Localization of nonlocal gradients in various topologies

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Abstract

In this paper, we study nonlocal gradients and their relationship to classical gradients. As the nonlocality vanishes we demonstrate the convergence of nonlocal gradients to their local analogue for Sobolev and BV functions. As a consequence of these localizations we give new characterizations of the Sobolev and BV spaces that are in the same spirit of Bourgain, Brezis, and Mironsecu's 2001 characterization. Integral functionals of the nonlocal gradient with proper growth are shown to converge to a corresponding functional of the classical gradient both pointwise and in the sense of Γ-convergence.

1 Introduction and main results

For a given $\Omega \subset \mathbb{R}^N$ that is open, bounded and sufficiently smooth, our interest in this paper is focused on the linear nonlocal operator

$$\begin{aligned} \mathcal{G}_\rho u(x) &:= p.v.\, N \int_\Omega \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho(x-y)\, dy \\ &= \lim_{\epsilon \to 0} N \int_{\Omega \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho(x-y)\, dy, \end{aligned} \tag{1.1}$$

whenever it exists for almost every $x \in \Omega$. Here, the kernel ρ is a nonnegative integrable radial function and $u \in L^1(\Omega)$ is a scalar valued function. By definition, $\mathcal{G}_\rho u(x)$ is a directed weighted difference quotient of u at x, and as such we call $\mathcal{G}_\rho u(x)$ a *nonlocal gradient of* u at x and the operator $\mathcal{G}_\rho$ the *nonlocal gradient operator*. The use of "gradient" for $\mathcal{G}_\rho$ is motivated by the fact that, as we will see shortly, it approximates the classical gradient operator as the nonlocality vanishes.

In recent years, nonlocal "differential" operators like (1.1) have appeared in a number of applications, most notably in the modeling of discontinuous physical, biological and social quantities. We mention, for instance, some in image processing [4] and [16, 17], the peridynamic model of continuum mechanics [13, 23, 24] and nonlocal diffusion [3, 18]. Various nonlocal operators have also been used implicitly in a number of works such as [5, 6], [8, 9, 25], [13, 14], and [3], for example. A basis to our work is the nonlocal vector calculus developed by Du et al in [14] that provides a framework to rigorously analyze nonlocal differential operators and their relationship to corresponding classical operators.

Our contribution in this paper is a further development of the analysis of nonlocal gradient operators that is initiated in [14]. Chief among them is a rigorous justification of the "localization" of $\mathcal{G}_\rho u(x)$ to the classical gradient $\nabla u(x)$, as one might expect, where the topology of localization depends on the functional space of the underlying field. In fact, we will demonstrate that such localization is possible for smooth function spaces and Sobolev spaces in their respective strong topologies (local uniform convergence in the former and strong L^p convergence in the latter), while for the space of functions of bounded variation, this localization is possible in the strict topology of measures. Not surprisingly, as a corollary of the localization result, we obtain yet another characterization of Sobolev and BV spaces. This characterization is in the same spirit of existing derivative-free characterizations of Sobolev and BV spaces, see [5, 6, 12, 19]. We will also present a Γ-convergence result for some integral functionals of nonlocal gradients.

To state our main results precisely, we adopt the framework of Bourgain, Brezis, and Mironsecu [5] and introduce a sequence of kernels forming an approximation of the identity that facilitate the localization mechanism. Let ρ_n be a sequence of radial functions, $\rho_n(x) = \hat{\rho}_n(|x|)$, that satisfy

$$\begin{cases} \rho_n \geq 0, \quad \int_{\mathbb{R}^N} \rho_n(x)\, dx = 1, \text{ and} \\ \lim_{n\to\infty} \int_{|x|>\delta} \rho_n(x)\, dx = 0 \quad \text{for all } \delta > 0. \end{cases} \tag{1.2}$$

For each $n \geq 1$, denote $\mathcal{G}_n u := \mathcal{G}_{\rho_n} u$. With formal computation Gilboa and Osher in [17] noted that

$$\mathcal{G}_n u(x) = \nabla u(x) + error,$$

while this relationship was made rigorous by Du, Gunzburger, Lehoucq and Zhou in [14], where it was shown that if $u \in H^1(\mathbb{R}^N;\mathbb{R}^d)$, then

$$\mathcal{G}_n u \to \nabla u \tag{1.3}$$

in $L^2(\mathbb{R}^N;\mathbb{R}^{d\times N})$ as $n \to \infty$. In addition, *distributional* localization of various nonlocal differential operators to their corresponding local differential operators is demonstrated in [14].

It is not clear how one would extend the techniques of [14] to obtain localization in strong topologies for functions in the Sobolev spaces $W^{1,p}(\Omega)$ $(p \neq 2)$ or the space of functions of bounded variation $BV(\Omega)$. In the case of (1.3), the use of the Fourier transform on the Hilbert space $H^1(\mathbb{R}^N;\mathbb{R}^d)$ seems to preclude application in the non-Hilbert setting. Meanwhile, the distributional localizations occur in a very weak topology, giving convergence for smooth functions which are compactly supported. In fact, this result, combined with a nonlocal integration by parts formula that relates $\mathcal{G}_\rho$ with a yet to be defined *nonlocal divergence* operator, allows one to deduce weak (or weak-star) convergence of the nonlocal gradients to their local counterpart when the underlying function is in a Sobolev or BV space. However, convergence in stronger topologies requires more subtle analysis.

We will shortly mention our localization results for smooth, Sobolev, and BV functions. First, a remark is in order as to the definition of the nonlocal gradient for functions in these spaces. If $u \in C^1(\overline{\Omega})$ it can easily be checked that $y \mapsto \frac{|u(y)-u(x)|}{|x-y|}\rho(y-x) \in L^1(\Omega)$ for almost all $x \in \Omega$, and therefore not only is the principle value integral (1.1) well-defined, it agrees with the Lebesgue integral. More generally, it is a consequence of the analysis of Bourgain, Brezis, and Mironescu [5] (see also Lemma 2.1) that this continues to hold for functions in $W^{1,p}(\Omega)$ and $BV(\Omega)$. Therefore, the following theorems recording the localization properties of the nonlocal gradient for functions in these spaces can be understood with the nonlocal gradient as a Lebesgue integral.

Theorem 1.1 *Suppose that $\Omega \subset \mathbb{R}^N$ is open, bounded, and sufficiently smooth. Assume also that $1 \leq p < \infty$ and ρ_n satisfy (1.2). Then the following holds.*

a) For any $u \in C^1(\overline{\Omega})$, $\mathcal{G}_n u \to \nabla u$ locally uniformly as $n \to \infty$. If $u \in C_c^1(\Omega)$, then the convergence is uniform.

b) For any $u \in W^{1,p}(\Omega)$, $\mathcal{G}_n u \to \nabla u$ in $L^p(\Omega; \mathbb{R}^N)$ as $n \to \infty$.

A similar result holds for BV functions which is stated below.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, and smooth. Assume ρ_n satisfy (1.2) and that $u \in BV(\Omega)$. Consider the sequence of vector-valued Radon measures $\boldsymbol{\mu}_n = \mathcal{G}_n u \, \mathcal{L}^N$. Then $\boldsymbol{\mu}_n \to Du$ strictly as measures. That is,*

$$\boldsymbol{\mu}_n \overset{*}{\rightharpoonup} Du$$

weakly-star in $\left(C_0(\Omega; \mathbb{R}^N)\right)'$ and

$$|\boldsymbol{\mu}_n|(\Omega) \to |Du|(\Omega).$$

In the above theorem the variation measure $|\boldsymbol{\mu}|$ is defined for any open subset A of Ω as

$$|\boldsymbol{\mu}|(A) = \sup \left\{ \int_A \boldsymbol{\phi} \cdot d\boldsymbol{\mu} : \boldsymbol{\phi} \in C_0(\Omega; \mathbb{R}^N),\ \mathrm{Supp}(\boldsymbol{\phi}) \subset A,\ \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

The above localization results will be used to characterize Sobolev and BV functions in terms of the asymptotics of their nonlocal gradients. For one direction of this characterization, we make use of the assumption that $u \in W^{1,p}(\Omega)$ or $u \in BV(\Omega)$ and deduce that $\mathcal{G}_\rho u$ is well-defined before applying the above theorems. However, for the converse, we relax our assumptions on the existence of the nonlocal gradient as a Lebesgue integral or even as a principle value integral and make use of a broader distributional definition of the operator $\mathcal{G}_\rho u$ for arbitrary $u \in L^1(\Omega)$. To make this precise, let us introduce the nonlocal operator defined by

$$(\mathfrak{D}_\rho)_i \phi_i(x) = -p.v.\ N \int_\Omega \frac{\phi_i(x) + \phi_i(y)}{|x-y|} \frac{x_i - y_i}{|x-y|} \rho(x-y)\, dy$$

for any $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$ measurable. Then we define *the nonlocal divergence* as

$$\mathfrak{D}_\rho \boldsymbol{\phi}(x) = \sum_{i=1}^N (\mathfrak{D}_\rho)_i \phi_i(x) = -p.v.\ N \int_\Omega \frac{\boldsymbol{\phi}(x) + \boldsymbol{\phi}(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho(x-y)\, dy.$$

For smooth vector fields this operator is related to the classical divergence operator (hence the name) and is the adjoint of the nonlocal gradient operator. Moreover, as will be shown in the next section, for all $x \in \Omega$, and $\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$,

$$|\mathfrak{D}_\rho \boldsymbol{\phi}(x)| \leq 3N\|\nabla\boldsymbol{\phi}\|_{L^\infty(\Omega)}\|\rho\|_{L^1(\mathbb{R}^d)}. \tag{1.4}$$

Definition 1.3 *Given $u \in L^1(\Omega)$, we define the* distribution nonlocal gradient $\mathfrak{G}_\rho u$ *as*

$$\langle \mathfrak{G}_\rho u, \boldsymbol{\phi}\rangle := -\int_\Omega u(x)\mathfrak{D}_\rho\boldsymbol{\phi}(x)\,dx, \quad \forall \boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^N).$$

This definition echoes the notion of distributional derivatives, and in fact, inequality (1.4) implies that $\mathfrak{G}_\rho u$ is a distribution, since

$$|\langle \mathfrak{G}_\rho u, \boldsymbol{\phi}\rangle| \leq 3N\|\nabla\boldsymbol{\phi}\|_{L^\infty(\Omega)}\|\rho\|_{L^1(\mathbb{R}^d)}\|u\|_{L^1(\Omega)}.$$

The following theorem shows that this notion of distributional nonlocal gradient agrees with the nonlocal gradient whenever it is well-defined as a Lebesgue integral, and therefore for Sobolev and BV functions.

Theorem 1.4 (Nonlocal integration by parts) *Suppose that $u \in L^1(\Omega)$ and $\frac{|u(x)-u(y)|}{|x-y|}\rho(x-y) \in L^1(\Omega\times\Omega)$. Then $\mathcal{G}_\rho u \in L^1(\Omega;\mathbb{R}^N)$ and for any $\boldsymbol{\phi} \in C_c^1(\Omega;\mathbb{R}^N)$*

$$\int_\Omega \mathcal{G}_\rho u(x)\cdot\boldsymbol{\phi}(x)\,dx = -\int_\Omega u(x)\,\mathfrak{D}_\rho\boldsymbol{\phi}(x)\,dx. \tag{1.5}$$

In this case, by definition, the distribution $\mathfrak{G}_\rho u$ is precisely the function $\mathcal{G}_\rho u$.

As a consequence of the above theorem we have the following theorem characterizing Sobolev spaces, along with the continuity of (nonlinear) integral functionals of the nonlocal gradient.

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, and sufficiently smooth. Assume ρ_n satisfy* (1.2) *and that $u \in L^p(\Omega)$ for some $1 < p < \infty$. Then $u \in W^{1,p}(\Omega)$ if and only if the sequence of distributions $\{\mathfrak{G}_n u\}$ is a bounded sequence in $L^p(\Omega;\mathbb{R}^N)$. Moreover, if f is continuous satisfying the growth condition $|f(z)| \leq C(1+|z|^p)$ $(C>0)$, then*

$$\lim_{n\to\infty}\int_\Omega f(\mathfrak{G}_n u)\,dx = \int_\Omega f(\nabla u)\,dx. \tag{1.6}$$

We remark that when $\frac{|u(x)-u(y)|}{|x-y|}\rho_n(x-y) \in L^1(\Omega\times\Omega)$ for all n, by Theorem 1.4, $\mathfrak{G}_n u = \mathcal{G}_n u$. Moreover, applying Holder's inequality

$$\int_\Omega |\mathcal{G}_n u|^p\,dx \leq N^p \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(x-y)\,dydx. \tag{1.7}$$

In [5] and [19], the finiteness of the limit of the right hand side in (1.7) was shown to be a necessary and sufficient condition for $u \in W^{1,p}(\Omega)$. Theorem 1.5 states that the finiteness of the limit of the left hand side (which is smaller) can used to test if $u \in W^{1,p}(\Omega)$. Moreover, when using the quantity on the right hand side, Bourgain, Brezis and Mironescu [5] had shown that in the limit one recovers a constant multiple of the L^p-norm of ∇u, while we recover the exact L^p-norm of ∇u by using the left hand side. In these aspects our characterization is tighter.

When $p = 1$, the corresponding theorem characterizes the space of functions of bounded variation, see also [12].

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, and sufficiently smooth. Assume ρ_n satisfy* (1.2)*, and that $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if the sequence of distributions $\{\mathfrak{G}_n u\}$ is a bounded sequence in $L^1(\Omega;\mathbb{R}^N)$. In this case, in addition to strict convergence, $\mathfrak{G}_n u$ converges to Du in the stronger sense*

$$\lim_{n\to\infty} \int_\Omega \sqrt{1+|\mathfrak{G}_n u|^2}\, dx = \int_\Omega \sqrt{1+|D^a u|^2}\, dx + |D^s u|(\Omega). \tag{1.8}$$

As a consequence,

$$\lim_{n\to\infty} \int_\Omega f(\mathfrak{G}_n u(x))dx = \int_\Omega f(D^a u)\, dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| \tag{1.9}$$

for any continuous f that is convex (or concave) with at most linear growth, $|f(z)| \leq C(1+|z|)$, for some $C > 0$ and for all $z \in \mathbb{R}^N$.

In the above theorem we have used the Radon-Nikodým decomposition $Du = D^a u\,\mathcal{L}^N + D^s u$ where $D^a u$ is the approximate gradient of u and $D^s u$ is singular with respect to the Lebesgue measure. We have also used the notation $\frac{dD^s u}{d|D^s u|}$ to represent the Radon-Nikodým derivative of $D^s u$ with respect to its total variation $|D^s u|$. Finally, the function f^∞ is the recession function of f, for any $z \in \mathbb{R}^d$ is defined as

$$f^\infty(z) := \limsup_{t\to\infty} \frac{f(tz)}{t}.$$

The convergence (1.8) is precisely $\langle\cdot\rangle-$ strict convergence of measures introduced in [20]. Equation (1.9) holds for bounded continuous functions f (in this case $f^\infty \equiv 0$) as well as f continuous and 1-homogeneous where $f^\infty \equiv f$, see [20, Theorem 5] and [1, Lemma 2.2].

As the above results demonstrate, the consideration of integral functionals of the nonlocal gradient allows one to obtain general theorems concerning the convergence of energies of the nonlocal gradient to the corresponding local energy as the nonlocality vanishes. More than this pointwise convergence of energies, we also have the following result on the Γ-convergence of the nonlocal energies.

Theorem 1.7 *Let $1 < p < \infty$ and suppose $\Omega \subset \mathbb{R}^N$ is open, bounded, and smooth. Assume ρ_n satisfy* (1.2) *and f_p satisfies*

$$c|z|^p \leq f_p(z) \leq C(1+|z|^p). \tag{1.10}$$

Then

$$\Gamma^-_{L^p(\Omega)}\text{-}\lim_{n\to\infty} \int_\Omega f_p(\mathcal{G}_n u)\, dx = \begin{cases} \displaystyle\int_\Omega f_p^{**}(\nabla u)\, dx, & u \in W^{1,p}(\Omega) \\ +\infty & \textit{otherwise} \end{cases}$$

where f^{**} is the greatest convex function on $\mathbb{R}^N$ majorized by f, and the Γ-limit is taken with respect to the strong topology of $L^p(\Omega)$.

In the case $p = 1$, a similar closed form Γ-limit will also be obtained, see Section 4.

Remark 1.8 *The above localization results for nonlocal gradients give the complete analogy to the localization and characterization results for nonlocal functionals initiated by Bourgain, Brezis, and Mironescu [5]. In particular, in this paper we show the nonlocal gradient analogy to localization in and characterization of Sobolev spaces [5], of BV spaces [12], and in the sense of* Γ*-convergence [22].*

Finally, we remark that nonlocal gradients can be defined for vector fields as well, naturally extending the definition (1.1) as (see [14] for more)

$$\mathcal{G}_n\mathbf{u}(x) := Np.v.\int_\Omega \frac{[\mathbf{u}(x)-\mathbf{u}(y)]}{|x-y|} \otimes \frac{[x-y]}{|x-y|}\rho_n(x-y)\,dy.$$

All of the above theorems remain valid for vector fields, with the exception of Theorem 1.7. This is particularly interesting, since there have been no nonlocal characterizations of the space of functions of Bounded Variation in the vector-valued setting. We should note, however, that there is a nonlocal characterization of the space of functions of Bounded Deformation, see [21].

The organization of the remainder of the paper is as follows. In Section 2, we show that the principle value integral definition of nonlocal gradients can be understood as a Lebesgue integral for weakly differentiable functions. We will also prove several results concerning the nonlocal divergence operator, as well as to demonstrate a proof of Theorem 1.4, nonlocal integration by parts. In Section 3, we will prove the localization theorems stated in the introduction, as well as our results concerning the characterization of Sobolev and BV spaces. Finally, in Section 4, we conclude the paper with the proof of two theorems asserting the Γ-convergence of the nonlocal energies to the relaxation of the local energy, treating the cases $1 < p < +\infty$ and $p = 1$ separately.

2 Nonlocal calculus

2.1 Nonlocal gradient operator

Our approach to localization is based on approximation by smooth functions of Sobolev and BV functions in the strong and strict topology, respectively. For Sobolev functions, this involves taking advantage of the uniform boundedness of the linear operator $\mathcal{G}_n u$ and the density of smooth functions, while for BV functions the result becomes more technical, requiring a careful upper bound of the total variation. We record a lemma that is essentially in [5], if not for the presence of $\mathcal{G}_\rho$.

Lemma 2.1 *Suppose that* $\Omega \subset \mathbb{R}^N$ *is open, bounded, and sufficiently smooth,* $\rho(z) \in L^1(\mathbb{R}^N)$ *and* $1 \leq p < \infty$*. Then the operator* $\mathcal{G}_\rho : W^{1,p}(\Omega) \to L^p(\Omega;\mathbb{R}^N)$ *is a bounded operator with the estimate*

$$\|\mathcal{G}_\rho u\|_{L^p} \leq C\|\rho\|_{L^1}\|\nabla u\|_{L^p}, \text{ for all } u \in W^{1,p}(\Omega)$$

Similarly $\mathcal{G}_\rho : BV(\Omega) \to L^1(\Omega;\mathbb{R}^N)$ *with the estimate*

$$\|\mathcal{G}_\rho u\|_{L^1(\Omega)} \leq C\|\rho\|_{L^1}|Du|(\Omega), \text{ for all } u \in BV(\Omega).$$

Here, $C = C(p, N, \Omega) > 0$.

Proof. Applying Hölder's inequality and [5][Theorem 1] we obtain

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho(x-y)\, dy \right)^p dx \\
&\le \int_\Omega \left[\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho(x-y)\, dy \right] \left[\int_\Omega \rho(x-y)\, dy \right]^{\frac{p}{p'}} dx \\
&\le \|\rho\|_{L^1}^{p-1} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho(x-y)\, dy dx \\
&\le C(p,\Omega) \|\rho\|_{L^1}^p \|\nabla u\|_{L^p(\Omega)}^p,
\end{aligned}$$

The above estimate implies that

$$\int_\Omega |\mathcal{G}_\rho u(x)|^p\, dx \le N^p \int_\Omega \left(\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\, \rho(x-y)\, dy \right)^p dx \le N^p C(p,\Omega) \|\rho\|_{L^1}^p \|\nabla u\|_{L^p(\Omega)}^p.$$

The subsequent statement for BV functions follows from the density of $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ in BV with respect to the strict convergence. ■

In the case of functions satisfying a Lipschitz condition, we have the following estimate.

Lemma 2.2 *If $u \in Lip(\Omega)$, the set of Lipschitz functions, then $\mathcal{G}_\rho u \in L^\infty(\Omega; \mathbb{R}^N)$ and*

$$||\mathcal{G}_\rho u||_{L^\infty} \le N \|\rho\|_{L^1}\, Lip(u; \Omega).$$

We remark that the notion of nonlocal gradient is not restricted to functions that have some form of classical derivatives. Depending on the severity of the singularity on ρ, $\mathcal{G}_\rho u$ may be well-defined as a Lebesgue integral even for discontinuous functions. Indeed, the space $BV(\Omega)$ includes discontinuous functions. More generally, for $s \in (0,1)$ one may take $\rho(z) = |z|^{-N+(1-s)}$. For this ρ, an estimate similar to the proof of Lemma 2.1 shows that $\mathcal{G}_\rho u \in L^1(\Omega; \mathbb{R}^N)$ for any $u \in W^{s,1}(\Omega)$ with the estimate

$$\|\mathcal{G}_\rho u\|_{L^1(\Omega)} \le C|u|_{W^{s,1}}.$$

2.2 Nonlocal divergence and integration by parts

We recall the definition of the nonlocal divergence operator given in the introduction:

$$(\mathfrak{D}_\rho)\, \boldsymbol{\phi}(x) = -p.v.\ N \int_\Omega \frac{\boldsymbol{\phi}(x) + \boldsymbol{\phi}(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho(x-y)\, dy.$$

Lemma 2.3 *Suppose that ϕ_i is measurable. Then for $x \in \Omega$ such that $\phi_i(x) < \infty$, we have $(\mathfrak{D}_\rho)_i \phi_i(x) < \infty$ if and only if $(\mathcal{G}_\rho)_i \phi_i(x) < \infty$.*

Proof. Fix $x \in \Omega$ and let $0 < \delta(x) = dist(x, \partial\Omega)$. For any $\epsilon \in (0, \delta(x))$, we have that

$$\begin{aligned}
&-\chi_{[\epsilon,\infty)}(|x-y|) \frac{\phi_i(x) + \phi_i(y)}{|x-y|} \frac{x_i - y_i}{|x-y|} \rho(x-y) + 2\phi_i(x) \chi_{[\epsilon,\delta(x))}(|x-y|) \frac{x_i - y_i}{|x-y|^2} \rho(x-y) \\
&= \chi_{[\epsilon,\infty)}(|x-y|) \frac{\phi_i(x) - \phi_i(y)}{|x-y|} \frac{x_i - y_i}{|x-y|} \rho(x-y) - 2\phi_i(x) \chi_{[\delta(x),\infty)}(|x-y|) \frac{x_i - y_i}{|x-y|^2} \rho(x-y)
\end{aligned}$$

Multiplying by N, integrating both sides in y we obtain

$$-N\int_{\Omega\setminus B(x,\epsilon)} \frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\,dy = N\int_{\Omega\setminus B(x,\epsilon)} \frac{\phi_i(x)-\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\,dy \\ - 2N\int_{\Omega}\phi_i(x)\chi_{[\delta(x),\infty)}(|x-y|)\frac{x_i-y_i}{|x-y|^2}\rho(x-y)\,dy. \tag{2.1}$$

Now we let $\epsilon\to 0$ to obtain

$$(\mathfrak{D}_\rho)_i\,\phi_i(x) = (\mathcal{G}_\rho)_i\,\phi_i(x) - 2N\,\phi_i(x)\int_\Omega \chi_{[\delta(x),\infty)}(|x-y|)\frac{x_i-y_i}{|x-y|^2}\rho(x-y)dy \tag{2.2}$$

■

Equation (2.2) gives a formula for the nonlocal divergence operator in terms of the nonlocal gradient and will be crucial in establishing some useful estimates later. When $\Omega = \mathbb{R}^N$ and ρ is compactly supported in a ball, (2.2) is derived in [14, Lemma 5.1] and used to establish localization in $H^1(\mathbb{R}^N)$.

Corollary 2.4 *For any $\boldsymbol{\phi}\in C_c^1(\Omega;\mathbb{R}^N)$, $\mathfrak{D}_\rho\boldsymbol{\phi}\in L^\infty(\Omega)$. Moreover,*

$$\|(\mathfrak{D}_\rho)_i\phi_i\|_{L^\infty(\Omega)} \le 3N\|\nabla\phi_i\|_{L^\infty(\Omega)}\|\rho\|_{L^1}.$$

Proof. We use equation (2.2) to prove the corollary. By Lemma 2.2, $|(\mathcal{G}_\rho)_i\phi_i| \le N\|\nabla\phi_i\|_{L^\infty(\Omega)}\|\rho\|_{L^1}$, and therefore it suffices to estimate the second term in the right hand side of (2.2). To that end, corresponding to $x\in\Omega$, choose $\xi(x)\in[\Omega\setminus supp(\phi_i)]\cap B(x,\delta(x))$. Then $\phi_i(\xi(x))=0$ and so

$$\phi_i(x)\int_\Omega \chi_{[\delta(x),\infty)}(|x-y|)\frac{x_i-y_i}{|x-y|^2}\rho(x-y)dy = \int_\Omega \chi_{[\delta(x),\infty)}(|x-y|)\phi_i(x)-\phi_i(\xi(x))\frac{x_i-y_i}{|x-y|^2}\rho(x-y)dy$$

Using the estimate $|x-\xi(x)|\le\delta(x)\le|x-y|$ when $\delta(x)\le|x-y|$, we have

$$\left|\int_\Omega \chi_{[\delta(x),\infty)}(|x-y|)\phi_i(x)-\phi_i(\xi(x))\frac{x_i-y_i}{|x-y|^2}\rho(x-y)dy\right| \\ \le \|\nabla\phi_i\|_{L^\infty(\Omega)}\int_\Omega \chi_{[\delta(x),\infty)}(|x-y|)\frac{|x-\xi(x)||x_i-y_i|}{|x-y|^2}\rho(x-y)dy \le \|\nabla\phi_i\|_{L^\infty(\Omega)}\|\rho\|_{L^1}.$$

This completes the proof. ■

Remark 2.5 *Equation* (2.1) *also give us the useful estimate*

$$\sup_{\epsilon>0,x\in\Omega}\left|-N\int_{\Omega\setminus B(x,\epsilon)}\frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\,dy\right| \le 3N\|\nabla\phi_i\|_{L^\infty(\Omega)}\|\rho\|_{L^1}. \tag{2.3}$$

The next lemma state a further relation between the nonlocal divergence and classical divergence as the nonlocality vanishes.

Lemma 2.6 *Suppose that ρ_n are given by* (1.2)*. Define $\mathfrak{D}_n := \mathfrak{D}_{\rho_n}$. Then for each $\boldsymbol{\phi} = (\phi_1,\phi_2,\dots,\phi_N)\in C_c^1(\Omega;\mathbb{R}^N)$, we have the following convergences.*

a) (Uniform convergence)

$$\lim_{n\to\infty} \|\mathfrak{D}_n \boldsymbol{\phi} - \mathit{Tr}(\mathcal{G}_n \boldsymbol{\phi})\|_{L^\infty(\Omega)} = 0, \ \text{where } \mathit{Tr}(\mathcal{G}_n \boldsymbol{\phi}) = \sum_{i=1}^N (\mathcal{G}_n)_i \phi_i \ .$$

b) (Localization) As a consequence, for any $1 \le p \le \infty$

$$\lim_{n\to\infty} \|\mathfrak{D}_n \boldsymbol{\phi} - \operatorname{div} \boldsymbol{\phi}\|_{L^p(\Omega)} = 0.$$

Proof. To prove part a) we recall equation (2.2) that relates $(\mathfrak{D}_n)_i \phi_i$ and $(\mathcal{G}_n)_i \phi_i$, we see that for all $x \in \Omega \setminus \operatorname{Supp}(\boldsymbol{\phi})$,

$$(\mathfrak{D}_n)_i \, \phi_i(x) - (\mathcal{G}_n)_i \, \phi_i(x) = 0. \tag{2.4}$$

On the other hand, for all $x \in Supp(\boldsymbol{\phi})$, using the fact that $\delta(x) = \operatorname{dist}(x, \partial\Omega) \ge \gamma = \operatorname{dist}(Supp(\boldsymbol{\phi}), \partial\Omega)$, and

$$\int_\Omega \chi_{[\delta(x),\infty)}(|x-y|) \frac{x_i - y_i}{|x-y|^2} \rho_n(x-y) dy = \int_\Omega \chi_{[\gamma,\infty)}(|x-y|) \frac{x_i - y_i}{|x-y|^2} \rho_n(x-y) dy,$$

we obtain that

$$|\,(\mathfrak{D}_n)_i \, \phi_i(x) - (\mathcal{G}_n)_i \, \phi_i(x)| \le \frac{2N}{\gamma} \|\phi\|_{L^\infty(\Omega)} \int_{|z|>\gamma} \rho_n(z) dz.$$

It then follows from above that

$$\|\,(\mathfrak{D}_n)_i \, \phi_i - (\mathcal{G}_n)_i \, \phi_i\|_{L^\infty(\Omega)} \le \frac{2N}{\gamma} \|\phi\|_{L^\infty(\Omega)} \int_{|z|>\gamma} \rho_n(|z|) dz \to 0, \ \text{as } n \to \infty \tag{2.5}$$

To prove part b) we note that the case $p = +\infty$ follows from (2.5), equation (2.4), and part a) of Theorem 1.1 which will be proved in the next section, while the fact that Ω is bounded implies that this holds for all $1 \le p < \infty$. ■

Finally, we prove the nonlocal integration by parts given in Theorem 1.4 that gives an important relationship between the two nonlocal operators.

Proof of Theorem 1.4. Let us begin by observing that, on the one hand, since $\frac{|u(y)-u(x)|}{|x-y|}\rho(y-x) \in L^1(\Omega\times\Omega)$, Lebesgue's dominated convergence theorem and Fubini's theorem imply that

$$\begin{aligned}\frac{1}{N}\int_\Omega \mathcal{G}_\rho u(x) \cdot \boldsymbol{\phi}(x) \, dx &= \lim_{\epsilon\to 0} \int_\Omega \int_{\Omega\setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho(x-y) \cdot \boldsymbol{\phi}(x) \, dydx \\ &= \lim_{\epsilon\to 0} \int_{\Omega\times\Omega\setminus\{|x-y|<\epsilon\}} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho(x-y) \cdot \boldsymbol{\phi}(x) \, d\mathcal{L}^{N^2}(x,y).\end{aligned}$$

On the other hand, using the estimate (2.3), again by Lebesgue's dominated convergence theorem,

$$-\frac{1}{N}\int_{\Omega} u(x)\mathfrak{D}_{\rho}\boldsymbol{\phi}(x)\,dx = \lim_{\epsilon\to 0}\int_{\Omega}\int_{\Omega\setminus B(x,\epsilon)} u(x)\sum_{i=1}^{N}\frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\,dydx$$
$$= \lim_{\epsilon\to 0}\int_{\Omega\times\Omega\setminus\{|x-y|<\epsilon\}} u(x)\sum_{i=1}^{N}\frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\,d\mathcal{L}^{N^2}(x,y).$$

Thus, to prove the lemma, it suffices to establish equality on the set $\{\Omega\times\Omega\setminus\{|x-y|<\epsilon\}\}$, where the singularity of the kernel is removed, and for clarity of presentation we omit this from the integral symbols that follow.

Working with the ith component of the nonlocal gradient, we have

$$\int \frac{u(x)-u(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\phi_i(x)\,d\mathcal{L}^{N^2}(x,y)$$
$$= \int \frac{u(x)\phi_i(x)}{|x-y|}\frac{x_i-y_i}{|x-y|}\,\rho(x-y)\,d\mathcal{L}^{N^2}(x,y) - \int \frac{u(y)\phi_i(x)}{|x-y|}\frac{x_i-y_i}{|x-y|}\,\rho(x-y)\,d\mathcal{L}^{N^2}(x,y).$$

Now, interchanging the roles of x and y in the second integral, using symmetry of the integration domain, we have

$$-\int \frac{u(y)\phi_i(x)}{|x-y|}\frac{x_i-y_i}{|x-y|}\,\rho(x-y)\,d\mathcal{L}^{N^2}(x,y) = \int \frac{u(x)\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\,\rho(x-y)\,d\mathcal{L}^{N^2}(x,y),$$

and therefore,

$$\int \frac{u(x)-u(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)\phi_i(x)\,d\mathcal{L}^{N^2}(x,y) = \int \frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho(x-y)u(x)\,d\mathcal{L}^{N^2}(x,y).$$

Summing from $i=1,\ldots,N$, completes the proof of the lemma. ■

3 Localization of nonlocal gradients

In this section we will study the localization results for nonlocal gradients asserted in the introduction and their application in characterizing Sobolev and BV spaces.

3.1 The convergence of nonlocal gradients to their local analogue

We assume that we have a sequence of radial function ρ_n satisfying (1.2). Let us first prove the following useful lemma.

Lemma 3.1 *Let*

$$c_n^i(x) := \int_{\Omega}\frac{(x_i-y_i)^2}{|x-y|^2}\rho_n(x-y)\,dy. \tag{3.1}$$

Then $Nc_n^i(x)\to 1$ pointwise, and the convergence is uniform on compact sets.

Proof. Let $K \subset \Omega$ be compact, and define $\gamma := dist(K, \partial\Omega)$. We compute, for $x \in K$,

$$\begin{aligned}
\left|Nc_n^i(x) - 1\right| &= \left|N \int_\Omega \frac{(x_i - y_i)^2}{|x-y|^2} \rho_n(x-y)\, dy - 1\right| \\
&\leq \left|N \int_{|x-y|\leq\gamma} \frac{(x_i - y_i)^2}{|x-y|^2} \rho_n(x-y)\, dy - 1\right| + \left|N \int_{|x-y|>\gamma} \frac{(x_i - y_i)^2}{|x-y|^2} \rho_n(x-y)\, dy\right| \\
&\leq \left|\int_{|h|\leq\gamma} \rho_n(h)\, dh - 1\right| + \left|N \int_{|h|>\gamma} \rho_n(h)\, dh\right|. \qquad (3.2)
\end{aligned}$$

Note that in the third inequality, we have used the fact that ρ is radial to write as

$$\int_{B(x,\gamma)} \frac{(x_i - y_i)^2}{|x-y|^2} \rho_n(x-y)\, dy = \frac{1}{N} \int_{B(\mathbf{0},\gamma)} \rho_n(h)\, dh.$$

The right hand side of (3.2) is now independent of x, and so letting $n \to \infty$, we obtain uniform convergence on K. ■

We now proceed to prove Theorem 1.1, which asserts the local uniform convergence of the nonlocal gradient to the gradient whenever the underlying function is smooth on the closed set.

Proof of part a) of Theorem 1.1. Let $u \in C^1(\overline{\Omega})$ and $K \subset \Omega$ be compact. We will show that $\mathcal{G}_n u(x) \to \nabla u(x)$ uniformly for $x \in K$ as $n \to \infty$. By Lemma 3.1, we know that $Nc_n^i(x) \to 1$ uniformly for $x \in K$ as $n \to \infty$, which implies that it suffices to show that for each i from 1 to N, the quantity

$$J_n^i(x) := \left|N \int_\Omega \frac{u(x) - u(y)}{|x-y|} \frac{(x_i - y_i)}{|x-y|} \rho_n(x-y)\, dy - Nc_n^i(x) \frac{\partial u}{\partial x_i}(x)\right|$$

tends to zero uniformly for $x \in K$ as $n \to \infty$. Rewriting J_n^i using the definition (3.1) of c_n^i, we have

$$J_n^i(x) = \left|N \int_\Omega \frac{u(x) - u(y) - \frac{\partial u}{\partial x_i}(x)(x_i - y_i)}{|x-y|} \frac{(x_i - y_i)}{|x-y|} \rho_n(x-y)\, dy\right|.$$

Let $\epsilon > 0$ be arbitrary. Choose δ small so that

$$\left|\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y)\right| < \epsilon \text{ whenever } |x-y| < \delta$$

Then

$$\begin{aligned}
J_n^i(x) \leq& \left|N \int_{B(x,\delta)} \frac{u(x) - u(y) - \frac{\partial u}{\partial x_i}(x)(x_i - y_i)}{|x-y|} \frac{(x_i - y_i)}{|x-y|} \rho_n(x-y)\, dy\right| \\
&+ \frac{2N}{\delta} \left(\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}\right) \int_{|x-y|\geq\delta} \rho_n(x-y)\, dy
\end{aligned}$$

Using mean value theorem, for every $y \in B(x,\delta)$, there exist $\xi(x,y) \in B(x,\delta)$ such that

$$u(x) - u(y) = \frac{\partial u}{\partial x_i}(\xi(x,y))(x_i - y_i),$$

which implies the estimate

$$J_n^i(x) \leq N\epsilon \int_{|z|<\delta} \rho_n(|z|)\, dz + \frac{2N}{\delta} \left(\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} \right) \int_{|z|\geq\delta} \rho_n(|z|)\, dz$$

From the last estimate we obtain that

$$\lim_{n\to\infty} \sup_{x\in K} J_n^i(x) \leq N\epsilon,$$

and sending $\epsilon \to 0$, the result is demonstrated.

If $u \in C_c^1(\Omega)$, then denoting $\gamma = \operatorname{dist}(\partial\Omega, \operatorname{Supp}(u))$, we have from what we established above

$$\sup_{x\in\Omega_{\gamma/2}} J_n^i(x) \to 0, \text{as } n \to \infty$$

where $\Omega_{\gamma/2}$ is the set of points in Ω that are at least $\gamma/2$ distance away from the boundary. For points $x \in \Omega \setminus (\Omega_{\gamma/2})$, we have

$$J_n^i(x) \leq \frac{2N}{\gamma} \|u\|_{L^\infty(\Omega)} \int_{|z|\geq\gamma/2} \rho(|z|)\, dz$$

which converges to 0, uniformly in x, as $n \to \infty$. ■

In the proof of the above theorem, we have used the differentiability and boundedness of u, along with the fact that the derivative is continuous to establish local uniform convergence. Assuming only that $u \in Lip(\Omega)$, a similar proof implies that the nonlocal gradient localizes to the differential of u at any point of differentiability, i.e. pointwise almost everywhere by Rademacher's theorem.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^N$ be open, assume ρ_n satisfy* (1.2)*, and that $u \in Lip(\Omega)$. Then*

$$\mathcal{G}_n u \to \nabla u$$

for $\mathcal{L}^N$ almost every $x \in \Omega$ as $n \to \infty$.

Having proven the localization result for smooth functions, along with the density estimate established in Lemma 2.1, we can now prove part b) of Theorem 1.1, which asserts the L^p strong convergence of the nonlocal gradients to the weak gradient of a Sobolev function.

Proof of part b) of Theorem 1.1. We will use standard density arguments to prove the theorem.

Step 1. The theorem holds true for $v \in C^2(\overline{\Omega})$. Indeed, for any $\tau > 0$, choose δ small enough that $|\Omega \setminus \Omega_\delta| < \tau$, where $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Then

$$\|\mathcal{G}_n v - \nabla v\|_{L^p(\Omega)}^p = \int_{\Omega_\delta} |\mathcal{G}_n v(x) - \nabla v(x)|^p dx + \int_{\Omega\setminus\Omega_\delta} |\mathcal{G}_n v(x) - \nabla v(x)|^p dx.$$

Applying part a) of Theorem 1.1, the first integral goes to 0, as $n \to \infty$. On the other hand, applying Lemma 2.2,

$$\int_{\Omega\setminus\Omega_\delta} |\mathcal{G}_n v(x) - \nabla v(x)|^p dx \leq 2^p \|\nabla v\|_{L^\infty}^p |\Omega \setminus \Omega_\delta| \leq 2^p \|\nabla v\|_{L^\infty}^p \tau.$$

That is, for any $\tau > 0$

$$\lim_{n\to\infty} \|\mathcal{G}_n v - \nabla v\|^p_{L^p(\Omega)} \leq 2^p \|\nabla v\|^p_{L^\infty} \tau,$$

obtaining the convergence.

Step 2. Let $u \in W^{1,p}(\Omega)$. We use the fact that $C^2(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, since Ω is sufficiently regular. Given $\tau > 0$ small, we can find $v \in C^2(\overline{\Omega})$ such that

$$\|\nabla u - \nabla v\|^p_{L^p(\Omega)} \leq \tau.$$

Then by Lemma 2.1, there exists C independent of n such that

$$\|\mathcal{G}_n u - \mathcal{G}_n v\|^p_{L^p(\Omega)} = \|\mathcal{G}_n (u - v)\|^p_{L^p(\Omega)} \leq C\tau$$

for all n. Then for each n we have

$$\|\mathcal{G}_n u - \nabla u\|_{L^p(\Omega)} \leq \|\mathcal{G}_n u - \mathcal{G}_n v\|_{L^p(\Omega)} + \|\mathcal{G}_n v - \nabla v\|_{L^p(\Omega)} + \|\nabla v - \nabla u\|_{L^p(\Omega)}.$$

Taking the limit as $n \to \infty$ and applying Step 1, combined with the above estimates, we obtain

$$\lim_{n\to\infty} \|\mathcal{G}_n u - \nabla u\|_{L^p(\Omega)} \leq (C+1)\tau,$$

completing the proof of the theorem. ■

For BV functions the following lemma gives an estimate for the variation measure $\boldsymbol{\mu}_n$ associated with $\mathcal{G}_n u$ applied on open subset of Ω. We notice that this estimate is a tighter one than that obtained in Lemma 2.1.

Lemma 3.3 *Suppose that Ω is open and bounded with sufficiently smooth boundary, and $A \Subset \Omega$ is an open subset. Then if $u \in BV(\Omega)$, then there exists a sequence of positive numbers $\alpha_n = \alpha_n(u, \Omega, A)$, such that for each $n \geq 1$,*

$$|\boldsymbol{\mu}_n|(A) = \sup_{\substack{\boldsymbol{\phi} \in C_c(A;\mathbb{R}^N) \\ \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \leq 1}} \left| \int_A \mathcal{G}_n u(x) \cdot \boldsymbol{\phi}(x)\, dx \right| \leq (|Du|(\Omega) + \alpha_n,).$$

Moreover, the sequence

$$\alpha_n := C(\Omega) \frac{N}{\gamma^N} \|u\|_{BV} \int_\gamma^\infty \hat{\rho}_n(t)\, t^{N-1}\, dt \to 0, \text{ as } n \to \infty,$$

where $\gamma = dist(A, \partial\Omega) > 0$.

Proof. It suffices to demonstrate that if $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ then the inequality

$$\left| \int_A \mathcal{G}_n u(x) \cdot \boldsymbol{\phi}(x)\, dx \right| \leq (|Du|(\Omega) + \alpha_n) \|\boldsymbol{\phi}\|_{L^\infty(\Omega)}, \tag{3.3}$$

holds for each n. If we can show this, the result will follow for general $u \in BV(\Omega)$ by density. To see this, note that the right hand side is continuous with respect to the strict convergence while it is a consequence of the integration by parts formula in Theorem 1.4 (and the fact that $\mathfrak{D}_n\boldsymbol{\phi}$ is uniformly bounded for smooth vector fields) that

$$u \mapsto \int_A \mathcal{G}_n u(x) \cdot \boldsymbol{\phi}(x)\, dx$$

is continuous with respect to strong convergence in $L^1(\Omega)$.

We now proceed to verify (3.3). First, we notice that

$$\begin{aligned}\mathcal{G}_n u(x) &= N \lim_{\epsilon\to 0} \int_{\Omega\setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|}\frac{x-y}{|x-y|}\rho_n(x-y)\, dy \\ &= N \lim_{\epsilon\to 0} \int_{\Omega\setminus B(x,\epsilon)} [u(x)-u(y)]\nabla_y \left(\int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dt\right) dy.\end{aligned}$$

Thus, for $\epsilon > 0$ (and we can take a sequence $\epsilon = \epsilon_j$) we may integrate by parts to obtain

$$\begin{aligned}&\int_{\Omega\setminus B(x,\epsilon)} [u(x)-u(y)]\nabla_y \left(\int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dt\right) dy \\ &\quad= \int_{\Omega\setminus B(x,\epsilon)} \nabla u(y) \int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dtdy + \int_{\partial\Omega} [u(x)-u(\xi)]\nu \int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dtd\mathcal{H}^{N-1}(\xi) \\ &\quad\quad - \int_{\partial B(0,\epsilon)} [u(x+h)-u(x)]\frac{h}{|h|}\int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t}\, dtd\mathcal{H}^{N-1}(h).\end{aligned}$$

Therefore, we have

$$\left|\int_A \mathcal{G}_n u\cdot\boldsymbol{\phi}\, dx\right| \leq I(\boldsymbol{\phi}) + II(\boldsymbol{\phi}) + III(\boldsymbol{\phi}),$$

where we have defined

$$\begin{aligned}I(\boldsymbol{\phi}) &:= N\left|\int_A \boldsymbol{\phi}(x)\cdot \lim_{\epsilon\to 0}\int_{\Omega\setminus B(x,\epsilon)} \nabla u(y)\int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dtdydx\right|, \\ II(\boldsymbol{\phi}) &:= N\left|\int_A \boldsymbol{\phi}(x)\cdot \lim_{\epsilon\to 0}\int_{\partial B(0,\epsilon)} [u(x+h)-u(x)]\frac{h}{|h|}\int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t}\, dtd\mathcal{H}^{N-1}(h)dx\right|, \\ III(\boldsymbol{\phi}) &:= N\left|\int_A \boldsymbol{\phi}\cdot\int_{\partial\Omega}[u(x)-u(y)]\nu\int_{|x-y|}^\infty \frac{\hat{\rho}_n(t)}{t}\, dtd\mathcal{H}^{N-1}(y)dx\right|.\end{aligned}$$

We claim that

$$I(\boldsymbol{\phi}) \leq |Du|(\Omega)\|\boldsymbol{\phi}\|_{L^\infty(\Omega)}, \qquad II(\boldsymbol{\phi}) = 0,$$

and there are a sequence of positive numbers α_n such that

$$III(\boldsymbol{\phi}) \leq \alpha_n\|\boldsymbol{\phi}\|_{L^\infty(\Omega)},$$

We begin by showing the bound for $I(\boldsymbol{\phi})$. We have

$$I(\boldsymbol{\phi}) = \left| N \int_A \boldsymbol{\phi}(x) \cdot \lim_{\epsilon \to 0} \int_{\Omega \setminus B(x,\epsilon)} \nabla u(y) \int_1^\infty \frac{\hat{\rho}_n(s|x-y|)}{s} \, dsdydx \right|,$$

where we have changed variables in the inner integral. Now, since u is smooth and $\rho_n \in L^1(\mathbb{R}^N)$, letting $\epsilon \to 0$ (along a sequence) and applying Lebesgue's dominated convergence theorem, we obtain

$$I(\boldsymbol{\phi}) \leq N \int_A |\boldsymbol{\phi}(x)| \int_\Omega |\nabla u(y)| \int_1^\infty \frac{\hat{\rho}_n(s|x-y|)}{s} \, dsdydx = \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla u(y)| \eta_n(y) dy,$$

where we have interchanged the order of integration, and introduced

$$\eta_n(y) = N \int_A \int_1^\infty \frac{\hat{\rho}_n(s|x-y|)}{s} \, ds\, dx.$$

Note that $\|\eta_n\|_{L^\infty(\Omega)} \leq 1$. Indeed, by changing variables $z = sx - sy$, we have

$$\eta_n(y) \leq N \int_1^\infty \frac{1}{s^{N+1}} \left(\int_{sA-sy} \hat{\rho}_n(|z|) dz \right) ds \leq N \int_1^\infty \frac{1}{s^{N+1}} ds = 1.$$

We then conclude that

$$I(\boldsymbol{\phi}) \leq \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla u(y)| \, dy = \|\phi\|_{L^\infty(\Omega)} |Du|(\Omega).$$

Let us show next that $II(\boldsymbol{\phi}) = 0$. Recall that

$$II(\boldsymbol{\phi}) = N \left| \int_A \boldsymbol{\phi}(x) \cdot \lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} [u(x+h) - u(x)] \frac{h}{|h|} \int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t} \, dt d\mathcal{H}^{N-1}(h) dx \right|.$$

Now, for small ϵ and all $x \in A$, $B(x, \epsilon) \Subset \Omega$. Therefore, we can find a constant $L = L(A, u)$ such that $|u(x+h) - u(x)| \leq L\epsilon$, for all $h \in \overline{B(0,\epsilon)}$. As a consequence,

$$\begin{aligned} II(\boldsymbol{\phi}) &\leq NL|\Omega| \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \lim_{\epsilon \to 0} \left[\epsilon \left(\int_{\partial B(0,\epsilon)} d\mathcal{H}^{N-1}(h) \right) \int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t} \, dt \right] \\ &= NL\mathcal{H}^{N-1}(S^{N-1}) |\Omega| \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \lim_{\epsilon \to 0} \epsilon^N \int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t} \, dt. \end{aligned}$$

since the integral over $\partial B(0, \epsilon)$ makes a contribution of $\mathcal{H}^{N-1}(S^{N-1})\epsilon^{N-1}$. Thus $II(\boldsymbol{\phi}) = 0$ if we show that

$$\lim_{\epsilon \to 0} \epsilon^N \int_\epsilon^\infty \frac{\hat{\rho}_n(t)}{t} \, dt = 0.$$

This follows from Lebesgue's dominated convergence theorem, since for almost every $t \in (0, \infty)$ fixed we have that

$$0 \leq \epsilon^N \frac{\hat{\rho}_n(t)}{t} \chi_{[\epsilon,\infty]}(t) \to 0$$

as $\epsilon \to 0$, while

$$\epsilon^N \frac{\hat{\rho}_n(t)}{t} \chi_{[\epsilon,\infty]}(t) \leq \hat{\rho}_n(t) t^{N-1} \in L^1(0,\infty).$$

Finally, we show that $III(\boldsymbol{\phi}) \leq \alpha_n \|\phi\|_{L^\infty(\Omega)}$. We recall

$$III(\boldsymbol{\phi}) = N \left| \int_A \boldsymbol{\phi} \cdot \int_{\partial\Omega} [u(x) - u(\xi)] \nu(\xi) \int_{|x-\xi|}^\infty \frac{\hat{\rho}_n(t)}{t} \, dt d\mathcal{H}^{N-1}(\xi) dx \right|.$$

Now, we estimate the above as

$$\begin{aligned} III(\boldsymbol{\phi}) \leq N \int_A |\boldsymbol{\phi}(x)| |u(x)| \int_{\partial\Omega} \left(\int_{|x-\xi|}^\infty \frac{\hat{\rho}_n(t)}{t} \, dt \right) d\mathcal{H}^{N-1}(\xi) \, dx \\ + N \int_A |\boldsymbol{\phi}(x)| \int_{\partial\Omega} |u(\xi)| \left(\int_{|x-\xi|}^\infty \frac{\hat{\rho}_n(t)}{t} \, dt \right) d\mathcal{H}^{N-1}(\xi) \, dx \end{aligned}$$

Now, since $dist(A, \partial\Omega) = \gamma > 0$, the integral involving $\hat{\rho}_n$ is bounded, and in fact, for all $x \in A$ and $\xi \in \partial\Omega$

$$\int_{|x-\xi|}^\infty \frac{\hat{\rho}_n(t)}{t} \, dt \leq \int_\gamma^\infty \frac{\hat{\rho}_n(t)}{t} \, dt \leq \frac{1}{\gamma^N} \int_\gamma^\infty \hat{\rho}_n(t) \, t^{N-1} \, dt.$$

As a result,

$$\begin{aligned} III(\boldsymbol{\phi}) \leq \frac{N}{\gamma^N} \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \left(\int_\gamma^\infty \rho_n(t) \, t^{N-1} \, dt \right) \times \\ \left[|\partial\Omega| \int_\Omega |u(x)| \, dx + |\Omega| \int_{\partial\Omega} |u(\xi)| d\mathcal{H}^{N-1}(\xi) \right]. \end{aligned}$$

Using the trace theorem for BV functions, we have that

$$|\partial\Omega| \int_\Omega |u(x)| \, dx + |\Omega| \int_{\partial\Omega} |u(\xi)| d\mathcal{H}^{N-1}(\xi) \leq C(\Omega) \|u\|_{BV}$$

from which it follows that

$$III(\boldsymbol{\phi}) \leq \alpha_n \|\boldsymbol{\phi}\|_{L^\infty(\Omega)}, \text{ where } \alpha_n = C(\Omega) \frac{N}{\gamma^N} \|u\|_{BV} \int_\gamma^\infty \hat{\rho}_n(t) \, t^{N-1} \, dt.$$

Observe that from the property of the sequence ρ_n, $\alpha_n \to 0$ as $n \to \infty$. ■

The next lemma is a careful estimation of an upper bound for the integral of a sequence of nonlocal gradients integrated over open subsets of Ω. It is very close in spirit to the work of Bourgain, Brezis, and Mironescu [5] and Lemma 2.1.

Lemma 3.4 *Suppose that $u \in BV(\Omega)$. Then for any open subset $A \subset \Omega$ such that $|Du|(\partial A) = 0$ we have*

$$\limsup_{n\to\infty} \int_A |\mathcal{G}_n u| \, dx \leq N |Du|(A).$$

Proof. Given $u \in BV(\Omega)$, [2, Proposition 3.21] implies that we may find an extension $\tilde{u} \in BV(\mathbb{R}^N)$ of $u \in BV(\Omega)$ with the property that $\tilde{u} = u$ in Ω, $|D\tilde{u}|(\partial\Omega) = 0$, and $\tilde{u} \in BV(\mathbb{R}^N)$. Then given any $\eta > 0$, we have

$$\begin{aligned}
\int_A |\mathcal{G}_n u|\, dx = \int_A |\mathcal{G}_n \tilde{u}| dx &\leq N \int_A \int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\, \rho_n(x-y)\, dydx \\
&\leq N \int_A \int_{\Omega\setminus B(x,\eta)} \frac{|u(x)-u(y)|}{|x-y|}\, \rho_n(x-y)\, dydx \\
&\quad + N \int_A \int_{B(x,\eta)} \frac{|\tilde{u}(x)-\tilde{u}(y)|}{|x-y|}\, \rho_n(x-y)\, dydx, \\
&=: I_n^1(u) + I_n^2(u).
\end{aligned}$$

We show that $\limsup_{n\to\infty} I_n^1(u) = 0$, while $\limsup_{n\to\infty} I_n^2(u) \leq N|Du|(A)$, and therefore we will have demonstrated (3.8). To see the former, notice that

$$N \int_A \int_{\Omega\setminus B(x,\eta)} \frac{|u(x)-u(y)|}{|x-y|}\, \rho_n(x-y)\, dydx \leq \frac{N}{\eta} \left(2\int_\Omega |u(x)| dx \right) \int_{|z|\geq\eta} \rho_n(z) dz$$

and since $u \in L^1(\Omega)$, this tends to zero a $n \to \infty$. Let us now estimate $I_n^2(u)$. Note that we can find a sequence of smooth functions $u_k \in C^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ such that $u_k \to \tilde{u}$ in $L^1(\mathbb{R}^N)$ and that $\int_U |\nabla u_k| dx \to |D\tilde{u}|(U)$, for any open subset U of $\mathbb{R}^N$ such that $|Du|(\partial U) = 0$. Then by Fatou's lemma we have that

$$\begin{aligned}
I_n^2(u) &= N \int_A \int_{B(0,\eta)} \frac{|\tilde{u}(x+h)-\tilde{u}(x)|}{|h|}\, \rho_n(h)\, dhdx \\
&\leq \liminf_{k\to\infty} N \int_A \int_{B(0,\eta)} \frac{|u_k(x+h)-u_k(x)|}{|h|}\, \rho_n(h)\, dhdx. \qquad (3.4)
\end{aligned}$$

We estimate the right hand side to obtain that

$$\begin{aligned}
N \int_A \int_{B(0,\eta)} \frac{|u_k(x+h)-u_k(x)|}{|h|}\, \rho_n(h)\, dhdx &\leq N \int_A \int_{B(0,\eta)} \int_0^1 |\nabla u_k(x+th)|\, dt\, \rho_n(h)\, dhdx \\
= N \int_{B(0,\eta)} \int_0^1 \int_A |\nabla u_k(x+th)|\, dxdt\, \rho_n(h)\, dh &\leq N \int_{(A)^\eta} |\nabla u_k(z)|\, dz = N|\nabla u_k|((A)^\eta).
\end{aligned}$$

where we denoted the open subset $A + B(0,\eta)$ of $\mathbb{R}^N$ by $(A)^\eta$. Letting $k \to \infty$, utilizing inequality (3.4) and the convergence $|\nabla u_k|((A)^\eta) \to |D\tilde{u}|((A)^\eta)$ for any η such that $|D\tilde{u}|(\partial(A)^\eta) = 0$, which is again true for all but at most countably many η, we obtain

$$I_n^2(u) \leq N|D\tilde{u}|((A)^\eta), \text{ and so } \limsup_{n\to\infty} I_n^2(u) \leq N|D\tilde{u}|((A)^\eta).$$

We note that $\cap_{\eta>0}(A)^\eta = \overline{A}$, and $|Du|(\partial A) = 0$ by hypothesis, thus completing the proof. ■

We are now ready to provide the proof of Theorem 1.2.

Proof of Theorem 1.2. We divide the proof in two steps.

Step 1: Weak-star convergence.

The weak-star convergence of the measures can be proved applying Theorem 1.4 and Lemma 2.6. To begin with, again, it suffices to show that

$$\lim_{n\to\infty}\int_\Omega \mathcal{G}_n u\cdot\boldsymbol{\phi}\,dx=\int_\Omega\boldsymbol{\phi}\cdot dDu, \tag{3.5}$$

for all $\boldsymbol{\phi}\in C_c^\infty(\Omega;\mathbb{R}^N)$. Now on the one hand, using the uniform bounds on $\boldsymbol{\mu}_n=\mathcal{G}_n u\,\mathcal{L}^N$ given by Lemma 2.1, we can deduce that up to a subsequence $\boldsymbol{\mu}_n\overset{*}{\rightharpoonup}\boldsymbol{\mu}$ weakly-star in $\big(C_0(\Omega;\mathbb{R}^N)\big)'$ to some $\boldsymbol{\mu}\in\big(C_0(\Omega;\mathbb{R}^N)\big)'$. That is,

$$\lim_{n\to\infty}\int_\Omega \mathcal{G}_n u(x)\cdot\boldsymbol{\phi}(x)\,dx=\int_\Omega\boldsymbol{\phi}\cdot d\boldsymbol{\mu}.$$

On the other hand, from the integration by parts formula, we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \mathcal{G}_n u(x)\cdot\boldsymbol{\phi}(x)\,dx&=-\lim_{n\to\infty}\int_\Omega u(x)\mathfrak{D}_n\boldsymbol{\phi}(x)\,dx\\&=-\int_\Omega u(x)\mathrm{div}\boldsymbol{\phi}(x)\,dx\\&=\int_\Omega\boldsymbol{\phi}(x)\cdot dDu,\end{aligned}$$

where we have used Lebesgue dominated theorem to pass to the limit, after noting that $\|\mathfrak{D}_n\boldsymbol{\phi}\|_{L^\infty(\Omega)}$ is uniformly bounded in n by Corollary 2.4. That completes the proof of the weak-star convergence of measures.

Step 2. The convergence of the total variations.

Recalling that the BV-seminorm is lower semicontinuous with the weak-star convergence, it follows that

$$|Du|(\Omega)\le\liminf_{n\to\infty}\,|\boldsymbol{\mu}_n|(\Omega).$$

Thus to complete the second half of the proof of the theorem, it is enough to show that

$$\limsup_{n\to\infty}|\boldsymbol{\mu}_n|(\Omega)\le|Du|(\Omega). \tag{3.6}$$

To that end, let us define the open sets $\Omega_\delta=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\delta\}$. Then we can write

$$|\boldsymbol{\mu}_n|(\Omega)=|\boldsymbol{\mu}_n|(\Omega_\delta)+|\boldsymbol{\mu}_n|(\Omega\setminus\Omega_\delta).$$

The desired result will be proven if we demonstrate that

$$\limsup_{n\to\infty}|\boldsymbol{\mu}_n|(\Omega_\delta)\le|Du|(\Omega),\ \text{and,} \tag{3.7}$$

$$\limsup_{n\to\infty}|\boldsymbol{\mu}_n|(\Omega\setminus\Omega_\delta)\le N|Du|(\Omega\setminus\overline{\Omega_\delta}) \tag{3.8}$$

for any $\delta > 0$ sufficiently small such that $|\partial\Omega_\delta| = 0$ and $|Du|(\partial\Omega_\delta) = 0$. Inequality (3.6) will then follow by sending $\delta \to 0$ and using the fact that $Du = D\tilde{u}$ is a Radon measure with $|D\tilde{u}|(\partial\Omega) = 0$.

However, inequality (3.7) follows from Lemma 3.3, by taking $A = \Omega_\delta$, while the inequality (3.8) follows from the fact the assumption $|\partial\Omega_\delta| = 0$ so that

$$|\boldsymbol{\mu}_n|(\Omega \setminus \Omega_\delta) = \int_{\Omega\setminus\Omega_\delta} |\mathcal{G}_n u|\,dx = \int_{\Omega\setminus\overline{\Omega_\delta}} |\mathcal{G}_n u|\,dx = |\boldsymbol{\mu}_n|(\Omega \setminus \overline{\Omega_\delta}),$$

to which we can apply Lemma 3.4 with $A = \Omega \setminus \overline{\Omega_\delta}$. ∎

3.2 Characterizations of Sobolev and BV Spaces

In this subsection we prove the nonlocal characterization of the Sobolev spaces $W^{1,p}(\Omega)$ for $1 < p < \infty$ and $BV(\Omega)$ when $p = 1$.

Proof of Theorem 1.5. We remark that if $u \in W^{1,p}(\Omega)$ for some $1 \le p < \infty$, then we obtain that (see [5] or Lemma 2.1 for the proof) for all n, $\mathfrak{G}_n u = \mathcal{G}_n u$ and that $\sup_{n\ge1} \|\mathfrak{G}_n u\|_{L^p} < \infty$. Conversely suppose $u \in L^p(\Omega)$ and

$$\sup_{n\ge1} \|\mathfrak{G}_n u\|_{L^p} < \infty.$$

Then since $p > 1$, up to a subsequence, $\mathfrak{G}_n u \rightharpoonup \mathbf{v}$ weakly in $L^p(\Omega;\mathbb{R}^N)$ for some $\mathbf{v} \in L^p(\Omega;\mathbb{R}^N)$. To complete the proof, we will just need to demonstrate that $\mathbf{v} = \nabla u$, the distributional derivative. To that end, for $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^N)$, by definition and Lemma 2.6

$$\int_\Omega \mathbf{v}(x)\cdot\boldsymbol{\phi}(x)\,dx = \lim_{n\to\infty}\int_\Omega \mathfrak{G}_n u\cdot\boldsymbol{\phi}\,dx = -\lim_{n\to\infty}\int_\Omega u\,\mathfrak{D}_n\boldsymbol{\phi}\,dx = -\int_\Omega u\,\mathrm{div}\,\boldsymbol{\phi}(x)\,dx.$$

Once we know that $u \in W^{1,p}(\Omega)$, the proof of the convergence of the limit (1.6) follows from generalized Lebesgue's dominated convergence theorem using the continuity of f, the strong convergence of $\mathfrak{G}_n u = \mathcal{G}_n u$ to ∇u in $L^p(\Omega;\mathbb{R}^N)$ and the inequality

$$|f(\mathcal{G}_n u(x))| \le C(1 + |\mathcal{G}_n u(x)|^p).$$

That completes the proof. ∎

We now prove the characterization of BV functions.

Proof of Theorem 1.6. Suppose that $u \in L^1(\Omega)$ and that $\sup_{n\ge1}\int_\Omega |\mathfrak{G}_n u|\,dx < \infty$. Then up to a subsequence, $\boldsymbol{\mu}_n \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ weakly-star in $\big(C_0(\Omega;\mathbb{R}^N)\big)'$ to some $\boldsymbol{\mu} \in \big(C_0(\Omega;\mathbb{R}^N)\big)'$. Arguing in a similar manner as the proof of Step 1 of Theorem 1.2, one can show that $\boldsymbol{\mu} = Du$.

Once we know that $u \in BV$, we may invoke Theorem 1.2 to conclude that $\boldsymbol{\mu}_n \to Du$ strictly as measures. Then we would like to apply the version of Reshetnyak Continuity Theorem proved in [20, Theorem 5] to obtain that

$$\lim_{n\to\infty}\int_\Omega f(\mathcal{G}_n u(x))\,dx = \int_\Omega f(D^a u)\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right)\,d|D^s u|$$

for any continuous f that is convex (or concave) with at most linear growth, see also [1, Lemma 2.2]. To apply [20, Theorem 5] however, we must first verify (1.8): $\boldsymbol{\mu}_n \to Du$ in the $\langle\cdot\rangle$-strict convergence of measures as introduced by [20]. To that end, it suffices to show that

$$\lim_{n\to\infty} \int_\Omega g(\mathcal{G}_n u)\, dx = \int_\Omega g(D^a u)\, dx, \tag{3.9}$$

where $g(z) := \sqrt{1+|z|^2} - |z|$ is non-negative, bounded, and Lipschitz continuous with constant one. Indeed, (3.9) implies

$$\begin{aligned}\int_\Omega \sqrt{1+|D^a u|^2} - |D^a u|\, dx &= \lim_{n\to\infty} \int_\Omega \sqrt{1+|\mathcal{G}_n u|^2} - |\mathcal{G}_n u|\, dx \\ &= \lim_{n\to\infty} \int_\Omega \sqrt{1+|\mathcal{G}_n u|^2}\, dx - |Du|(\Omega),\end{aligned}$$

and equation (1.8) would follow from rearranging terms. We remark that the proof of the convergence (3.9) is essentially a vector version of Lemma 5 of Ponce's paper [22], and we give a proof in a similar spirit, mutatis mutandis, by taking advantage of the specific structure of g.

We begin by letting $\tilde{u} \in BV(\mathbb{R}^N)$ be an extension of $u \in BV(\Omega)$. Now, we want to demonstrate (3.9), which can be accomplished by showing that $g(\mathcal{G}_n u) \rightharpoonup g(D^a u)$ weakly in $L^p(\Omega)$ for some $1 < p < +\infty$ (since $|\Omega| < +\infty$ implies that $\chi_\Omega(x) \in L^{p'}(\Omega)$ for every $1 < p < \infty$). However, from the fact that g is bounded above by one, we deduce that

$$\|g(\mathcal{G}_n u)\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}}.$$

This then implies that up to a subsequence $g(\mathcal{G}_n u) \rightharpoonup v$ weakly in $L^p(\Omega)$, and it remains to show that $v(x_0) = g(D^a u(x_0))$ for Lebesgue almost every $x_0 \in \Omega$. This is a local property, and so it suffices to show that

$$\lim_{\epsilon\to 0} \left| \fint_{B(x_0,\epsilon)} v(x) - g(D^a u(x))\, dx \right| = 0.$$

Now weak convergence in $L^p(\Omega)$ together with the Lipschitz continuity of g imply that

$$\begin{aligned}\lim_{\epsilon\to 0} \left| \fint_{B(x_0,\epsilon)} v(x) - g(D^a u(x))\, dx \right| &= \lim_{\epsilon\to 0} \lim_{n\to\infty} \left| \fint_{B(x_0,\epsilon)} g(\mathcal{G}_n u) - g(D^a u(x))\, dx \right| \\ \leq \lim_{\epsilon\to 0} \lim_{n\to\infty} \fint_{B(x_0,\epsilon)} &\left| \int_\Omega N \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_n(x-y)\, dy - D^a u(x) \right| dx\end{aligned}$$

Recalling Lemma 3.1, which says that $Nc_n^i(x) \to 1$ uniformly for $x \in B(x_0,\epsilon)$ if $\epsilon > 0$ is small, we have

$$\begin{aligned}&\lim_{\epsilon\to 0} \left| \fint_{B(x_0,\epsilon)} v(x) - g(D^a u(x))\, dx \right| \\ &= \lim_{\epsilon\to 0} \lim_{n\to\infty} \fint_{B(x_0,\epsilon)} \left| \int_\Omega N \frac{u(x)-u(y)-D^a u(x)\cdot(x-y)}{|x-y|} \frac{x-y}{|x-y|} \rho_n(x-y)\, dy \right| dx, \\ &\leq \lim_{\epsilon\to 0} \lim_{n\to\infty} \fint_{B(x_0,\epsilon)} \int_{\mathbb{R}^N} N \frac{|\tilde{u}(x)-\tilde{u}(y)-D^a u(x)\cdot(x-y)|}{|x-y|} \rho_n(x-y)\, dydx.\end{aligned}$$

Now, Step 2 in Lemma 5 from [22] demonstrates that

$$\lim_{n\to\infty} \fint_{B(x_0,\epsilon)} \int_{\mathbb{R}^N} N \frac{|\tilde{u}(x) - \tilde{u}(y) - D^a u(x) \cdot (x-y)|}{|x-y|} \rho_n(x-y)\, dydx \le N \frac{1}{|B(x_0,\epsilon)|} |D^s u|(B(x_0,\epsilon)),$$

and so we conclude that almost every $x_0 \in \mathbb{R}^N$,

$$\lim_{\epsilon\to 0} \left| \fint_{B(x_0,\epsilon)} v(x) - g(D^a u(x))\, dx \right| \le \lim_{\epsilon \to 0} N \frac{1}{|B(x_0,\epsilon)|} |D^s u|(B(x_0,\epsilon)) = 0,$$

where we applied the Besicovitch derivation theorem, using the fact that $D^s u$ is singular with respect to Lebesgue measure. ∎

We conclude this section by stating and proving a possible characterization of Sobolev and BV functions as weak limits of bounded sequence u_n in L^p with a uniformly bounded nonlocal gradients.

Theorem 3.5 *Suppose that $1 < p < \infty$ and u_n is a bounded sequence in $L^p(\Omega)$ such that the distribution $\mathfrak{G}_n u_n \in L^p(\Omega;\mathbb{R}^N)$ for each n and satisfies the uniform estimate*

$$\sup_{n\ge 1} \int_\Omega |\mathfrak{G}_n u_n(x)|^p\, dx = K < \infty.$$

Then any weak limit u of u_n is in $W^{1,p}(\Omega)$. Moreover,

$$\|\nabla u\|_{L^p(\Omega)} \le K.$$

Proof. Let u be a weak limit of the sequence u_n in $L^p(\Omega)$, and $\mathbf{v}$ be a weak limit of $\mathfrak{G}_n u_n$ in $L^p(\Omega;\mathbb{R}^N)$. We claim that $\mathbf{v} = \nabla u$, the distributional derivative of u. To that end, it follows from application of integration by parts, Theorem 1.4, and Lemma 2.6 that for any $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}\int_\Omega \mathbf{v}(x)\cdot\boldsymbol{\phi}(x)\, dx &= -\lim_{n\to\infty}\int_\Omega \mathfrak{G}_n u_n(x)\cdot\boldsymbol{\phi}(x)\, dx \\ &= -\lim_{n\to\infty}\int_\Omega u_n(x)\, \mathfrak{D}_n\boldsymbol{\phi}(x)\, dx = -\int_\Omega u(x)\mathrm{div}\boldsymbol{\phi}(x)\, dx\end{aligned}$$

where in the last equality we used the strong convergence $\mathfrak{D}_n\boldsymbol{\phi} \to \mathrm{div}\boldsymbol{\phi}$ in $L^{p'}(\Omega)$ proved in Lemma 2.6 (part b) (ii)) and the weak convergence of u_n to u in L^p. The estimate for the seminorm $\|\nabla u\|_{L^p}$ follows from

$$\left|\int_\Omega u(x)\,\mathrm{div}\boldsymbol{\phi}(x)\, dx\right| \le \lim_{n\to\infty}\left|\int_\Omega \mathfrak{G}_n u_n(x)\cdot\boldsymbol{\phi}(x)\, dx\right| \le K\|\boldsymbol{\phi}\|_{L^{p'}}$$

∎

For BV functions a similar results holds true, though it requires a little more subtlety. Before we state the theorem for the BV case, let us recall the following lemma, a fact about Radon measures possessing weak derivatives that are also Radon measures (see [2][Exercise 3.2]).

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^N$ is an open set. Suppose that μ and $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_N)$ are finite Radon measures in Ω. If for all $\boldsymbol{\phi} \in C_c^1(\Omega; \mathbb{R}^N)$,*

$$-\int_\Omega div(\boldsymbol{\phi}) d\mu = \int_\Omega \boldsymbol{\phi} \cdot d\boldsymbol{\nu}, \tag{3.10}$$

then there exists $u \in BV(\Omega)$ such that $d\,\mu = u(x)\,dx$, and $d\boldsymbol{\nu} = dDu$.

The equality (3.10) defines $\boldsymbol{\nu}$ as a distributional gradient of μ. The lemma, then, says that if both μ and its distributional gradient are finite Radon measures, then μ is absolutely continuous with respect to the Lebesgue measure and that its density is actually in $BV(\Omega)$. Then we can prove the following theorem extending the previous result to the BV case.

Theorem 3.7 *Suppose that $u_n \in L^1(\Omega)$ is a bounded sequence in $L^1(\Omega)$ such that $\mathfrak{G}_n u_n \in L^1(\Omega; \mathbb{R}^N)$ and satisfies the uniform estimate*

$$\sup_{n\geq 1} \int_\Omega |\mathfrak{G}_n u_n|\, dx = K < \infty.$$

Then there exists $u \in BV(\Omega)$ such that $u_n \overset{*}{\rightharpoonup} u$ in the sense of measures and that

$$|Du|(\Omega) \leq K.$$

Proof. By assumption, there exists a Radon measures μ and $\boldsymbol{\nu}$ such that $u_n \overset{*}{\rightharpoonup} \mu$, and $\mathfrak{G}_n u_n \overset{*}{\rightharpoonup} \boldsymbol{\nu}$, in the sense of measures. Moreover, $|\mu|(\Omega) \leq \sup_n \|u_n\|_{L^1(\Omega)} < \infty$ and for any $\boldsymbol{\phi} \in C_c^1(\Omega; \mathbb{R}^N)$

$$\begin{aligned} -\int_\Omega \mathrm{div}\boldsymbol{\phi}(x)\, d\mu &= -\lim_{n\to\infty} \int_\Omega u_n(x)[\mathrm{div}\boldsymbol{\phi}(x) - \mathfrak{D}_n \boldsymbol{\phi}(x)]dx - \lim_{n\to\infty} \int_\Omega u_n(x) \mathfrak{D}_n \boldsymbol{\phi}(x) dx \\ &= \lim_{n\to\infty} \int_\Omega \boldsymbol{\phi}(x) \cdot \mathfrak{G}_n u_n\, dx = \int_\Omega \boldsymbol{\phi}(x) \cdot d\boldsymbol{\nu} \end{aligned}$$

where we have used the uniform convergence $\|\mathrm{div}\boldsymbol{\phi}(x) - \mathfrak{D}_n \boldsymbol{\phi}\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, and the boundedness of the sequence u_n in $L^1(\Omega)$. Applying the previous Lemma 3.6, we see that there exists $u \in BV(\Omega)$ with $d\boldsymbol{\nu} = dDu$. Moreover, as weak-star limit of $\mathfrak{G}_n u_n$, $|Du|(\Omega) \leq K$. ■

4 Gamma convergence of the nonlocal gradient

We conclude the paper with the proof of the Γ-convergence of the nonlocal energies to the relaxation of their local analogues. Let us first recall the definition of Γ-lower and upper limits (see [11]), with respect to the $L^p(\Omega)$ topology, where we have taken advantage of the fact that $L^p(\Omega)$ is a metric space to give the equivalent sequential definition.

Definition 4.1 *Given a bounded open set $A \subset \mathbb{R}^N$, let F_n be any sequence of functionals $F_n : L^p(A) \to [0, \infty]$. For each $u \in L^1(A)$ we set*

$$\Gamma^-_{L^p(A)}\text{-}\liminf_{n\to\infty} F_n(u) := \min\{\liminf_{n\to\infty} F_n(u_n) : u_n \to u \text{ in } L^p(A)\},$$
$$\Gamma^-_{L^p(A)}\text{-}\limsup_{n\to\infty} F_n(u) := \min\{\limsup_{n\to\infty} F_n(u_n) : u_n \to u \text{ in } L^p(A)\}.$$

If $\Gamma^-_{L^p(A)}$-$\limsup_{n\to\infty} F_n(u) = \Gamma^-_{L^p(A)}$-$\liminf_{n\to\infty} F_n(u)$ *for some* $u \in L^p(A)$ *we say that the sequence* F_n Γ*-converges at* u *and denote this common number* $\Gamma^-_{L^p(A)}$-$\lim_{n\to\infty} F_n(u)$*.*

Given a continuous function f_p that satisfies the condition (1.10), $1 < p < \infty$, let us introduce the functionals F_n and F as

$$F_n(u,p) = \begin{cases} \displaystyle\int_\Omega f_p(\mathfrak{G}_n u)\, dx, & \text{if } \mathfrak{G}_n u \in L^p(\Omega;\mathbb{R}^N) \\ \infty & \text{otherwise,} \end{cases}$$

and

$$F(u,p) = \begin{cases} \displaystyle\int_\Omega f_p(\nabla u(x))\, dx, & \text{if } u \in W^{1,p}(\Omega) \\ \infty, \text{ otherwise.} \end{cases}$$

By definition (see [11][Chapter 3], for example) the relaxation of F is given by

$$sc^-_{L^p(\Omega)} F(u,p) := \min \left\{ \liminf_{n\to\infty} F(u_n,p) : u_n \to u \text{ in } L^p(\Omega) \right\}.$$

Under the hypothesis we have placed on f_p, when $1 < p < \infty$ a representation $sc^-_{L^p(\Omega)} F(u,p)$ is given by, [11][Chapter 3],

$$sc^-_{L^p(\Omega)} F(u,p) = \begin{cases} \displaystyle\int_\Omega f_p^{**}(\nabla u)\, dx, & \text{when } u \in W^{1,p}(\Omega) \\ \qquad +\infty, \text{ otherwise,} \end{cases}$$

where f^{**} is the greatest convex function on $\mathbb{R}^N$ majorized by f. We now proceed to prove the Γ-convergence result claimed in the introduction in Theorem 1.7.

Proof of Theorem 1.7. As usual, the Γ-limit consists of two inequalities, a lower bound and an upper bound. Our technique follows that developed by Ponce in [22], whereby the problem of verifying a Γ-limit is reduced to the problem of studying the relaxation of a functional. Let us first show the lower bound

$$sc^-_{L^p(\Omega)} F(u,p) \le \Gamma^-_{L^p(\Omega)}\text{-}\liminf_{n\to\infty} F_n(u,\Omega). \tag{4.1}$$

Without loss of generality we may assume that there exists a sequence $u_n \to u$ in $L^p(\Omega)$ for which

$$\liminf_{n\to\infty} F_n(u_n,p) < \infty.$$

For any such sequence we may utilize the growth condition (1.10) of f_p to deduce that, up to a subsequence, which we will not relabel,

$$c \int_\Omega |\mathfrak{G}_n u_n|^p\, dx \le F_n(u_n,p) \le C. \tag{4.2}$$

Since $p > 1$, applying Theorem 3.5 we obtain that $u \in W^{1,p}(\Omega)$, and $\mathfrak{G}_n u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega;\mathbb{R}^N)$. Now, using the fact that $f_p^{**} \leq f_p$, we have that

$$\liminf_{n\to\infty} \int_\Omega f_p^{**}(\mathfrak{G}_n u_n)\, dx \leq \Gamma^-_{L^p(\Omega)}\text{-}\liminf_{n\to\infty} F_n(u,p).$$

Moreover, since f_p^{**} is convex, it is lower semicontinuous with respect to weak convergence, and thus when $p > 1$ we have

$$\int_\Omega f_p^{**}(\nabla u)\, dx \leq \Gamma^-_{L^p(\Omega)}\text{-}\liminf_{n\to\infty} F_n(u,p),$$

which implies the lower bound, recalling the representation for $sc^-_{L^p(\Omega)}F(u,p)$.

To prove the upper bound

$$\Gamma^-_{L^p(\Omega)}\text{-}\limsup_{n\to\infty} F_n(u,p) \leq sc^-_{L^p(\Omega)}F(u,p), \tag{4.3}$$

we pick $u \in W^{1,p}(\Omega)$, so that by Theorem 1.6, we have the convergence

$$\lim_{n\to\infty} \int_\Omega f_p(\mathfrak{G}_n u)\, dx = \int_\Omega f_p(\nabla u)\, dx.$$

Then choosing $u_n = u$ we conclude

$$\Gamma^-_{L^p(\Omega)}\text{-}\limsup_{n\to\infty} F_n(u,p) \leq \lim_{n\to\infty} \int_\Omega f_p(\mathcal{G}_n u)\, dx = \int_\Omega f_p(\nabla u)\, dx.$$

Now, since $\Gamma^-_{L^p(\Omega)}$- lim sup is lower semicontinuous on $L^p(\Omega)$ (c.f. [11][Proposition 6.8]), we may take the lower semicontinuous envelope of the above equation to arrive at the inequality

$$\Gamma^-_{L^p(\Omega)}\text{-}\limsup_{n\to\infty} F_n(u,p) \leq sc^-_{L^p(\Omega)}F(u,p),$$

which is precisely inequality (4.3). ■

We have a similar result when $p = 1$, though we must define our energies appropriately. We define

$$F_n(u,1) = \begin{cases} \displaystyle\int_\Omega f_1(\mathfrak{G}_n u)\, dx, & \text{if } \mathfrak{G}_n u \in L^1(\Omega;\mathbb{R}^N) \\ \infty & \text{otherwise,} \end{cases}$$

and

$$F(u,1) = \begin{cases} \displaystyle\int_\Omega f_1(\nabla u(x))\, dx, & \text{if } u \in C^1(\overline{\Omega}) \\ \infty, \text{ otherwise.} \end{cases}$$

Then we have the following theorem connecting the Γ-limit of F_n and the $sc^-_{L^1(\Omega)}F(u,1)$.

Theorem 4.2 *Suppose $\Omega \subset \mathbb{R}^N$ is open, bounded, and smooth. Assume ρ_n satisfy* (1.2) *and f_1 satisfies*

$$c|z| \leq f_1(z) \leq C(1+|z|). \tag{4.4}$$

Then

$$\Gamma^-_{L^1(\Omega)}\text{-}\lim_{n\to\infty} F_n(u,1) = sc^-_{L^1(\Omega)} F(u,1)$$

where the Γ-limit is taken with respect to the strong topology of $L^1(\Omega)$.

Let us first note that if $u \in BV(\Omega)$, then the same argument as Ponce [22][Theorem 8], which is a consequence of [7][Theorem 4.4.1 and Remark 4.4.4] and [10][Theorem 4.7], implies the representation

$$sc^-_{L^1(\Omega)} F(u,1) = \int_\Omega f_1^{**}(D^a u)\, dx + \int_\Omega (f_1^{**})^\infty \left(\frac{dD^s u}{d|D^s u|}\right)\, d|D^s u|.$$

Further, the assumption (4.4) implies that $sc^-_{L^1(\Omega)} F(u,1) = +\infty$ if $u \in L^1(\Omega) \setminus BV(\Omega)$.

Proof. We should prove again the inequalities (4.1) and (4.3). Arguing as previously, from equation (4.2), this time applying Theorem 3.7, we deduce that up to a subsequence $\boldsymbol{\mu}_n := \mathfrak{G}_n u_n \overset{*}{\rightharpoonup} Du$ for some $u \in BV(\Omega)$. Then applying Theorem 2.34 in [2] with this choice of $\boldsymbol{\mu}_n$ and $\nu = \mathcal{L}^N$ allows us to deduce that

$$\int_\Omega f_1^{**}\left(D^a u\right)\ dx + \int_\Omega (f_1^{**})^\infty \left(\frac{dD^s u}{d|D^s u|}\right)\ d|D^s u| \leq \liminf_{n\to\infty} \int_\Omega f_1^{**}(\mathfrak{G}_n u_n)\, dx.$$

This estimate, along with the inequality

$$\liminf_{n\to\infty} \int_\Omega f_1^{**}(\mathfrak{G}_n u_n)\, dx \leq \Gamma^-_{L^1(\Omega)}\text{-}\liminf_{n\to\infty} F_n(u,1)$$

and the representations for $sc^-_{L^1(\Omega)} F(u,1)$ recalled above, we have that the inequality (4.1) is demonstrated.

To prove inequality (4.3), we note that for $u \in C^1(\overline{\Omega})$, by Theorem 1.1, we have the convergence

$$\lim_{n\to\infty} \int_\Omega f_1(\mathcal{G}_n u)\, dx = \int_\Omega f_1(\nabla u)\, dx.$$

Then choosing $u_n = u$ we conclude

$$\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{n\to\infty} F_n(u,1) \leq \lim_{n\to\infty} \int_\Omega f_1(\mathcal{G}_n u)\, dx = \int_\Omega f_1(\nabla u)\, dx.$$

Again taking the lower semicontinuous envelope, this time with respect to $L^1(\Omega)$ strong convergence, using $\Gamma^-_{L^1(\Omega)}$- lim sup is lower semicontinuous on $L^1(\Omega)$ (c.f. [11][Proposition 6.8]), we deduce that

$$\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{n\to\infty} F_n(u,1) \leq sc^-_{L^1(\Omega)} F(u,1),$$

which along with the representation formula we recorded for the right hand side completes the proof. ■

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References

[1] J.J. Alibert, and G. Bouchitté. Non-uniform integrability and generalized Young measures. *J. Convex Anal.* 4:129-147, 1997.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] F. Andreu-Vaillo, J. M. Mazón, Julio D. Rossi , and J. J. Toledo-Melero. Nonlocal Diffusion Problems. American Mathematical Society. Mathematical Surveys and Monographs 2010. Vol. 165.

[4] G. Aubert and P. Kornprobst. Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? *SIAM J. Numer. Anal.*, 47:844-860, 2009.

[5] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In: Menaldi, J.L., et al. (eds.) *Optimal control and partial differential equations*, pp 439-455 (A volume in honour of A. Benssoussan's 60th birthday). IOS Press, 2001.

[6] H. Brezis. How to recognize constant functions. Connections with Sobolev spaces. *Russ. Math. Surv.*, 57:693-708, 2002.

[7] G. Buttazzo. Semicontinuity, relaxation and integral representation in the calculus of variations. (Pitman Research Notes in Mathematics Series, Vol. 207). Harlow: Longman 1989.

[8] L. Caffarelli and L. Silvestre. The Evans-Krylov theorem for nonlocal fully nonlinear equations. *Ann. of Math.* 174:1163-1187, 2011.

[9] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62:597-638, 2009.

[10] A. Corbo Esposito and R. De Arcangelis. Comparison Results for Some Types of Relaxation of Variational Integral Functionals. *Ann. Mat. Pura Appl.*, 164:155-193, 1993.

[11] G. Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[12] J. Dávila. On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations*, 15:519-527, 2002.

[13] Q. Du, M. Gunzburger, R.B. Lehoucq, and K. Zhou. Analysis and Approximation of Nonlocal Diffusion Problems with Volume Constraints. *SIAM Rev.*, 54:667-696, 2012.

[14] Q. Du, M. Gunzburger, R.B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume constrained problems, and nonlocal balance laws. *Math. Models Methods Appl. Sci.*, 23:493-540, 2013.

[15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of Functions*, CRC Press, Boca Ration, Fl, 1992.

[16] G. Gilboa and S. Osher. Nonlocal linear image regularization and supervised segmentation. *Multiscale Model. Simul.*, 6:595-630, 2007.

[17] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7:1005-1028, 2008.

[18] M. Gunzburger and R.B. Lehoucq. A nonlocal vector calculus with application to nonlocal boundary value problems, *Multiscale Model. Simul.*, 8:1581-1598, 2010.

[19] G. Leoni and D. Spector. Characterization of Sobolev and BV spaces, *J. Funct. Anal.* 261:2926-2958, 2011.

[20] J. Kristensen and F. Rindler. Relaxation of signed integral functionals in BV. *Calc. Var. Partial Differential Equations*, 37:29-62, 2010.

[21] T. Mengesha. Nonlocal Korn-type characterization of Sobolev vector fields, *Comm. Cont. Math.*, 14:1250028, 28pp., 2012.

[22] A. Ponce. A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations*, 19:229-255, 2004.

[23] S.A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48:175-209, 2000.

[24] S.A. Silling and R.B. Lehoucq. Convergence of peridynamics to classical elasticity theory. *J. Elasticity* 93:13-37, 2008.

[25] L. Silvestre. Hölder estimates for solutions of integro differential equations like the fractional laplace, *Indiana Univ. Math. J.*, 55:1155-1174, 2006.