LORENTZ ESTIMATES FOR DEGENERATE AND SINGULAR EVOLUTIONARY SYSTEMS

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ABSTRACT. We prove estimates of Calderón-Zygmund type for evolutionary p-Laplacian systems in the setting of Lorentz spaces. We suppose the coefficients of the system to satisfy only a VMO condition with respect to the space variable. Our results hold true, *mutatis mutandis*, also for stationary p-Laplacian systems.

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1. INTRODUCTION

The object of this note is the study of the possibility of Calderón-Zygmund type estimates in the scale of Lorentz spaces for weak solutions to parabolic systems of p-Laplacian type:

$$u_t - \operatorname{div}\big[a(x,t)|Du|^{p-2}Du\big] = \operatorname{div}\big[|F|^{p-2}F\big], \qquad p > \frac{2n}{n+2}, \tag{1.1}$$

where (1.1) is considered in the parabolic cylinder $\Omega_T := \Omega \times (-T, 0)$, being $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a bounded connected open set, $T > 0$ and $u : \Omega_T \to R^N$, $N \geq 1$. More precisely, we want to study the integrability of the spatial gradient Du in terms of the integrability of the datum appearing on the right-hand side in the scale of Lorentz spaces. In particular we shall prove that for equation (1.1) the implication

$$|F| \in L(\gamma, q) \quad \text{locally in } \Omega_T \quad \Longrightarrow \quad |Du| \in L(\gamma, q) \quad \text{locally in } \Omega_T \tag{1.2}$$

holds for $\gamma > p$ and $q \in (0, \infty]$. We recall that the Lorentz space $L(\gamma, q)(A)$, for $A \subset R^k$, $k \in \mathbb{N}$, open set and for parameters $1 \leq \gamma < \infty$ and $0 < q < \infty$, is defined by requiring, for a measurable function $g : A \to \mathbb{R}$, the quantity

$$\|g\|^q_{L(\gamma,q)(A)} := q \int_0^\infty \Big(\lambda^\gamma \big|\{\xi \in A : |g(\xi)| > \lambda\}\big|\Big)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} \tag{1.3}$$

to be finite. For the case $q = \infty$, the space $L(\gamma, \infty)(A)$, $1 \leq \gamma < \infty$, is nothing else than the Marcinkiewicz space $\mathcal{M}^\gamma(A)$, that is the space of measurable functions g such that

$$\|g\|_{L(\gamma,\infty)(A)} = \|g\|_{\mathcal{M}^\gamma(A)} := \sup_{\lambda>0} \Big(\lambda^\gamma \big|\{\xi \in A : |g(\xi)| > \lambda\}\big|\Big)^{\frac{1}{\gamma}} < \infty. \tag{1.4}$$

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The local variant of such spaces is defined in the usual way; see Paragraph 3.2 for some more details about Lorentz spaces. We anticipate here that we are not necessarily going to consider continuous coefficient, but rather ones having a controlled deviation from their averages on balls; i.e., we shall consider a natural VMO condition just with respect to the spatial variable x, see (3.3)-(3.4). Note that the lower bound in (1.1) naturally appears in the regularity theory of the parabolic p-Laplacian operator (see [18, 41, 30, 8]) and it is unavoidable for the type of regularity we are considering here.

Even if we refer to the following Section 2 for a more detailed, but at the same time just partial, description of the history of the problem, we want here to stress that the starting point of our work is the paper [2] by Acerbi & Mingione, where the (Lebesgue) case $q = \gamma$ has been settled: the authors proved the estimate

$$F \in L^{\gamma}_{\text{loc}}(\Omega_T, \mathbb{R}^N) \qquad \Longrightarrow \qquad Du \in L^{\gamma}_{\text{loc}}(\Omega_T, \mathbb{R}^N), \tag{1.5}$$

for solution to (1.1), with $\gamma > p$. This (quite recent) paper has earned several extensions in the last years, and more in general estimates of this kind (which we call of *Calderón-Zygmund type*) have gained a very strong interest: for instance see the results in elliptic domains with rough boundaries [12, 14, 15], the global estimates in parabolic domains done in [10, 11, 13], and also the results related to the obstacle problem [8, 9].

Surprisingly enough, the very natural extension to the Lorentz spaces setting of (1.5) has remained unproved up to now, we think essentially due to technical reasons. We settle here this natural fragment of the theory by exploiting the tools we developed in [3, 4, 6]; we stress our approach also allows to treat parabolic obstacle problems of p-Laplacian type, see the forthcoming [5].

Elliptic estimates. The approach we are developing here gives as a byproduct also the elliptic version of (1.2), where the objects in play are the ones one expects. In particular we consider here the elliptic system

$$\operatorname{div}\left[c(x)|Du|^{p-2}Du\right] = \operatorname{div}\left[|G|^{p-2}G\right] \qquad \text{in } \Omega, \tag{1.6}$$

where Ω is again a bounded connected open set of $\mathbb{R}^n$ and u maps Ω into $\mathbb{R}^N$; we can consider here the full range $p > 1$, since in the elliptic case the restriction in (1.1) is not anymore necessary. Implication (1.2) reads here as

$$|G| \in L(\gamma, q) \quad \text{locally in } \Omega \qquad \Longrightarrow \qquad |Du| \in L(\gamma, q) \quad \text{locally in } \Omega.$$

We preferred to focus our approach on the parabolic problem mainly since techniques we employ here were born as parabolic ones, see the forthcoming Section 2; hence we opt to present in a more detailed way the proof in the time-dependent setting, in order to show the basic points of the argument, and then show how to modify such proof to get the (simpler) elliptic one.

2. A SKETCHY HISTORY OF THE PROBLEM

The starting point of the so-called *nonlinear Calderón-Zygmund theory* is the paper [24] of Iwaniec where it is proved that for solutions to the equation (1.6) (i.e. we take $N = 1$ here), for $\Omega = \mathbb{R}^n$, the implication

$$G \in L^{\gamma}(\mathbb{R}^n) \qquad \Longrightarrow \qquad Du \in L^{\gamma}(\mathbb{R}^n) \tag{2.1}$$

holds true for $\gamma > p$. DiBenedetto & Manfredi in [19] extended this result to systems and also caught the borderline case

$$|G|^{p-2}G \in BMO(\mathbb{R}^n, \mathbb{R}^N) \qquad \Longrightarrow \qquad Du \in BMO(\mathbb{R}^n, \mathbb{R}^N).$$

In the same [19] can also be found a localized version of the first implication, valid also for systems: for Ω bounded open set,

$$G \in L^{\gamma}(\Omega, \mathbb{R}^N) \qquad \Longrightarrow \qquad Du \in L^{\gamma}(\Omega, \mathbb{R}^N) \tag{2.2}$$

for $\gamma > p$, see also [1], where more general structures are considered. This last result can also be recovered using the elliptic theorem of this paper, since by Fubini's Theorem there holds $L(\gamma,\gamma) \equiv L^{\gamma}$.

The parabolic analogue of (2.2), for solutions to (1.1),

$$F \in L^{\gamma}(\Omega_T, \mathbb{R}^N) \qquad \Longrightarrow \qquad Du \in L^{\gamma}(\Omega_T, \mathbb{R}^N) \tag{2.3}$$

for all $\gamma > p$ was settled, as we already said, in [2] by Acerbi & Mingione. Their technique of proof, which we shall follow, had to be necessarily different from both these of Iwaniec and DiBenedetto & Manfredi, since the lack of homogeneity of evolutionary p-Laplacian rules out approaches based on maximal functions and Harmonic Analysis tools. Indeed, estimates for solution to p-Laplacian type equations and systems have a non-homogeneous character when considered on standard parabolic cylinders $Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^2, t_0)$; therefore, usual iteration and covering arguments do not apply. Following DiBenedetto's approach, in the case of parabolic p-Laplacian problems one has to work with cylinders of the form

$$Q_R^{\lambda}(x_0,t_0) = B_R(x_0) \times (t_0 - \lambda^{2-p}R^2, t_0)$$

(when $p \geq 2$) for $\lambda \geq 1$ and $R > 0$, where the average of the gradient of the solution considered is approximatively λ, i.e.

$$\fint_{Q_R^{\lambda}} |Du|^p \, dz \approx \lambda^p.$$

On such cylinders parabolic p-Laplacian problems behave heuristically as non-degenerate ones and therefore estimates show homogeneity, allowing for (appropriate) covering and iteration arguments. Following [2, 25, 26], the next estimate for the measure of the super-level set of the gradient holds true:

$$\begin{aligned} \big|Q_R \cap \{|Du| > T\lambda\}\big| \lesssim c(T)\frac{\epsilon}{\lambda^p} \int_{Q_{2R}\cap\{|Du|>\lambda\}} |Du|^p \, dz \\ + c(T)\,\frac{1}{\epsilon\lambda^{p\gamma}} \int_{Q_{2R}\cap\{|F|>\lambda\}} |F|^{p\gamma} \, dz, \end{aligned} \tag{2.4}$$

where T is an *a priori defined* constant and $\epsilon \in (0,1)$ is to be chosen. Having this estimate at hand and using Fubini's Theorem leads quickly to local estimates on cylinders of the $L^{p\gamma}$ norm of Du and therefore to (2.3). The presence of the *weight* ϵ in the last display follows from the appropriate choice of *intrinsic cylinders* associated to the problem (1.1):

$$\fint_{Q_R^{\lambda}} |Du|^p \, dz + \frac{1}{\epsilon}\left(\fint_{Q_R^{\lambda}} |F|^{p\gamma} \, dz\right)^{1/\gamma} \approx \lambda^p.$$

This technique - which allows to re-prove the elliptic results without use of Harmonic Analysis-based arguments - shows to be particularly flexible and has been applied also to both elliptic and parabolic measure data problems [31, 34, 3]. Indeed estimates as (2.4) allow to deal with almost every kind of rearrangement invariant function spaces, see also [32, 6, 7], and in this paper we show the implementation of this technique to the setting of Lorentz spaces; see also the forthcoming [4] for an application of this machinery to the setting of (parabolic and degenerate) measure data problems.

2.1. **Introducing coefficients.** In the elliptic linear case

$$\operatorname{div}\big[A(x)Du\big] = \operatorname{div} F, \tag{2.5}$$

if the matrix $A(x) \equiv A$ is constant, Calderón-Zygmund estimates follow from the classic work of Stampacchia [35, 36]. Similar linear interpolation techniques apply when the x-dependence of the matrix is continuous, via perturbation techniques, see Campanato [16]. This Functional Analysis approach has been pursuit on the other hand, for the p-Laplacian operator, by Kinnunen & Zhou in [27], where they prove (2.2), for $N = 1$, under the assumption that the coefficient matrix is (bounded and) VMO continuous, that is

$$\lim_{R\searrow 0} \omega_A(R) = 0, \quad \text{where} \quad \omega_A(R) = \sup_{\substack{B_\rho(x)\subset\Omega\\ 0<\rho\le R}} \fint_{B_\rho(x)} \big|A-(A)_{B_\rho(x)}\big|\,dy;$$

$(A)_{B_\rho}$ denotes here the average of A over the ball B_ρ, see Paragraph 3.1. Note that a continuous function is VMO continuous, but the converse implication does not in general hold true. In a sense, VMO condition prescribes that the oscillation of A goes to zero not pointwise, but in an integral sense.

One now can expect that for parabolic equations the correct condition to impose on the coefficients is the *global* VMO regularity, as done in [2], in the sense that one would require that the excess of the coefficients over parabolic cylinders goes to zero: referring to (1.1)

$$\lim_{R\searrow 0} \omega_a(R) = 0 \quad \text{where} \quad \omega_a(R) = \sup_{\substack{Q_\rho(z)\subset\Omega_T\\ 0<\rho\le R}} \fint_{Q_\rho(z)} \big|a-(a)_{Q_\rho(z)}\big|\,dz.$$

As noticed by Krylov in [28], however, since we are dealing with *spatial gradient regularity*, only VMO regularity with respect to the *spatial variable* is sufficient: this is to say that if we consider product coefficients $a(x,t) = d(x)h(t)$ in (1.1), we need to require boundedness and VMO regularity of d but just *boundedness and measurability* of h, see the next Section 3.

2.2. **General vectorial structures.** Note that in the case $N = 1$ (2.1) together with its local variants (2.2) and (2.3) can be extended to general structures of p-Laplacian type, that is vector fields $a(\cdot)$ satisfying

$$\begin{cases} \langle \partial a(\xi)\lambda,\lambda\rangle \ge \nu\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2,\\ |a(\xi)| + \big(s^2+|\xi|^2\big)^{\frac12}|\partial a(\xi)| \le L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.6}$$

for all $\lambda,\xi\in\mathbb{R}^n$, with $s\in[0,1]$ the degeneracy parameter and for $0<\nu\le 1\le L<\infty$.

Such a generalization for systems is not anymore possible, and the reason in clear once looking at the elliptic case: if Du would belong to L^γ, with $\gamma\gg 1$, then this would imply the boundedness of u by Sobolev's embedding. However, for the case $0 \equiv F \in L^\gamma(\Omega_T, \mathbb{R}^N)$ for all γ, the counterexample of Šverák and Yan [38] shows that solution to

$$\operatorname{div} a(Du) = 0, \qquad u : B_1 \subset \mathbb{R}^5 \to \mathbb{R}^{14}$$

and $a(\cdot)$ satisfying (2.6) for $p = 2$, adapted to the vectorial case, are not necessarily bounded. An analog of implications (2.2) and (2.3) would on the other hand hold for systems enjoying a peculiar structure, and this is called *quasi-diagonal p-Laplacian (or Uhlenbeck) structure*:

$$a\big(Du\big) = g\big(|Du|^2\big)Du \qquad \text{where} \qquad g\big(|Du|^2\big) \approx |Du|^{p-2}.$$

In other words, in order to prove (2.3) in the full range $\gamma > p$ (and here one could spend some words why should this be the full range, and not $\gamma > p-1$, as in the case of linear operators as (2.5), but this would go far beyond the purposes of this short introduction), additional structure must be imposed on the vector field, see for instance the last part of [2]; that is, asking that the gradient non-linearity depends on Du via its modulus $|Du|$, or $|Du|^2$. This allows to prove that the function $v = |Du|^2$ is a non-negative sub-solution of a certain PDE, and it is precisely such a property, which for elliptic systems is called “quasi-subharmonicity”, that makes possible to prove gradient boundedness (again here Šverák and Yan counterexample shows that it is not to be expected in the general case) and everywhere $C^{1,\alpha}$ regularity, see the basic work of Uhlenbeck [40] and the one of Tolksdorf [39]. A generalization of our work for these structures, also encoding in a genuine non-linear way VMO regularity as done in [13, 29], is possible, but shall not be considered here, where we focus on the genuine p-Laplacian structure.

Finally we mention that the way to match (2.3) with general p-Laplacian structures, keeping into account the previous counterexample, is to consider exponents $\gamma \not\gg 1$, that is, when considering general vector fields satisfying (2.6), to prove (2.3) for the range

$$p \le \gamma \le p + \frac{4}{n} + \epsilon,$$

with ϵ a small constant, depending on n, p, ν, L.

3. Assumptions, statement of the results, notation, tools

For the parabolic coefficient $a : \Omega_T \to \mathbb{R}$ we assume that it is measurable and that

$$\nu \le a(x,t) \le L \tag{3.1}$$

holds for any $(x,t) \in \Omega_T$ and for constants $0 < \nu \le 1 \le L < \infty$. With regard to its regularity, we will assume that it satisfies a VMO condition with respect the spatial variable. More precisely, denoting

$$(a)_{B_\rho(x)}(t) := \fint_{B_\rho(x)} a(x,t)\,dx \qquad \text{for } B_\rho(x) \subset \Omega, \tag{3.2}$$

we define $\omega_a \colon [0,\infty) \to [0,1]$ in the following way:

$$\omega_a(R) := \frac{1}{2L} \sup_{t\in(-T,0)} \sup_{\substack{B_\rho(x)\subset\Omega\\ 0<\rho\le R}} \fint_{B_\rho(x)} \big|a(\cdot,t) - (a)_{B_\rho(x)}(t)\big|\,dy \tag{3.3}$$

for any $R > 0$ and we suppose that

$$\lim_{R\searrow 0} \omega_a(R) = 0. \tag{3.4}$$

Here, we stress that we assume not more than measurability and boundedness with respect to the time variable; moreover, our assumptions on a allow product coefficients of the type $a(x,t) = d(x)h(t)$, with $d \in VMO(\Omega) \cap L^\infty(\Omega)$ and $h \in L^\infty(0,T)$. Note moreover that here we are considering "something closer" to $\mathrm{VMO_{loc}}$ than to the classic definition of VMO, that would involve also balls intersecting $\partial\Omega$. However, since here we are interested in interior regularity, this definition is sufficiently general for our purposes, which are encoded in the following

Theorem 3.1. *Let u be a weak solution to (1.1), with $a(\cdot)$ satisfying (3.1) and (3.4) and with $|F| \in L(\gamma,q)$ locally in Ω_T, for $\gamma > p$ and $0 < q \le \infty$; then $|Du| \in L(\gamma,q)$ locally in Ω_T. Moreover there exists a radius R_0, depending on n,N,p,ν,L,γ,q such that the following local estimate holds, for cylinders $Q_{2R}(z_0) \equiv Q_{2R} \subset \Omega_T$ with $R \le R_0$:*

$$|Q_R|^{-\frac{1}{\gamma}}\|Du\|_{L(\gamma,q)(Q_R)} \le c\left(\fint_{Q_{2R}} |Du|^p\,dz\right)^{\frac{d}{p}} \\ + c\,|Q_{2R}|^{-\frac{d}{\gamma}}\big\||F|+1\big\|^d_{L(\gamma,q)(Q_{2R})}, \tag{3.5}$$

for a constant depending on n,N,p,ν,L,γ,q (except in the case $q = \infty$, where the constant and R_0 depend only on n,N,p,ν,L,γ) and where the scaling deficit $d \ge 1$ is defined by

$$d \equiv d(p) := \begin{cases} \dfrac{p}{2} & \text{if} \quad p \ge 2, \\[2ex] \dfrac{2p}{p(n+2)-2n} & \text{if} \quad \dfrac{2n}{n+2} < p < 2. \end{cases} \tag{3.6}$$

We remark here that the constant depends critically on $\gamma - p$, as it blows up when $\gamma \to p$. The same will happen for the elliptic problem. Here, referring to (1.6), we shall suppose $c : \Omega \to \mathbb{R}$ bounded and VMO regular:

$$\nu \le c(x) \le L, \qquad \text{for } x \in \Omega \quad \text{and} \quad \lim_{R\searrow 0} \omega_c(R) = 0, \tag{3.7}$$

where now, with $(c)_{B_\rho(x)} := \fint_{B_\rho(x)} c(y)\,dy$,

$$\omega_c(R) := \frac{1}{2L} \sup_{\substack{B_\rho(x)\subset\Omega \\ 0<\rho\le R}} \fint_{B_\rho(x)} \big|c(\cdot) - (c)_{B_\rho(x)}\big|\,dy. \tag{3.8}$$

In this case the Calderón-Zygmund result takes the following form; note that here the deficit scaling d is not anymore present.

Theorem 3.2. *Let u be a solution to (1.6), with $c(\cdot)$ satisfying (3.7) and $|G| \in L(\gamma,q)$ locally in Ω, for $\gamma > p$ and $0 < q \le \infty$; then $|Du| \in L(\gamma,q)$ locally in Ω. Moreover there exists a radius R_0 and a constant c such that the following estimate holds, for balls $B_{2R}(x_0) \equiv B_{2R} \subset \Omega$ with $R \le R_0$:*

$$|B_R|^{-\frac{1}{\gamma}}\|Du\|_{L(\gamma,q)(B_R)}$$

$$\leq c\left(\fint_{B_{2R}}|Du|^p\,dx\right)^{\frac{1}{p}}+c\,|B_{2R}|^{-\frac{1}{\gamma}}\|G\|_{L(\gamma,q)(B_{2R})}.$$

The constant and R_0 have the same dependencies as in Theorem 3.1, except from the fact that they depend on ν, L only through the ellipticity ratio L/ν.

3.1. **Notation.** By *weak solution* to (1.1), following [18], we mean a map

$$u\in C(-T,0;L^2(\Omega,\mathbb{R}^N))\cap L^p(-T,0;W^{1,p}(\Omega,\mathbb{R}^N))$$

such that for any $-T\leq t_1<t_2\leq 0$ the integral formulation

$$-\int_\Omega u\cdot\varphi(\cdot,\tau)\,dx\Big|_{\tau=t_1}^{t_2}+\int_{\Omega\times(t_1,t_2)}\big[u\cdot\varphi_t-\big\langle a(\cdot)|Du|^{p-2}Du,D\varphi\big\rangle\big]\,dz$$
$$=\int_{\Omega\times(t_1,t_2)}\big\langle|F|^{p-2}F,D\varphi\big\rangle\,dz \tag{3.9}$$

holds for every test function $\varphi\in C_c^\infty(\Omega_T,\mathbb{R}^N)$; here $\cdot$ denotes the scalar product in $\mathbb{R}^N$ while $\langle\cdot,\cdot\rangle$ denotes that in R^{nN}. Usually one is lead to consider a slicewise reformulation of (3.9) in terms of the so-called Steklov averages, in order to overcome the problems that could appear once using the solution itself as test function, due to its lack of regularity with respect to the time variable. However, this is quite standard and here we will proceed formally, referring to appropriate papers for the rigorous computations.

Analogously with *weak solution* to (1.6) we mean a function $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega\big\langle c(\cdot)|Du|^{p-2}Du,D\phi\big\rangle\,dx=\int_\Omega\big\langle|G|^{p-2}G,D\phi\big\rangle\,dx$$

holds for every test function $\phi\in C_c^\infty(\Omega,\mathbb{R}^N)$.

In the parabolic setting $\mathbb{R}^{n+1}$ will always be thought as $\mathbb{R}^n\times\mathbb{R}$, so a point $z\in\mathbb{R}^{n+1}$ will be often also denoted as (x,t), z_0 as (x_0,t_0) and so on. Being $B_R(x_0)$ the ball $\{x\in\mathbb{R}^n:|x-x_0|<R\}$, we shall consider parabolic cylinders of the form

$$Q_R(z_0):=B_R(x_0)\times(t_0-R^2,t_0),$$

but we shall also deal with scaled cylinders of the form

$$Q_R^\lambda(z_0):=\begin{cases}B_R(x_0)\times\big(t_0-\lambda^{2-p}R^2,t_0\big) & \text{if } p\geq 2,\\ B_{\lambda^{\frac{p-2}{2}}R}(x_0)\times\big(t_0-R^2,t_0\big) & \text{if } p<2,\end{cases}$$

where the stretching parameter *will be always greater than one*: $\lambda\geq 1$; hence in both cases $Q_R^\lambda(z_0)\subset Q_R^1(z_0)=Q_R(z_0)$. We shall denote $\Lambda_R^\lambda(t_0)=:\big(t_0-\lambda^{2-p}R^2,t_0\big)$ and $B_R^\lambda(x_0):=B_{\lambda^{\frac{p-2}{2}}R}(x_0)$, and we shall drop the λ when it will be one: $\Lambda_R(t_0)=:\big(t_0-R^2,t_0\big)$ and $B_R^\lambda(x_0):=B_R(x_0)$.

With $\chi B_R(x_0)$, for a constant $\chi>1$, we will denote the χ-times enlarged ball, i.e. $\chi B_R(x_0):=B_{\chi R}(x_0)$, and the same for cylinders: $\chi Q_R^\lambda(z_0):=Q_{\chi R}^\lambda(z_0)$. By parabolic boundary of a cylinder $\mathcal{K}:=C\times I$ in $\mathbb{R}^{n+1}$, we mean $\partial_{\mathcal{P}}\mathcal{K}:=C\times\{\inf I\}\cup\partial C\times I$. Being $A\in\mathbb{R}^k$ a measurable set with positive measure and

$f : A \to \mathbb{R}^m$ an integrable map, with $k, m \geq 1$, we denote with $(f)_A$ the averaged integral

$$(f)_A := \fint_A f(\xi)\, d\xi := \frac{1}{|A|}\int_A f(\xi)\, d\xi.$$

We will denote with c a generic constant always greater than one, possibly varying from line to line; however, the ones we shall need to recall will be denoted with special symbols, such as $c_{DiB}, \tilde{c}, c_*, c_\ell$. We finally remark that by sup we shall always mean essential supremum.

3.2. **Lorentz spaces.** The reader might recall the definition of Lorentz spaces in (1.3)-(1.4). Since here we assume A to have finite measure, the spaces $L(\gamma, q)$ decrease in the first parameter γ; this means that for $1 \leq \gamma_1 \leq \gamma_2 < \infty$ and $0 < q \leq \infty$ we have a continuous embedding $L(\gamma_2, q)(A) \hookrightarrow L(\gamma_1, q)(A)$ with

$$\|g\|_{L(\gamma_1,q)(A)} \leq |A|^{\frac{1}{\gamma_1}-\frac{1}{\gamma_2}} \|g\|_{L(\gamma_2,q)(A)}.$$

On the other hand the Lorentz-spaces increase in the second parameter q, i.e. we have for $0 < q_1 \leq q_2 \leq \infty$ the continuous embedding $L(\gamma, q_1)(A) \hookrightarrow L(\gamma, q_2)(A)$ with

$$\|g\|_{L(\gamma,q_2)(A)} \leq c(\gamma, q_1, q_2)\|g\|_{L(\gamma,q_1)(A)}$$

when $q_2 < \infty$, while the constant clearly does not depend on q_2 when $q_2 = \infty$; see, essentially, Lemma 3.5 for $\lambda = 0$ and an appropriate choice of the quantities involved. Note moreover that by Fubini's theorem we have

$$\|g\|^\gamma_{L^\gamma(A)} = \gamma \int_0^\infty \lambda^\gamma \big|\{\xi \in A : |g(\xi)| > \lambda\}\big| \frac{d\lambda}{\lambda} = \|g\|^\gamma_{L(\gamma,\gamma)(A)}\,,$$

so that $L^\gamma(A) = L(\gamma, \gamma)(A)$. Note moreover that $L(\gamma, q)(A) \subset L^p(A)$ for any $\gamma > p$ and all $0 < q \leq \infty$, see for instance (5.19).

Remark 3.3. Note that the notation we use might be misleading, since, due to the lack of sub-additivity, the quantity $\|\cdot\|_{L(\gamma,q)(A)}$ is just a quasi-norm. Nevertheless, the mapping $g \mapsto \|g\|_{L(\gamma,q)(A)}$ is lower semi-continuous with respect to a.e. convergence, see [32, Remark 3] or [7, Section 3].

3.3. **Technical tools.** The first inequality we shall need is a variant of the classic Hardy's inequality; see [22, Theorem 330] or also [23].

Lemma 3.4. *Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^\infty f(\lambda)\, d\lambda < \infty; \tag{3.10}$$

then for any $\alpha \geq 1$ and for any $r > 0$ there holds

$$\int_0^\infty \lambda^r \Big(\int_\lambda^\infty f(\mu)\, d\mu\Big)^\alpha \frac{d\lambda}{\lambda} \leq \Big(\frac{\alpha}{r}\Big)^\alpha \int_0^\infty \lambda^r \big[\lambda f(\lambda)\big]^\alpha \frac{d\lambda}{\lambda}.$$

The following reverse-Hölder inequality is also classic; we propose it in a suitable form.

Lemma 3.5. *Let $h : [0, +\infty) \to [0, +\infty)$ be a non-increasing, measurable function and let $\alpha_1 \leq \alpha_2 \leq \infty$ and $r > 0$. Then, if $p_2 < \infty$*

$$
\begin{aligned}
\left[\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_2} \frac{d\mu}{\mu}\right]^{1/\alpha_2} &\leq \varepsilon \lambda^r h(\lambda) \\
&\quad + \frac{c}{\varepsilon^{\alpha_2/\alpha_1 - 1}} \left[\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu}\right]^{1/\alpha_1} \qquad (3.11)
\end{aligned}
$$

for every $\varepsilon \in (0,1]$ and for any $\lambda \geq 0$; if $\alpha_2 = \infty$ then

$$
\sup_{\mu > \lambda} \left[\mu^r h(\mu)\right] \leq c\, \lambda^r h(\lambda) + c \left(\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu}\right)^{1/\alpha_1}. \qquad (3.12)
$$

The constant c depends only on α_1, α_2, r except in the case $\alpha_2 = \infty$. In this case $c \equiv c(\alpha_1, r)$.

Proof. We sketch the very simple proof, which is a variant of that in [37, Appendix B.3], given for $\lambda = 0$. Clearly we can suppose the right-hand side quantities finite. We first face the case $\alpha_2 = \infty$: for $\tilde{\mu} > \lambda$ fixed, being $\mu \to h(\mu)$ non-increasing, we have

$$
\begin{aligned}
\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu} \geq \int_\lambda^{\tilde{\mu}} \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu} &\geq [h(\tilde{\mu})]^{\alpha_1} \int_\lambda^{\tilde{\mu}} \mu^{r\alpha_1} \frac{d\mu}{\mu} \\
&= \frac{1}{r\alpha_1} [h(\tilde{\mu})]^{\alpha_1} \left[\tilde{\mu}^{r\alpha_1} - \lambda^{r\alpha_1}\right].
\end{aligned}
$$

Taking the supremum with respect to $\tilde{\mu} > \lambda$ and relabeling variable give

$$
\sup_{\mu > \lambda} [\mu^r h(\mu)]^{\alpha_1} \leq [\lambda^r h(\lambda)]^{\alpha_1} + c(\alpha_1, r) \int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu}
$$

since, again, $\mu \to h(\mu)$ is non-increasing. Now, using the previous estimate

$$
\begin{aligned}
\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_2} \frac{d\mu}{\mu} &\leq \sup_{\mu > \lambda} \left[\mu^r h(\mu)\right]^{\alpha_2 - \alpha_1} \int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu} \\
&\leq [\lambda^r h(\lambda)]^{\alpha_2 - \alpha_1} \int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu} \\
&\quad + c(\alpha_1, \alpha_2, r) \left[\int_\lambda^\infty \left[\mu^r h(\mu)\right]^{\alpha_1} \frac{d\mu}{\mu}\right]^{\frac{\alpha_2 - \alpha_1}{\alpha_1} + 1}.
\end{aligned}
$$

At this point, given $\varepsilon \in (0,1]$ as in the statement, an appropriate use of Young's inequality with conjugate exponents $\alpha_2/(\alpha_2 - \alpha_1)$ and α_2/α_1 yields (3.11). □

Note that the previous Lemma gives the proof of the second embedding in Paragraph 3.2. Finally, the following Lemma can be deduced from [17, Lemma 2.2].

Lemma 3.6. *Let $p > 1$. Then there exists a constant $c_\ell \equiv c_\ell(n, N, p)$ such that for any A, $B \in \mathbb{R}^{Nn}$, not both zero, there holds*

$$
|A|^p \leq c_\ell |B|^p + c_\ell \left(|A|^2 + |B|^2\right)^{\frac{p-2}{2}} |A - B|^2.
$$

4. The setting of the proof

In the first part of this section we shall describe the setting of the proof of Theorem 3.1 and we shall also collect several results, stated directly in the form we need. For more general statements, one can refer to the papers we shall mention.

First we state the higher integrability result of Kinnunen & Lewis [25] in a form fitting our aims.

Theorem 4.1. *Let $\mathcal{K} = C \times I \subset \Omega_T$ and let $\tilde u \in L^p_{\rm loc}(I; W^{1,p}_{\rm loc}(C))$ be a local weak solution to*

$$\tilde u_t - \operatorname{div}\left[a(x,t)|D\tilde u|^{p-2}D\tilde u\right] = -\operatorname{div}\left[|H|^{p-2}H\right] \qquad \text{in } \mathcal{K}, \tag{4.1}$$

with $H \in L^{(1+\sigma)p}_{\rm loc}(\mathcal{K})$ for some $\sigma > 0$. Then there exist two constants $\epsilon_0 \in (0,\sigma]$ and $c \geq 1$, both depending on n, N, p, ν, L, σ, such that $D\tilde u \in L^{p(1+\epsilon_0)}_{\rm loc}(\mathcal{K})$ and

$$\fint_{\tilde Q} |D\tilde u|^{p(1+\epsilon)}\,dz \leq c\,R^{p\epsilon\frac{p-2}{2}}\left(\fint_{2\tilde Q}|D\tilde u|^p\,dz\right)^{1+\frac{p\epsilon}{2}} + c\,R^{-p(1+\epsilon)} + c\fint_{2\tilde Q}|H|^{p(1+\epsilon)}\,dz \tag{4.2}$$

for any $\epsilon \in [0,\epsilon_0]$ and for all $2\tilde Q \equiv B_{2R}(x_0) \times (t_0-(2R)^p, t_0) \subset \mathcal{K}$.

The reader should pay attention here to the particular form of the cylinders $\tilde Q$. Notice, moreover, that in the case $p = 2$ estimate (4.2) has the homogeneous character on the standard parabolic cylinders Q_R one could expect, except for the term $cR^{-2(1+\epsilon)}$ which, on the other hand, is required in the proof for the general case $p \neq 2$. However, once considered on intrinsic cylinders, this estimate shows back the homogeneous form it has in the elliptic case also in the case $p \neq 2$:

Corollary 4.2. *Let $\mathcal{K} \subset \Omega_T$ and $\tilde u$ as in the Theorem above. Then if $Q^\lambda_{2R} \equiv Q^\lambda_{2R}(z_0) \subset \mathcal{K}$ for some $\lambda \geq 1$ and moreover*

$$\frac{\lambda}{\kappa} \leq \left(\fint_{Q^\lambda_{2R}}|D\tilde u|^p\,dz\right)^{1/p} + \left(\tilde M\fint_{Q^\lambda_{2R}}|H|^{p(1+\epsilon)}\,dz\right)^{1/[p(1+\epsilon)]} \leq \kappa\lambda \tag{4.3}$$

holds for some constant $\kappa \geq 1$ and $\tilde M \geq 1$, then

$$\fint_{Q^\lambda_R}|D\tilde u|^{p(1+\epsilon)}\,dz \leq c\,\lambda^{p(1+\epsilon)} \tag{4.4}$$

for any $\epsilon \in [0,\epsilon_0]$, ϵ_0 as above and the constant c depending on $n, N, p, \nu, L, \sigma, \kappa$.

Proof. The proof follows in the case $p \geq 2$ from [2, Lemma 3] and in the case $p < 2$ from [2, Lemma 4], once considered also the stronger, in our case, bound from above in (4.3), which allows to deduce plainly (4.4). □

4.1. **Comparisons.** We start with a solution to equation (1.1) and a cylinder $Q^\lambda_R(z_0)$ such that $Q^\lambda_{20R}(z_0) \subset \Omega_T$ and

$$\frac{\lambda^p}{\kappa} \leq \fint_{Q^\lambda_{20R}(z_0)}|Du|^p\,dz \leq \lambda^p, \qquad M\fint_{Q^\lambda_{20R}(z_0)}|F|^p\,dz \leq \lambda^p \tag{4.5}$$

for $M \geq 1$ to be chosen and for a constant κ depending on n, p; we shall show later how to deduce the existence of such a cylinder. Next, on the same cylinder we define the comparison function

$$v \in u + L^p(\Lambda^\lambda_{20R}(t_0); W^{1,p}_0(B_{20R}(x_0), \mathbb{R}^N)) \qquad \text{if } p \geq 2$$

($v+L^p(\Lambda_{20R}(t_0);W_0^{1,p}(B^\lambda_{20R}(x_0),\mathbb{R}^N))$ in the case $p<2$) solution to the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t v - \operatorname{div}\left[a(x,t)|Dv|^{p-2}Dv\right] = 0 & \text{in } Q^\lambda_{20R}(z_0), \\ v = u & \text{on } \partial_{\mathcal{P}} Q^\lambda_{20R}(z_0). \end{cases} \tag{4.6}$$

Existence of such a function is a classic fact since u belongs to the energy space. Taking as a test function $u-v$, eventually smoothened, and subtracting the weak formulation of $(4.6)_1$ to that of (3.9), after some simple algebraic manipulations (essentially, Young's inequality; see [2, Section 4, Step 4] or [6, Section 7]), discarding the term coming from the parabolic part, averaging, we get the comparison estimate

$$\fint_{Q^\lambda_{20R}(z_0)} \left(|Du|^2+|Dv|^2\right)^{\frac{p-2}{2}} |Du-Dv|^2\,dz \le c_1\,\frac{\lambda^p}{M^{p-1}} \tag{4.7}$$

with $c_1\equiv c_1(n,N,p,\nu,L)$. As a byproduct of the proof of the previous inequality, we also get the energy estimate

$$\fint_{Q^\lambda_{20R}(z_0)} |Dv|^p\,dz \le c_2\,\lambda^p \tag{4.8}$$

for a constant having the same dependencies of c_1. Note now that, using Lemma 3.6, we have

$$\begin{aligned} \fint_{Q^\lambda_{20R}(z_0)} |Dv|^p\,dz &\ge \frac{1}{c_\ell}\fint_{Q^\lambda_{20R}(z_0)} |Du|^p\,dz \\ &\qquad - \fint_{Q^\lambda_{20R}(z_0)} \left(|Du|^2+|Dv|^2\right)^{\frac{p-2}{2}} |Du-Dv|^2\,dz \\ &\ge \frac{\lambda^p}{c_\ell\kappa} - c_1\,\frac{\lambda^p}{M^{p-1}} \ge \frac{\lambda^p}{2c_\ell\kappa}, \end{aligned}$$

provided we choose $M\equiv M(n,N,p,\nu,L,\kappa)$ big enough, that is

$$M^{p-1}\ge 2c_\ell c_1\kappa. \tag{4.9}$$

Therefore we have

$$\frac{\lambda^p}{\kappa_1} \le \fint_{Q^\lambda_{20R}(z_0)} |Dv|^p\,dz \le \kappa_1\lambda^p \tag{4.10}$$

with $\kappa_1\equiv\kappa_1(n,N,p,\nu,L):=\max\{c_2,2c_\ell\kappa\}$.

Now we call $\tilde a(t) := (a)_{B_{10R}(x)}(t)$ for a.e. $t\in\Lambda^\lambda_{10R}(t_0)$ (or for a.e. $t\in\Lambda_{10R}(t_0)$, in the case $p<2$), where we defined the averaged coefficient in (3.2), and we define the second comparison function

$$w\in v+L^p(\Lambda^\lambda_{10R}(t_0);W_0^{1,p}(B_{10R}(x_0),\mathbb{R}^N))$$

(respectively in $v+L^p(\Lambda_{10R}(t_0);W_0^{1,p}(B^\lambda_{10R}(x_0),\mathbb{R}^N))$) as the solution to

$$\begin{cases} \partial_t w - \operatorname{div}\left[\tilde a(t)|Dw|^{p-2}Dw\right] = 0 & \text{in } Q^\lambda_{10R}(z_0), \\ w = v & \text{on } \partial_{\mathcal{P}} Q^\lambda_{10R}(z_0). \end{cases}$$

Again taking as test function $v-w$ and again subtracting, the first inequality we get is the following correspondent to (4.7):

$$\fint_{Q^\lambda_{10R}(z_0)} |Dw|^p\,dz \le c\,\lambda^p, \tag{4.11}$$

$c\equiv c(n,N,p,\nu,L,\kappa)$. Then we have to focus our attention to the inequality, see [6, Display (7.17)] (we suppress the point z_0 in the notation from here on)

$$\begin{aligned}&\fint_{Q^\lambda_{10R}} \big(|Dv|^2+|Dw|^2\big)^{\frac{p-2}{2}}|Dv-Dw|^2\,dz \\ &\le c\fint_{Q^\lambda_{10R}} \big|\tilde a(t)-a(\cdot)\big||Dv|^{p-1}|Dv-Dw|\,dz;\end{aligned} \tag{4.12}$$

we estimate, using Hölder's inequality twice and (4.8)-(4.11)

$$\begin{aligned}&\le c\left(\fint_{Q^\lambda_{10R}} \big|\tilde a(t)-a(\cdot)\big|^{\frac{p}{p-1}}|Dv|^p\,dz\right)^{\frac{p-1}{p}}\left(\fint_{Q^\lambda_{10R}} \big(|Dv|^p+|Dw|^p\big)\,dz\right)^{\frac1p}\\ &\le c\left(\fint_{Q^\lambda_{10R}} \big|\tilde a(t)-a(\cdot)\big|^{\frac{p(1+\epsilon_1)}{(p-1)\epsilon_1}}\,dz\right)^{\frac{(p-1)\epsilon_1}{p(1+\epsilon_1)}}\left(\fint_{Q^\lambda_{10R}} |Dv|^{p(1+\epsilon_1)}\,dz\right)^{\frac{p-1}{p(1+\epsilon_1)}}\lambda,\end{aligned}$$

for $c\equiv c(n,N,p,\nu,L)$ and $\epsilon_1>0$ being the higher integrability exponent from Corollary 4.2. Indeed v is a solution to (4.1), with $H\equiv 0$, in $\mathcal{K}=Q^\lambda_{20R}(z_0)$. The first term is estimate using the fact $a,\tilde a\le L$ and the definition of ω_a in (3.3), in the case $p\ge 2$:

$$\left(\fint_{\Lambda^\lambda_{10R}}\fint_{B_{10R}} \big|\tilde a(t)-a(\cdot)\big|^{\frac{p(1+\epsilon_1)}{(p-1)\epsilon_1}}\,dx\,dt\right)^{\frac{(p-1)\epsilon_1}{p(1+\epsilon_1)}} \le c(p,L,\epsilon_1)\big[\omega_a(R)\big]^{\bar\varepsilon}, \tag{4.13}$$

where $\bar\varepsilon=\frac{(p-1)\epsilon_1}{p(1+\epsilon_1)}$ is a constant depending only upon n,N,p,ν,L and c ultimately depends on these parameters. A completely analogously estimate holds in the case $p<2$, since $B_{\lambda^{(p-2)/2}R}\subset B_R$. Taking into account (4.10), we have by (4.4) of Corollary 4.2

$$\left(\fint_{Q^\lambda_{10R}} |Dv|^{p(1+\epsilon_1)}\,dz\right)^{\frac{p-1}{p(1+\epsilon_1)}} \le c\,\lambda^{p-1};$$

these two last estimates lead to

$$\fint_{Q^\lambda_{10R}} \big(|Dv|^2+|Dw|^2\big)^{\frac{p-2}{2}}|Dv-Dw|^2\,dz \le c\,\big[\omega_a(R)\big]^{\bar\varepsilon}\lambda^p, \tag{4.14}$$

$c\equiv c(n,N,p,\nu,L)$. This is the last comparison estimate we were looking for.

5. The proof in the parabolic case

In this Section we give the proof of Theorem 3.1.

5.1. **The exit time.** Here we show how to build intrinsic cylinders in the sense of (4.5). Consider a standard cylinder as in the statement of Theorem 3.1 and recall that, due to Paragraph 3.2, $F \in L^{\gamma-\varepsilon}$ locally, for $\varepsilon > 0$. Define, for $M \geq 1$ which will be defined later but only depending on $n, N, p, \nu, L, \gamma, q$, the quantities

$$\lambda_0 := \left(\fint_{Q_{2R}} |Du|^p \, dz \right)^{\frac{d}{p}} + \left(M^{\eta/p} \fint_{Q_{2R}} \big(|F|+1\big)^\eta \, dz \right)^{\frac{d}{\eta}} \geq 1, \quad (5.1)$$

where d is given by (3.6), $\eta = p(1+\epsilon_0)$, where ϵ_0 is the higher integrability exponent appearing in Theorem 4.1 for the choice $\sigma := (p+\gamma)/2$ and $B^{1/d} := 40^{(n+2)/p} \cdot 2^{1-1/d}$. Note that in particular $\eta \in (p, \gamma)$. Again here we will be quite sloppy, since this technique is by-now standard and we refer to [2, 3, 6] for its detailed description in different contexts. Define moreover the Calderón-Zygmund operator

$$CZ\big(Q_r^\lambda(\bar z)\big) := \left(\fint_{Q_r^\lambda(\bar z)} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(M^{\eta/p} \fint_{Q_r^\lambda(\bar z)} |F|^\eta \, dz \right)^{\frac{1}{\eta}}$$

for cylinders $Q_r^\lambda(\bar z) \subset Q_{2R}$ and with $\bar z \in Q_R$. We stress here that we choose the exponent for M just in order to make the computations of the previous Paragraph as similar as possible to those in [6].

For fixed $\lambda > B\lambda_0$ and for radii $R/20 \leq r \leq R/2$, enlarging the domain of integration from $Q_r^\lambda(\bar z)$ to Q_{2R} (note that this is possible since $r \leq R/2$), we have

$$CZ\big(Q_r^\lambda(\bar z)\big) < 2^{1-1/d} \left[\frac{|Q_{2R}|}{|Q_r^\lambda(\bar z)|} \right]^{1/p} \lambda_0^{1/d}.$$

In the case $p \geq 2$ we estimate the ratio of volumes in the following way:

$$\left[\frac{|Q_{2R}|}{|Q_r^\lambda(\bar z)|} \right]^{1/p} \lambda_0^{1/d} < 40^{(n+2)/p} \lambda^{1-2/p} \lambda^{1/d} B^{-1/d} = 2^{1/d-1}\lambda,$$

by the definition of d (3.6) and the bound on r. If $p < 2$ on the other hand

$$\left[\frac{|Q_{2R}|}{|Q_r^\lambda(\bar z)|} \right]^{1/p} \lambda_0^{1/d} < 40^{(n+2)/p} \lambda^{[(p-2)n]/(2p)} \lambda^{1/d} B^{-1/d} = 2^{1/d-1}\lambda,$$

for the same reason. Hence in both cases

$$CZ\big(Q_r^\lambda(\bar z)\big) < \lambda \qquad \text{for} \quad \frac{R}{20} \leq r \leq \frac{R}{2} \quad \text{and} \quad \lambda > B\lambda_0. \quad (5.2)$$

Now for points $\bar z$ of the super-level

$$E(\lambda, Q_R) := \{ z \in Q_R : |Du(z)| > \lambda \},$$

by Lebesgue's Theorem, we have $CZ(Q_r^\lambda(\bar z)) > \lambda$ for small radii $0 < r \ll 1$. Therefore, once fixed $\lambda > B\lambda_0$, for any point $\bar z \in E(\lambda, Q_R)$, due to the absolutely continuity of the integral, we can pick the maximal radius $r_{\bar z}$ such that

$$CZ\big(Q_{r_{\bar z}}^\lambda(\bar z)\big) = \left(\fint_{Q_r^\lambda(\bar z)} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(M^{\eta/p} \fint_{Q_r^\lambda(\bar z)} |F|^\eta \, dz \right)^{\frac{1}{\eta}} = \lambda,$$

in the sense that for any $r \in (r_{\bar z}, R/2]$, $CZ(Q_r^\lambda(\bar z)) < \lambda$. Note that by (5.2) we have $r_{\bar z} < R/20$ and therefore $Q^\lambda_{20r_{\bar z}}(\bar z) \subset Q_{2R}$. Moreover, we have

$$\frac{\lambda}{20^{(n+2)/p}} \le \left(\fint_{Q^\lambda_{20r_{\bar z}}(\bar z)} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(M^{\eta/p}\fint_{Q^\lambda_{20r_{\bar z}}(\bar z)} |F|^\eta\,dz\right)^{\frac{1}{\eta}} \le \lambda \tag{5.3}$$

the left-hand side inequality following from the reduction of the domain of integration from $Q^\lambda_{20r_{\bar z}}(\bar z)$ to $Q^\lambda_{r_{\bar z}}(\bar z)$, the right-hand side from the maximality of $r_{\bar z}$, as previously explained. The reader can see here which is the choice of κ performed in (4.5), i.e. $\kappa = 20^{(n+2)/p}$; at this point we have (4.7) and (4.14) for $R = r_{\bar z}$, and the constants depending on n, p instead of κ. The task of the following Paragraph 5.2 will be to show how to match this Paragraph and the previous one in order to get a level-set estimate similar to (2.4).

First, however, a density estimate which will be fundamental in what follows. Single out one of the previously defined cylinders, say $Q \equiv Q^\lambda_{r_{\bar z}}(\bar z)$, for fixed λ, such that $CZ(Q) = \lambda$. We then have that one of the following alternatives must hold:

$$\left(\frac{\lambda}{2}\right)^p \le \fint_Q |Du|^p\,dz \qquad \text{or} \qquad \left(\frac{\lambda}{2}\right)^\eta \le M^{\eta/p}\fint_Q |F|^\eta\,dz. \tag{5.4}$$

Suppose we are in the first case; we split the average as follows:

$$\begin{aligned}
&\fint_Q |Du|^p\,dz\\
&\le \frac{|Q \setminus E(\lambda/4, Q_{2R})|}{|Q|}\left(\frac{\lambda}{4}\right)^p + \frac{1}{|Q|}\int_{Q\cap E(\lambda/4,Q_{2R})} |Du|^p\,dz\\
&\le \left(\frac{\lambda}{4}\right)^p + c\left(\frac{|Q\cap E(\lambda/4,Q_{2R})|}{|Q|}\right)^{1-\frac{1}{1+\epsilon_2}}\left(\fint_Q |Du|^{p(1+\epsilon_2)}\,dz\right)^{\frac{1}{1+\epsilon_2}},
\end{aligned}$$

for $\epsilon_2 > 0$ being, this time, the exponent ϵ_0 from Corollary 4.2 for the choices $\tilde u = u$, $H \equiv F$, $\mathcal{K} = \Omega_T$, $\sigma = \eta/p - 1$. Thus, taking into account (5.3), we have a constant depending on n, N, p, ν, L, γ *but not on* M such that

$$\fint_Q |Du|^{p(1+\epsilon_1)}\,dx \le c\,\lambda^{p(1+\epsilon_1)}. \tag{5.5}$$

Therefore using $(5.4)_1$ and reabsorbing

$$\left(\frac{\lambda}{4}\right)^p \le c\left(\frac{|Q\cap E(\lambda/4,Q_{2R})|}{|Q|}\right)^{1-\frac{1}{1+\epsilon_2}}\lambda^p;$$

now dividing by λ^p and recalling that $Q = Q^\lambda_{r_{\bar z}}(\bar z)$ we infer

$$|Q^\lambda_{r_{\bar z}}(\bar z)| \le c\left|Q^\lambda_{r_{\bar z}}(\bar z)\cap E(\lambda/4, Q_{2R})\right| \tag{5.6}$$

with the constant depending on n, N, p, ν, L, γ.

If on the other hand $(5.4)_2$ holds, take

$$\varsigma = \frac{1}{4M^{1/p}}; \tag{5.7}$$

then using Fubini's Theorem and splitting the integral

$$\left(\frac{\lambda}{2}\right)^{\eta}\frac{1}{M^{\eta/p}} \le \fint_Q |F|^{\eta}\,dz = \frac{\eta}{|Q|}\int_0^{\infty}\mu^{\eta}\big|\{z\in Q : |F(z)|>\mu\}\big|\frac{d\mu}{\mu}$$
$$\le (\varsigma\lambda)^{\eta} + \frac{\eta}{|Q|}\int_{\varsigma\lambda}^{\infty}\mu^{\eta}\big|\{z\in Q : |F(z)|>\mu\}\big|\frac{d\mu}{\mu}.$$

The choice of ς allows to reabsorb the first term of the right-hand side and to infer, dividing by λ^{η} and recalling the expression for ς

$$|Q| \le \frac{\eta}{(\varsigma\lambda)^{\eta}}\int_{\varsigma\lambda}^{\infty}\mu^{\eta}\big|\{z\in Q : |F(z)|>\mu\}\big|\frac{d\mu}{\mu}.$$

Putting together the estimate in the last display with that in (5.6) we get

$$|Q^{\lambda}_{r_{\bar z}}(\bar z)| \le c\,\big|Q^{\lambda}_{r_{\bar z}}(\bar z)\cap E(\lambda/4, Q_{2R})\big|$$
$$+\frac{c}{(\varsigma\lambda)^{\eta}}\int_{\varsigma\lambda}^{\infty}\mu^{\eta}\big|\{z\in Q^{\lambda}_{r_{\bar z}}(\bar z) : |F(z)|>\mu\}\big|\frac{d\mu}{\mu}. \tag{5.8}$$

5.2. **Level-set estimates.** We move here toward the end of the proof.

Take a point $\bar z \in E(A\lambda, Q_R)$, for $A \ge 1$ to be chosen; hence $|Du(\bar z)| > A\lambda$ and in particular $\bar z \in E(\lambda, Q_R)$. Therefore we can consider the cylinder $Q^{\lambda}_{r_{\bar z}}(\bar z)$ previously defined, where $CZ(Q^{\lambda}_{r_{\bar z}}(\bar z)) = \lambda$ and (5.3) hold. Define the comparison functions v and w, respectively, over the cylinders $Q^{\lambda}_{20r_{\bar z}}(\bar z)$ and $Q^{\lambda}_{10r_{\bar z}}(\bar z)$, as done in Paragraph 4.1. One of the *focal points* of the proof is that, since w is solution to a systems with just time-dependent coefficients, Dw turns out to be locally bounded in $Q^{\lambda}_{10r_{\bar z}}(\bar z)$, see [18, Chapter VIII], and moreover, since estimate (4.11) holds with a constant c depending on n, N, p, ν, L, we also have the explicit formula

$$\sup_{Q^{\lambda}_{5r_{\bar z}}(\bar z)} |Dw| \le c_{DiB}\lambda, \tag{5.9}$$

with c_{DiB} just depending on n, N, p, ν, L but not on the cylinder, neither on λ. This will be here used to prove that

$$|Dw(z)|^p \le \big(|Du(z)|^2+|Dv(z)|^2\big)^{\frac{p-2}{2}}|Du(z)-Dv(z)|^2$$
$$+\big(|Dv(z)|^2+|Dw(z)|^2\big)^{\frac{p-2}{2}}|Dv(z)-Dw(z)|^2 \tag{5.10}$$

holds for any $z \in Q^{\lambda}_{5r_{\bar z}}(\bar z)\cap E(A\lambda, Q_{2R})$, for an appropriate choice of A. Indeed applying Lemma 3.6 twice yields

$$|Du(z)|^p \le c_\ell^2\,|Dw(z)|^p + c_\ell^2\left(|Dv(z)|^2+|Dw(z)|^2\right)^{\frac{p-2}{2}}|Dv(z)-Dw(z)|^2$$
$$+\,c_\ell\left(|Du(z)|^2+|Dv(z)|^2\right)^{\frac{p-2}{2}}|Du(z)-Dv(z)|^2. \tag{5.11}$$

Suppose now that (5.10) fails: then also by (5.9) and the fact that $|Du(z)| > A\lambda$

$$|Dw(z)|^p \le c^p_{DiB}\,\lambda^p < c^p_{DiB}\frac{|Du(z)|^p}{A^p} < \frac{2c_\ell^2 c^p_{DiB}}{A^p}|Dw(z)|^p,$$

which is clearly a contradiction for the choice of

$$A \equiv A(n, N, p, \nu, L), \qquad A^p := 2c_\ell^2 c^p_{DiB} \ge 1.$$

Therefore, combining (5.11) and (5.10), we get

$$
\begin{aligned}
|Du(z)|^p \le 2c_\ell^2 \left(|Du(z)|^2 + |Dv(z)|^2\right)^{\frac{p-2}{2}} |Du(z) - Dv(z)|^2 \\
+ 2c_\ell^2 \left(|Dv(z)|^2 + |Dw(z)|^2\right)^{\frac{p-2}{2}} |Dv(z) - Dw(z)|^2
\end{aligned}
$$

for $z \in Q^\lambda_{5r_{\bar z}}(\bar z) \cap E(A\lambda, Q_{2R})$. Hence

$$
\begin{aligned}
&\big|\{z \in Q^\lambda_{5r_{\bar z}}(\bar z) : |Du(z)| > A\lambda\}\big| \\
&\quad \le \big|\{z \in Q^\lambda_{5r_{\bar z}}(\bar z) : \\
&\qquad\qquad \left(|Du(z)|^2 + |Dv(z)|^2\right)^{\frac{p-2}{2}} |Du(z) - Dv(z)|^2 > \frac{(A\lambda)^p}{4c_\ell^2}\}\big| \\
&\quad + \big|\{z \in Q^\lambda_{5r_{\bar z}}(\bar z) : \\
&\qquad\qquad \left(|Dv(z)|^2 + |Dw(z)|^2\right)^{\frac{p-2}{2}} |Dv(z) - Dw(z)|^2 > \frac{(A\lambda)^p}{4c_\ell^2}\}\big| \\
&\quad \le \frac{c}{\lambda^p} \int_{Q^\lambda_{20r_{\bar z}}(\bar z)} \left(|Du|^2 + |Dv|^2\right)^{\frac{p-2}{2}} |Du - Dv|^2\, dz \\
&\qquad\qquad + \frac{c}{\lambda^p} \int_{Q^\lambda_{10r_{\bar z}}(\bar z)} \left(|Dv|^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dv - Dw|^2\, dz \\
&\quad \le c\left[\frac{1}{M^{p-1}} + [\omega_a(r_{\bar z})]^{\bar\varepsilon}\right] |Q^\lambda_{r_{\bar z}}(\bar z)| \\
&\quad \le c\left[\frac{1}{M^{p-1}} + [\omega_a(r_{\bar z})]^{\bar\varepsilon}\right] \Bigg[\big|Q^\lambda_{r_{\bar z}}(\bar z) \cap E(\lambda/4, Q_{2R})\big| \\
&\qquad\qquad + \frac{1}{(\varsigma\lambda)^\eta} \int_{\varsigma\lambda}^\infty \mu^\eta \big|\{z \in Q^\lambda_{r_{\bar z}}(\bar z) : |F(z)| > \mu\}\big| \frac{d\mu}{\mu}\Bigg], \qquad (5.12)
\end{aligned}
$$

first by (4.7) and (4.14), then by (5.8).

Now consider the collection $\mathcal{E}_\lambda$ of cylinders $Q^\lambda_{r_{\bar z}}(\bar z)$, when $\bar z$ varies in $E(A\lambda, Q_R)$. By a Vitali-type argument, we extract a countable sub-collection $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda$ such that the 5-times enlarged cylinders cover almost all $E(A\lambda, Q_R)$ and the cylinders are pairwise disjoints. I.e., if we denote the cylinders of $\mathcal{F}_\lambda$ by $Q^0_i := Q^\lambda_{r_{\bar z_i}}(\bar z_i)$, for $i \in \mathcal{I}_\lambda$, being eventually $\mathcal{I}_\lambda = \mathbb{N}$, with their "vertices" $\bar z_i \in E(A\lambda, Q_R)$, we have

$$
Q^0_i \cap Q^0_j = \emptyset \quad \text{whenever } i \neq j \quad \text{and} \quad E(A\lambda, Q_R) \subset \bigcup_{i \in \mathcal{I}_\lambda} Q^1_i \cup \mathcal{N}_\lambda,
$$

with $|\mathcal{N}_\lambda| = 0$ and where we denoted $Q^1_i := 5Q^0_i = Q^\lambda_{5r_{\bar z_i}}(\bar z_i)$. Using the two facts in the previous display we can extend (5.12) to the full level set: indeed considering (5.12) just over the cylinders $Q^\lambda_{5r_{\bar z}} = Q^1_i$ and summing over $\mathcal{I}_\lambda$ (recall that now $\lambda > B\lambda_0$ is fixed and that the $Q^\lambda_{r_{\bar z}} = Q^0_i$ are disjoint) we get

$$
\begin{aligned}
\big|E(A\lambda, Q_R)\big| \le c\, G(R, M) \Bigg[\big|E(\lambda/4, Q_{2R})\big| \\
+ \frac{1}{(\varsigma\lambda)^\eta} \int_{\varsigma\lambda}^\infty \mu^\eta \big|\{z \in Q_{2R} : |F(z)| > \mu\}\big| \frac{d\mu}{\mu}\Bigg], \qquad (5.13)
\end{aligned}
$$

where $G(R, M) := M^{1-p} + [\omega_a(R)]^{\bar\varepsilon}$, since $r_{\bar z_i} \le R$ and $R \to \omega_a(R)$ is increasing.

5.3. **Conclusion, case $q<\infty$.** Multiply inequality (5.13), for $\gamma>p$ and $q<\infty$ as in the statement of Theorem 3.1, by $(A\lambda)^\gamma$, then raise both sides to the power q/γ and integrate with respect to the measure $d\lambda/(A\lambda)$ over $B\lambda_0$; recall indeed that inequality (5.13) holds true just for λ varying in this range. This yields, recalling that $A\geq 1$ is a constant depending on n,N,p,ν,L and ς depends on p,M

$$
\begin{aligned}
&\int_{B\lambda_0}^{\infty}\Big((A\lambda)^\gamma\big|\{z\in Q_R:|Du(z)|>A\lambda\}\big|\Big)^{\frac{q}{\gamma}}\frac{d\lambda}{A\lambda}\\
&\leq c\,[G(R,M)]^{\frac{q}{\gamma}}\bigg[\int_0^\infty\Big(\lambda^\gamma\big|\{z\in Q_{2R}:|Du(z)|>\lambda/4\}\big|\Big)^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda}\\
&\quad+c(p,\gamma,q,M)\times\\
&\qquad\qquad\times\int_0^\infty\lambda^{q(1-\frac{\eta}{\gamma})}\Big(\int_{\varsigma\lambda}^\infty\mu^\eta\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|\frac{d\mu}{\mu}\Big)^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda}\bigg]\\
&=:c\,[G(R,M)]^{\frac{q}{\gamma}}\big[I+II\big],
\end{aligned}\tag{5.14}
$$

where c depends on n,N,p,ν,L,γ,q. At this point a simple change of variable yields

$$
I=c(q)\|Du\|^q_{L(\gamma,q)(Q_{2R})}.
$$

For II the situation is a bit more involved. First we examine **the case $q\geq\gamma$**; here we make the change of variables $\tilde\lambda=\varsigma\lambda$, recalling the definition of ς in (5.7), and then we use Lemma 3.4 with $f(\mu)=\mu^{\eta-1}\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|$, $\alpha=q/\gamma\geq 1$ and $r=q(1-\eta/\gamma)>0$ to infer

$$
\begin{aligned}
II&=c(M)\int_0^\infty\tilde\lambda^{q(1-\frac{\eta}{\gamma})}\Big(\int_{\tilde\lambda}^\infty\mu^\eta\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|\frac{d\mu}{\mu}\Big)^{\frac{q}{\gamma}}\frac{d\tilde\lambda}{\tilde\lambda}\\
&\leq\frac{c}{\gamma-p}\int_0^\infty\tilde\lambda^{q(1-\frac{\eta}{\gamma})+\eta\frac{q}{\gamma}}\big|\{z\in Q_{2R}:|F(z)|>\tilde\lambda\}\big|^{\frac{q}{\gamma}}\frac{d\tilde\lambda}{\tilde\lambda}\\
&=c\|F\|^q_{L(\gamma,q)(Q_{2R})}
\end{aligned}
$$

with $c\equiv c(p,\gamma,q,M)$. Note that (3.10) is satisfied since $F\in L^\eta(Q_{2R})$.

In **the case $0<q<\gamma$**, on the other hand, we use Lemma 3.5 with $h(\mu)=\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|^{\frac{q}{\gamma}}$, $r=\eta q/\gamma$, $\alpha_1=1<\gamma/q=\alpha_2$ and $\varepsilon=1$:

$$
\begin{aligned}
&\bigg[\int_\lambda^\infty\mu^\eta\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|\frac{d\mu}{\mu}\bigg]^{\frac{q}{\gamma}}\\
&\leq\lambda^{\eta\frac{q}{\gamma}}\big|\{z\in Q_{2R}:|F(z)|>\lambda\}\big|^{\frac{q}{\gamma}}\\
&\qquad\qquad+c\int_\lambda^\infty\mu^{\eta\frac{q}{\gamma}}\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|^{\frac{q}{\gamma}}\frac{d\mu}{\mu}.
\end{aligned}
$$

Therefore in this case, again after changing variable $\varsigma\lambda\leftrightarrow\lambda$

$$
\begin{aligned}
II&\leq c\int_0^\infty\lambda^{q(1-\frac{\eta}{\gamma})}\bigg[\lambda^{\eta\frac{q}{\gamma}}\big|\{z\in Q_{2R}:|F(z)|>\lambda\}\big|^{\frac{q}{\gamma}}\\
&\qquad\qquad+c\int_\lambda^\infty\mu^{\eta\frac{q}{\gamma}-1}\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|^{\frac{q}{\gamma}}\,d\mu\bigg]\frac{d\lambda}{\lambda}\\
&\leq c\,\|F\|^q_{L(\gamma,q)(Q_{2R})}
\end{aligned}
$$

$$
\begin{aligned}
&+ c\int_0^\infty \lambda^{q(1-\frac{\eta}{\gamma})}\left[\int_\lambda^\infty \mu^{\eta\frac{q}{\gamma}-1}\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|^{\frac{q}{\gamma}}\,d\mu\right]\frac{d\lambda}{\lambda}\\
&\le c\,\|F\|^q_{L(\gamma,q)(Q_{2R})},
\end{aligned}
$$

by Fubini's Theorem, $c\equiv c(p,\gamma,q,M)$. Therefore, all in all, putting all these estimates in (5.14), after simple manipulations, we have that for all $\gamma>p$ and $0<q<\infty$,

$$
\begin{aligned}
\|Du\|_{L(\gamma,q)(Q_R)} \le \tilde{c}\,[G(R,M)]^{\frac{1}{\gamma}}\Big[&\|Du\|_{L(\gamma,q)(Q_{2R})}\\
&+c(p,\gamma,q,M)\|F\|_{L(\gamma,q)(Q_{2R})}\Big]+c\,B\lambda_0|Q_{2R}|^{\frac{1}{\gamma}}
\end{aligned}
\tag{5.15}
$$

with $\tilde{c}$ depending on n,N,p,ν,L,γ,q. Recall the definition of $G(R,M)$: it is now enough to choose first M big enough, also satisfying (4.9), and then R_0 small enough so that $\tilde{c}[G(R,M)]^{\frac{1}{\gamma}}\le\frac{1}{2}$ for all $R\le R_0$. Note that it is possible to do this just making M and R_0 depend on n,N,p,ν,L,γ,q since the constant $\tilde{c}$ depends on the same parameters. M is now a constant depending on these quantities. Therefore we have

$$
\begin{aligned}
\|Du\|_{L(\gamma,q)(Q_R)} \le \frac{1}{2}&\|Du\|_{L(\gamma,q)(Q_{2R})}\\
&+c\,\|F\|_{L(\gamma,q)(Q_{2R})}+c\,B\lambda_0|Q_{2R}|^{\frac{1}{\gamma}}.
\end{aligned}
\tag{5.16}
$$

At this point, if we knew $\|Du\|^q_{L(\gamma,q)(Q_{2R})}<\infty$, a standard iteration argument would be enough to get an estimate similar to (3.5) in the case $\gamma>p$ and $0<q<\infty$. However, this boundedness is what we want to prove here: therefore we shall need to be a bit more careful in estimating the terms containing Du. We however preferred to not overcharge the proof of technicalities in order to focus our attention on the treatment of the Lorentz norms, which is the main point of this paper together with the estimates of Paragraph 5.1.

Here we then show how to refine estimates about Du. Consider the truncated gradients

$$
|Du(z)|_k := \min\big\{|Du(z)|,k\big\} \qquad \text{for } z\in\Omega_T \text{ and } k\in\mathbb{N}\cap[B\lambda_0,\infty)
$$

and note that from (5.13) we have, calling $E_k(\lambda,Q_\rho):=\{z\in Q_\rho:|Du(z)|_k>\lambda\}$

$$
\begin{aligned}
\big|E_k(A\lambda,Q_R)\big|\le c\,G(R,M)\Bigg[&\big|E_k(\lambda/4,Q_{2R})\big|\\
&+\frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta\big|\{z\in Q_{2R}:|F(z)|>\mu\}\big|\frac{d\mu}{\mu}\Bigg],
\end{aligned}
$$

for $k\in\mathbb{N}\cap[B\lambda_0,\infty)$. Indeed in the case $k\le A\lambda$ we have $E_k(A\lambda,Q_R)=\emptyset$ and therefore the previous estimate holds trivially. In the case $k>A\lambda$ on the other hand it follows since $E_k(A\lambda,Q_R)=E(A\lambda,Q_R)=\{z\in Q_R:|Du(z)|>A\lambda\}$ and $E_k(\lambda/4,Q_{2R})=E(\lambda/4,Q_{2R})$. At this point, working exactly as in the previous lines, we get that (5.16) holds with $|Du|_k$ in place of Du. Now finally we can finally use a well-know iteration argument (see [31] for the rigorous computations; we should, instead of Q_R and Q_{2R}, consider Q_{ρ_1} and Q_{ρ_2} for $R\le\rho_1<\rho_2\le 2R$ and change accordingly B and the radii involved) since $\||Du|_k\|_{L(\gamma,q)(Q_{2R})}<\infty$ and, recalling the definition of B and λ_0, we get

$$|Q_R|^{-\frac{1}{\gamma}}\||Du|_k\|_{L(\gamma,q)(Q_R)} \le c\left(\fint_{Q_{2R}} |Du|^p\,dz + \fint_{Q_{2R}} (|F|+1)^p\,dz\right)^{\frac{d}{p}} + c\,|Q_{2R}|^{-\frac{1}{\gamma}}\|F\|_{L(\gamma,q)(Q_{2R})}. \tag{5.17}$$

A standard Hölder's inequality in Marcinkiewicz spaces, see [31, Lemma 2.8], yields

$$\int_{Q_{2R}} (|F|+1)^p\,dz \le \frac{\gamma}{\gamma-p}|Q_{2R}|^{1-\frac{p}{\gamma}}\||F|+1\|^p_{\mathcal{M}^\gamma(Q_{2R})}; \tag{5.18}$$

therefore using (3.12) we finally get

$$\left(\fint_{Q_{2R}} (|F|+1)^p\,dz\right)^{\frac{1}{p}} \le \frac{c(p,\gamma,q)}{(\gamma-p)^{1/p}}|Q_{2R}|^{-\frac{1}{\gamma}}\||F|+1\|_{L(\gamma,q)(Q_{2R})}. \tag{5.19}$$

Keeping in mind that $d \ge 1$, from (5.17) we then infer (3.5) in the case $q < \infty$, after taking the limit $k \to \infty$ and using the lower semi-continuity of the Lorentz quasi-norm with respect to almost everywhere convergence, see Remark 3.3. For the case $q = \infty$ only minor modifications have to be done, see the following step.

5.4. **Conclusion, case $q = \infty$.** We come back to the second alternative in (5.4). This time we split, for τ small to be chosen

$$\begin{aligned}\left(\frac{\lambda}{2}\right)^\eta &\le M^{\eta/p}\fint_Q |F|^\eta\,dz\\ &\le M^{\eta/p}(\tau\lambda)^\eta + \frac{M^{\eta/p}}{|Q|}\int_{\{z\in Q:|F(z)|>\tau\lambda\}} |F|^\eta\,dz.\end{aligned}$$

Hence, using a Hölder's inequality similar to (5.18), we have, calling for shortness $F(\tau\lambda, Q)$ the set $\{z \in Q : |F(z)| > \tau\lambda\}$ and $F(\mu, Q) := \{z \in Q : |F(z)| > \mu\}$

$$\begin{aligned}&\left(\frac{\lambda}{2}\right)^\eta - M^{\eta/p}(\tau\lambda)^\eta \le \frac{M^{\eta/p}}{|Q|}\int_{F(\tau\lambda,Q)} |F|^\eta\,dz\\ &\le \frac{\gamma M^{\eta/p}}{\gamma-\eta}\frac{|F(\tau\lambda,Q)|^{1-\frac{\eta}{\gamma}}}{|Q|}\sup_{\mu>0}\mu^\eta\big|\{z\in F(\tau\lambda,Q):|F(z)|>\mu\}\big|^{\frac{\eta}{\gamma}}\\ &\le \frac{\gamma M^{\eta/p}}{\gamma-\eta}\frac{|F(\tau\lambda,Q)|^{1-\frac{\eta}{\gamma}}}{|Q|}\left[(\tau\lambda)^\eta|F(\tau\lambda,Q)|^{\frac{\eta}{\gamma}} + \sup_{\mu>\tau\lambda}\mu^\eta|F(\mu,Q)|^{\frac{\eta}{\gamma}}\right]\\ &\le \frac{\gamma M^{\eta/p}}{\gamma-p}\left[\frac{|F(\tau\lambda,Q)|}{|Q|}(\tau\lambda)^\eta + \frac{|F(\tau\lambda,Q)|^{1-\frac{\eta}{\gamma}}}{|Q|}\sup_{\mu>\tau\lambda}\mu^\eta|F(\mu,Q)|^{\frac{\eta}{\gamma}}\right].\end{aligned}$$

Now choosing τ appropriate, such that

$$\frac{1}{2^\eta} - M^{\eta/p}\tau^\eta\frac{2\gamma-p}{\gamma-p} \ge \frac{1}{4^\eta}, \qquad \text{i.e.} \qquad \tau = \frac{c(\gamma,p)}{M^{1/p}},$$

we have

$$|Q| \le \frac{c}{\gamma-p}\frac{|F(\tau\lambda,Q)|^{1-\frac{\eta}{\gamma}}}{(\tau\lambda)^\eta}\left[\sup_{\mu>\tau\lambda}\mu^\gamma|F(\mu,Q)|\right]^{\frac{\eta}{\gamma}}$$

$$
\begin{aligned}
&\le c\,(\tau\lambda)^{-\gamma}\Big[(\tau\lambda)^{\gamma}|F(\tau\lambda,Q)|\Big]^{1-\frac{\eta}{\gamma}}\Big[\sup_{\mu>\tau\lambda}\mu^{\gamma}|F(\mu,Q)|\Big]^{\frac{\eta}{\gamma}}\\
&\le \frac{c}{\gamma-p}(\tau\lambda)^{-\gamma}\sup_{\mu>\tau\lambda}\mu^{\gamma}|F(\mu,Q)|.
\end{aligned}
$$

Now taking into account the previous estimate, which follows if we suppose $(5.4)_2$, together with (5.6), which follows from $(5.4)_1$ exactly as in the case $q<\infty$, estimating as in (5.12) and then summing up as in Paragraph 5.2 we get

$$
\begin{aligned}
&\big|E(A\lambda,Q_R)\big|\\
&\qquad\le c\,G(R,M)\Big[\big|E(\lambda/4,Q_{2R})\big|+(\tau\lambda)^{-\gamma}\sup_{\mu>\tau\lambda}\mu^{\gamma}|F(\mu,Q_{2R})|\Big].
\end{aligned}
$$

At this point, we multiply inequality (5.13) by $(A\lambda)^{\gamma}$ and then we take the supremum with respect to λ over $(B\lambda_0,\infty)$; this gives, after changing variable again,

$$
\begin{aligned}
&\sup_{\lambda>B\lambda_0}(A\lambda)^{\gamma}\big|\{z\in Q_R:|Du(z)|>A\lambda\}\big|\\
&\qquad\le c\,G(R,M)\Big[\sup_{\lambda>B\lambda_0}\lambda^{\gamma}\big|\{z\in Q_{2R}:|Du(z)|>\lambda/4\}\big|\\
&\qquad\qquad+c(p,\gamma,q,M)\sup_{\lambda>B\tau\lambda_0}\sup_{\mu>\lambda}\mu^{\gamma}|F(\mu,Q_{2R})|\Big].
\end{aligned}
$$

Since $\sup_{\lambda>B\tau\lambda_0}\sup_{\mu>\lambda}\mu^{\gamma}|F(\mu,Q_{2R})|\le\|F\|^{\gamma}_{\mathcal{M}^{\gamma}(Q_{2R})}$ similarly as in (5.15), using also (5.18) we get, after some simple algebraic manipulations and recalling the definition of $B\lambda_0$ in (5.1)

$$
\begin{aligned}
\|Du\|_{\mathcal{M}^{\gamma}(Q_R)}&\le\tilde c\,\big[G(R,M)\big]^{1/\gamma}\Big[\|Du\|_{\mathcal{M}^{\gamma}(Q_{2R})}\\
&\qquad+c(p,\gamma,q,M)\|F\|_{\mathcal{M}^{\gamma}(Q_{2R})}\Big]+\tau|Q_{2R}|^{\frac{1}{\gamma}}B\lambda_0\\
&\le\frac12\|Du\|_{\mathcal{M}^{\gamma}(Q_{2R})}+c\,\|F\|_{\mathcal{M}^{\gamma}(Q_{2R})}\\
&\qquad+c(p,\gamma,M)\,|Q_{2R}|^{\frac{1}{\gamma}}\Big[\Big(\fint_{Q_{2R}}|Du|^p\,dz\Big)^{\frac1p}\\
&\qquad\qquad+\Big(\fint_{Q_{2R}}(|F|+1)^{\eta}\,dz\Big)^{\frac1\eta}\Big]^{d}
\end{aligned}
$$

where we chose again M big and then R_0 small enough to satisfy (4.9) and to ensure that $G(R,M)\le 1/(2\tilde c)^{\gamma}$. As in (5.18) we have

$$
\Big(\fint_{Q_{2R}}(|F|+1)^{\eta}\,dz\Big)^{\frac1\eta}\le\frac{c(p,\gamma)}{(\gamma-p)^{1/\eta}}|Q_{2R}|^{-\frac1\gamma}\big\||F|+1\big\|_{\mathcal{M}^{\gamma}(Q_{2R})}
$$

and this finally leads, up to the truncation argument of the previous section, to (3.5) in the case $q=\infty$.

6. The elliptic proof

The proof in the elliptic case is much easier, since estimates already have the homogeneous character needed and therefore there is no need for intrinsic arguments. We shall sketch just the first part, since the modifications to be done with respect to the parabolic argument are quite straightforward. Take $B_{2R}(x_0) \equiv B_{2R} \subset \Omega$ and subsequently define

$$\lambda_0 := \left(\fint_{B_{2R}} |Du|^p \, dx \right)^{\frac{1}{p}} + \left(M^{\eta/p} \fint_{B_{2R}} |F|^\eta \, dx \right)^{\frac{1}{\eta}}$$

for η defined similarly as after (5.1), taking this time into account the elliptic higher integrability exponent (see the following (6.5)); take a point $\bar{x} \in B_R(x_0)$ and consider, for radii $0 < r < R/2$

$$CZ\big(B_r(\bar{x})\big) := \left(\fint_{B_r(\bar{x})} |Du|^p \, dx \right)^{\frac{1}{p}} + \left(M^{\eta/p} \fint_{B_r(\bar{x})} |F|^\eta \, dx \right)^{\frac{1}{\eta}}.$$

Simply enlarging the domain of integration we get $CZ\big(B_r(\bar{x})\big) \leq (2R/r)^{n/p}\lambda_0$. Hence if we consider points in the super-levels

$$E(\lambda, B_{2R}) := \big\{ x \in B_{2R} : |Du(z)| > \lambda \big\}$$

for $\lambda > B\lambda_0$ and radii $R/40 \leq r \leq R/2$, then we have $CZ(B_r(\bar{x})) < \lambda$; at the same time, by Lebesgue's differentiation Theorem we get that for small radii $0 < r \ll 1$

$$CZ\big(B_r(\bar{x})\big) > \lambda.$$

Hence we get the existence of a maximal radius $r_{\bar{x}}$ such that $CZ(B_{r_{\bar{x}}}(\bar{x})) = \lambda$ and

$$\frac{\lambda}{20^{n/p}} \leq \left(\fint_{B_{20 r_{\bar{x}}}(\bar{x})} |Du|^p \, dx \right)^{\frac{1}{p}} + \left(M^{\eta/p} \fint_{B_{20 r_{\bar{x}}}(\bar{x})} |F|^\eta \, dx \right)^{\frac{1}{\eta}} \leq \lambda. \tag{6.1}$$

Call B the ball $B_{r_{\bar{x}}}(\bar{x})$. Next, we build the function $v \in u + W_0^{1,p}(20B, \mathbb{R}^N)$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}\big[a(x,t)|Dv|^{p-2}Dv\big] = 0 & \text{in } 20B, \\ v = u & \text{on } \partial(20B). \end{cases} \tag{6.2}$$

Again taking $u - v$ as test function and subtracting the weak formulations, as suggested for (4.7) – recall this estimate is inferred essentially from the elliptic part – we get

$$\fint_{20B} \big(|Du|^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Du - Dv|^2 \, dx \leq c \, \frac{\lambda^p}{M^{p-1}}$$

and

$$\fint_{20B} |Dv|^p \, dx \leq c \, \lambda^p$$

with $c \equiv c(n, N, p, L/\nu)$. Now we build the second comparison function $w \in v + W_0^{1,p}(10B, \mathbb{R}^N)$ as the unique solution of

$$\begin{cases} \operatorname{div}\big[|Dw|^{p-2}Dw\big] = 0 & \text{in } 10B, \\ w = v & \text{on } \partial(10B). \end{cases} \tag{6.3}$$

Using the test function $v - w$, multiplying (6.3)$_1$ by $\tilde{a} := \fint_{10B} c(x)\,dx$ and subtracting the weak formulations for v and w we first get

$$\fint_{10B} |Dw|^p\,dx \le c\,\lambda^p \tag{6.4}$$

and then, as in (4.12), for a constant depending on $n, N, p, L/\nu$

$$\begin{aligned}&\fint_{10B} \left(|Dv|^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dv - Dw|^2\,dx\\ &\le c\left(\fint_{10B} \left|a(\cdot) - \tilde{a}\right|^{\frac{p(1+\epsilon_3)}{(p-1)\epsilon_3}}\,dx\right)^{\frac{(p-1)\epsilon_3}{p(1+\epsilon_3)}} \left(\fint_{10B} |Dv|^{p(1+\epsilon_3)}\,dx\right)^{\frac{p-1}{p(1+\epsilon_3)}} \lambda,\end{aligned}$$

for $\epsilon_3 > 0$ the elliptic higher integrability; indeed from classic elliptic theory, see [21], a solution to (6.2)$_1$ belongs to $L^{p(1+\epsilon_3)}(10B)$, for some $\epsilon_3 \equiv \epsilon_3(n, p, L/\nu)$ and moreover there holds

$$\left(\fint_{10B} |Dv|^{p(1+\epsilon_3)}\,dx\right)^{\frac{1}{1+\epsilon_3}} \le c \fint_{20B} |Dv|^p\,dx \le c\,\lambda^p$$

by (6.1), with the constant c depending on $n, N, p, L/\nu$. Hence we have

$$\fint_{Q^\lambda_{10R}} \left(|Dv|^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dv - Dw|^2\,dz \le c\left[\omega_c(R)\right]^{\bar{\varepsilon}} \lambda^p,$$

$c \equiv c(n, N, p, L/\nu)$, where $\bar{\varepsilon}$ is as after (4.13), starting from the elliptic higher integrability exponent ϵ_3 instead of ϵ_1 and ω_c is the elliptic modulus of integral oscillation defined in (3.8).

Now the proof goes on as in the parabolic case, except for some slight modifications. In particular the stationary analogue of (5.5) can for instance be found in [20]:

$$\begin{aligned}\left(\fint_{B} |Du|^{p(1+\epsilon_4)}\,dx\right)^{\frac{1}{1+\epsilon_4}} &\le c \fint_{2B} |Du|^p\,dx + c\left(\fint_{2B} |F|^{p(1+\epsilon_4)}\,dx\right)^{\frac{1}{1+\epsilon_4}}\\ &\le c\,\lambda^p\end{aligned} \tag{6.5}$$

for some $\epsilon_4(n, p, L/\nu, \gamma) \le \eta/p-1$ and for a constant depending on $n, N, p, L/\nu, \gamma$. Moreover, the sup bound for w takes the form (see [19, 33, 39, 40])

$$\sup_{5B} |Dw| \le c(n, p, L/\nu)\left(\fint_{10B} |Dw|^p\,dx\right)^{\frac{1}{p}} \le c\,\lambda$$

keeping into account also (6.4); in this case the estimate is more neat than (5.9), since it does not anymore involve intrinsic cylinders.

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