In the following we perform all the computations in the chart given by the stereographic projection (which is conformal); we denote by ω the inverse of the stereographic projection.

Before proceeding with the proof, we need to make a small adjustment to the immersions. We claim that there exist $a^\varepsilon \in \mathbb{R}^2$, $b^\varepsilon \in \mathbb{R}^2$, $R^\varepsilon \in SO(3)$ and $z^\varepsilon \in \mathbb{C}$ satisfying

$$(57) \quad a^\varepsilon = o(1), \quad b^\varepsilon = o(1), \quad |Id - R^\varepsilon| = o(1), \quad z^\varepsilon = o(1) \quad ,$$

such that, up to replacing Φ^ε by $\Phi^\varepsilon(a^\varepsilon + z^\varepsilon \cdot)$, and $\Omega^\varepsilon = \omega^\varepsilon + \varepsilon^2 \rho^\varepsilon$, where ρ^ε is given by (52), by $R^\varepsilon[\omega(\cdot + b^\varepsilon) + \varepsilon^2 \rho^\varepsilon(\cdot + b^\varepsilon)]$ we get

$$(58) \quad \begin{aligned} |\nabla \Phi^\varepsilon| \text{ and } |\nabla \Omega^\varepsilon| \text{ are maximal at } 0, \quad \text{Vect}\{\Phi_x^\varepsilon(0), \Phi_y^\varepsilon(0)\} &= \text{Vect}\{\Omega_x^\varepsilon(0), \Omega_y^\varepsilon(0)\}, \\ \text{and } \Phi_x^\varepsilon(0) &= \Omega_x^\varepsilon(0). \end{aligned}$$

This is a simple consequence of the $C_{loc}^2(\mathbb{R}^2)$ convergence of Φ^ε to ω . Indeed, we choose first a^ε and b^ε such that $|\nabla \Phi^\varepsilon|$ and $|\nabla \Omega^\varepsilon|$ are maximal at 0, then R^ε such that the tangent plane of Φ^ε and $R^\varepsilon \Omega^\varepsilon$ coincide at 0 and finally we find z^ε in order to adjust the first derivatives.

Therefore from now on we will assume that (58) is satisfied.

Now we prove Theorem 1.2. We set

$$\bar{\Phi}^\varepsilon = \Omega^\varepsilon + r^\varepsilon$$

for some function r^ε and, thanks to the computations of Section 3.3, we see that r^ε satisfies

$$(59) \quad L_\omega(r^\varepsilon) = O(\varepsilon^3) + o(|\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|).$$

Moreover, combining (53) and (58), we get that

$$(60) \quad g^\varepsilon(\nabla r^\varepsilon, \nabla r^\varepsilon)(0) = O(\varepsilon^6).$$

Indeed the error terms of $r_x^\varepsilon(0)$ and $r_y^\varepsilon(0)$ lie in the plane generated by $\Omega_x^\varepsilon(0)$ and $\Omega_y^\varepsilon(0)$. So it suffices to estimate their projection against $\Omega_x^\varepsilon(0)$ and $\Omega_y^\varepsilon(0)$. But this one vanish up to the ε^3 order thanks to (53). Observe that we also have

$$(61) \quad g^\varepsilon(\nabla^2 r^\varepsilon, \nabla \omega^\varepsilon)(0) = O(\varepsilon^3).$$

Claim : $\sup_{\mathbb{R}^2} |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon| = O(\varepsilon^3)$.

Proof of the Claim: let us denote $\mu_\varepsilon := |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|$ and assume by contradiction that $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^3}{\mu_\varepsilon} = 0$. Up to a reparametrization we can assume that this sup is achieved at some point z_ε which is confined in a fixed compact subset of \mathbb{R}^2 . In fact, we can do a reparametrization in order to make this requirement satisfied before performing the adjustments of the previous page. Then we set

$$\tilde{r}_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\mu_\varepsilon}.$$

By construction, \tilde{r}^ε is bounded in C^4 -norm on every compact subset of \mathbb{R}^2 and therefore, by Arzelà-Ascoli's Theorem, it converges up to subsequences to a limit function \tilde{r} in C_{loc}^3 -topology. Thanks to (59), \tilde{r} is a solution of the linearized equation (62) and, recalling (60)-(61), it satisfies (63) with $\nabla \tilde{r}(0) = 0$ and $\langle \nabla^2 \tilde{r}, \nabla \omega \rangle(0) = 0$. Then, applying Lemma A.1, we get that $\nabla \tilde{r} \equiv 0$, which is in contradiction with the fact that $|\nabla \tilde{r}| + |\nabla^2 \tilde{r}| + |\nabla^3 \tilde{r}| + |\nabla^4 \tilde{r}| = 1$ at some point at finite distance. This proves the claim. \square

Mimicking the proof of the claim above, one can prove that setting

$$\tilde{r}_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\varepsilon^3},$$

then, up to subsequences, \tilde{r}_ε converges to a function \tilde{r} in $C_{loc}^3(\mathbb{R}^2)$ which, using (41), (45) and (46), satisfies the linearized Willmore equation

$$L_\omega(\tilde{r}) = \Delta \left(\frac{1}{6|\nabla \omega|^2} R_{\alpha\beta\gamma\mu, \nu}(p_k) \omega^\beta \omega^\gamma \omega^\nu \langle \nabla \omega^\alpha, \nabla \omega^\mu \rangle + \frac{1}{6} \text{Ric}_{\alpha\beta, \gamma}(p_k) \omega^\alpha \omega^\beta \omega^\gamma \right).$$

Recalling identity (50), the last equation can be rewritten as follows

$$L_\omega(\tilde{r}) = \Delta \left(\frac{1}{12} \text{Ric}_{\alpha\beta, \gamma}(p_k) \omega^\alpha \omega^\beta \omega^\gamma \right).$$

Finally, integrating this relation against the ω^α , for $\alpha = 1, \dots, 3$, which are solutions of the linearized equation, we get

$$\int_{\mathbb{R}^2} \Delta \omega \left(\frac{1}{12} \text{Ric}_{\alpha\beta, \gamma}(p_k) \omega^\alpha \omega^\beta \omega^\gamma \right) dz = 0.$$

Let us note that the integration by parts above has been possible thanks to the decay of ω and its derivatives at infinity. The last identity gives

$$\int_{\mathbb{R}^2} (\text{Ric}_{\alpha\beta, \gamma}(p_k) \omega^\alpha \omega^\beta \omega^\gamma) \omega \frac{|\nabla \omega|^2}{2} dz = 0.$$

Then, by a change of variable, we get

$$\int_{\mathbb{S}^2} (\text{Ric}_{\alpha\beta, \gamma}(p_k)(p_k) y^\alpha y^\beta y^\gamma) y dvol_h = 0,$$

where h is the standard metric on \mathbb{S}^2 and y^α are the position coordinates of \mathbb{S}^2 in \mathbb{R}^3 . Finally, using the following relation

$$\int_{\mathbb{S}^2} y^\alpha y^\beta y^\gamma y^\mu dvol_h = \frac{4\pi}{15} (\delta^{\alpha\beta} \delta^{\mu\gamma} + \delta^{\alpha\mu} \delta^{\beta\gamma} + \delta^{\alpha\gamma} \delta^{\beta\mu}),$$

and the second Bianchi identity, we obtain

$$\nabla \text{Scal}(\bar{p}) = 0,$$

Which proves the theorem. \square

A The linearized Willmore operator

The aim of this appendix is to derive the linearized Willmore equation and to classify its solution. The Willmore equation for a conformal immersion Φ into \mathbb{R}^3 can be written as

$$W'(\Phi) = \Delta_{\bar{g}}(H) + H|A^\circ|_{\bar{g}}^2 = 0,$$

where $\Delta_{\bar{g}} = \frac{2}{|\nabla\Phi|^2}\Delta$, H is the mean curvature and A° is the trace-less second fundamental form. Equivalently one has

$$H = \frac{1}{2}\langle\Delta_{\bar{g}}\Phi, \vec{\nu}\rangle$$

where $\vec{\nu}$ is the inward pointing unit normal of the immersion Φ . Hence, by multiplying the first equation by $\frac{|\nabla\Phi|^2}{2}$, we can consider the equivalent equation

$$\widetilde{W}'(\Phi) = \Delta H + \langle\Delta\Phi, \vec{\nu}\rangle \frac{|A^\circ|_{\bar{g}}^2}{2} = 0.$$

Of course any conformal parametrization, ω , of a round sphere is a solution. Then, expanding $\widetilde{W}'(\omega + t\rho)$ for some function ρ and using the fact that $A^\circ \equiv 0$ for a round sphere, we get

$$(62) \quad L_\omega(\rho) := \delta\widetilde{W}_\omega(\rho) = -\Delta \left(\frac{\langle\Delta\rho, \omega\rangle + 2\langle\nabla\omega, \nabla\rho\rangle}{|\nabla\omega|^2} \right) = 0.$$

Let also consider the linearization of the conformality condition, which gives

$$(63) \quad \begin{cases} \langle\omega_x, \rho_x\rangle - \langle\omega_y, \rho_y\rangle = 0 \\ \langle\omega_x, \rho_y\rangle + \langle\omega_y, \rho_x\rangle = 0 \end{cases}$$

In the following lemma we classify the solutions of the linearized operator following the previous work [17] of the first author concerning the linearized operator for the constant mean curvature equation.

Lemma A.1. *Let $\rho \in \mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)^2$ be a solution of the linearized equation (62) which satisfies (63) and the additional normalizing conditions*

$$\nabla\rho(0) = 0 \quad \text{and} \quad \langle\nabla^2\rho, \nabla\omega\rangle(0) = 0 \quad .$$

Then $\nabla\rho \equiv 0$.

Proof. First we remark that thanks to the definition of $\mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$, we have

$$\frac{\langle\Delta\rho, \omega\rangle + 2\langle\nabla\omega, \nabla\rho\rangle}{|\nabla\omega|^2} \in L^2(\mathbb{R}^2).$$

Hence, using Liouville's theorem, we get that

$$(64) \quad \langle\Delta\rho, \omega\rangle + 2\langle\nabla\omega, \nabla\rho\rangle = 0.$$

Then, thanks to the fact that $(\omega_x, \omega_y, \omega)$ is a basis of \mathbb{R}^3 and (63), there exists $a, b, c, d : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$(65) \quad \begin{cases} \rho_x = a\omega_x + b\omega_y + c\omega \\ \rho_y = -b\omega_x + a\omega_y + d\omega \end{cases} .$$

²the pushforward of $H^2(S^2)$ on \mathbb{R}^2 via the stereographic projection.

Then, plugging (65) into (64) and using the relation $\rho_{xy} = \rho_{yx}$, we see that a, b, c, d satisfy the following equations

$$\begin{aligned} (66) \quad & a_y + b_x = d \\ (67) \quad & b_y - a_x = -c \\ & c_y - d_x = b|\nabla\omega|^2 \\ & c_x + d_y = -a|\nabla\omega|^2 \quad . \end{aligned}$$

These equations imply that a and b satisfy

$$\Delta a = -a|\nabla\omega|^2, \quad \Delta b = -b|\nabla\omega|^2.$$

Since $\rho \in \dot{H}^1(\mathbb{R}^2, \mathbb{R}^3)$, then a and b can be seen as functions in $H^1(S^2)$ satisfying $\Delta\alpha = 2\alpha$, therefore a and b are linear combinations of the first non vanishing eigenfunctions of Δ_{S^2} (see also Lemma C.1 of [17]), that is to say

$$a = \sum_{i=0}^2 a_i \psi_i \text{ and } b = \sum_{i=0}^2 b_i \psi_i$$

where

$$\psi_i(x) = \frac{x_i}{(1+|x|^2)} \text{ for } i = 1, 2 \text{ and } \psi_0(x) = \frac{1-|x|^2}{1+|x|^2}.$$

Finally using the fact that $\nabla\rho(0) = 0$ and $\langle \nabla^2\rho, \nabla\omega \rangle(0) = 0$, (66) and (67), we can conclude that $a \equiv b \equiv c \equiv d \equiv 0$, which proves the lemma. \square

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