ON THE TRANSVERSALITY CONDITIONS AND THEIR GENERICITY

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Abstract. In this note we review some results on the *transversality conditions* for a smooth Fredholm map $f : \mathsf{X} \times (0,T) \to \mathsf{Y}$ between two Banach spaces X, Y. These conditions are well-known in the realm of bifurcation theory and commonly accepted as "generic". Here we show that under the transversality assumptions the sections $\mathcal{C}(t) = \{x : f(x,t) = 0\}$ of the zero set of f are discrete for every $t \in (0,T)$ and we discuss a somehow explicit family of perturbations of f along which transversality holds up to a residual set.

The application of these results to the case when f is the X-differential of a time-dependent energy functional $\mathscr{E} : \mathsf{X} \times (0,T) \to \mathbb{R}$ and $\mathcal{C}(t)$ is the set of the critical points of $\mathscr{E}$ provides the motivation and the main example of this paper.

1. Introduction

Let X, Y be a couple of Banach spaces and let $f : X \times (0,T) \to \mathsf{Y}$ be a C^2-map defined in the open set $X \subset \mathsf{X}$.

In this note, we investigate the so-called *transversality conditions* for the set $\mathcal{S}$ of the singular points of f, i.e. the subset of the zero set $\mathcal{C}$ of f where its (partial) differential $\mathrm{D}_x f$ with respect to $x \in \mathsf{X}$ is not invertible:

$$\mathcal{C} := \Big\{(x,t) \in X \times (0,T) : f(x,t) = 0\Big\}, \quad \mathcal{S} := \Big\{(x,t) \in \mathcal{C} : \mathrm{D}_x f(x,t) \text{ is non invertible}\Big\}. \tag{1.1}$$

In particular we will focus on two important features related to transversality. The first one concerns the topology of the sections $\mathcal{C}(t) := \Big\{x \in X : (x,t) \in \mathcal{C}\Big\}$ of $\mathcal{C}$: it is not difficult to show that for every $t \in (0,T)$ the sets $\mathcal{C}(t)$ are discrete, so that each point in $\mathcal{C}(t)$ is isolated. A second property we will discuss in some detail is the *generic* character of transversality.

We first illustrate our results, and their role for applications, in the simpler finite dimensional setting.

The transversality conditions in the finite-dimensional case. Let us consider the case when $\mathsf{X} = \mathsf{Y} = \mathbb{R}^n$. The symbol $\langle \cdot, \cdot \rangle$ represents the scalar product in $\mathbb{R}^n$, and $\mathbb{M}^{k \times m}$ is the space of matrixes with k rows and m columns. If $K \in \mathbb{M}^{k \times m}$, we denote by $\mathsf{N}(K)$ its null space in $\mathbb{R}^k$, by $\mathsf{R}(K)$ its range in $\mathbb{R}^m$, and by K^* its transposed.

Definition 1.1 (Transversality conditions in $\mathbb{R}^n$)**.** *We say that f satisfies at a point $(x_0, t_0) \in \mathcal{S}$ the transversality conditions if*

(T1) $\dim(\mathsf{N}(\mathrm{D}_x f(x_0,t_0))) = 1$ (*and therefore* $\dim(\mathsf{N}(\mathrm{D}_x f(x_0,t_0)^*)) = 1$).
(T2) *If* $0 \neq w^* \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0)^*)$ *then* $\langle \partial_t f(x_0,t_0), w^* \rangle \neq 0$.
(T3) *If* $0 \neq v \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0))$ *then* $\langle \mathrm{D}_x^2 f(x_0,t_0)[v,v], w^* \rangle \neq 0$.

We say that f satisfies the transversality conditions if they hold at every point of $\mathcal{S}$.

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While referring to Definition 2.2 ahead for the precise statement of the transversality conditions in an infinite-dimensional setting, let us point out that, there it is crucial, as well as natural, to require in addition that $f(\cdot,t)$ is a Fredholm map of index 0 for every $t \in (0,T)$.

Conditions (T2) and (T3) look particularly simple when $n = 1$: in this case they read

$$\partial_t f(x_0,t_0) \neq 0, \qquad \partial^2_{xx} f(x_0,t_0) \neq 0.$$

Singular perturbation of gradient flows and the structure of $\mathcal{C}(t)$. Our motivation for getting further insight on the topology of $\mathcal{C}(t)$ stems from the study of the limit as $\varepsilon \downarrow 0$ of the differential equation

$$\varepsilon x'(t) + f(x(t),t) = 0, \quad t \in (0,T), \tag{1.2a}$$

in particular in the case of a gradient flow, when

$$f = \mathrm{D}_x \mathscr{E}, \quad \mathscr{E} : X \times (0,T) \to \mathbb{R} \quad \text{is a } \mathrm{C}^3 \text{ functional.} \tag{1.2b}$$

The study of the *singular perturbation* problem (1.2a,b) was carried out in finite dimension in [12], where it was shown that a family $\{x_\varepsilon\}_\varepsilon$ of solutions to (1.2a) converges to a piecewise regular curve obtained by connecting smooth branches of solutions to the equilibrium equation

$$f(x(t),t) = 0, \quad \text{i.e. } x(t) \in \mathcal{C}(t), \quad t \in (0,T), \tag{1.3}$$

by means of heteroclinic solutions of the gradient flow

$$\theta'(s) + f(\theta(s),t) = 0, \quad \lim_{s\to\pm\infty} \theta(s) = x_\pm(t) \quad \text{at every jump point } t \text{ for the limit curve } x. \tag{1.4}$$

The analysis in [12] hinges on the assumption that $f = \mathrm{D}_x\mathscr{E}$ satifies the transversality conditions at its singular points (which are the degenerate critical points of $\mathscr{E}$). In addition, a crucial role is played by the condition that at every time $\underline{\mathrm{t}} \in (0,T)$ there exists at most one degenerate critical point $(\underline{\mathrm{x}},\underline{\mathrm{t}})$ in $\mathcal{S}$, $\mathrm{D}^2_x\mathscr{E}$ is positive semidefinite at $(\underline{\mathrm{x}},\underline{\mathrm{t}})$ and the (unique) heteroclinic starting from $\underline{\mathrm{x}}$ ends in a local minimum of $\mathscr{E}(\cdot,\underline{\mathrm{t}})$. This allows the author to tackle this singular limit by means of refined techniques from bifurcation theory, see also [2], where the quasistatic limit of a *second-order* system was addressed in the finite-dimensional case.

In the forthcoming paper [1], we develop a different *variational* approach to the limit $\varepsilon \to 0$ of (1.2a), even in the setting of an infinite-dimensional space X. Under general coercivity assumptions on $\mathscr{E}$, we prove that, up to a subsequence, any family $\{x_\varepsilon\}_\varepsilon$ of solutions to (1.2a) pointwise converges to a curve x fulfilling the equilibrium equation (1.3) and a variational jump condition. In fact, the crucial property that lies at the core of the analysis in [1] is that $\mathcal{C}(t)$ is discrete for every $t \in (0,T)$.

The next result shows that if f satisfies the transversality conditions then $\mathcal{C}(t)$ is a discrete set for every $t \in (0,T)$.

Theorem 1.2. *If f satisfies the transversality conditions then for every $t \in (0,T)$ the set $\mathcal{C}(t)$ only consists of isolated points.*

We present here the simple proof in the finite dimensional case, postponing the discussion of the general case to the next section.

Proof. Let us fix $x_0 \in \mathcal{C}(t_0)$. If $\mathrm{D}_x f(x_0,t_0)$ is invertible, then the thesis follows immediately by the implicit function theorem. It is thus sufficient to consider the case when $(x_0,t_0) \in \mathcal{S}$.

Let us first show that $\mathsf{N}(\mathrm{d}f(x_0,t_0))$ has dimension 1 so that the differential of f at (x_0,t_0) has rank n and is surjective. If $(v,\alpha) \in \mathsf{N}(\mathrm{d}f(x_0,t_0))$, then taking the duality with w^* as in (T2) we get

$$\begin{aligned} 0 &= \langle \mathrm{D}_x f(x_0,t_0)[v], w^*\rangle + \alpha\langle \partial_t f(x_0,t_0), w^*\rangle = \langle \mathrm{D}_x f(x_0,t_0)^*[w^*], v\rangle + \alpha\langle \partial_t f(x_0,t_0), w^*\rangle \\ &= \alpha\langle \partial_t f(x_0,t_0), w^*\rangle, \end{aligned}$$

so that $\alpha = 0$ by (T2). Choosing now an arbitrary vector $w^* \in \mathbb{R}^n$ we get $v \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0))$, which has dimension 1 thanks to (T1).

Since the differential $\mathrm{d}f(x_0,t_0)$ has rank n, the implicit function theorem implies that there exists an open neighborhood U of (x_0,t_0) and a smooth curve

$$\gamma : (-\varepsilon,\varepsilon) \to X \times (0,T), \qquad s \mapsto \gamma(s) = (\mathsf{x}(s),\mathsf{t}(s)),$$

such that

$$(\mathsf{x}(0),\mathsf{t}(0)) = (x_0,t_0), \quad \dot\gamma(s) \neq 0 \quad \text{in } (-\varepsilon,\varepsilon), \quad \mathcal{C} \cap U = \gamma((-\varepsilon,\varepsilon)). \tag{1.5}$$

Differentiating w.r.t s the identity $f(\mathsf{x}(s),\mathsf{t}(s)) = 0$ we then obtain

$$\mathrm{D}_x f(\mathsf{x}(s),\mathsf{t}(s))[\dot{\mathsf{x}}(s)] + \dot{\mathsf{t}}(s)\partial_t f(\mathsf{x}(s),\mathsf{t}(s)) = 0 \quad \forall\, s \in (-\varepsilon,\varepsilon). \tag{1.6}$$

Multiplying (1.6) at $s=0$ by w^* we find

$$\begin{aligned} 0 &= \langle \mathrm{D}_x f(x_0,t_0)[\dot{\mathsf{x}}(0)], w^*\rangle + \dot{\mathsf{t}}(0)\langle \partial_t f(x_0,t_0), w^*\rangle \\ &= \langle \mathrm{D}_x f(x_0,t_0)^*[w^*], \dot{\mathsf{x}}(0)\rangle + \dot{\mathsf{t}}(0)\langle \partial_t f(x_0,t_0), w^*\rangle = 0 + \dot{\mathsf{t}}(0)\langle \partial_t f(x_0,t_0), w^*\rangle. \end{aligned}$$

Since $\langle \partial_t f(x_0,t_0), w^*\rangle \neq 0$, we then have $\dot{\mathsf{t}}(0) = 0$, which implies $\dot{\mathsf{x}}(0) \neq 0$ by the second of (1.5). Evaluating (1.6) at $s=0$ we therefore get $\mathrm{D}_x f(x_0,t_0)[\dot{\mathsf{x}}(0)] = 0$, i.e. $\dot{\mathsf{x}}(0) \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0))$.

Differentiating w.r.t s twice, we have

$$\begin{aligned} \mathrm{D}_x^2 f(\mathsf{x}(s),\mathsf{t}(s))[\dot{\mathsf{x}}(s)]^2 + 2\dot{\mathsf{t}}(s)\partial_t \mathrm{D}_x f(\mathsf{x}(s),\mathsf{t}(s))[\dot{\mathsf{x}}(s)] + \mathrm{D}_x f(\mathsf{x}(s),\mathsf{t}(s))[\ddot{\mathsf{x}}(s)] \\ + \dot{\mathsf{t}}(s)^2\partial_t^2 f(\mathsf{x}(s),\mathsf{t}(s)) + \ddot{\mathsf{t}}(s)\partial_t f(\mathsf{x}(s),\mathsf{t}(s)) = 0 \quad \forall\, s \in (-\varepsilon,\varepsilon). \end{aligned}$$

Multiplying the above relation at $s=0$ by w^* and taking into account that $\dot{\mathsf{x}}(0) \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0)) \setminus \{0\}$ and $\dot{\mathsf{t}}(0) = 0$, we then conclude

$$\ddot{\mathsf{t}}(0)\langle \partial_t f(x_0,t_0), w^*\rangle = \langle \mathrm{D}_x^2 f(x_0,t_0)[\dot{\mathsf{x}}(0),\dot{\mathsf{x}}(0)], w^*\rangle \neq 0,$$

thanks to (T3) from Definition 1.1. The above relation then gives

$$\ddot{\mathsf{t}}(0) \neq 0.$$

This yields the existence of a neighborhood V of (x_0,t_0) such that $V \cap \mathcal{C}(t_0)$ contains only the point x_0, since a simple Taylor expansion of t around $s=0$ shows that $\mathsf{t}(s) \neq t_0$ in a sufficiently small neighborhood of 0. □

Genericity of the transversality conditions. As the results from [12], [2], and [1] reveal their crucial role, it is interesting to investigate to what extent the transversality conditions can be assumed to hold for "generic" vector fields. In the realm of bifurcation theory, this is commonly accepted, see e.g. [4, 5, 11]. Nonetheless, it is not always easy to find, in the literature, an explicit result stating the precise meaning of genericity, especially in the infinite-dimensional setting (see [10] for a *finite-dimensional* version of the genericity result).

We address this issue in Section 3. Its main result, Theorem 3.2, states that a certain class of perturbations of f satisfy the transversality conditions up to a *meagre* subset (or, in finite dimension, up to a Lebesgue negligible set). The key idea at the basis of our proof is that the transversality conditions are equivalent to the property that the "augmented" operator

$$g : X \times (0,T) \times (\mathsf{X} \setminus \{0\}) \to \mathsf{Y} \times \mathsf{Y} \qquad g(x,t,v) := (f(x,t), \mathrm{D}_x f(x,t)[v]), \tag{1.7}$$

has $(0,0)$ as a regular value, namely that the total differential $\mathrm{d}g$ is surjective at each point of $g^{-1}(0,0)$. This is proved in Lemma 2.8 below. We will combine this fact with a classical result from the paper [8]. Theorem 1.1 therein indeed provides conditions ensuring that, up to a meagre set, the perturbations of a given operator have $(0,0)$ as a regular value.

Our finite-dimensional genericity result reads

Theorem 1.3. *Let $f \in \mathrm{C}^3(X\times(0,T);\mathbb{R}^n)$. There exists a meagre (or with zero Lebesgue measure) set $N \subset \mathbb{R}^n \times \mathbb{M}^{n\times n}$ such that for every $(y,K) \in (\mathbb{R}^n \times \mathbb{M}^{n\times n}) \setminus N$ the map*

$$\tilde f : X \times (0,T) \to \mathbb{R}^n, \qquad \tilde f(x,t) := f(x,t) + y + K[x] \tag{1.8}$$

satisfies the transversality conditions of Definition 1.1.

The proof of this result, based on the classical Sard's theorem, and the motivation for the enhanced regularity requirement on f, are postponed to Section 3.1, Remark 3.4.

Plan of the paper. In Section 2, we analyze the transversality conditions in the setting of an infinite-dimensional Banach space X. We prove that they imply that the zeroes of f are isolated (cf. Thm. 2.5). In Section 2.3 we also obtain the characterization of the transversality conditions in terms of the operator g (1.7), which is the milestone for our genericity result, Theorem 3.2, proved in Sec. 3.1. Finally, in Section 3.2, in view of the application to the singular perturbation problem (1.2a)–(1.2b) we specialize the discussion and our results to the case when $f = \mathrm{D}_x\mathscr{E}$, with $\mathscr{E} : \mathsf{X} \times (0,T) \to \mathbb{R}$ a smooth energy functional.

2. The transversality conditions in the infinite-dimensional case

In the infinite-dimensional context, it is natural to study the transversality conditions for the class of vector fields $f : X \times (0,T) \to \mathsf{Y}$ which are *Fredholm maps* of index 0 between two Banach spaces X, Y (see below for a definition). We prove Lemma 2.4 which provides a characterization of the first two transversality conditions in terms of the surjectivity of the operator $\mathrm{d}f$. We rely on this in the proof of the main result of this section, viz. Theorem 2.5, which states that the transversality conditions imply that the zeroes of f are isolated. Finally, in Section 2.3, we obtain a characterization of the *full* set of transversality conditions in Lemma 2.8. We will exploit this result in order to investigate their genericity in Section 3.

2.1. Assumptions and preliminary results.

Notation and preliminary definitions. Let X be a Banach space; we shall denote by $\|\cdot\|_{\mathsf{X}}$ its norm, by X^* its dual, and by ${}_{\mathsf{X}^*}\langle\cdot,\cdot\rangle_{\mathsf{X}}$ the duality pairing between X^* and X.

Given two Banach spaces X and Y, we denote by $\mathcal{L}(\mathsf{X};\mathsf{Y})$ the space of linear bounded operators from X to Y and by $\mathcal{L}^k(\mathsf{X};\mathsf{Y})$ the space of the continuous k-linear forms from X^k to Y. If $L \in \mathcal{L}(\mathsf{X};\mathsf{Y})$ we denote by L^* its adjoint operator in $\mathcal{L}(\mathsf{Y}^*;\mathsf{X}^*)$. We also set

$$\text{kernel of } L: \ \mathsf{N}(L) = \big\{x \in \mathsf{X} : Lx = 0\big\}, \quad \text{range of } L: \ \mathsf{R}(L) = \big\{Lx : x \in \mathsf{X}\big\}. \tag{2.1}$$

A subspace $\mathsf{R} \subset \mathsf{Y}$ has finite codimension if there exists a finite-dimensional space $\mathsf{S} \subset \mathsf{Y}$ such that $\mathsf{R} + \mathsf{S} = \mathsf{Y}$. In this case R is closed and we can always choose S so that $\mathsf{S} \cap \mathsf{R} = \{0\}$; the codimension of R is then defined as $\mathrm{codim}(\mathsf{R}) := \dim(\mathsf{S})$. Moreover, $\mathsf{R}(L^*)$ is closed and we have the adjoint relations

$$\mathsf{R}(L)^\perp = \mathsf{N}(L^*), \qquad \mathsf{N}(L)^\perp = \mathsf{R}(L^*). \tag{2.2}$$

We recall that $L \in \mathcal{L}(\mathsf{X};\mathsf{Y})$ is a *Fredholm operator* if

$$\dim(\mathsf{N}(L)),\ \mathrm{codim}(\mathsf{R}(L)) \quad \text{are finite.}$$

In particular the range of a Fredholm operator is closed. The *index* of the operator L is defined as

$$\mathrm{ind}(L) := \dim(\mathsf{N}(L)) - \mathrm{codim}(\mathsf{R}(L)). \tag{2.3}$$

For Fredholm operators (2.2) yields

$$\mathrm{ind}(L) = 0 \quad \Longleftrightarrow \quad \dim(\mathsf{N}(L)) = \dim(\mathsf{N}(L^*)), \quad \mathrm{codim}(\mathsf{R}(L)) = \mathrm{codim}(\mathsf{R}(L^*)). \tag{2.4}$$

The stability theorem shows that the collection of all Fredholm operators is open in $\mathcal{L}(\mathsf{X};\mathsf{Y})$ and the index ind is a locally constant function. Let $X \subset \mathsf{X}$ be a connected open subset of X. A map $f \in \mathrm{C}^1(X;\mathsf{Y})$ is a Fredholm map if for every $x \in X$ the differential $\mathrm{D}f(x) \in \mathcal{L}(\mathsf{X},\mathsf{Y})$ is a

Fredholm operator. The *index* of f is defined as the index of $\mathrm{D}f(x)$ for some $x \in X$. By the stability theorem and the connectedness of X this definition is independent of x.

In this infinite-dimensional context, our basic assumption on the vector field $f : X \times (0,T) \to \mathsf{Y}$ is the following.

Assumption 2.1. *We require that $f \in \mathrm{C}^2(X \times (0,T); \mathsf{Y})$, and that*

$$f(\cdot,t) : X \to \mathsf{Y} \text{ is a Fredholm map of index } 0 \text{ for every } t \in (0,T). \tag{2.5}$$

We shall denote by $\mathrm{D}_x f : X \times (0,T) \to \mathcal{L}(\mathsf{X};\mathsf{Y})$ and by $\partial_t f : X \times (0,T) \to \mathsf{Y}$ the partial Gâteau derivatives of f, whereas $\mathrm{d}f$ is the total differential of f. As usual, the second order (partial) differential $\mathrm{D}^2_x f : X \times (0,T) \to \mathcal{L}^2(\mathsf{X};\mathsf{Y})$ at a point (x,t) is identified with its canonical bilinear form. As in the previous section we set

$$\mathcal{C} := \Big\{(x,t) \in X \times (0,T) : f(x,t) = 0\Big\}, \quad \mathcal{S} := \Big\{(x,t) \in \mathcal{C} : \mathrm{D}_x f(x,t) \text{ is not invertible}\Big\}, \tag{2.6}$$

and we denote by $\mathcal{C}(t)$ and $\mathcal{S}(t)$ their sections at the time $t \in (0,T)$. Let us now state the infinite-dimensional version of Definition 1.1.

Definition 2.2 (Transversality conditions in the infinite-dimensional case)**.** *Let f comply with Assumption 2.1. We say that f satisfies at a point $(x_0,t_0) \in \mathcal{S}$ the transversality conditions if*

(T1) $\dim(\mathsf{N}(\mathrm{D}_x f(x_0,t_0))) = 1$*;*
(T2) *If* $0 \neq \ell^* \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0)^*)$ *then* ${}_{\mathsf{Y}^*}\langle \ell^*, \partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} \neq 0$*;*
(T3) *If* $0 \neq v \in \mathsf{N}(\mathrm{D}_x f(x_0,t_0))$ *then* ${}_{\mathsf{Y}^*}\langle \ell^*, \mathrm{D}^2_x f(x_0,t_0)[v,v]\rangle_{\mathsf{Y}} \neq 0$.

If the above properties hold at every $(x_0,t_0) \in \mathcal{S}$, we say that f satisfies the transversality conditions.

Remark 2.3. Thanks to (2.4) and (2.5), the above condition (T1) yields that $\mathsf{N}(\mathrm{D}_x f(x_0,t_0)^*)$ has also dimension 1. Note that Assumption 2.1 implies that $f : X \times (0,T) \to \mathsf{Y}$ is a Fredholm map of index 1 with respect to the variable (x,t). Indeed,

$$\mathrm{d}f(x,t)[v,\tau] = \mathrm{D}_x f(x,t)[v] + \tau \partial_t f(x,t); \tag{2.7}$$

if $\partial_t f(x,t) \in \mathsf{R}\big(\mathrm{D}_x f(x,t)\big)$ then $\mathrm{codim}\big(\mathsf{R}(\mathrm{d}f(x,t))\big) = \mathrm{codim}\big(\mathsf{R}(\mathrm{D}_x f(x,t))\big)$ but $\dim\big(\mathsf{N}(\mathrm{d}f(x,t))\big) = \dim\big(\mathsf{N}(\mathrm{D}_x f(x,t))\big) + 1$. On the other hand, if $\partial_t f(x,t) \notin \mathsf{R}\big(\mathrm{D}_x f(x,t)\big)$ then $\mathrm{codim}\big(\mathsf{R}(\mathrm{d}f(x,t))\big) = \mathrm{codim}\big(\mathsf{R}(\mathrm{D}_x f(x,t))\big) - 1$ and $\dim\big(\mathsf{N}(\mathrm{d}f(x,t))\big) = \dim\big(\mathsf{N}(\mathrm{D}_x f(x,t))\big)$.

Let us now get further insight into the transversality conditions (T1) and (T2).

Lemma 2.4. *Suppose that $f : X \times (0,T) \to \mathsf{Y}$ satisfies Assumption 2.1 and let $(x_0,t_0) \in \mathcal{S}$ be fixed. Then conditions (T1) and (T2) of Definition 2.2 hold if and only if $\mathrm{d}f(x_0,t_0)$ is onto.*

Proof. Let us suppose that $\mathrm{d}f(x_0,t_0)$ is onto. The inequality $\dim(\mathsf{N}(\mathrm{D}_x f(x_0,t_0))) \leq 1$ follows immediately by Remark 2.3 and from the fact that $\mathrm{codim}\big(\mathrm{d}f(x_0,t_0)\big) = 0$, because $\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)\big) \times \{0\} \subseteq \mathsf{N}\big(\mathrm{d}f(x_0,t_0)\big)$. Since $(x_0,t_0) \in \mathcal{S}$, we conclude that (T1) holds.

Suppose now that $0 \neq \ell^* \in \mathsf{N}\big(\mathrm{D}_x f(x_0,t_0))^*\big)$ and let $\xi \in \mathsf{Y}$ be such that ${}_{\mathsf{Y}^*}\langle \ell^*, \xi\rangle_{\mathsf{Y}} \neq 0$.Since $\mathrm{d}f(x_0,t_0)$ is onto, there exists $(v,\tau) \in \mathsf{X} \times \mathbb{R}$ such that

$$\xi = \mathrm{D}_x f(x_0,t_0)[v] + \tau \partial_t f(x_0,t_0).$$

Therefore

$$\begin{aligned} 0 \neq {}_{\mathsf{Y}^*}\langle \ell^*, \xi\rangle_{\mathsf{Y}} &= {}_{\mathsf{Y}^*}\langle \ell^*, \mathrm{D}_x f(x_0,t_0)[v]\rangle_{\mathsf{Y}} + \tau\, {}_{\mathsf{Y}^*}\langle \ell^*, \partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} \\ &= {}_{\mathsf{X}^*}\langle \mathrm{D}_x f(x_0,t_0)^*[\ell^*], v\rangle_{\mathsf{X}} + \tau\, {}_{\mathsf{Y}^*}\langle \ell^*, \partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} = 0 + \tau\, {}_{\mathsf{Y}^*}\langle \ell^*, \partial_t f(x_0,t_0)\rangle_{\mathsf{Y}}, \end{aligned} \tag{2.8}$$

so that ${}_{\mathsf{Y}^*}\langle \ell^*, \partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} \neq 0$.

In order to prove the converse implication, let us suppose that (T1-2) hold and let $\ell^* \in \mathsf{R}\big(\mathrm{d}F(x_0,t_0)\big)^\perp \subset \mathsf{Y}^*$. Since for every $v \in \mathsf{X}$ and $\tau \in \mathbb{R}$

$$ {}_{\mathsf{Y}^*}\langle \ell^*, \mathrm{D}_x f(x_0,t_0)[v] + \tau\partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} = 0, $$

by choosing $\tau = 0$ we immediately get $\ell^* \in \mathsf{R}(\mathrm{D}_x f(x_0,t_0))^\perp = \mathsf{N}(\mathrm{D}_x f(x_0,t_0)^*)$; property (T2) then yields $\ell^* = 0$, so that $\mathrm{d}f(x_0,t_0)$ is onto. □

2.2. **The transversality conditions imply that the zeroes are isolated.** We are now in the position to state and prove the analogue of Theorem 1.2:

Theorem 2.5. *Suppose that f satisfies Assumptions 2.1 and the transversality conditions of Definition 2.2. Then, for every $t \in (0,T)$ the set $\mathcal{C}(t) := \{x \in \mathsf{X} : f(x,t) = 0\}$ consists of isolated points.*

The proof follows the same outline as for Theorem 1.2. Indeed, we first of all observe that, for $t_0 \in (0,T)$ fixed, *non-singular* points x (i.e., such that $\mathsf{N}(\mathrm{D}_x f(x,t_0))$ is trivial) are isolated. In the singular case, after some preliminary discussion we again apply the implicit function theorem in order to deduce that, in a neighborhood of (x_0,t_0) with x_0 (singular) zero, the zero set of f is the graph of a suitable function. This allows us to exploit the transversality conditions and deduce by the same arguments as in the finite-dimensional case, that x_0 is isolated.

Proof. Let us fix $(x_0,t_0) \in \mathcal{C}$ and consider first the case when $\mathsf{N}(\mathrm{D}_x f(x_0,t_0)) = \{0\}$. Since $\mathrm{D}_x f(x_0,t_0)$ is a Fredholm operator of index 0, $\mathrm{D}_x f(x_0,t_0)$ is invertible and we can apply the infinite-dimensional version of the implicit function theorem (see e.g. [3, Theorem 2.3]) and conclude that x_0 is isolated in $\mathcal{C}(t_0)$.

Now suppose that $\mathsf{N}_0 := \mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)\big)$ is non-trivial. Then, by (T1) of Definition 2.2, $\mathsf{N}_0 = \mathrm{span}(n_0)$ for some $n_0 \in \mathsf{X} \setminus \{0\}$, so that we can write $\mathsf{X} = \mathsf{Z} + \mathsf{N}_0$, where Z is the topological supplement of N_0 in X. Thus, every $x \in \mathsf{X}$ can be uniquely expressed as $x = z + sn_0$, with $z \in \mathsf{Z}$ and $s \in \mathbb{R}$. In particular, it is not restrictive to assume $x_0 = z_0 + n_0$, $z_0 \in \mathsf{Z}$.

Let us now consider the function $\tilde f(z,s,t) := f(z+sn_0,t)$ defined in a neighborhood of $(z_0,1,t_0)$ in $\mathsf{Z} \times \mathbb{R} \times \mathbb{R}$. In analogy to the proof of Theorem 1.2, we now show that the differential of the function $(z,t) \to \tilde f(z,1,t)$ at the point (z_0,t_0), denoted by $\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)$, is invertible. Let us note first that by construction and Lemma 2.4

$$ \mathsf{R}\big(\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)\big) = \mathsf{R}\big(\mathrm{d}f(x_0,t_0)\big) = \mathsf{Y}. \tag{2.9} $$

To check injectivity of $\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)$, suppose that $(v,\tau) \in \mathsf{N}\big(\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)\big) \subseteq \mathsf{Z} \times \mathbb{R}$, i.e.

$$ 0 = \mathrm{D}_z \tilde f(z_0,1,t_0)[v] + \tau\partial_t \tilde f(x_0,1,t_0) = \mathrm{D}_x f(x_0,t_0)[v] + \tau\partial_t f(x_0,t_0). $$

Taking the duality with $0 \neq \ell^* \in \mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)^*\big) = \mathsf{R}\big(\mathrm{D}_x f(x_0,t_0)\big)^\perp$ and recalling (T2) we get $\tau = 0$. Then $\mathrm{D}_x f(x_0,t_0)[v] = 0$ so that $v \in \mathsf{Z} \cap \mathsf{N}_0 = \{0\}$. This concludes the proof of the injectivity of $\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)$.

Since $\mathrm{D}_{(z,t)}\tilde f(z_0,1,t_0)$ is invertible, we can now apply the implicit function theorem: there exists a neighborhood U of (x_0,t_0), an open interval $I \ni 0$, and a C^2-curve $(\mathsf{z},\mathsf{t}) : I \to \mathsf{Z} \times (0,T)$ such that setting $\gamma(s) := (\mathsf{z}(s) + (1+s)n_0, \mathsf{t}(s))$ we have

$$ \gamma(0) = (x_0,t_0), \quad U \cap \mathcal{C} = \gamma(I). \tag{2.10} $$

Differentiating the identity $f(\gamma(s)) = 0$ w.r.t. s and evaluating at $s = 0$ we obtain

$$ 0 = \mathrm{D}_x f(x_0,t_0)[\dot{\mathsf{z}}(0) + n_0] + \dot{\mathsf{t}}(0)\partial_t f(x_0,t_0). \tag{2.11} $$

Testing by $0 \neq \ell^* \in \mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)^*\big) = \mathsf{R}\big(\mathrm{D} f(x_0,t_0)\big)^\perp$ and recalling (T2) as well as $\dot{\mathsf{z}} \in \mathsf{Z}$, we then obtain

$$ \dot{\mathsf{t}}(0) = 0, \quad \dot{\mathsf{z}}(0) = 0. \tag{2.12} $$

A further differentiation w.r.t. s gives at $s = 0$

$$0 = \mathrm{D}^2_x f(x_0, t_0)[n_0, n_0] + \mathrm{D}_x f(x_0, t_0)[\ddot{\mathsf{z}}(0)] + \ddot{\mathsf{t}}(0)\partial_t f(x_0, t_0),$$

also in view of (2.12). Testing the last expression by ℓ^* as before, we have

$$0 = {}_{\mathsf{Y}}\langle \mathrm{D}_x f(x_0, t_0)[n_0, n_0], \ell^*\rangle_{\mathsf{Y}^*} + \ddot{\mathsf{t}}(0)\, {}_{\mathsf{Y}}\langle \partial_t f(x_0, t_0), \ell^*\rangle_{\mathsf{Y}^*}\,.$$

From (T3) we deduce that $\ddot{\mathsf{t}}(0) \neq 0$ and we conclude that 0 is the only solution of the equation $\mathsf{t}(s) = t_0$ in a small neighborhood of 0, so that x_0 is an isolated point in $\mathcal{C}(t_0)$ by (2.10). □

It is interesting to note that when f is *analytic* then conditions (T1-T2) of Definition 2.2 are still sufficient to provide a useful property of $\mathcal{C}(t)$.

Theorem 2.6. *Suppose that f is an analytic map satisfying Assumptions* 2.1 *and* (T1-T2) *of Definition 2.2, and that any connected component of $\mathcal{C}(t)$ is compact. Then the set $\mathcal{C}(t)$ is the disjoint union of a discrete set and of an analytic manifold of dimension* 1*, whose connected components are compact curves. Moreover, for every connected curve $C \subset \mathcal{C}(t)$ there exist $\varepsilon > 0$ and an open neighborhood V such that $\mathcal{C}(s) \cap V = \emptyset$ for every $s \in (0, T)$ with $0 < |t - s| < \varepsilon$.*

Proof. Let us keep the same notation of the proof of Theorem 2.5 and let x_0 be a singular point of $\mathcal{C}(t_0)$. Since f is analytic, the curve γ parameterizing the set $\mathcal{C}$ in the neighborhood U of the singular point (x_0, t_0) is analytic. If x_0 is an accumulation point of $\mathcal{C}(t_0)$ then the t-component of γ takes the value t_0 in a set accumulating at 0, so that t has to be identically constant. It follows that $\gamma(I)$ is contained in $\mathcal{C}(t_0)$ so that we conclude that the connected component C of $\mathcal{C}(t_0)$ containing x_0 is a (non-degenerate) compact analytic curve. The last assertion follows by the fact that every point of $C \subset \mathcal{C}(t_0)$ has a neighborhood U in $X \times (0, T)$ such that $\mathcal{C} \cap U \subset C \times \{t_0\}$ and that C is compact. □

2.3. **A characterization of the transversality conditions.** Lemma 2.4 sheds light onto the relation between the first two transversality conditions, and the surjectivity of the operator $\mathrm{d}f(x_0, t_0)$. In order to get a characterization of the *full* set of transversality conditions of Definition 2.2, one has to bring into play the "augmented" operator

$$g : X \times (0, T) \times \mathsf{X} \to \mathsf{Y} \times \mathsf{Y} \qquad g(x, t, v) := (f(x, t), \mathrm{D}_x f(x, t)[v]). \tag{2.13}$$

Let us first prove a preliminary result.

Lemma 2.7. *If $f : X \times (0, T) \to \mathsf{Y}$ satisfies Assumption 2.1, then $g(\cdot, t, \cdot) : X \times \mathsf{X} \to \mathsf{Y} \times \mathsf{Y}$ is a Fredholm map of index* 0 *for every $t \in (0, T)$.*

Proof. In order to simplify the notation, we omit to indicate the explicit dependence on the time variable. For a fixed $(x, v) \in X \times \mathsf{X}$, we can write

$$\mathrm{d}g(x, v)[\tilde{x}, \tilde{v}] = (H[\tilde{x}], K[\tilde{x}] + H[\tilde{y}]) \quad \text{for every } (\tilde{x}, \tilde{v}) \in \mathsf{X} \times \mathsf{X},$$

where

$$H := \mathrm{D}_x f(x), \qquad\qquad K := \mathrm{D}^2_x f(x)[v, \cdot].$$

Thus the thesis follows if we show that for every Fredholm operator $H \in \mathcal{L}(\mathsf{X}; \mathsf{Y})$ of index 0 and for every $K \in \mathcal{L}(\mathsf{X}; \mathsf{Y})$ the operator

$$A : \mathcal{L}(\mathsf{X} \times \mathsf{X}; \mathsf{Y} \times \mathsf{Y}), \quad A[x_1, x_2] = (H[x_1], H[x_2] + K[x_1])$$

is a Fredholm operator of index 0.

We first observe that

$$(x_1, x_2) \in \mathsf{N}(A) \quad \Leftrightarrow \quad x_1 \in \mathsf{N}(H), \quad K[x_1] = -H[x_2],$$

so that, by introducing the finite-dimensional space

$$\mathsf{M} := \Big\{ x \in \mathsf{N}(H) : K[x]s \in \mathsf{R}(H) = \big(\mathsf{N}(H^*)\big)^\perp \Big\},$$

it is easy to check that

$$\dim\big(\mathsf{N}(A)\big) = \dim\mathsf{M}\cdot\dim(\mathsf{N}(H)) < \infty. \tag{2.14}$$

Let now S be a finite-dimensional subspace of Y such that $\mathsf{S}+\mathsf{R}(H)=\mathsf{Y}$. it is immediate to check that $(\mathsf{S}\times\mathsf{S})+\mathsf{R}(A)=\mathsf{Y}\times\mathsf{Y}$ so that $\mathrm{codim}(\mathsf{R}(A))$ is finite and A is a Fredholm operator.

We consider now the continuous perturbation $[0,1]\ni\vartheta\mapsto A_\vartheta := (H, H+\vartheta K)$: every A_ϑ is a Fredholm operator so that the index of $A=A_1$ coincides with the index of $A_0=(H,H)$ since the index is a continuous function. On the other hand it is straightforward to check that A_0 has index 0. □

Let us now consider the restriction (still denoted by g) of g from (2.13) to the open domain

$$\Sigma := X\times(0,T)\times\big(\mathsf{X}\setminus\{0\}\big) \tag{2.15}$$

and notice that

$$g^{-1}(0,0) = \Big\{(x,t,v)\in\Sigma : (x,t)\in\mathcal{S},\ v\in\mathsf{N}\big(\mathrm{D}_x f(x,t)\big)\Big\}. \tag{2.16}$$

According to the common terminology, we say that $(0,0)$ is a regular value of g if $\mathrm{d}g$ is surjective at each point of $g^{-1}(0,0)$.

Lemma 2.8. *Let us suppose that f satisfies Assumption 2.1 and let $g:\Sigma\to\mathsf{Y}\times\mathsf{Y}$ be defined as in* (2.13), (2.15)*. Then f satisfies the transversality conditions if and only if $(0,0)$ is a regular value of g.*

Proof. A direct calculation shows that $\mathrm{d}g(x_0,t_0,v_0)[\tilde x,\tilde t,\tilde v]=(y_1,y_2)$ if and only if

$$\begin{cases} y_1 = \mathrm{D}_x f(x_0,t_0)[\tilde x] + \tilde t\partial_t f(x_0,t_0) = \mathrm{d}f(x_0,t_0)[\tilde x,\tilde t] \\ y_2 = \mathrm{D}_x^2 f(x_0,t_0)[v_0,\tilde x] + \tilde t\,\mathrm{D}_x\partial_t f(x_0,t_0)[v_0] + \mathrm{D}_x f(x_0,t_0)[\tilde v]. \end{cases} \tag{2.17}$$

Let us first suppose that $(0,0)$ is a regular value of g, let $(x_0,t_0)\in\mathcal{S}$ and $0\neq v_0\in\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)\big)$. It follows that $(x_0,t_0,v_0)\in \mathrm{g}^{-1}(0,0)$ so that $\mathrm{d}g(x_0,t_0,v_0)$ is onto and, in view of Lemma 2.4, conditions (T1-2) hold.

Let us now choose $0\neq\ell^*\in\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)^*\big)$, $y_2\in\mathsf{Y}$ such that ${}_{\mathsf{Y}^*}\langle\ell^*,y_2\rangle_{\mathsf{Y}}\neq 0$, $y_1=0$, and a solution $(\tilde x,\tilde t,\tilde v)$ of (2.17). From the first line of (2.17) we get

$$0 = {}_{\mathsf{Y}^*}\langle\ell^*,\mathrm{D}_x f(x_0,t_0)[\tilde x]\rangle_{\mathsf{Y}} + \tilde t\,{}_{\mathsf{Y}^*}\langle\ell^*,\partial_t f(x_0,t_0)\rangle_{\mathsf{Y}} = 0 + \tilde t\,{}_{\mathsf{Y}^*}\langle\ell^*,\partial_t f(x_0,t_0)\rangle_{\mathsf{Y}}$$

so that $\tilde t=0$, since ${}_{\mathsf{Y}^*}\langle\ell^*,\partial_t f(x_0,t_0)\rangle_{\mathsf{Y}}\neq 0$. The same relation then yields $\tilde x\in\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)\big)$, whence $\tilde x=\lambda v_0$, for some $\lambda\in\mathbb{R}$.

From the second line of (2.17) we obtain (taking into account that $\tilde t=0$)

$$\begin{aligned} 0\neq {}_{\mathsf{Y}^*}\langle\ell^*,y_2\rangle_{\mathsf{Y}} &= \lambda\,{}_{\mathsf{Y}^*}\langle\ell^*,\mathrm{D}_x^2 f(x_0,t_0)[v_0,v_0]\rangle_{\mathsf{Y}} + {}_{\mathsf{Y}^*}\langle\ell^*,\mathrm{D}_x f(x_0,t_0)[\tilde v]\rangle_{\mathsf{Y}} \\ &= \lambda\,{}_{\mathsf{Y}^*}\langle\ell^*,\mathrm{D}_x^2 f(x_0,t_0)[v_0,v_0]\rangle_{\mathsf{Y}} + 0. \end{aligned}$$

Therefore ${}_{\mathsf{Y}^*}\langle\ell^*,\mathrm{D}_x^2 f(x_0,t_0)[v_0,v_0]\rangle_{\mathsf{Y}}\neq 0$, which shows the validity of condition (T3).

Let us now prove the converse implication. We assume the validity of (T1-2-3), we choose a point $(x_0,t_0,v_0)\in g^{-1}(0,0)$ and we want to prove that $\mathrm{d}g(x_0,t_0,v_0)$ is onto; equivalently, if $(\ell_1^*,\ell_2^*)\in\big(\mathsf{R}(\mathrm{d}g(x_0,t_0,v_0))\big)^\perp$ then $\ell_1^*=\ell_2^*=0$.

The fact that $\ell_1^*=0$ is an immediate consequence of the surjectivity of $\mathrm{d}f(x_0,t_0)$, which is the first component of $\mathrm{d}g(x_0,t_0,v_0)$ as showed by (2.17).

Choosing $\tilde x=0$, $\tilde t=0$ and an arbitrary $\tilde v$ in the second component of (2.17) we see that $\ell_2^*\in\big(\mathsf{R}(\mathrm{D}_x f(x_0,t_0))\big)^\perp=\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)^*\big)$. Choosing now $\tilde v=0$, $\tilde t=0$, and $\tilde x=v_0$ we get ${}_{\mathsf{Y}^*}\langle\ell_2^*,\mathrm{D}_x^2 f(x_0,t_0)[v_0,v_0]\rangle_{\mathsf{Y}}=0$. Since $0\neq v_0\in\mathsf{N}\big(\mathrm{D}_x f(x_0,t_0)\big)$, (T3) yields $\ell_2^*=0$. □

Notice that for $(x_0,t_0)\in\mathcal{S}$ the operator $\mathrm{d}g(x_0,t_0,0)$ cannot be onto, since the second component of (2.17) reduces to $\mathrm{D}_x f(x_0,t_0)[\cdot]$ whose range has codimension 1.

3. On the genericity of the transversality conditions

In this section we discuss the genericity of the transversality conditions of Definition 2.2. In the following Section 3.1, we give our main result in this direction, Theorem 3.2. It states that, up to a *small* (in the topological sense) set of operators within a certain class, it is always possible to perturb a (suitably smooth) map $f : X \times (0,T) \to \mathsf{Y}$ with such operators, in such a way as to obtain a map $\tilde{f}$ which complies with Definition 2.2. The proof of Theorem 3.2 relies on the characterization of the transversality conditions provided in Section 2.3, and on a well-known genericity result from the seminal paper [8]. In Section 3.2 we revisit the genericity result, as well as Theorem 2.5, in the case where f is the space differential of a time-dependent energy functional.

Notice that the genericity of conditions (T1-T2) of Definition 2.2 with respect to a simpler class of perturbations is a direct consequence of the results of [8].

3.1. **The genericity results.**

In order to make precise in which sense we are going to prove that the transversality conditions hold "generically", let us recall that a set N in a topological space $\mathcal{T}$ is said to be *nowhere dense* if the interior of its closure is empty. Equivalently, its complement $\mathcal{T} \setminus A$ has dense interior, i.e. A is contained in the complement of an open and dense set.

A set is said to be *meagre* if it is contained in a countable union of nowhere dense sets. Conversely, a set is *residual* if it is the complement of a meagre set, i.e. it contains the intersection of a countable collection of open and dense sets.

We will suppose that

$$\mathsf{X}, \mathsf{Y}, \mathsf{Z} \text{ are separable Banach spaces}, \quad X \subset \mathsf{X}, \ 0 \in Z \subset \mathsf{Z} \quad \text{open and connected}, \tag{3.1}$$

and we will deal with a sufficiently large class of additive perturbations of the map $f : X \times (0,T) \to \mathsf{Y}$ of the type

$$\tilde{f}(x,t) := f(x,t) + y + j(x,z), \quad y \in \mathsf{Y}, \ z \in Z, \tag{3.2}$$

obtained by means of $j : X \times Z \to \mathsf{Y}$ of class C^2 with $j(x,0) = 0$. We will suppose that for every $x \in X$, $t \in (0,T)$, $z \in Z$ and $v \in \mathsf{X} \setminus \{0\}$ the following admissibility conditions are satisfied by j:

$$\mathrm{D}_x \tilde{f}(x,t) = \mathrm{D}_x f(x,t) + \mathrm{D}_x j(x,z) \text{ is a Fredholm operator in } \mathcal{L}(\mathsf{X};\mathsf{Y}), \tag{J1}$$

$$(\tilde{v},\tilde{z}) \mapsto \mathrm{D}_x \tilde{f}(x,t)[\tilde{v}] + \mathrm{D}^2_{xz} j(x,z)[v,\tilde{z}] \text{ is surjective}, \tag{J2}$$

i.e. for every $w \in \mathsf{Y}$ there exist $\tilde{v}, \tilde{z} \in \mathsf{X} \times \mathsf{Z}$ such that

$$\mathrm{D}_x f(t,x)[\tilde{v}] + \mathrm{D}_x j(x,z)[\tilde{v}] + \mathrm{D}^2_{xz} j(x,z)[v,\tilde{z}] = w. \tag{J2'}$$

Notice that when j is a bilinear map, (J2') takes the simpler form

$$\mathrm{D}_x f(t,x)[\tilde{v}] + j(\tilde{v},z) + j(v,\tilde{z}) = w. \tag{3.3}$$

In Example 3.1 below, we exhibit a particular case of admissible mapping j, with values in a suitable space of compact operators.

Example 3.1. Let us consider a separable Banach space $\mathcal{K}$ continuously included in $\mathcal{L}(\mathsf{X};\mathsf{Y})$ such that

$$\text{every } K \in \mathcal{K} \text{ is a compact operator}, \tag{K1}$$

$$\text{for every } v \in \mathsf{X} \setminus \{0\}, w \in \mathsf{Y} \text{ there exists } K \in \mathcal{K} : \quad K[v] = w. \tag{K2}$$

As a typical example, we can choose $\mathcal{K}$ to be the closure in $\mathcal{L}(\mathsf{X},\mathsf{Y})$ of the nuclear operators of the form

$$K[x] = \sum_{n=1}^{N} {}_{\mathsf{X}^*}\langle \ell_n, x \rangle_{\mathsf{X}} \, y_n, \quad \ell_n \in \mathsf{X}_0^*, \ y_n \in \mathsf{Y}.$$

where X_0^* is a separable subspace of X^* that separates the points of X. Observe that condition (K2) holds. Indeed, whenever $v \in \mathsf{X} \setminus \{0\}$, $w \in \mathsf{Y}$ are given, by choosing $\ell \in \mathsf{X}_0^*$ so that ${}_{\mathsf{X}^*}\langle \ell, v\rangle_{\mathsf{X}} = 1$ we can simply set

$$K[x] := {}_{\mathsf{X}^*}\langle \ell, x\rangle_{\mathsf{X}}\, w. \tag{3.4}$$

Hence, we let $\mathsf{Z} := \mathcal{K}$ and take as j the bilinear map

$$j : \mathsf{X} \times \mathcal{K} \to \mathsf{Y}, \quad j(x, K) := K[x] \tag{3.5}$$

Then, condition (J1) is satisfied, since for every $K \in \mathcal{K}$ the differential $\mathrm{D}_x j(x, K) = K$ is a compact operator and thus, when added to a Fredholm operator, gives rise to a Fredholm operator with the same index. It is immediate to check that (K2) guarantees the validity of (J2), since $\mathrm{D}^2_{xz} j(x, K)[\tilde v, \tilde K] = \tilde K[\tilde v]$.

We then have the following theorem.

Theorem 3.2. *Let us assume that* (3.1) *and the admissibility conditions* (J1-2) *hold, for a map* $f \in \mathrm{C}^3(X \times (0, T); \mathsf{Y})$ *complying with Assumption 2.1 and for* $j \in \mathrm{C}^2(X \times Z; \mathsf{Y})$.

Every open neighborhood U of the origin in $\mathsf{Y} \times \mathsf{Z}$ contains a residual subset U_r such that for every $(y, z) \in U_r$ the map

$$\tilde f : \mathsf{X} \times (0, T) \longrightarrow \mathsf{Y} \qquad \tilde f(x, t) := f(x, t) + y + j(x, z). \tag{3.6}$$

satisfies the transversality conditions of Definition 2.2.

Proof. First of all, it is useful to pass from f to the functional

$$\mathcal{F} : X \times (0, T) \times \mathsf{Y} \times \mathsf{Z} \longrightarrow \mathsf{Y} \qquad \mathcal{F}(x, t, y, z) := f(x, t) + y + j(x, z),$$

which incorporates the perturbation terms, so that the map $\tilde f$ of (3.6) coincides with $\mathcal{F}(\cdot, \cdot, y, z)$.

In accord with (2.13), we consider the *augmented* functional $\mathcal{G} : \Sigma \times \mathsf{Y} \times Z \to \mathsf{Y} \times \mathsf{Y}$ (recall that $\Sigma := X \times (0, T) \times \big(\mathsf{X} \setminus \{0\}\big)$)

$$\mathcal{G}(x, t, v, y, z) := \begin{pmatrix} \mathcal{F}(x, t, y, z) \\ \mathrm{D}_x \mathcal{F}(x, t, y, z)[v] \end{pmatrix} = \begin{pmatrix} f(x, t) + y + j(x, z) \\ \mathrm{D}_x f(x, t)[v] + \mathrm{D}_x j(x, z)[v] \end{pmatrix}, \tag{3.7}$$

which for every $(y, z) \in \mathsf{Y} \times Z$ gives raise to the perturbed functionals

$$\tilde g(x, t, v) = \mathcal{G}(x, t, v, y, z) = \Big(f(x, t) + y + j(x, z), \big(\mathrm{D}_x f(x, t) + \mathrm{D}_x j(x, z)\big)[v] \Big). \tag{3.8}$$

By Lemma 2.8, $\tilde f$ satisfies the transversality conditions if and only if $(0, 0)$ is a regular value for $\tilde g$. We conclude by applying the next result. □

Proposition 3.3. *Under the same assumptions of Theorem 3.2, the set*

$$\mathfrak{R} := \{(y, z) \in \mathsf{Y} \times \mathsf{Z} \,:\, (0, 0) \text{ is a regular value of the map } (x, t, v) \mapsto \mathcal{G}(x, t, v, y, z)\}$$

is residual *in* $\mathsf{Y} \times \mathsf{Z}$.

Proof. We are going to apply [8, Theorem 1.1, Remark A.1] and thus check that the corresponding assumptions hold, namely

(a) the space $\mathsf{Y} \times \mathsf{Z}$ is separable;
(b) $\mathcal{G} \in \mathrm{C}^2(\Sigma \times \mathsf{Y} \times Z; \mathsf{Y} \times \mathsf{Y})$ and for every $(y, z) \in \mathsf{Y} \times Z$ the map $(x, t, v) \mapsto \mathcal{G}(x, t, v, y, z)$ is Fredholm with index 1;
(c) $(0, 0)$ is a regular value for $\mathcal{G}$.

Condition (a) clearly holds since we have assumed Y and Z separable.

As for (b), clearly $\mathcal{G}$ is of class C^2 thanks to the enhanced C^3-regularity required of f and to the C^2-regularity of j. Moreover, condition (J1), Lemma 2.7 and the same argument of Remark 2.3 yield that every perturbed functional $\tilde g$ as in (3.8) is Fredholm of index 1.

Let us now focus on the last property (c), and let us set $h(t,x,z) := f(t,x) + j(z,x)$. We have to check that, if $\mathcal{G}(x_0,t_0,v_0,y_0,z_0) = (0,0)$, then $\mathrm{d}\mathcal{G}(x_0,t_0,v_0,y_0,z_0)$ is onto, namely that for every $(w_1,w_2) \in \mathsf{Y}\times\mathsf{Y}$ there exists $(\tilde{x},\tilde{t},\tilde{v},\tilde{y},\tilde{z}) \in \mathsf{X}\times\mathbb{R}\times\mathsf{X}\times\mathsf{Y}\times\mathsf{Z}$ such that

$$w_1 = \mathrm{D}_x h(x_0,t_0,z_0)[\tilde{x}] + \tilde{t}\,\partial_t f(x_0,t_0) + \tilde{y} + \mathrm{D}_z j(x_0,z_0)[\tilde{z}], \tag{3.9a}$$

$$w_2 = \mathrm{D}^2_x h(x_0,t_0,z_0)[v_0,\tilde{x}] + \tilde{t}\,\partial_t \mathrm{D}_x f(x_0,t_0)[v_0] + \mathrm{D}_x h(x_0,t_0,v_0)[\tilde{v}] + \mathrm{D}^2_{xz} j(x_0,z_0)[v_0,\tilde{z}]. \tag{3.9b}$$

For this, we choose $\tilde{x}=0$, $\tilde{t}=0$ and we use condition (J2) to find $\tilde{v}\in\mathsf{X}$ and $\tilde{z}\in\mathsf{Z}$ such that (3.9b) is satisfied. In order to fulfill (3.9a), we take $\tilde{y} := w_1 - \mathrm{D}j(z_0)(\tilde{z})[x_0]$. With this, we conclude that (c) holds. □

Remark 3.4 (The finite-dimensional case)**.** Our genericity result in the finite-dimensional setting of Sec. 1, Theorem 1.3, derives from Theorem 3.2.

Indeed, the perturbed map $\tilde{f}$ in (1.8) is of the form (3.6), where we have taken as admissible perturbation j the mapping (3.5) from Example 3.1. We accordingly introduce the finite-dimensional analogues of the functionals $\mathcal{F}$ and $\mathcal{G}$, to which Lemma 2.8 clearly applies. In this case, the set Σ defined in (2.15) reduces to $X\times(0,T)\times(\mathbb{R}^n\setminus\{0\})$ and $\mathcal{K}$ is $\mathbb{M}^{n\times n}$. Also, note that since $f\in\mathrm{C}^3(X\times(0,T);\mathbb{R}^n)$, then $\mathcal{G}$ is of class C^2. Proceeding as in the proof of [8, Theorem 1.1] it is possible to show that proving that the set

$$\mathfrak{R} := \{(y,K)\in\mathbb{R}^n\times\mathbb{M}^{n\times n} : (0,0) \text{ is a regular value of the map } \mathcal{G}(\cdot,\cdot,\cdot,y,K)\} \tag{3.10}$$

has full Lebesgue measure, is equivalent to proving that the set $\mathfrak{V}$ of the regular values of the function $\pi : \mathcal{G}^{-1}(0,0)\to\mathbb{R}^n\times\mathbb{M}^{n\times n}$ has full Lebesgue measure in $\mathbb{R}^n\times\mathbb{M}^{n\times n}$, where π is the projection onto the last two components in $\Sigma\times\mathbb{R}^n\times\mathbb{M}^{n\times n}$. This property follows from the regularity of $\mathcal{G}$, the fact that $(0,0)$ is a regular value of $\mathcal{G}$ so that π is a C^2 map, and the classical Sard's Theorem, since $2 > \dim(\mathcal{G}^{-1}(0,0)) - \dim(\mathbb{R}^n\times\mathbb{M}^{n\times n}) = n^2+n+1-(n+n^2) = 1$. Once (3.10) is established, we can conclude as done in the proof of Theorem 3.2.

3.2. **Critical points of an energy functional.** In this last section we consider the particular case when

$$\mathsf{X}\subset\mathsf{H},\ \mathsf{Y}\subset\mathsf{H}^* \quad \text{with continuous and dense inclusions,}$$

H is a separable Hilbert space and f is the space differential of a time-dependent functional $\mathscr{E} : X\times(0,T)\to\mathbb{R}$, i.e.

$$f = \mathrm{D}_x\mathscr{E}, \qquad \mathscr{E}\in\mathrm{C}^3(X\times(0,T)), \qquad X\subset\mathsf{X} \quad \text{connected and open.} \tag{3.11}$$

We are thus assuming that the differential of the energy takes values (and is regular) in a possibly smaller Banach space Y contained in X^*. On the other hand (see [8, Remark 1.1]) for every $(x,t)\in X\times(0,T)$ we will suppose that the linear operator L associated with the second order differential $\mathrm{D}^2_x\mathscr{E}$ admits a unique continuous extension $\tilde{L}\in\mathcal{L}(\mathsf{H},\mathsf{H}^*)$ satisfying

$$D(\tilde{L};\mathsf{Y}) = \big\{h\in\mathsf{H} : \tilde{L}h\in\mathsf{Y}\big\}\subset\mathsf{X}. \tag{3.12}$$

Notice that for every $v,w\in X$

$${}_{\mathsf{H}^*}\langle Lv,w\rangle_{\mathsf{H}} = \mathrm{D}^2_x\mathscr{E}(x,t)[v,w] = \mathrm{D}^2_x\mathscr{E}(x,t)[w,v] = {}_{\mathsf{H}^*}\langle Lw,v\rangle_{\mathsf{H}},$$

so that $\tilde{L}$ is selfadjoint.

In this setting, $\mathcal{C}$ is the set of critical points of $\mathscr{E}$ and $\mathcal{S}$ is the corresponding singular subset

$$\mathcal{C} := \Big\{(x,t)\in X\times(0,T) : \mathrm{D}_x\mathscr{E}(x,t) = 0\Big\}, \quad \mathcal{S} := \Big\{(x,t)\in\mathcal{C} : \mathrm{D}^2_x\mathscr{E}(x,t) \text{ is not invertible.}\Big\}. \tag{3.13}$$

We will assume that $\mathrm{D}_x\mathscr{E}$ is a Fredholm map of index 0. In particular we can identify the kernel $N = \mathsf{N}(\mathrm{D}^2_x\mathscr{E}(x,t))$ in X with $N^* = \mathsf{N}(\mathrm{D}^2_x\mathscr{E}(x,t)^*)$ in Y^*: it is in fact easy to check that the canonical inclusion $\mathsf{X}\subset\mathsf{Y}^*$ induced by the scalar product of H yields $N\subset N^*$, since (still using the notation L for the second order differential $\mathrm{D}^2_x\mathscr{E}$), $Lv=0$ and $v\in\mathsf{X}$ yield

$${}_{\mathsf{X}^*}\langle L^*v,w\rangle_{\mathsf{X}} = {}_{\mathsf{H}}\langle v,Lw\rangle_{\mathsf{H}^*} = {}_{\mathsf{H}^*}\langle Lv,w\rangle_{\mathsf{H}} = 0 \quad \text{if } v\in\mathsf{N}(L).$$

This implies that $\mathsf{N}(L)$ and $\mathsf{N}(L^*)$ coincide, since they have the same dimension by the index property.

The transversality conditions read

Definition 3.5 (Transversality conditions for a time-dependent functional)**.** *We say that $\mathscr{E}$ satisfies at a point $(x_0, t_0) \in \mathcal{S}$ the transversality conditions if*

(E1) $\dim(\mathsf{N}(\mathrm{D}_x^2\mathscr{E}(x_0,t_0))) = 1$;
(E2) *If* $0 \neq v \in \mathsf{N}(\mathrm{D}_x^2\mathscr{E}(x_0,t_0))$ *then* ${}_{\mathsf{X}^*}\langle \partial_t \mathrm{D}_x\mathscr{E}(x_0,t_0), v\rangle_{\mathsf{X}} \neq 0$;
(E3) *If* $0 \neq v \in \mathsf{N}(\mathrm{D}_x^2\mathscr{E}(x_0,t_0))$ *then* $\mathrm{D}_x^3\mathscr{E}(x_0,t_0)[v,v,v] \neq 0$.

The functional $\mathscr{E}$ satisfies the transversality conditions if (E1-2-3) *hold for every* $(x_0,t_0) \in \mathcal{S}$.

Theorem 2.5 immediately yields:

Corollary 3.6. *If $\mathscr{E} \in \mathrm{C}^3(X \times (0,T))$ is a time-dependent functional with Fredholm differentials $\mathrm{D}\mathscr{E}_x(\cdot,t)$ which satisfies the transversality conditions* (E1-2-3)*, then the sets $\mathcal{C}(t) := \big\{x \in X : \mathrm{D}_x\mathscr{E}(x,t) = 0\big\}$ are discrete for every $t \in (0,T)$.*

We now address the genericity of the transversality conditions from Definition 3.5. In Corollary 3.7 below we rephrase, in terms of the functional $\mathscr{E}$, the statement of Theorem 3.2, considering here only the simple case of Example 3.1. Accordingly, we consider the set $\mathcal{N}_{sym}$ obtained by taking the closure in $\mathcal{L}^2(\mathsf{X};\mathbb{R})$ of all the symmetric bilinear forms of the type

$$\mathscr{K}(x,y) = \sum_{j=1}^{n} {}_{X^*}\langle \ell_j, x\rangle_X \ {}_{X^*}\langle \ell_j, y\rangle_X\,. \tag{3.14}$$

Corollary 3.7. *Let $\mathscr{E} \in \mathrm{C}^4(X \times (0,T))$ be a time-dependent functional with Fredholm differentials. Every open neighborhood U of the origin in $\mathsf{X}^* \times \mathcal{N}_{sym}$ contains a residual subset U_r such that for every $(\ell, \mathscr{K}) \in U_r$ the functionals*

$$\tilde{\mathscr{E}}(x,t) = \mathscr{E}(x,t) + {}_{X^*}\langle \ell, x\rangle_X + \frac{1}{2}\mathscr{K}(x,x) \tag{3.15}$$

satisfy the transversality conditions (E1-2-3)*.*

Proof. We apply Theorem 3.2: notice that the perturbations (3.15) of $\mathscr{E}$ correspond to the family of perturbations for $f = \mathrm{D}_x\mathscr{E}$

$$\tilde{f}(x,t) = f(x,t) + \ell + K[x], \tag{3.16}$$

where $K \in \mathcal{L}(\mathsf{X},\mathsf{X}^*)$ is associated with $\mathscr{K}$ by

$${}_{X^*}\langle K\, x_1, x_2\rangle_X = \mathscr{K}(x_1,x_2), \qquad \text{for every} \quad x_1, x_2 \in \mathsf{X}. \tag{3.17}$$

Clearly, the collection of all such operators satisfies the admissibility conditions (K1-2); in order to check (K3), we fix $x \in \mathsf{X} \setminus \{0\}$, $\ell \in \mathsf{X}^*$, and we consider the bilinear forms

$$\mathscr{K}(x_1,x_2) = \frac{{}_{X^*}\langle \ell, x_1\rangle_X \ {}_{X^*}\langle \ell, x_2\rangle_X}{{}_{X^*}\langle \ell, x\rangle_X} \qquad \text{if} \ {}_{X^*}\langle \ell, x\rangle_X \neq 0,$$
$$\mathscr{K}(x_1,x_2) = {}_{X^*}\langle \ell, x_1\rangle_X \ {}_{X^*}\langle x^*, x_2\rangle_X + {}_{X^*}\langle x^*, x_1\rangle_X \ {}_{X^*}\langle \ell, x_2\rangle_X \qquad \text{if} \ {}_{X^*}\langle \ell, x\rangle_X = 0,$$

where $x^* \in \mathsf{X}^*$ satisfies ${}_{X^*}\langle x^*, x\rangle_X = 1$. It is immediate to check that the associate linear operator K satisfies $K[x] = \ell$. □

We conclude by exhibiting an integral energy functional $\mathscr{E}$, whose critical points (i.e. the zeroes of its space differential) solve a semilinear elliptic equation on a bounded domain Ω. Therefore, as customary we will use the letter x to denote the points in Ω, and write $\mathscr{E}(u,t)$ in place of $\mathscr{E}(x,t)$.

Example 3.8. Let Ω be a bounded connected open set in $\mathbb{R}^d$, $d \le 3$, and let

$$\mathsf{X} = H^2(\Omega) \cap H^1_0(\Omega), \qquad \mathsf{Y} = L^2(\Omega),$$

and

$$\mathscr{E}(u,t) := \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 + \mathcal{W}(u(x)) - \ell(t)u(x) \right) \mathrm{d}x$$

with $\ell \in \mathrm{C}^4(0,T;L^2(\Omega))$ and $\mathcal{W}$ the usual double-well potential $\mathcal{W}(u) = (u^2-1)^2/4$. Observe that $\mathscr{E} \in \mathrm{C}^4(\mathsf{X} \times (0,T))$ thanks to the continuous imbedding of X in $L^\infty(\Omega)$.

We have that

$$f = \mathrm{D}_u \mathscr{E} : \mathsf{X} \times (0,T) \to \mathsf{Y} \text{ is given by } f(u,t) = Au + \mathcal{W}'(u) - \ell(t)$$

with $A : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ the Laplacian operator with homogeneous Dirichlet boundary conditions. Note that A is Fredholm with index 0. It follows from [7, Thm. 5.26, p. 238] that also $f(\cdot,t)$ is a Fredholm map with index 0 for every $t \in (0,T)$. We have that

$$\mathrm{D}_u f(u_0,t_0)[v] = Av + \mathcal{W}''(u_0)v \qquad \text{for all } v \in \mathsf{X}.$$

Let us now construct an explicit perturbation of f to which Theorem 3.2 applies. We take

$$\mathsf{Z} = \mathrm{C}(\overline{\Omega}) \text{ and } j : \mathsf{Z} \times \mathsf{X} \to \mathsf{Y} \text{ defined by } j(u,z) := zu, \tag{3.18}$$

so that $j(u,z)$ is a bilinear operator. Observe that the perturbed field $\tilde f(u,t) = f(u,t) + y + zu$ is the space differential of the perturbed energy

$$\tilde{\mathscr{E}}(u,t) = \mathscr{E}(u,t) + \int_\Omega y\, u \,\mathrm{d}x + \frac{1}{2}\int_\Omega z\, u^2 \,\mathrm{d}x. \tag{3.19}$$

It follows again from [7, Thm. 5.26, p. 238] that $\tilde f(\cdot,t)$ is a Fredholm map of index 0 for every $t \in (0,T)$.

In order to check that Theorem 3.2 applies in this setting, it would remain to verify that j in (3.18) complies with condition (J2') in the form (3.3). Therefore, for given $u_0 \in \mathsf{X}, t_0 \in (0,T), v_0 \in \mathsf{X} \setminus \{0\}$ and $z_0 \in \mathrm{C}(\overline{\Omega})$ we want to show that for every $w \in L^2(\Omega)$ the equation

$$A\tilde v + (W''(u_0) + z_0)\tilde v + v_0 \tilde z = w \tag{3.20}$$

admits at least a solution $\tilde v \in \mathsf{X}$, $\tilde z \in \mathrm{C}(\overline{\Omega})$. Since the operator $\tilde v \mapsto A\tilde v + (W''(u_0) + z_0)\tilde v$ is Fredholm of index 0 from X to Y (and thus its range is closed with finite codimension), it is sufficient to prove that the only element $\xi \in L^2(\Omega)$ such that

$$\int_\Omega \xi\Big(A\tilde v + (W''(u_0) + z_0)\tilde v + v_0\tilde z\Big)\,\mathrm{d}x = 0 \quad \text{for every } \tilde v \in \mathsf{X},\ \tilde z \in \mathrm{C}(\overline{\Omega})$$

is $\xi \equiv 0$.

Choosing $\tilde z = 0$ and an arbitrary $\tilde v$ we get

$$\int_\Omega \xi\Big(A\tilde v + (W''(u_0) + z_0)\tilde v\Big)\,\mathrm{d}x = 0 \quad \text{for every } \tilde v \in \mathsf{X},$$

so that $\xi \in \mathsf{X}$ and

$$A\xi + (W''(u_0) + z_0)\xi = 0. \tag{3.21}$$

On the hand, choosing $\tilde v = 0$ and arbitrary $\tilde z$ we get $\xi = 0$ on the set $\omega = \{x \in \Omega : v_0(x) \neq 0\}$, which is open and not empty since $v_0 \in \mathrm{C}(\overline{\Omega}) \neq 0$. Then, (3.21) and the unique continuation principle (see e.g. [6]) imply that $\xi = 0$. Therefore, condition (3.20) is satisfied

Theorem 3.9. *Every open neighborhood U of the origin in $L^2(\Omega) \times \mathrm{C}(\overline{\Omega})$ contains a (dense) residual set U_r such that the functional $\tilde{\mathscr{E}}$ defined by* (3.19) *satisfies the transversality conditions for every $(y,z) \in U_r$. In particular, for every $t \in (0,T)$ and $(y,z) \in U_r$ the solutions $u \in H^2(\Omega) \cap H^1_0(\Omega)$ of the equation*

$$-\Delta u + \mathcal{W}'(u) + zu = y + \ell(t) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{3.22}$$

are isolated in $H^2(\Omega) \cap H^1_0(\Omega)$.

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