

**CALDERÓN-ZYGMUND ESTIMATES FOR PARABOLIC
 $p(x, t)$ -LAPLACIAN SYSTEMS**

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ABSTRACT. We prove local Calderón-Zygmund estimates for weak solutions of the evolutionary $p(x, t)$ -Laplacian system

$$\partial_t u - \operatorname{div} (a(x, t)|Du|^{p(x, t)-2}Du) = \operatorname{div} (|F|^{p(x, t)-2}F)$$

under the classical hypothesis of logarithmic continuity for the variable exponent $p(x, t)$. More precisely, we show that the spatial gradient Du of the solution is as integrable as the right-hand side F , i.e.

$$|F|^{p(\cdot)} \in L_{\text{loc}}^q \implies |Du|^{p(\cdot)} \in L_{\text{loc}}^q \quad \text{for any } q > 1$$

together with quantitative estimates. Thereby, we allow the presence of eventually discontinuous coefficients $a(x, t)$, only requiring a VMO condition with respect to the spatial variable x .

1. INTRODUCTION

The aim in this paper is to provide a Calderón-Zygmund theory for weak solutions to the parabolic $p(x, t)$ -Laplacian system

$$(1.1) \quad \partial_t u - \operatorname{div} (a(z)|Du|^{p(z)-2}Du) = \operatorname{div} (|F|^{p(z)-2}F) \quad \text{in } \Omega_T := \Omega \times (0, T).$$

Here, Ω is an open set in \mathbb{R}^n with $n \geq 2$ and Ω_T denotes the parabolic cylinder over Ω . Since we consider the case of systems, the solution is a possibly vector valued function $u: \Omega_T \rightarrow \mathbb{R}^N$ with $N \geq 1$. With respect to the variable exponent $p(x, t)$ we assume logarithmic continuity which is classical in the theory of variable exponent problems, while the regularity assumption on $a(x, t)$ allows to catch a large class of not necessarily continuous coefficients, including the ones of the splitting form $a(x, t) = b(x)c(t)$, where $b(\cdot)$ belongs to the class of VMO functions and $c(\cdot)$ is merely a measurable bounded function. For the precise assumptions we refer to Section 2 below. As usual we consider weak solutions u of (1.1), meaning that they belong to a certain parabolic Sobolev space, in particular $|Du|^{p(\cdot)} \in L^1$, see Definition 2.1 below. The existence of such weak solutions is ensured by a result of Antontsev and Shmarev [7, 8]. Our intention in this paper is to establish a local Calderón-Zygmund theory for weak solutions to (1.1) and can be summarized as follows: For any $q > 1$ we prove the inclusion

$$|F|^{p(\cdot)} \in L_{\text{loc}}^q(\Omega_T) \implies |Du|^{p(\cdot)} \in L_{\text{loc}}^q(\Omega_T)$$

together with quantitative gradient estimates. At this stage we mention that when the exponent $p(\cdot)$ is constant the result has been obtained in [4]; the stationary case of our result

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has been treated in [3] before. We stress that, apart from the parabolic scaling deficit, our quantitative gradient estimate is homogeneous in the sense that the constant is independent of the $p(\cdot)$ -energy of Du . To our knowledge this fact is new even for the stationary elliptic case.

Parabolic systems of the type considered in (1.1) are simplified versions of systems arising in the mathematical modeling of certain phenomena in fluid dynamics, such as models for non-Newtonian fluids, especially electro-rheological fluids. The peculiarity of these fluids is that their viscosity strongly depends on the external electromagnetic field and therefore varies in space and time. A mathematical model for electro-rheological fluids has been developed by Růžička in [31] and admits a $p(x, t)$ -growth structure in the non linear diffusion term. For simplified versions of this model partial regularity results can be found in [5, 21] and for the stationary case in [2, 13]. Other applications (in the case of equations) are models for flows in porous media [6, 25].

Compared to the stationary case there are only a few regularity results for parabolic problems with non-standard growth. The first one we mention, which in turn is the starting point for almost any other regularity result in this area, is the self improving property of higher integrability, i.e. the existence of some $\varepsilon > 0$, depending only on the structural constants, such that

$$|Du|^{p(\cdot)} \in L_{\text{loc}}^{1+\varepsilon}(\Omega_T).$$

This result was first established in the case of the $p(x, t)$ -Laplacian equation by Antontsev and Zhikov [9], and later for a quite general class of parabolic systems with $p(x, t)$ -growth independently by Zhikov and Pastukhova [33] and Bögelein and Duzaar [10]. With regard to Hölder regularity, Chen and Xu [15] proved in the scalar case that weak solutions of the parabolic $p(x, t)$ -Laplacian equation are locally bounded and Hölder continuous. The local Hölder continuity of the spatial gradient Du for the parabolic $p(x, t)$ -Laplacian system has recently been established by Bögelein and Duzaar [11]. As already mentioned before, for more general parabolic systems with non-standard growth partial regularity results can be found in [5, 21].

The history of Calderón-Zygmund estimates for non linear problems starts in the elliptic setting. The result for the p -Laplacian equation, i.e. the scalar case $N = 1$, has been obtained by Iwaniec [26], while the vectorial case $N > 1$ has been treated by DiBenedetto and Manfredi [20]. The extension to elliptic equations with VMO coefficients has been achieved by Kinnunen and Zhou [29]. General elliptic equations, also involving non-standard growth conditions, have been treated by Acerbi and Mingione [3] who built on previous ideas of Caffarelli and Peral [14] valid for homogeneous equations with highly oscillating coefficients. For the case of higher order systems with non-standard growth conditions we refer to Habermann [24]. The anisotropic character of the evolutionary p -Laplacian system makes impossible to use the elliptic techniques in the parabolic setting. Indeed it was initially not clear how to transfer such results to the parabolic setting. The result has finally been achieved by Acerbi and Mingione [4] who introduced the necessary new tools for developing a local Calderón-Zygmund theory for the time dependent, parabolic case (see also Misawa [30] for the special case $F \in \text{BMO}$). Later on, extensions to general parabolic systems have been obtained by Duzaar, Mingione and Steffen [22] while a Calderón-Zygmund theory for evolutionary obstacle problems can be found in [12, 32].

The main difficulty when considering the time dependent parabolic case comes from the non homogeneous scaling behavior of the system, in the sense that the solution multiplied by a constant is in general not anymore a solution. Note that this problem appears already in the standard growth case when $p(\cdot) \equiv p$ with $p \neq 2$. As a consequence, all local

estimates for the solution (such as energy estimates or reverse Hölder inequalities) become inhomogeneous and the use of maximal operators, which are typically used in the proof of Calderón-Zygmund estimates, becomes delicate. The technique how to overcome this problem goes back to the pioneering work of DiBenedetto and Friedman [19]. The idea is to choose a system of parabolic cylinders different from the standard one, whose space-time scaling depends on the local behavior of the solution itself. In a certain sense this allows to rebalance the non homogeneous scaling of the parabolic p -Laplacian system. This technique is by now classical and the core of the proof of almost any regularity result for degenerate parabolic problems. The strategy is to find parabolic cylinders of the form

$$(1.2) \quad Q_\varrho^{(\lambda)}(z_o) := B_\varrho(x_o) \times (t_o - \lambda^{2-p}\varrho^2, t_o + \lambda^{2-p}\varrho^2), \quad z_o = (x_o, t_o)$$

such that the scaling parameter $\lambda > 0$ and the average of $|Du|^p$ over $Q_\varrho^{(\lambda)}(z_o)$ are coupled in the following way:

$$\int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^p dz \approx \lambda^p.$$

Such cylinders are called intrinsic cylinders or cylinders with intrinsic coupling. The delicate aspect in this coupling clearly relies in the fact that the value of the integral average must be comparable to the scaling factor λ , which itself is involved in the construction of its support. On such intrinsic cylinders, i.e. when $|Du|$ is comparable to λ in the above sense, the parabolic p -Laplacian system $\partial_t u = \operatorname{div}(|Du|^{p-2} Du)$ behaves in a certain sense like $\partial_t u = \lambda^{p-2} \Delta u$. Therefore, using intrinsic cylinders of the type $Q_\varrho^{(\lambda)}(z_o)$ we can rebalance the occurring multiplicative factor λ^{p-2} for instance by rescaling u in time by a factor λ^{2-p} .

When dealing with the case of non-standard growth the construction of such a uniform system of intrinsic cylinders is not anymore possible, since the exponent p appears in the scaling parameter λ^{2-p} and therefore the scaling will depend on the particular point z_o . This means that the scaling of the intrinsic cylinder will in fact depend on space and time, so that we have to deal with a *non uniform intrinsic geometry*. More precisely, we consider cylinders of the type

$$(1.3) \quad Q_\varrho^{(\lambda)}(z_o) := B_\varrho(x_o) \times (t_o - \lambda^{\frac{2-p_o}{p_o}} \varrho^2, t_o + \lambda^{\frac{2-p_o}{p_o}} \varrho^2), \quad p_o := p(z_o),$$

with an intrinsic coupling of the form (where for simplicity we omit the role of the right-hand side F):

$$\int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(z)} dz \approx \lambda.$$

Note that compared to (1.2) we performed a change of parameter $\lambda^{p_o} \leftrightarrow \lambda$, so that the right-hand side is independent of p_o and hence independent of z_o . The main difficulty now comes from the fact that the heuristics we described above for the standard growth case do not apply for the case of non-standard growth, i.e. on $Q_\varrho^{(\lambda)}(z_o)$ the parabolic $p(x, t)$ -Laplacian system behaves like $\partial_t u = \lambda^{\frac{p(z)-2}{p(z)}} \Delta u$ such that the multiplicative factor $\lambda^{\frac{p(z)-2}{p(z)}}$ does not cancel out with the scaling factor $\lambda^{\frac{2-p_o}{p_o}}$. This problem will be solved by a parabolic localization argument which has its origin in [10], see Section 4.

Now, we briefly describe the strategy of proof of our main result. Since we have to work on a system of non uniform intrinsic cylinders of the type (1.3) there is no uniform maximal function available. For this reason we remove the use of any maximal operator in

the proof by a technique which goes back to [4]. Instead of maximal operators we construct a covering of the super level sets

$$\{|Du(z)|^{p(z)} > \lambda\}, \quad \lambda \gg 1$$

by exit cylinders $Q_{\varrho_i}^{(\lambda)}(z_i)$, $i = 1, \dots, \infty$ defined according to (1.3) on which we have

$$\int_{Q_{\varrho_i}^{(\lambda)}(z_i)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \approx \lambda.$$

Thereby, $M \gg 1$ is a suitably chosen parameter depending on the structural constants of the problem. Then, we know that

$$\int_{Q_{\varrho_i}^{(\lambda)}(z_i)} |Du|^{p(\cdot)} dz \lesssim \lambda \quad \text{and} \quad \int_{Q_{\varrho_i}^{(\lambda)}(z_i)} (|F| + 1)^{p(\cdot)} dz \lesssim \frac{\lambda}{M}.$$

Therefore, if M is large u solves approximately (here we suppose $a \equiv 1$ for simplicity)

$$\partial_t u - \operatorname{div}(|Du|^{p(z)-2} Du) \approx 0 \quad \text{on } Q_{\varrho_i}^{(\lambda)}(z_i).$$

This heuristic suggests to compare u to the solution w of

$$\begin{cases} \partial_t w - \operatorname{div}(|Dw|^{p(z_i)-2} Dw) = 0 & \text{in } Q_{\varrho_i}^{(\lambda)}(z_i), \\ w = u & \text{on } \partial_{\mathcal{P}} Q_{\varrho_i}^{(\lambda)}(z_i). \end{cases}$$

To be precise, this will be done in a two step comparison argument. Here, we stress that the comparison argument strongly relies on the parabolic localization technique, since we replaced the variable exponent $p(z) - 2$ by the constant exponent $p(z_i) - 2$. The advantage now is that the theory of DiBenedetto and Friedman [18] ensures that Dw satisfies an a priori L^∞ -estimate. Via the comparison argument this L^∞ -estimate can be transferred into estimates for Du on the super level sets. At this stage the final result follows by a standard argument using Fubini's theorem.

2. STATEMENT OF THE RESULT

As already mentioned before, Ω will be a bounded open set in \mathbb{R}^n , $n \geq 2$, and $\Omega_T := \Omega \times (0, T)$, $T > 0$ will denote the space-time cylinder over Ω . Moreover, we shall consider an exponent function $p : \Omega_T \rightarrow (\frac{2n}{n+2}, \infty)$. For $k \in \mathbb{N}$ we define $L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$ to be the set of those measurable functions $v : \Omega_T \rightarrow \mathbb{R}^k$ such that $|v|^{p(\cdot)} \in L^1(\Omega_T)$, i.e.

$$L^{p(\cdot)}(\Omega_T, \mathbb{R}^k) := \left\{ v : \Omega_T \rightarrow \mathbb{R}^k : \int_{\Omega_T} |v|^{p(\cdot)} dz < \infty \right\}.$$

As usual we shall deal with weak solutions to (1.1) which are specified in the following

Definition 2.1. *We identify a map $u \in L^2(\Omega_T, \mathbb{R}^N) \cap L^1(0, T; W^{1,1}(\Omega, \mathbb{R}^N))$ as a weak solution of the parabolic system (1.1) if and only if $Du \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^{Nn})$ and*

$$(2.1) \quad \int_{\Omega_T} u \cdot \varphi_t - \langle a(\cdot) |Du|^{p(\cdot)-2} Du, D\varphi \rangle dz = \int_{\Omega_T} \langle |F|^{p(\cdot)-2} F, D\varphi \rangle dz$$

holds for every test function $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$.

Since our problem is of local nature, it is not restrictive to assume the existence of γ_1, γ_2 such that

$$(2.2) \quad \frac{2n}{n+2} < \gamma_1 \leq p(z) \leq \gamma_2 < \infty \quad \text{for all } z \in \Omega_T.$$

The lower bound $\gamma_1 > 2n/(n+2)$ is unavoidable even in the constant exponent case $p(\cdot) \equiv p$, cf. [17, Chapters 5, 8]. With respect to the regularity of $p(\cdot)$ we will assume the following strong logarithmic continuity condition to hold:

$$(2.3) \quad |p(z) - p(\tilde{z})| \leq \omega(d_{\mathcal{P}}(z, \tilde{z})) \quad \text{for any } z, \tilde{z} \in \Omega_T,$$

where the parabolic metric $d_{\mathcal{P}}$ is given by

$$d_{\mathcal{P}}(z, \tilde{z}) := \max \left\{ |x - \tilde{x}|, \sqrt{|t - \tilde{t}|} \right\} \quad \text{for } z = (x, t), \tilde{z} = (\tilde{x}, \tilde{t}) \in \mathbb{R}^{n+1}$$

and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ is a nondecreasing modulus of continuity satisfying

$$(2.4) \quad \limsup_{\varrho \downarrow 0} \omega(\varrho) \log\left(\frac{1}{\varrho}\right) = 0$$

an assumptions which has been first considered in [1] and then used in [3] and [13]. By virtue of (2.4) we may assume that there exists $R_1 \in (0, 1]$ depending on $\omega(\cdot)$ such that

$$(2.5) \quad \omega(\varrho) \log\left(\frac{1}{\varrho}\right) \leq 1 \quad \text{for all } \varrho \in (0, R_1].$$

For the coefficient function $a : \Omega_T \rightarrow \mathbb{R}$ we will assume its measurability and that

$$(2.6) \quad \nu \leq a(z) \leq L \quad \text{for any } z \in \Omega_T$$

holds with some constants $0 < \nu \leq 1 \leq L$. With regard to its regularity, we will only assume that it satisfies a VMO condition with respect the spatial variable. More precisely, denoting

$$(a)_{x_o, \varrho}(t) := \int_{B_{\varrho}(x_o)} a(x, t) dx \quad \text{for } B_{\varrho}(x_o) \subset \Omega,$$

we assume that there exists $\tilde{\omega} : [0, \infty) \rightarrow [0, 1]$ such that

$$(2.7) \quad \sup_{B_{\varrho}(x_o) \subset \Omega, 0 < \varrho \leq r} \int_{B_{\varrho}(x_o)} |a(x, t) - (a)_{x_o, \varrho}(t)| dx \leq \tilde{\omega}(r)$$

for a.e. $t \in (0, T)$ any $r > 0$ and

$$(2.8) \quad \lim_{r \downarrow 0} \tilde{\omega}(r) = 0.$$

Here, we stress that with respect to time we assume not more than measurability. Moreover, our assumptions on a allow product coefficients of the type $a(x, t) = b(x)c(t)$, with $b \in VMO(\Omega) \cap L^{\infty}(\Omega)$ and $c \in L^{\infty}(0, T)$. Now we are ready to state our main result.

Theorem 2.2. *Let u be a weak solution of the parabolic system (1.1) where $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$ and $a : \Omega_T \rightarrow [\nu, L]$ satisfy the assumptions (2.2) – (2.4) and (2.6) – (2.8). Moreover, assume that $|F|^{p(\cdot)} \in L^q_{\text{loc}}(\Omega_T)$ for some $q > 1$. Then we have*

$$|Du|^{p(\cdot)} \in L^q_{\text{loc}}(\Omega_T).$$

Moreover, for $K \geq 1$ there exist a radius $R_o = R_o(n, N, \nu, L, \gamma_1, \gamma_2, K, \omega(\cdot), \tilde{\omega}(\cdot), q) > 0$ and a constant $c = c(n, N, \nu, L, \gamma_1, \gamma_2, q)$ such that the following holds: If

$$(2.9) \quad \int_{\Omega_T} |Du|^{p(\cdot)} + (|F| + 1)^{p(\cdot)} dz \leq K,$$

then for every parabolic cylinder $Q_{2R} \equiv Q_{2R}(z_o) \Subset \Omega_T$ with $R \in (0, R_o]$, there holds

$$\int_{Q_R} |Du|^{p(\cdot)q} dz$$

$$(2.10) \quad \leq c \left[\int_{Q_{2R}} |Du|^{p(\cdot)} dz + \left(\int_{Q_{2R}} (|F| + 1)^{p(\cdot)q} dz \right)^{\frac{1}{q}} \right]^{1+d(p_o)(q-1)},$$

where we denoted

$$(2.11) \quad d(p_o) := \begin{cases} \frac{p_o}{2} & \text{if } p_o \geq 2, \\ \frac{2p_o}{p_o(n+2) - 2n} & \text{if } p_o < 2, \end{cases} \quad \text{with } p_o := p(z_o).$$

We note that the constant c in Theorem 2.2 remains stable when $q \downarrow 1$ and it blows up, i.e. $c \rightarrow \infty$ when $q \rightarrow \infty$. Moreover, due to an improved localization technique we are able to prove the gradient estimate (2.10) with a constant c independent of the parameter K , which was even not known in the elliptic case, cf. [3, 24].

Remark 2.3. The same result holds true if we assume, instead of the VMO condition (2.8), that the BMO seminorm of a with respect to x is small, i.e. that

$$[a]_{\text{BMO}} := \sup_{r>0} \tilde{\omega}(r) \leq \varepsilon_{\text{BMO}}$$

with some constant $\varepsilon_{\text{BMO}} > 0$ depending on $n, N, \nu, L, \gamma_1, \gamma_2, q$.

3. PRELIMINARIES AND NOTATION

3.1. Notation. For a point $z_o \in \mathbb{R}^{n+1}$ we shall always write $z_o = (x_o, t_o)$ with $x_o \in \mathbb{R}^n$ and $t_o \in \mathbb{R}$ and we shall consider – as we did for instance in the statement of Theorem 2.2 – symmetric parabolic cylinders around z_o of the form $Q_\varrho(z_o) := B_\varrho(x_o) \times (t_o - \varrho^2, t_o + \varrho^2)$. Moreover, in the course of the proof of our main result, in order to rebalance the non homogeneity of the parabolic system, we shall also deal with scaled cylinders of the form

$$(3.1) \quad Q_\varrho^{(\lambda)}(z_o) := B_\varrho(x_o) \times \Lambda_\varrho^{(\lambda)}(z_o),$$

where $\lambda > 0$ and

$$\Lambda_\varrho^{(\lambda)}(z_o) := (t_o - \lambda^{\frac{2-p_o}{p_o}} \varrho^2, t_o + \lambda^{\frac{2-p_o}{p_o}} \varrho^2).$$

In any case, when considering a certain cylinder $Q_\varrho^{(\lambda)}(z_o)$ with center z_o , by p_o we denote the value of $p(\cdot)$ at the center of the cylinder, i.e. $p_o \equiv p(z_o)$. Note that such a system of scaled cylinders is non-uniform in the sense that the scaling $\lambda^{\frac{2-p_o}{p_o}}$ depends on the particular point z_o via $p_o \equiv p(z_o)$. In the particular case $\lambda = 1$ the cylinders $Q_\varrho^{(1)}(z_o)$ reduce to the standard parabolic ones, i.e. $Q_\varrho^{(1)}(z_o) \equiv Q_\varrho(z_o)$. By $\chi Q_\varrho^{(\lambda)}(z_o)$, for a constant $\chi > 1$, we denote the χ -times enlarged cylinder, i.e. $\chi Q_\varrho^{(\lambda)}(z_o) := Q_{\chi\varrho}^{(\lambda)}(z_o)$. For $g \in L^1_{\text{loc}}(C, \mathbb{R}^k)$, with $C \subset \Omega_T$ of strictly positive measure, we shall write

$$(g)_C := \int_C g(z) dz := \frac{1}{|C|} \int_C g(z) dz$$

for the mean value of g on C . Finally, α_n denotes the Lebesgue measure of the unit ball $B_1(0)$ in \mathbb{R}^n . We will denote with c a generic constant greater than one, possibly varying from line to line. We will highlight the dependencies of the constants in between parentheses. For shortness of notation we will denote by the word *data* exactly the set of parameters $n, N, \nu, L, \gamma_1, \gamma_2$, so that writing $c(\text{data}, M)$ we will mean that the constant c depends on $n, N, \nu, L, \gamma_1, \gamma_2$ and moreover upon M . Constants we need to recall will be denoted with special symbols, such as $c_{D_i B}, \tilde{c}, c_*, c_\ell$.

3.2. Preliminaries. The following lemma is a standard iteration lemma and can for instance be found in [23, Lemma 6.1].

Lemma 3.1. *Let $\phi : [R, 2R] \rightarrow [0, \infty)$ be a function such that*

$$\phi(r_1) \leq \frac{1}{2}\phi(r_2) + \mathcal{A} + \frac{\mathcal{B}}{(r_2 - r_1)^\beta} \quad \text{for every } R \leq r_1 < r_2 \leq 2R,$$

where $\mathcal{A}, \mathcal{B} \geq 0$ and $\beta > 0$. Then

$$\phi(R) \leq c(\beta) \left[\mathcal{A} + \frac{\mathcal{B}}{R^\beta} \right].$$

The next lemma is a useful tool when dealing with p -growth problems. The continuous dependence of the constant with respect to p allows to instead consider a constant depending on γ_1, γ_2 when $p \in [\gamma_1, \gamma_2]$.

Lemma 3.2. *Let $p \in [\gamma_1, \gamma_2]$. Then there exists a constant $c \equiv c(n, N, \gamma_1, \gamma_2)$ such that for any $A, B \in \mathbb{R}^{Nn}$, not both zero, there holds*

$$(|A|^2 + |B|^2)^{(p-2)/2} |B - A|^2 \leq c \langle |B|^{p-2} B - |A|^{p-2} A, B - A \rangle.$$

The following Lemma can be deduced from [16, Lemma 2.2]. Note that the dependence of the constant on γ_2 instead of p can be deduced from the proof of the lemma.

Lemma 3.3. *Let $p \in [\gamma_1, \gamma_2]$. Then there exists a constant $c_\ell \equiv c_\ell(\gamma_2)$ such that for any $A, B \in \mathbb{R}^{Nn}$ there holds*

$$|A|^p \leq c_\ell |B|^p + c_\ell (|A|^2 + |B|^2)^{\frac{p-2}{2}} |B - A|^2.$$

Finally, we state a useful estimate which is a consequence of Iwaniec's inequality for Orlicz spaces [27], see also [3, inequality (28)]. Let $\beta > 0$, $Q \subset \mathbb{R}^{n+1}$ and $g \in L^\sigma(Q)$ for some $\sigma > 1$. Then, there holds

$$(3.2) \quad \int_Q |g| \log^\beta \left(e + \frac{|g|}{(g)_Q} \right) dz \leq c(\sigma, \beta) \left(\int_Q |g|^\sigma dz \right)^{\frac{1}{\sigma}} \quad \text{for all } \sigma > 1.$$

Thereby, the constant $c(\sigma, \beta)$ blows up when $\sigma \downarrow 1$. Moreover, $c(\sigma, \beta)$ depends continuously on β and therefore it can be replaced by a constant $c(\sigma, \gamma_1, \gamma_2)$ if $\beta \in [\gamma_2', \gamma_1']$.

4. NON UNIFORM INTRINSIC GEOMETRY

In the following lemma we provide a parabolic localization technique. Obviously the difficulty stems from the necessity to couple the technique of intrinsic geometry with the localization needed to treat the variable exponent growth conditions. As we already pointed out in the introduction, this will be achieved by a *non uniform intrinsic geometry*, i.e. a system of cylinders as defined in (3.1) whose scaling depends on the particular point considered. Most of this technique goes back to [10].

Lemma 4.1. *Let $\kappa, K, M \geq 1$ and $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$ satisfy (2.3) and (2.5). Then there exists a radius $\varrho_o \equiv \varrho_o(n, \gamma_1, \kappa, K, M, \omega(\cdot)) \in (0, R_1]$ such that the following holds: whenever $Du, F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^{Nn})$ satisfy (2.9) and $Q_\varrho^{(\lambda)}(z_o) \subset \Omega_T$ is a parabolic cylinder with $\varrho \in (0, \varrho_o]$ and $\lambda \geq 1$ such that*

$$(4.1) \quad \lambda \leq \kappa \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz,$$

then we have

$$(4.2) \quad \lambda \leq \left(\frac{\Gamma}{4\varrho^{n+2}} \right)^{\frac{p_o}{2}}, \quad p_2 - p_1 \leq \omega(\Gamma\varrho^\alpha) \quad \text{and} \quad \lambda^{\omega(\Gamma\varrho^\alpha)} \leq e^{\frac{3np_o}{\alpha}},$$

where

$$p_o := p(z_o), \quad p_1 := \inf_{Q_\varrho^{(\lambda)}(z_o)} p(\cdot), \quad p_2 := \sup_{Q_\varrho^{(\lambda)}(z_o)} p(\cdot)$$

and

$$(4.3) \quad \Gamma := 4\beta_n \kappa K M, \quad \beta_n := \max\{1, (2\alpha_n)^{-1}\}, \quad \alpha := \min\left\{1, \gamma_1 \frac{n+2}{4} - \frac{n}{2}\right\}.$$

Proof. We first deduce from (4.1), (2.9) (recall that $Q_\varrho^{(\lambda)}(z_o) \subset \Omega_T$) and the definitions of Γ and β_n in (4.3) the following bound for λ :

$$\lambda \leq \frac{\kappa K M}{|Q_\varrho^{(\lambda)}(z_o)|} = \frac{\kappa K M}{2\alpha_n \varrho^{n+2}} \lambda^{\frac{p_o-2}{p_o}} \leq \frac{\beta_n \kappa K M}{\varrho^{n+2}} \lambda^{\frac{p_o-2}{p_o}} = \frac{\Gamma}{4\varrho^{n+2}} \lambda^{\frac{p_o-2}{p_o}}.$$

Rewriting this inequality we obtain (4.2)₁. Now, we come to the proof of (4.2)₂. We define

$$(4.4) \quad \varrho_o := R_1^{\frac{1}{\alpha}} \Gamma^{-\frac{2}{\alpha}} \leq R_1 \leq 1$$

and assume that $\varrho \leq \varrho_o$. Keeping in mind the definition of α and Γ this determines ϱ_o as a constant depending on $n, \gamma_1, K, M, \kappa, \omega(\cdot)$. From (2.3) and the fact that $\lambda \geq 1$ we obtain the following preliminary bound for the oscillation of $p(\cdot)$ on $Q_\varrho^{(\lambda)}(z_o)$:

$$p_2 - p_1 \leq \omega(2\varrho + \sqrt{2}\lambda^{\frac{2-p_o}{2p_o}}\varrho) \leq \omega(2\varrho + \sqrt{2}\lambda^{\frac{2-\gamma_1}{2p_o}}\varrho).$$

In the case $\gamma_1 \geq 2$ this leads us to

$$p_2 - p_1 \leq \omega(4\varrho),$$

while in the case $\frac{2n}{n+2} < \gamma_1 < 2$ we infer from (4.2)₁ that

$$p_2 - p_1 \leq \omega\left(4\lambda^{\frac{2-\gamma_1}{2p_o}}\varrho\right) \leq \omega\left(4\left(\frac{\Gamma}{4}\right)^{\frac{2-\gamma_1}{4}}\varrho^{1-\frac{(2-\gamma_1)(n+2)}{4}}\right) \leq \omega\left(\Gamma\varrho^{\gamma_1 \frac{n+2}{4} - \frac{n}{2}}\right).$$

Note that the restriction $\gamma_1 > \frac{2n}{n+2}$ ensures that the exponent of ϱ is positive, i.e. $\gamma_1 \frac{n+2}{4} - \frac{n}{2} > 0$. Combining the estimates from the cases $\gamma_1 \geq 2$ and $\gamma_1 < 2$ and recalling that $\varrho \leq 1$ we arrive at:

$$p_2 - p_1 \leq \omega(\Gamma\varrho^\alpha),$$

which proves (4.2)₂. Finally, we come to the proof of (4.2)₃. Using the definition of ϱ_o in (4.4) and the logarithmic bound (2.5) (which is applicable since $R_1/\Gamma \leq R_1$) we obtain

$$\Gamma^{\omega(\Gamma\varrho^\alpha)} \leq \Gamma^{\omega(\Gamma\varrho_o^\alpha)} \leq \Gamma^{\omega(R_1/\Gamma)} \leq \left(\frac{\Gamma}{R_1}\right)^{\omega(R_1/\Gamma)} = \exp\left[\omega\left(\frac{R_1}{\Gamma}\right) \log\left(\frac{\Gamma}{R_1}\right)\right] \leq e.$$

Moreover, by a similar reasoning and using the last inequality we get

$$\begin{aligned} \varrho^{-\omega(\Gamma\varrho^\alpha)} &= \Gamma^{\frac{\omega(\Gamma\varrho^\alpha)}{\alpha}} (\Gamma\varrho^\alpha)^{-\frac{\omega(\Gamma\varrho^\alpha)}{\alpha}} \leq e^{\frac{1}{\alpha}} (\Gamma\varrho^\alpha)^{-\frac{\omega(\Gamma\varrho^\alpha)}{\alpha}} \\ &= e^{\frac{1}{\alpha}} \exp\left[\frac{\omega(\Gamma\varrho^\alpha)}{\alpha} \log\frac{1}{\Gamma\varrho^\alpha}\right] \leq e^{\frac{2}{\alpha}}. \end{aligned}$$

At this stage (4.2)₃ follows from (4.2)₁ and the previous two inequalities since

$$\lambda^{\omega(\Gamma\varrho^\alpha)} \leq (\Gamma\varrho^{-(n+2)})^{\frac{p_o\omega(\Gamma\varrho^\alpha)}{2}} \leq e^{\frac{p_o}{2} + \frac{p_o(n+2)}{\alpha}} \leq e^{\frac{3np_o}{\alpha}}.$$

This completes the proof of the lemma. \square

Since the family of intrinsic cylinders is non uniform – in the sense that the scaling depends on the center of the cylinder – we need the following non uniform version of Vitali’s covering theorem, which can be found in [10, Lemma 7.1]. Note that we may choose $L_1 = 1$ due to assumption (2.5) and that we replaced M by KM which is more suitable in our setting.

Lemma 4.2. *Let $K, M, \lambda \geq 1$ and let $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$ fulfill assumptions (2.3) and (2.5). Then there exists $\chi \equiv \chi(n, \gamma_1) \geq 5$ and $\varrho_1 = \varrho_1(n, \gamma_1, K, M) \in (0, 1]$ such that the following is true: Let $\mathcal{F} = \{Q_i\}_{i \in \mathcal{I}}$ be a family of axially parallel parabolic cylinders of the form*

$$Q_i \equiv Q_{\varrho_i}^{(\lambda)}(z_i) \equiv B_{\varrho_i}(x_i) \times \left(t_i - \lambda \frac{2-p(z_i)}{p(z_i)} \varrho_i^2, t_i + \lambda \frac{2-p(z_i)}{p(z_i)} \varrho_i^2 \right)$$

with uniformly bounded radii, in the sense that there holds

$$(4.5) \quad \varrho_i \leq \min \left\{ \varrho_1, \left[\beta_n K M \lambda^{-\frac{2}{p(z_i)}} \right]^{\frac{1}{n+2}} \right\} \quad \forall i \in \mathcal{I}$$

with β_n defined in (4.3). Then there exists a countable subcollection $\mathcal{G} \subset \mathcal{F}$ of disjoint parabolic cylinders, such that

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{Q \in \mathcal{G}} \chi Q.$$

5. HIGHER INTEGRABILITY

In this Section we provide a higher integrability result for solutions to homogeneous parabolic $p(x, t)$ -Laplacian systems that will be crucial later in the proof of certain comparison estimates. We consider the parabolic system

$$(5.1) \quad \partial_t v - \operatorname{div} (a(z) |Dv|^{p(z)-2} Dv) = 0 \quad \text{on } A \times (t_1, t_2) =: \mathfrak{A},$$

where $A \subset \mathbb{R}^n$ is an open set and $t_1 < t_2$. Then, we have the following higher integrability result from [10, Theorem 2.2].

Theorem 5.1. *Suppose that $p : \mathfrak{A} \rightarrow [\gamma_1, \gamma_2]$ satisfies (2.3) and (2.5) and that $a : \mathfrak{A} \rightarrow \mathbb{R}$ satisfies (2.6). Then there exists $\varepsilon_o \equiv \varepsilon_o(\text{data}) > 0$ such that the following holds: whenever a function $v \in L^2(\mathfrak{A}, \mathbb{R}^N) \cap L^1(t_1, t_2; W^{1,1}(A, \mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(\mathfrak{A}, \mathbb{R}^{Nn})$ is a weak solution to the parabolic system (5.1) on \mathfrak{A} , we have that*

$$(5.2) \quad Dv \in L_{\text{loc}}^{p(\cdot)(1+\varepsilon_o)}(\mathfrak{A}, \mathbb{R}^{Nn}).$$

Moreover, for any $K \geq 1$ there exists a radius $\varrho_2 \equiv \varrho_2(n, \gamma_1, \gamma_2, K, \omega(\cdot)) \in (0, R_1]$ such that there holds: If

$$(5.3) \quad \int_{\mathfrak{A}} (|Dv| + 1)^{p(\cdot)} dz \leq K$$

and $\varepsilon \in (0, \varepsilon_o]$, then for any parabolic cylinder $Q_{2\varrho}(z_o) \subset \mathfrak{A}$ with $\varrho \in (0, \varrho_2]$ we have

$$(5.4) \quad \int_{Q_\varrho(z_o)} |Dv|^{p(\cdot)(1+\varepsilon)} dz \leq c \left(\int_{Q_{2\varrho}(z_o)} |Dv|^{p(\cdot)} dz \right)^{1+\varepsilon d(p(z_o))} + c$$

for a constant $c \equiv c(\text{data})$ and with $d(\cdot)$ defined in (2.11).

Note that the quantitative higher integrability estimate (5.4) is non homogeneous, in the sense that the exponents of $|Dv|$ on both sides of the inequality are different. In the following Corollary we deduce a homogeneous version of this estimate valid on intrinsic cylinders of the type (5.5). In order to understand that inequality (5.7) is homogeneous one has to interpret $\lambda \approx \int |Dv|^{p(\cdot)} dz$ in a heuristic sense which will become clear later on.

Corollary 5.2. *Let $K, c_*, \hat{c} \geq 1$ and suppose that $p: \mathfrak{A} \rightarrow [\gamma_1, \gamma_2]$ satisfies (2.3) and (2.5) and that $a: \mathfrak{A} \rightarrow \mathbb{R}$ fulfills (2.6). Then, there exist $\varepsilon_o \equiv \varepsilon_o(\text{data}) > 0$, $c \equiv c(\text{data}, c_*, \hat{c}) \geq 1$ and $\varrho_2 \equiv \varrho_2(n, \gamma_1, \gamma_2, K, \omega(\cdot)) \in (0, R_1]$ such that the following holds: whenever $v \in L^2(\mathfrak{A}, \mathbb{R}^N) \cap L^1(t_1, t_2; W^{1,1}(A, \mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(\mathfrak{A}, \mathbb{R}^{Nn})$ is a weak solution to the parabolic system (5.1) satisfying (5.3) and*

$$(5.5) \quad \int_{Q_{2\varrho}^{(\lambda)}(z_o)} |Dv|^{p(\cdot)} dz \leq c_* \lambda$$

for some cylinder $Q_{2\varrho}^{(\lambda)}(z_o) \subset \mathfrak{A}$ with $\varrho \in (0, \varrho_2]$ and $\lambda \geq 1$ satisfying

$$(5.6) \quad \lambda^{p_2 - p_1} \leq \hat{c}, \quad \text{where } p_1 := \inf_{Q_{2\varrho}^{(\lambda)}(z_o)} p(\cdot), \quad p_2 := \sup_{Q_{2\varrho}^{(\lambda)}(z_o)} p(\cdot),$$

then we have (5.2) and

$$(5.7) \quad \int_{Q_{\varrho}^{(\lambda)}(z_o)} |Dv|^{p(\cdot)(1+\varepsilon_o)} dz \leq c \lambda^{1+\varepsilon_o}.$$

Proof. Without loss of generality we assume that $z_o = 0$. We let ε_o and ϱ_2 be the constants appearing in Theorem 5.1. The strategy now is to rescale the problem from $Q_{\varrho}^{(\lambda)}, Q_{2\varrho}^{(\lambda)}$ to the standard parabolic cylinders $Q_{\varrho}, Q_{2\varrho}$ via a transformation in time and then apply Theorem 5.1. We start with the case $p_o := p(0) \geq 2$ and define for $(x, t) \in Q_{2\varrho}$ the rescaled exponent

$$\tilde{p}(x, t) := p\left(x, \lambda^{\frac{2-p_o}{p_o}} t\right),$$

the rescaled function

$$\tilde{v}(x, t) := \lambda^{-\frac{1}{p_o}} v\left(x, \lambda^{\frac{2-p_o}{p_o}} t\right)$$

and the rescaled coefficient

$$\tilde{a}(x, t) := \lambda^{\frac{\tilde{p}(x,t)-p_o}{p_o}} a\left(x, \lambda^{\frac{2-p_o}{p_o}} t\right).$$

Then, \tilde{v} is a weak solution of the parabolic system

$$(5.8) \quad \partial_t \tilde{v} - \operatorname{div}(\tilde{a}(\cdot) |D\tilde{v}|^{\tilde{p}(\cdot)-2} D\tilde{v}) = 0 \quad \text{in } Q_{2\varrho}.$$

In order to apply the higher integrability Theorem 5.1 to \tilde{v} we have to ensure that the hypotheses on \tilde{p} and \tilde{a} are satisfied. Since $p_o \geq 2$ and $\lambda \geq 1$ we have

$$(5.9) \quad \begin{aligned} |\tilde{p}(x_1, t_1) - \tilde{p}(x_2, t_2)| &= \left| p\left(x_1, \lambda^{\frac{2-p_o}{p_o}} t_1\right) - p\left(x_2, \lambda^{\frac{2-p_o}{p_o}} t_2\right) \right| \\ &\leq \omega\left(\max\{|x_1 - x_2|, \lambda^{\frac{2-p_o}{2p_o}} \sqrt{|t_1 - t_2|}\}\right) \\ &\leq \omega\left(\max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}\right) \\ &= \omega\left(d_{\mathcal{P}}((x_1, t_1), (x_2, t_2))\right). \end{aligned}$$

Moreover by (2.6) and (5.6) it holds that

$$(5.10) \quad \frac{\nu}{\hat{c}} \leq \nu \lambda^{-\frac{p_2-p_1}{p_o}} \leq \tilde{a}(x, t) \leq L \lambda^{\frac{p_2-p_1}{p_o}} \leq \hat{c} L.$$

Therefore, we are allowed to apply Theorem 5.1 with $(\nu/\hat{c}, \hat{c}L)$ instead of (ν, L) to the function \tilde{v} on $Q_\varrho, Q_{2\varrho}$ to infer that $D\tilde{v} \in L_{\text{loc}}^{p(\cdot)(1+\varepsilon_o)}(Q_{2\varrho}, \mathbb{R}^{Nn})$ and moreover the following quantitative estimate holds:

$$\int_{Q_\varrho} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\varepsilon_o)} dz \leq c \left(\int_{Q_{2\varrho}} |D\tilde{v}|^{\tilde{p}(\cdot)} dz \right)^{1+\varepsilon_o d(p_o)} + c$$

for a constant $c \equiv c(\text{data})$. Note that $p_o = p(0) = \tilde{p}(0)$. Scaling back from v to \tilde{v} and back and using the preceding estimate, (5.5) and (5.6) several times we find that

$$\begin{aligned} \int_{Q_\varrho^{(\lambda)}} |Dv|^{p(\cdot)(1+\varepsilon_o)} dz &= \int_{Q_\varrho} \lambda^{\frac{\tilde{p}(\cdot)}{p_o}(1+\varepsilon_o)} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\varepsilon_o)} dz \\ &\leq c \lambda^{1+\varepsilon_o} \int_{Q_\varrho} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\varepsilon_o)} dz \\ &\leq c \lambda^{1+\varepsilon_o} \left(\int_{Q_{2\varrho}} |D\tilde{v}|^{\tilde{p}(\cdot)} dz \right)^{1+\varepsilon_o d(p_o)} + c \lambda^{1+\varepsilon_o} \\ &= c \lambda^{1+\varepsilon_o} \left(\int_{Q_{2\varrho}^{(\lambda)}} \lambda^{-\frac{p(\cdot)}{p_o}} |Dv|^{p(\cdot)} dz \right)^{1+\varepsilon_o d(p_o)} + c \lambda^{1+\varepsilon_o} \\ &\leq c \lambda^{\varepsilon_o(1-d(p_o))} \left(\int_{Q_{2\varrho}^{(\lambda)}} |Dv|^{p(\cdot)} dz \right)^{1+\varepsilon_o d(p_o)} + c \lambda^{1+\varepsilon_o} \\ (5.11) \quad &\leq c(\text{data}, c_*, \hat{c}) \lambda^{1+\varepsilon_o}. \end{aligned}$$

This proves the lemma in the case $p_o \geq 2$. In the case $p_o < 2$ we define similarly as above

$$\tilde{p}(x, t) := p\left(\lambda^{\frac{p_o-2}{2p_o}} x, t\right), \quad \tilde{v}(x, t) := \lambda^{-\frac{1}{2}} v\left(\lambda^{\frac{p_o-2}{2p_o}} x, t\right)$$

and

$$(5.12) \quad \tilde{a}(x, t) := \lambda^{\frac{\tilde{p}(x,t)-p_o}{p_o}} a\left(\lambda^{\frac{p_o-2}{2p_o}} x, t\right)$$

for $(x, t) \in Q_{2\tilde{\varrho}}$, where $\tilde{\varrho} := \lambda^{\frac{2-p_o}{2p_o}} \varrho$. A straightforward computation shows that

$$D\tilde{v}(x, t) = \lambda^{-\frac{1}{p_o}} Dv\left(\lambda^{\frac{p_o-2}{2p_o}} x, t\right) \quad \text{in } Q_{2\tilde{\varrho}}$$

and that \tilde{v} is a weak solution of the system (5.8) in $Q_{2\tilde{\varrho}}$, where $\tilde{a}, \tilde{v}, \tilde{p}$ are this time the quantities defined just above. Notice that estimate (5.10) holds also for the vector field defined in (5.12), while the verification of (5.9) in this case is analogous to the previous one. Applying again Theorem 5.1 and repeating the computations in (5.11) we obtain the assertion of the lemma also in the case $p_o < 2$. \square

6. A PRIORI ESTIMATES

In the next Theorem we state the gradient bound of DiBenedetto and Friedman [17, 18] for parabolic standard growth problems. Later on, we will transfer these a priori estimates via a comparison argument to our non-standard growth problem. Therefore, in this Section we consider parabolic systems with constant p -growth of the type

$$(6.1) \quad w_t - \operatorname{div}(\tilde{a}(t)|Dw|^{p-2}Dw) = 0 \quad \text{on } A \times (t_1, t_2) =: \mathfrak{A},$$

with $p > 2n/(n+2)$ and $\tilde{a}: (t_1, t_2) \rightarrow \mathbb{R}$. Thereby, A is an open set in \mathbb{R}^n and $t_1 < t_2$. Moreover, we denote

$$C_\varrho^{(\lambda)}(z_o) := B_\varrho(x_o) \times (t_o - \lambda^{\frac{2-p}{p}} \varrho^2, t_o + \lambda^{\frac{2-p}{p}} \varrho^2).$$

Note that the scaling of this system of cylinders does not depend on the center z_o . Later on, we will apply the subsequent Theorem with the choice $p = p_o \equiv p(z_o)$, and hence the cylinders $C_\varrho^{(\lambda)}(z_o)$ will coincide with the ones defined in (3.1). As mentioned above, the next Theorem is a consequence of the gradient bounds proved in [18, 17]. The precise statement for the case $p \geq 2$ can be found in [4, Lemma 1] (replacing λ by $\lambda^{\frac{1}{p}}$), and for the case $2n/(n+2) < p < 2$ in [4, Lemma 2] (replacing ϱ by $\lambda^{\frac{p-2}{2}}\varrho$ and subsequently λ by $\lambda^{\frac{1}{p}}$).

Theorem 6.1. *Let $w \in C^0(t_1, t_2; L^2(A, \mathbb{R}^N)) \cap L^p(t_1, t_2; W^{1,p}(A, \mathbb{R}^N))$ be a weak solution to (6.1) in \mathfrak{A} with $\tilde{a}: (t_1, t_2) \rightarrow \mathbb{R}$ satisfying $\nu \leq \tilde{a} \leq L$ for some constants $0 < \nu \leq 1 \leq L$. Moreover suppose that*

$$\int_{C_{2\varrho}^{(\lambda)}(z_o)} |Dw|^p dz \leq c_* \lambda$$

holds for some cylinder $C_{2\varrho}^{(\lambda)}(z_o) \in \mathfrak{A}$, where c_ is a given positive constant. Then there exists a constant $c_{D_i B} \geq 1$, depending on n, N, p, ν, L and c_* such that*

$$\sup_{C_\varrho^{(\lambda)}(z_o)} |Dw| \leq c_{D_i B} \lambda^{\frac{1}{p}}.$$

7. COMPARISON ESTIMATES

In this Section we prove two different comparison estimates. The first one compares the weak solution u of the original inhomogeneous parabolic system (1.1) to the solution v of the associated homogeneous parabolic system (7.2) below. The second one compares v to the solution w of the frozen parabolic system (7.14). Both, the parabolic localization Lemma 4.1 and the homogeneous version of the higher integrability estimate from Corollary 5.2 will be crucial in order to achieve homogeneous comparison estimates.

Now, we let $K \geq 1$ and suppose that (2.9) is satisfied. Next, we fix $\kappa, M \geq 1$ to be specified later. In the following we consider a cylinder $Q := Q_\varrho^{(\lambda)}(z_o)$ with $z_o = (x_o, t_o) \in \Omega_T$, $\varrho \in (0, 1]$ and $\lambda \geq 1$ defined according to (3.1) and which satisfies $2Q := Q_{2\varrho}^{(\lambda)}(z_o) \in \Omega_T$ and

$$(7.1) \quad \frac{\lambda}{\kappa} \leq \int_{2Q} |Du|^{p(\cdot)} dz + \int_{2Q} M(|F| + 1)^{p(\cdot)} dz \leq \lambda.$$

Moreover, we abbreviate $B := B_\varrho(x_o)$ and $\Lambda := \Lambda_\varrho^{(\lambda)}(t_o)$ so that $Q \equiv B \times \Lambda$ and define

$$p_o := p(z_o), \quad p_1 := \inf_{2Q} p(\cdot) \quad \text{and} \quad p_2 := \sup_{2Q} p(\cdot).$$

By $v \in L^2(2Q, \mathbb{R}^N) \cap L^1(2\Lambda; W^{1,1}(2B, \mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(2Q, \mathbb{R}^{Nn})$ we denote the unique solution of the homogeneous initial-boundary value problem

$$(7.2) \quad \begin{cases} \partial_t v - \operatorname{div}(a(z)|Dv|^{p(z)-2}Dv) = 0 & \text{in } 2Q, \\ v = u & \text{on } \partial_P 2Q. \end{cases}$$

Thereby, the parabolic boundary $\partial_P 2Q$ is given by $\partial_P 2Q := (\partial 2B \times 2\Lambda) \cup (\overline{2B} \times \{t_o - \lambda^{2-p_o}(2\varrho)^2\})$. Note that the existence of v can be inferred from [8] by small modifications. Our first aim is to prove suitable energy and comparison estimates for the comparison function v . We hence subtract the weak formulation of the parabolic system (7.2) from the one of (1.1) given in (2.1). This yields

$$\begin{aligned} \int_{2Q} (u-v) \cdot \partial_t \varphi \, dz - \int_{2Q} a(\cdot) \langle |Du|^{p(\cdot)-2} Du - |Dv|^{p(\cdot)-2} Dv, D\varphi \rangle \, dz \\ = \int_{2Q} \langle |F|^{p(\cdot)-2} F, D\varphi \rangle \, dz \end{aligned}$$

for any $\varphi \in C_0^\infty(2Q, \mathbb{R}^N)$. For $\theta > 0$ and $\tau := t_\theta + \lambda^{\frac{2-p_\theta}{p_\theta}} (2\varrho)^2$ we define

$$(7.3) \quad \chi_\theta(t) := \begin{cases} 1 & \text{on } (-\infty, \tau - \theta], \\ -\frac{1}{\theta}(t - \tau) & \text{on } (\tau - \theta, \tau), \\ 0 & \text{on } [\tau, \infty). \end{cases}$$

Since $Du - Dv \in L^{p(\cdot)}(2Q, \mathbb{R}^{Nn})$ and $u = v$ on $\partial_P 2Q$ in the sense of traces, we are (formally) allowed to choose $\varphi = (u-v)\chi_\theta$ in the preceding identity. We note that the argument can be made rigorous via the use of Steklov averages and an approximation argument; since this is standard we omit the details. This choice of φ together with the observation that

$$\begin{aligned} \int_{2Q} (u-v) \cdot \partial_t [(u-v)\chi_\theta] \, dz &= - \int_{2Q} \partial_t (u-v) \cdot (u-v)\chi_\theta \, dz \\ &= -\frac{1}{2} \int_{2Q} \partial_t |u-v|^2 \chi_\theta \, dz \\ &= \frac{1}{2} \int_{2Q} |u-v|^2 \partial_t \chi_\theta \, dz \\ &= -\frac{1}{2\theta} \int_{\tau-\theta}^\tau \int_{2B} |u-v|^2 \, dz \\ (7.4) \quad &\xrightarrow{\theta \downarrow 0} -\frac{1}{2} \int_{2B} |u-v|^2(\cdot, \tau) \, dx \leq 0 \end{aligned}$$

leads us after letting $\theta \downarrow 0$ to

$$(7.5) \quad \begin{aligned} \int_{2Q} a(\cdot) \langle |Du|^{p(\cdot)-2} Du - |Dv|^{p(\cdot)-2} Dv, D(u-v) \rangle \, dz \\ \leq - \int_{2Q} \langle |F|^{p(\cdot)-2} F, D(u-v) \rangle \, dz. \end{aligned}$$

This inequality will be used in the following in two different directions. The first one will lead to an energy inequality for Dv . Rearranging terms and taking into account that $\nu \leq a(\cdot) \leq L$ we find

$$\begin{aligned} \nu \int_{2Q} |Dv|^{p(\cdot)} \, dz &\leq L \int_{2Q} (|Du|^{p(\cdot)-1} |Dv| + |Dv|^{p(\cdot)-1} |Du|) \, dz \\ &\quad + \int_{2Q} |F|^{p(\cdot)-1} (|Du| + |Dv|) \, dz \\ &\leq \frac{\nu}{2} \int_{2Q} |Dv|^{p(\cdot)} \, dz + c \int_{2Q} (|Du|^{p(\cdot)} + |F|^{p(\cdot)}) \, dz, \end{aligned}$$

where in the last line we applied Young's inequality and $c = c(\nu, L, \gamma_1, \gamma_2)$. Reabsorbing the first integral of the right-hand side into the left and subsequently using (7.1) we get the

following *energy estimate* for Dv :

$$(7.6) \quad \int_{2Q} |Dv|^{p(\cdot)} dz \leq c \int_{2Q} (|Du|^{p(\cdot)} + |F|^{p(\cdot)}) dz \leq c(\text{data}) \lambda |Q|.$$

We now come to the proof of the comparison estimate. Starting again from (7.5) we use Lemma 3.2 and Young's inequality to infer

$$\begin{aligned} & \nu \int_{2Q} (|Du|^2 + |Dv|^2)^{\frac{p(\cdot)-2}{2}} |Du - Dv|^2 dz \\ & \leq c \int_{2Q} |F|^{p(\cdot)-1} (|Du| + |Dv|) dz \\ & \leq c M^{-\frac{\gamma_1-1}{\gamma_1}} \int_{2Q} (|Du|^{p(\cdot)} + |Dv|^{p(\cdot)} + M|F|^{p(\cdot)}) dz, \end{aligned}$$

where the constant c depends on $n, N, L, \gamma_1, \gamma_2$. Finally, using (7.1) and the energy estimate (7.6) this leads us to the *first comparison estimate* we were looking for:

$$(7.7) \quad \int_{2Q} (|Du|^2 + |Dv|^2)^{\frac{p(\cdot)-2}{2}} |Du - Dv|^2 dz \leq c(\text{data}) M^{-\frac{\gamma_1-1}{\gamma_1}} \lambda |Q|.$$

We now let $\varepsilon_o = \varepsilon_o(\text{data}) > 0$ be the higher integrability exponent from Corollary 5.2 and set

$$\varrho_3 := \min\{\varrho_o/2, \varrho_2\} \in (0, 1],$$

where ϱ_o is the radius appearing in the localization Lemma 4.1 and ϱ_2 the one for the higher integrability from Corollary 5.2. Note that ϱ_3 depends on $\text{data}, \kappa, K, M, \omega(\cdot)$. In the course of the proof we shall further reduce the value of ϱ_3 when necessary, but without changing its dependencies. In the following we assume that

$$\varrho \leq \varrho_3.$$

Thanks to assumption (7.1) we are allowed to apply Lemma 4.1 on $2Q$ which yields that

$$(7.8) \quad p_2 - p_1 \leq \omega(\Gamma(2\varrho)^\alpha) \quad \text{and} \quad \lambda^{p_2-p_1} \leq \lambda^{\omega(\Gamma(2\varrho)^\alpha)} \leq e^{\frac{3n\varrho_o}{\alpha}} \leq e^{\frac{3n\gamma_2}{\alpha}},$$

where Γ and α are defined according to (4.3). Note that for the second estimate we also used that $\lambda \geq 1$. Therefore, assumption (5.6) of Corollary 5.2 is satisfied with $\hat{c} \equiv \hat{c}(n, \gamma_1, \gamma_2) := e^{\frac{3n\gamma_2}{\alpha}}$. Due to the energy estimate (7.6) we know that also assumption (5.5) is satisfied with c_* replaced by the constant $c(\text{data})$ from (7.6). The application of the Corollary then ensures that $Dv \in L^{p(\cdot)(1+\varepsilon_o)}(Q, \mathbb{R}^{Nn})$ and moreover

$$(7.9) \quad \int_Q |Dv|^{p(\cdot)(1+\varepsilon_o)} dz \leq c(\text{data}) \lambda^{1+\varepsilon_o}.$$

Next, we reduce the value of ϱ_3 in such a way that

$$(7.10) \quad \omega(\Gamma(2\varrho_3)^\alpha) \leq \frac{\varepsilon_1}{\gamma_1'}, \quad \text{where } \varepsilon_1 := \sqrt{1+\varepsilon_o} - 1 \leq \varepsilon_o$$

is satisfied. Then, by (7.8), for any $z \in 2Q$ there holds

$$\begin{aligned} p_o(1+\varepsilon_1) & \leq p(z)(1+\omega(\Gamma(2\varrho)^\alpha))(1+\varepsilon_1) \leq p(z)(1+\omega(\Gamma(2\varrho_3)^\alpha))(1+\varepsilon_1) \\ & < p(z)(1+\varepsilon_1)^2 = p(z)(1+\varepsilon_o) \end{aligned}$$

and therefore we have $Dv \in L^{p_o(1+\varepsilon_1)}(Q, \mathbb{R}^{Nn})$ together with the estimate

$$\int_Q |Dv|^{p_o(1+\varepsilon_1)} dz \leq \int_Q |Dv|^{p(\cdot)(1+\omega(\Gamma(2\varrho)^\alpha))(1+\varepsilon_1)} dz + 1$$

$$\begin{aligned}
&\leq \left(\int_Q |Dv|^{p(\cdot)(1+\varepsilon_o)} dz \right)^{\frac{(1+\omega(\Gamma(2\varrho)^\alpha))(1+\varepsilon_1)}{1+\varepsilon_o}} + 1 \\
&\leq c \lambda^{(1+\omega(\Gamma(2\varrho)^\alpha))(1+\varepsilon_1)} + 1 \\
&= c \lambda^{1+\varepsilon_1} \lambda^{\omega(\Gamma(2\varrho)^\alpha)(1+\varepsilon_1)} + 1 \\
(7.11) \quad &\leq c(\text{data}) \lambda^{1+\varepsilon_1},
\end{aligned}$$

where we used Hölder's inequality, (7.9), (7.8) and the fact that $\lambda \geq 1$. For later reference we also provide the following estimate using (7.8) and (7.10):

$$\begin{aligned}
p'_o(p_2 - 1) &= p_o \left(1 + \frac{p_2 - p_o}{p_o - 1} \right) \leq p_o \left(1 + \frac{\omega(\Gamma(2\varrho_3)^\alpha)}{\gamma_1 - 1} \right) \\
(7.12) \quad &\leq p_o \left(1 + \frac{\varepsilon_1}{\gamma_1} \right) < p_o(1 + \varepsilon_1).
\end{aligned}$$

Together with (7.11), Hölder's inequality and (7.8) this implies

$$\begin{aligned}
\int_Q |Dv|^{p'_o(p_2-1)} dz &\leq \left(\int_Q |Dv|^{p_o(1+\varepsilon_1)} dz \right)^{\frac{p_2-1}{(p_o-1)(1+\varepsilon_1)}} \\
(7.13) \quad &\leq c \lambda^{\frac{p_2-1}{p_o-1}} = c \lambda^{1+\frac{p_2-p_o}{p_o-1}} \leq c(\text{data}) \lambda.
\end{aligned}$$

We now define

$$\tilde{a}(t) := (a)_{x_o, \varrho}(t) := \int_{B_\varrho(x_o)} a(\cdot, t) dx \quad \text{for any } t \in (0, T).$$

Note that $\nu \leq \tilde{a}(t) \leq L$ for any $t \in (0, T)$ as a consequence of (2.6). By

$$w \in C^0(\Lambda, L^2(B; \mathbb{R}^N)) \cap L^{p_o}(\Lambda, W^{1, p_o}(B; \mathbb{R}^N))$$

we denote the unique solution to the initial-boundary value problem

$$(7.14) \quad \begin{cases} \partial_t w - \operatorname{div}(\tilde{a}(t)|Dw|^{p_o-2}Dw) = 0 & \text{in } Q, \\ w = v & \text{on } \partial_P Q. \end{cases}$$

We now start deriving energy and comparison estimates for w . As before, we subtract the weak formulations of (7.2) and (7.14) and test the result with $\varphi := (v - w)\chi_\theta$, where χ_θ is defined in (7.3). Here, we recall that $Dv \in L^{p_o}(Q, \mathbb{R}^{Nn})$ by (7.11) and therefore φ is (formally) admissible as a test function. Proceeding as before, i.e. treating the terms involving the time derivatives with the argument performed in (7.4) and passing to the limit $\theta \downarrow 0$ we obtain

$$(7.15) \quad \int_Q \langle a(\cdot)|Dv|^{p(\cdot)-2}Dv - \tilde{a}(t)|Dw|^{p_o-2}Dw, D(v - w) \rangle dz \leq 0.$$

Firstly, we shall use this inequality to get an energy estimate for Dw . Rearranging terms, taking into account that $\nu \leq a(\cdot), \tilde{a}(\cdot) \leq L$ and applying Young's inequality we find

$$\begin{aligned}
\nu \int_Q |Dw|^{p_o} dz &\leq L \int_Q (|Dw|^{p_o-1}|Dv| + |Dv|^{p(\cdot)-1}|Dw|) dz \\
&\leq \frac{\nu}{2} \int_Q |Dw|^{p_o} dz + c \int_Q (|Dv|^{p_o} + |Dv|^{p'_o(p(\cdot)-1)}) dz
\end{aligned}$$

with a constant $c = c(\nu, L, \gamma_1, \gamma_2)$. Reabsorbing the first integral of the right-hand side into the left and using Hölder's inequality, (7.11), (7.13) and the fact that $\lambda \geq 1$ we get the following *energy estimate for Dw* :

$$(7.16) \quad \int_Q |Dw|^{p_o} dz \leq c \left[\int_Q |Dv|^{p_o} dz + \int_Q |Dv|^{p'_o(p_2-1)} dz + 1 \right] \leq c(\text{data}) \lambda.$$

In order to obtain a comparison estimate we once again start from (7.15) which can be rewritten as follows:

$$\begin{aligned} & \int_Q \tilde{a}(t) \langle |Dv|^{p_o-2} Dv - |Dw|^{p_o-2} Dw, D(v-w) \rangle dz \\ & \leq \int_Q (\tilde{a}(t) - a(\cdot)) \langle |Dv|^{p_o-2} Dv, D(v-w) \rangle dz \\ & \quad + \int_Q a(\cdot) \langle |Dv|^{p_o-2} Dv - |Dv|^{p(\cdot)-2} Dv, D(v-w) \rangle dz. \end{aligned}$$

Using Lemma 3.2 and the fact that $\nu \leq \tilde{a}(\cdot) \leq L$ we obtain

$$(7.17) \quad \begin{aligned} & \int_Q (|Dv|^2 + |Dw|^2)^{\frac{p_o-2}{2}} |Dv - Dw|^2 dz \\ & \leq c \left[\int_Q |\tilde{a}(t) - a(\cdot)| |Dv|^{p_o-1} |Dv - Dw| dz \right. \\ & \quad \left. + \int_Q ||Dv|^{p_o-1} - |Dv|^{p(\cdot)-1}| |Dv - Dw| dz \right] =: c[\text{I} + \text{II}], \end{aligned}$$

where $c = c(\nu, L, \gamma_1, \gamma_2)$. Now we estimate separately the two terms. For the first one we use Hölder's inequality several times, (7.11), (7.16), the fact that $a(\cdot), \tilde{a}(t) \leq L$, (2.7) and $\tilde{\omega} \leq 1$ to infer that

$$\begin{aligned} \text{I} & \leq c \left(\int_Q |a(t) - a(\cdot)|^{p'_o} |Dv|^{p_o} dz \right)^{\frac{1}{p'_o}} \left(\int_Q (|Dv|^{p_o} + |Dw|^{p_o}) dz \right)^{\frac{1}{p_o}} \\ & \leq c \left(\int_Q |a(t) - a(\cdot)|^{\frac{p'_o(1+\varepsilon_1)}{\varepsilon_1}} dz \right)^{\frac{\varepsilon_1}{p'_o(1+\varepsilon_1)}} \left(\int_Q |Dv|^{p_o(1+\varepsilon_1)} dz \right)^{\frac{1}{p_o(1+\varepsilon_1)}} \lambda^{\frac{1}{p_o}} \\ & \leq c \lambda^{\frac{1}{p'_o} + \frac{1}{p_o}} [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{\gamma_1(1+\varepsilon_1)}} \leq c(\text{data}) [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_1}} \lambda. \end{aligned}$$

In order to estimate II we first use (7.8) to find that for any $z \in Q$ and $b \geq 0$ there holds

$$\begin{aligned} |b^{p_o-1} - b^{p(z)-1}| & \leq |p_o - p(z)| \sup_{\sigma \in [p_1-1, p_2-1]} b^\sigma |\log b| \\ & \leq \omega(\Gamma(2\varrho)^\alpha) \left[b^{p_2-1} \log(e + b^{p'_o(p_2-1)}) + \frac{1}{e^{(\gamma_1-1)}} \right], \end{aligned}$$

where in the last line we used $b^\sigma |\log b| \leq \frac{1}{e^{(\gamma_1-1)}}$ for $b \in [0, 1]$ and $\sigma \in [p_1 - 1, p_2 - 1]$ and $b^\sigma |\log b| \leq b^{p_2-1} \log(e + b^{p'_o(p_2-1)})$ for $b > 1$ and $\sigma \in [p_1 - 1, p_2 - 1]$. This together with Hölder's inequality, (7.11) and (7.16) yields

$$\begin{aligned} \text{II} & \leq c\omega(\Gamma(2\varrho)^\alpha) \int_Q \left[|Dv|^{p_2-1} \log(e + |Dv|^{p'_o(p_2-1)}) + 1 \right] |Dv - Dw| dz \\ & \leq c\omega(\Gamma(2\varrho)^\alpha) \left(\int_Q \left[|Dv|^{p_2-1} \log(e + |Dv|^{p'_o(p_2-1)}) + 1 \right]^{p'_o} dz \right)^{\frac{1}{p'_o}} \end{aligned}$$

$$\begin{aligned} & \cdot \left(\int_Q |Dv - Dw|^{p_o} dz \right)^{\frac{1}{p_o}} \\ & \leq c\omega(\Gamma(2\varrho)^\alpha) \left(\int_Q \left[|Dv|^{p_2-1} \log(e + |Dv|^{p'_o(p_2-1)}) + 1 \right]^{p'_o} dz \right)^{\frac{1}{p'_o}} \lambda^{\frac{1}{p'_o}}, \end{aligned}$$

where $c = c(\text{data})$. Next, we note that the monotonicity of the logarithm implies

$$\log(e + ab) \leq \log(e + a) + \log(e + b) \quad \forall a, b \geq 0,$$

which together with Young's inequality allows to further estimate Π as follows:

$$\begin{aligned} \Pi & \leq c\omega(\Gamma(2\varrho)^\alpha) \lambda^{\frac{1}{p_o}} \left[\int_Q |Dv|^{p'_o(p_2-1)} \log^{p'_o} \left(e + \frac{|Dv|^{p'_o(p_2-1)}}{(|Dv|^{p'_o(p_2-1)})_Q} \right) dz \right. \\ & \quad \left. + \log^{p'_o} \left(e + (|Dv|^{p'_o(p_2-1)})_Q \right) \int_Q |Dv|^{p'_o(p_2-1)} dz + 1 \right]^{\frac{1}{p'_o}} \\ (7.18) \quad & = c(\text{data}) \omega(\Gamma(2\varrho)^\alpha) \lambda^{\frac{1}{p_o}} [\Pi_1 + \Pi_2 + 1]^{\frac{1}{p'_o}}, \end{aligned}$$

with the obvious meaning of Π_1 and Π_2 . In order to estimate Π_1 we apply inequality (3.2) with the choices $g = |Dv|^{p'_o(p_2-1)}$ and

$$\sigma := \frac{1 + \varepsilon_1}{1 + \frac{\varepsilon_1}{\gamma_1}} = c(\text{data}) > 1$$

to infer that

$$\Pi_1 \leq c(\text{data}) \left(\int_Q |Dv|^{p'_o(p_2-1)\sigma} dz \right)^{\frac{1}{\sigma}}.$$

To the integral on the right-hand side we apply Hölder's inequality (which is justified by (7.12)). Subsequently using (7.11) and (7.8) we obtain

$$\begin{aligned} \Pi_1 & \leq c \left(\int_Q |Dv|^{p_o(1+\varepsilon_1/\gamma_1)\sigma} dz \right)^{\frac{1}{\sigma} \cdot \frac{p_2-1}{(p_o-1)(1+\varepsilon_1/\gamma_1)}} \\ & = c \left(\int_Q |Dv|^{p_o(1+\varepsilon_1)} dz \right)^{\frac{p_2-1}{(p_o-1)(1+\varepsilon_1)}} \\ & \leq c \lambda^{\frac{p_2-1}{p_o-1}} = c \lambda^{1 + \frac{p_2-p_o}{p_o-1}} \leq c(\text{data}) \lambda. \end{aligned}$$

Now, we come to the estimate for Π_2 in (7.18). From (7.13) and (4.2)₁ we get

$$\left(|Dv|^{p'_o(p_2-1)} \right)_Q = \int_Q |Dv|^{p'_o(p_2-1)} dz \leq c \lambda \leq c(\text{data}, \kappa) \left(\frac{KM}{\varrho^{n+2}} \right)^{\frac{p_o}{2}}.$$

Using this estimate, again (7.13), the fact that $\log(cx) \leq c \log(x)$ for $c \geq 1$ and that we can always assume $c(KM/\varrho^{n+2})^{p_o/2} \geq e$ by possibly reducing the value of ϱ_3 we find

$$\Pi_2 \leq \log^{p'_o} \left(e + c \left(\frac{KM}{\varrho^{n+2}} \right)^{\frac{p_o}{2}} \right) \int_Q |Dv|^{p'_o(p_2-1)} dz \leq c M^{p'_o} \log^{p'_o} \left(\frac{K}{\varrho} \right) \lambda$$

with $c = c(\text{data}, \kappa)$. Joining the estimates for Π_1 and Π_2 with (7.18) we end up with

$$\Pi \leq c(\text{data}, \kappa) \omega(\Gamma(2\varrho)^\alpha) M \log \left(\frac{K}{\varrho} \right) \lambda.$$

Merging the preceding estimates for I and II into (7.17) we get

$$\begin{aligned} & \int_Q (|Dv|^2 + |Dw|^2)^{\frac{p_o-2}{2}} |Dv - Dw|^2 dz \\ & \leq c(\text{data}, \kappa) \left[\omega(\Gamma(2\varrho)^\alpha) M \log\left(\frac{K}{\varrho}\right) + [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_1}} \right] \lambda. \end{aligned}$$

Here, we still want to replace the exponent p_o in the integral on the left-hand side by $p(\cdot)$. This is achieved with the help of Hölder's inequality as follows:

$$\begin{aligned} & \int_{\frac{1}{2}Q} (|Dv|^2 + |Dw|^2)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2 dz \\ & \leq \left(\int_{\frac{1}{2}Q} (|Dv|^2 + |Dw|^2)^{\frac{p_o-2}{2}} |Dv - Dw|^2 dz \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\int_{\frac{1}{2}Q} (|Dv|^2 + |Dw|^2)^{\frac{2p(\cdot)-p_o-2}{2}} |Dv - Dw|^2 dz \right)^{\frac{1}{2}} \\ & \leq c \left[\omega(\Gamma(2\varrho)^\alpha) M \log\left(\frac{K}{\varrho}\right) + [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_1}} \right]^{\frac{1}{2}} \lambda^{\frac{1}{2}} \\ & \quad \cdot \left(\int_{\frac{1}{2}Q} |Dv|^{2p(\cdot)-p_o} + |Dw|^{2p(\cdot)-p_o} dz \right)^{\frac{1}{2}}. \end{aligned}$$

In order to further estimate the integral on the right-hand side we use that fact that $2p(\cdot) - p_o \leq p(\cdot)(1 + \omega(\Gamma(2\varrho)^\alpha)) \leq p(\cdot)(1 + \omega(\Gamma\varrho_3^\alpha)) \leq p(\cdot)(1 + \varepsilon_o)$ which is a consequence of (7.10), Hölder's inequality, (7.9), (7.8) and $\lambda \geq 1$ to infer that

$$\begin{aligned} & \int_{\frac{1}{2}Q} |Dv|^{2p(\cdot)-p_o} dz \leq 2^{n+2} \int_Q |Dv|^{p(\cdot)(1+\omega(\Gamma(2\varrho)^\alpha))} dz + 1 \\ & \leq 2^{n+2} \left(\int_Q |Dv|^{p(\cdot)(1+\varepsilon_o)} dz \right)^{\frac{1+\omega(\Gamma(2\varrho)^\alpha)}{1+\varepsilon_o}} + 1 \\ & \leq c \lambda^{1+\omega(\Gamma(2\varrho)^\alpha)} + 1 \leq c(\text{data}, \kappa) \lambda. \end{aligned}$$

Moreover, since the parabolic system (7.14)₁ is of the same type as (6.1) we are allowed by (7.16) to apply Theorem 6.1 which yields that

$$(7.19) \quad \sup_{\frac{1}{2}Q} |Dw| \leq c_{DiB} \lambda^{\frac{1}{p_o}}.$$

Note that c_{DiB} initially depends on n, N, ν, L, p_o . Since the dependence upon p_o is continuous it can be replaced by a larger constant depending on γ_1 and γ_2 instead of p_o , i.e. $c_{DiB} = c_{DiB}(\text{data})$. Therefore, using (7.19) and (7.8) we can bound also the integral involving Dw in terms of λ . Inserting this above we deduce the *second comparison estimate* we were looking for:

$$(7.20) \quad \begin{aligned} & \int_{\frac{1}{2}Q} (|Dv|^2 + |Dw|^2)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2 dz \\ & \leq c(\text{data}, \kappa) \left[\omega(\Gamma(2\varrho)^\alpha) M \log\left(\frac{K}{\varrho}\right) + [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_1}} \right]^{\frac{1}{2}} \lambda |Q|. \end{aligned}$$

Note that this estimate holds for any cylinder $\frac{1}{2}Q \equiv Q_{\varrho/2}^{(\lambda)}(z_o)$ with $\lambda \geq 1$ and $\varrho \in (0, \varrho_3]$ such that $2Q$ satisfies the intrinsic relation (7.1) and $2Q \Subset \Omega_T$. We recall that $\varrho_3 \in (0, 1]$ depends on $data, \kappa, K, M, \omega(\cdot)$.

8. PROOF OF THE CALDERÓN-ZYGMUND ESTIMATE

This Section is devoted to the proof of Theorem 2.2. We shall proceed in several steps.

8.1. A stopping-time argument. Here, we shall construct a covering of the upper level set of $|Du|^{p(\cdot)}$ with respect to some parameter λ by intrinsic cylinders. The argument uses a certain *stopping time argument* which takes its origin in [28] together with the non uniform version of Vitali's covering argument from Lemma 4.2.

We let $K \geq 1$ and suppose that (2.9) is satisfied and consider a standard parabolic cylinder $Q_R \equiv Q_R(\mathfrak{z}_o)$ such that $Q_{2R} \Subset \Omega_T$. Then, we fix $M \geq 1$ to be specified later and define

$$(8.1) \quad \lambda_o := \left[\int_{Q_{2R}} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \right]^d \geq 1, \quad \text{where } d := \sup_{Q_{2R}} d(p(\cdot))$$

and $d(\cdot)$ is defined according to (2.11). Next, as in [4, Section 4], we fix two numbers $R \leq r_1 < r_2 \leq 2R$ such that $Q_R \subset Q_{r_1} \subset Q_{r_2} \subset Q_{2R}$, all the cylinders sharing the same center \mathfrak{z}_o . In the following we shall consider λ such that

$$(8.2) \quad \lambda > B\lambda_o, \quad \text{where } B := \left(\frac{8\chi R}{r_2 - r_1} \right)^{(n+2)d}$$

and for $z_o \in Q_{r_1}$ we consider radii ϱ satisfying

$$(8.3) \quad \min \left\{ 1, \lambda^{\frac{p_o-2}{2p_o}} \right\} \frac{r_2 - r_1}{4\chi} \leq \varrho \leq \min \left\{ 1, \lambda^{\frac{p_o-2}{2p_o}} \right\} \frac{r_2 - r_1}{2},$$

where $p_o := p(z_o)$ and $\chi \equiv \chi(n, \gamma_1) \geq 5$ denotes the constant appearing in Lemma 4.2. Note that these choices of λ and ϱ ensure that $Q_\varrho^{(\lambda)}(z_o) \subset Q_{r_2}$ for any $z_o \in Q_{r_1}$. Next, we want to prove that for any $z_o \in Q_{r_1}$ there holds

$$(8.4) \quad \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz < \lambda.$$

Indeed, enlarging the domain of integration from $Q_\varrho^{(\lambda)}(z_o)$ to Q_{2R} and recalling the definition of λ_o from (8.1) we infer that

$$\begin{aligned} & \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \\ & \leq \frac{|Q_{2R}|}{|Q_\varrho^{(\lambda)}(z_o)|} \int_{Q_{2R}} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \\ & = \left(\frac{2R}{\varrho} \right)^{n+2} \lambda^{\frac{p_o-2}{p_o}} \lambda_o^{\frac{1}{d}}. \end{aligned}$$

Now we distinguish the cases $p_o \geq 2$ and $p_o < 2$. If $p_o \geq 2$, then $1/d \leq 1/d(p_o) = 2/p_o$ and $\min \left\{ 1, \lambda^{\frac{p_o-2}{2p_o}} \right\} = 1$, so that, using also the choice of ϱ from (8.3) we obtain

$$\begin{aligned} \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz & \leq \left(\frac{8\chi R}{r_2 - r_1} \right)^{n+2} \lambda^{\frac{p_o-2}{p_o}} \lambda_o^{\frac{1}{d}} \\ & < B^{\frac{1}{d}} \lambda^{\frac{p_o-2}{p_o}} B^{-\frac{1}{d}} \lambda^{\frac{1}{d}} = \lambda^{\frac{p_o-2}{p_o}} \lambda^{\frac{1}{d}} \leq \lambda. \end{aligned}$$

If $\gamma_1 \leq p_o < 2$, we have $1/d \leq 1/d(p_o) = 1 - n(2 - p_o)/(2p_o)$ and $\min\{1, \lambda^{\frac{p_o-2}{2p_o}}\} = \lambda^{\frac{p_o-2}{2p_o}}$ and therefore we get

$$\begin{aligned} \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz &\leq \left(\frac{8\chi R \lambda^{\frac{2-p_o}{2p_o}}}{r_2 - r_1} \right)^{n+2} \lambda^{\frac{p_o-2}{p_o}} \lambda_o^{\frac{1}{d}} \\ &= B^{\frac{1}{d}} \lambda^{\frac{n(2-p_o)}{2p_o}} \lambda_o^{\frac{1}{d}} < B^{\frac{1}{d}} \lambda^{\frac{n(2-p_o)}{2p_o}} B^{-\frac{1}{d}} \lambda_o^{\frac{1}{d}} = \lambda^{\frac{n(2-p_o)}{2p_o}} \lambda_o^{\frac{1}{d}} \leq \lambda. \end{aligned}$$

Hence, in any case we proved that (8.4) holds.

For λ as in (8.2) we consider the upper level set

$$E(\lambda, r_1) := \{z \in Q_{r_1} : z \text{ is a Lebesgue point of } |Du| \text{ and } |Du(z)|^{p(z)} > \lambda\}.$$

In the following we show that also a converse inequality holds true for small radii and for points $z_o \in E(\lambda, r_1)$. Indeed, by Lebesgue's differentiation theorem (see [10, (7.9)]) we infer for any $z_o \in E(\lambda, r_1)$ that

$$\lim_{\varrho \downarrow 0} \int_{Q_\varrho^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \geq |Du(z_o)|^{p(z_o)} > \lambda.$$

From the preceding reasoning we conclude that the last inequality yields a radius for which the considered integral takes a value larger than λ , while (8.4) states that the integral is smaller than λ for any radius satisfying (8.3). Therefore, the continuity of the integral yields the existence of a maximal radius ϱ_{z_o} in between, i.e.

$$(8.5) \quad 0 < \varrho_{z_o} < \min\left\{1, \lambda^{\frac{p_o-2}{2p_o}}\right\} \frac{r_2 - r_1}{4\chi}$$

such that

$$(8.6) \quad \int_{Q_{\varrho_{z_o}}^{(\lambda)}(z_o)} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz = \lambda.$$

By saying that ϱ_{z_o} is maximal we mean that for every $\varrho \in (\varrho_{z_o}, \min\{1, \lambda^{\frac{p_o-2}{2p_o}}\}(r_2 - r_1)/2)$ inequality (8.4) holds. With this choice of ϱ_{z_o} we define concentric parabolic cylinders centered at $z_o \in E(\lambda, r_1)$ as follows:

$$(8.7) \quad Q_{z_o}^0 := Q_{\varrho_{z_o}}^{(\lambda)}(z_o), \quad Q_{z_o}^1 := Q_{\chi\varrho_{z_o}}^{(\lambda)}(z_o), \quad Q_{z_o}^2 := Q_{2\chi\varrho_{z_o}}^{(\lambda)}(z_o), \quad Q_{z_o}^3 := Q_{4\chi\varrho_{z_o}}^{(\lambda)}(z_o).$$

Then, we have $Q_{z_o}^0 \subset Q_{z_o}^1 \subset Q_{z_o}^2 \subset Q_{z_o}^3 \subset Q_{r_2}$ and for $j \in \{0, \dots, 3\}$ there holds

$$(8.8) \quad \frac{\lambda}{(4\chi)^{n+2}} \leq \int_{Q_{z_o}^j} |Du|^{p(\cdot)} + M(|F| + 1)^{p(\cdot)} dz \leq \lambda.$$

Note that the upper bound follows from (8.6) and the maximal choice of the stopping radius ϱ_{z_o} , while the lower bound follows from (8.6) by enlarging the domain of integration from $Q_{z_o}^0$ to $Q_{z_o}^j$ and taking into account that $|Q_{z_o}^j|/|Q_{z_o}^0| \leq (4\chi)^{n+2}$.

8.2. Estimates on intrinsic cylinders. We now fix one particular cylinder $Q_{z_o}^0$ and define the comparison functions v and w as the unique solutions to the initial-boundary value problems (7.2) and (7.14) with $Q_{z_o}^3$ and $Q_{z_o}^2$ instead of $2Q$ and Q . Thanks to (8.8) we know that (7.1) is satisfied with $\kappa = \kappa(n, \gamma_1) = (4\chi)^{n+2}$. Moreover, we assume that

$$R \leq R_o \leq \varrho_3,$$

where $\varrho_3 = \varrho_3(\text{data}, K, M, \omega(\cdot)) \in (0, 1]$ denotes the radius introduced after (7.20) for the choice $\kappa = (4\chi)^{n+2}$. This ensures that we may apply (7.7), (7.19) and (7.20) with $\kappa =$

$(4\chi)^{n+2}$ for any radius smaller than ϱ_3 . Therefore, from (7.19) applied with $\kappa = (4\chi)^{n+2}$ we infer that

$$(8.9) \quad \sup_{Q_{z_o}^1} |Dw| \leq c_{DiB} \lambda^{\frac{1}{p_o}},$$

where $c_{DiB} = c_{DiB}(\text{data}) \geq 1$. In the following by $c_\ell = c_\ell(\gamma_2) \geq 1$ we denote the constant from Lemma 3.3. For A chosen in dependence on data according to

$$A \geq 2c_\ell^2 c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}} \geq 1$$

we now consider $z \in Q_{z_o}^1 \cap E(A\lambda, r_1)$. Our aim now is to deduce a suitable estimate for $|Du(z)|^{p(z)}$. Applying Lemma 3.3 twice yields

$$(8.10) \quad \begin{aligned} |Du(z)|^{p(z)} &\leq c_\ell^2 |Dw(z)|^{p(z)} \\ &\quad + c_\ell^2 (|Dv(z)|^2 + |Dw(z)|^2)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2 \\ &\quad + c_\ell (|Du(z)|^2 + |Dv(z)|^2)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2. \end{aligned}$$

Next, we prove that

$$(8.11) \quad \begin{aligned} |Dw(z)|^{p(z)} &\leq (|Du(z)|^2 + |Dv(z)|^2)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2 \\ &\quad + (|Dv(z)|^2 + |Dw(z)|^2)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2 \end{aligned}$$

holds. Indeed, if (8.11) fails to hold we obtain from (8.9), (4.2)_{2,3} from Lemma 4.1 (which is applicable due to (8.8)), the fact that $z \in E(A\lambda, r_1)$ and (8.10) that

$$\begin{aligned} |Dw(z)|^{p(z)} &\leq c_{DiB}^{p(z)} \lambda^{\frac{p(z)}{p_o}} \leq c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}} \lambda \\ &< \frac{c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}}}{A} |Du(z)|^{p(z)} \leq \frac{2c_\ell^2 c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}}}{A} |Dw(z)|^{p(z)}. \end{aligned}$$

But this contradicts the choice of A and hence (8.11) is proved. Therefore, combining (8.10) and (8.11) we get

$$\begin{aligned} |Du(z)|^{p(z)} &\leq 2c_\ell^2 (|Du(z)|^2 + |Dv(z)|^2)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2 \\ &\quad + 2c_\ell^2 (|Dv(z)|^2 + |Dw(z)|^2)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2. \end{aligned}$$

Integrating over $Q_{z_o}^1 \cap E(A\lambda, r_1)$ and using the comparison estimates (7.7) and (7.20) applied with $\kappa = (4\chi)^{n+2}$ we obtain

$$(8.12) \quad \begin{aligned} \int_{Q_{z_o}^1 \cap E(A\lambda, r_1)} |Du|^{p(\cdot)} dz &\leq 2c_\ell^2 \int_{Q_{z_o}^1} (|Du|^2 + |Dv|^2)^{\frac{p(\cdot)-2}{2}} |Du - Dv|^2 dz \\ &\quad + 2c_\ell^2 \int_{Q_{z_o}^1} (|Dv|^2 + |Dw|^2)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2 dz \\ &\leq c(\text{data}) G(M, R) \lambda |Q_{z_o}^0|, \end{aligned}$$

where

$$(8.13) \quad G(M, R) := \sup_{\varrho \in (0, R]} \left[\frac{1}{M^{2-\frac{2}{\gamma_1}}} + \omega(\Gamma(2\varrho)^\alpha) M \log \left(\frac{K}{\varrho} \right) + [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_2}} \right]^{\frac{1}{2}}.$$

Note that $M \geq 1$ is yet to be chosen and α and Γ are defined according to (4.3). Moreover, we recall that this estimate holds for any $\lambda > B\lambda_o$ and $z_o \in E(\lambda, r_1)$.

Next, we will infer a bound for the measure of the cylinder $Q_{z_o}^0$. From (8.6) we have

$$(8.14) \quad |Q_{z_o}^0| = \frac{1}{\lambda} \int_{Q_{z_o}^0} |Du|^{p(\cdot)} dz + \frac{1}{\lambda} \int_{Q_{z_o}^0} M(|F| + 1)^{p(\cdot)} dz.$$

We split the first integral of (8.14) as follows:

$$\begin{aligned} \int_{Q_{z_o}^0} |Du|^{p(\cdot)} dz &= \int_{Q_{z_o}^0 \cap \{|Du|^{p(\cdot)} \leq \lambda/4\}} |Du|^{p(\cdot)} dz + \int_{Q_{z_o}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} dz \\ &\leq \frac{\lambda}{4} |Q_{z_o}^0| + \int_{Q_{z_o}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} dz, \end{aligned}$$

and similarly the second one

$$\int_{Q_{z_o}^0} M(|F| + 1)^{p(\cdot)} dz \leq \frac{\lambda}{4} |Q_{z_o}^0| + \int_{Q_{z_o}^0 \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F| + 1)^{p(\cdot)} dz.$$

Inserting the last two estimates into (8.14) we can reabsorb the term $|Q_{z_o}^0|/2$ from the right-hand side into the left, yielding the following estimate:

$$\begin{aligned} |Q_{z_o}^0| &\leq \frac{2}{\lambda} \int_{Q_{z_o}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} dz \\ &\quad + \frac{2}{\lambda} \int_{Q_{z_o}^0 \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F| + 1)^{p(\cdot)} dz. \end{aligned}$$

Using this estimate in (8.12) we obtain for a constant $c \equiv c(\text{data})$ that

$$(8.15) \quad \begin{aligned} \int_{Q_{z_o}^1 \cap E(A\lambda, r_1)} |Du|^{p(\cdot)} dz &\leq cG(M, R) \int_{Q_{z_o}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} dz \\ &\quad + cG(M, R) \int_{Q_{z_o}^0 \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F| + 1)^{p(\cdot)} dz. \end{aligned}$$

8.3. Estimates on level sets. Here, we will extend estimate (8.15) to the super level set $E(A\lambda, r_1)$. To this aim we first construct a suitable covering of $E(\lambda, r_1)$ by intrinsic cylinders of the type as considered in the preceding steps. Here, we recall from the preceding two steps that for every $z_o \in E(\lambda, r_1)$ there exists a radius ϱ_{z_o} satisfying (8.5) such that on the cylinders $Q_{z_o}^j$, $j \in \{0, \dots, 3\}$ the estimates (8.8) and (8.15) hold. Next, we want to apply the Vitali-type covering argument from Lemma 4.2. For this aim we note that (8.6) and (4.2)₁ (with $\kappa = 1$) imply that

$$\lambda \leq \left(\frac{\beta_n MK}{\varrho_{z_o}^{n+2}} \right)^{\frac{p(z_o)}{2}}.$$

This ensures that assumption (4.5) of Lemma 4.2 is satisfied for the family $\mathcal{F} := \{Q_{z_o}^0\}$ of parabolic cylinders with center $z_o \in E(\lambda, r_1)$ (note that by possibly reducing the value of R_o we can ensure that $\varrho_{z_o} \leq R \leq R_o \leq \varrho_1$, where ϱ_1 is the radius from Lemma 4.2). The application of the Lemma then yields the existence of a countable subfamily $\{Q_{z_i}^0\}_{i=1}^\infty \subset \mathcal{F}$ of pairwise disjoint parabolic cylinders, such that the χ -times enlarged cylinders $Q_{z_i}^1$ cover the set $E(A\lambda, r_1)$, i.e.

$$E(A\lambda, r_1) \subset E(\lambda, r_1) \subset \bigcup_{i \in \mathbb{N}} Q_{z_i}^1.$$

Moreover, for the 4χ -times enlarged cylinders $Q_{z_i}^3$ we know that $Q_{z_i}^3 \subset Q_{r_2}$. Here, we have used the notation from (8.7) with z_o replaced by z_i . Since we know that on any of the cylinders $Q_{z_i}^1$, $i \in \mathbb{N}$ estimate (8.15) holds, we obtain after summing over $i \in \mathbb{N}$ that

$$(8.16) \quad \int_{E(A\lambda, r_1)} |Du|^{p(\cdot)} dz \leq cG(M, R) \int_{E(\lambda/4, r_2)} |Du|^{p(\cdot)} dz \\ + cG(M, R) \int_{Q_{r_2} \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F|+1)^{p(\cdot)} dz,$$

where $c = c(\text{data})$. We recall that this estimate holds for every $\lambda > B\lambda_o$.

8.4. Raising the integrability exponent. Having arrived at this stage we would like to multiply both sides of (8.16) by λ^{q-2} and subsequently integrate with respect to λ over $(B\lambda_o, \infty)$. This, formally would lead to an $L^{p(\cdot)q}$ estimate of Du after reabsorbing $\int |Du|^{p(\cdot)q} dz$ on the left-hand side. However, this step is not allowed since the integral might be infinite. This problem will be overcome in the following by a truncation argument. For $k \geq B\lambda_o$ we define the truncation operator

$$T_k : [0, +\infty) \rightarrow [0, k], \quad T_k(\sigma) := \min\{\sigma, k\}$$

and

$$E_k(A\lambda, r_1) := \{z \in Q_{r_1} : T_k(|Du(z)|^{p(z)}) > A\lambda\}.$$

Then, from inequality (8.16) we deduce that

$$(8.17) \quad \int_{E_k(A\lambda, r_1)} |Du|^{p(\cdot)} dz \leq cG(M, R) \int_{E_k(\lambda/4, r_2)} |Du|^{p(\cdot)} dz \\ + cG(M, R) \int_{Q_{r_2} \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F|+1)^{p(\cdot)} dz.$$

This can be seen as follows: In the case $k \leq A\lambda$ we have $E_k(A\lambda, r_1) = \emptyset$ and therefore (8.17) holds trivially. In the case $k > A\lambda$ inequality (8.17) follows since $E_k(A\lambda, r_1) = E(A\lambda, r_1)$ and $E_k(\lambda/4, r_2) = E(\lambda/4, r_2)$. Therefore, multiplying both sides of (8.17) by λ^{q-2} and integrating with respect to λ over $(B\lambda_o, +\infty)$, we obtain

$$(8.18) \quad \int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{E_k(A\lambda, r_1)} |Du|^{p(\cdot)} dz d\lambda \\ \leq cG(M, R) \int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{E_k(\lambda/4, r_2)} |Du|^{p(\cdot)} dz d\lambda \\ + cG(M, R) \int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{Q_{r_2} \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F|+1)^{p(\cdot)} dz d\lambda.$$

Using Fubini's theorem we get for the integral on the left-hand side of (8.18) that

$$\int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{E_k(A\lambda, r_1)} |Du|^{p(\cdot)} dz d\lambda \\ = \int_{E_k(AB\lambda_o, r_1)} |Du|^{p(\cdot)} \int_{B\lambda_o}^{T_k(|Du(z)|^{p(z)})/A} \lambda^{q-2} d\lambda dz \\ = \frac{1}{q-1} \left[\frac{1}{A^{q-1}} \int_{E_k(AB\lambda_o, r_1)} |Du|^{p(\cdot)} T_k(|Du|^{p(\cdot)})^{q-1} dz \right]$$

$$\begin{aligned}
& - (B\lambda_o)^{q-1} \int_{E_k(AB\lambda_o, r_1)} |Du|^{p(\cdot)} dz \Big] \\
& \geq \frac{1}{q-1} \left[\frac{1}{A^{q-1}} \int_{Q_{r_1}} |Du|^{p(\cdot)} T_k(|Du|^{p(\cdot)})^{q-1} dz \right. \\
& \quad \left. - (B\lambda_o)^{q-1} \int_{Q_{r_1}} |Du|^{p(\cdot)} dz \right],
\end{aligned}$$

where in the last line we used the decomposition $Q_{r_1} = E_k(AB\lambda_o, r_1) \cup (Q_{r_1} \setminus E_k(AB\lambda_o, r_1))$ and the fact that $T_k(|Du|^{p(\cdot)}) \leq AB\lambda_o$ on $Q_{r_1} \setminus E_k(AB\lambda_o, r_1)$. Again by Fubini's theorem we obtain for the first integral on the right-hand side of (8.18)

$$\begin{aligned}
& \int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{E_k(\lambda/4, r_2)} |Du|^{p(\cdot)} dz d\lambda \\
& = \int_{E_k(B\lambda_o/4, r_2)} |Du|^{p(\cdot)} \int_{B\lambda_o}^{4T_k(|Du|^{p(\cdot)})} \lambda^{q-2} d\lambda dz \\
& \leq \frac{4^{q-1}}{q-1} \int_{Q_{r_2}} |Du|^{p(\cdot)} T_k(|Du|^{p(\cdot)})^{q-1} dz
\end{aligned}$$

and analogously for the integral involving the right-hand side F :

$$\begin{aligned}
& \int_{B\lambda_o}^{\infty} \lambda^{q-2} \int_{Q_{r_2} \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F|+1)^{p(\cdot)} dz d\lambda \\
& = \int_{Q_{r_2} \cap \{M(|F|+1)^{p(\cdot)} > B\lambda_o/4\}} M(|F|+1)^{p(\cdot)} \int_{B\lambda_o}^{4M(|F|+1)^{p(\cdot)}} \lambda^{q-2} d\lambda dz \\
& \leq \frac{4^{q-1} M^q}{q-1} \int_{Q_{r_2}} (|F|+1)^{p(\cdot)q} dz.
\end{aligned}$$

Hence, joining the preceding estimates with (8.18) we get

$$\begin{aligned}
& \int_{Q_{r_1}} |Du|^{p(\cdot)} T_k(|Du|^{p(\cdot)})^{q-1} dz \\
& \leq (AB\lambda_o)^{q-1} \int_{Q_{r_1}} |Du|^{p(\cdot)} dz \\
& \quad + \bar{c} A^{q-1} G(M, R) \int_{Q_{r_2}} |Du|^{p(\cdot)} T_k(|Du|^{p(\cdot)})^{q-1} dz \\
(8.19) \quad & \quad + \bar{c} A^{q-1} M^q G(M, R) \int_{Q_{r_2}} (|F|+1)^{p(\cdot)q} dz,
\end{aligned}$$

where $\bar{c} = \bar{c}(\text{data})$. Note that the estimate stays stable as $q \downarrow 1$.

8.5. Choice of the parameters. We now perform the choices of the parameters M and R_o in such a way that $\bar{c} A^{q-1} G(M, R) \leq \frac{1}{2}$ whenever $R \leq R_o$. First, we choose $M = M(\text{data}, q) \geq 1$ large enough to have

$$\frac{\bar{c} A^{q-1}}{M^{1-\frac{1}{\gamma_1}}} \leq \frac{1}{4}.$$

Next, we reduce the value of R_o , now depending on $data, K, \omega(\cdot), \tilde{\omega}(\cdot), q$, in such a way that for any $\varrho \leq R_o$ we have

$$(8.20) \quad \bar{c} A^{q-1} \left[\omega(\Gamma(2\varrho)^\alpha) M \log \left(\frac{K}{\varrho} \right) + [\tilde{\omega}(\varrho)]^{\frac{\varepsilon_1}{2\gamma_1}} \right]^{\frac{1}{2}} \leq \frac{1}{4}.$$

Note that this is possible due to the assumptions (2.4) and (2.8). Recalling the definition of G in (8.13) we therefore have $\bar{c} A^{q-1} G(M, R) \leq \frac{1}{2}$ for any $R \leq R_o$. Using this in (8.19) we get

$$\begin{aligned} & \int_{Q_{r_1}} |Du|^{p(\cdot)} T_k (|Du|^{p(\cdot)})^{q-1} dz \\ & \leq \frac{1}{2} \int_{Q_{r_2}} |Du|^{p(\cdot)} T_k (|Du|^{p(\cdot)})^{q-1} dz \\ & \quad + c \left(\frac{R}{r_2 - r_1} \right)^\beta \lambda_o^{q-1} \int_{Q_{2R}} |Du|^{p(\cdot)} dz + c \int_{Q_{2R}} (|F| + 1)^{p(\cdot)q} dz, \end{aligned}$$

where $\beta \equiv (n+2)(q-1)d$ and with a constant $c = c(data, q)$. At this point we apply Lemma 3.1 with

$$\phi(r) \equiv \int_{Q_r} |Du|^{p(\cdot)} T_k (|Du|^{p(\cdot)})^{q-1} dz,$$

and

$$\mathcal{A} \equiv c \int_{Q_{2R}} (|F| + 1)^{p(\cdot)q} dz \quad \text{and} \quad \mathcal{B} \equiv c R^\beta \lambda_o^{q-1} \int_{Q_{2R}} |Du|^{p(\cdot)} dz,$$

yielding that

$$\int_{Q_R} |Du|^{p(\cdot)} T_k (|Du|^{p(\cdot)})^{q-1} dz \leq c(\beta) \left[\mathcal{A} + \frac{\mathcal{B}}{R^\beta} \right].$$

Passing to the limit $k \rightarrow \infty$ which is possible by Fatou's lemma and taking averages we find that

$$(8.21) \quad \int_{Q_R} |Du|^{p(\cdot)q} dz \leq c \left[\lambda_o^{q-1} \int_{Q_{2R}} |Du|^{p(\cdot)} dz + \int_{Q_{2R}} (|F| + 1)^{p(\cdot)q} dz \right].$$

Note that $c = c(data, q)$, since β depends continuously on $p(\cdot)$, i.e. the dependence upon $p(\cdot)$ via the parameter d can be replaced by a dependence on γ_1 and γ_2 . Since $Q_{2R} \Subset \Omega_T$ was arbitrary, we have thus proved the first assertion in Theorem 2.2, i.e. that $|Du|^{p(\cdot)} \in L_{loc}^q(\Omega_T)$. It now remains to show the estimate (2.10).

8.6. Adjusting the exponent. Here, we first observe that (8.21) together with the definition of λ_o in (8.1) lead to estimate (2.10) in Theorem 2.2, but with d instead of $d(p_o)$, with $p_o := p(\mathfrak{z}_o)$ and \mathfrak{z}_o is the center of the cylinder $Q_{2R} \equiv Q_{2R}(\mathfrak{z}_o)$. We recall that d was defined in (2.11) and $d \geq d(p_o)$. In order to reduce the exponent from d to $d(p_o)$ we need to show a bound of the following form

$$(8.22) \quad \mathfrak{E}^{d-d(p_o)} \leq c(n, \gamma_1), \quad \text{where} \quad \mathfrak{E} := \int_{Q_{2R}} |Du|^{p(\cdot)} + (|F| + 1)^{p(\cdot)q} dz.$$

To this aim we first shall deduce an upper bound for $d - d(p_o)$ in terms of $\omega(R)$. Since $d(p(\cdot))$ is continuous there exists $\hat{z} \in \bar{Q}_R$ such that $d = d(p(\hat{z}))$. From the definition of $d(\cdot)$ in (2.11) we observe that

$$d(p_o) \geq \max \left\{ \frac{p_o}{2}, \frac{2p_o}{p_o(n+2) - 2n} \right\}.$$

In the following we distinguish the cases where $p(\hat{z})$ is larger, respectively smaller than 2. In the case $p(\hat{z}) \geq 2$ we get from (2.5) that

$$d - d(p_o) = \frac{p(\hat{z})}{2} - d(p_o) \leq \frac{p(\hat{z})}{2} - \frac{p_o}{2} \leq \frac{1}{2} \omega(R),$$

while in the case $p(\hat{z}) < 2$ we have $p(\hat{z}) \leq p_o$ and therefore we find in a similar way that

$$\begin{aligned} d - d(p_o) &\leq \frac{2p(\hat{z})}{p(\hat{z})(n+2) - 2n} - \frac{2p_o}{p_o(n+2) - 2n} \\ &= \frac{4n(p_o - p(\hat{z}))}{[p(\hat{z})(n+2) - 2n][p_o(n+2) - 2n]} \leq \frac{4n}{[\gamma_1(n+2) - 2n]^2} \omega(R). \end{aligned}$$

Hence, in any case we have proved that $d - d(p_o) \leq c(n, \gamma_1)\omega(R)$. Recalling the definition of \mathfrak{E} from (8.22) and using (2.9) we thus obtain

$$\mathfrak{E}^{d-d(p_o)} \leq c(n, \gamma_1) [R^{-(n+2)} K]^{c(n, \gamma_1)\omega(R)} \leq c(n, \gamma_1).$$

We note that the last inequality is a consequence of the logarithmic continuity of ω from (2.5), since $R^{-\omega(R)} \leq e$ and

$$K^{\omega(R)} = \exp[\omega(R) \log K] \leq \exp[\omega(R) \log(\frac{1}{R})] \leq e$$

provided $R \leq R_o \leq \min\{R_1, 1/K\}$, where R_1 is the radius from (2.5). This finishes the proof of (8.22) and by the reasoning from above we therefore obtain the asserted estimate (2.10). We have thus completed the proof of Theorem 2.2. \square

8.7. Proof of Remark 2.3. Here, it is enough to ensure that we can choose $R_o > 0$ and ε_{BMO} in such a way that (8.20) is satisfied. Assuming for instance

$$[a]_{\text{BMO}} \leq \varepsilon_{\text{BMO}} := \left(\frac{1}{8\bar{c}A^{q-1}}\right)^{\frac{4\gamma'_1}{\varepsilon_1}} \text{ and } \omega(\Gamma(2\varrho)^\alpha)M \log\left(\frac{K}{\varrho}\right) \leq \left(\frac{1}{8\bar{c}A^{q-1}}\right)^2$$

for any $\varrho \leq R_o$ we conclude that (8.20) holds, since $\tilde{\omega}(\varrho) \leq [a]_{\text{BMO}}$. The rest of the proof is completely the same as the one of Theorem 2.2.

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