

MONGE PROBLEM IN METRIC MEASURE SPACES WITH RIEMANNIAN CURVATURE-DIMENSION CONDITION

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ABSTRACT. We prove the existence of solutions for the Monge minimization problem, addressed in a metric measure space (X, d, m) enjoying the Riemannian curvature-dimension condition $\text{RCD}^*(K, N)$, with $N < \infty$. For the first marginal measure, we assume that $\mu_0 \ll m$. As a corollary, we obtain that the Monge problem and its relaxed version, the Monge-Kantorovich problem, attain the same minimal value.

Moreover we prove a structure theorem for d -cyclically monotone sets: neglecting a set of zero m -measure they do not contain any branching structures, that is, they can be written as the disjoint union of the image of a disjoint family of geodesics.

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1. INTRODUCTION

Let (X, d, m) be a metric measure space verifying the Riemannian curvature dimension condition $\text{RCD}^*(K, N)$ for $K, N \in \mathbb{R}$ with $N \geq 1$. In this note we prove the existence of a solution for the following Monge problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$, the space of Borel probability measures over X , solve the following minimization problem

$$(1.1) \quad \inf_{T_{\#}\mu_0 = \mu_1} \int_X d(x, T(x)) \mu_0(dx),$$

provided $\mu_0 \ll m$. In more detail, the minimization of the functional runs over the set of μ_0 -measurable maps $T : X \rightarrow X$ such that $T_{\#}\mu_0 = \mu_1$, that is

$$\mu_0(T^{-1}(A)) = \mu_1(A), \quad \forall A \in \mathcal{B}(X),$$

where $\mathcal{B}(X)$ denotes the σ -algebra of all Borel subsets of X .

On the way to the proof of the existence of an optimal map, we will also prove a structure theorem for branching structures inside d -cyclically monotone sets. Before giving the statements of the two main results of this note and an account on the strategies to prove them, we recall some of the (extensive) literature on the Monge minimization problem.

The first formulation for (1.1) (Monge in 1781) was addressed in \mathbb{R}^n with the cost given by the Euclidean norm and the measures $\mu_0, \mu_1 \ll \mathcal{L}^n$ were supposed to be supported on two disjoint compact sets. The original problem remained unsolved for a long time. In 1978 Sudakov in [28] proposed a solution for any distance cost induced by a norm, but an argument about disintegration of measures contained

in his proof was not correct, see [21] for details. Then the Euclidean case was correctly solved by Evans and Gangbo in [16], under the assumptions that $\text{spt } \mu_0 \cap \text{spt } \mu_1 = \emptyset$, $\mu_0, \mu_1 \ll \mathcal{L}^n$ and their densities are Lipschitz functions with compact support. After that, many results reduced the assumptions on the supports of μ_0, μ_1 , see [7] and [29]. The result on manifolds with geodesic cost is obtained in [17]. The case of a general norm as cost function on \mathbb{R}^n has been solved first in the particular case of crystalline norms in [3], and then in full generality independently by L. Caravenna in [8] and by T. Champion and L. De Pascale in [14].

The study of the geodesic metric space framework started with [6], where the metric space was assumed to be also non-branching. There the existence of solutions to (1.1) was obtained for metric spaces verifying the measure-contraction property $\text{MCP}(K, N)$ (for instance the Heisenberg group). An application of the results of [6] to the Wiener space can be found in [9]. Then in [10] the problem was studied removing the non-branching assumption but obtaining existence of solutions only in a particular case.

Non-branching metric measure spaces enjoying $\text{CD}^*(K, N)$ also verify $\text{MCP}(K, N)$, see [13]. Then from [6] the Monge problem is solved also in that case. So with respect to the most general known case, we impose a stronger curvature information (namely $\text{RCD}^*(K, N)$) and we remove the non-branching assumption.

1.1. The results. The nowadays classical strategy to show existence of optimal maps is to relax the integral functional to the larger class of transport plans

$$\Pi(\mu_0, \mu_1) := \{\pi \in \mathcal{P}(X \times X) : (P_1)_\# \pi = \mu_0, (P_2)_\# \pi = \mu_1\},$$

over where the functional we want to minimize has now the following expression

$$\int d(x, y) \eta(dx dy).$$

For $i = 1, 2$, $P_i : X \times X \rightarrow X$ denotes the projection map on the i -th component. Assuming that the functional is finite at least on one element of $\Pi(\mu_0, \mu_1)$, we have the existence of $\eta_{opt} \in \Pi(\mu_0, \mu_1)$ so that

$$\int d(x, y) \eta_{opt}(dx dy) = \inf_{\eta \in \Pi(\mu_0, \mu_1)} \int_X d(x, y) \eta(dx),$$

by linearity in η and tightness of $\Pi(\mu_0, \mu_1)$. Then the central question, whose positive answer would prove existence of a solution to Monge problem, is whether η_{opt} is supported on the graph of a m -measurable map $T : X \rightarrow X$.

A property of η_{opt} inside $\Pi(\mu_0, \mu_1)$ is the fact that is concentrated on a d -cyclically monotone set. We shall build an optimal map starting from this monotonicity. But while the Riemannian curvature-dimension condition $\text{RCD}^*(K, N)$ gives crucial information on d^2 -cyclically monotone sets (neglecting a set of measure zero, they are the graph of a measurable map, see Section 2 and references therein), nothing is known under this curvature assumption on the structure of d -cyclically monotone sets. In particular what we would like to exclude is the presence of branching structures. Note that the first result proving absence of branching geodesics assuming a curvature condition, in that case strong $\text{CD}(K, \infty)$, is contained in [24]. The same type of result, but only for L^2 -Wasserstein geodesics with end point a Dirac delta, was already present in an earlier work of Rajala, see [23].

The strategy we will follow is: prove that d -cyclically monotone sets do not have branching structures m -almost everywhere; then use the approach with Disintegration Theorem (see for instance [6] and references therein) to reduce the Monge problem to a family of 1-dimensional Monge problem. There one can apply the 1-dimensional theory. Thanks to the curvature assumption we can prove a suitable property for the first marginal measures and obtain the existence of the 1-dimensional optimal maps, one for each 1-dimensional Monge problem. Then gluing together all the one-dimensional optimal maps, one gets an optimal map $T : X \rightarrow X$ solving the Monge problem (1.1). A more precise program on the use of Disintegration Theorem in the Monge problem will be given in Section 3.

We conclude this introductory part stating the two main results we will prove. The first is about the structure of the d -cyclically monotone set associated to a Kantorovich potential φ^d for the problem (1.1).

Theorem 1.1. *Let (X, d, m) be a metric measure space verifying $\text{RCD}^*(K, N)$ for some $K, N \in \mathbb{R}$, with $N \geq 1$. Let moreover Γ be a d -cyclically monotone set as (3.1) and let \mathcal{T}_e be the set of all points moved*

by Γ as in Definition 3.2. Then there exists $\mathcal{T} \subset \mathcal{T}_e$ that we call the transport set such that

$$m(\mathcal{T}_e \setminus \mathcal{T}) = 0,$$

and for all $x \in \mathcal{T}$, the transport ray $R(x)$ is formed by a single geodesic and for $x \neq y$, both in \mathcal{T} , either $R(x) = R(y)$ or $R(x) \cap R(y)$ is contained in the set of initial points $a \cup b$ as defined in Definition 3.2.

All the terminology used in Theorem 1.1 will be introduced in Section 3. Taking advantage of Theorem 1.1 we then obtain the following

Theorem 1.2. *Let (X, d, m) be a metric measure space verifying $\text{RCD}^*(K, N)$ for $N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $W_1(\mu_0, \mu_1) < \infty$ and $\mu_0 \ll m$. Then there exists a Borel map $T : X \rightarrow X$ such that $T_{\#}\mu_0 = \mu_1$ and*

$$\int_X d(x, T(x))\mu_0(dx) = \int_{X \times X} d(x, y)\eta_{\text{opt}}(dxdy).$$

In the previous theorem, W_1 denotes the L^1 -Wasserstein distance on the space of probability measures on (X, d) .

A straightforward corollary of Theorem 1.2 is that the relaxation to the set of transference plan $\Pi(\mu_0, \mu_1)$ does not lower the value of the minimum:

$$\begin{aligned} \inf_{T_{\#}\mu_0=\mu_1} \int_X d(x, T(x))\mu_0(dx) &\leq \int d(x, T(x))\mu_0(dx) \\ &= \min_{\eta \in \Pi(\mu_0, \mu_1)} \int_X d(x, y)\eta(dx) \\ &\leq \inf_{T_{\#}\mu_0=\mu_1} \int_X d(x, T(x))\mu_0(dx). \end{aligned}$$

Hence

$$(1.2) \quad \min_{T_{\#}\mu_0=\mu_1} \int_X d(x, T(x))\mu_0(dx) = \min_{\eta \in \Pi(\mu_0, \mu_1)} \int_X d(x, y)\eta(dx).$$

As it will be clear from their proofs, the results contained in Theorem 1.1 and Theorem 1.2 can be obtained omitting the $\text{RCD}^*(K, N)$ condition and assuming instead the metric measure space to satisfy the strong $\text{CD}^*(K, N)$ condition. Even if strong $\text{CD}^*(K, N)$ is a more general condition than $\text{RCD}^*(K, N)$, the latter is stable with respect to measured Gromov-Hausdorff convergence. Hence we have decided in its favor to state and prove the results contained in this note.

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2. $\text{RCD}^*(K, N)$ SPACES

Here we briefly give some references for $\text{RCD}^*(K, N)$ and state some of the main properties of metric measure spaces verifying it.

Few notations: we will denote with $\text{Geo}(X) \subset C([0, 1], X)$ the space of geodesics endowed with uniform topology and for a Borel set $F \subset X \times X$, we will often use the notation $F(x)$ for $P_2(F \cap \{x\} \times X)$.

For $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, the space of probability measures with finite second moment, we consider the following set of optimal geodesics:

$$\text{OptGeo}(\mu_0, \mu_1) := \left\{ \nu \in \mathcal{P}(\text{Geo}(X)) : \int d(x, y)^2(e_s, e_t)_{\#}\nu = (t-s)^2 W_2^2(\mu_0, \mu_1), 0 \leq s \leq t \leq 1 \right\},$$

where W_2 is the L^2 -Wasserstein distance. We write $\nu \in \text{OptGeo}$, if $\nu \in \text{OptGeo}(e_0_{\#}\nu, e_1_{\#}\nu)$.

Sturm and independently Lott and Villani, using the L^2 -Wasserstein space, introduced a class of metric measure space verifying a generalized curvature condition called curvature-dimension condition $\text{CD}(K, N)$, with $K, N \in \mathbb{R}$ and $N \geq 2$. The condition models a lower bound on Ricci curvature and an upper bound on the dimension. See [26, 27] and [22] for the precise definitions. Then a variant called reduced curvature-dimension condition, denoted with $\text{CD}^*(K, N)$, has been introduced in [5].

The Riemannian Curvature Dimension condition $\text{RCD}(K, \infty)$, has been introduced by L. Ambrosio, N. Gigli and G. Savaré in [2], as the reinforcement of $\text{CD}(K, \infty)$ with the assumption of infinitesimally Hilbertian structure, see [1].

The finite dimensional case has been studied in [18, 19] and the precise $\text{RCD}^*(K, N)$ has been given in [15] and [4] with two different approaches. We refer to these fundamental papers for the precise definitions. Here we will make use of some property enjoyed by this class of spaces.

The following theorem is taken from [24].

Theorem 2.1. *Let (X, d, m) be a $\text{RCD}(K, \infty)$ space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be two measures absolutely continuous w.r.t. m . Then there exists a unique $\nu \in \text{OptGeo}(\mu_0, \mu_1)$ and this plan is induced by a map and is concentrated on a set of non-branching geodesics.*

For $\nu \in \text{OptGeo}(\mu_0, \mu_1)$ to be concentrated on a set of non-branching geodesic means that for any $t \in [0, 1]$ the evaluation map e_t restricted to $\text{supp}(\pi)$ is invertible, that is there exists a Borel map $(e_t)^{-1} : \text{supp}(\mu_t) \rightarrow \text{Geo}(X)$ such that

$$\nu = ((e_t)^{-1})_{\#} (e_t)_{\#} \nu,$$

for every $t \in [0, 1]$.

Theorem 2.1 has been used in [20] to prove the following localization result for entropy inequality of $\text{RCD}^*(K, N)$ spaces.

Proposition 2.2. *Let (X, d, m) be an $\text{RCD}^*(K, N)$ space and $\mu_i = \varrho_i m \in \mathcal{P}_2(X)$, $i = 0, 1$ be two given measures. Let $\nu \in \text{OptGeo}(\mu_0, \mu_1)$ be the unique optimal geodesic plan of Theorem 2.1. If $\mu_t = (e_t)_{\#} \pi$, then $\mu_t \ll m$ for all $t \in [0, 1]$ and if $\mu_t = \varrho_t m$ then*

$$\varrho_r(\gamma_r)^{-\frac{1}{N}} \geq \varrho_s(\gamma_s)^{-\frac{1}{N}} \sigma_{K, N}^{\left(\frac{t-r}{t-s}\right)}(d(\gamma_s, \gamma_t)) + \varrho_t(\gamma_t)^{-\frac{1}{N}} \sigma_{K, N}^{\left(\frac{r-s}{t-s}\right)}(d(\gamma_s, \gamma_t)), \quad \nu - a.e. \gamma,$$

for all $0 \leq s \leq r \leq t \leq 1$.

Again in [20] it is proven that if $N < \infty$ and the first marginal is absolutely continuous with respect to m , then there exists a unique optimal plan and it is concentrated on the graph of a Borel function. The optimal plan is also induced by an element of $\pi \in \text{OptGeo}$ concentrated on a set of non-branching geodesics. Note that all these results are for geodesics in the L^2 -Wasserstein space, while the object of our investigation are d -cyclically monotone sets, usually having lower ‘‘regularity’’ than d^2 -cyclically monotone sets. Finally note that from $\text{RCD}^*(K, N)$ it follows that $(X, d, m) = (\text{supp}(m), d, m)$, and the metric space (X, d) is geodesic and proper (provided $N < \infty$).

From now on we will assume (X, d, m) to verify $\text{RCD}^*(K, N)$ for some $K, N \in \mathbb{R}$ with $N \geq 2$.

3. d -GEODESICS AND d^2 -GEODESICS

To avoid the trivial case we can assume that the two marginal measures have finite L^1 -Wasserstein distance, $W_1(\mu_0, \mu_1) < \infty$. Consequently we infer the existence of $\eta \in \Pi(\mu_0, \mu_1)$, such that

$$\int_{X \times X} d(x, y) \eta(dxdy) = \inf \left\{ \int_{X \times X} d(x, y) \pi(dxdy) : \pi \in \Pi(\mu_0, \mu_1) \right\} = W_1(\mu_0, \mu_1),$$

where $\Pi(\mu_0, \mu_1)$ is the set of transport plans,

$$\Pi(\mu_0, \mu_1) := \{ \pi \in \mathcal{P}(X \times X) : (P_1)_{\#} \pi = \mu_0, (P_2)_{\#} \pi = \mu_1 \}.$$

The set of optimal transport plans, i.e. plans realizing the previous identity, will be denoted with $\Pi_{\text{opt}}(\mu_0, \mu_1)$. Since the cost is finite, we can also assume the existence of a Kantorovich potential, that is a 1-Lipschitz function $\varphi^d : X \rightarrow \mathbb{R}$, such that

$$\eta \in \Pi_{\text{opt}}(\mu_0, \mu_1) \iff \eta(\{(x, y) \in X \times X : \varphi^d(x) - \varphi^d(y) = d(x, y)\}) = 1.$$

We also use the notation:

$$(3.1) \quad \Gamma := \{(x, y) \in X \times X : \varphi^d(x) - \varphi^d(y) = d(x, y)\}.$$

Almost by definition, the set Γ is a d -cyclically monotone set.

The following is a standard fact of d -cyclically monotone sets.

Lemma 3.1. *Let $(x, y) \in X \times X$ be an element of Γ . Let $\gamma \in \text{Geo}(X)$ be such that $\gamma_0 = x$ and $\gamma_1 = y$. Then*

$$(\gamma_s, \gamma_t) \in \Gamma,$$

for all $0 \leq s \leq t \leq 1$.

Proof. Take $0 \leq s \leq t \leq 1$ and note that

$$\begin{aligned} & \varphi^d(\gamma_s) - \varphi^d(\gamma_t) \\ &= \varphi^d(\gamma_s) - \varphi^d(\gamma_t) + \varphi^d(\gamma_0) - \varphi^d(\gamma_0) + \varphi^d(\gamma_1) - \varphi^d(\gamma_1) \\ &\geq d(\gamma_0, \gamma_1) - d(\gamma_0, \gamma_s) - d(\gamma_t, \gamma_1) \\ &= d(\gamma_s, \gamma_t). \end{aligned}$$

The claim follows. \square

It is therefore natural to consider the set of geodesics $G \subset \text{Geo}(X)$ such that

$$\gamma \in G \iff \{(\gamma_s, \gamma_t) : 0 \leq s \leq t \leq 1\} \subset \Gamma,$$

that is $G := \{\gamma \in \text{Geo}(X) : (\gamma_0, \gamma_1) \in \Gamma\}$.

We now recall some definitions, already given in [6], that will be needed to describe the structure of Γ .

Definition 3.2. We define the set of *transport rays* by

$$R = \Gamma \cup \Gamma^{-1},$$

where $\Gamma^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma\}$. The set of *initial points* and *final points* respectively by

$$\begin{aligned} a &:= \{z \in X : \exists x \in X, (x, z) \in \Gamma, d(x, z) > 0\}, \\ b &:= \{z \in X : \exists x \in X, (z, x) \in \Gamma, d(x, z) > 0\}. \end{aligned}$$

The set of *end points* is $a \cup b$. We also define the *transport set with end points*:

$$\mathcal{T}_e = P_1(\Gamma \setminus \{x = y\}) \cup P_1(\Gamma^{-1} \setminus \{x = y\}).$$

where $\{x = y\}$ stands for $\{(x, y) \in X^2 : d(x, y) = 0\}$.

Remark 3.3. Here we discuss the measurability of the sets introduced in Definition 3.2. Since φ^d is 1-Lipschitz, Γ is closed and therefore Γ^{-1} and R are closed as well. Moreover thanks to curvature assumption the space is proper, hence the sets Γ, Γ^{-1}, R are σ -compact.

Then we look at the set of initial and final points:

$$a = P_2(\Gamma \cap \{(x, z) \in X \times X : d(x, z) > 0\})^c, \quad b = P_1(\Gamma \cap \{(x, z) \in X \times X : d(x, z) > 0\})^c.$$

Since $\{(x, z) \in X \times X : d(x, z) > 0\} = \cup_n \{(x, z) \in X \times X : d(x, z) \geq 1/n\}$, it follows that both a and b are the complement of σ -compact sets. Hence a and b are Borel sets. Reasoning as before, it follows that \mathcal{T}_e is a σ -compact set.

The next lemma permits to reduce the analysis of the existence of solutions of the Monge problem on the whole X to the same problem restricted to the transport set with end points.

Lemma 3.4. *Let $\eta \in \Pi_{opt}(\mu_0, \mu_1)$, then*

$$\eta(\mathcal{T}_e \times \mathcal{T}_e \cup \{x = y\}) = 1.$$

Proof. It is enough to observe that if $(z, w) \in \Gamma$ with $z \neq w$, then $w \in \Gamma(z)$ and $z \in \Gamma^{-1}(w)$ and therefore

$$(z, w) \in \mathcal{T}_e \times \mathcal{T}_e.$$

Hence $\Gamma \setminus \{x = y\} \subset \mathcal{T}_e \times \mathcal{T}_e$. Since $\eta(\Gamma) = 1$, the claim follows. \square

As a consequence, $\mu_0(\mathcal{T}_e) = \mu_1(\mathcal{T}_e)$ and any optimal map T such that $T_{\#}\mu_0 \llcorner \mathcal{T}_e = \mu_1 \llcorner \mathcal{T}_e$ can be extended to an optimal map T' with $T'_{\#}\mu_0 = \mu_1$ with the same cost by setting

$$(3.2) \quad T'(x) = \begin{cases} T(x), & \text{if } x \in \mathcal{T}_e \\ x, & \text{if } x \notin \mathcal{T}_e. \end{cases}$$

Using the terminology introduced so far, we explain the strategy we will follow to prove existence of an optimal map: first we need to find a suitable subset \mathcal{T} of \mathcal{T}_e called the *transport set*, with $m(\mathcal{T}_e \setminus \mathcal{T}) = 0$, where no branching occurs. Then

- (1) prove that for every $x \in \mathcal{T}$ there exists only one unparametrized geodesic passing through x and contained in \mathcal{T}_e ;
- (2) reduce the L^1 optimal transport problem to a 1-dimensional L^1 optimal transport problem along each unparametrized geodesic;
- (3) prove regularity (i.e. absence of atoms) of conditional probability for the 1-dimensional L^1 optimal transport problem.

Once these three points have been accomplished, one obtains the existence of an optimal map for each 1-dimensional L^1 optimal transport problem, by considering for instance the unique monotone rearrangement between the two 1-dimensional measures. Then gluing all the 1-dimensional optimal maps, one obtains a global optimal map.

We recall the one dimensional result for the Monge problem, Theorem 2.18 of [30], that will be used as a building block.

Theorem 3.5. *Let μ_0, μ_1 be probability measures on \mathbb{R} , μ_0 with no atoms, and let*

$$H(s) := \mu((-\infty, s)), \quad F(t) := \nu((-\infty, t)),$$

be the left-continuous distribution functions of μ_0 and μ_1 respectively. Then the following holds.

- (1) *The non decreasing function $T : \mathbb{R} \rightarrow \mathbb{R} \cup [-\infty, +\infty)$ defined by*

$$T(s) := \sup \{t \in \mathbb{R} : F(t) \leq H(s)\}$$

maps μ_0 to μ_1 . Moreover any other non decreasing map T' such that $T'_\# \mu_0 = \mu_1$ coincides with T on the support of μ_0 up to a countable set.

- (2) *If $\phi : [0, +\infty] \rightarrow \mathbb{R}$ is non decreasing and convex, then T is an optimal transport relative to the cost $c(s, t) = \phi(|s - t|)$. Moreover T is the unique optimal transference map if ϕ is strictly convex.*

4. THE TRANSPORT SET

We now prove that the set of transport rays R is an equivalence relation on a subset of \mathcal{T}_e . In order to do so, we study the branching geodesics in Γ . The presence of branching structures inside Γ can be modeled by the existence of $x, z, w \in \mathcal{T}_e$ such that

$$(x, z), (x, w) \in \Gamma, \quad (z, w) \notin R.$$

Actually the previous condition only describes branching in the direction given by Γ . Branching in the direction of Γ^{-1} will be treated analogously.

In the next Lemma, using Lemma 3.1, we prove that, once a branching happens, there exist two distinct geodesics, both contained in $\Gamma(x)$, that are not in relation in the sense of R . Recall that $G \subset \text{Geo}(X)$ is the set of geodesics γ such that

$$(\gamma_s, \gamma_t) \in \Gamma, \quad 0 \leq s \leq t \leq 1.$$

Lemma 4.1. *Let $x \in \mathcal{T}_e$ and $z, w \in \mathcal{T}_e$ be such that $z, w \in \Gamma(x)$ and $(z, w) \notin R$. Then there exist two distinct geodesics $\gamma^1, \gamma^2 \in \text{Geo}(X)$ such that*

- $\gamma^1, \gamma^2 \in G$;
- $(x, \gamma_s^1), (x, \gamma_s^2) \in \Gamma$ for all $s \in [0, 1]$;
- $(\gamma_s^1, \gamma_s^2) \notin R$ for all $s \in [0, 1]$;
- $\varphi^d(\gamma_s^1) = \varphi^d(\gamma_s^2)$ for all $s \in [0, 1]$.

Moreover both geodesics are non-constant.

Proof. Since $z, w \in \Gamma(x)$, from Lemma 3.1 there exist two geodesics $\gamma^1, \gamma^2 \in G$ such that

$$\gamma_0^1 = \gamma_0^2 = x, \quad \gamma_1^1 = z, \quad \gamma_1^2 = w.$$

Since $(z, w) \notin R$, necessarily both z and w are different from x and x is not a final point, that is $x \notin b$. So the previous geodesics are not constant. Since z and w can be exchanged, we can also assume that $\varphi^d(z) \geq \varphi^d(w)$. Since $z \in \Gamma(x)$, $\varphi^d(x) \geq \varphi^d(z)$ and by continuity there exists $s_2 \in (0, 1]$ such that

$$\varphi^d(z) = \varphi^d(\gamma_{s_2}^2).$$

Note that $z \neq \gamma_{s_2}^2$, otherwise $w \in \Gamma(z)$ and therefore $(z, w) \in R$. Moreover still $(z, \gamma_{s_2}^2) \notin R$. Indeed if the contrary was true, then

$$0 = |\varphi^d(z) - \varphi^d(\gamma_{s_2}^2)| = d(z, \gamma_{s_2}^2),$$

that is a contradiction with $z \neq \gamma_{s_2}^2$.

So by continuity there exists $\delta > 0$ such that

$$\varphi^d(\gamma_{1-s}^1) = \varphi^d(\gamma_{s_2(1-s)}^2), \quad d(\gamma_{1-s}^1, \gamma_{s_2-s}^2) > 0,$$

for all $0 \leq s \leq \delta$.

Hence reapplying the previous argument $(\gamma_{1-s}^1, \gamma_{s_2(1-s)}^2) \notin R$. The curve γ^1 and γ^2 of the claim are then obtained properly restricting and rescaling the geodesic γ^1 and γ^2 considered so far. \square

There is a measurable correspondence between points of branching and couples of geodesics. To prove it we need the following selection result, Theorem 5.5.2 of [25], page 198.

Theorem 4.2. *Let X and Y be Polish spaces, $F \subset X \times Y$ analytic, and \mathcal{A} the σ -algebra generated by the analytic subsets of X . Then there is an \mathcal{A} -measurable section $u : P_1(F) \rightarrow Y$ of F .*

Recall that given $F \subset X \times Y$, a section u of F is a function from $P_1(F)$ to Y such that $\text{graph}(u) \subset F$.

Lemma 4.3. *Consider the set of possible branching points defined as*

$$A_+ := \{x \in \mathcal{T}_e : \exists z, w \in \Gamma(x), (z, w) \notin R\}.$$

Then there exists an m -measurable map $u : A_+ \mapsto G \times G$ such that if $u(x) = (\gamma^1, \gamma^2)$ then

- $(x, \gamma_s^1), (x, \gamma_s^2) \in \Gamma$ for all $s \in [0, 1]$;
- $(\gamma_s^1, \gamma_s^2) \notin R$ for all $s \in [0, 1]$;
- $\varphi^d(\gamma_s^1) = \varphi^d(\gamma_s^2)$ for all $s \in [0, 1]$.

Moreover both geodesics are non-constant.

Proof. Since $G = \{\gamma \in \text{Geo}(X) : (\gamma_0, \gamma_1) \in \Gamma\}$, and that $\Gamma \subset X \times X$ is closed, the set G is a complete and separable metric space. Consider now the set

$$\begin{aligned} F := & \{(x, \gamma^1, \gamma^2) \in \mathcal{T}_e \times G \times G : (x, \gamma_0^1), (x, \gamma_0^2) \in \Gamma\} \\ & \cap (X \times \{(\gamma^1, \gamma^2) \in G \times G : d(\gamma_1^1, \gamma_1^2) > 0\}) \\ & \cap (X \times \{(\gamma^1, \gamma^2) \in G \times G : d(\gamma_0^1, \gamma_0^2) > 0\}) \\ & \cap (X \times \{(\gamma^1, \gamma^2) \in G \times G : d(\gamma_0^1, \gamma_1^1) > 0\}) \\ & \cap (X \times \{(\gamma^1, \gamma^2) \in G \times G : \varphi^d(\gamma_i^1) = \varphi^d(\gamma_i^2), i = 0, 1\}). \end{aligned}$$

It follows from Remark 3.3 that F is σ -compact. To avoid possible intersections in interior points of γ^1 with γ^2 we consider the following map:

$$\begin{aligned} h : G \times G & \rightarrow [0, \infty) \\ (\gamma^1, \gamma^2) & \mapsto h(\gamma^1, \gamma^2) := \min_{s \in [0, 1]} d(\gamma_s^1, \gamma_s^2). \end{aligned}$$

From compactness of $[0, 1]$, we deduce the continuity of h . Therefore

$$\hat{F} := F \cap \{(x, \gamma^1, \gamma^2) \in X \times G \times G : h(\gamma^1, \gamma^2) > 0\}$$

is a Borel set and from Lemma 4.1,

$$\hat{F} \cap (\{x\} \times G \times G) \neq \emptyset$$

for all $x \in A_+$. By Theorem 4.2 we infer the existence of an \mathcal{A} -measurable selection u of \hat{F} . Then $A_+ = P_1(\hat{F})$ and in particular u is m -measurable. Moreover if $u(x) = (\gamma^1, \gamma^2)$, then

$$d(\gamma_s^1, \gamma_s^2) > 0, \quad \varphi^d(\gamma_s^1) = \varphi^d(\gamma_s^2),$$

for all $s \in [0, 1]$, and therefore $(\gamma_s^1, \gamma_s^2) \notin R$ for all $s \in [0, 1]$. The claim follows. \square

Note that if we assume (X, d, m) to satisfy $\text{RCD}^*(K, N)$, in the proof of Lemma 4.3 we have also shown that A_+ is a σ -compact set. Indeed since (X, d) will be proper, than any open set is σ -compact and therefore also \hat{F} σ -compact set and consequently the same holds for A_+ .

We recall here the crucial construction, already introduced in [11], that permits to apply the known results on the structure of d^2 -cyclically monotone sets to d -cyclically monotone one.

Lemma 4.4. *Let $\Delta \subset \Gamma$ be any set so that:*

$$(x_0, y_0), (x_1, y_1) \in \Delta \quad \Rightarrow \quad (\varphi^d(y_1) - \varphi^d(y_0)) \cdot (\varphi^d(x_1) - \varphi^d(x_0)) \geq 0.$$

Then Δ is d^2 -cyclically monotone.

Proof. It follows directly from the hypothesis of the Lemma that the set

$$\{(\varphi^d(x), \varphi^d(y)) : (x, y) \in \Delta\} \subset \mathbb{R} \times \mathbb{R}$$

is $|\cdot|^2$ -cyclically monotone, where $|\cdot|$ denotes the modulus. Then for $\{(x_i, y_i)\}_{i \leq N} \subset \Delta$, since $\Delta \subset \Gamma$, it holds

$$\begin{aligned} \sum_{i=1}^N d^2(x_i, y_i) &= \sum_{i=1}^N |\varphi^d(x_i) - \varphi^d(y_i)|^2 \\ &\leq \sum_{i=1}^N |\varphi^d(x_i) - \varphi^d(y_{i+1})|^2 \\ &\leq \sum_{i=1}^N d^2(x_i, y_{i+1}), \end{aligned}$$

where the last inequality is given by the 1-Lipschitz regularity of φ^d . The claim follows. \square

The first consequence of Lemma 4.4 is the following

Proposition 4.5. *Let (X, d, m) satisfy $\text{RCD}^*(K, N)$. Then the set*

$$A_+ := \{x \in \mathcal{T}_e : \exists z, w \in \Gamma(x), (z, w) \notin R\}$$

has m -measure zero.

Proof. Step 1. Suppose by contradiction that $m(A_+) > 0$. By definition of A_+ , thanks to Lemma 4.1 and Lemma 4.3, for every $x \in A_+$ there exist two non-constant geodesics $\gamma^1, \gamma^2 \in \text{Geo}(X)$ such that

- $\gamma^1, \gamma^2 \in G$;
- $(x, \gamma_s^1), (x, \gamma_s^2) \in \Gamma$ for all $s \in [0, 1]$;
- $(\gamma_s^1, \gamma_s^2) \notin R$ for all $s \in [0, 1]$;
- $\varphi^d(\gamma_s^1) = \varphi^d(\gamma_s^2)$ for all $s \in [0, 1]$.

Moreover the map $A_+ \ni x \mapsto u(x) := (\gamma^1, \gamma^2) \in G^2$ is m -measurable.

By inner regularity of compact sets, we can assume that the previous map is continuous and in particular the functions

$$A_+ \ni x \mapsto \varphi^d(\gamma_j^i) \in \mathbb{R}, \quad i = 1, 2, j = 0, 1$$

are all continuous. Put $\alpha_x = \varphi^d(\gamma_0^1)$ and $\beta_x = \varphi^d(\gamma_1^1)$ and note that $\alpha_x > \beta_x$. Now we want to show the existence of a subset $B \subset A_+$, still with $m(B) > 0$, such that

$$\sup_{x \in B} \beta_x < \inf_{x \in B} \alpha_x.$$

By continuity of α and β , a set B verifying the previous inequality can be obtained considering the set $A_+ \cap B_r(x)$, for $x \in A_+$ and for r sufficiently small. Since $m(A_+) > 0$, for m -a.e. $x \in A_+$ the set $A_+ \cap B_r(x)$ has positive m -measure. So the existence of $B \subset A_+$ enjoying the aforementioned properties follows.

Step 2. Let $I = [c, d]$ be a non trivial interval such that

$$\sup_{x \in B} \beta_x < c < d < \inf_{x \in B} \alpha_x.$$

Then by construction for all $x \in B$ the image of the composition of the geodesics γ^1 and γ^2 with φ^d contains the interval I :

$$I \subset \{\varphi^d(\gamma_s^i) : s \in [0, 1]\}, \quad i = 1, 2.$$

Then fix any point inside I , say c . For any $x \in B$ there exists $s(x)$ such that

$$\varphi^d(\gamma_{s(x)}^1) = \varphi^d(\gamma_{s(x)}^2) = c.$$

We can now define on B two transport maps T^1 and T^2 by

$$B \ni x \mapsto T^i(x) := \gamma_{s(x)}^i, \quad i = 1, 2.$$

Accordingly we define the transport plan

$$\eta := \frac{1}{2} ((Id, T^1)_\# m_B + (Id, T^2)_\# m_B),$$

where $m_B := m(B)^{-1} m_{\perp B}$.

Step 3. The support of η is d^2 -cyclically monotone. To prove it we will use Lemma 4.4. The measure η is concentrated on the set

$$\Delta := \{(x, \gamma_{s(x)}^1) : x \in B\} \cup \{(x, \gamma_{s(x)}^2) : x \in B\} \subset \Gamma.$$

Take any two couples $(x_0, y_0), (x_1, y_1) \in \Delta$ and by definition:

$$\varphi^d(y_1) - \varphi^d(y_0) = 0,$$

and therefore trivially $(\varphi^d(y_1) - \varphi^d(y_0))(\varphi^d(x_1) - \varphi^d(x_0)) = 0$, and Lemma 4.4 can be applied to Δ . Hence η is optimal with $(P_1)_\# \eta \ll m$ and this is in contradiction with the fact that curvature property $\text{RCD}^*(K, N)$ implies: every L^2 -optimal transportation is induced by a map, see [20]. The claim follows. \square

Thanks to the symmetry of the statement of Proposition 4.5, it can be proven that also the set

$$A_- := \{x \in \mathcal{T}_e : \exists z, w \in \Gamma(x)^{-1}, (z, w) \notin R\}$$

has m -measure zero. Regarding measurability, also A_- is a σ -compact set. Being the difference of two σ -compact sets, the set $\mathcal{T}_e \setminus (A_+ \cup A_-)$ is σ -compact as well.

The next Theorem clarifies the importance of absence of branching geodesic in the sense of Proposition 4.5.

Theorem 4.6. *Let (X, d, m) satisfy $\text{RCD}^*(K, N)$. Then the set of transport rays $R \subset X \times X$ is an equivalence relation on the set*

$$\mathcal{T}_e \setminus (A_+ \cup A_-).$$

Proof. First, for all $x \in P_1(\Gamma)$, $(x, x) \in R$. If $x, y \in \mathcal{T}_e$ with $(x, y) \in R$, then by definition of R , it follows straightforwardly that $(y, x) \in R$.

So the only property needing a proof is transitivity. Let $x, z, w \in \mathcal{T}_e \setminus (A_+ \cup A_-)$ be such that $(x, z), (z, w) \in R$ with x, z and w distinct points. The claim is $(x, w) \in R$. So we have 4 different possibilities: the first one is

$$z \in \Gamma(x), \quad w \in \Gamma(z).$$

This immediately implies $w \in \Gamma(x)$ and therefore $(x, w) \in R$. The second possibility is

$$z \in \Gamma(x), \quad z \in \Gamma(w),$$

that can be rewritten as $(z, x), (z, w) \in \Gamma^{-1}$. Since $z \notin A_-$, necessarily $(x, w) \in R$. Third possibility:

$$x \in \Gamma(z), \quad w \in \Gamma(z),$$

and since $z \notin A_+$ it follows that $(x, w) \in R$. The last case is

$$x \in \Gamma(z), \quad z \in \Gamma(w),$$

and therefore $x \in \Gamma(w)$, hence $(x, w) \in R$ and the claim follows. \square

5. STRUCTURE OF d -MONOTONE SETS

Theorem 4.6 says that the right set to look at in order to perform a reduction of the Monge problem to a family of 1-dimensional Monge problem is

$$\mathcal{T} := \mathcal{T}_e \setminus (A_+ \cup A_-).$$

We will refer to \mathcal{T} as the *transport set*. From now on we will always assume (X, d, m) to satisfy $\text{RCD}^*(K, N)$.

The next step is to show that each equivalence class of R is formed by a single geodesic.

Lemma 5.1. *Fix any $x \in \mathcal{T}$. Then for any $z, w \in R(x)$ there exists $\gamma \in G \subset \text{Geo}(X)$ such that*

$$\{x, z, w\} \subset \{\gamma_s : s \in [0, 1]\}.$$

If $\hat{\gamma} \in G$ enjoys the same property, then

$$\{\hat{\gamma}_s : s \in [0, 1]\} \cup \{\gamma_s : s \in [0, 1]\} \subset \{\tilde{\gamma}_s : s \in [0, 1]\}$$

for some $\tilde{\gamma} \in \text{Geo}(X)$.

Since $G = \{\gamma \in \text{Geo}(X) : (\gamma_0, \gamma_1) \in \Gamma\}$, Lemma 5.1 states that as soon as we fix an element x in $\mathcal{T}_e \setminus (A_+ \cup A_-)$ and we pick two elements z, w in the same equivalence class of x , then these three points are aligned on a geodesic γ whose image is again all contained in the same equivalence class $R(x)$.

Proof. The proof is quite similar to the proof of Theorem 4.6. First assume that x, z are w all distinct points otherwise the claim follows trivially. Consider different cases.

First case: $z \in \Gamma(x)$ and $w \in \Gamma^{-1}(x)$. Then by d -cyclical monotonicity

$$d(z, w) \leq d(z, x) + d(x, w) = \varphi^d(w) - \varphi^d(z) \leq d(z, w).$$

Hence z, x and w lie on a geodesic.

Second case: $z, w \in \Gamma(x)$. Without loss of generality $\varphi^d(x) \geq \varphi^d(w) \geq \varphi^d(z)$. Since in the proof of Lemma 4.1 we have already excluded the case $\varphi^d(w) = \varphi^d(z)$, we assume $\varphi^d(x) > \varphi^d(w) > \varphi^d(z)$. Then if there would not exist any geodesic $\gamma \in G$ with $\gamma_0 = x$ and $\gamma_1 = z$ and $\gamma_s = w$, there will be $\gamma \in G$ with $(\gamma_0, \gamma_1) = (x, z)$ and $s \in (0, 1)$ such that

$$\varphi^d(\gamma_s) = \varphi^d(w), \quad \gamma_s \in \Gamma(x), \quad \gamma_s \neq w.$$

As observed in the proof of Lemma 4.1, this would imply that $(\gamma_s, w) \notin R$ and since $x \notin A_+$ this would be a contradiction. Hence the second case follows.

The remaining two cases follow with the same reasoning, exchanging the role of $\Gamma(x)$ with the one of $\Gamma^{-1}(x)$. The second part of the statement follows now easily. \square

The next step is to decompose the reference measure m restricted to \mathcal{T} with respect to the partition given by R , that is

$$\{[x]\}_{x \in \mathcal{T}} = \{y \in \mathcal{T} : (x, y) \in R\}_{x \in \mathcal{T}}.$$

In order to use Disintegration Theorem, we need to construct the quotient map

$$f : \mathcal{T} \rightarrow \{[x]\}_{x \in \mathcal{T}}$$

associated to the equivalence relation R . To give a precise statement we need to introduce some terminology.

A *cross-section of an equivalence relation E* is a set $D \subset X$ such that the intersection of D with each equivalence class of E is a singleton. A *section of an equivalence relation E* is a map $f : X \rightarrow X$ such that for any $x, y \in X$ it holds

$$(x, f(x)) \in E, \quad (x, y) \in E \Rightarrow f(x) = f(y).$$

Note that to each section f is canonically associated a cross-section

$$D = \{x \in X : x = f(x)\}.$$

The following result is taken from [6], first part of Section 4. There the result is proved under the additional assumption of non-branching. That assumption is only used to deduce that each equivalence class of R is a single geodesic. Since we have proved this property in Lemma 5.1, we give the proposition here without a proof.

Proposition 5.2. *There exists an m -measurable section*

$$f : \mathcal{T} \rightarrow \mathcal{T}$$

for the equivalence relation R .

Since

$$D = f(\mathcal{T}) = \{x \in \mathcal{T} : d(x, f(x)) = 0\},$$

it follows that D is m -measurable. We can also consider the quotient measure in the following way

$$q := f_{\#} m_{\perp \mathcal{T}}.$$

By inner regularity of compact sets, there exists a σ -compact set $\mathcal{S} \subset D$ such that $q(D \setminus \mathcal{S}) = 0$. Since \mathcal{S} a Borel set, the Disintegration of m restricted to $f^{-1}(\mathcal{S})$ is strongly consistent:

$$m_{\perp f^{-1}(\mathcal{S})} = \int_{\mathcal{S}} m_{\alpha} q(d\alpha), \quad m_{\alpha}(f^{-1}(\alpha)) = \|m_{\alpha}\|, \quad q - a.e. \alpha \in \mathcal{S}.$$

Since $q(D \setminus \mathcal{S}) = 0$ reads also as

$$m(\mathcal{T} \setminus f^{-1}(\mathcal{S})) = 0,$$

the previous disintegration formula becomes

$$(5.1) \quad m_{\perp \mathcal{T}} = \int_{\mathcal{S}} m_{\alpha} q(d\alpha), \quad m_{\alpha}(f^{-1}(\alpha)) = \|m_{\alpha}\|, \quad q - a.e. \alpha \in \mathcal{S}.$$

We conclude this section by recalling a definition from [6], Section 4.

Definition 5.3 (Ray map). Define the *ray map* $g : \text{Dom}(g) \subset \mathcal{S} \times \mathbb{R} \rightarrow \mathcal{T}$ via the formula

$$\begin{aligned} \text{graph}(g) &:= \left\{ (y, t, x) : y \in \mathcal{S}, t \in [0, +\infty), x \in \Gamma(y) \cap \mathcal{T} \cap \{d(x, y) = t\} \right\} \\ &\cup \left\{ (y, t, x) : y \in \mathcal{S}, t \in (-\infty, 0), x \in \Gamma^{-1}(y) \cap \mathcal{T} \cap \{d(x, y) = -t\} \right\} \\ &= \text{graph}(g^+) \cup \text{graph}(g^-). \end{aligned}$$

Hence the ray map associate to each $y \in \mathcal{S}$ and t the unique element in $\Gamma(y) \cap \mathcal{T}$ at distance t from y if t is positive or the unique element in $\Gamma^{-1}(y) \cap \mathcal{T}$ at distance $-t$, if t is negative. Thanks to Theorem 4.6 and Lemma 5.1, the ray map g is well defined and we can take

$$\text{Dom}(g) := g^{-1}(\mathcal{T}).$$

Next we list few regularity properties enjoyed by g .

Proposition 5.4. *The following holds.*

- (1) *The restriction of $\text{graph}(g)$ to $\mathcal{S} \times \mathbb{R}$ is analytic, and therefore the map is Borel.*
- (2) *The range of g is $\mathcal{T} \cup a \cup b$.*
- (3) *$t \mapsto g(y, t)$ is a d 1-Lipschitz Γ -order preserving for $y \in \mathcal{S}$.*
- (4) *$\text{Dom}(g) \ni (y, t) \mapsto g(y, t)$ is bijective on \mathcal{T} , and its inverse is*

$$x \mapsto g^{-1}(x) = (f(x), \pm d(x, f(x)))$$

where f is the quotient map of Proposition 5.2 and the positive/negative sign depends on $x \in \Gamma(f(x))$ or $x \in \Gamma^{-1}(f(x))$.

In this Section we have obtained the first result of this note. In particular we have shown that given a d -monotone set Γ , neglecting a set of m -measure zero, the set of all those points moved by Γ , denoted with \mathcal{T}_e , can be written as the union of a family of disjoint geodesics.

We include the result in the next theorem.

Theorem 5.5. *Let (X, d, m) verify $\text{RCD}^*(K, N)$ for some $K, N \in \mathbb{R}$, with $1 \leq N < \infty$. Let moreover Γ be a d -cyclically monotone set as (3.1) and let \mathcal{T}_e be the set of all points moved by Γ as in Definition 3.2. Then there exists $\mathcal{T} \subset \mathcal{T}_e$ that we call transport set such that*

$$m(\mathcal{T}_e \setminus \mathcal{T}) = 0,$$

and for all $x \in \mathcal{T}$, the transport ray $R(x)$ is formed by a single geodesic and for $x \neq y$, both in \mathcal{T} , either $R(x) = R(y)$ or $R(x) \cap R(y)$ is contained in the set of initial points $a \cup b$ as defined in Definition 3.2.

6. REGULARITY OF DISINTEGRATION

Now we show that for q -a.e. $y \in \mathcal{S}$

$$(6.1) \quad m_y \ll (g(y, \cdot))_{\#} \mathcal{L}^1.$$

Property (6.1) is linked to the behavior in time of the measure of evolving subsets of \mathcal{T} , where the “evolving subsets” has to be made precise.

Since in $\text{RCD}^*(K, N)$ -spaces a concavity estimate for densities of L^2 -geodesics in $\mathcal{P}_2(X, d, m)$ holds, it is natural to look for a definition of evolution inside the transport set where an L^2 -structure can come into play.

Lemma 6.1. *For each $C \subset \mathcal{T}$ and $\delta \in \mathbb{R}$ the set*

$$(C \times \{\varphi^d = \delta\}) \cap \Gamma,$$

is d^2 -cyclically monotone.

Proof. The proof follows from Lemma 4.4. Indeed the set $(C \times \{\varphi^d = c\}) \cap \Gamma$ is trivially a subset of Γ and whenever

$$(x_0, y_0), (x_1, y_1) \in (C \times \{\varphi^d = \delta\}) \cap \Gamma,$$

then $(\varphi^d(y_1) - \varphi^d(y_0)) \cdot (\varphi^d(x_1) - \varphi^d(x_0)) = 0$. \square

We can deduce the following

Corollary 6.2. *For each $C \subset \mathcal{T}$ and $\delta \in \mathbb{R}$ define*

$$C_\delta := P_1((C \times \{\varphi^d = \delta\}) \cap \Gamma).$$

Then if $m(C_\delta) > 0$, there exists a unique $\nu \in \text{OptGeo}$ such that

$$(6.2) \quad (e_0)_{\#} \nu = m(C_\delta)^{-1} m_{\perp C_\delta}, \quad (e_0, e_1)_{\#}(\nu) ((C \times \{\varphi^d = \delta\}) \cap \Gamma) = 1.$$

From Corollary 6.2 and $\text{RCD}^*(K, N)$, we infer the existence of a map $T_{C, \delta}$ depending on C and δ such that

$$(Id, T_{C, \delta})_{\#} (m(C_\delta)^{-1} m_{\perp C_\delta}) = (e_0, e_1)_{\#} \nu.$$

Taking advantage of the ray map g , we define a convex combination between the identity map and $T_{C, \delta}$ as follows:

$$C_\delta \ni x \mapsto (T_{C, \delta})_t(x) \in \{z \in \Gamma(x) : d(x, z) = t \cdot d(x, T_{C, \delta}(x))\}.$$

Since $C \subset \mathcal{T}$, the map $(T_{C, \delta})_t$ is well defined for all $t \in [0, 1]$. We then define the evolution of any subset A of C_δ in the following way:

$$[0, 1] \ni t \mapsto (T_{C, \delta})_t(A).$$

In particular from now on we will adopt the following notation:

$$A_t := (T_{C, \delta})_t(A), \quad \forall A \subset C_\delta, A \text{ compact}.$$

So for any $C \subset \mathcal{T}$ compact and $\delta \in \mathbb{R}$ we have defined an evolution for compact subsets of C_δ . The definition of the evolution depends both on C and δ .

Remark 6.3. Here we spend a few lines on the measurability of the maps involved in the definition of evolution of sets. First note that since Γ is closed and C is compact, we can prove that also C_δ is compact. Indeed from compactness of C we obtain that φ^d is bounded on C and then, since C is bounded, it follows that also $C \times \{\varphi^d = c\} \cap \Gamma$ is bounded. From $\text{RCD}^*(K, N)$, X is proper and therefore compactness follows.

Moreover

$$\text{graph}(T_{C, \delta}) = (C \times \{\varphi^d = \delta\}) \cap \Gamma,$$

hence $T_{C, \delta}$ is continuous. Moreover

$$(T_{C, \delta})_t(A) = P_2(\{(x, z) \in \Gamma \cap (A \times X) : d(x, z) = t \cdot d(x, T_{C, \delta}(x))\}),$$

hence if A is compact, the same holds for $(T_{C, \delta})_t(A)$. It is also possible to show that

$$[0, 1] \ni t \mapsto m((T_{C, \delta})_t(A))$$

is m -measurable. We refer to [6], Lemma 5.2, for its proof.

The next result gives quantitative information on the behavior of the map $t \mapsto m(A_t)$. The statement will be given assuming the lower bound on the generalized Ricci curvature K to be positive. Analogous estimates holds for any $K \in \mathbb{R}$.

Proposition 6.4. *For each $C \subset \mathcal{T}$ and $\delta \in \mathbb{R}$ such that $m(C_\delta) > 0$, it holds*

$$(6.3) \quad m(A_t) \geq (1-t) \cdot \inf_{x \in C_\delta} \left(\frac{\sin\left((1-t)d(x, T_{C,\delta}(x))\sqrt{K/(N-1)}\right)}{\sin\left(d(x, T_{C,\delta}(x))\sqrt{K/(N-1)}\right)} \right)^{N-1} m(A),$$

for all $t \in [0, 1]$ and $A \subset C_\delta$ compact set.

Proof. The proof of (6.3) is obtained by the standard method of approximation with Dirac deltas of the second marginal. More precisely: consider a sequence $\{y_i\}_{i \in \mathbb{N}} \subset \{\varphi^d = \delta\}$ dense in $T_{C,\delta}(C_\delta)$. For each $I \in \mathbb{N}$, define the family of sets

$$E_{i,I} := \{x \in C_\delta : d(x, y_i) \leq d(x, y_j), j = 1, \dots, I\},$$

for $i = 1, \dots, I$. Then for all $I \in \mathbb{N}$, by the same argument of Lemma 6.1, the set

$$\bigcup_{i=1}^I E_{i,I} \times \{y_i\} \subset X \times X,$$

is d^2 -cyclically monotone. Since $\text{RCD}^*(K, N)$ implies $\text{MCP}(K, N)$, see [20] and [13], and the L^2 optimal plans are unique, the estimate (6.3) is proved letting $I \rightarrow \infty$. Similar arguments already appeared many times in literature so we don't include all the details. See last section of [6] and reference therein or the proof of Theorem 2.2 in [12]. \square

6.1. Absolute continuity of conditional measures. We are now ready to prove that for q -a.e. $y \in \mathcal{S}$ (6.1) holds. For each $y \in \mathcal{S}$ we consider the Radon-Nikodym derivative of m_y with respect to $g(y, \cdot)_\# \mathcal{L}^1$:

$$m_y = r(y, \cdot)g(y, \cdot)_\# \mathcal{L}^1 + \omega_y, \quad \omega_y \perp g(y, \cdot)_\# \mathcal{L}^1$$

Lemma 6.5. *There exists a Borel set $C \subset X$ such that*

$$\mathcal{L}^1(g^{-1}(C) \cap (\{y\} \times \mathbb{R})) = 0, \quad \omega_y = m_{y \llcorner C},$$

for q -a.e. $y \in \mathcal{S}$.

Proof. Consider the measure

$$\lambda = g_\#(q \otimes \mathcal{L}^1),$$

and compute the Radon-Nikodym decomposition

$$m = \frac{Dm}{D\lambda} \lambda + \omega.$$

Then there exists a Borel set C such that $\omega = m_{\llcorner C}$ and $\lambda(C) = 0$. The set C proves the Lemma. Indeed $C = \cup_{y \in \mathcal{S}} C_y$ where $C_y = C \cap f^{-1}(y)$ is such that $m_{y \llcorner C_y} = \omega_y$ and $(g(y, \cdot)_\# \mathcal{L}^1)(C_y) = 0$ for q -a.e. $y \in \mathcal{S}$. \square

Theorem 6.6. *For q -a.e. $y \in \mathcal{S}$, the conditional probabilities m_y are absolutely continuous w.r.t. $g(y, \cdot)_\# \mathcal{L}^1$.*

Proof. Step 1. Take as C the set constructed in Lemma 6.5 and suppose by contradiction that

$$m(C) > 0,$$

and we already know that $q \otimes \mathcal{L}^1(g^{-1}(C)) = 0$.

We want to find $\hat{C} \subset C$ compact set with $m(\hat{C}) > 0$ and $\delta \in \mathbb{R}$ such that for each $z \in \hat{C}$ there exists $w \in \Gamma(z)$ such that $\varphi^d(w) = \delta$. Possibly localizing, we can assume that for each $z \in C$

$$\sup\{d(x, y) : x, y \in R(z)\} \geq M,$$

and $C \subset B_\varepsilon(\bar{z})$ for some $\bar{z} \in \mathcal{T}$ and $\varepsilon \leq M$. Then the 1-Lipschitz property of φ^d implies the existence of such δ and \hat{C} compact, such that $m(\hat{C}) > 0$ with

$$\hat{C} = \hat{C}_\delta.$$

We can therefore consider the evolution in time of \hat{C} with respect to δ , $(\hat{C})_t$ and for $t \in [0, 1]$ the inequality (6.3) holds.

Step 2. Since $\hat{C} \subset C$, it still holds that

$$q \otimes \mathcal{L}^1(g^{-1}(\hat{C})) = 0.$$

In particular, for all $t \in [0, 1]$ it follows that

$$q \otimes \mathcal{L}^1(g^{-1}((\hat{C})_t)) \leq q \otimes \mathcal{L}^1(g^{-1}(\hat{C})) = 0.$$

Indeed since the evolution of \hat{C} runs along the transport rays, $f(\hat{C}_t) = f(\hat{C})$ where f is the quotient map. Moreover on each single transport ray, the evolution of \hat{C} is just the linear contraction to a single point. Hence the inequality follows.

We also need the following object: for each $y \in f(\hat{C}) = P_1(g^{-1}(\hat{C}))$ there exists only one $\tau \in \mathbb{R}$, say $\tau(y)$ such that

$$g(y, \tau) \in \{\varphi^d = \delta\}.$$

To underline the (m -measurable) dependence of τ on y and δ , we will denote it with $\tau(y, \delta)$. With this notation, we can express \hat{C}_t in the following way:

$$\hat{C}_t = g \left(\left\{ (y, \tau + (\tau(y, \delta) - \tau)t) : (y, \tau) \in g^{-1}(\hat{C}) \right\} \right),$$

and consequently

$$g^{-1}(\hat{C}_t) = \left\{ (y, \tau + (\tau(y, \delta) - \tau)t) : (y, \tau) \in g^{-1}(\hat{C}) \right\}.$$

Then by Fubini-Tonelli Theorem and Proposition 6.4

$$\begin{aligned} 0 &< \int_{(0,1/2)} m(\hat{C}_t) dt = \int_{(0,1/2)} \left(\int_{g^{-1}(\hat{C}_t)} ((g^{-1})_{\#} m) (dy d\tau) \right) dt \\ &= ((g^{-1})_{\#} m) \otimes \mathcal{L}^1 \left(\left\{ (y, \tau, t) \in \mathcal{S} \times \mathbb{R} \times [0, 1/2] : (y, \tau) \in g^{-1}(\hat{C}_t) \right\} \right) \\ &= \int_{\mathcal{S} \times \mathbb{R}} \mathcal{L}^1 \left(\left\{ t \in [0, 1/2] : (y, \tau) \in g^{-1}(\hat{C}_t) \right\} \right) (g_{\#}^{-1} m) (dy d\tau), \\ (6.4) \quad &= \int_{\mathcal{S} \times \mathbb{R}} \mathcal{L}^1 \left(\left\{ t \in [0, 1/2] : \left(y, \frac{\tau - \tau(y, \delta)t}{1-t} \right) \in g^{-1}(\hat{C}) \right\} \right) (g_{\#}^{-1} m) (dy d\tau). \end{aligned}$$

Now by definition of C , for q -a.e. $y \in \mathcal{S}$,

$$\mathcal{L}^1 \left(\{t \in \mathbb{R} : (y, t) \in g^{-1}(\hat{C})\} \right) = 0.$$

Since the function $[0, 1/2] \ni t \mapsto (\tau - \tau(y, \delta)t) / (1-t)$ is smooth, also the following holds

$$\mathcal{L}^1 \left(\left\{ t \in [0, 1/2] : \left(y, \frac{\tau - \tau(y, \delta)t}{1-t} \right) \in g^{-1}(\hat{C}) \right\} \right) = 0,$$

for q -a.e. $y \in \mathcal{S}$.

Since $(P_1)_{\#}((g^{-1})_{\#} m) = q$ it follows that the last integral in (6.4) is null, giving a contradiction with the strictly positive sign of the first one. \square

7. EXISTENCE OF SOLUTION TO THE MONGE PROBLEM

Using Theorem 6.6 we prove the existence of an m -measurable map $\hat{T} : X \rightarrow X$ such that

$$\int_X d(x, \hat{T}(x)) \mu_0(dx) = \inf_{T_{\#} \mu_0 = \mu_1} \int_X d(x, T(x)) \mu_0(dx),$$

with $\hat{T}_{\#} \mu_0 = \mu_1$, provided μ_0 is absolutely continuous with respect to m . So assume $\mu_0 = \varrho_0 m$.

Justified by Lemma 3.4, extension (3.2) and Proposition 4.5, we assume that $\mu_0(\mathcal{T}) = \mu_1(\mathcal{T}_e) = 1$. Then (5.1) gives that

$$(7.1) \quad \mu_0 = \varrho_0 m_{\mathcal{T}} = \int_{\mathcal{S}} \varrho_0 m_{\alpha} q(d\alpha) = \int_{\mathcal{S}} \mu_{0,y} q_{\mu_0}(dy),$$

where $\mu_{0,y} = c(y)\varrho_0 m_\alpha$ with $c(y)$ normalizing constant, and $q_{\mu_0} = c(y)^{-1}q$.

Since R is an equivalence relation only on \mathcal{T} and a priori $\mu_1(\mathcal{T}_e \setminus \mathcal{T}) > 0$ is not excluded, (7.1) is not automatically true for μ_1 . To get a disintegration for μ_1 we pass through a disintegration of a given $\eta \in \Pi_{opt}(\mu_0, \mu_1)$.

Lemma 7.1. *Let $\eta \in \Pi_{opt}(\mu_0, \mu_1)$ be given, then the following disintegration formula holds:*

$$\eta = \int_{\mathcal{S}} \eta_y q_{\mu_0}(dy), \quad \eta_y \in \mathcal{P}((R(y) \cap \mathcal{T}) \times R(y)),$$

with $P_{1\#}\eta_y = \mu_{0,y}$.

Proof. Since $\eta(\mathcal{T} \times X \cap \Gamma) = 1$, we want to find the right partition of $(\mathcal{T} \times X) \cap \Gamma$ and that can be done via the partition $\{R(y)\}_{y \in \mathcal{S}}$ of \mathcal{T} :

$$(\mathcal{T} \times X) \cap \Gamma = \bigcup_{y \in \mathcal{S}} [(R(y) \cap \mathcal{T}) \times X] \cap \Gamma.$$

Then by Disintegration Theorem it follows that

$$\eta = \int_{\mathcal{S}} \eta_y q_\eta(dy), \quad \eta_y \in \mathcal{P}(((R(y) \cap \mathcal{T}) \times X) \cap \Gamma),$$

where q_η is the quotient measure of η associated to the equivalence relation (or partition) described in the previous lines. Since $((R(y) \cap \mathcal{T}) \times X) \cap \Gamma \subset (R(y) \cap \mathcal{T}) \times R(y)$, to prove the claim we need to show that:

$$q_\eta = q_{\mu_0}.$$

Since for $I \subset \mathcal{S}$ it holds that

$$\mu_0(f^{-1}(I)) = \eta(f^{-1}(I) \times X) = \eta(\{(x, y) \in \Gamma : x \in \mathcal{T}, f(x) \in I\}),$$

the claim follows. \square

We can obtain a dimensional reduction also for μ_1 :

$$(7.2) \quad \mu_1 = P_{2\#}\eta = \int_{\mathcal{S}} (P_2)_\# \eta_y q_{\mu_0}(dy) = \int_{\mathcal{S}} \mu_{1,y} q_{\mu_0}(dy).$$

In particular $\eta_y \in \Pi(\mu_{0,y}, \mu_{1,y})$ is d -cyclically monotone (and hence optimal, because $R(y)$ is one dimensional) for q_{μ_0} -a.e. y . If $\mu_1(\mathcal{T}) = 1$, then (7.2) is the disintegration of μ_1 w.r.t. R .

Remark 7.2. Since $\mu_0(\mathcal{T}) = 1$ and for each $y \in \mathcal{T}$ the set $R(y)$ is 1-dimensional and for $y \in \mathcal{S}$ we have proved that $R(y) = g(y, \mathbb{R})$, without loss of generality we can refer to $\mu_{0,y}$ and $\mu_{1,y}$ as Borel probability measures over \mathbb{R} . It will be clear from the context whether we still refer to them as measures over X .

We now prove the existence of a solution to Monge minimization problem.

Theorem 7.3. *Let (X, d, m) be a metric measure space verifying $\text{RCD}^*(K, N)$ for $N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $W_1(\mu_0, \mu_1) < \infty$ and $\mu_0 \ll m$. Then there exists a Borel map $T : X \rightarrow X$ such that $T_\# \mu_0 = \mu_1$ and*

$$\int_X d(x, T(x)) \mu_0(dx) = \int_{X \times X} d(x, y) \eta(dxdy),$$

for any $\eta \in \Pi_{opt}(\mu_0, \mu_1)$.

Proof. Step 1. By means of the map g^{-1} , we reduce to a transport problem on $\mathcal{S} \times \mathbb{R}$, with cost

$$c((y, s), (y', t)) = \begin{cases} |t - s|, & \text{if } y = y', \\ +\infty, & \text{if } y \neq y'. \end{cases}$$

It is enough to prove the theorem in this setting under the following assumptions: \mathcal{S} compact and $\mathcal{S} \ni y \mapsto (\mu_{0,y}, \mu_{1,y})$ weakly continuous.

From the weak continuity of the map $y \mapsto (\mu_{0,y}, \mu_{1,y})$, it follows that the maps

$$(y, t) \mapsto H(y, t) := \mu_{0,y}((-\infty, t)), \quad (y, t) \mapsto F(y, t) := \mu_{1,y}((-\infty, t))$$

are lower semi-continuous. Both are increasing in t and H is continuous in t for q_{μ_0} -a.e. $y \in \mathcal{S}$. The map T defined as in Theorem 3.5 by

$$T(y, s) := \left(y, \sup \{ t : F(y, t) \leq H(y, s) \} \right)$$

is Borel. In fact, for A Borel,

$$T^{-1}(A \times [t, +\infty)) = \{ (y, s) : y \in A, H(y, s) \geq F(y, t) \} \in \mathcal{B}(\mathcal{S} \times \mathbb{R}).$$

Step 2. Since $\mu_{0,y}$ has no atoms for q_{μ_0} -a.e. $y \in \mathcal{T}$, $T(y, \cdot)$ is optimal for the transport problem between $\mu_{0,y}$ and $\mu_{1,y}$ with cost $|\cdot|$. By d -cyclical monotonicity, the same holds for η_y . Then using Lemma 7.1, it follows that

$$\begin{aligned} \int d(x, T(x))\mu_0(dx) &= \int_{\mathcal{S}} \int_{\mathbb{R}} |t - T(y, t)|\mu_{0,y}(dt)q_{\mu_0}(dy) \\ &= \int_{\mathcal{S}} \int_{X \times X} d(x, z)\eta_y(dxdz)q_{\mu_0}(dy) \\ &= \int d(x, z)\eta(dxdz). \end{aligned}$$

The claim follows. \square

Since a priori

$$\inf_{\eta \in \Pi(\mu_0, \mu_1)} \int d(x, z)\eta(dxdz) \leq \inf_{T_{\sharp} \mu_0 = \mu_1} \int d(x, T(x))\mu_0(dx),$$

we have also proved (1.2).

Corollary 7.4. *Let (X, d, m) be a metric measure space verifying $\text{RCD}^*(K, N)$ for $N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $W_1(\mu_0, \mu_1) < \infty$ and $\mu_0 \ll m$. Then*

$$\min_{\eta \in \Pi(\mu_0, \mu_1)} \int d(x, z)\eta(dxdz) = \min_{T_{\sharp} \mu_0 = \mu_1} \int d(x, T(x))\mu_0(dx).$$

APPENDIX: ESTIMATE ON THE ONE DIMENSIONAL DENSITY

Here we include an interesting regularity result concerning the density along optimal transports in $\text{RCD}^*(K, N)$ spaces. This result was not necessary for proving Theorem 7.3 and Corollary 7.4, but it will be used later in a future publication to extend the result of [11] from non-branching $\text{CD}^*(K, N)$ spaces to the case $\text{RCD}^*(K, N)$.

Using the curvature property $\text{RCD}^*(K, N)$ verified by (X, d, m) and the estimate (6.3), we can prove a regularity property for the density of $g(y, \cdot)_{\sharp}^{-1}m_y$ with respect to \mathcal{L}^1 . We introduce the function $h : \mathcal{S} \times \mathbb{R} \rightarrow [0, \infty)$ so that

$$m = g_{\sharp} (h q \otimes \mathcal{L}^1).$$

We will prove an estimate for the map $t \mapsto h(y, t)$ for q -a.e. $y \in \mathcal{S}$. Again the estimates proved here are obtained assuming $K > 0$, anyway analogous calculations hold for any $K \in \mathbb{R}$ after suitable modifications.

Each ray $R(y)$ for q -a.e. $y \in \mathcal{S}$ is invariant for the evolution for compact subsets of the transport set \mathcal{T} , introduced in Section 6. Then, using standard arguments, estimate (6.3) can be localized at the level of the density h : for each compact set $A \subset \mathcal{T}$

$$\begin{aligned} &\int_{P_2(g^{-1}(A_t))} h(y, s)\mathcal{L}^1(ds) \\ &\geq (1-t) \left(\inf_{\tau \in P_2(g^{-1}(A))} \frac{\sin((1-t)|\tau - \sigma|\sqrt{K/(N-1)})}{\sin(|\tau - \sigma|\sqrt{K/(N-1)})} \right)^{N-1} \int_{P_2(g^{-1}(A))} h(y, s)\mathcal{L}^1(ds), \end{aligned}$$

for q -a.e. $y \in \mathcal{S}$ such that $g(y, \sigma) \in \mathcal{T}$.

Then using change of variable, one obtains that for q -a.e. $y \in \mathcal{S}$:

$$h(y, s + |s - \sigma|t) \geq \left(\frac{\sin((1-t)|s - \sigma|\sqrt{K/(N-1)})}{\sin(|s - \sigma|\sqrt{K/(N-1)})} \right)^{N-1} h(y, s),$$

for \mathcal{L}^1 -a.e. $s \in P_2(g^{-1}(R(y)))$ and $\sigma \in \mathbb{R}$ such that $s + |\sigma - s| \in P_2(g^{-1}(R(y)))$. We can rewrite the estimate in the following way:

$$h(y, \tau) \geq \left(\frac{\sin((\sigma - \tau)\sqrt{K/(N-1)})}{\sin((\sigma - s)\sqrt{K/(N-1)})} \right)^{N-1} h(y, s),$$

for \mathcal{L}^1 -a.e. $s \leq \tau \leq \sigma$ such that $g(y, s), g(y, \tau), g(y, \sigma) \in \mathcal{T}$.

Since evolution can be also defined backwardly, we have proved the next

Proposition 7.5. *For q -a.e. $y \in \mathcal{S}$ it holds:*

$$\left(\frac{\sin((\sigma_+ - \tau)\sqrt{K/(N-1)})}{\sin((\sigma_+ - s)\sqrt{K/(N-1)})} \right)^{N-1} \leq \frac{h(y, \tau)}{h(y, s)} \leq \left(\frac{\sin((\tau - \sigma_-)\sqrt{K/(N-1)})}{\sin((s - \sigma_-)\sqrt{K/(N-1)})} \right)^{N-1},$$

for $\sigma_- < s \leq \tau < \sigma_+$ such that their image via $g(y, \cdot)$ is contained in $R(y)$.

The proof follows just inverting the direction of the evolution of A_t as defined in Section 6, that is replace Γ with Γ^{-1} . Then, adapting (6.3) to the new evolution and repeating the calculations done in the Appendix, we obtain the other inequality proving Proposition 7.5. For details see [6].

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