On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of $\mathsf{RCD}^*(K,N)$ metric measure spaces

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Abstract

We prove higher summability and regularity of $\Gamma(f)$ for functions f in spaces satisfying the Bakry-Émery condition $\mathsf{BE}(K,\infty)$.

As a byproduct, we obtain various equivalent weak formulations of $\mathsf{BE}(K,N)$ and we prove the Local-to-Global property of the $\mathsf{RCD}^*(K,N)$ condition in locally compact metric measure spaces $(X,\mathsf{d},\mathfrak{m})$, without assuming a priori the non-branching condition on the metric space.

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1 Introduction

Curvature-dimension conditions for metric-measure spaces. The theory of synthetic Ricci lower bounds has been so far developed along two lines: the Bakry-Émery approach [9], see also [8, 10], uses the formalism of Dirichlet forms (and the heat flow associated with the Dirichlet form) and it is based on the so-called $\mathsf{BE}(K,N)$ condition, formally expressed in differential terms by

$$\Gamma_2(f) \geq K\,\Gamma\big(f\big) + \frac{1}{N}(\Delta f)^2, \quad \text{where} \quad \Gamma_2(f) := \frac{1}{2}\Delta\Gamma\big(f\big) - \Gamma\big(f,\Delta f\big). \tag{1.1}$$

Here Γ is the *Carré du Champ* representing the energy density of a strongly local Dirichlet form

$$\mathcal{E}(f,g) := \int_X \Gamma\big(f,g\big)\,\mathrm{d}\mathfrak{m} \quad f,\, g \in \mathrm{D}(\mathcal{E}) \subset L^2(X,\mathfrak{m}), \tag{1.2}$$

and Δ is the associated selfadjoint linear operator in the Lebesgue space $L^2(X,\mathfrak{m})$ (see e.g. [11, 19]). A fundamental example is of course given by Euclidean measure spaces endowed with the classical Dirichlet energy.

The more recent approach of Lott-Villani [25] and Sturm [32, 33], based on the so-called $\mathsf{CD}(K,N)$ condition, makes sense for metric measure spaces and it is based on convexity inequalities fulfilled by suitable "entropies" along geodesics for the Wasserstein distance. In the case $N<\infty$, also the more recent variant $\mathsf{CD}^*(K,N)$, see [6], should be considered, which provides an (a priori) weaker condition when $K\neq 0$.

Since these theories formalize "local" conditions (namely the lower bound on the Ricci tensor and the upper bound on the dimension) with "nonlocal" tools, for both theories it is important to ascertain the validity of the so-called Global-to-Local and Local-to-Global properties. Since this theme has been more investigated on the $\mathsf{CD}(K,N)$ side, we confine our discussion to this theory, although the equivalence results that we shall mention later on could be used to read some results also from the $\mathsf{BE}(K,N)$ side. Typically, the Global-to-Local property requires some strong convexity property (either of the entropy or of the subdomain under consideration), see for instance [35, Proposition 30.1], while the Local-to-Global property has been established under the non-branching assumption, first in $\mathsf{CD}(K,\infty)$ spaces [32], then in $\mathsf{CD}(0,N)$ spaces [35, Theorem 30.37] and eventually in $\mathsf{CD}^*(K,N)$ spaces [6] (see also [13] for recent progress on the globalization for $\mathsf{CD}(K,N)$). On the other hand, since the non-branching assumption is unstable under Gromov-Hausdorff convergence, it is desirable to have stronger axiomatizations of the $\mathsf{CD}(K,N)$ theory which retain stability and Local-to-Global properties and do not involve the non-branching assumption. Actually, some results of the $\mathsf{CD}(K,N)$ theory initially proved under the non-branching assumption have been recently proved by Rajala without making use of this assumption [27], [28]. But, a recent remarkable paper by the same author [29] provides for all $K\in\mathbb{R}$ and $N\geq 1$ a (highly branching) compact metric measure space satisfying $\mathsf{CD}^*(0,4)=\mathsf{CD}(0,4)$ locally, but not $\mathsf{CD}(K,\infty)$ (and therefore not even $\mathsf{CD}(K,N)$).

The case of $\mathsf{RCD}^*(K,N)$ ***spaces.*** The main goal of this paper is to prove the Local-to-Global property in the class of locally compact *Riemannian* metric measure spaces $\mathsf{RCD}^*(K,N)$ independently of non-branching assumptions. Local compactness is in fact always true in the finite dimensional case $N<\infty$ [33, Corollary 2.4], so that it is a restrictive assumption only in the case $N=\infty$.

The RCD axiomatization can be obtained from the CD one by just adding the requirement that the metric measure structure is infinitesimally Hilbertian (an assumption suggested by CHEEGER-COLDING in [14, Appendix 2]); formally, this translates into the assumption that the so-called CHEEGER's energy Ch is a quadratic form.

The class of $\mathsf{RCD}(K,\infty)$ has been introduced in [3] (and then improved in [1]), while its dimensional counterparts $\mathsf{RCD}^*(K,N)$ have been studied in the more recent papers [18], [5]. Since in the RCD spaces Ch can also be viewed as a Dirichlet form, a more precise connection between the RCD^* and the BE sides of the theory is possible and can indeed be established: without entering here in too many technical details, we just mention that in [3] it was proved that $\mathsf{BE}(K,\infty)$ holds for $\mathsf{RCD}(K,\infty)$ spaces, while the implication from $\mathsf{BE}(K,\infty)$ to $\mathsf{RCD}(K,\infty)$ has been established in [4] under mild regularity assumptions on the metric measure structure; the dimensional counterparts of this equivalence are given in [18], [5].

Using these connections and partitions of unity, we can read the local RCD property as a local BE property and then use partitions of unity to globalize it; eventually we use the equivalence in the converse direction to obtain the RCD property globally.

Integral formulation of $\mathsf{BE}(K,\infty)$ ***and gradient estimates.*** Even if the classical differential formulation (1.1) of the Bakry-Émery condition is clearly local, the weak-integral $\mathsf{BE}(K,N)$ condition introduced in [4] has a global character: the corresponding Γ_2 tensor also involves a test function φ in the multilinear form

$$\mathbf{\Gamma}_2(f;\varphi) := \int_X \Big(\frac{1}{2}\Gamma(f)\Delta\varphi - \Gamma\big(f,\Delta f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m}, \quad f,\ \Delta f\in \mathrm{D}(\mathcal{E}),\ \varphi,\Delta\varphi\in L^\infty(X,\mathfrak{m}), \tag{1.3}$$

and the resulting $\mathsf{BE}(K,N)$ condition

$$\mathbf{\Gamma}_2(f;\varphi) \ge \int_X \Big(K\,\Gamma\big(f\big) + \frac{1}{N}(\Delta f)^2\Big)\varphi\,\mathrm{d}\mathfrak{m} \quad \text{for every } \varphi\ge 0, \tag{1.4}$$

is thus of global type and involves test functions φ which belong to the domain of Δ in L^∞. The formulation based on (1.3) and (1.4) is carefully adapted to deal with the lowest regularity and summability properties of f, φ, that should be both sufficient to give sense to the $\mathbf{\Gamma}_2$ tensor and invariant with respect to the action of the Markov semigroup. The latter is a crucial requirement that is intrinsically global, not satisfied by the stronger differential formulation as in (1.1) which would impose $\Gamma\big(f\big)\in \mathrm{D}(\Delta)$. In fact, the typical approach requiring the existence of an algebra of sufficiently smooth functions where all the relevant computations can be carried on, is quite useful to deal with many concrete examples but it does not seem to be well adapted to the non-smooth framework of general metric measure spaces.

Therefore finding useful localizations of $\mathsf{BE}(K,N)$ is not a trivial issue, since it involves the summability of $\Delta\varphi$ and the regularity of Δf and of $\Gamma\big(f\big)$. Recall that a product with a test function χ affects Δf through the Leibniz formula

$$\Delta(f\chi) = \chi\Delta f + f\Delta\chi + 2\Gamma\big(f,\chi\big), \tag{1.5}$$

thus showing the importance to secure existence of good classes of cutoff functions χ with $\chi, \Gamma\big(\chi\big),\ \Delta\chi\in L^\infty(X,\mathfrak{m})$ (a problem addressed in Lemma 6.7) and general conditions ensuring $\Gamma\big(f\big)\in \mathrm{D}(\mathcal{E})$. Similar problems arise with the chain rule

$$\Delta\eta(f) = \eta'(f)\Delta f + \eta''(f)\Gamma\big(f\big), \quad \eta\in \mathrm{C}^2(\mathbb{R}). \tag{1.6}$$

It is therefore natural to investigate higher integrability and regularity properties of $\Gamma(f)$ (see Theorems 3.1 and 5.5) that are interesting by themselves and will also play a role in our forthcoming paper [5].

Improving some results of [31], in this paper we show that in $\mathsf{BE}(K,\infty)$ spaces functions $f \in L^2 \cap L^\infty(X,\mathfrak{m})$ with $\Delta f \in L^2(X,\mathfrak{m})$ satisfy the extra integrability property for $\Gamma\big(f\big)$

$$\Gamma\big(f\big) \in L^2(X,\mathfrak{m}), \quad \int_X \Gamma\big(f\big)^2 \,\mathrm{d}\mathfrak{m} \le A_K \|f\|^2_{L^\infty(X,\mathfrak{m})} \int_X \big(f-\Delta f\big)^2 \,\mathrm{d}\mathfrak{m}. \tag{1.7}$$

In addition, if f and Δf belong to $L^4(X,\mathfrak{m})$ then $\Gamma\big(f\big)$ belongs to the domain of the Dirichlet form $\mathcal{E}$ and satisfies

$$\Gamma\big(f\big) \in \mathrm{D}(\mathcal{E}), \quad \int_X \Big(\Gamma\big(f\big)^2 + \Gamma\big(\Gamma\big(f\big)\big)\Big) \,\mathrm{d}\mathfrak{m} \le B_K \int_X \big(f-\Delta f\big)^4 \,\mathrm{d}\mathfrak{m}. \tag{1.8}$$

The constants A_K, B_K in the previous estimates depend only on K. These properties allow for simpler formulation of (1.4) and are the starting points for studying its localization, since we will show that the same estimates hold even if X is covered by a collection of spaces satisfying $\mathsf{BE}(K,\infty)$.

Plan of the paper. The paper is organized as follows: in Section 2 we will work in the general framework of Dirichlet spaces, without assuming that the Dirichlet form $\mathcal{E}$ is induced by the Cheeger energy and actually avoiding any reference to a metric structure (so that the role of modulus of the weak gradient is played by $\sqrt{\Gamma(f)}$); more precisely we just assume that (X,τ) is a Polish topological space endowed with a σ-finite reference Borel measure $\mathfrak{m}$ and a strongly local and symmetric Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$ enjoying a *Carré du Champ* $\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(X,\mathfrak{m})$ and a Γ-calculus (see e.g. [4, § 2]). In this framework, under the $\mathsf{BE}(K,\infty)$ condition, we establish useful higher integrability properties for $\Gamma(f)$ as (1.7) by an interpolation argument (Section 3) and the extra-regularity property (1.8) (Section 5). These properties will be used in [5] and in the second part of the paper to prove the Local-to-Global property. Still in the same framework, in Section 4 we provide equivalent formulations and implications of the $\mathsf{BE}(K,N)$ property that play a role in this and in the companion paper [5].

In the second part, composed by Sections 6 and Section 7, we will work instead with metric measure spaces and we will use the previous estimates to prove the Local-to-Global property. In Section 6 we discuss basic localization properties of gradients and Laplacians and show how curvature lower bounds can be used to obtain existence of cutoff functions with bounded Laplacian.

In Section 7 we recall the precise definitions of $\mathsf{RCD}^*(K,N)$ spaces, the equivalence results with $\mathsf{BE}(K,N)$, and we carry on the proof of the Local-to-Global property in case the space $(X,\mathsf{d},\mathfrak{m})$ is locally compact.

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2 Notation, preliminaries and the Bakry-Émery condition

In this section we will recall the basic assumptions related to the Bakry-Émery condition.

Strongly local Dirichlet forms and Γ***-calculus.*** The natural setting is provided by a Polish topological space (X,τ) endowed with a σ-finite reference Borel measure $\mathfrak{m}$ with full support (i.e. $\mathrm{supp}(\mathfrak{m}) = X$) and

$$\begin{gathered}\text{a strongly local, symmetric Dirichlet form } \mathcal{E} \text{ on } L^2(X,\mathfrak{m}) \text{ enjoying}\\ \text{a \textit{Carré du Champ} } \Gamma : \mathrm{D}(\mathcal{E}) \times \mathrm{D}(\mathcal{E}) \to L^1(X,\mathfrak{m}) \text{ and}\\ \text{generating a mass-preserving Markov semigroup } (\mathsf{P}_t)_{t\geq 0} \text{ on } L^2(X,\mathfrak{m}),\end{gathered} \tag{2.1}$$

(see e.g. [4, §2]). None of the estimates we are discussing in this section really needs an underlying compatible metric structure, as the one discussed in [4, §3]. We refer to [4, §2] for the basic notation and assumptions; in any case, we will apply all the results to the case of the Cheeger energy (thus assumed to be quadratic) of the metric measure space $(X,\mathsf{d},\mathfrak{m})$ and we will use the calculus properties of the Dirichlet form that are related to the Γ-formalism.

In the following we call $\mathbb{V}$ the Hilbert space made by $\mathrm{D}(\mathcal{E}) \subset L^2(X,\mathfrak{m})$ endowed with the scalar product

$$(f,g)_{\mathbb{V}} := \int_X \Big(fg + \Gamma(f,g)\Big)\,\mathrm{d}\mathfrak{m}.$$

The Laplace operator $-\Delta : \mathbb{V} \to \mathbb{V}'$ and its perturbation $-\Delta_\lambda$ are respectively defined as

$$\langle -\Delta f, g\rangle := \mathcal{E}(f,g), \quad -\Delta_\lambda f = \lambda f - \Delta f \quad \text{for every } f,\, g \in \mathbb{V}. \tag{2.2}$$

The operator Δ is the generator of the Markov semigroup $(\mathsf{P}_t)_{t\geq 0}$ and its realization in $L^2(X,\mathfrak{m})$ is an unbounded selfadjoint nonnegative operator with domain $\mathrm{D}_{L^2}(\Delta)$.

We will denote by

$$\mathrm{D}_{L^p}(\Delta) := \Big\{f \in \mathbb{V} \cap L^p(X,\mathfrak{m}) : \Delta f \in L^2 \cap L^p(X,\mathfrak{m})\Big\}, \quad p \in [1,\infty], \tag{2.3}$$

its domain as unbounded operator in $L^p(X,\mathfrak{m})$, endowed with the norm $\|f\|_{\mathbb{V}} + \|f - \Delta f\|_{L^2\cap L^p}$. This choice of the norm is justified by the inequalities

$$\lambda f - \Delta f = g \in L^p(X,\mathfrak{m}) \quad \Longrightarrow \quad \lambda\|f\|_{L^p} \leq \|g\|_{L^p}, \quad \|\Delta f\|_{L^p} \leq 2\|g\|_{L^p} = 2\|\Delta_\lambda f\|_{L^p} \tag{2.4}$$

for all $\lambda \geq 0$. In turn, the implication (2.4) follows by the fact that the resolvents $\lambda(\lambda - \Delta)^{-1}$, $\lambda > 0$, associated to a Dirichlet form are sub-Markovian (see e.g. [26, Def. 4.1 and Thm. 4.4]) and therefore contractive in every $L^p(X,\mathfrak{m})$.

The Γ_2 ***tensor and the Bakry-Émery condition.*** We introduce the multilinear form $\mathbf{\Gamma}_2$ given by

$$\mathbf{\Gamma}_2(f,g;\varphi) := \frac{1}{2}\int_X \Big(\Gamma\big(f,g\big)\,\Delta\varphi - \big(\Gamma\big(f,\Delta g\big) + \Gamma\big(g,\Delta f\big)\big)\varphi\Big)\,\mathrm{d}\mathfrak{m} \quad (f,g,\varphi) \in \mathrm{D}(\mathbf{\Gamma}_2), \tag{2.5}$$

where $\mathrm{D}(\mathbf{\Gamma}_2) := \mathrm{D}_{\mathbb{V}}(\Delta) \times \mathrm{D}_{\mathbb{V}}(\Delta) \times \mathrm{D}_{L^\infty}(\Delta)$, and $\mathrm{D}_{\mathbb{V}}(\Delta) = \big\{f \in \mathbb{V} : \Delta f \in \mathbb{V}\big\}$. When $f = g$ we also set

$$\mathbf{\Gamma}_2(f;\varphi) := \mathbf{\Gamma}_2(f,f;\varphi) = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\,\Delta\varphi - \Gamma\big(f,\Delta f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m}, \tag{2.6}$$

so that

$$\mathbf{\Gamma}_2(f,g;\varphi) = \frac{1}{4}\mathbf{\Gamma}_2(f+g;\varphi) - \frac{1}{4}\mathbf{\Gamma}_2(f-g;\varphi). \tag{2.7}$$

The multilinear form $\mathbf{\Gamma}_2$ provides a weak version (inspired by [8, 10]) of the Bakry-Émery condition [9, 7]. In the sequel, given $f : X \to \mathbb{R}$, we denote by $\mathrm{supp}(f)$ the smallest closed set $C \subset X$ such that $f = 0$ $\mathfrak{m}$-a.e. in $X \setminus C$; this way, the definition of support is independent of modifications of f in $\mathfrak{m}$-negligible sets.

Definition 2.1 (Bakry-Émery condition)**.** *Let $K \in \mathbb{R}$, $N \in [1, \infty]$, and $\nu := \frac{1}{N} \in [0, 1]$. We say that the strongly local Dirichlet form $\mathcal{E}$ satisfies the $\mathsf{BE}(K, N)$ condition, if it admits a Carré du Champ $\Gamma : \mathrm{D}(\mathcal{E}) \times \mathrm{D}(\mathcal{E}) \to L^1(X, \mathfrak{m})$ and for every $(f, \varphi) \in \mathrm{D}_{\mathbb{V}}(\Delta) \times \mathrm{D}_{L^\infty}(\Delta)$ with $\varphi \geq 0$ there holds*

$$\mathbf{\Gamma}_2(f; \varphi) \geq K \int_X \Gamma\big(f\big)\, \varphi \,\mathrm{d}\mathfrak{m} + \nu \int_X (\Delta f)^2 \varphi \,\mathrm{d}\mathfrak{m}. \tag{2.8}$$

We say that $\mathcal{E}$ satisfies the $\mathsf{BE}_{\mathrm{loc}}(K, N)$ condition if (2.8) *holds for all $(f, \varphi) \in \mathrm{D}_{\mathbb{V}}(\Delta) \times \mathrm{D}_{L^\infty}(\Delta)$ with $\varphi \geq 0$ compactly supported.*

Remark 2.2 (On the global character of the $\mathsf{BE}(K, N)$ condition)**.** Notice that $\mathsf{BE}(K, N)$ has a global nature, related to the fact that an integration by parts is understood in the weak formulation (2.6) of Bochner's inequality; for this reason, even the issue of the Global-to-Local property is delicate in this framework, since the passage to a smaller open set $U \subset X$ changes the Dirichlet form and the action of the Laplacian operator (unless one deals with functions compactly supported in U, compare with Remark 6.6). As a matter of fact, the localization seems to involve some "metric" assumption on U, relative to the distance $\mathsf{d}_{\mathcal{E}}$ induced by $\mathcal{E}$, see Proposition 6.4(c). For this reason, in the discussion of the Local-to-Global and Global-to-Local properties, we will deal with metric measure spaces under a metric version of the $\mathsf{BE}(K, N)$ condition (see Definition 6.1), although the equivalence results of [4], [18], [5] could be used to translate back the result to the BE formalism. ■

Remark 2.3 (Pointwise gradient estimates for the heat flow under $\mathsf{BE}(K, \infty)$)**.** When $N = \infty$, condition (2.8) is in fact equivalent (see [4, Corollary 2.3]) to either of the following pointwise gradient estimates

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt}\, \mathsf{P}_t\big(\Gamma\big(f\big)\big) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathbb{V}, \tag{2.9}$$

$$2\mathrm{I}_{2K}(t)\Gamma\big(\mathsf{P}_t f\big) \leq \mathsf{P}_t f^2 - \big(\mathsf{P}_t f\big)^2 \quad \mathfrak{m}\text{-a.e. in } X, \quad \text{for every } t > 0,\ f \in L^2(X, \mathfrak{m}), \tag{2.10}$$

where I_K denotes the real function

$$\mathrm{I}_K(t) := \int_0^t \mathrm{e}^{Kr}\,\mathrm{d}r = \begin{cases} \frac{1}{K}(\mathrm{e}^{Kt} - 1) & \text{if } K \neq 0, \\ t & \text{if } K = 0. \end{cases} \quad ■$$

3 Interpolation estimates: extra integrability of $\Gamma(f)$

Let us now consider the semigroup $(\mathsf{P}^\lambda_t)_{t\geq 0}$ generated by the operator $\Delta_\lambda := \Delta - \lambda$, $\lambda \geq 0$,

$$\mathsf{P}^\lambda_t f := \exp(t\Delta_\lambda) f = \mathrm{e}^{-\lambda t} \exp(t\Delta) f = \mathrm{e}^{-\lambda t}\, \mathsf{P}_t f. \tag{3.1}$$

Since $\sqrt{\mathrm{I}_{2K}(t)} \geq \sqrt{t}\,\mathrm{e}^{Kt}$ if $K \leq 0$, choosing $\lambda \geq K_-$ and $p \geq 2$, $\mathsf{BE}(K, \infty)$ and the contractivity of P_t in L^p yield by (2.10)

$$\big\|\Gamma\big(\mathsf{P}^\lambda_t f\big)^{1/2}\big\|_{L^p(X,\mathfrak{m})} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^p(X,\mathfrak{m})} \quad \text{for every } f \in L^p(X, \mathfrak{m}),\ t > 0. \tag{3.2}$$

We prove now a useful estimate for $\Gamma(f)$ in $L^p(X,\mathfrak{m})$ when $\Delta f\in L^p(X,\mathfrak{m})$ and f is bounded. For the sake of simplicity, we will only focus on the cases $p=2$ and $p=\infty$, that will also play a role in [5]. Analogous results (in $X=\mathbb{R}^d$) when only a one-sided bound on f is available have been proved with completely different proofs in [24, 20].

Theorem 3.1 (Gradient interpolation)**.** *Assume* $\mathsf{BE}(K,\infty)$*, let* $\lambda\ge K_-$*,* $p\in\{2,\infty\}$*, and* $f\in L^2\cap L^\infty(X,\mathfrak{m})$ *with* $\Delta f\in L^p(X,\mathfrak{m})$*. Then* $\Gamma(f)\in L^p(X,\mathfrak{m})$ *and*

$$\|\Gamma(f)\|_{L^p(X,\mathfrak{m})}\le c\|f\|_{L^\infty(X,\mathfrak{m})}\,\|\Delta_\lambda f\|_{L^p(X,\mathfrak{m})} \tag{3.3}$$

for a universal constant c *independent of* $\lambda,X,\mathfrak{m},f$ *(*$c=\sqrt{2\pi}$ *when* $p=\infty$*).*

Moreover, if $f_n\in \mathrm{D}_{L^2}(\Delta)\cap L^\infty(X,\mathfrak{m})$ *with* $\sup_n\|f_n\|_{L^\infty(X,\mathfrak{m})}<\infty$ *and* $f_n\to f$ *strongly in* $\mathrm{D}_{L^2}(\Delta)$*, then* $\Gamma(f_n)\to\Gamma(f)$ *and* $\Gamma(f_n-f)\to 0$ *strongly in* $L^2(X,\mathfrak{m})$*.*

Proof. Let us first consider the case $p=\infty$ (here we follow the argument of [16, Prop. 3.6]): recalling the identity

$$f=\int_0^\infty \mathrm{e}^{-s}\,\mathsf{R}_{ts}(f-tAf)\,\mathrm{d}s$$

valid for nonnegative selfadjoint semigroups R with infinitesimal generator A, by applying (3.2) with $p=\infty$ to $f_t:=f-t\Delta_\lambda f$ we get

$$\begin{aligned}\Big\|\Gamma(f)^{1/2}\Big\|_{L^\infty(X,\mathfrak{m})}&\le\|f_t\|_{L^\infty(X,\mathfrak{m})}\int_0^\infty \mathrm{e}^{-s}(2ts)^{-1/2}\,\mathrm{d}s=\sqrt{\frac{\pi}{2t}}\,\|f_t\|_{L^\infty(X,\mathfrak{m})}\\&\le\sqrt{\frac{\pi}{2t}}\Big(\|f\|_{L^\infty(X,\mathfrak{m})}+t\|\Delta_\lambda f\|_{L^\infty(X,\mathfrak{m})}\Big).\end{aligned}$$

Choosing $t=\|f\|_\infty\,\|\Delta_\lambda f\|_\infty^{-1}$ we obtain (3.3).

In order to prove the formula (3.2) in the case $p=2$, by an elementary approximation, suffices to show the inequality under the additional assumption $f\in\mathbb{V}$, $\Delta f\in\mathbb{V}$. We use the Leibniz formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(\mathsf{P}^\lambda_t f)=2\Gamma\big(\mathsf{P}^\lambda_t f,\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big)$$

to get

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(\mathsf{P}^\lambda_t f)^{1/2}\Big|^2=\frac{\big(\Gamma(\mathsf{P}^\lambda_t f,\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f)\big)^2}{\Gamma(\mathsf{P}^\lambda_t f)}\le\Gamma\big(\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big). \tag{3.4}$$

Commuting Δ with $\frac{\mathrm{d}}{\mathrm{d}t}$, using the identity $\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f=\Delta_\lambda\mathsf{P}^\lambda_t f$ together with the fact that $\lambda\ge0$ we get

$$\begin{aligned}\int_0^\infty\int_X\Gamma\big(\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t&=-\int_0^\infty\int_X\big(\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big)\big(\frac{\mathrm{d}}{\mathrm{d}t}\Delta\mathsf{P}^\lambda_t f\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\\&=-\int_0^\infty\int_X\big(\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big)\big(\frac{\mathrm{d}}{\mathrm{d}t}\Delta_\lambda\mathsf{P}^\lambda_t f\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t-\lambda\int_0^\infty\int_X\big(\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}^\lambda_t f\big)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\\&\le-\frac12\int_0^\infty\int_X\frac{\mathrm{d}}{\mathrm{d}t}|\Delta_\lambda\mathsf{P}^\lambda_t f|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\le\frac12\int_X\big(\Delta_\lambda f\big)^2\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{3.5}$$

Setting $g_t:=\Gamma(\mathsf{P}^\lambda_t f)^{1/2}$, inserting (3.4) into (3.5) it follows that

$$\int_0^\infty\|t^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}g_t\|^2_{L^2(X,\mathfrak{m})}\,\frac{\mathrm{d}t}{t}\le\frac12\|\Delta_\lambda f\|^2_{L^2(X,\mathfrak{m})}. \tag{3.6}$$

According to the J.L. Lions Trace interpolation method (here we follow the notation of [34, 1.8.1]), the estimates (3.2) and (3.6) show that g_t belongs to the weighted functional space $V_1(\infty, \frac{1}{2}, L^\infty(X,\mathfrak{m}); 2, 1/2, L^2(X,\mathfrak{m}))$, so that its trace at $t = 0$ belongs to the K-interpolation space

$$g_0 \in (L^\infty(X,\mathfrak{m}), L^2(X,\mathfrak{m}))_{\theta,p} \quad \text{with} \quad \theta = \frac{1}{2}, \quad p = 4,$$

with

$$\|g_0\|_{(L^\infty(X,\mathfrak{m}),L^2(X,\mathfrak{m}))_{\theta,p}} \le c\|f\|^{1/2}_{L^\infty(X,\mathfrak{m})}\|\Delta_\lambda f\|^{1/2}_{L^2(X,\mathfrak{m})}.$$

Since $g_0 = \Gamma\big(f\big)^{1/2}$ we get (3.3) also in the case $p = 2$.

Let us now consider the last statement. The fact that $\Gamma\big(f_n - f\big) \to 0$ strongly in $L^2(X,\mathfrak{m})$ follows immediately from the interpolation inequality (3.3) by replacing f with $f_n - f$ and observing that $\Delta_\lambda(f_n - f) \to 0$ strongly in $L^2(X,\mathfrak{m})$ since $f_n \to f$ in $\mathrm{D}_{L^2}(\Delta)$. Recalling that

$$\big|\Gamma\big(f_n\big) - \Gamma\big(f\big)\big|^2 = \big|\Gamma\big(f_n - f, f_n + f\big)\big|^2 \le \Gamma\big(f_n - f\big)\Gamma\big(f_n + f\big)$$

we also get $\Gamma\big(f_n\big) \to \Gamma\big(f\big)$ strongly in $L^2(X,\mathfrak{m})$. □

4 Equivalent formulations of $\mathsf{BE}(K,N)$

Let $f \in \mathrm{D}_{\mathbb{V}}(\Delta)$ and $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$ and let us consider the expression (2.6) of $\mathbf{\Gamma}_2(f;\varphi)$; under the additional assumption $f \in \mathrm{D}_{L^\infty}(\Delta)$, by "integrating by parts" the term $\Gamma\big(f, \Delta f\big)$ it is possible to write $\mathbf{\Gamma}_2(f;\varphi)$ in a different form:

Lemma 4.1. *If $f \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathrm{D}_{L^\infty}(\Delta)$ and $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$ then*

$$\mathbf{\Gamma}_2(f;\varphi) = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\Delta\varphi + \Delta f\,\Gamma\big(f,\varphi\big) + \varphi(\Delta f)^2\Big)\,\mathrm{d}\mathfrak{m}. \tag{4.1}$$

Proof. Starting from the Leibniz formula

$$\Gamma\big(f,g\big)\varphi = \Gamma\big(f, g\varphi\big) - \Gamma\big(f,\varphi\big)g \quad \text{for every } f \in \mathbb{V},\ g,\varphi \in \mathbb{V} \cap L^\infty(X,\mathfrak{m}),$$

if $f \in \mathrm{D}_{L^2}(\Delta)$ one has by integration

$$\int_X \Gamma\big(f,g\big)\varphi\,\mathrm{d}\mathfrak{m} = -\int_X \Big(g\varphi\,\Delta f + \Gamma\big(f,\varphi\big)g\Big)\,\mathrm{d}\mathfrak{m}.$$

Choosing $g = \Delta f \in \mathbb{V} \cap L^\infty(X,\mathfrak{m})$ we immediately see that (2.6) yields (4.1). □

Recalling the polarization identity (2.7), if $f,\ g \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathrm{D}_{L^\infty}(\Delta)$ and $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$, from (4.1) we also get

$$\mathbf{\Gamma}_2(f,g;\varphi) = \frac{1}{2}\int_X \Big(\Gamma\big(f,g\big)\Delta\varphi + \Delta f\,\Gamma\big(g,\varphi\big) + \Delta g\,\Gamma\big(f,\varphi\big) + 2\varphi\,\Delta f\,\Delta g\Big)\,\mathrm{d}\mathfrak{m}. \tag{4.2}$$

In the passage from (2.5) to (4.1) (or, equivalently, (4.2)) we used the additional regularity assumption $f \in \mathrm{D}_{L^\infty}(\Delta)$; therefore the following approximation result will be useful in the verification of the $\mathsf{BE}(K,N)$ property.

Lemma 4.2 (Approximation of $\mathrm{D}_{\mathbb{V}}(\Delta)$ by $\mathrm{D}_{\mathbb{V}}(\Delta)\cap \mathrm{D}_{L^\infty}(\Delta)$)**.** *Let $f\in \mathrm{D}_{\mathbb{V}}(\Delta)$. Then there exist $f_n\in \mathrm{D}_{\mathbb{V}}(\Delta)\cap \mathrm{D}_{L^\infty}(\Delta)$ such that*

$$f_n\to f \text{ in } \mathbb{V} \quad \text{and} \quad \Delta f_n\to \Delta f \text{ in } \mathbb{V}.$$

In particular, $\mathsf{BE}(K,N)$ holds if and only if (2.8) holds for all $f\in \mathrm{D}_{\mathbb{V}}(\Delta)\cap \mathrm{D}_{L^\infty}(\Delta)$ and all $\varphi\in \mathrm{D}_{L^\infty}(\Delta)$ nonnegative.

Proof. Let $f\in \mathrm{D}_{\mathbb{V}}(\Delta)$ and define $h\in\mathbb{V}$ by $h:=f-\Delta f$. Consider the truncated functions

$$h_n:=\max\{\min\{h,n\},-n\}. \tag{4.3}$$

Clearly $h_n\in \mathbb{V}\cap L^\infty(X,\mathfrak{m})$ and $h_n\to h$ in $\mathbb{V}$. Let $f_n\in\mathbb{V}$ be the unique (weak) solution of

$$f_n-\Delta f_n=h_n. \tag{4.4}$$

Of course $f_n\to f\in\mathbb{V}$ and the variational maximum principle implies that $|f_n|\le |h_n|\le n$ $\mathfrak{m}$-a.e. in X; let us briefly recall the argument, well known in literature as *Stampacchia's truncation.* The solution f_n of (4.4) is the unique minimum point of the strictly convex functional

$$\mathbb{V}\ni g\mapsto \frac12\int_X |g-h_n|^2\,\mathrm{d}\mathfrak{m}+\frac12\mathcal{E}(g,g);$$

replacing f_n by the truncated function $\tilde f_n:=\max\{\min\{f_n,n\},-n\}\in\mathbb{V}$ neither of the integrals above increase. It follows that $f_n=\tilde f_n$ $\mathfrak{m}$-a.e. in X, as desired.

We conclude by observing that, since f_n belong to $\mathbb{V}\cap L^\infty(X,\mathfrak{m})$, we have $\Delta f_n=f_n-h_n$ belong to $\mathbb{V}\cap L^\infty(X,\mathfrak{m})$ as well and, since $h_n\to h$ and $f_n\to f$ in $\mathbb{V}$, we get $\Delta f_n\to\Delta f$ in $\mathbb{V}$. □

Thanks to the improved integrability of Γ, provided by the $\mathsf{BE}(K,\infty)$ condition, we can now somehow extend the domain of $\mathbf{\Gamma}_2(f,g;\varphi)$ to $\Big(\mathrm{D}_{L^2}(\Delta)\cap L^\infty(X,\mathfrak{m})\Big)^3$, i.e. neither requiring Δf, Δg to be in $\mathbb{V}$ nor requiring $\Delta\varphi$ to be in $L^\infty(X,\mathfrak{m})$.

Corollary 4.3. *If $\mathsf{BE}(K,\infty)$ holds then the right hand side of (4.2) makes sense in the space $\Big(\mathrm{D}_{L^2}(\Delta)\cap L^\infty(X,\mathfrak{m})\Big)^3$. In addition, if $\mathsf{BE}(K,N)$ holds then*

$$\int_X\Big(\frac12\Gamma(f)\Delta\varphi+\Delta f\,\Gamma\big(f,\varphi\big)+\varphi(\Delta f)^2\Big)\,\mathrm{d}\mathfrak{m}\ge K\int_X\Gamma\big(f\big)\,\varphi\,\mathrm{d}\mathfrak{m}+\nu\int_X(\Delta f)^2\varphi\,\mathrm{d}\mathfrak{m} \tag{4.5}$$

is satisfied by every choice of $f,\,\varphi\in \mathrm{D}_{L^2}(\Delta)\cap L^\infty(X,\mathfrak{m})$ with $\varphi\ge0$.

Proof. Notice that the right hand side of (4.2) makes sense if $f,\,g,\varphi\in \mathrm{D}_{L^2}(\Delta)\cap L^\infty(X,\mathfrak{m})$ since $\Gamma\big(f\big)$, $\Gamma\big(g\big)$, $\Gamma\big(\varphi\big)\in L^2(X,\mathfrak{m})$ by Theorem 3.1. Under the assumption $\mathsf{BE}(K,N)$, in order to check (4.5) we introduce the mollified heat flow

$$\mathfrak{H}^\varepsilon f:=\frac1\varepsilon\int_0^\infty \mathsf{P}_r f\,\kappa(r/\varepsilon)\,\mathrm{d}r, \tag{4.6}$$

where $\kappa\in \mathrm{C}^\infty_c(0,\infty)$ is a nonnegative regularization kernel with $\int_0^\infty\kappa(r)\,\mathrm{d}r=1$.

Setting $f^\varepsilon := \mathfrak{H}^\varepsilon f$, $\varphi^\varepsilon := \mathfrak{H}^\varepsilon \varphi$, it is not difficult to check that if $f, \varphi \in \mathrm{D}_{L^2}(\Delta) \cap L^\infty(X, \mathfrak{m})$ then $f^\varepsilon, \varphi^\varepsilon \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathrm{D}_{L^\infty}(\Delta)$; morever, $\varphi^\varepsilon \geq 0$ if $\varphi \geq 0$, so that (2.8) and (4.1) yield

$$\int_X \Big(\frac{1}{2}\Gamma\big(f^\varepsilon\big)\Delta\varphi^\varepsilon + \Delta f^\varepsilon\,\Gamma\big(f^\varepsilon, \varphi^\varepsilon\big) + (1-\nu)\varphi^\varepsilon(\Delta f^\varepsilon)^2\Big)\,\mathrm{d}\mathfrak{m} \geq K \int_X \Gamma\big(f^\varepsilon\big)\,\varphi^\varepsilon\,\mathrm{d}\mathfrak{m}.$$

Since

$$f^\varepsilon \to f,\ \varphi^\varepsilon \to \varphi \quad \text{strongly in } \mathrm{D}_{L^2}(\Delta), \quad \|f^\varepsilon\|_{L^\infty(X,\mathfrak{m})} \leq \|f\|_{L^\infty(X,\mathfrak{m})}, \quad \|\varphi^\varepsilon\|_{L^\infty(X,\mathfrak{m})} \leq \|\varphi\|_{L^\infty(X,\mathfrak{m})}$$

we can apply the continuity properties of Γ stated in Theorem 3.1 to pass to the limit in the previous inequality as $\varepsilon \downarrow 0$. □

5 Further regularity for $\Gamma(f)$ in $\mathsf{BE}(K,\infty)$ spaces and the measure-valued Γ_2-tensor

5.1 Quasi-regular Dirichlet forms and the measure-valued Γ_2-tensor

In this section we will assume that the Dirichlet form $\mathcal{E}$ is *quasi-regular*, according to Ma and Röckner: we refer to [26, III.2, III.3, IV.3] (covering the more general case of a possibly non-symmetric Dirichlet form) and [15, 1.3] for the precise definition and for the related notions of $\mathcal{E}$-polar sets and $\mathcal{E}$-quasi-continuous functions, see also the concise account of [31, § 2.3]. Here we just recall that this setting covers the main example of *regular* Dirichlet forms in *locally compact and separable metric spaces*, which is sufficient for our main applications in the next sections. Still the results presented here, at least in the case $\mathsf{BE}(K,\infty)$, could be interesting in more general situations where (X,τ) is not locally compact: this is the reason why we state them in greater generality.

Remark 5.1 (Regular Dirichlet forms)**.** When (X,τ) is also locally compact, we recall that a Dirichlet form $\mathcal{E}$ is *regular* if $\mathrm{D}(\mathcal{E}) \cap \mathrm{C_c(X)}$ is dense both in $\mathrm{D}(\mathcal{E})$ (w.r.t. the $\mathbb{V}$-norm) and in $\mathrm{C}_c(X)$ (w.r.t. uniform convergence).

If $\mathcal{E}$ is quasi-regular, then [15, Remark 1.3.9(ii)]

$$\text{every function } f \in \mathbb{V} \text{ admits an } \mathcal{E}\text{-quasi-continuous representative } \tilde{f}.$$

The function $\tilde{f}$ is uniquely determined up to a $\mathcal{E}$-polar set. We introduce the convex set

$$\mathbb{V}_+ := \big\{\varphi \in \mathbb{V} : \varphi \geq 0 \ \mathfrak{m}\text{-a.e. in } X\big\}$$

and we denote by $\mathbb{V}'$ the set of continuous linear functionals $\ell : \mathbb{V} \to \mathbb{R}$, while $\mathbb{V}'_+$ denotes the convex subset of all continuous linear functionals ℓ such that $\langle \ell, \phi\rangle \geq 0$ for all $\phi \in \mathbb{V}_+$; we also set $\mathbb{V}'_\pm := \mathbb{V}'_+ - \mathbb{V}'_+$.

The next result provides an important characterization of functionals in $\mathbb{V}'_+$, that motivates our interest for quasi-regular Dirichlet forms (see [26, Ch. VI, Prop. 2.1] and also [11, Ch. I, § 9.2] in the case of a finite measure $\mathfrak{m}(X) < \infty$ for the proof).

Proposition 5.2. *Let us assume that $\mathcal{E}$ is quasi-regular. Then for every $\ell \in \mathbb{V}'_+$ there exists a unique σ-finite and nonnegative Borel measure μ_ℓ in X such that*

(1) *every $\mathcal{E}$-polar set is μ_ℓ-negligible;*

(2) *for all $f \in \mathbb{V}$ the $\mathcal{E}$-q.c. representative $\tilde{f}$ belongs to $L^1(X, \mu_\ell)$ and*

$$\langle \ell, f \rangle = \int_X \tilde{f} \, \mathrm{d}\mu_\ell; \tag{5.1}$$

(3) *if $\langle \ell, \varphi \rangle \leq M < \infty$ for every $\varphi \in \mathbb{V}_+$ with $\varphi \leq 1$ $\mathfrak{m}$-a.e. in X, then μ_ℓ is a finite measure and $\mu_\ell(X) \leq M$.*

We will often identify $\ell \in \mathbb{V}'_+$ with the corresponding measure μ_ℓ. Notice that if $\ell \in \mathbb{V}'$ and there exists $h \in L^1 \cap L^2(X, \mathfrak{m})$ such that

$$\langle \ell, \varphi \rangle \geq \int_X h\varphi \, \mathrm{d}\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}_+ \tag{5.2}$$

then there exists a measure $\mu_+ \in \mathbb{V}'_+$ such that ℓ can be represented by the signed meassure $\mu_\ell = h\mathfrak{m} + \mu_+$. When for some $f \in \mathbb{V}$ the functional $\ell = \Delta f$ can be identified with a signed measure μ_ℓ, we will use the notation $\mu_\ell = \Delta^* f$.

The next result collects a few useful properties that have been proved in [31, § 3]; we introduce the space $\mathbb{L}_\Gamma := \big\{ f \in \mathbb{V} : \ f, \Gamma\big(f\big) \in L^\infty(X, \mathfrak{m}) \big\}$.

Theorem 5.3. *Let us suppose that $\mathcal{E}$ satisfies the $\mathsf{BE}(K, \infty)$ condition.*

(1) *For every $f \in \mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$ we have $\Gamma\big(f\big) \in \mathbb{V}$ with*

$$\mathcal{E}(\Gamma\big(f\big)) \leq -\int_X \Big(2K\Gamma\big(f\big)^2 + 2\Gamma\big(f\big)\Gamma\big(f, \Delta f\big)\Big) \, \mathrm{d}\mathfrak{m}. \tag{5.3}$$

(2) $\mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$ *is an algebra (i.e. closed w.r.t. pointwise multiplication) and, more generally, if $\boldsymbol{f} = (f_i)_{i=1}^n \in (\mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma)^n$ then $\Phi(\boldsymbol{f}) \in \mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$ for every smooth function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with $\Phi(0) = 0$.*

(3) *If $\mathcal{E}$ is also quasi-regular and $f \in \mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$, then $\Delta\Gamma\big(f\big)$ can be represented by a signed measure vanishing on $\mathcal{E}$-polar sets and, defining*

$$\Gamma^*_{2,K}[f] := \frac{1}{2}\Delta^\star\Gamma\big(f\big) - \Big(\Gamma\big(f, \Delta f\big) + K\Gamma\big(f\big)\Big)\mathfrak{m}, \tag{5.4}$$

the measure $\Gamma^*_{2,K}[f]$ is nonnegative, satisfies

$$\Gamma^*_{2,K}[f](X) \leq \int_X \Big(\big(\Delta f\big)^2 - K\Gamma\big(f\big)\Big) \, \mathrm{d}\mathfrak{m} \tag{5.5}$$

and provides a representation of the $\mathbf{\Gamma}_2$ multilinear form as follows:

$$\mathbf{\Gamma}_2(f; \varphi) = \int_X \tilde{\varphi} \, \mathrm{d}\,\Gamma^\star_{2,K}[f] + K\int_X \Gamma\big(f\big)\varphi \, \mathrm{d}\mathfrak{m}, \quad \textit{for every } \varphi \in \mathrm{D}_{L^\infty}(\Delta). \tag{5.6}$$

(4) *There exists a continuous, symmetric and bilinear map $\gamma_{2,K} : \big(\mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma\big)^2 \to L^1(X, \mathfrak{m})$ such that for every $f \in \mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$ (so that $\Gamma\big(f\big) \in \mathbb{V} \cap L^\infty(X, \mathfrak{m})$) there holds*

$$\Gamma^\star_{2,K}[f] = \gamma_{2,K}[f, f]\mathfrak{m} + \Gamma_2^\perp[f], \quad \textit{with} \quad \Gamma_2^\perp[f] \geq 0, \quad \Gamma_2^\perp[f] \perp \mathfrak{m}. \tag{5.7}$$

Setting $\gamma_{2,K}[f] := \gamma_{2,K}[f, f] \geq 0$, one has for every $f \in \mathrm{D}_\mathbb{V}(\Delta) \cap \mathbb{L}_\Gamma$

$$\Gamma\big(\Gamma\big(f\big)\big) \leq 4\gamma_{2,K}[f]\,\Gamma\big(f\big) \quad \mathfrak{m}\text{-a.e. in } X. \tag{5.8}$$

Notice that the measures $\Gamma^\star_{2,K}[f]$, $K \in \mathbb{R}$, just differ by a multiple of $\Gamma(f)\mathfrak{m}$, so the (nonnegative) singular part in the Lebesgue decomposition (5.7) is independent of K. In all the relevant estimates, it would be sufficient to consider the Lebesgue density $\gamma_{2,K}[\cdot]$, but it is still useful to think in terms of measures to recover all the information coded inside $\mathbf{\Gamma}_2(\cdot;\cdot)$.

If $\mathsf{BE}(K,N)$ holds and $\mathcal{E}$ is quasi-regular, we have the refined inequalities

$$\Gamma^\star_{2,K}[f] \geq \gamma_{2,K}[f]\mathfrak{m} \geq \nu(\Delta f)^2\mathfrak{m}. \tag{5.9}$$

5.2 Measure-valued Γ_2 tensor under lower regularity assumptions

In this section we want to show that the regularity assumptions in Theorem 5.3 can be considerably relaxed: in particular we will give a meaning to $\Gamma^\star_{2,K}[f]$ for every $f \in \mathrm{D}_{L^4}(\Delta)$. The main tools are the a-priori estimates of Theorem 5.5 and the following simple approximation result.

Lemma 5.4. *Let us assume* $\mathsf{BE}(K,\infty)$ *and* $p \in (1,\infty)$*. For every* $f \in \mathrm{D}_{L^p}(\Delta)$ *there exist* $f_n \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathbb{L}_\Gamma$ *converging to* f *in* $\mathrm{D}_{L^p}(\Delta)$*.*

Proof. We argue as in the proof of Lemma 4.2: by setting $h = f - \Delta f \in L^2 \cap L^p(X,\mathfrak{m})$ and considering the truncated functions $h_n \in L^2 \cap L^\infty(X,\mathfrak{m})$ as in (4.3) and f_n given by (4.4), it is immediate to see that f_n converge to f in $\mathrm{D}_{L^p}(\Delta)$ thanks to (2.4). On the other hand, $f_n \in \mathrm{D}_{L^\infty}(\Delta)$, so that $f_{n,\varepsilon} := \mathfrak{H}_\varepsilon f_n$ (where $\mathfrak{H}_\varepsilon$ is given by (4.6)) belong to $\mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathbb{L}_\Gamma$ and converge to f_n in $\mathrm{D}_{L^p}(\Delta)$ as $\varepsilon \downarrow 0$. A simple diagonal argument exhibits a sequence f_{n,ε_n} satisfying the thesis of the Lemma. □

Theorem 5.5. *Let us assume that* $\mathsf{BE}(K,\infty)$ *holds and let* $f, g \in \mathrm{D}_{L^4}(\Delta)$*. Then* $\Gamma(f,g) \in \mathbb{V}$ *and for every* $\lambda \geq (K-1/2)_-$ *and* $f, g \in \mathrm{D}_{L^4}(\Delta)$ *we have (with non-optimal constants)*

$$\|\Gamma(f)\|_{\mathbb{V}} \leq 2\sqrt{10}\,\|\Delta_\lambda f\|^2_{L^4}, \quad \|\Gamma(f,g)\|_{\mathbb{V}} \leq 4\sqrt{10}\,\|\Delta_\lambda f\|_{L^4}\,\|\Delta_\lambda g\|_{L^4}. \tag{5.10}$$

In particular, if $\mathsf{BE}(K,N)$ *holds, for every* $f \in \mathrm{D}_{L^4}(\Delta)$ *and* $\varphi \in \mathbb{V}_+$ *we have*

$$\int_X \Big(-\frac{1}{2}\Gamma(\Gamma(f),\varphi) + \Delta f\,\Gamma(f,\varphi) + \varphi(\Delta f)^2\Big)\,\mathrm{d}\mathfrak{m} \geq \int_X \Big(K\,\Gamma(f) + \nu(\Delta f)^2\Big)\varphi\,\mathrm{d}\mathfrak{m} \tag{5.11}$$

and for every $\varphi \in \mathbb{V}_+$ *with* $\Gamma(\varphi) \in L^\infty(X,\mathfrak{m})$*, setting* $K_\lambda := 2K + 2\lambda \geq 1$*, there holds*

$$\begin{aligned}\int_X \Big(\Gamma(\Gamma(f)) + K_\lambda\Gamma(f)^2\Big)\varphi\,\mathrm{d}\mathfrak{m} \leq 2\int_X \Big(\Delta_\lambda f\Gamma(f,\Gamma(f)) + \Delta f\,\Delta_\lambda f\Gamma(f)\Big)\varphi\,\mathrm{d}\mathfrak{m}\\ + \int_X \Big(-\Gamma(f)\Gamma(\Gamma(f),\varphi) + 2\Delta_\lambda f\,\Gamma(f)\Gamma(f,\varphi)\Big)\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{5.12}$$

Proof. For every $f \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathbb{L}_\Gamma$, (5.3) and an integration by parts immediately yield

$$\begin{aligned}K_\lambda \int_X \Gamma(f)^2\,\mathrm{d}\mathfrak{m} + \mathcal{E}(\Gamma(f)) &\leq -2\int_X \Gamma(f)\Gamma(f,\Delta_\lambda f)\,\mathrm{d}\mathfrak{m}\\ &= 2\int_X \Big(\Delta_\lambda f\,\Delta f\,\Gamma(f) + \Delta_\lambda f\,\Gamma(f,\Gamma(f))\Big)\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{5.13}$$

By (2.4) and the Hölder, inequality the right hand side of (5.13) can be bounded by

$$4\|\Delta_\lambda f\|_{L^4}^2 \|\Gamma(f)\|_{L^2} + 2\|\Delta_\lambda f\|_{L^4}\ \|\Gamma(f)\|_{L^2}^{1/2} \|\Gamma(\Gamma(f))\|_{L^1}^{1/2}$$
$$\le \frac{1}{4}\|\Gamma(f)\|_{L^2}^2 + 16\|\Delta_\lambda f\|_{L^4}^4 + \frac{1}{2}\mathcal{E}(\Gamma(f)) + \frac{1}{4}\|\Gamma(f)\|_{L^2}^2 + 4\|\Delta_\lambda f\|_{L^4}^4$$

which yields the first estimate of (5.10). The second one can be obtained by polarization, see the next Remark 5.6.

Now we use Lemma 5.4 to approximate any $f \in \mathrm{D}_{L^4}(\Delta)$ with a sequence f_n in $\mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathbb{L}_\Gamma$ converging to f in $\mathrm{D}_{L^4}(\Delta)$ and we pass to the limit in (5.10) by using the obvious bounds

$$\big\|\Gamma(f_n) - \Gamma(f_m)\big\|_{\mathbb{V}} = \big\|\Gamma(f_n - f_m, f_n + f_m)\big\|_{\mathbb{V}} \le 4\sqrt{10}\big\|\Delta_\lambda(f_n - f_m)\big\|_{L^4}\big\|\Delta_\lambda(f_n + f_m)\big\|_{L^4}. \tag{5.14}$$

By the regularity of $\Gamma(f)$ we can easily integrate by parts (4.5) obtaining

$$\mathbf{\Gamma}_2(f;\varphi) = \int_X -\frac{1}{2}\Gamma\big(\Gamma(f),\varphi\big) + \Delta f\,\Gamma(f,\varphi) + \varphi(\Delta f)^2\,\mathrm{d}\mathfrak{m} \tag{5.15}$$

and thus, if $\mathsf{BE}(K,N)$ holds, (5.11). If $\varphi \in \mathbb{V}_+$ is bounded with $\Gamma(\varphi) \in L^\infty(X,\mathfrak{m})$, the inequality (5.12) is an immediate consequence of (5.11) with $\nu = 0$, by replacing φ with $\Gamma(f)\varphi$. In the general case $\varphi \in \mathbb{V}_+$ with $\Gamma(\varphi) \in L^\infty(X,\mathfrak{m})$ we use a truncation argument. □

Remark 5.6. If A, B are normed spaces and $G : A \times A \to B$ is a symmetric bilinear map satisfying $\|G(a,a)\|_B \le C\|a\|_A^2$ for every $a \in A$, then G is continuous and satisfies

$$\|G(a_0,a_1)\|_B \le 2C\,\|a_0\|_A\|a_1\|_A \quad \text{for every } a_0, a_1 \in A. \tag{5.16}$$

It is sufficient to apply the polarization identity to G to obtain the estimate

$$\|G(a_0,a_1)\|_B \le \frac{C}{4}\Big(\|a_0+a_1\|_A^2 + \|a_0-a_1\|_A^2\Big) \le C\Big(\|a_0\|_A^2 + \|a_1\|_A^2\Big).$$

Then, substituting a_0 by λa_0 and a_1 by $\lambda^{-1}a_1$ and optimizing w.r.t. the parameter $\lambda > 0$ the inequality (5.16) follows.

Corollary 5.7. *Assume that* $\mathsf{BE}(K,\infty)$ *holds. Then, for every* $f \in \mathrm{D}_{L^4}(\Delta)$ *the linear functional*

$$\mathbb{V} \ni \varphi \mapsto \int_X -\frac{1}{2}\Gamma\big(\Gamma(f),\varphi\big) + \Delta f\,\Gamma(f,\varphi) + \big((\Delta f)^2 - K\Gamma(f)\big)\varphi\,\mathrm{d}\mathfrak{m} \tag{5.17}$$

belongs to $\mathbb{V}_+'$ *and can be represented by a measure that satisfies* (5.5) *and* (5.8), *with* $\gamma_{2,K}[f]$ *defined as in* (5.7). *We still denote this measure by* $\Gamma^\star_{2,K}[f]$.

Proof. Let us denote by $\ell_f \in \mathbb{V}'$ the functional in (5.17). By (5.11) it is immediate to see that $\ell_f \in \mathbb{V}_+'$ so that we can apply the representation result stated in Proposition 5.2; moreover, the first inequality in (5.10) gives

$$\begin{aligned} \Big|\langle \ell_f,\varphi\rangle\Big| &\le \|\varphi\|_{\mathbb{V}}\Big(\frac{1}{2}\|\Gamma(f)\|_{\mathbb{V}} + \|\Delta f\|_{L^4}\|\Gamma(f)\|_{L^2}^{1/2} + \|\Delta f\|_{L^4}^2 + |K|\|\Gamma(f)\|_{L^2}\Big) \quad (5.18)\\ &\le C_K\|\varphi\|_{\mathbb{V}}\,\|f-\Delta f\|_{L^4}^2 \end{aligned}$$

for all $\varphi \in \mathbb{V}$.

In order to prove (5.8) and (5.5) we apply Lemma 5.4, obtaining $f_n \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathbb{L}_\Gamma$ converging to f in $\mathrm{D}_{L^4}(\Delta)$. We first observe that Theorem 5.3, the convergence of Δf_n in $L^4(X, \mathfrak{m})$ and the convergence of $\Gamma(f_n)$ in $\mathbb{V}$ coming from (5.14), together with (5.18), yield

$$\lim_{n\to\infty} \int_X \varphi \, \mathrm{d}\Gamma^*_{2,K}[f_n] = \int_X \varphi \, \mathrm{d}\Gamma^*_{2,K}[f] \quad \text{for every } \varphi \in \mathbb{V}. \tag{5.19}$$

Passing to the limit as $n \to \infty$ in the inequality (derived from the fact that f_n satisfy (5.5))

$$\int_X \varphi \, \mathrm{d}\Gamma^*_{2,K}[f_n] \le \int_X \Big(\big(\Delta f_n\big)^2 - K\Gamma\big(f_n\big) \Big) \, \mathrm{d}\mathfrak{m}$$

we obtain the same inequality with f in place of f_n; then, (5.18) and Proposition 5.2(3) provide (5.5) for f. In addition, we can still use the strong convergence of $\Gamma(f_n)$ to $\Gamma(f)$ in $\mathbb{V}$ to show that $\big(\Gamma(f_n)\big)^{1/2}$ and $\big(\Gamma(\Gamma(f_n))\big)^{1/2}$ converge to $\big(\Gamma(f)\big)^{1/2}$ and to $\big(\Gamma(\Gamma(f))\big)^{1/2}$ in $L^2(X, \mathfrak{m})$ respectively. Since the functions $g_n := (\gamma_{2,K}[f_n])^{1/2}$ are uniformly bounded in $L^2(X, \mathfrak{m})$ thanks to (5.5), up to extracting a weakly converging subsequence, it is not restrictive to assume that $g_n \rightharpoonup g$ in $L^2(X, \mathfrak{m})$ as $n \to \infty$ so that for every essentially bounded $\varphi \in \mathbb{V}_+$

$$\begin{aligned} \int_X \big(\Gamma(\Gamma(f))\big)^{1/2} \varphi \, \mathrm{d}\mathfrak{m} &= \lim_{n\to\infty} \int_X \big(\Gamma(\Gamma(f_n))\big)^{1/2} \varphi \, \mathrm{d}\mathfrak{m} \\ &\le 2 \lim_{n\to\infty} \int_X g_n \Gamma\big(f_n\big)^{1/2} \varphi \, \mathrm{d}\mathfrak{m} = 2 \int_X g \Gamma\big(f\big)^{1/2} \varphi \, \mathrm{d}\mathfrak{m}. \end{aligned} \tag{5.20}$$

On the other hand, for every essentially bounded $\psi \in \mathbb{V}'_+$ we obtain from (5.19)

$$\begin{aligned} \int_X g^2 \psi \, \mathrm{d}\mathfrak{m} \le \liminf_{n\to\infty} \int_X g_n^2 \psi \, \mathrm{d}\mathfrak{m} \le \lim_{n\to\infty} \int_X \psi \, \mathrm{d}\Gamma^\star_{2,K}[f_n] = \int_X \psi \, \mathrm{d}\Gamma^\star_{2,K}[f] \\ = \int_X \psi \gamma_{2,k}[f] \, \mathrm{d}\mathfrak{m} + \int_X \psi \, \mathrm{d}\Gamma^\perp_{2,K}[f], \end{aligned} \tag{5.21}$$

so that $g^2 \le \gamma_{2,k}[f]$ $\mathfrak{m}$-a.e. in X. Combining with (5.20) and taking the squares we eventually get (5.8). □

6 Metric measure spaces and their localization

6.1 Metric measure spaces, weak gradients and Cheeger energy

We refer to the papers [2], [3], [4] for the basic facts and terminology on calculus in metric measure spaces; we will use the notation $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for the Sobolev space, Ch for the Cheeger energy arising from the relaxation in $L^2(X, \mathfrak{m})$ of the local Lipschitz constant

$$|\mathrm{D}f|(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}, \quad f : X \to \mathbb{R}, \tag{6.1}$$

of Lipschitz maps, $|\mathrm{D}f|_w$ for the so-called minimal relaxed gradient.

From now on, we shall denote by $\mathbf{X}$ the class of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ satisfying the following three conditions:

(a) (X, d) is complete and separable;

(b) $\mathfrak{m}$ is a nonnegative Borel measure with $\mathrm{supp}(\mathfrak{m}) = X$, satisfying

$$\mathfrak{m}(B_r(x)) \le c\,\mathrm{e}^{Ar^2} \tag{6.2}$$

for suitable constants $c \ge 0$, $A \ge 0$.

(c) $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian according to the terminology introduced in [21], i.e., the Cheeger energy Ch is a quadratic form.

As explained in [3], [4], the quadratic form Ch canonically induces a strongly regular Dirichlet $\mathcal{E}$ form in (X, τ), where τ is the topology induced by d. In addition, but this fact is less elementary (see [3, §4.3]), the formula

$$\Gamma\big(f\big) = |\mathrm{D}f|_w^2, \quad \Gamma\big(f, g\big) = \lim_{\epsilon \downarrow 0} \frac{|\mathrm{D}(f + \epsilon g)|_w^2 - |\mathrm{D}f|_w^2}{2\epsilon} \qquad f,\, g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

(where the limit takes place in $L^1(X, \mathfrak{m})$) provides an explicit expression of the *Carré du Champ* $\Gamma : \mathrm{D}(\mathcal{E}) \times \mathrm{D}(\mathcal{E}) \to L^1(X, \mathfrak{m})$ and yields the pointwise upper estimate

$$\Gamma\big(f\big) \le |\mathrm{D}f|^2 \quad \mathfrak{m}\text{-a.e. in } X, \text{ whenever } f \in \mathrm{Lip}(X) \cap L^2(X, \mathfrak{m}), \quad |\mathrm{D}f| \in L^2(X, \mathfrak{m}). \tag{6.3}$$

Eventually, (6.2) ensures that the generated Markov semigroup $(\mathsf{P}_t)_{t \ge 0}$ is mass-preserving, so that (2.1) and the formalism of Section 2 applies to the class of metric measure spaces in $\mathbf{X}$; in particular we can identify $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\mathbb{V}$.

The above discussions justify the following natural definition (equivalent to the $\mathsf{RCD}^*(K, N)$ condition, see the next Section 7).

Definition 6.1 (Metric $\mathsf{BE}(K, N)$ condition for metric measure spaces)**.** *We say that* $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$ *satisfies the* metric $\mathsf{BE}(K, N)$ condition *if the Dirichlet form associated to the Cheeger energy of* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies* $\mathsf{BE}(K, N)$ *according to Definition 2.1 and any*

$$f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \text{ with } \big\|\Gamma\big(f\big)\big\|_{L^\infty} \le 1 \text{ has a 1-Lipschitz representative.} \tag{6.4}$$

It is worth noticing that if $(X, \mathsf{d}, \mathfrak{m})$ satisfies the metric $\mathsf{BE}(K, \infty)$ condition then d coincides with the intrinsic distance $\mathsf{d}_\mathcal{E}$ induced by $\mathcal{E}$ and (X, d) is a length space (recall that (X, d) is a length space if the distance between two arbitrary points in X is the infimum of the length of the absolutely continuous curves connecting them). More precisely, the inequality $\mathsf{d}_\mathcal{E} \le \mathsf{d}$ is a direct consequence of (6.4), while the curvature condition is involved in the proof of the converse inequality.

In this section we see how these concepts can be localized, building suitable cutoff functions with good second order regularity properties and a partition of unity subordinated to an open covering. As an application, we see how the metric $\mathsf{BE}(K, N)$ condition can, to some extent, be globalized (at least in locally compact metric spaces).

The results of this section could be put in a more abstract setup, as we did in §2, assuming the existence of cutoff functions f with $\Gamma(f) \in L^\infty(X, \mathfrak{m})$ separating sets with positive distance. However, since the results we aim to are relative to metric measure spaces, we prefer to state them in this setting, where, as a simple but useful application of (6.3), we can easily construct cutoff functions with bounded weak gradient. To this aim, we consider the distance-functions and the corresponding neighbourhoods

$$\mathsf{d}(x,F) := \inf_{y\in F} \mathsf{d}(x,y), \quad F^{[h]} := \big\{x\in X : \mathsf{d}(x,F)\le h\big\}, \quad \text{for } F\subset X,\ x\in X,\ h\ge 0, \tag{6.5}$$

and we compose them with a function $\eta \in \mathrm{Lip_c}(\mathbb{R})$ with bounded support, so that $\chi := \eta\circ \mathsf{d}(\cdot,F)$ has bounded support; it is immediate to see that

$$\chi = \eta\circ\mathsf{d}(\cdot,F) \text{ belongs to } \mathbb{V}, \text{ with } \quad \Gamma\big(\chi\big) \le \big|\eta'\big(\mathsf{d}(\cdot,F)\big)\big|^2 \quad \mathfrak{m}\text{-a.e. in } X. \tag{6.6}$$

Let us conclude this introductory part with two simple remarks concerning proper metric spaces and and the regularity of the Cheeger energy. Recall that a metric space (X,d) is called *proper* if every closed bounded subset is compact.

Remark 6.2 (Proper metric spaces and the length condition)**.** Every complete and locally compact metric space (X,d) is also proper if it satisfies a length condition (see e.g. [12, Prop. 2.5.22]). Properness immediately yields that this infimum is also attained so that a locally compact length space is proper and geodesic. This characterization is well adapted to our situation, since every m.m.s. $(X,\mathsf{d},\mathfrak{m})$ satisfying the metric $\mathsf{BE}(K,N)$ condition is a length space and it is also locally compact (thus proper and geodesic) if $N<\infty$.

Remark 6.3 (Regularity of the Cheeger energy in proper metric spaces)**.** Recalling Remark 5.1, it is immediate to check that the Cheeger energy is a regular (thus a fortiori quasi-regular) Dirichlet form whenever (X,d) is proper. In fact, it is easy to see that every function with finite energy can be approximated in $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ by functions with bounded support (see e.g. [2, Lemma 4.11]) and those functions are limits in $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ of sequences of Lipschitz functions with bounded support. The same approximation property holds for uniform convergence and any function in $\mathrm{C_c}(X)$ by Arzelà-Ascoli Theorem.

6.2 Localization of metric measure spaces

In connection with localization-globalization of spaces $(X,\mathsf{d},\mathfrak{m})\in\mathbf{X}$, the following properties of the relaxed gradient will be useful (see [3, Theorem 4.19] for the proof).

Proposition 6.4 (Localization of relaxed gradients and $\mathbf{X}$)**.** *For $U\subset X$ open, let us consider the metric measure space $(\bar U,\mathsf{d},\mathfrak{m}\llcorner\bar U)$ and let us denote by $|\mathrm{D}f|_{w,\bar U}$ the minimal relaxed gradient in the new space. Then:*

(a) $f\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ implies $f\in W^{1,2}(\bar U,\mathsf{d},\mathfrak{m}\llcorner\bar U)$ and $|\mathrm{D}f|_w = |\mathrm{D}f|_{w,\bar U}$ $\mathfrak{m}$-a.e. in U. Conversely, if $f\in W^{1,2}(\bar U,\mathsf{d},\mathfrak{m}\llcorner\bar U)$ and $\mathrm{supp}(f)$ has positive distance from $X\setminus U$, then f extended with the 0 value to the whole of X belongs to $f\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$.

(b) If $\mathfrak{m}(\partial U)=0$ then $(\bar U,\mathsf{d},\mathfrak{m}\llcorner\bar U)\in\mathbf{X}$.

For $U\subset X$ open, $W^{1,2}_{\mathrm c}(U,\mathsf{d},\mathfrak{m})$ will denote the subspace of $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ whose functions have compact support in U. We will similarly consider $\mathrm{Lip_c}(U)$. We will occasionally identify a measurable function $f:U\to\mathbb{R}$ with compact support in U with its trivial extension $\tilde f$ to X and viceversa.

We can also introduce the localized versions of Lebesgue and Sobolev spaces on open subsets of X. Even if not explicitly assumed, these notions are interesting when X is locally compact.

Definition 6.5 (Local L^p and Sobolev spaces)**.** *Let $U \subset X$ be open and non-empty and let $f : U \to \mathbb{R}$ be a $\mathfrak{m}$-measurable map. We say that $f \in L^p_{\rm loc}(U, \mathfrak{m})$, $p \in [1, \infty]$, if $f_{|E} \in L^p(E, \mathfrak{m} \llcorner E)$ for every compact subset $E \subset U$. We say that $f \in W^{1,2}_{\rm loc}(U, \mathsf{d}, \mathfrak{m})$ if for every compact set $E \subset U$ there exists $f_E \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that $f = f_E$ $\mathfrak{m}$-a.e. in E. For every $f, g \in W^{1,2}_{\rm loc}(U, \mathsf{d}, \mathfrak{m})$ we can then define $\Gamma\big(f, g\big) \in L^1_{\rm loc}(U, \mathfrak{m})$ by $\Gamma\big(f, g\big)_{|E} := \Gamma\big(f_E, g_E\big)$.*

It is not difficult to check that the above definition is consistent thanks to the locality property of Γ. We can also easily check the equivalent characterization in terms of cutoff function (used for instance in [22]):

$$f \in W^{1,2}_{\rm loc}(U, \mathsf{d}, \mathfrak{m}) \quad \text{iff} \quad \widetilde{f\chi} \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}) \quad \text{for every } \chi \in \mathrm{Lip_c}(U). \tag{6.7}$$

Notice that if $f \in W^{1,2}_{\rm loc}(U, \mathsf{d}, \mathfrak{m})$ and $h \in W^{1,2}_{\rm c}(U, \mathsf{d}, \mathfrak{m})$ we have $\widetilde{\Gamma}(f, h) \in L^1(X, \mathfrak{m})$ with

$$\widetilde{\Gamma}(f, h) = \Gamma\big(\widetilde{f\chi}, h\big) \quad \text{whenever } \chi \in \mathrm{Lip_c}(U),\ \chi \equiv 1 \text{ on } \operatorname{supp}(h). \tag{6.8}$$

Let us now consider the localization property of the Laplace operator.

Lemma 6.6 (Global to local for the Laplacian of compactly supported functions)**.** *Let $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$ and, given an open set $U \subset X$, assume that $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ has support with positive distance from $X \setminus U$. Then $\Delta f \in L^2(X, \mathfrak{m})$ if and only if $\Delta_{\bar{U}} f \in L^2(\bar{U}, \mathfrak{m} \llcorner \bar{U})$. In addition $\Delta_{\bar{U}} f = \Delta f$ $\mathfrak{m}$-a.e. in $\bar{U}$ and $\operatorname{supp}(\Delta f) \subset \operatorname{supp}(f)$.*

Proof. Let us assume that $\Delta f \in L^2(X, \mathfrak{m})$. First of all, choosing Lipschitz functions g in (2.2) with compact support in $G := X \setminus \operatorname{supp}(f)$ (these functions are dense in $L^2(G, \mathfrak{m} \llcorner G)$ by a simple truncation argument) we see that $\Delta f = 0$ $\mathfrak{m}$-a.e. in $X \setminus \operatorname{supp}(f)$, i.e. $\operatorname{supp}(\Delta f) \subset \operatorname{supp}(f)$. Now, for every $\psi \in W^{1,2}(\bar{U}, \mathsf{d}, \mathfrak{m} \llcorner \bar{U})$ we can apply Proposition 6.4 and a multiplication by a cutoff function as in (6.6) with $F = \operatorname{supp}(f)$ to find another function $\tilde{\psi} \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ coinciding with ψ in a neighbourhood of $\operatorname{supp}(f)$; thanks to the locality of Γ we have then

$$\int_{\bar{U}} \Gamma(\psi, f)\, \mathrm{d}\mathfrak{m} = \int_X \Gamma(\tilde{\psi}, f)\, \mathrm{d}\mathfrak{m} = -\int_X \tilde{\psi}\, \Delta f\, \mathrm{d}\mathfrak{m} = -\int_{\bar{U}} \psi\, \Delta f\, \mathrm{d}\mathfrak{m}.$$

The proof of the converse implication is similar. □

Lemma 6.7 (Construction of smoother cutoff functions)**.** *Let $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$ and let $U \subset X$ be an open subset such that $(\bar{U}, \mathsf{d}, \mathfrak{m} \llcorner \bar{U}) \in \mathbf{X}$ satisfies the metric $\mathsf{BE}(K, \infty)$ condition.*

Then, for all $E \subset U$ compact and $G \subset X$ open and relatively compact with $E \subset G$ and $\bar{G} \subset U$ there exists a Lipschitz function $\hat{\chi} : X \to \mathbb{R}$ satisfying:

(i) $0 \le \hat{\chi} \le 1$, $\hat{\chi} \equiv 1$ on a neighbourhood $E^{[h]}$ of E for some $h > 0$, and $\operatorname{supp}(\hat{\chi}) \subset G$;

(ii) $\Delta \hat{\chi} \in L^\infty(X, \mathfrak{m})$ and $\Gamma\big(\hat{\chi}\big) \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

Proof. Notice that the point is to construct a cut off function with L^∞ Laplacian, indeed the existence of a Lipschitz cut off function $f : X \to [0, 1]$ satisfying (i) is trivial by (6.6), since $\varepsilon := \inf_{x \notin G} \mathsf{d}(x, E) > 0$; we can thus suppose that $f \equiv 1$ on $E^{[\varepsilon/3]}$ and $\operatorname{supp}(f) \subset E^{[2\varepsilon/3]}$.

At first we regularize f via the mollified heat flow $\mathfrak{H}^t_U$ of the m.m.s. $(\bar{U}, \mathsf{d}, \mathfrak{m} \llcorner \bar{U})$ defined as in (4.6) by setting

$$f_t := \mathfrak{H}^t_U f.$$

Since by assumption $(\bar{U}, \mathsf{d}, \mathfrak{m} \llcorner \bar{U})$ satisfies the metric $\mathsf{BE}(K, \infty)$ condition, by the pointwise gradient estimate (2.9) together with the maximum principle and (6.4), we know that $\{f_t\}_{t\in[0,\delta]}$ are uniformly Lipschitz on $\bar{U}$ for any $\delta > 0$, and moreover $\Delta f_t \in L^\infty(\bar{U}, \mathfrak{m} \llcorner \bar{U})$. Combining Arzelà-Ascoli theorem and the fact that $f_t \to f$ in $L^2(\bar{U}, \mathfrak{m})$ as $t \to 0$ it follows that $f_t \to f$ uniformly on $\bar{G}$. Therefore, recalling also the maximum principle for the heat flow, for $\delta > 0$ small enough we have

$$\frac{3}{4} \le f_\delta \le 1 \text{ on } E^{[\varepsilon/3]} \quad \text{ and } \quad 0 \le f_\delta \le \frac{1}{4} \text{ on } \bar{G} \setminus E^{[2\varepsilon/3]}. \tag{6.9}$$

Now let $\eta \in C^2([0,1],[0,1])$ be such that $\eta([0,1/4]) = \{0\}$ and $\eta([3/4,1]) = \{1\}$. It is immediate to check, using Lemma 6.6, Proposition 6.4 and Theorem 5.5 (applied to the m.m.s. $(\bar{U}, \mathsf{d}, \mathfrak{m} \llcorner \bar{U})$) that the trivial extension of $\hat{\chi} := \eta \circ f_\delta$ outside $\bar{G}$ has the desired properties with $h := \varepsilon/3$. □

Remark 6.8. Notice that whenever $E \subset U$, with E compact and U open and locally compact, we can always find an open and relatively compact neighbourhood G of E as in the above Lemma: since for every $x \in E$ there exists a relatively compact open ball B_x with $\overline{B_x} \subset U$, it is sufficient to set $G := \cup_{x\in E_0} B_x$ where $E_0 \subset E$ is a finite set such that $(B_x)_{x\in E_0}$ is an open cover of E.

The lemma above easily provides the following proposition, stating the existence of a regular partition of unity.

Proposition 6.9 (Partition of unity)**.** *Let $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$, $E \subset X$ compact and let $U = \cup_{i\in I} U_i$ where $\{U_i\}_{i\in I}$ is a covering of E by non-empty locally compact open subsets such that the m.m.s. $(\bar{U}_i, \mathsf{d}, \mathfrak{m} \llcorner \bar{U}_i) \in \mathbf{X}$ satisfy the metric $\mathsf{BE}(K, \infty)$ condition. Then there exist Lipschitz functions $\chi_i : X \to [0, \infty)$, null for all but finitely many i and satisfying:*

(i) $\mathrm{supp}(\chi_i) \subset U_i$ *is compact and* $\sum_i \chi_i \equiv 1$ *on a neighbourhood of* E*;*

(ii) $\Delta \chi_i \in L^\infty(X, \mathfrak{m})$ *and* $\Gamma(\chi_i) \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

In particular $\psi := \sum_i \chi_i$ *satisfies* $\psi \equiv 1$ *on* E *and*

$$\psi \in \mathrm{Lip_c}(U), \quad 0 \le \psi \le 1, \quad \Delta\psi \in L^\infty(X, \mathfrak{m}), \quad \Gamma(\psi) \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}). \tag{6.10}$$

Proof. By the compactness of E we can assume with no loss of generality that I is finite. Since the continuous function

$$E \ni x \mapsto \max_{i\in I} \mathsf{d}(x, X \setminus U_i)$$

has a minimum $a > 0$ in E, considering the sets

$$E_i := \{x \in E : \ \mathrm{dist}(x, X \setminus U_i) \ge a/2\}, \qquad G_i := \{x \in X : \ \mathrm{dist}(x, E_i) < b\}$$

for $b > 0$ sufficiently small, we provide compact sets $E_i \subset G_i$ and open relatively compact (by Remark 6.8) sets $G_i \subset X$ with $E_i \subset G_i$ and $\bar{G}_i \subset U_i$, in such a way that $\cup_{i\in I} E_i = E$ and the neighbourhood $E^{[h]}$ of E is contained in $\cup_i G_i$ for $h < b$.

With this choice of E_i and G_i, if we consider the cutoff functions $\hat\chi_i$ provided by Lemma 6.7, we clearly have $\sum_i \hat\chi_i \geq 1$ on $E^{[h]}$ for some positive $h < b$ sufficiently small. By Leibniz and chain rule, it is therefore clear that the functions

$$\chi_i := \frac{\hat\chi_i}{\eta(\sum_i \hat\chi_i)} = 2\hat\chi_i + \big(\frac{1}{\eta(\sum_i \hat\chi_i)} - 2\big)\hat\chi_i$$

satisfy (i) and (ii) above, provided that we choose a smooth nondecreasing function $\eta(s)$ identically equal to s on $[1,\infty)$ and identically equal to $1/2$ on $[0,1/2]$.

In order to prove the regularity for $\Gamma\big(\psi\big)$ of (6.10) it is sufficient to recall Proposition 6.4 and

$$\Gamma\big(\sum_i \chi_i\big) = \sum_{i,j} \Gamma\big(\chi_i,\chi_j\big), \qquad \Gamma\big(\chi_i,\chi_j\big) = \frac14\Gamma\big(\chi_i+\chi_j\big) - \frac14\Gamma\big(\chi_i-\chi_j\big). \qquad \square \tag{6.11}$$

Remark 6.10. Let $U = \cup_{i\in I} U_i$ as in the previous Proposition 6.9. Then a measurable function $f : U \to \mathbb{R}$ belongs to $W^{1,2}_{\rm loc}(U,\mathsf{d},\mathfrak{m})$ if and only if $\widetilde{f\psi} \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ for every function ψ as in (6.10). Thus we can use better cutoff functions in (6.7).

We can now use the partitions of unity to prove a local higher integrability and regularity of Γ.

Lemma 6.11 (Improved local integrability and regularity of $\Gamma(f)$)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a locally compact m.m.s. in $\mathbf{X}$ and let $X = \cup_{i\in I} U_i$ where $\{U_i\}_{i\in I}$ are non-empty open subsets such that $(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i) \in \mathbf{X}$ satisfy the metric $\mathsf{BE}(K,\infty)$ condition for all $i \in I$.*

Then for every $f \in \mathrm{D}_{L^4}(\Delta)\cap L^\infty(X,\mathfrak{m})$ one has $\Gamma\big(f\psi\big) \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ and $f\psi \in D_{L^4}(\Delta)$ for every cutoff function ψ satisfying (6.10). In particular, $\Gamma(f) \in W^{1,2}_{\rm loc}(X,\mathsf{d},\mathfrak{m})$.

Proof. Let us consider the compact set $E := \mathrm{supp}(\psi)$ and let $\{\chi_i\}$, $I = \{1,\dots,n\}$, be the partition of unity, subordinated to E and to the open covering $\{U_i\}$, constructed in Proposition 6.9; let $\hat\chi_i$ be a cutoff function provided by Lemma 6.7 corresponding to the compact set $\mathrm{supp}(\chi_i)$, the open set U_i and G_i as in Remark 6.8. We define

$$\hat f_i := \hat\chi_i f, \quad f_i := \chi_i f, \quad \text{so that} \quad f_i = \chi_i \hat f_i. \tag{6.12}$$

Recalling Lemma 6.6 and the Leibniz formula (1.5), it is easy to check that $\hat f_i \in L^\infty \cap W^{1,2}(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)$ and that $\Delta\hat f_i \in L^2(\bar U_i,\mathfrak{m}\llcorner\bar U_i)$. Since by assumption the m.m.s. $(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)$ satisfies the $\mathsf{BE}(K,\infty)$ condition, by the gradient interpolation Theorem 3.1 with $p = 2$ we obtain that $\Gamma(\hat f_i) \in L^2(\bar U_i,\mathfrak{m}\llcorner\bar U_i)$.

Still applying (1.5), now with $f := \hat f_i$ and $\chi := \chi_i$, we obtain that $f_i \in W^{1,2}(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)\cap L^\infty(\bar U_i,\mathfrak{m})$ with $\Delta_{\bar U_i} f_i \in L^4(\bar U_i,\mathfrak{m}\llcorner\bar U_i)$: Theorem 5.5 provides $\Gamma\big(f_i\big) \in W^{1,2}(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)$ and since $\Gamma\big(f_i\big)$ has compact support in U_i we conclude that (the trivial extension of) $\Gamma\big(f_i\big)$ belongs to $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ and that $\Delta f_i \in L^4(X,\mathfrak{m})$. More generally, since χ_j is globally Lipschitz and with bounded Laplacian in X, we obtain that $f_i\chi_j \in \mathrm{D}_{L^4}(\Delta)$ for all i, $j = 1,\dots,n$. Since both $f_i\chi_j$ and $f_j\chi_i$ have compact support in U_i we obtain $\Delta_{\bar U_i}(f_i\chi_j \pm f_j\chi_i) \in L^4(\bar U_i,\mathfrak{m}\llcorner\bar U_i)$. By applying Theorem 5.5 in U_i we get $\Gamma\big(f_i\chi_j \pm f_j\chi_i\big) \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, so that by polarization

$$\Gamma\big(f_i,f_j\big) = \Gamma\big(f_i\chi_j, f_j\chi_i\big) \in W^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

The bilinearity of Γ yields

$$\Gamma\big(\sum_i f_i\big) = \Gamma\big(\sum_i f_i, \sum_j f_j\big) = \sum_{ij} \Gamma\big(f_i, f_j\big) \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

Using that $\sum_i \chi_i \equiv 1$ on E and the identity $f\psi = \sum_i (f\psi)\chi_i = \psi \sum_i f_i$ we conclude that $\Gamma\big(f\psi\big) \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and $\Delta(f\psi) \in L^4(X, \mathfrak{m})$.

Finally, $\Gamma\big(f\big) \in W^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ follows by Remark 6.10. □

Theorem 6.12. *Under the same assumptions of the previous Lemma 6.11, every function* $f \in \mathrm{D}_{L^4}(\Delta) \cap L^\infty(X, \mathfrak{m})$ *satisfies* (5.11) *for every nonnegative* $\varphi \in W^{1,2}_{\mathrm{c}}(X, \mathsf{d}, \mathfrak{m})$.

If, moreover, (X, d) *is a proper metric space then there exists a nonnegative Radon measure* $\Gamma^\star_{2,K}[f]$ *vanishing on* $\mathcal{E}$*-polar sets and representing the linear functional*

$$W^{1,2}_{\mathrm{c}}(X, \mathsf{d}, \mathfrak{m}) \ni \varphi \mapsto \int_X -\frac{1}{2}\Gamma\big(\Gamma\big(f\big), \varphi\big) + \Delta f\, \Gamma\big(f, \varphi\big) + \big((\Delta f)^2 - K\Gamma\big(f\big)\big)\varphi \,\mathrm{d}\mathfrak{m}. \tag{6.13}$$

Finally, the density $\gamma_{2,K}[f]$ *of the measure* $\Gamma^\star_{2,K}[f]$ *still satisfies* (5.8).

Notice that both (5.11) and (5.12) make sense under the above assumptions thanks to Lemma 6.11.

Proof. Let $f \in \mathrm{D}_{L^4}(\Delta) \cap L^\infty(X, \mathfrak{m})$ and $\varphi \in W^{1,2}_{\mathrm{c}}(X, \mathsf{d}, \mathfrak{m})$ be fixed and let us prove (5.11).

Let $E = \mathrm{supp}(\varphi)$ and let $\{\chi_i\}$ be the cutoff functions constructed in Proposition 6.9, subordinated to the open covering $\{U_i\}$, whose sum is identically 1 in a neighbourhood of E, null for all but finitely many i; since χ_i have support contained in U_i, for all i such that χ_i is not null we apply Lemma 6.7 to obtain Lipschitz function $\hat\chi_i$ with compact support in U_i, bounded Laplacian, identically equal to 1 on a neighbourhood of $\mathrm{supp}(\chi_i)$. We can also find $\psi \in \mathrm{Lip}_{\mathrm{c}}(U)$ satisfying (6.10) and $\psi \equiv 1$ on $\mathrm{supp}(\sum_i \hat\chi_i)$.

It is easy to check, using Lemma 6.11 and Lemma 6.6, that the functions $\varphi_i := \chi_i\varphi$ and $f_i := \hat\chi_i f$ satisfy the assumptions of Theorem 5.5 in the metrically $\mathsf{BE}(K, N)$ m.m.s. $(\bar U_i, \mathsf{d}, \mathfrak{m}\llcorner\bar U_i)$. We thus get

$$\int_{\bar U_i} \Big(-\frac{1}{2}\Gamma\big(\Gamma\big(f_i\big), \varphi_i\big) + \Delta f_i\, \Gamma(f_i, \varphi_i) + \varphi_i(\Delta f_i)^2\Big) \,\mathrm{d}\mathfrak{m} \geq K \int_{\bar U_i} \Gamma(f_i)\, \varphi_i \,\mathrm{d}\mathfrak{m} + \nu \int_{\bar U_i} (\Delta f_i)^2 \varphi_i \,\mathrm{d}\mathfrak{m}. \tag{6.14}$$

Since Lemma 6.11 gives that $\Gamma(f) \in W^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, recalling also that $\sum_i \varphi_i \equiv \varphi$ and

$\hat\chi_i \equiv 1$ on $\mathrm{supp}(\chi_i)$, we can write

$$\begin{aligned}
&\int_X \Big(-\frac{1}{2}\Gamma\big(\Gamma(f),\varphi\big) + \Delta f\, \Gamma\big(f,\varphi\big) + \varphi(\Delta f)^2\Big)\,\mathrm{d}\mathfrak{m}\\
&= \int_X \bigg(-\frac{1}{2}\Gamma\big(\Gamma(f)\,\psi,\sum_i \varphi_i\big) + \Delta f\, \Gamma\big(f,\sum_i \varphi_i\big) + \big(\sum_i \varphi_i\big)(\Delta f)^2\bigg)\,\mathrm{d}\mathfrak{m}\\
&= \sum_i \int_{\bar U_i} \Big(-\frac{1}{2}\Gamma\big(\Gamma(f)\,\psi,\varphi_i\big) + \Delta f\, \Gamma\left(f,\varphi_i\right) + \varphi_i(\Delta f)^2\Big)\,\mathrm{d}\mathfrak{m}\\
&= \sum_i \int_{\bar U_i} \Big(-\frac{1}{2}\Gamma\big(\Gamma(f_i),\varphi_i\big) + \Delta f_i\, \Gamma\left(f_i,\varphi_i\right) + \varphi_i(\Delta f_i)^2\Big)\,\mathrm{d}\mathfrak{m}\\
&\ge \sum_i K\int_{\bar U_i} \Gamma(f_i)\,\varphi_i\,\mathrm{d}\mathfrak{m} + \nu \int_{\bar U_i} (\Delta f_i)^2\varphi_i\,\mathrm{d}\mathfrak{m} \quad \text{by (6.14)}\\
&= \sum_i K\int_{\bar U_i} \Gamma(f)\,\varphi_i\,\mathrm{d}\mathfrak{m} + \nu \int_{\bar U_i} (\Delta f)^2\varphi_i\,\mathrm{d}\mathfrak{m}\\
&= K\int_X \Gamma(f)\,\varphi\,\mathrm{d}\mathfrak{m} + \nu \int_X (\Delta f)^2\varphi\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

In order to prove the second part of the statement, in the case when (X,d) is proper (recall Remark 6.2), let us call ℓ_f the linear functional defined by (6.13), let us fix $x_0 \in X$ with the collection of the open balls $B_R := \{x \in X : \mathsf{d}(x,x_0) < R\}$ and let us consider the Hilbert space $\mathbb{V}_R$ obtained by taking the closure in $\mathbb{V}$ of the set $\{\varphi \in W^{1,2}(X,\mathsf{d},\mathfrak{m}) : \mathrm{supp}\,\varphi \subset B_R\}$. It is easy to check that the restriction of $\mathcal{E}$ to $\mathbb{V}_R$ is a regular Dirichlet form (recall Remark 6.3) on $L^2(B_R,\mathfrak{m}\llcorner B_R)$ and the restriction of ℓ_f to $\mathbb{V}_R$ is a nonnegative continuous functional, which also satisfies (3) of Proposition 5.2: in fact, if $\varphi \in \mathbb{V}_R$ with $0 \le \varphi \le 1$ $\mathfrak{m}$-a.e., by taking a cutoff function ψ as in (6.10) with $U \supset \overline{B}_R$ and $\psi \equiv 1$ on $\overline{B}_R$, the inequality $0 \le \psi - \varphi \in W^{1,2}_{\mathrm{c}}(X,\mathsf{d},\mathfrak{m})$ and (5.11) yield

$$\langle \ell_f,\varphi\rangle \le \int_X -\frac{1}{2}\Gamma\big(\Gamma(f),\psi\big) + \Delta f\, \Gamma\big(f,\psi\big) + \big((\Delta f)^2 - K\Gamma(f)\big)\psi\,\mathrm{d}\mathfrak{m}.$$

Thus, the action of ℓ_f on $\mathbb{V}_R$ can be represented by a finite nonnegative Borel measure μ_R on B_R not charging $\mathcal{E}$-polar subsets of B_R. It is easy to check that $S < R$ yields $(\mu_R)\llcorner B_S = \mu_S$, so that we can eventually find a nonnegative Radon measure $\Gamma^\star_{2,K}[f]$ as stated in the Theorem.

Let us now take an arbitrary compact set $E \subset U_i$ and a Lipschitz function χ_i with compact support in U_i, bounded Laplacian and identically 1 on a neighbourhood of $\mathrm{supp}(\varphi_i)$. The function $\hat f_i = \chi_i f$ belongs to $W^{1,2}(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)$ with $f_i, \Delta_{U_i} f_i \in L^4(\bar U_i,\mathfrak{m}\llcorner\bar U_i)$ so that applying Corollary 5.7 we get a finite nonnegative Borel measure $\mu_i = \Gamma^\star_{2,K}[f_i]$ (relative to U_i) with Lebesgue density γ_i satisfying

$$\langle \ell_{f_i},\psi\rangle = \int_{\bar U_i} \psi\,\mathrm{d}\mu_i \quad \text{for every} \quad \psi \in W^{1,2}_{\mathrm{c}}(U_i,\mathsf{d},\mathfrak{m}), \quad \Gamma\big(\Gamma(f_i)\big) \le 4\gamma_i\Gamma\big(f_i\big) \quad \mathfrak{m}\text{-a.e. on } U_i. \tag{6.15}$$

For every function $\varphi \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ with support in E we then have

$$\int_E \varphi\,\mathrm{d}\Gamma^\star_{2,K}[f] = \langle \ell_f,\varphi\rangle = \langle \ell_{f_i},\varphi\rangle = \int_E \varphi\,\mathrm{d}\mu_i$$

so that $\Gamma^\star_{2,K}[f]$ coincide with μ_i on E. It follows that its Lebesgue density γ coincides with γ_i and (6.15) yields

$$\Gamma\big(\Gamma(f)\big) = \Gamma\big(\Gamma(f_i)\big) \le 4\gamma_i\,\Gamma(f_i) = 4\gamma\,\Gamma(f) \quad \mathfrak{m}\text{-a.e. on } E.$$

Since we can cover X with a sequence of compact sets contained in some set U_i, we conclude. □

Theorem 6.13. *Let $(X,\mathsf{d},\mathfrak{m}) \in \mathbf{X}$ be a proper m.m.s. and let $X = \cup_{i\in I} U_i$ where $\{U_i\}_{i\in I}$ are non-empty open sets such that $(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner \bar U_i) \in \mathbf{X}$ satisfy the metric $\mathsf{BE}(K,\infty)$ condition for all $i \in I$. For every $f \in \mathrm{D}_{L^4}(\Delta)\cap L^\infty(X,\mathfrak{m})$ the following properties hold:*

(a) *the measure $\Gamma^\star_{2,K}[f]$ is finite and satisfies* (5.5)*;*

(b) *$\Gamma(f) \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ and satisfies* (5.12) *for any nonnegative φ with $\Gamma(\varphi) \in L^\infty(X,\mathfrak{m})$.*

Proof. Let us fix $x_0 \in X$ and let us consider the Lipschitz cutoff function

$$\varphi_n(x) = \begin{cases} 1 & \text{on } B_{2^n}(x_0),\\ 0 & \text{on } X \setminus B_{2^{n+1}}(x_0),\\ 2 - \mathsf{d}(x,x_0)2^{-n} & \text{on } B_{2^{n+1}}(x_0)\setminus B_{2^n}(x_0), \end{cases} \tag{6.16}$$

which satisfies $|\mathrm{D}\varphi_n| \le 2^{-n}$. By Theorem 6.12, if $f \in \mathrm{D}_{L^4}(\Delta)\cap L^\infty(X,\mathfrak{m})$ and γ is the Lebesgue density of $\Gamma^\star_{2,K}[f]$, we get

$$\begin{aligned} -\frac12\int_X \Gamma\big(\Gamma(f),\varphi_n^2\big)\,\mathrm{d}\mathfrak{m} &= -\int_X \Gamma\big(\Gamma(f),\varphi_n\big)\,\varphi_n\,\mathrm{d}\mathfrak{m} \le \int_X \varphi_n\sqrt{\Gamma\big(\Gamma(f)\big)}\sqrt{\Gamma(\varphi_n)}\,\mathrm{d}\mathfrak{m}\\ &\le 2\int_X \varphi_n\sqrt{\gamma\Gamma(f)\Gamma(\varphi_n)}\,\mathrm{d}\mathfrak{m} \le \varepsilon\int_X \gamma\varphi_n^2\,\mathrm{d}\mathfrak{m} + \frac{1}{4^n\varepsilon}\int_X \Gamma(f)\,\mathrm{d}\mathfrak{m},\\ \int_X \Delta f\,\Gamma\big(f,\varphi_n^2\big)\,\mathrm{d}\mathfrak{m} &\le 2^{-n}\|\Delta f\|_{L^2}\sqrt{\mathcal{E}(f)}. \end{aligned}$$

In addition, the integrability of $\Gamma(f)$ gives

$$\limsup_{n\to\infty}\int_X \Big((\Delta f)^2 - K\Gamma(f)\Big)\varphi_n^2\,\mathrm{d}\mathfrak{m} \le \int_X \Big((\Delta f)^2 - K\Gamma(f)\Big)\,\mathrm{d}\mathfrak{m}.$$

Applying the definition of $\Gamma^\star_{2,K}[f]$ we get

$$(1-\varepsilon)\int_X \varphi_n^2\,\mathrm{d}\Gamma^\star_{2,K}[f] \le \frac{1}{4^n\varepsilon}\mathcal{E}(f) + 2^{-n}\|\Delta f\|_{L^2}\sqrt{\mathcal{E}(f)} + \int_X \Big((\Delta f)^2 - K\Gamma(f)\Big)\varphi_n^2\,\mathrm{d}\mathfrak{m}.$$

Passing to the limit first as $n\to\infty$ and then as $\varepsilon \downarrow 0$ we get the bound of statement (a).

Concerning (b), let us first remark that, setting $g := \Gamma(f) \in W^{1,2}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ and $g_k := \min\{g,k\}$, by Theorem 6.12 we have

$$\Gamma(g_k) \le 4\gamma\,g_k\,\chi_k, \quad \text{where} \quad \chi_k = \begin{cases} 1 & \text{in the set } \{g \le k\},\\ 0 & \text{in the set } \{g > k\} \end{cases}$$

so that $g_k \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$.

Integrating now the nonnegative function $g_k\varphi_n^2 \in W^{1,2}_{\mathrm{c}}(X,\mathsf{d},\mathfrak{m})$ with respect to $\Gamma^\star_{2,K}[f]$ and observing that $\Gamma\big(g,g_k\big)=\Gamma\big(g_k\big)\chi_k=\Gamma\big(g_k\big)$, from (5.12) of Theorem 5.5 we get

$$\int_X \Big(K_\lambda\, g\, g_k+\Gamma\big(g_k\big)\Big)\varphi_n^2\,\mathrm{d}\mathfrak{m} \le 2\int_X \Big(\Delta_\lambda f\,\Gamma\big(f,g_k\big)+\Delta f\Delta_\lambda f g_k\Big)\varphi_n^2\,\mathrm{d}\mathfrak{m}$$
$$+\,2\int_X\Big(-\,g_k\Gamma\big(g,\varphi_n\big)+2\Delta_\lambda f\, g_k\Gamma\big(f,\varphi_n\big)\Big)\varphi_n\,\mathrm{d}\mathfrak{m},$$

where $K_\lambda=2K+2\lambda$ and we choose λ in such a way that $K_\lambda\ge 1$. We can now estimate from above the integrals on the right hand side:

$$I=\int_X \Delta_\lambda f\,\Gamma\big(f,g_k\big)\varphi_n^2\,\mathrm{d}\mathfrak{m}\le \|\Delta_\lambda f\|_{L^4}\,\|g_k^{1/2}\varphi_n\|_{L^4}\,\|\Gamma\big(g_k\big)^{1/2}\varphi_n\|_{L^2}$$
$$\le 2\|\Delta_\lambda f\|^4_{L^4}+\frac{1}{8}\int_X g_k^2\,\varphi_n^2\,\mathrm{d}\mathfrak{m}+\frac{1}{4}\int_X\Gamma\big(g_k\big)\varphi_n^2\,\mathrm{d}\mathfrak{m},$$

since $|\Gamma\big(f,g_k\big)|\le\sqrt{g\,\Gamma\big(g_k\big)}\chi_k=\sqrt{g_k\,\Gamma\big(g_k\big)}\chi_k$ and $|\varphi_n|\le 1$;

$$II=\int_X\Delta f\,\Delta_\lambda f\,g_k\,\varphi_n^2\,\mathrm{d}\mathfrak{m}\le\|\Delta f\|_{L^4}\|\Delta_\lambda f\|_{L^4}\,\|g_k\varphi_n^2\|_{L^2}$$
$$\le 8\|\Delta_\lambda f\|^4_{L^4}+\frac{1}{8}\int_X g_k^2\,\varphi_n^2\,\mathrm{d}\mathfrak{m}\qquad\text{by (2.4) and } |\varphi_n|\le 1,$$
$$III=-\int_X g_k\Gamma\big(g,\varphi_n\big)\varphi_n\,\mathrm{d}\mathfrak{m}\le\frac{k}{2^n}\int_X\sqrt{\Gamma\big(g\big)}\,\mathrm{d}\mathfrak{m}\le\frac{k}{2^{n-1}}\Big(\Gamma^\star_{2,K}[f](X)\cdot\mathcal{E}(f)\Big)^{1/2}$$

where we used (5.8) and the finiteness of $\Gamma^\star_{2,K}[f]$,

$$IV=2\int_X\Delta_\lambda f\,g_k\Gamma\big(f,\varphi_n\big)\varphi_n\,\mathrm{d}\mathfrak{m}\le\frac{k}{2^{n-1}}\|\Delta_\lambda f\|_{L^2}\,\mathcal{E}(f)^{1/2}.$$

Since $g_k^2\le g\,g_k$, summing the contribution of the four terms and using (5.10), we get

$$\frac{1}{2}\int_X\Big(K_\lambda\,g\,g_k+\Gamma\big(g_k\big)\Big)\varphi_n^2\,\mathrm{d}\mathfrak{m}\le 20\|\Delta_\lambda f\|^2_{L^4}+\frac{k}{2^{n-2}}\mathcal{E}(f)^{1/2}\Big(\|\Delta_\lambda f\|_{L^2}+\Gamma^\star_{2,K}[f](X)^{1/2}\Big).$$

Passing first to the limit as $n\to\infty$ we obtain

$$\int_X\Big(K_\lambda\,g\,g_k+\Gamma\big(g_k\big)\Big)\,\mathrm{d}\mathfrak{m}\le 40\|\Delta_\lambda f\|^2_{L^4}.$$

We then pass to the limit as $k\to\infty$ and we obtain $g\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$. Finally, (5.12) can be obtained as in the proof of Theorem 5.5. □

Theorem 6.14. *Let $(X,\mathsf{d},\mathfrak{m})\in\mathbf{X}$, with (X,d) length and locally compact. Assume that there exists a covering $\{U_i\}_{i\in I}$ of X by non-empty open sets U_i such that $(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)\in\mathbf{X}$ satisfy the metric $\mathsf{BE}(K,N)$ condition.*

Then also $(X,\mathsf{d},\mathfrak{m})$ satisfies the metric $\mathsf{BE}(K,N)$ condition.

Proof. By applying Lemma 4.2, for every $f \in \mathrm{D}_{\mathbb{V}}(\Delta)$ we can find $f_k \in \mathrm{D}_{\mathbb{V}}(\Delta) \cap \mathrm{D}_{L^\infty}(\Delta) \subset \mathrm{D}_{L^4}(\Delta) \cap L^\infty(X, \mathfrak{m})$ with $f_k \to f$ and $\Delta f_k \to \Delta f$ in $\mathbb{V}$ as $k \to \infty$. If $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$ is nonnegative, an integration by parts and Theorem 6.13 give

$$\mathbf{\Gamma}_2(f_k; \varphi) = \int_X \Big(-\frac{1}{2}\Gamma\big(\Gamma\big(f_k\big), \varphi\big) + \Delta f_k \Gamma\big(f_k, \varphi\big) + (\Delta f_k)^2 \varphi \Big) \,\mathrm{d}\mathfrak{m}.$$

Therefore, if φ_n is defined by (6.16), we can apply Theorem 6.12 to get

$$\begin{aligned} \mathbf{\Gamma}_2(f_k; \varphi) &= \lim_{n\to\infty} \int_X \Big(-\frac{1}{2}\Gamma\big(\Gamma\big(f_k\big), \varphi\varphi_n\big) + \Delta f_k \Gamma\big(f_k, \varphi\varphi_n\big) + (\Delta f_k)^2 \varphi\varphi_n \Big) \,\mathrm{d}\mathfrak{m} \\ &\overset{(5.11)}{\geq} \lim_{n\to\infty} \int_X \Big(K\Gamma\big(f_k\big) + \nu(\Delta f_k)^2 \Big) \varphi\varphi_n \,\mathrm{d}\mathfrak{m} = \int_X \Big(K\Gamma\big(f_k\big) + \nu(\Delta f_k)^2 \Big) \varphi \,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{6.17}$$

We can use the convergence of f_k and Δf_k in $\mathbb{V}$ to obtain that $\mathbf{\Gamma}_2(f_k; \varphi)$ converges to $\mathbf{\Gamma}_2(f; \varphi)$. Therefore, passing to the limit as $k \to \infty$ in (6.17) we obtain the $\mathsf{BE}(K, N)$ condition.

In order to conclude, it suffices to show that any essentially bounded $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\||\mathrm{D}f|_w\|_{L^\infty(X,\mathfrak{m})} \leq 1$ has a 1-Lipschitz representative. Clearly, the fact that $(\bar{U}_i, \mathsf{d}, \mathfrak{m}\llcorner\bar{U}_i)$ satisfy the metric $\mathsf{BE}(K, \infty)$ condition implies that f has a 1-Lipschitz representative on $\bar{U}_i$, and therefore f has a locally 1-Lipschitz representative $\tilde{f}$ on $\cup_i U_i$. If we consider an absolutely continuous curve γ connecting x to y, this easily yields (by a covering argument)

$$|\tilde{f}(x) - \tilde{f}(y)| \leq \mathrm{length}(\gamma).$$

Since (X, d) is a length space (in fact geodesic), we conclude. □

7 $\mathsf{RCD}^*(K, N)$ spaces and their localization and globalization

In the next two subsections we introduce the $\mathsf{RCD}(K, \infty)$ and $\mathsf{RCD}^*(K, N)$ spaces, and discuss their equivalent characterizations as well as their localization and globalization properties; the case $N = \infty$ is by now well established [4], while the dimensional case is more recent [18], [5].

7.1 The case $N = \infty$

We say that $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$ is a $\mathsf{RCD}(K, \infty)$ space if the Shannon entropy $\mathcal{U}_\infty : \mathscr{P}_2(X) \to (-\infty, +\infty]$

$$\mathcal{U}_\infty(\mu) := \begin{cases} \int_X \rho \log \rho \,\mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m}; \\ \\ +\infty & \text{otherwise} \end{cases} \tag{7.1}$$

is convex along Wasserstein geodesics. More precisely, here $\mathscr{P}_2(X)$ stands for the space of Borel probability measures with finite quadratic moments and condition (6.2) guarantees that the negative part of $\rho \log \rho$ is integrable for any $\mu = \rho\mathfrak{m} \in \mathscr{P}_2(X)$, see [3] for details. Hence, (7.1) makes sense.

If we endow $\mathscr{P}_2(X)$ with the quadratic Wasserstein distance W_2, we say that $(X, \mathsf{d}, \mathfrak{m}) \in \mathbf{X}$ is a $\mathsf{RCD}(K, \infty)$ space if for all $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m}$ in $\mathscr{P}_2(X)$ and for every constant speed geodesic μ_t in $\mathscr{P}_2(X)$ from μ_0 to μ_1, for all $t \in [0, 1]$ there holds $\mu_t = \rho_t\mathfrak{m}$ and

$$\int_X \rho_t \log \rho_t \,\mathrm{d}\mathfrak{m} \leq (1-t) \int_X \rho_0 \log \rho_0 \,\mathrm{d}\mathfrak{m} + t \int_X \rho_1 \log \rho_1 \,\mathrm{d}\mathfrak{m} - \frac{K}{2} t(1-t) W_2^2(\mu_0, \mu_1). \tag{7.2}$$

This class of spaces has been introduced in [3], where one of the main results is also the equivalence with another entropic formulation, based on the so-called EVI_K property of the Shannon entropy along the heat flow. The definition adopted here has been later on improved in [1] (asking the convexity inequality along *some* geodesic, and then recovering convexity along *all* geodesics out of the EVI_K theorem [17]), see also [30] for new recent developments. While this characterization is extremely useful in the proof of stability properties [3, 4, 23], in the proof of localization or globalization properties it suffers the same limitations described in Remark 2.2. It is instead crucial for us the following connection between $\mathsf{RCD}(K,\infty)$ and $\mathsf{BE}(K,\infty)$, obtained in [4].

Theorem 7.1 (Equivalence of $\mathsf{RCD}(K,\infty)$ and $\mathsf{BE}(K,\infty)$)**.** *Let $(X,\mathsf{d},\mathfrak{m})\in\mathbf{X}$. Then $(X,\mathsf{d},\mathfrak{m})$ is $\mathsf{RCD}(K,\infty)$ if and only if it satisfies the metric $\mathsf{BE}(K,\infty)$ condition.*

Notice that the assumption that functions with bounded relaxed gradient have a continuous representative is necessary, in conjunction with $\mathsf{BE}(K,\infty)$, to have $\mathsf{RCD}(K,\infty)$: this way simple examples where $\mathsf{Ch}\equiv 0$ and $\mathsf{BE}(K,\infty)$ obviously holds (see for instance [3, Remark 4.12]) are ruled out. For the reader's convenience, we state the Global-to-Local property, see [3, Theorem 6.20] for the proof, relying on the fact that one can find geodesics connecting probability measures in $\bar U$ lying entirely in $\bar U$.

Proposition 7.2 (Global-to-Local for $RCD(K,\infty)$)**.** *Let $(X,\mathsf{d},\mathfrak{m})\in\mathbf{X}$ be $\mathsf{RCD}(K,\infty)$ and let $U\subset X$ be open. If $\mathfrak{m}(\partial U)=0$ and $(\bar U,\mathsf{d})$ is geodesic, then $(\bar U,\mathsf{d},\mathfrak{m}\llcorner\bar U)$ is $\mathsf{RCD}(K,\infty)$.*

The proof of the Local-to-Global property, established under the non-branching condition in [32], heavily relies on the $\mathsf{BE}(K,\infty)$ characterization of Theorem 7.1. Notice that the only global assumptions are (6.2) and the length property (necessary already for subsets of Euclidean spaces).

Theorem 7.3 (Local-to-Global for $\mathsf{RCD}(K,\infty)$)**.** *Let $(X,\mathsf{d},\mathfrak{m})\in\mathbf{X}$ be a length and locally compact space and assume that there exists a covering $\{U_i\}_{i\in I}$ of X by non-empty open subsets such that $(\bar U_i,\mathsf{d},\mathfrak{m}\llcorner\bar U_i)\in\mathbf{X}$ satisfy $\mathsf{RCD}(K,\infty)$.*

Then $(X,\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}(K,\infty)$ space.

The *proof* is an immediate consequence of Theorem 7.1 and Theorem 6.14.

7.2 The case $N<\infty$

For $N\geq 1$, let $U_N:[0,\infty)\to\mathbb{R}$ be defined by $U_N(r):=N(r-r^{1-1/N})$. The induced dimension-dependent Rényi entropy functionals $\mathcal{U}_N$ (whose limit as $N\to\infty$ is the Shannon entropy $\mathcal{U}_\infty$ in (7.1)) is defined by

$$\mathcal{U}_N(\mu):=\int_X U_N(\varrho)\,\mathrm{d}\mathfrak{m}+N\mu^\perp(X)\quad\text{if }\mu=\varrho\mathfrak{m}+\mu^\perp,\quad \mu^\perp\perp\mathfrak{m}. \tag{7.3}$$

Since $U_N(0)=0$ and the negative part of U grows at most linearly, $\mathcal{U}_N$ is well defined and with values in $\mathbb{R}$ if μ has bounded support.

We now introduce, for $\kappa \in \mathbb{R}$, the distortion coefficients

$$\sigma_\kappa^{(t)}(\delta) := \begin{cases} +\infty & \text{if } \kappa \geq \pi^2, \\ \dfrac{\sin(t\sqrt{\kappa}\delta)}{\sin(\sqrt{\kappa}\delta)} & \text{if } 0 < \kappa < \pi^2 \\ t & \text{if } \kappa = 0, \\ \dfrac{\sinh(t\sqrt{-\kappa}\delta)}{\sin(\sqrt{-\kappa}\delta)} & \text{if } \kappa < 0. \end{cases} \tag{7.4}$$

The so-called $\mathsf{CD}^*(K,N)$ condition introduced by Bacher and Sturm in [6] is based, in analogy with the case $N = \infty$, on a convexity inequality of $\mathcal{U}_N$ along Wasserstein geodesics; it is a variant of the $\mathsf{CD}(K,N)$ condition originally introduced by Sturm and studied in [25], [32, 33] (based on a different choice of the distortion coefficients in (7.5) below). Here we just mention that $\mathsf{CD}_{\rm loc}(K,N)$ is equivalent to $\mathsf{CD}^*(K,N)$, and this fact strongly suggests that the latter should have better globalization/localization properties. For the purpose of this paper, we just define the "Riemannian" $\mathsf{CD}^*(K,N)$ condition, adding the condition that Ch is a quadratic form.

Definition 7.4 ($\mathsf{RCD}^*(K,N)$ condition). *For $K \in \mathbb{R}$ and $N \in [1,\infty)$, we say that $(X,\mathsf{d},\mathfrak{m}) \in \mathbf{X}$ satisfies the $\mathsf{RCD}^*(K,N)$ condition if for every $\mu_i = \varrho_i\mathfrak{m} \in \mathscr{P}(X)$, $i = 0,1$, with bounded support, for all constant speed geodesic $\mu_s : [0,1] \to \mathscr{P}_2(X)$ from μ_0 to μ_1 and for every $M \geq N$ there holds*

$$\mathcal{U}_M(\mu_s) \leq \int \Big(\sigma_{K/M}^{(1-s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0(\gamma_0)^{-1/M} + \sigma_{K/M}^{(s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1(\gamma_1)^{-1/M}\Big)\,\mathrm{d}\boldsymbol{\pi}(\gamma), \tag{7.5}$$

where σ_κ is defined in (7.4) and $\mathcal{U}_M$ is defined in (7.3).

The following result, extending [4] to the dimensional case, has been proved in [5] using, from this paper, only the "abstract" regularity estimates in $\mathsf{BE}(K,N)$ Dirichlet spaces derived in §3; see also Remark 7.6 below for the closely related result [18].

Theorem 7.5 (Equivalence of $\mathsf{RCD}^*(K,N)$ and $\mathsf{BE}(K,N)$). *If $(X,\mathsf{d},\mathfrak{m}) \in \mathbf{X}$ satisfies the metric $\mathsf{BE}(K,N)$ condition then $(X,\mathsf{d},\mathfrak{m})$ is $\mathsf{RCD}^*(K,N)$. The converse holds if $\mathfrak{m}(X)$ is finite or $K \geq 0$.*

Remark 7.6 (Other characterizations of $\mathsf{RCD}^*(K,N)$). As in the case $N = \infty$, another characterization of $\mathsf{RCD}^*(K,N)$ has been given in [5] in terms of suitable Evolution Variation Inequalities (EVI) satisfied by the gradient flow of the Reny entropy $\mathcal{U}_N$, with a modulus of continuity proportional to K and dependent on N (in the limit case $N = \infty$ the modulus is proportional to the squared Wasserstein distance). This requires a detailed analysis of the gradient flow of the Reny entropy $\mathcal{U}_N$, a nonlinear diffusion equation. In this connection, a remarkable result obtained in [18] is the characterization of the $\mathsf{BE}(K,N)$ property in terms of an EVI property fulfilled, along the heat flow, by the modified Shannon entropy

$$\tilde{\mathcal{U}}_N(\mu) := \exp\Big(-\frac{1}{N}\mathcal{U}_\infty(\mu)\Big).$$

This has the advantage of avoiding many technical difficulties related to nonlinear diffusion equations in metric measure spaces. The definition of $\mathsf{RCD}^*(K,N)$ adopted in [18] is actually based on this EVI property and, due to the equivalence with $\mathsf{BE}(K,N)$ proved in that paper, our Local-to-Global result applies to this definition as well.

However, as we discussed in the previous subsection, all these $\mathsf{EVI}_{K,N}$ formulations, while technically important to get stability and convexity properties on *all* geodesics, are less relevant in the study of localization/globalization properties. ■

Proposition 7.7 (Global-to-Local for $\mathsf{RCD}^*(K,N)$). *Let $(X,\mathsf{d},\mathfrak{m}) \in \mathbf{X}$ be $\mathsf{RCD}^*(K,N)$ and let $U \subset X$ be open. If $\mathfrak{m}(\partial U) = 0$ and $(\bar{U},\mathsf{d})$ is geodesic, then $(\bar{U},\mathsf{d},\mathfrak{m}\llcorner\bar{U})$ is $\mathsf{RCD}^*(K,N)$.*

Proof. The proof follows the same lines of Proposition 7.2: first (independently of curvature assumptions) we obtain from Proposition 6.4(b) that the condition $(X,\mathsf{d},\mathfrak{m})$ localizes to U. Then, we use the fact that one can find geodesics connecting probability measures in $\bar{U}$ lying entirely in $\bar{U}$. □

Theorem 7.8 (Local-to-Global for $\mathsf{RCD}^*(K,N)$). *Let $(X,\mathsf{d},\mathfrak{m}) \in \mathbf{X}$ be a length space and assume that there exists a covering $\{U_i\}_{i\in I}$ of X by non-empty open subsets such that $\mathfrak{m}(U_i) < \infty$ if $K < 0$, and $(\bar{U}_i,\mathsf{d},\mathfrak{m}\llcorner\bar{U}_i) \in \mathbf{X}$ satisfy $\mathsf{RCD}^*(K,N)$.*

Then $(X,\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}^*(K,N)$ space.

Proof. Since $\mathfrak{m}(\bar{U}_i) < \infty$ if $K < 0$, we know from Theorem 7.5 that all spaces $(\bar{U}_i,\mathsf{d},\mathfrak{m}\llcorner\bar{U}_i)$ are metrically $\mathsf{BE}(K,N)$ and they are also locally compact, so that (X,d) is locally compact. Therefore Theorem 6.14 applies and shows that $(X,\mathsf{d},\mathfrak{m})$ is a metrically $\mathsf{BE}(K,N)$ space and we conclude applying Theorem 7.5 once more. □

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure*, Arxiv preprint, arXiv 1207.4924, (2012). To appear in Transactions of the AMS.

[2] L. Ambrosio, N. Gigli, and G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.

[3] ——, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J., 163 (2014), pp. 1405–1490.

[4] ——, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, Arxiv preprint, arXiv 1209.5786, (2012), pp. 1–61.

[5] L. Ambrosio, A. Mondino, and G. Savaré, *Nonlinear diffusion equations and curvature conditions in metric measure spaces*, In progress, (2013).

[6] K. Bacher and K.-T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal., 259 (2010), pp. 28–56.

[7] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, in Lectures on probability theory (Saint-Flour, 1992), vol. 1581 of Lecture Notes in Math., Springer, Berlin, 1994, pp. 1–114.

[8] ——, *Functional inequalities for Markov semigroups*, in Probability measures on groups: recent directions and trends, Tata Inst. Fund. Res., Mumbai, 2006, pp. 91–147.

[9] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, vol. 1123, Springer, Berlin, 1985, pp. 177–206.

[10] D. Bakry and M. Ledoux, *A logarithmic Sobolev form of the Li-Yau parabolic inequality*, Rev. Mat. Iberoamericana, 22 (2006), pp. 683-702.

[11] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener sapces*, vol. 14 of De Gruyter studies in Mathematics, De Gruyter, 1991.

[12] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[13] F. Cavalletti, *Decomposition of geodesics in the Wasserstein space and the globalization problem*, Geom. Funct. Anal., 24 (2014), pp. 493–551.

[14] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom., 46 (1997), pp. 406–480.

[15] Z.-Q. Chen and M. Fukushima, *Symmetric Markov processes, time change, and boundary theory*, vol. 35 of London Mathematical Society Monographs Series, Princeton University Press, Princeton, NJ, 2012.

[16] T. Coulhon and A. Sikora, *Riesz meets Sobolev*, Colloq. Math., 118 (2010), pp. 685–704.

[17] S. Daneri and G. Savaré, *Eulerian calculus for the displacement convexity in the Wasserstein distance*, SIAM J. Math. Anal., 40 (2008), pp. 1104–1122.

[18] M. Erbar, K. Kuwada, and K.T Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Arxiv 1303.4382, (2013).

[19] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, vol. 19 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, extended ed., 2011.

[20] U. Gianazza, G. Savaré, and G. Toscani, *The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation*, Arch. Ration. Mech. Anal., 194 (2009), pp. 133–220.

[21] N. Gigli, *On the differential structure of metric measure spaces and applications*, preprint arXiv:1205.6622 (2012), To appear on Mem. Amer. Math. Soc.

[22] N. Gigli, A. Mondino, *A PDE approach to nonlinear potential theory in metric measure spaces*, J. Math. Pures Appl., 100 (2013), pp. 505–534.

[23] N. Gigli, A. Mondino, and G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Arxiv preprint, arXiv:1311.4907, (2013), pp. 1–62.

[24] P.-L. Lions and C. Villani, *Régularité optimale de racines carrées*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), pp. 1537–1541.

[25] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2), 169 (2009), pp. 903–991.

[26] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms*, Springer, New York, 1992.

[27] T. Rajala, *Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm*, J. Funct. Anal., 263 (2012), pp. 896–924.

[28] ——, *Local Poincaré inequalities from stable curvature conditions on metric spaces*, Calc. Var. Partial Differential Equations, 44 (2012), pp. 477–494.

[29] ——, *Failure of the local-to-global property for $CD(K,N)$ spaces*, (2013), To appear on Ann. Sc. Norm. Super. Pisa Cl. Sci., DOI Number: 10.2422/2036 − 2145.201307_004.

[30] T. Rajala and K.-T. Sturm, *Non-branching geodesics and optimal maps in strong $CD(K,\infty)$-spaces*, Calc. Var. Partial Differential Equations, 50 (2014), pp. 831–846.

[31] G. Savaré, *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $RCD(K,\infty)$ metric measure spaces*, Disc. Cont. Dyn. Sist. A, 34 (2014), pp. 1641–1661.

[32] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math., 196 (2006), pp. 65–131.

[33] ——, *On the geometry of metric measure spaces. II*, Acta Math., 196 (2006), pp. 133–177.

[34] H. Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosiusu Barth Verlag, Heidelberg, Leipzig, 1995.

[35] C. Villani, *Optimal transport. Old and new*, vol. 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2009.