A NOTE ON A RESIDUAL SUBSET OF LIPSCHITZ FUNCTIONS ON METRIC SPACES

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Abstract. Let (X, d) be a quasi-convex, complete and separable metric space with reference probability measure m. We prove that the set of of real valued Lipschitz function with non zero point-wise Lipschitz constant m-almost everywhere is residual, and hence dense, in the Banach space of Lipschitz and bounded functions. The result is the metric analogous of a result proved for real valued Lipschitz maps defined on $\mathbb{R}^2$ by Alberti, Bianchini and Crippa in [1].

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1. Introduction

In the context of metric spaces, say (X, d), it is possible to look at the point-wise variation of a real valued map considering

$$\mathrm{Lip}\, f(x) := \limsup_{y\to x, y\neq x} \frac{|f(x) - f(y)|}{d(x, y)}, \tag{1.1}$$

that is called *point-wise Lipschitz constant.* In the smooth framework $\mathrm{Lip}\, f$ corresponds to the modulus of ∇f: if (X, d) is an open subset of $\mathbb{R}^d$ endowed with the euclidean norm and f is locally Lipschitz, then $\mathrm{Lip}\, f = |\nabla f|$ almost everywhere with respect to the Lebesgue measure. Or more in general if (X, d, m) is a metric measure space admitting a differentiable structure in the sense of Cheeger, see [5], [6] for the definitions, and f is Lipschitz, then $\mathrm{Lip}\, f = |df|$ m-a.e. where df is the Cheeger's differential of f.

Once a point-wise information is given we are interested at looking at those points where the "differential" vanishes: define the singular set of f as follows

$$S(f) := \{x \in X : \mathrm{Lip}\, f(x) = 0\}.$$

The classical Sard's Theorem states that if $f : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth then the Lebesgue measure of $f(S(f))$ is 0. As soon as the regularity assumption on f is dropped, the conclusion of Sard's Theorem does not hold anymore and one may look for weaker properties to hold.

The question, inspired by a similar problem in [4], Section 6, is if it is possible to approximate any Lipschitz function with functions having negligible $S(f)$ with respect to a given reference measure.

For real valued Lipschitz functions defined on $\mathbb{R}^2$, with Lebesgue measure playing the role of the reference measure, a positive answer is contained in [1], see Proposition 4.10. We prove the following.

Theorem 1.1. *Assume (X, d) is a quasi-convex, complete and separable metric space and let m be a Borel probability measure over it. The set of those $f \in D^\infty(X)$ so that $m(S(f)) = 0$ is residual, and therefore dense, in $D^\infty(X)$.*

The Banach space $D^\infty(X)$ will be the space of bounded functions with bounded point-wise Lipschitz constant, endowed with the uniform norm. See below for a precise definition. Recall that a set in a topological space is residual if it contains a countable intersection of open dense set. By Baire Theorem, a residual set in a complete metric space is dense.

2. Setting

Let (X, d) be a metric space and m is a Borel probability measure over X so that X coincides with its support. For $f : X \to \mathbb{R}$ the *Lipschitz constant* of f is defined as usual by

$$\mathrm{LIP}(f) := \sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

and we say that f is Lipschitz if $\mathrm{LIP}(f)$ is a finite number. Accordingly denote by $\mathrm{LIP}^\infty(X)$ the space of bounded Lipschitz functions. The natural norm on $\mathrm{LIP}^\infty(X)$ is given by

$$\|f\|_{\mathrm{LIP}^\infty(X)} = \|f\|_\infty + \mathrm{LIP}(f),$$

where $\|\cdot\|_\infty$ is the uniform norm. The space of bounded Lipschitz functions endowed with $\|f\|_{\mathrm{LIP}^\infty(X)}$ turns out to be a Banach space. The point-wise version of $\mathrm{LIP}(f)$ is given by the point-wise Lipschitz constant as defined in 1.1. The corresponding space of bounded functions with bounded point-wise Lipschitz constant can be considered:

$$D^\infty(X) := \{f : X \to \mathbb{R} : \|f\|_\infty + \|\mathrm{Lip}\, f\|_\infty < \infty\}.$$

A study of $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ can be found in [3]. The following results are taken from [3].

It is straightforward to note that $\mathrm{LIP}^\infty(X) \subset D^\infty(X)$ and for a general metric space this is the only valid inclusion. Examples of metric spaces and functions in $D^\infty(X)$ not satisfying a global Lipschitz bound can be constructed, see [3]. If (X, d) is quasi-convex also the other inclusion holds and $\mathrm{LIP}^\infty(X) = D^\infty(X)$ and the two semi-norms are comparable: there exists $C \geq 1$ so that

$$\|\mathrm{Lip}\, f\|_\infty \leq \mathrm{LIP}(f) \leq C\|\mathrm{Lip}\, f\|_\infty.$$

Hence $D^\infty(X)$, or equivalently $\mathrm{LIP}^\infty(X)$, endowed with the norm $\|\cdot\|_\infty + \|\mathrm{Lip}\,(\cdot)\|_\infty$ is a Banach space. We will denote this norm with $\|\cdot\|_{D^\infty}(X)$.

Recall that a metric space (X, d) is quasi-convex if there exists a constant $C \geq 1$ such that for each pair of points $x, y \in X$ there exists a curve γ connecting the two points such that $l(\gamma) \leq Cd(x, y)$, where $l(\gamma)$ denotes the length of γ defined with the usual "affine" approximation: for $\gamma : [a, b] \to X$ its length $l(\gamma)$ is defined by

$$l(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(x_i, x_{i+1}) : a = x_1 < x_2 < \cdots < x_{n+1} = b, n \in \mathbb{N} \right\}.$$

Associated to the length $l(\gamma)$ there is the distance obtained minimizing it:

$$d_L(x, y) = \inf\{l(\gamma) : \gamma_0 = x, \gamma_1 = y\}.$$

The function d_L is indeed a distance on each component of accessibility by rectifiable paths, i. e. those paths having finite l. By quasi-convexity it follows that

$$d(x, y) \leq d_L(x, y) \leq Cd(x, y),$$

with $C > 1$. Hence (X, d_L) is a complete and separable metric space that is also a length space. Clearly (X, d_L) has the same open sets of (X, d). For a more detailed discussion on length spaces see [2].

We will use the following notation. For $r > 0$ and $z \in X$, we will denote with $B_r(z)$ the ball of radius r centered in z. The complement in X of a set A will be denoted by A^c and ∂A denotes the topological boundary of A. The closure of A is $cl(A)$ and the interior part $int(A)$. Associated to a set we can consider the distance from it: for $x \in X$ and $A \subset X$

$$d(x, A) := \inf_{w \in A} d(x, w).$$

3. The Result

Lemma 3.1. *For any Borel function $f : X \to \mathbb{R}$, the function $\mathrm{Lip}\, f : X \to \bar{\mathbb{R}}$ is universally measurable.*

Proof. In order to prove the claim we just have to show that the set $\{x \in X : \mathrm{Lip}\, f(x) \geq a\}$ is Souslin for any $a \in \mathbb{R}$. Since f is a Borel map then

$$\bigcap_{n \in \mathbb{N}} \left\{ (x, y) \in X \times X : 0 < d(x, y) \leq \frac{1}{n},\ \frac{|f(x) - f(y)|}{d(x, y)} \geq a \right\}$$

is a Borel set. Note that

$$\{x \in X : \mathrm{Lip}\, f(x) \geq a\} = P_1 \left(\bigcap_{n \in \mathbb{N}} \left\{ (x, y) \in X \times X : 0 < d(x, y) \leq \frac{1}{n},\ \frac{|f(x) - f(y)|}{d(x, y)} \geq a \right\} \right),$$

where $P_1 : X \times X \to X$ denotes the projection on the first element. It follows from the definition of Souslin set that $\{x \in X : \mathrm{Lip}\, f(x) \geq a\}$ is Souslin and the claim follows. □

Then after Lemma 3.1 it makes sense to look at those functions f so that $m(S(f)) = 0$. We will need the following

Lemma 3.2. *Let $K \subset X$ be a closed set and consider the length distance function from K that is $g(x) := d_L(x, K)$. Then*

$$1 \leq \mathrm{Lip}\, g(x) \leq C, \qquad \textit{for } x \in K^c,$$

Proof. Step 1. Assume that $d = d_L$ so that (X, d) is also a length space and $g = d(x, K)$. Then fix $x \in K^c$: for any $z \in K$ and $y \in K^c$ it holds

$$d(x, z) - d(y, z) \leq d(x, y)$$

hence trivially $\mathrm{Lip}\, g(x) \leq 1$.

Consider now a minimizing sequence $z_n \in K$ for x, that is that $g(x) \geq d(x, z_n) - 1/n$. From the length structure it follows that for any n there exists $\gamma^n : [0, 1] \to X$ rectifiable curve starting in x and arriving in z_n so that $d(x, z_n) \geq l(\gamma^n) - 1/n$. So for any y_n in the image of γ^n

$$\frac{g(x) - g(y_n)}{d(x, y_n)} \geq \frac{l(\gamma_n) - d(y_n, z_n) - 2/n}{d(x, y_n)}.$$

Since $l(\gamma^n) \geq d(x, y_n) + d(y_n, z_n)$ it follows that

$$\frac{g(x) - g(y_n)}{d(x, y_n)} \geq \frac{d(x, y_n) - 2/n}{d(x, y_n)}.$$

Since the only constrain on y_n was to belong to the image of γ^n, we can choose y_n so that the previous ratio converges to 1. Hence $\mathrm{Lip}\, g(x) = 1$.

Step 2. We now drop the assumption on the length structure of the space. Let (X, d) be quasi-convex and $g(x) = d_L(x, K)$. Since (X, d_L) is a length space for any $x \in K^c$

$$\limsup_{y \to x, y \neq x} \frac{|g(x) - g(y)|}{d_L(x, y)} = 1.$$

Having (X, d_L) and (X, d) the same open set, K^c does not depend on the metric. Since $d \leq d_L \leq Cd$ the claim follows. □

We can now prove Theorem 1.1. The proof uses now the ideas contained in Proposition 4.10 in [1].

Theorem 3.3. *Assume (X, d) is a quasi-convex, complete and separable space and let m be a Borel probability measure over it. The set of those $f \in D^\infty(X)$ so that $m(S(f)) = 0$ is residual in $D^\infty(X)$ and therefore dense.*

Proof. Consider the following sets

$$G := \{f \in D^\infty(X) : m(S(f)) = 0\}, \qquad G_r := \{f \in D^\infty(X) : m(S(f)) < r\}.$$

The claim is then to prove that G is a residual set. Since $G = \cap G_r$, where the intersection runs over a sequence of r converging to 0, the claim is proved once it is proved that each G_r is open and dense in $D^\infty(X)$.

Step 1. The set G_r is open in $D^\infty(X)$. Fix $f \in G_r$. Then there exists $\delta > 0$ so that

$$m\left(\{x \in X : \mathrm{Lip}\, f(x) \leq \delta\}\right) < r.$$

Since for any $g \in D^\infty(X)$ it holds that

$$\mathrm{Lip}\, f(x) \leq \mathrm{Lip}\, g(x) + \mathrm{Lip}\, (f - g)(x),$$

for any $g \in D^\infty(X)$ so that $\|g - f\|_{D^\infty}(X) \leq \delta$ it holds that

$$S(g) \subset \{x \in X : \mathrm{Lip}\, f(x) \leq \delta\},$$

and therefore $m(S(g)) < r$ and consequently $g \in G_r$.

Step 2. The set G_r is dense in $D^\infty(X)$. Given $f \in D^\infty(X)$ and $\delta > 0$ we have to find $g \in G_r$ so that $\|f - g\|_{D^\infty(X)} \leq \delta$. Without loss of generality we can assume $m(S(f)) \geq r$.

For every $\varepsilon > 0$ denote with $S(f)^\varepsilon$ the ε-neighborhood of the set of singular points of f, i.e.

$$S(f)^\varepsilon = \{z \in X : d(z, S(f)) < \varepsilon\}.$$

The set $S(f)^\varepsilon$ is open and denote by K its complementary in X. Associated to K we consider the distance function $\hat{g}$ as defined in Lemma 3.2 that is $\hat{g}(x) := d_L(x, K)$. A rough bound on $\hat{g}(x)$ can be given in terms of the "diameter" of $S(f)$:

$$\hat{g}(x) \leq C \sup\{d(x,z) : cl(S(f)^\varepsilon)\},$$

where $cl(S(f)^\varepsilon)$ stands for the closure of $S(f)^\varepsilon$. Since to approximate with functions in G_r we can make an error in measure strictly less than r and since m is a probability measure, we can assume $S(f)$ to have finite diameter and by inner regularity we can even assume it to be closed. Therefore

$$\|\hat{g}\|_\infty \leq M, \qquad M > 0.$$

From Lemma 3.2 we have $\mathrm{Lip}\, \hat{g}(x) > 0$ for $x \in S(f)^\varepsilon$ and clearly $\mathrm{Lip}\, \hat{g}(x) = 0$ for $x \in int(K)$, where $int(K)$ stands for the interior part of K.

Note that the boundary of $S(f)^\varepsilon$ is contained in the set $\{z : d(z, S(f)) = \varepsilon\}$. Indeed $z \in \partial S(f)^\varepsilon$ if and only if $d(z, S(f)) \geq \varepsilon$ and for every $\eta > 0$ there exists a point $w \in X$ so that

$$d(z,w) \leq \eta, \qquad d(w, S(f)) < \varepsilon.$$

Let η_n be a sequence converging to 0 and w_n the corresponding sequence converging to z. To each w_n associate $x_n \in S(f)$ so that $d(w_n, x_n) < \varepsilon$. Then

$$d(z, x_n) \leq d(z, w_n) + d(w_n, x_n) < \eta_n + \varepsilon.$$

Passing to the limit $d(z, S(f)) \leq \varepsilon$ and therefore necessarily $d(z, S(f)) = \varepsilon$.

Moreover for $\varepsilon \neq \varepsilon'$

$$\{z : d(z, S(f)) = \varepsilon\} \cap \{z : d(z, S(f)) = \varepsilon'\} = \emptyset,$$

hence there exists at most countably many ε so that $m(\{z : d(z, S(f)) = \varepsilon\}) > 0$. Hence for any $r > 0$ there exists $\varepsilon > 0$ so that

$$m(\{z : d(z, S(f)) = \varepsilon\}) = 0, \qquad m(S(f)^\varepsilon \setminus S(f)) < r,$$

where the second expression holds because $S(f)$ is closed. From what said so far, denoting $g := f + (\delta/2M)\hat{g}$ is such that

$$\|f - g\|_{D^\infty(X)} \leq \delta.$$

To conclude the proof observe that $S(g) \subset S(f)^\varepsilon \setminus S(f)$, hence by construction $g \in G_r$. □

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