

LQG Mean-Field Games with ergodic cost

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Abstract—We consider stochastic differential games with N players, linear-Gaussian dynamics in arbitrary state-space dimension, and long-time-average cost with quadratic running cost. Admissible controls are feedbacks for which the system is ergodic. We first study the existence of affine Nash equilibria by means of an associated system of N Hamilton-Jacobi-Bellman and N Kolmogorov-Fokker-Planck partial differential equations. We give necessary and sufficient conditions for the existence and uniqueness of quadratic-Gaussian solutions in terms of the solvability of suitable algebraic Riccati and Sylvester equations. Under a symmetry condition on the running costs and for nearly identical players we study the large population limit, N tending to infinity, and find a unique quadratic-Gaussian solution of the pair of Mean Field Game HJB-KFP equations. This extends some of the classical results on Mean Field Games by Huang, Caines, and Malhame and by Lasry and Lions, and the more recent paper by one of the authors in the 1-dimensional case.

I. INTRODUCTION

We consider a system of N stochastic differential equations

$$dX_t^i = (A^i X_t^i - \alpha_t^i) dt + \sigma^i dW_t^i, \quad X_0^i = x^i \in \mathbb{R}^d, \quad (1)$$

$i = 1, \dots, N$, where A^i, σ^i are given $d \times d$ matrices, with $\det(\sigma^i) \neq 0$, (W_t^1, \dots, W_t^N) are N independent Brownian motions, $\alpha_t^i: [0, +\infty[\rightarrow \mathbb{R}^d$ is a process adapted to W_t^i such that the corresponding trajectory X_t^i is ergodic, and it represents the control of the i -th player in the differential game that we now describe. For each initial positions $X = (x^1, \dots, x^N) \in \mathbb{R}^{Nd}$ we consider for the i -th player the long-time-average cost functional

$$J^i(X, \alpha^1, \dots, \alpha^N) := \liminf_{T \rightarrow \infty} \frac{\mathbb{E}}{T} \left[\int_0^T \frac{\alpha_t^{iT} R^i \alpha_t^i}{2} + (X_t - \bar{X}_i)^T Q^i (X_t - \bar{X}_i) dt \right] \quad (2)$$

where $\mathbb{E}[\cdot]$ denotes the expected value, R^i are positive definite symmetric $d \times d$ matrices, Q^i are symmetric $Nd \times Nd$ matrices, and $\bar{X}_i \in \mathbb{R}^{Nd}$ are given reference positions. For this N -persons game we are interested in the synthesis of Nash equilibrium strategies in feedback form from the solutions of a system of N Hamilton-Jacobi-Bellman and N Kolmogorov-Fokker-Planck partial differential equations,

and in the limit behavior of these solutions and strategies as $N \rightarrow +\infty$, within the theory of Mean-Field Games as formulated by Lasry and Lions [18], [19], [20]. In particular, we produce a solution of the limit pair of Mean-Field Games HJB-KFP equations and give additional conditions for its uniqueness. In the case $d = 1$ of one-dimensional state space, the problem was solved explicitly in [2] under a condition that the players be almost identical that we will use here as well. In the multi-dimensional case $d > 1$, however, the solvability of the HJB equations among quadratic functions and of the KFP equations among Gaussian densities leads to some nontrivial Algebraic Riccati Equations and Sylvester equations whose solution is not explicit, in general.

Large population limits for multi-agent systems were studied by Huang, Caines and Malhame, independently of Lasry-Lions. They introduced a method named Nash certainty equivalence principle [14], [15], [16]. We cannot review here the number of papers inspired by their approach, but let us mention [22], [5], [4] for LQ problems, [24] for risk-sensitive games, [23] for recent progress on nonlinear systems, and the references therein. Some of these papers also deal with ergodic cost functionals, e.g., [16], [22], but their assumptions and methods differ from ours.

Concerning the Lasry-Lions approach to MFG let us mention the lecture notes [7] and [13], [1] on numerical methods, [12] on discrete games, [8] on the long time behaviour of solutions, and [11] on the large population limit for nonlinear ergodic control of several populations. A very recent survey on both approaches is [6].

There is a wide spectrum of applications of Mean-Field Games that we do not try to list here for lack of space and refer instead to the quoted literature.

The paper is organised as follows. In Section II, we define admissible strategies, introduce the system of HJB and KFP equations associated to the N -persons game and recall some known facts about algebraic Riccati equations. In Section III, we define almost identical players, and give a characterization for existence and uniqueness of solutions to the system of HJB-KFP equations, in the class of quadratic value functions and multivariate Gaussian invariant measures, in terms of Algebraic Riccati and Sylvester equations. An analogous result holds also without assuming that players are almost identical, but due to lack of space we refer to our forthcoming paper [3] for the general case.

Section IV is devoted to the analysis of the limit when the number of players tends to infinity, under natural rescaling assumptions on the matrix coefficients of the game.

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Uniqueness for the limit pair of MFG HJB-KFP equations is also discussed. Finally, in Section V we show some simple sufficient conditions for existence and uniqueness of a Nash equilibrium strategy in the N -player game, as well as for its convergence. In these examples the solutions are explicit.

II. PRELIMINARIES

We consider strategies whose corresponding solution to (1) is ergodic.

Definition 2.1: A strategy α^i is said to be *admissible* (for the i -th player) if it is a bounded process adapted to the Brownian motion W_t^i such that the corresponding solution X_t^i to (1) satisfies

- $\mathbb{E}[(X_t^i)(X_t^i)^T]$ is bounded,
- X_t^i is *ergodic* in the following sense: there exists a probability measure $m^i = m^i(\alpha^i)$ on \mathbb{R}^d such that

$$\int_{\mathbb{R}^d} |x|^2 dm^i(x) < \infty$$

and

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \left[\int_{\mathbb{R}^d} g(X_t^i) dt \right] = \int_{\mathbb{R}^d} g(x) dm^i(x),$$

locally uniformly w.r.t. the initial state of X_t^i , for all functions g which are polynomials of degree at most 2. One can prove that affine strategies are admissible. Namely, the following proposition follows by standard arguments in stochastic differential equations, see [9], [17].

Proposition 2.1: For the affine feedback

$$\alpha^i(x) = K^i x + c^i, \quad x \in \mathbb{R}^d, \quad (3)$$

with $K^i \in \text{Mat}_{d \times d}(\mathbb{R})$ such that the matrix $A^i - K^i$ has only eigenvalues with negative real part, and $c^i \in \mathbb{R}^d$, consider the process $\alpha_t^i := \alpha^i(X_t^i)$ where X_t^i solves

$$dX_t^i = [(A^i - K^i)X_t^i - c^i]dt + \sigma^i dW_t^i. \quad (4)$$

Then, α_t^i is admissible. Namely, the process X_t^i has a unique invariant measure m^i given by a multivariate Gaussian $\mathcal{N}(\mu, V)$ with mean $\mu = -(A^i - K^i)^{-1}c^i$ and variance matrix V which satisfies the algebraic relation

$$(A^i - K^i)V + V(A^i - K^i)^T + \sigma^i(\sigma^i)^T = 0, \quad (5)$$

and such a measure m^i is exactly the one which makes X_t^i ergodic.

Next we write the system of HJB-KFP equations associated to the game (1)–(2), as in [2], [18], [20]. We start by remarking that the part of the cost depending on the state of the game can be also written as

$$\begin{aligned} F^i(X^1, \dots, X^N) &:= (X - \bar{X}_i)^T Q^i (X - \bar{X}_i) \\ &= \sum_{j,k=1}^N (X^j - \bar{X}_i^j)^T Q_{jk}^i (X^k - \bar{X}_i^k) \end{aligned} \quad (6)$$

where the matrices Q_{jk}^i are $d \times d$ blocks of Q^i . The standing assumptions on the game are summed up in the following condition.

- (H) Assume that σ^i in (1) are invertible matrices, that R^i in (2) are symmetric positive definite matrices and that Q^i in (2) are symmetric matrices. Moreover, assume that blocks Q_{ii}^i are symmetric positive definite.

For the game (1)–(2) under consideration, we observe that the i -th Hamiltonian takes the form

$$\begin{aligned} H^i(x, p) &:= \min_{\omega} \left\{ -\omega^T \frac{R^i}{2} \omega - p^T (A^i x - \omega) \right\} \\ &= -p^T A^i x + \min_{\omega} \left\{ -\omega^T \frac{R^i}{2} \omega - p^T \cdot \omega \right\}. \end{aligned}$$

Since the minimum is attained at $(R^i)^{-1}p$, we conclude

$$H^i(x, p) = -((R^i)^{-1}p)^T \frac{R^i}{2} ((R^i)^{-1}p) - p^T (A^i x - (R^i)^{-1}p)$$

Introducing the notations

$$\begin{aligned} f^i(x; m^1, \dots, m^N) &:= \\ &\int_{\mathbb{R}^{d(N-1)}} F^i(\xi^1, \dots, \xi^{i-1}, x, \xi^{i+1}, \dots, \xi^N) \prod_{j \neq i} dm^j(\xi^j), \end{aligned} \quad (7)$$

for any N -vector of probability measures (m^1, \dots, m^N) , and

$$\nu^i := \frac{(\sigma^i)(\sigma^i)^T}{2} \in \text{Mat}_{d \times d}(\mathbb{R}),$$

the system of HJB-KFP equations associated to the game is given by

$$\begin{cases} -\text{tr}(\nu^i D^2 v^i) + H^i(x, \nabla v^i) + \lambda^i = f^i(x; m^1, \dots, m^N) \\ -\text{tr}(\nu^i D^2 m^i) - \text{div} \left(m^i \frac{\partial H^i}{\partial p}(x, \nabla v^i) \right) = 0 \\ \int_{\mathbb{R}^d} m^i(x) dx = 1, \quad m^i > 0 \end{cases} \quad (8)$$

$i = 1, \dots, N$, where tr denotes the trace of the matrix in brackets. The unknowns of the PDE system are the scalar functions v^i , the real numbers λ^i , and the measures m^i , where we have slightly abused the notation and denoted with m^i the density of the measure as well.

In the forthcoming paper [3] we study the solvability of this system among quadratic v^i and Gaussian m^i . We give necessary and sufficient conditions for existence in terms of solvability of some algebraic Riccati equations (ARE in the following) and Sylvester equations. We also prove that the invertibility of a suitable matrix constructed from the data is necessary and sufficient for uniqueness. In the next section we describe only the special case of nearly identical players that is the relevant one for the large population limit $N \rightarrow \infty$. Our main tool are some basic facts about algebraic Riccati equations that we recall next, referring to [10], [21] and its bibliography for the proof of the results.

Proposition 2.2: Consider the ARE

$$X \mathcal{R} X - Q = 0 \quad (9)$$

with $\mathcal{R}, \mathcal{Q} \in \text{Mat}_{d \times d}(\mathbb{R})$ symmetric and positive definite. Let X be a $d \times d$ real matrix and denote by Ξ and \mathcal{H} the following real matrices

$$\Xi := \begin{bmatrix} I_d \\ X \end{bmatrix} \in \text{Mat}_{2d \times d}(\mathbb{R}), \quad (10)$$

$$\mathcal{H} := \begin{pmatrix} \mathbf{0} & \mathcal{R} \\ \mathcal{Q} & \mathbf{0} \end{pmatrix} \in \text{Mat}_{2d \times 2d}(\mathbb{R}).$$

Then the following facts hold.

- (i) X is a solution of (9) if and only if the d -dimensional linear subspace $\text{Im } \Xi$ is \mathcal{H} -invariant, i.e. if and only if $\mathcal{H}\xi \in \text{Im } \Xi$ for all $\xi \in \text{Im } \Xi$.
- (ii) If the matrix \mathcal{H} has no purely imaginary nonzero eigenvalues, then equation (9) has solutions X such that $X = X^T$.
- (iii) If equation (9) has symmetric solutions, then there exists a unique symmetric solution which is also positive definite.

III. NEARLY IDENTICAL PLAYERS

Our main assumption, as in [2], is the following *Symmetry Condition*

- (S) every player is influenced in the same way by pair of other players, i.e. for each $i \in \{1, \dots, N\}$ and each $j, k \neq i$

$$F^i(X^1, \dots, X^j, \dots, X^k, \dots, X^N) = F^i(X^1, \dots, X^k, \dots, X^j, \dots, X^N) \quad (11)$$

We can easily prove the following lemma.

Lemma 3.1: Assumption (S) holds if and only if there exist matrices B_i, C_i, D_i and vectors Δ_i such that

$$Q_{ij}^i = \frac{B_i}{2}, \quad Q_{jj}^i = C_i, \quad \bar{X}_i^j = \Delta_i, \quad \forall j \neq i,$$

$$Q_{jk}^i = D_i, \quad \forall j, k \neq i, j \neq k.$$

Under assumption (S), the quadratic costs F^i take the following form

$$F^i(X^1, \dots, X^N) = (X^i - \bar{X}_i^i)^T Q_{ii}^i (X^i - \bar{X}_i^i) + (X^i - \bar{X}_i^i)^T \frac{B_i}{2} \left(\sum_{k \neq i} (X^k - \Delta_i) \right) + \left(\sum_{j \neq i} (X^j - \Delta_i)^T \right) \frac{B_i}{2} (X^i - \bar{X}_i^i) + \sum_{j \neq i} (X^j - \Delta_i)^T C_i (X^j - \Delta_i) + \sum_{j, k \neq i, j \neq k} (X^j - \Delta_i)^T D_i (X^k - \Delta_i) \quad (12)$$

In particular, they can be written in the form arising in the Lasry-Lions formulation of mean field games, namely

$$F^i(X^1, \dots, X^N) = V_i \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X^j} \right] (X^i),$$

where δ_{X^j} is the Dirac measure on \mathbb{R}^d centered in the point X^j and V_i is the operator, mapping probability measures on \mathbb{R}^d into quadratic polynomials, defined by the expression

$$V_i[m](X) := (X - \bar{X}_i^i)^T Q_{ii}^i (X - \bar{X}_i^i) + (N-1) \int_{\mathbb{R}^d} ((X - \bar{X}_i^i)^T \frac{B_i}{2} (\xi - \Delta_i) + (\xi - \Delta_i)^T \frac{B_i}{2} (X - \bar{X}_i^i)) dm(\xi) + (N-1) \int_{\mathbb{R}^d} (\xi - \Delta_i)^T (C_i - D_i) (\xi - \Delta_i) dm(\xi) + \left((N-1) \int_{\mathbb{R}^d} (\xi - \Delta_i) dm(\xi) \right)^T D_i \left((N-1) \int_{\mathbb{R}^d} (\xi - \Delta_i) dm(\xi) \right) \quad (13)$$

Definition 3.1: We say that the players are *almost identical* if the costs F^i satisfy (S) and if all players have the same:

- control systems, i.e. $A^i = A$ and $\sigma^i = \sigma$ (and therefore $\nu^i = \nu$) for all i ,
- costs of the control, i.e. $R^i = R$ for all i ,
- reference positions, i.e. $\bar{X}_i^i = H$ (own reference position, or happy place) and $\Delta_i = \Delta$ (reference position of the other players) for all i ,
- primary costs of displacement, i.e. $Q_{ii}^i = Q$ and $B_i = B$ for all i .

As in [2], we say that players are *almost identical* because the secondary costs of displacement, i.e. the matrices C_i and D_i , can still be different among the players.

Since we are interested in solutions to (8) among functions v^i that are quadratic and measures m^i that are Gaussian, we specialize the system of HJB-KFP equations to this case. Focusing our attention to identically distributed solutions for the various players, i.e. considering measures of the form $m^1 = \dots = m^N = \mathcal{N}(\mu, \Sigma^{-1})$ for some common variance matrix Σ^{-1} and mean vector μ , we obtain from (12) and (7) that

$$f^i(X^i; m^1, \dots, m^N) = (X^i - H)^T Q (X^i - H) + (N-1) (X^i - H)^T \frac{B}{2} (\mu - \Delta) + (N-1) (\mu - \Delta)^T \frac{B}{2} (X^i - H) + \sum_{j \neq i} \int_{\mathbb{R}^d} (\xi^j - \Delta_i)^T C_i (\xi^j - \Delta_i) dm^j(\xi^j) + (N-1) \text{tr}((C_i - D_i) \Sigma) + (N-1) (\mu - \Delta)^T C_i (\mu - \Delta) + (N-1)(N-2) (\mu - \Delta)^T D_i (\mu - \Delta) \quad (14)$$

Hence, searching for explicit solutions given by the following expressions

$$m^i(x) = \gamma \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma (x - \mu) \right\} \quad (15)$$

$$v^i(x) = x^T \frac{\Lambda}{2} x + \rho x, \quad (16)$$

for suitable symmetric matrices Λ, Σ , with Σ positive definite, and suitable vectors μ, ρ and γ given by

$$\gamma = \frac{1}{(2\pi)^{d/2}(\det \Sigma^{-1})^{1/2}},$$

the system (8) of $2N$ partial differential equation in \mathbb{R}^d reduces to an algebraic system. Namely, one finds that (8) rewrites as

$$g(x; \Sigma, \mu, \Lambda, \rho) + \lambda^i = f^i(x; m^1, \dots, m^N) \quad (17)$$

$$m^i(x) \cdot h(x; \Sigma, \mu, \Lambda, \rho) = 0 \quad (18)$$

where equalities must hold for all $x \in \mathbb{R}^d$ and g, h are quadratic functions of the variable x . Since $m^i > 0$ in \mathbb{R}^d , the second equation implies that h must vanish identically, and this results in the following matrix relations

$$\Lambda = R(\nu\Sigma + A), \quad \rho = -R\nu\Sigma\mu \quad (19)$$

among coefficients of (15) and (16). Hence, the first equation can be rewritten as

$$\tilde{g}(x; \Sigma, \mu) + \lambda^i = f^i(x; m^1, \dots, m^N) \quad \forall x \in \mathbb{R}^d$$

and, by interpreting it as an equality between quadratic forms, we find the conditions

$$\Sigma \frac{\nu R\nu}{2} \Sigma - \frac{A^T R A}{2} = Q \quad (20)$$

$$\mathcal{B} \mu = -QH - (N-1) \frac{B}{2} \Delta \quad (21)$$

$$\mu^T \frac{\Sigma \nu R \nu \Sigma}{2} \mu - \text{tr}(\nu R \nu \Sigma + \nu R A) + \lambda^i = \tilde{F}_o^i \quad (22)$$

with

$$\mathcal{B} := Q + \frac{A^T R A}{2} + (N-1) \frac{B}{2} \quad (23)$$

and

$$\begin{aligned} \tilde{F}_o^i := & H^T Q H + (N-1)(N-2) (\mu - \Delta)^T D_i (\mu - \Delta) \\ & - (N-1) \left(H^T \frac{B}{2} (\mu - \Delta) + (\mu - \Delta)^T \frac{B}{2} H \right) \\ & + (N-1) \text{tr}((C_i - D_i)\Sigma) + (N-1) (\mu - \Delta)^T C_i (\mu - \Delta). \end{aligned} \quad (24)$$

Under hypotheses **(H)**, equation (20) is an ARE of the form (9), with $\mathcal{R} := \nu R \nu / 2$ and $\mathcal{Q} := Q + A^T R A / 2$ both positive definite. In the notations of Proposition 2.2, if we prove that the corresponding matrix \mathcal{H} has no purely imaginary nonzero eigenvalues, then the proposition ensures that (20) admits a unique symmetric and positive definite solution Σ . But ℓ is an eigenvalue of \mathcal{H} if and only if ℓ is a solution of the equation

$$0 = \det(\mathcal{H} - \ell I_{2d}) = \det(\ell^2 I_d - \mathcal{R}\mathcal{Q}),$$

i.e. if and only if ℓ^2 is an eigenvalue of the matrix $\mathcal{R}\mathcal{Q}$. Since the eigenvalues of $\mathcal{R}\mathcal{Q}$ are positive, all eigenvalues of \mathcal{H} are real. This implies the existence and uniqueness of the symmetric and positive definite matrix Σ .

Moreover, the second equation is just a linear system which admits a unique solution μ if and only if the matrix \mathcal{B} is

invertible. Finally, once Σ and μ have been found, they can be used in the third equation and in (19) to obtain uniquely the values λ_i , the matrix Λ and the vector ρ . Hence, it remains to verify whether the matrix Λ obtained in (19) is symmetric as requested in our ansatz (16). Owing to (19), we observe that Λ is symmetric if and only if

$$R(\nu\Sigma + A) = (\nu\Sigma + A)^T R \quad (25)$$

which in turn can be interpreted as a Sylvester equation for the matrix Σ .

Therefore, it is possible to collect in the next condition the required properties for the existence of solutions $(\Sigma, \mu, \lambda^1, \dots, \lambda^N)$ to (20)–(22).

(E_N) Every symmetric and positive definite solution X of the algebraic Riccati equation

$$X \frac{\nu R \nu}{2} X = \frac{A^T R A}{2} + Q, \quad (26)$$

provides also a solution to the following Sylvester equation

$$X \nu R - R \nu X = R A - A^T R. \quad (27)$$

Moreover, the matrices $\mathcal{B} \in \text{Mat}_{d \times d}(\mathbb{R})$ and $[\mathcal{B}, P] \in \text{Mat}_{d \times (d+1)}(\mathbb{R})$ have the same rank, where \mathcal{B} is the matrix defined in (23), $P := -QH + (1-N) \frac{B}{2} \Delta$ and $[\mathcal{B}, P]$ is the matrix whose columns are the columns of \mathcal{B} and the vector P , i.e.

$$[\mathcal{B}, P] := (\mathcal{B}^1, \dots, \mathcal{B}^d, P), \quad (28)$$

being \mathcal{B}^j the columns of the matrix \mathcal{B} .

In view of the previous analysis, we have proved the following theorem about games with nearly identical players.

Theorem 1: Assume that the N -player game having dynamics (1) and costs (2) satisfies assumption **(H)** and that the players are almost identical in the sense of Definition 3.1. Then, the associated system of $2N$ HJB–KFP equations (8) admits solutions $(v, m, \lambda^1, \dots, \lambda^N)$ of the form v quadratic function with $v(0) = 0$ and m multivariate Gaussians $\mathcal{N}(\mu, \Sigma^{-1})$, i.e. of the form (15)–(16), if and only if **(E_N)** is satisfied. Moreover, such solutions are also unique if and only if the matrix \mathcal{B} defined in (23) is invertible and, if this is the case, the affine feedbacks

$$\bar{\alpha}^i(x) = \bar{\alpha}(x) := R^{-1} \nabla v(x), \quad x \in \mathbb{R}^d, \quad i = 1, \dots, N$$

provide a Nash equilibrium strategy for all initial positions $X_0 \in \mathbb{R}^{Nd}$, among the admissible strategies, and $J^i(X_0, \bar{\alpha}) = \lambda_i$ for all X_0 and all i .

IV. THE LIMIT AS $N \rightarrow +\infty$

Next we study the convergence of Nash equilibria when the population of players becomes very large, i.e., when $N \rightarrow +\infty$. Assume for simplicity that the control system, the costs of the control and the reference positions are always the same, i.e. that A, σ, R, H and Δ are all independent from the number of players N . We denote with

$$Q^N, \quad B^N, \quad C_i^N, \quad D_i^N,$$

the primary and secondary costs of displacement, respectively, which are assumed to depend on N . We assume that these quantities, when $N \rightarrow +\infty$, tend to suitable matrices $\hat{Q}, \hat{B}, \hat{C}, \hat{D}$ with their natural scaling, i.e. as $N \rightarrow +\infty$ there holds

$$Q^N \rightarrow \hat{Q}, \quad B^N(N-1) \rightarrow \hat{B}, \\ C_i^N(N-1) \rightarrow \hat{C}, \quad D_i^N(N-1)^2 \rightarrow \hat{D}, \quad \forall i. \quad (29)$$

If we define an operator acting on probability measures of \mathbb{R}^d by setting for all $\mathfrak{m} \in \mathcal{P}(\mathbb{R}^d)$

$$\begin{aligned} \hat{V}[\mathfrak{m}](X) &:= (X-H)^T \hat{Q} (X-H) + \\ &\int_{\mathbb{R}^d} ((X-H)^T \frac{\hat{B}}{2} (\xi - \Delta) + (\xi - \Delta)^T \frac{\hat{B}}{2} (X-H)) d\mathfrak{m}(\xi) \\ &\quad + \int_{\mathbb{R}^d} (\xi - \Delta)^T \hat{C} (\xi - \Delta) d\mathfrak{m}(\xi) \\ &\quad + \left(\int_{\mathbb{R}^d} (\xi - \Delta) d\mathfrak{m}(\xi) \right)^T \hat{D} \left(\int_{\mathbb{R}^d} (\xi - \Delta) d\mathfrak{m}(\xi) \right) \end{aligned} \quad (30)$$

then it is easy to verify that, as $N \rightarrow +\infty$, for all i and all $\mathfrak{m} \in \mathcal{P}(\mathbb{R}^d)$

$$V_i^N[\mathfrak{m}](X) \rightarrow \hat{V}[\mathfrak{m}](X), \quad \text{locally uniformly in } X.$$

By denoting with λ_i^N , v^N and m^N the solutions found in Section III, we expect that the limits of these solutions satisfy, like in [2], [18], [20], the system of two mean field equations

$$\begin{cases} -\text{tr}(\nu D^2 v) + \nabla v^T \frac{R^{-1}}{2} \nabla v - \nabla v^T A x + \lambda = \hat{V}[m](x) \\ -\text{tr}(\nu D^2 m) - \text{div}(m \cdot (R^{-1} \nabla v - A x)) = 0 \\ \int_{\mathbb{R}^d} m(x) dx = 1, \quad m > 0 \end{cases} \quad (31)$$

As before, we look for solutions such that

$$v(x) = x^T \frac{\Lambda}{2} x + \rho x, \\ m(x) = \gamma \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma (x - \mu) \right\}, \quad (32)$$

for suitable symmetric matrices Λ, Σ , with Σ positive definite, suitable vectors μ, ρ and a normalization constant γ depending only on the matrix Σ and on the dimension of the space. By computations similar to those of Section III we verify that the equation for the measure reduces, as before, to the matrix relations

$$\Lambda = R(\nu \Sigma + A), \quad \rho = -R\nu \Sigma \mu. \quad (33)$$

Concerning the equation for the value function, we proceed as in the previous sections and obtain the conditions

$$\Sigma \frac{\nu R \nu}{2} \Sigma - \frac{A^T R A}{2} = \hat{Q} \quad (34)$$

$$-\frac{\Sigma \nu R \nu \Sigma}{2} \mu = -\hat{Q} H + \frac{\hat{B}}{2} (\mu - \Delta) \quad (35)$$

$$\mu^T \frac{\Sigma \nu R \nu \Sigma}{2} \mu - \text{tr}(\nu R \nu \Sigma + \nu R A) + \lambda = \hat{F}_o \quad (36)$$

with

$$\begin{aligned} \hat{F}_o &= H^T \hat{Q} H - \left(H^T \frac{\hat{B}}{2} (\mu - \Delta) + (\mu - \Delta)^T \frac{\hat{B}}{2} H \right) \\ &\quad + \text{tr}(\hat{C} \Sigma) + (\mu - \Delta)^T (\hat{C} + \hat{D}) (\mu - \Delta) \end{aligned}$$

In particular, the first equation has exactly the same form as (20), and it hence admits a unique symmetric and positive definite solution Σ . Also, we can rewrite the second equality in the form

$$-\left(\hat{Q} + \frac{A^T R A}{2} + \frac{\hat{B}}{2} \right) \mu = -\hat{Q} H - \frac{\hat{B}}{2} \Delta,$$

which admits a unique solution μ whenever the matrix

$$\mathcal{B}_\infty := \hat{Q} + \frac{A^T R A}{2} + \frac{\hat{B}}{2} \quad (37)$$

is invertible. Finally, once Σ and μ have been found, one can insert them into the third equation and (33) to obtain the value λ , the matrix Λ and the vector ρ required by (32).

Based on the previous analysis, we can pass to the limit as $N \rightarrow \infty$ in (19), (20), (21), and (22); the argument for the Riccati equation (20) is nontrivial and it is given in detail in [3]. The result we get is the following.

Theorem 2: Assume **(H)**, almost identical players, **(E_N)** for all N , the convergence of the data (29), and the invertibility of the matrix \mathcal{B}_∞ defined in (37). Then the solutions $(v^N, m^N, \lambda_1^N, \dots, \lambda_N^N)$ found in Theorem 1 converge to a solution (v, m, λ) of the Mean-Field system of HJB–KFP equations (31) as $N \rightarrow \infty$ in the following sense: $v^N \rightarrow v$ in $C_{loc}^1(\mathbb{R}^d)$ with second derivative converging uniformly in \mathbb{R}^d , $m^N \rightarrow m$ in $C^k(\mathbb{R}^d)$ for all k , and $\lambda_i^N \rightarrow \lambda$ for all i . Moreover such solution has v quadratic and m multivariate Gaussian, i.e. it is of the form (32) with parameters satisfying (33), (34), (35), (36). Finally, the solution is unique among functions of the form (32).

A natural question is whether the PDE system (31) has other solutions that are not quadratic-Gaussian. We add a normalization condition on v , to avoid addition of constants, and make a simple assumption that ensures the monotonicity of \hat{V} with respect to the scalar product in the Lebesgue space L^2 . Then an argument of Lasry and Lions implies uniqueness [18], [20].

Theorem 3: The integral operator \hat{V} satisfies $\int_{\mathbb{R}^d} (\hat{V}[\mathfrak{m}] - \hat{V}[\mathfrak{n}])(x) d(\mathfrak{m} - \mathfrak{n})(x) \geq 0$ for all probability measures $\mathfrak{m}, \mathfrak{n}$ on \mathbb{R}^d if and only if the matrix \hat{B} is positive semidefinite. Then, under the assumptions of Theorem 2 and for $\hat{B} \geq 0$, (v, m, λ) is the unique solution of (31) such that $v(0) = 0$.

V. EXPLICIT SUFFICIENT CONDITIONS

We have already stressed that condition **(E_N)** has the important role to translate the existence of quadratic-Gaussian solutions to the system (8) into algebraic matrix relations. In

this section we show that such conditions can also be easily verified in some special cases.

Consider a N -player game with dynamics (1) and costs (2) and assume that **(H)** holds, that players are almost identical in the sense of Definition 3.1, and also that

- (a) dynamics (1) are given by a drift matrix A which is symmetric and a diffusion matrix $\sigma = sI_d$ with $s \in \mathbb{R} \setminus \{0\}$,
- (b) costs (2) are given by a diagonal matrix $R = rI_d$, with $r > 0$, and by matrices Q^i such that, in the notations of Lemma 3.1, the $d \times d$ square blocks $B_i = B$ are null and only the blocks $Q_{ii}^i = Q$, C_i and D_i can be nonzero.

Then, it is easy to see that the relation in **(E_N)** about solutions of (26) and (27) is automatically satisfied. Indeed, both matrices $\nu = s^2 I_d / 2 =: \bar{\nu}$ and R commute with any other matrix and thus (27) reduces to

$$r(A - A^T) = r \frac{s^2}{2} (X - X) = 0,$$

which is always satisfied under the symmetry assumption on A . Moreover we can calculate the explicit expression of the matrix Σ . Indeed, the matrix $\frac{2}{r}Q + A^2$ is symmetric and positive definite and thus admits a positive definite square root in $\text{Mat}_{d \times d}(\mathbb{R})$, that we denote with E . If we now consider the ARE (20), we find

$$r \frac{\bar{\nu}^2}{2} \Sigma^2 = Q + \frac{r}{2} A^2 = \frac{r}{2} E^2,$$

which implies

$$\Sigma = \frac{1}{\bar{\nu}} E.$$

To verify the part of condition **(E_N)** dealing with the matrix \mathcal{B} , we can rewrite (23) as

$$\mathcal{B} = Q + \frac{r}{2} A^2 = \frac{r}{2} E^2$$

because now $B = 0$. Hence, \mathcal{B} is invertible and the linear system (21) admits a unique solution, explicitly given by

$$\mu = -\frac{2}{r} (E^2)^{-1} QH,$$

where H is the reference position (*happy state*) for the players. In turn, the expressions found for Σ and μ can be used in (19) and (22) to obtain Λ , ρ and $\lambda^1, \dots, \lambda^N$, completing the construction of the unique solution of quadratic-Gaussian type.

In conclusion, for games with N nearly identical players which satisfy **(H)**, conditions (a) and (b) are sufficient to guarantee the conclusions of Theorems 1, 2 and 3, and also the following formula for the unique affine Nash equilibrium strategy

$$\bar{\alpha}(x) = (E + A)x + \frac{2}{r} E^{-1} QH.$$

Remark 5.1: In [3] we give some more general sufficient conditions for the validity of Theorems 1, 2 and 3. In particular, the symmetry of A can be replaced by the assumption that it is non-defective (i.e., each eigenvalue has multiplicity equal to the dimension of the corresponding eigenspace).

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