VARIATIONAL EVOLUTION OF ONE-DIMENSIONAL LENNARD-JONES SYSTEMS

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Abstract. We analyze Lennard-Jones systems from the standpoint of variational principles beyond the static framework. In a one-dimensional setting such systems have already been shown to be equivalent to energies of Fracture Mechanics. Here we show that this equivalence can also be given in dynamical terms using the notion of minimizing movements.

1. **Introduction.** The main scope of this paper is to analyze Lennard-Jones systems from the standpoint of variational principles beyond the static framework. In a one-dimensional setting such systems have already been shown to be equivalent to energies of Fracture Mechanics using the notion of equivalence by Γ-convergence [9, 10]. Here we show that this equivalence can also be given in dynamical terms using the notion of minimizing movements.

We start by briefly recalling the definition of a minimizing-movement scheme. Typically, we are given an 'energy functional' F, defined on a space X, whose (local) minimizers provide the stable configurations of the system. As an answer to the problem of modeling the evolution from a given initial state u^0, in [12] (see also [1, 5]) a general scheme is proposed, based on an iterative-minimization process. More precisely, in the particular case in which X is a Hilbert space, we fix a 'time step' $\tau > 0$ and consider the sequence $(u^k_\tau)_k$ recursively defined by letting $u^0_\tau = u^0$ and u^k_τ $(k \geq 1)$ be a minimizer of the penalized functional

$$v \mapsto F(v) + \frac{1}{2\tau}\|v - u^{k-1}_\tau\|^2_X \,. \tag{1}$$

We interpret u^k_τ as the state of the system at discrete times $t = k\tau$, and let $u_\tau\colon [0, +\infty) \to X$ be its piecewise-constant extension for all positive times: $u_\tau(t) = u^{\lfloor t/\tau \rfloor}_\tau$. A function $u\colon [0, +\infty) \to X$ is a *minimizing movement* for F from u^0 if

2010 *Mathematics Subject Classification.* Primary: 35K22, 74A45; Secondary: 49J45, 74R10.
Key words and phrases. Lennard-Jones systems, variational evolution, minimizing movements, Fracture Mechanics.

u is the pointwise limit of a (sub)sequence (u_{τ_n}). Note that the last term in (1) tends to constrain the minimizer u^k_τ on a $O(\tau)$-neighbourhood of u^{k-1}_τ, thus giving a X-continuous trajectory in the limit. As a standard example we mention the case $X = L^2(\Omega)$, with Ω an open subset of $\mathbb{R}^n$, and $F(u) = \int_\Omega |\nabla u|^2\,dx$ on the Sobolev space $H^1(\Omega)$, extended with value $+\infty$ otherwise; it turns out that the evolution of an initial datum u^0 is given by the (weak) solution of the heat equation $u_t = 2\Delta_x u$ with initial condition $u(\cdot, 0) = u^0$ and Neumann boundary conditions.

Consider now the case of an energy F_ε which depends on a small parameter ε, and assume that we know its limit F as $\varepsilon \to 0$ (technically, the Γ-limit in a suitable topology). The method of minimizing movements, applied to each F_ε, leads to functionals defined as in (1) but depending on both ε and τ. By letting simultaneously ε and τ tend to 0 we have a *minimizing movement along the sequence F_ε at time scale τ*. In general the evolution of the system from an initial state, driven by the functional F_ε according to the scheme above is not close, for ε and τ small, to the evolution ruled by the limit functional F. This is true (upon some bounds and equi-coerciveness assumptions) if ε is sufficiently small with respect to τ (see [7] Theorem 8.1 (i)), but it is in general false; e.g., for F with interfacial energies. In that case evolutions for F_ε constructed from discrete approximations are often *pinned* by local minimizers (see [8], [7] Section 9.5). We note that a condition which guarantees that the minimizing movement is independent of the mutual behaviour of ε and τ is the convexity of the functionals F_ε (see [7] Section 11.1 and [13]; see also [4] Section 3.2.4): as a heuristic motivation, consider that, in the convex case, energies as in (1) do not possess local minimizer other than the global one. Unfortunately, convexity conditions are often ruled out in many physical models, as the one we are considering here.

In this paper we focus the attention on a well-known family of non-convex energies defined through a Lennard-Jones potential, and prove in particular that the minimizing movements along this family coincide with that of their limit also when τ is sufficiently small with respect to ε (i.e., in a regime "opposite" to the one for which this always holds). More precisely, we consider the family $(F_\varepsilon)_{\varepsilon>0}$ of functionals defined on the set of functions $u\colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$ by (see (7))

$$F_\varepsilon(u) = \sum_{i=0}^{N_\varepsilon-1} \psi\left(\frac{u_{i+1}-u_i}{\sqrt{\varepsilon}}\right), \qquad (u_i := u(i\varepsilon),\ N_\varepsilon := \lfloor 1/\varepsilon \rfloor), \tag{2}$$

where $\psi\colon (-1,+\infty) \to \mathbb{R}$ is, up to a translation, a convex-concave potential of Lennard-Jones type with minimum in 0 (see Figure 2: we postpone the motivation for such a potential to the next section). The $\sqrt{\varepsilon}$-scaling considered here leads to the prototypical free-discontinuity functional, namely the Mumford-Shah functional (or Griffith fracture energy)

$$F(u) = \frac{1}{2}\psi''(0)\int_0^1 |u'|^2\,dx + a\#S(u),$$

where $a = \lim_{z\to+\infty}\psi(z)$, and $S(u)$ denotes the set of jump points of the function u (i.e., the set of the points x where the right and left limits $u^\pm$ are different). Moreover, the increasing-jump condition $u^+ > u^-$ on $S(u)$ has to be taken into account. It is known that the minimizing-movement scheme can be applied to the functional F, giving the heat equation with Neumann boundary conditions on the jump set (and on the boundary), with the constraint that $S(u(t))$ is decreasing (see, e.g., [7] Example 7.3).

For every $\varepsilon, \tau > 0$ we can define the discrete evolution $(u^k_{\varepsilon,\tau})_k$ from an initial datum, driven by the functional F_ε according to the scheme (1). As above, we denote by $u_{\varepsilon,\tau}$ its piecewise-constant extension for all positive times: $u_{\varepsilon,\tau}(t) = u^{\lfloor t/\tau \rfloor}_{\varepsilon,\tau}$. In Section 3 we prove a compactness result for sequences $\big(u_{\varepsilon_n,\tau_n}\big)$ and in Section 4 we characterize the minimizing movement along F_{ε_n} (with time step τ_n); namely, we prove that all limit points of $\big(u_{\varepsilon_n,\tau_n}\big)$ are weak solutions of the heat equation, independently of the particular sequences (ε_n) and (τ_n), with fixed jump set, hence obtaining the minimizing movement for the Mumford-Shah functional. This result is obtained under the assumption $\tau << \varepsilon^2$. From this in particular we deduce that the limit as $\varepsilon \to 0$ of the minimizing movements along the scaled Lennard-Jones functionals F_ε (computed at ε fixed) coincides with the minimizing movement of the limit Griffith energy F. This is a consequence of the general property that the minimizing movements along a sequence F_ε at time scale τ are limits of minimizing movements along functionals F_ε at fixed ε if τ is small enough with respect to ε (see [7] Theorem 8.1(ii)). A similar 'commutativity' result between Γ-convergence and gradient flow has been obtained for Ginzburg-Landau energies [15].

In the framework of the theory of minimizing movements for varying energies, this result gives an example of a family of non-convex energies for which the limit is independent of the ratio of ε and τ, at least for some cases comprising the 'extreme' regimes. Note however that the hypothesis $\tau << \varepsilon^2$ is required only for an argument used to compare finite-difference equations with related ordinary differential equations in the proof of Proposition 4 and seems to be only a technical assumption. It can be altogether dropped upon requiring some bounds on the Dirichlet energy of the initial data. Compared with other situations where the limit motion of interfaces does depend on the mutual behaviour of ε and τ due to pinning effects, it must be noted that for the Griffith fracture energy F interfaces are themselves automatically pinned. This is a characteristic feature of brittle fracture, where the crack site is supposed to be increasing in time.

2. Setting of the problem and preliminary results.

2.1. **Function spaces.** Let $I = (a, b)$ be a bounded open interval. We denote by $W^{k,p}(I)$ and $H^k(I) := W^{k,2}(I)$ the standard Sobolev spaces on I. Moreover, we say that a function $u\colon I \to \mathbb{R}$ is *piecewise*-$W^{1,p}(I)$ if there exist $a = x_0 < x_1 < \ldots < x_{m+1} = b$ such that

$$u \in W^{1,p}(x_k, x_{k+1}) \quad \text{for every } k = 0, \ldots, m. \tag{3}$$

It is well known that, considering the continuous representative of u in each interval, the limits

$$u^+(x_k) := \lim_{x \to x_k^+} u(x), \qquad u^-(x_k) := \lim_{x \to x_k^-} u(x)$$

exist and are finite. The minimal set $\{x_1, \ldots, x_m\}$ for which (3) holds coincides with the *jump set* $S(u)$ of the function u.

If $u \in BV(I)$; i.e., u is a function with *bounded variation* in I, then its distributional derivative Du is a measure which can be written as

$$Du = u'\mathrm{d}x + D^s u, \tag{4}$$

for a suitable function $u' \in L^1(I)$ and with $D^s u$ singular with respect to the Lebesgue measure $\mathrm{d}x$. It is well known that if $u \in BV(I)$ then the unilateral (approximate) limits $u^\pm(x)$ exist and are finite for every $x \in I$.

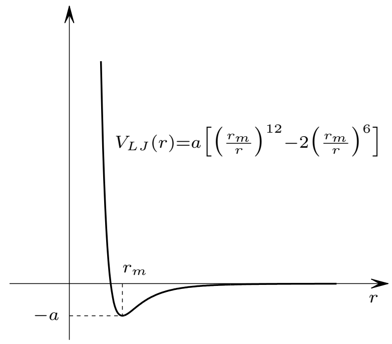

FIGURE 1. A potential of Lennard-Jones type

A relevant subspace of $BV(I)$ is the space $SBV(I)$ (*special* functions with bounded variation) determined by the condition that $D^s u$ is concentrated on the set $S(u)$ of discontinuity points of u (i.e., the points where $u^\pm$ are different). In this case,

$$D^s u = (u^+ - u^-)\mathrm{d}\mathcal{H}^0 \llcorner S(u),$$

(here $\mathcal{H}^0$ denotes the counting measure) and we refer to $D^s u$ as the *jump* part $D^j u$ of the derivative Du. It turns out that u is piecewise-$W^{1,p}(I)$ if and only if $u \in SBV(I)$, the set $S(u)$ is finite and $u' \in L^p(I)$ (see [3] Section 3.2). In this case, the density u' in the decomposition (4) is nothing but the usual weak derivative of u as a Sobolev function in each interval of the partition determined by $S(u)$. A crucial property of this space is given by the following compactness and closure results (see [3], Theorems 4.8 and 4.7, where the general n-dimensional setting is considered; see also [6], Theorem 7.3, for the one-dimensional case).

Theorem 2.1. *Let (u_n) be a sequence of piecewise-$H^1(I)$ functions, such that*

$$\sup_n \Big(\int_a^b |u_n'(x)|^2\,\mathrm{d}x + \#S(u_n) + \|u_n\|_\infty\Big) < +\infty.$$

Then there exist a subsequence (u_{n_k}) and a piecewise-$H^1(I)$ function u such that

$$u_{n_k} \to u, \quad u'_{n_k} \rightharpoonup u' \quad \textit{in } L^2(a,b).$$

Moreover, $D^j u_{n_k} \rightharpoonup D^j u$ weakly* in the sense of measures; i.e., for every $\varphi \in C^0([0,1])$ vanishing on 0 and 1 we have $\int_0^1 \varphi D^j u_{n_k} \to \int_0^1 \varphi D^j u$.

For a function $u = u(x,t)$ depending on both a space and a time variable, if $u(\cdot,t)$ is piecewise-$H^1(I)$ we denote by $u_x(\cdot,t)$ the (density of the absolutely continuous part of the) derivative of $u(\cdot,t)$.

Since in this paper we do not make use of any technical result about Γ-convergence, we refer the interested reader to [6] and [11] for a thorough presentation. In view of the arguments displayed in the next subparagraph, we only need to recall that the main feature of Γ-convergence for a sequence of functionals is that, under mild compactness assumptions, it leads to the convergence of minima and minimizers.

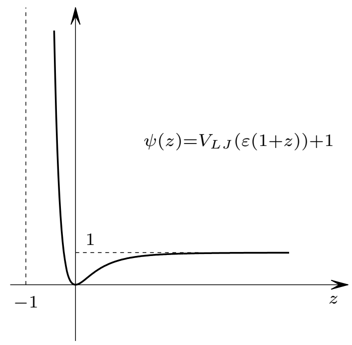

FIGURE 2. A possible function ψ in the definition of F_ε ($\varepsilon = r_m$ and $a = 1$)

2.2. **Lennard-Jones potentials.** Consider a one-dimensional array of particles whose mutual interactions can be described by a nearest-neighbour scheme ruled by a *potential of Lennard-Jones type*; i.e.,

$$V_{LJ}(r) = 4a\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] = a\left[\left(\frac{r_m}{r}\right)^{12} - 2\left(\frac{r_m}{r}\right)^{6}\right], \tag{5}$$

where: r denotes the distance between the particles, a is the depth of the potential well and $r_m = 2^{1/6}\sigma$ is the distance at which the minimum is attained (see Fig. 1). These parameters can be adjusted according to experimental data.

A specific configuration of the array of particles corresponds to assigning the distance of each particle from the neighbouring ones. Equivalently, denoting by N the number of particles, we define a *configuration* as a function $w\colon [0,l]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$, where ε is a fixed parameter, and $l = N\varepsilon$; thus, we prescribe the position of each point, labelled by an element of $[0,l]\cap\varepsilon\mathbb{Z}$. In this perspective, the identity function is the *reference configuration*. It will be convenient to choose r_m as the reference space-step ε. We will assume that $l = 1$ (thus taking a 'sample' of material into account), and denote by N_ε the number of elements in $[0,1]\cap\varepsilon\mathbb{Z}$.

If $w\colon [0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$ is a given configuration, we denote the value $w(i\varepsilon)$ simply by w_i. The energy corresponding to w is given by

$$\sum_{i=0}^{N_\varepsilon-1} V_{LJ}(w_{i+1}-w_i)\,,$$

with the constraint that $w_{i+1} > w_i$ (we agree that the energy takes value $+\infty$ if the configuration w does not satisfy this constraint). The effective configurations under given boundary data are obtained by minimizing this energy. In terms of the displacement $v = w - id$, and making the difference quotient $(v_{i+1}-v_i)/\varepsilon$ explicit, each term of the sum can be written as $V_{LJ}\big(\varepsilon(1+\frac{v_{i+1}-v_i}{\varepsilon})\big)$. Since the minimizers are not affected by the addition of a constant in the energy, we equivalently consider the following functional, whose absolute minimum is zero:

$$E_\varepsilon(v) = \sum_{i=0}^{N_\varepsilon-1} \psi\Big(\frac{v_{i+1}-v_i}{\varepsilon}\Big)\,,$$

where $\psi(t) = V_{LJ}(\varepsilon(1+t)) + a$ (independent of ε: indeed, V_{LJ} in (5) is a function of r_m/r; i.e., of ε/r). For a graph see Fig. 2.

When ε is small, the minimizers of E_ε can be qualitatively described by means of the minimizers of the Γ-limit functional for $\varepsilon \to 0$. In order to have the same functional domain for E_ε independently of ε, we consider each function $v\colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$ as a function in $L^1(0,1)$, defined by $v(x) = v(\lfloor x/\varepsilon \rfloor)$. Then it turns out (see [6] Theorem 11.7) that the Γ-limit of (E_ε) with respect to the L^1-convergence is given by

$$E(v) = \begin{cases} a\,\#S(v) & \text{if } v \text{ is piecewise constant on } (0,1) \\ & \qquad\qquad \text{and } v^+ > v^- \text{ on } S(v), \\ +\infty & \text{otherwise.} \end{cases}$$

A more refined analysis of the displacement v (i.e., of the "correction" term with respect to the identity) can be obtained by suitably rescaling the state variable, so as to obtain a non-trivial limit. By letting $v = \sqrt{\varepsilon} u$ we get the functionals:

$$F_\varepsilon(u) = \sum_{i=0}^{N_\varepsilon - 1} \psi\Big(\frac{u_{i+1} - u_i}{\sqrt{\varepsilon}}\Big).$$

In [9] (see also [10]) it is proved that as a Γ-limit we get the well-known Mumford-Shah functional

$$F(u) = \frac{1}{2}\psi''(0) \int_0^1 |u'|^2 \,\mathrm{d}x + a\,\#S(u),$$

with the constraint $u^+ > u^-$. Note that, in terms of the variable u, the initial configuration w can be written as $w = id + \sqrt{\varepsilon} u$.

In this paper we focus on the relationship between the asymptotic bahaviour of F_ε as $\varepsilon \to 0$ and the methods of *minimizing movements* described below.

2.3. **Setting of the problem and first results.** Let $\varepsilon > 0$ be given. If u is a function $[0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$, we denote the value $u(i\varepsilon)$ simply by u_i; therefore, we often write u as an indexed family $(u_i)_{i=0,1,\dots,N_\varepsilon}$ where $N_\varepsilon = \lfloor 1/\varepsilon \rfloor$. By u we will also denote the piecewise-constant extension defined by $u(x) = u_i$ with $i = \lfloor x/\varepsilon \rfloor$. The $L^p(0,1)$ norms of u are defined taking this extension into account.

Let $\psi\colon (-1,+\infty) \to [0,+\infty)$ be a C^1 function satisfying the following conditions (see the model example in Fig. 2):

A1) there exists $z_0 > 0$ such that ψ is C^3 and convex in $(-1, z_0)$ and is concave in $(z_0, +\infty)$;

A2) $\displaystyle\lim_{z\to -1^+} \psi(z) = +\infty, \qquad \lim_{z \to +\infty} \psi(z) = 1.$

A3) $\psi(0) = 0$, $\psi'(0) = 0$ and $\psi''(0) > 0$.

Remark 1. As regards the smoothness assumptions about ψ, we point out that the requirement that ψ is globally C^1 is needed to deduce the optimality conditions in the form of Proposition 2, while the assumption that ψ is C^3 on $(-1, z_0)$ is used in the proof of Theorem 3.2; otherwise, C^2 suffices.

Note, in particular, that the stated conditions imply that ψ is monotone on each of the intervals $(-1, 0]$ and $[0, +\infty)$; moreover, 0 is a minimum point and there exists a constant $\nu > 0$ such that

$$\psi(z) \geq \nu z^2 \qquad \text{for } z \leq z_0. \tag{6}$$

On the space of discrete functions $u\colon [0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$ we consider the functionals

$$F_\varepsilon(u)=\begin{cases}\displaystyle\sum_{i=0}^{N_\varepsilon-1}\psi\Big(\frac{u_{i+1}-u_i}{\sqrt{\varepsilon}}\Big) & \text{if } u_{i+1}-u_i>-\sqrt{\varepsilon} \text{ for all } i\\ +\infty & \text{otherwise.}\end{cases}\tag{7}$$

It will be useful to express F_ε in an "integral form" with explicit dependence on the difference quotient:

$$F_\varepsilon(u)=\begin{cases}\displaystyle\sum_{i=0}^{N_\varepsilon-1}\varepsilon\,\varphi_\varepsilon\Big(\frac{u_{i+1}-u_i}{\varepsilon}\Big) & \text{if } u_{i+1}-u_i>-\sqrt{\varepsilon} \text{ for all } i\\ +\infty & \text{otherwise,}\end{cases}\tag{8}$$

where

$$\varphi_\varepsilon(z)=\frac{1}{\varepsilon}\psi(\sqrt{\varepsilon}z).\tag{9}$$

Thus $\varphi_\varepsilon\colon(-1/\sqrt{\varepsilon},+\infty)\to[0,+\infty)$.

For a function $u\colon[0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$ a key role will be played by the "singular set" of the points i where the discrete gradient $(u_{i+1}-u_i)/\varepsilon$ exceeds the threshold given by the inflection point of ψ. More precisely, we define

$$I_\varepsilon^+(u)=\Big\{i\in\mathbb{Z}:\quad 0\le i\le N_\varepsilon-1,\ \frac{u_{i+1}-u_i}{\varepsilon}>\frac{z_0}{\sqrt{\varepsilon}}\Big\}.\tag{10}$$

For future reference we state the following lemma.

Lemma 2.2. *Let $u\colon[0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$ with $F_\varepsilon(u)<+\infty$. Then*

a) $\#I_\varepsilon^+(u)\le\dfrac{1}{\nu z_0^2}F_\varepsilon(u)$;

b) if $\zeta_0\in(-1,0)$ is such that $\psi(\zeta_0)\ge F_\varepsilon(u)$, then $\dfrac{u_{i+1}-u_i}{\varepsilon}>\dfrac{\zeta_0}{\sqrt{\varepsilon}}$ for every $i=0,\dots,N_\varepsilon-1$.

Proof. Estimate (a) immediately follows from (6), since

$$F_\varepsilon(u)\ge\sum_{i\in I_\varepsilon^+(u)}\nu z_0^2=\nu z_0^2\#I_\varepsilon^+(u).$$

As for (b), for every i we have

$$\psi\Big(\sqrt{\varepsilon}\frac{u_{i+1}-u_i}{\varepsilon}\Big)\le F_\varepsilon(u)\le\psi(\zeta_0),$$

and we conclude by the monotonicity of ψ in $(-1,0]$. □

For any given $u\colon[0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$ we define the extension $\hat u$ on $[0,1]$ obtained by linear interpolation outside the set $\varepsilon I_\varepsilon^+(u)$:

$$\hat u(x)=\begin{cases}u_i & \text{if } i:=\lfloor x/\varepsilon\rfloor\in I_\varepsilon^+(u) \text{ or } i=N_\varepsilon\\ (1-\lambda)u_i+\lambda u_{i+1} & \text{otherwise (here, } i:=\lfloor x/\varepsilon\rfloor\\ & \text{and } \lambda=x/\varepsilon-\lfloor x/\varepsilon\rfloor).\end{cases}\tag{11}$$

Remark 2. *a)* The extension $\hat u$ is right-continuous and

$$i\varepsilon\in S(\hat u)\quad\text{if and only if}\quad i-1\in I_\varepsilon^+(u).$$

Note that $\hat u^+(x)-\hat u^-(x)>0$ for every $x\in S(\hat u)$.

b) Recalling that by u we also denote the piecewise-constant function $[0,1]\to\mathbb{R}$ defined by $u(x)=u_i$ with $i=\lfloor x/\varepsilon\rfloor$, we have

$$|\hat{u}(x)-u(x)|\le z_0\sqrt{\varepsilon}\qquad\text{for every } x\in[0,1]. \tag{12}$$

An important compactness property for the extensions $\hat{u}$ is given by the following lemma.

Lemma 2.3. *Let (ε_n) be a positive infinitesimal sequence and let (v_n) be an equibounded sequence of functions $[0,1]\cap\varepsilon_n\mathbb{Z}\to\mathbb{R}$ such that*

$$F_{\varepsilon_n}(v_n)\le M$$

for some constant M. Let $\hat{v}_n$ be the extensions introduced according to (11). Then

$$\int_0^1|\hat{v}_n'(x)|^2\,\mathrm{d}x+\#S(\hat{v}_n)\le\frac{M}{\nu\min(z_0^2,1)}. \tag{13}$$

In particular, up to a subsequence, there exists a piecewise-$H^1(0,1)$ function v such that

$$\hat{v}_n\to v,\quad \hat{v}_n'\rightharpoonup v'\quad\text{in } L^2(0,1).$$

Moreover, $D^j\hat{v}_n\rightharpoonup D^jv$ weakly* in the sense of measures.

Proof. We have:

$$\begin{aligned}M\ge F_{\varepsilon_n}(v_n)&=\sum_{i\notin I^+_{\varepsilon_n}(v_n)}\varepsilon_n\varphi_{\varepsilon_n}\left(\frac{(v_n)_{i+1}-(v_n)_i}{\varepsilon_n}\right)\\&\qquad+\sum_{i\in I^+_{\varepsilon_n}(v_n)}\varepsilon_n\varphi_{\varepsilon_n}\left(\frac{(v_n)_{i+1}-(v_n)_i}{\varepsilon_n}\right)\\&\ge\nu\sum_{i\notin I^+_{\varepsilon_n}(v_n)}\varepsilon_n\left(\frac{(v_n)_{i+1}-(v_n)_i}{\varepsilon_n}\right)^2+\nu z_0^2\,\#I^+_{\varepsilon_n}(v_n)\\&\ge\nu\min(z_0^2,1)\left[\int_0^1|\hat{v}_n'(x)|^2\,\mathrm{d}x+\#S(\hat{v}_n)\right].\end{aligned}$$

We conclude by applying Theorem 2.1. □

Remark 3. By the uniform estimate (12), the $L^p(0,1)$ convergence of $(\hat{v}_n)$ is equivalent to the $L^p(0,1)$ convergence of the piecewise-constant functions v_n.

Lemma 2.4. *Let (v_n) and v be as in Lemma 2.3. Then*

a) $v^+-v^->0$ on $S(v)$;

b) up to a subsequence, (v_n) satisfies the following property: for every $\overline{x}\in S(v)$ there exists a sequence (x^n) converging to $\overline{x}$ and such that

$$x^n\in S(\hat{v}_n)\quad\textit{and}\quad\lim_{n\to\infty}\big(\hat{v}_n^+(x^n)-\hat{v}_n^-(x^n)\big)>0.$$

Proof. a) Since $D^j\hat{v}_n$ are positive measures which weakly* converge to D^jv, this latter is a positive measure, too.

b) Let $\overline{x}\in S(v)$ and let V be an open neighbourhood of $\overline{x}$ such that $S(v)\cap\overline{V}=\{\overline{x}\}$. By the weak* convergence of the measures $D^j\hat{v}_n$ to D^jv on V, we have (see, e.g., [3], Prop. 1.62) $D^jv(V)=\lim_{n\to\infty}D^j\hat{v}_n(V)$; i.e.,

$$v^+(\overline{x})-v^-(\overline{x})=\lim_{n\to\infty}\sum_{x\in S(\hat{v}_n)\cap V}\big(\hat{v}_n^+(x)-\hat{v}_n^-(x)\big). \tag{14}$$

By estimate (13) for every $n \in \mathbb{N}$, we can define $x_1^n, \ldots, x_m^n$, with m independent of n, such that

$$S(\hat{v}_n) \subseteq \{x_i^n : \ i = 1, \ldots, m\}.$$

Up to a subsequence we can assume that every sequence $(x_i^n)_n$ converges to a point in $[0,1]$: denote by S this set of points. It turns out that $S \cap \overline{V} \neq \emptyset$, otherwise $v^+(\overline{x}) - v^-(\overline{x}) = 0$ by (14). By the arbitrariness of V we must have $\overline{x} \in S$. Hence, we can choose V such that $V \cap S = \{\overline{x}\}$. From (14) it follows that there exists a sequence $(x_i^n)_n$ converging to $\overline{x}$ such that

$$x_i^n \in S(\hat{v}_n), \quad \text{and} \quad \limsup_{n\to\infty} \big(\hat{v}_n^+(x_i^n) - \hat{v}_n^-(x_i^n)\big) > 0,$$

otherwise $v^+(\overline{x}) - v^-(\overline{x}) = 0$. □

2.4. **Minimizing movements along F_ε.** As mentioned in the introduction, we apply the method of minimizing movements to the functionals F_ε, but we allow the space-discretization parameter ε to vary as the time-discretization step goes to zero.

For each $\varepsilon > 0$ let $u_\varepsilon^0 \colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$ be a given function and let $\tau > 0$ be fixed. We recursively define a sequence $u^k := u^k_{\varepsilon,\tau}$ ($k \in \mathbb{N}$) of real-valued functions on $[0,1] \cap \varepsilon\mathbb{Z}$, by requiring that u^0 is the initial datum u_ε^0 just fixed, while for any $k \geq 1$, the function u^k is a minimizer of

$$G^k_{\varepsilon,\tau}(v) := F_\varepsilon(v) + \frac{1}{2\tau} \sum_{i=0}^{N_\varepsilon} \varepsilon |v_i - u_i^{k-1}|^2 \,, \tag{15}$$

with respect to all functions $v \colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$. We state some easy consequences of this definition.

Proposition 1. *For every $k \in \mathbb{N}$ the following properties hold:*

a) $F_\varepsilon(u^k) \leq F_\varepsilon(u^{k-1})$,

b) $\displaystyle\sum_{i=0}^{N_\varepsilon} \varepsilon |u_i^k - u_i^{k-1}|^2 \leq 2\tau \big[F_\varepsilon(u^{k-1}) - F_\varepsilon(u^k)\big]$,

c) $\|u^k\|_\infty \leq \|u^{k-1}\|_\infty \leq \|u_\varepsilon^0\|_\infty$.

Proof. The minimality of u^k with respect to the test function $v = u^{k-1}$, implies that

$$F_\varepsilon(u^k) + \frac{1}{2\tau} \sum_{i=0}^{N_\varepsilon} \varepsilon |u_i^k - u_i^{k-1}|^2 \leq F_\varepsilon(u^{k-1}).$$

From this inequality, (a) and (b) follow immediately.

Moreover, if $M := \|u^{k-1}\|_\infty$, then for every u we have

$$G^k_{\varepsilon,\tau}((u \wedge M) \vee (-M)) \leq G^k_{\varepsilon,\tau}(u).$$

Therefore $\|u^k\|_\infty \leq \|u^{k-1}\|_\infty$. □

Since u^k is a solution of a minimum problem in finite dimension we get the classical optimality conditions.

Proposition 2. *Let u^k be defined recursively by (15). Then, the following equations hold:*

$$
\begin{aligned}
&-\varphi'_\varepsilon\left(\frac{u_1^k-u_0^k}{\varepsilon}\right)+\frac{\varepsilon}{\tau}(u_0^k-u_0^{k-1})=0\\
&\varphi'_\varepsilon\left(\frac{u_i^k-u_{i-1}^k}{\varepsilon}\right)-\varphi'_\varepsilon\left(\frac{u_{i+1}^k-u_i^k}{\varepsilon}\right)+\frac{\varepsilon}{\tau}(u_i^k-u_i^{k-1})=0 \qquad (0<i<N_\varepsilon)\\
&\varphi'_\varepsilon\left(\frac{u_{N_\varepsilon}^k-u_{N_\varepsilon-1}^k}{\varepsilon}\right)+\frac{\varepsilon}{\tau}(u_{N_\varepsilon}^k-u_{N_\varepsilon}^{k-1})=0.
\end{aligned}
$$

For any given $\varepsilon>0$ and $\tau>0$ and for every $k\in\mathbb{N}$ we interpret the values $(u^k_{\varepsilon,\tau})_i$ (for $i=0,\dots,N_\varepsilon$) as the discrete evolution, at the time $t=k\tau$, of the initial (discrete) datum $u^0_\varepsilon\colon[0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$. The goal is to detect the limit evolution as $\varepsilon,\tau\to0$.

Remark 4. The optimality conditions in the proposition above easily suggest the form of the evolution equation satisfied by a possible limit function u. Indeed, by dividing the i-th equation by ε and applying the mean-value theorem to $\varphi'_\varepsilon(z)=\psi'(\sqrt{\varepsilon}z)/\sqrt{\varepsilon}$, we get

$$\frac{u_i^k-u_i^{k-1}}{\tau}=\psi''(\sqrt{\varepsilon}\xi)\frac{u_{i+1}^k-2u_i^k+u_{i-1}^k}{\varepsilon^2},$$

where ξ is a suitable value between the two difference quotients. Hence, in the limit we obtain

$$u_t=\psi''(0)u_{xx}$$

at the points in which u is twice differentiable (see Theorem 4.1).

On the initial datum u^0_ε we make the following assumptions:

$B1$) $(u^0_\varepsilon)_\varepsilon$ is an equibounded set of functions $[0,1]\cap\varepsilon\mathbb{Z}\to\mathbb{R}$; i.e., we have $\sup\{(u^0_\varepsilon)_i : 0\le i\le N_\varepsilon,\ \varepsilon>0\}<+\infty$;

$B2$) there exists $M>0$ such that $F_\varepsilon(u^0_\varepsilon)\le M$ for every $\varepsilon>0$.

With in view the analysis of the limit, as $\varepsilon,\tau\to0$, of the discrete evolutions $(u^k_{\varepsilon,\tau})_k$ defined in the previous section, we introduce the piecewise-constant spatial-time extension $u_{\varepsilon,\tau}$ of these values to $[0,1]\times[0,+\infty)$ by defining

$$
\begin{aligned}
&u_{\varepsilon,\tau}\colon[0,1]\times[0,+\infty)\to\mathbb{R},\\
&u_{\varepsilon,\tau}(x,t)=\big(u^k_{\varepsilon,\tau}\big)_i \quad \text{with } k=\lfloor t/\tau\rfloor \text{ and } i=\lfloor x/\varepsilon\rfloor.
\end{aligned}
\tag{16}
$$

In the following section we give a compactness result (Theorem 3.1) for the family $u_{\varepsilon,\tau}$ as $\varepsilon,\tau\to0$.

3. **Compactness.** The compactness result contained in Theorem 3.1 follows a standard argument in the theory of minimizing movements (see [1], [2] and [7]). In Theorem 3.2 we prove a regularity result for the limit function.

Proposition 3. *For any $s,t\ge0$, with $s<t$, we have*

$$\|u_{\varepsilon,\tau}(\cdot,t)-u_{\varepsilon,\tau}(\cdot,s)\|_2\le\big(2F_\varepsilon(u^0_\varepsilon)\big)^{1/2}\sqrt{t-s+\tau}.$$

Proof. Let $x \in [0,1]$ and $0 \le s < t$ be fixed; set $h = \lfloor s/\tau \rfloor$ and $k = \lfloor t/\tau \rfloor$. For every i it turns out that:

$$\begin{aligned} |\big(u^k_{\varepsilon,\tau}\big)_i - \big(u^h_{\varepsilon,\tau}\big)_i| &\le \sum_{j=h}^{k-1} |\big(u^{j+1}_{\varepsilon,\tau}\big)_i - \big(u^j_{\varepsilon,\tau}\big)_i| \\ &\le \sqrt{k-h}\sqrt{\sum_{j=h}^{k-1} |\big(u^{j+1}_{\varepsilon,\tau}\big)_i - \big(u^j_{\varepsilon,\tau}\big)_i|^2}\,. \end{aligned}$$

Therefore, by Proposition 1, we have

$$\begin{aligned} \sum_{i=0}^{N_\varepsilon} \varepsilon|\big(u^k_{\varepsilon,\tau}\big)_i - \big(u^h_{\varepsilon,\tau}\big)_i|^2 &\le (k-h)\sum_{i=0}^{N_\varepsilon}\sum_{j=h}^{k-1} \varepsilon|\big(u^{j+1}_{\varepsilon,\tau}\big)_i - \big(u^j_{\varepsilon,\tau}\big)_i|^2 \\ &\le 2\tau(k-h)\sum_{j=h}^{k-1}\big(F_\varepsilon(u^j_{\varepsilon,\tau}) - F_\varepsilon(u^{j+1}_{\varepsilon,\tau})\big) \\ &\le 2\tau(k-h)(F_\varepsilon(u^h_{\varepsilon,\tau}) - F_\varepsilon(u^k_{\varepsilon,\tau})\big) \le 2(t-s+\tau)F_\varepsilon(u^0_\varepsilon). \quad \square \end{aligned}$$

Theorem 3.1. *Under assumptions* $(B1)$ *and* $(B2)$*, let* (ε_n) *and* (τ_n) *be positive infinitesimal sequences, and let* $v_n = u_{\varepsilon_n,\tau_n}$ *be the piecewise-constant functions defined in* (16)*. For every* $t \ge 0$ *denote by* $\hat v_n(\cdot,t)$ *the piecewise-affine extension of* $v_n(\cdot,t)$ *according to* (11)*. Then there exist a subsequence (not relabeled) of* (v_n) *and a function* $u \in C^{1/2}([0,+\infty);L^2(0,1))$ *such that*

$$v_n \to u, \quad \hat v_n \to u \quad \text{in } L^\infty([0,T];L^2(0,1)) \text{ and a.e. in } (0,1)\times(0,T)$$

for every $T \ge 0$*. Moreover, for every* $t \ge 0$*,*

$$\begin{aligned} &u(\cdot,t) \text{ is piecewise-}H^1(0,1) \\ &(\hat v_n)_x(\cdot,t) \rightharpoonup u_x(\cdot,t) \quad \text{in } L^2(0,1). \end{aligned}$$

Finally, for every $\overline{x} \in S\big(u(\cdot,t)\big)$ *there exist a subsequence* $(\hat v_{n_h})_h$*, possibly depending on* t*, and a sequence* $(x^h)_h$ *converging to* $\overline{x}$ *and such that*

$$x^h \in S(v_{n_h}) \quad \text{and} \quad \lim_{h\to\infty}\big(\hat v^+_{n_h}(x^h,t) - \hat v^-_{n_h}(x^h,t)\big) > 0.$$

Proof. Let $t \ge 0$ be fixed. By Proposition 1(a) we have that $F_{\varepsilon_n}\big(v_n(\cdot,t)\big)$ is a bounded sequence. Thus, we can apply Lemma 2.3 to the functions $v_n(\cdot,t)$: the sequence $(\hat v_n(\cdot,t))_n$ is pre-compact with respect to the $L^2(0,1)$ convergence; moreover, the limit is piecewise-$H^1(0,1)$, and we have weak-L^2 convergence of $(\hat v_n)_x(\cdot,t)$. Note that, by the uniform estimate (12), the $L^2(0,1)$ (or a.e.) convergence of $(\hat v_n)$ is equivalent to the corresponding convergence of (v_n).

By a diagonalization argument we can assume that, up to a subsequence, $\hat v_n(\cdot,t)$ converge in $L^2(0,1)$ for every $t \in \mathbb{Q}^+$: let $u(\cdot,t)$ be the limit function. The estimate in Proposition 3 allows to get the $L^2(0,1)$ convergence for every $t \ge 0$ (hence, $u(\cdot,t)$ is well defined for every $t \ge 0$). Moreover

$$\|u(\cdot,t) - u(\cdot,s)\|_2 \le C\sqrt{t-s}, \tag{17}$$

for any $s,t \ge 0$, with $s < t$ and for a suitable constant C, independent of s and t. Thus $u \in C^{1/2}([0,+\infty);L^2(0,1))$. Furthermore, by the uniqueness of the L^2 limit, the compactness result of Theorem 2.1 guarantees that $u(\cdot,t)$ is piecewise-$H^1(0,1)$ and that $(\hat v_n)_x(\cdot,t)$ weakly converges to $u_x(\cdot,t)$ in $L^2(0,1)$ for every $t \ge 0$.

We now prove the convergence of (v_n) to u in $L^\infty([0,T];L^2(0,1))$ (from which the analogous convergence of $(\hat{v}_n)$ follows as well). Let $T>0$ be fixed. For any given $N\in\mathbb{N}$, define $t_j = jT/N$ for $j=0,\dots,N$; then, for every $t\in[0,T]$ there exists $j=0,\dots,N-1$ with $t_j\le t\le t_{j+1}$. By Proposition 3 and estimate (17), we have:

$$\begin{aligned}\|v_n(\cdot,t)-u(\cdot,t)\|_2 &\le \|v_n(\cdot,t)-v_n(\cdot,t_j)\|_2+\|v_n(\cdot,t_j)-u(\cdot,t_j)\|_2\\ &\qquad +\|u(\cdot,t_j)-u(\cdot,t)\|_2\\ &\le 2C\sqrt{t-t_j+\tau_n}+\|v_n(\cdot,t_j)-u(\cdot,t_j)\|_2\,.\end{aligned}$$

Fix $\sigma>0$ and let $n_\sigma\in\mathbb{N}$ be such that

$$\|v_n(\cdot,t_j)-u(\cdot,t_j)\|_2\le\sigma\quad\text{for every } n\ge n_\sigma \text{ and } j=0,\dots,N-1.$$

Then

$$\sup_{t\in[0,T]}\|v_n(\cdot,t)-u(\cdot,t)\|_2\le 2C\sqrt{(T/N)+\tau_n}+\sigma\qquad\text{for every } n\ge n_\sigma,$$

and this yields

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}\|v_n(\cdot,t)-u(\cdot,t)\|_2\le 2C\sqrt{(T/N)}+\sigma.$$

By the arbitrariness of N and σ we deduce the convergence in $L^\infty([0,T];L^2(0,1))$. In particular, we have the convergence in $L^2((0,1)\times(0,T))$, and hence the convergence a.e. (up to a subsequence).

Finally, if $\overline{x}\in S\big(u(\cdot,t)\big)$ then we can apply Lemma 2.4(b) to the sequence $v_n=v_n(\cdot,t)$ (note that we are arguing for a t fixed, and the possible subsequence considered in Lemma 2.4(b) may depend on t). □

Remark 5. The weak-$L^2(0,1)$ convergence of the sections $(\hat{v}_n)_x(\cdot,t)$ and their uniform boundedness in $L^2(0,1)$ (see Lemma 2.3) allow to deduce the weak-$L^2\big((0,1)\times(0,T)\big)$ convergence of $(\hat{v}_n)_x$.

Theorem 3.2. *Let $v_n=u_{\varepsilon_n,\tau_n}$ be a sequence converging to a function u as in Theorem 3.1. Then $u_x(\cdot,t)\in H^1(0,1)$ for a.e. $t\ge 0$ and $(u_x)_x\in L^2\big((0,1)\times(0,T)\big)$ for every $T>0$. Moreover, for a.e. $t\ge0$, we have $u_x(0,t)=u_x(1,t)=0$ and $u_x(\cdot,t)=0$ on $S\big(u(\cdot,t)\big)$.*

For future reference it is useful to isolate from the proof a technical lemma.

Let $v_n=u_{\varepsilon_n,\tau_n}$ be a sequence converging to u according to Theorem 3.1. In the sequel we will drop the index n and simply write ε and τ in place of ε_n and τ_n. By (16) we have

$$v_n(x,t)=\big(u^k_{\varepsilon,\tau}\big)_i\quad\text{with } k=\lfloor t/\tau\rfloor \text{ and } i=\lfloor x/\varepsilon\rfloor. \tag{18}$$

We extend the definition by setting

$$\big(u^k_{\varepsilon,\tau}\big)_i=\begin{cases}\big(u^k_{\varepsilon,\tau}\big)_0 & \text{if } i\in\mathbb{Z},\ i<0,\\ \big(u^k_{\varepsilon,\tau}\big)_{N_\varepsilon} & \text{if } i\in\mathbb{Z},\ i>N_\varepsilon\,.\end{cases}$$

Thus, for every $x\in\mathbb{R}$ and $t\ge0$ we can define the piecewise-constant function

$$w_n(x,t)=\varphi'_\varepsilon\bigg(\frac{(u^k_{\varepsilon,\tau})_{i+1}-(u^k_{\varepsilon,\tau})_i}{\varepsilon}\bigg),\qquad\text{with}\quad\begin{array}{l} i=\lfloor x/\varepsilon\rfloor\\ k=\lfloor t/\tau\rfloor.\end{array} \tag{19}$$

Lemma 3.3. *For every $t \geq 0$*

$$w_n(\cdot,t) \rightharpoonup \psi''(0)u_x(\cdot,t) \qquad \text{in } L^2(0,1).$$

Moreover, for every $T>0$ the sequence $(w_n(\cdot,t))_n$ is uniformly bounded, with respect to $t\in[0,T]$, in $L^2(0,1)$. In particular, $u_x \in L^2\big((0,1)\times(0,T)\big)$.

Proof. Let $t \geq 0$ be fixed. Denote by χ_n the characteristic function of the set $\bigcup_{i\in I_\varepsilon^+} \varepsilon[i,i+1)$, where $I_\varepsilon^+ = I_\varepsilon^+\big(v_n(\cdot,t)\big)$. Consider the decomposition

$$w_n(\cdot,t) = \chi_n w_n(\cdot,t) + (1-\chi_n)w_n(\cdot,t).$$

By Lemma 2.2 and the decreasing monotonicity of $F_\varepsilon(u^k)$ with respect to k (see Proposition 1), it turns out that

$$\begin{aligned}\int_0^1 |\chi_n w_n(x,t)|^2 \,\mathrm{d}x &= \sum_{i\in I_\varepsilon^+} \varepsilon |w_n(i\varepsilon,t)|^2 \\ &\leq \varepsilon(\# I_\varepsilon^+)\varphi_\varepsilon'\Big(\frac{z_0}{\sqrt{\varepsilon}}\Big)^2 \leq \frac{M}{\nu z_0^2}\psi'(z_0)^2\,,\end{aligned} \tag{20}$$

so that $\big(\chi_n w_n(\cdot,t)\big)$ is bounded in $L^2(0,1)$. By the same argument we get

$$\int_0^1 |\chi_n w_n(x,t)| \,\mathrm{d}x \leq \varepsilon(\# I_\varepsilon^+)\varphi_\varepsilon'\Big(\frac{z_0}{\sqrt{\varepsilon}}\Big) \leq \sqrt{\varepsilon}\frac{M}{\nu z_0^2}\psi'(z_0) \to 0 \tag{21}$$

as $n \to +\infty$. We conclude that

$$\chi_n w_n(\cdot,t) \rightharpoonup 0 \quad \text{weakly in } L^2(0,1). \tag{22}$$

Let us now consider $(1-\chi_n)w_n(\cdot,t)$. Note that, in the notation of (19), if $i \notin I_\varepsilon^+$ and $x \in [i\varepsilon,(i+1)\varepsilon)$ we have

$$\frac{(u^k_{\varepsilon,\tau})_{i+1} - (u^k_{\varepsilon,\tau})_i}{\varepsilon} = (\hat v_n)_x(x,t),$$

where $\hat v_n(\cdot,t)$ is the extension of $v_n(\cdot,t)$ according to (11). If we take into account that $(\hat v_n)_x(x,t) = 0$ in $\big(i\varepsilon,(i+1)\varepsilon\big)$ if $i\in I_\varepsilon^+$, then

$$(1-\chi_n)w_n(\cdot,t) = \varphi_\varepsilon'\big((\hat v_n)_x(\cdot,t)\big). \tag{23}$$

Consider now the Taylor expansion of φ_ε' at 0; for every $x\in[i\varepsilon,(i+1)\varepsilon)$, with $i\notin I_\varepsilon^+$, we have

$$\varphi_\varepsilon'\big((\hat v_n)_x(x,t)\big) = \varphi_\varepsilon'(0) + \varphi_\varepsilon''(0)(\hat v_n)_x(x,t) + \frac12 \varphi_\varepsilon'''(\xi_n)\big((\hat v_n)_x(x,t)\big)^2$$

with ξ_n between 0 and $(\hat v_n)_x(x,t)$; hence,

$$\varphi_\varepsilon'\big((\hat v_n)_x(x,t)\big) = \psi''(0)(\hat v_n)_x(x,t) + \frac12\sqrt{\varepsilon}r_n\big((\hat v_n)_x(x,t)\big)^2 \tag{24}$$

with $r_n = \psi'''(\sqrt{\varepsilon}\xi_n)$. From Lemma 2.2 we deduce that

$$\frac{\zeta_0}{\sqrt{\varepsilon}} < (\hat v_n)_x(x,t) \leq \frac{z_0}{\sqrt{\varepsilon}}$$

where $\zeta_0 \in (-1,0)$ is such that $\psi(\zeta_0) \geq M$. Then (r_n) is a bounded sequence, and $(\sqrt{\varepsilon}(\hat v_n)_x)_n$ is bounded. This implies that

$$\big|\varphi_\varepsilon'\big((\hat v_n)_x(x,t)\big)\big| \leq C|(\hat v_n)_x(x,t)| \tag{25}$$

for a suitable constant C. Note that (24) and (25) hold for $i\in I_\varepsilon^+$, too (indeed $(\hat v_n)_x(x,t) = 0$ for such indices).

Let $T>0$ and $t\in[0,T]$ be fixed. By the boundedness of $\big((\hat v_n)_x(\cdot,t)\big)_n$ in $L^2(0,1)$ (see Lemma 2.3) we have the weak convergence of $\varphi'_\varepsilon\big((\hat v_n)_x(\cdot,t)\big)$ in $L^2(0,1)$, *up to a subsequence*. Let us show that the limit is $\psi''(0)u_x(\cdot,t)$, hence it is independent of the subsequence, and therefore the whole sequence converges:

$$\varphi'_\varepsilon\big((\hat v_n)_x(\cdot,t)\big)\rightharpoonup\psi''(0)u_x(\cdot,t)\quad\text{weakly in } L^2(0,1). \tag{26}$$

Indeed, by the weak-L^2 convergence of $(\hat v_n)_x(\cdot,t)$ (see Theorem 3.1), the right-hand side of (24) weakly converges to $\psi''(0)u_x(\cdot,t)$ in $L^1(0,1)$.

By (22), (23) and (26) we now have

$$w_n(\cdot,t))\rightharpoonup\psi''(0)u_x(\cdot,t)\quad\text{weakly in } L^2(0,1). \tag{27}$$

Finally, by the uniform boundedness (with respect to t) of $\big((\hat v_n)_x(\cdot,t)\big)_n$ in $L^2\big((0,1)\big)$ (see Lemma 2.3), from (25) we deduce the same boundedness of $\big(\varphi'_\varepsilon\big((\hat v_n)_x(\cdot,t)\big)\big)_n$, too. From this, (23) and the uniform L^2-bound of $\chi w_n(\cdot,t)$ (see (20)), we conclude with the uniform $L^2(0,1)$-bound for $w_n(\cdot,t)$. □

Proof of Theorem 3.2. Here we improve the convergence result of the previous lemma, showing that $\psi''(0)u_x(\cdot,t)$ is the weak limit in $H^1(0,1)$ of the *piecewise-affine* extension $\tilde w_n$ of the function defined in (19) on the nodes.

By Proposition 1

$$\sum_{i=0}^{N_\varepsilon}\varepsilon\big|\big(u^k_{\varepsilon,\tau}\big)_i-\big(u^{k-1}_{\varepsilon,\tau}\big)_i\big|^2\le 2\tau\big[F_\varepsilon\big(u^{k-1}_{\varepsilon,\tau}\big)-F_\varepsilon\big(u^k_{\varepsilon,\tau}\big)\big].$$

Let $T>0$ be fixed, and $M_\tau=\lfloor T/\tau\rfloor$. Then

$$\sum_{k=1}^{M_\tau}\sum_{i=0}^{N_\varepsilon}\tau\varepsilon\big|\big(u^k_{\varepsilon,\tau}\big)_i-\big(u^{k-1}_{\varepsilon,\tau}\big)_i\big|^2\le 2\tau^2F_\varepsilon(u^0_\varepsilon)\le 2\tau^2M$$

(where M is given in assumption $(B2)$). By Proposition 2 and the extension, defined above, of $\big(u^k_{\varepsilon,\tau}\big)_i$ for $i<0$ and $i>N_\varepsilon$, this estimate can be written as

$$\sum_{k=1}^{M_\tau}\tau\sum_{i\in\mathbb{Z}}\varepsilon\tau^2\left[\varepsilon^{-1}\left(\varphi'_\varepsilon\Big(\frac{(u^k_{\varepsilon,\tau})_{i+1}-(u^k_{\varepsilon,\tau})_i}{\varepsilon}\Big)-\varphi'_\varepsilon\Big(\frac{(u^k_{\varepsilon,\tau})_i-(u^k_{\varepsilon,\tau})_{i-1}}{\varepsilon}\Big)\right)\right]^2\le 2\tau^2M\,.$$

Let $\tilde w_n(\cdot,t)$ be the function obtained as the piecewise-affine extension of the values $w_n(\cdot,t)$ on the nodes $\varepsilon\mathbb{Z}$. By the previous estimate we have

$$\sum_{k=1}^{M_\tau}\tau\int_{\mathbb{R}}\big[(\tilde w_n)_x(x,k\tau)\big]^2\,\mathrm{d}x\le 2M,$$

and therefore, for every $\delta>0$ and $\tau<\delta$:

$$\int_\delta^T\mathrm{d}t\int_{\mathbb{R}}\big[(\tilde w_n)_x(x,t)\big]^2\,\mathrm{d}x\le 2M.$$

By Fatou's Lemma

$$\int_\delta^T\Big(\liminf_{n\to+\infty}\int_{\mathbb{R}}\big[(\tilde w_n)_x(x,t)\big]^2\,\mathrm{d}x\Big)\,\mathrm{d}t\le 2M. \tag{28}$$

We deduce that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}}\big[(\tilde w_n)_x(x,t)\big]^2\,\mathrm{d}x<+\infty\qquad\text{for a.e. } t\in(\delta,T).$$

We now fix t satisfying this condition; let (n_k) be a sequence such that

$$\lim_{k\to+\infty}\int_{\mathbb{R}}\big[(\tilde{w}_{n_k})_x(x,t)\big]^2\,\mathrm{d}x=\liminf_{n\to+\infty}\int_{\mathbb{R}}\big[(\tilde{w}_n)_x(x,t)\big]^2\,\mathrm{d}x. \tag{29}$$

Then

$$\int_{\mathbb{R}}\big[(\tilde{w}_{n_k})_x(x,t)\big]^2\,\mathrm{d}x\le C \tag{30}$$

for a suitable constant C independent of k.

The functions $w_n(\cdot,t)$ in (19) take the value 0 outside the interval $[0,\varepsilon N_\varepsilon]$. Therefore, the weak convergence stated in Lemma 3.3 yields

$$w_n(\cdot,t)\rightharpoonup w(\cdot,t):=\begin{cases}\psi''(0)\,u_x(\cdot,t) & \text{in } (0,1),\\ 0 & \text{otherwise in } \mathbb{R}\end{cases}\qquad \text{in } L^2(\mathbb{R}).$$

By (30) this implies the weak convergence in $L^2(\mathbb{R})$ of the piecewise-affine functions $\tilde{w}_{n_k}(\cdot,t)$. Indeed, $\sum_i\varepsilon\big|w_{n_k}\big((i+1)\varepsilon,t\big)-w_{n_k}(i\varepsilon,t)\big|^2\le\varepsilon^2C$. Thus

$$\tilde{w}_{n_k}(\cdot,t)\rightharpoonup w(\cdot,t)\qquad\text{in } L^2(\mathbb{R}). \tag{31}$$

At this point we have proved that for a.e. $t\ge0$ both (30) and (31) hold. Therefore, for any open interval $J\supset[0,1]$ we have $w\in H^1(J)$; in particular, $u_x(\cdot,t)\in H^1(0,1)$ and $u_x(0,t)=u_x(1,t)=0$ for a.e. $t\ge0$.

Moreover, $\big(\tilde{w}_{n_k}\big)_x(\cdot,t)$ weakly converges to $w_x(\cdot,t)$ il $L^2(0,1)$; therefore, by (29)

$$\begin{aligned}\int_0^1[w_x(x,t)]^2\,\mathrm{d}x&\le\liminf_{k\to+\infty}\int_{\mathbb{R}}\big[(\tilde{w}_{n_k})_x(x,t)\big]^2\,\mathrm{d}x\\&=\liminf_{n\to+\infty}\int_{\mathbb{R}}\big[(\tilde{w}_n)_x(x,t)\big]^2\,\mathrm{d}x.\end{aligned}$$

Let us now take the arbitrariness of t into account: by (28) it turns out that

$$\int_\delta^T\mathrm{d}t\int_0^1[w_x(x,t)]^2\,\mathrm{d}x\le2M$$

for every $\delta>0$. We conclude that $(u_x)_x\in L^2\big((0,1)\times(0,T)\big)$.

Finally, let us prove that $u_x(\cdot,t)$ vanishes on the jump points of $u(\cdot,t)$. Let t be such that (30) and (31) hold. For the sake of simplicity let us drop the subscript k from n_k. By the compact injection of $H^1(0,1)$ into $C([0,1])$, we deduce that

$$\tilde{w}_n(\cdot,t)\to\psi''(0)u_x(\cdot,t)\quad\text{in } C([0,1]).$$

Let $\overline{x}$ be a jump point of $u(\cdot,t)$; on account of Lemma 2.4(b) we can assume that there exist a sequence (x^n) converging to $\overline{x}$ and a value $\gamma>0$ such that for every n

$$x^n\in S\big(\hat{v}_n(\cdot,t)\big),\qquad \hat{v}_n^+(x^n,t)-\hat{v}_n^-(x^n,t)\ge\gamma>0.$$

Recall that x^n can be expressed as $i_n\varepsilon$, for a suitable i_n. By Remark 2(a), $x^n=i_n\varepsilon\in S\big(\hat{v}_n(\cdot,t)\big)$ if and only if $i_n-1\in I_\varepsilon^+\big(\hat{v}_n(\cdot,t)\big)$. Then

$$\begin{aligned}\tilde{w}_n\big((i_n-1)\varepsilon,t\big)=w_n\big((i_n-1)\varepsilon,t\big)&=\varphi_\varepsilon'\Big(\frac{\hat{v}_n^+(x^n,t)-\hat{v}_n^-(x^n,t)}{\varepsilon}\Big)\\&\le\varphi_\varepsilon'\Big(\frac{\gamma}{\varepsilon}\Big)=\frac{1}{\sqrt{\varepsilon}}\psi'\Big(\frac{\gamma}{\sqrt{\varepsilon}}\Big).\end{aligned}$$

Note now that $\lim_{z\to+\infty} z\psi'(z) = 0$; indeed, for every $z \geq 2z_0$ there exists a value $\xi_z \in (z/2, z)$ such that

$$\frac{\psi(z) - \psi(z/2)}{z/2} = \psi'(\xi_z) \geq \psi'(z) \geq 0,$$

from which $0 \leq z\psi'(z) \leq 2\big(\psi(z) - \psi(z/2)\big) \to 0$ as $z \to +\infty$. Therefore,

$$\lim_{n\to+\infty} \tilde{w}_n((i_n - 1)\varepsilon, t) = 0,$$

and the uniform convergence of $\tilde{w}_n(\cdot, t)$ to $\psi''(0)u_x(\cdot, t)$ imply that $u_x(\overline{x}, t) = 0$. □

4. **Limit equation and evolution of the singular set.**

4.1. **Limit equation.** Assume that $(u^0_\varepsilon)_{\varepsilon>0}$ is an indexed family of functions satisfying conditions $(B1)$ and $(B2)$ and converging a.e. (as piecewise-constant functions) to a function u^0. By the estimate of Lemma 2.3 we have that u^0 is piecewise-$H^1(0,1)$. For any fixed time step τ let $u_{\varepsilon,\tau}$ be the discrete evolution of the initial datum u^0_ε as in (16).

Theorem 4.1. *Let $v_n = u_{\varepsilon_n,\tau_n}$ be a sequence converging to a function u as in Theorem 3.1 (thus $u_x(\cdot, t) \in H^1(0,1)$ for a.e. $t \geq 0$ by Theorem 3.2). Then*

$$u_t = \psi''(0)(u_x)_x \tag{32}$$

in the distributional sense in $(0,1) \times (0,+\infty)$, i.e.,

$$\int_0^1 \int_0^T u(x,t)\phi_t(x,t)\,\mathrm{d}x\mathrm{d}t = \psi''(0) \int_0^1 \int_0^T \phi_x(x,t)u_x(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

for every $T > 0$ and $\phi \in C^\infty_c\big((0,1) \times (0,T)\big)$ (recall that u_x denotes the density of the absolutely continuous part of the derivative). Moreover

$$u(\cdot,0) = u^0 \quad \textit{a.e. in } (0,1);$$
$$u_x(\cdot,t) = 0 \quad \textit{on } S\big(u(\cdot,t)\big) \cup \{0,1\} \textit{ for a.e. } t \geq 0.$$

Proof. As above, we will drop the index n and simply write ε and τ in place of ε_n and τ_n.

Taking Theorem 3.1 and Theorem 3.2 into account, we only have to prove that u satisfies the equation $u_t = \psi''(0)(u_x)_x$ in the distributional sense and that $u(\cdot,0) = u^0$. Note that $u(\cdot,0)$ is well defined since $u \in C^{1/2}([0,+\infty); L^2(0,1))$. As for the initial datum, we have:

$$\begin{aligned}\|u(\cdot,0) - u^0\|_{L^2(0,1)} &\leq \|u(\cdot,0) - u_{\varepsilon,\tau}(\cdot,0)\|_2 + \|u_{\varepsilon,\tau}(\cdot,0) - u^0\|_2 \\ &= \|u(\cdot,0) - u_{\varepsilon,\tau}(\cdot,0)\|_2 + \|u^0_\varepsilon - u^0\|_2\,.\end{aligned}$$

Both terms on the right-hand side tend to 0 since for every $T > 0$ we have $u_{\varepsilon,\tau} \to u$ in $L^\infty\big([0,T]; L^2(0,1)\big)$ (see Theorem 3.1), and (u^0_ε) is an equibounded sequence converging a.e. to u^0.

We now address the evolution equation. Fix $T > 0$ and let $M_\tau = \lfloor T/\tau \rfloor$. Let $\phi \in C^\infty_c\big((0,1) \times (0,T)\big)$ be fixed, and define

$$\phi^k_i = \phi(i\varepsilon, k\tau) \qquad \text{with } k, i \in \mathbb{Z}.$$

Recall the summation by parts formula:

$$\sum_{j=0}^{l-1} a_j(b_{j+1} - b_j) = a_l b_l - a_0 b_0 - \sum_{j=0}^{l-1}(a_{j+1} - a_j)b_{j+1}\,.$$

Then ($l = M_\tau$) we have:

$$\begin{aligned}A &:= \sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1} \varepsilon\tau (u^k_{\varepsilon,\tau})_i \,\frac{\phi_i^{k+1}-\phi_i^k}{\tau}\\ &= \varepsilon\sum_{i=0}^{N_\varepsilon}\left[(u^{M_\tau}_{\varepsilon,\tau})_i\phi_i^{M_\tau} - (u^0_{\varepsilon,\tau})_i\phi_i^0 - \sum_{k=0}^{M_\tau-1}\left((u^{k+1}_{\varepsilon,\tau})_i-(u^k_{\varepsilon,\tau})_i\right)\phi_i^{k+1}\right].\end{aligned}$$

Since ϕ has compact support in $(0,1)\times(0,T)$ we have $\phi_i^0=\phi_0^k=0$ and, for ε and τ sufficiently small we can assume that $\phi_i^{M_\tau}=\phi^k_{N_\varepsilon}=0$.

The optimality conditions now yield:

$$A = -\tau\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\left[\varphi_\varepsilon'\left(\frac{(u^{k+1}_{\varepsilon,\tau})_{i+1}-(u^{k+1}_{\varepsilon,\tau})_i}{\varepsilon}\right)-\varphi_\varepsilon'\left(\frac{(u^{k+1}_{\varepsilon,\tau})_{i}-(u^{k+1}_{\varepsilon,\tau})_{i-1}}{\varepsilon}\right)\right]\phi_i^{k+1}.$$

Apply again the summation by parts formula, with

$$a_j=\phi_j^{k+1},\qquad b_j=\varphi_\varepsilon'\Bigg(\frac{(u^{k+1}_{\varepsilon,\tau})_{j}-(u^{k+1}_{\varepsilon,\tau})_{j-1}}{\varepsilon}\Bigg);$$

then

$$A=\tau\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\big(\phi_{i+1}^{k+1}-\phi_i^{k+1}\big)\varphi_\varepsilon'\Bigg(\frac{(u^{k+1}_{\varepsilon,\tau})_{i+1}-(u^{k+1}_{\varepsilon,\tau})_{i}}{\varepsilon}\Bigg).$$

We conclude that

$$\begin{aligned}&\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\varepsilon\tau(u^k_{\varepsilon,\tau})_i\,\frac{\phi_i^{k+1}-\phi_i^k}{\tau}\\ &\qquad=\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\varepsilon\tau\,\frac{\phi_{i+1}^{k+1}-\phi_i^{k+1}}{\varepsilon}\,\varphi_\varepsilon'\Bigg(\frac{(u^{k+1}_{\varepsilon,\tau})_{i+1}-(u^{k+1}_{\varepsilon,\tau})_{i}}{\varepsilon}\Bigg).\end{aligned}\tag{33}$$

Let now $\phi^{(0,1)}_{\varepsilon,\tau}$ and $\phi^{(1,0)}_{\varepsilon,\tau}$ be the piecewise-constant functions on $\mathbb{R}^2$ defined by

$$\phi^{(0,1)}_{\varepsilon,\tau}(x,t)=\frac{\phi_i^{k+1}-\phi_i^k}{\tau},\quad \phi^{(1,0)}_{\varepsilon,\tau}(x,t)=\frac{\phi_{i+1}^{k}-\phi_i^k}{\varepsilon}$$

where $i=\lfloor x/\varepsilon\rfloor$ and $k=\lfloor t/\tau\rfloor$. Then, we can write the left-hand side of (33) in the following form (as $\phi=0$ in a neighbourhood of $\partial([0,1]\times[0,T])$):

$$\int_0^1\int_0^T u_{\varepsilon,\tau}(x,t)\phi^{(0,1)}_{\varepsilon,\tau}(x,t)\,\mathrm{d}x\mathrm{d}t.$$

In the limit as $n\to+\infty$ we get

$$\int_0^1\int_0^T u(x,t)\phi_t(x,t)\,\mathrm{d}x\mathrm{d}t.$$

We now examine the right-hand side of (33). By means of the functions w_n introduced in equation (19), this term can be written as

$$\int_0^1\mathrm{d}x\int_0^T\phi^{(1,0)}_{\varepsilon,\tau}(x,t)w_n(x,t)\,\mathrm{d}t.$$

By Lemma 3.3, in the limit as $n\to+\infty$ we get

$$\psi''(0)\int_0^1\mathrm{d}t\int_0^T\phi_x(x,t)u_x(x,t)\,\mathrm{d}x\,,$$

which concludes the proof. □

4.2. **Evolution of the singular set.** Let $u^k_{\varepsilon,\tau}$ $(k = 0, 1, 2, \dots)$ be the discrete evolution of the initial datum u^0_ε as introduced in Section 2 through the minimization of the functional in (15). In what follows we analyse the evolution of the singular set $I^+_\varepsilon(u^k_{\varepsilon,\tau})$ (see (10)) with respect to the index k. The key tool will be estimate (35) below and the subsequent lemma, which are a discrete version of the argument applied in [14] (Lemma 4.10 and Proposition 4.11); this will require a condition on the ratio τ/ε^2.

We simply write u^k_i in place of $\big(u^k_{\varepsilon,\tau}\big)_i$. Fix $0 \le i < N_\varepsilon$ and define

$$v^k_i := \frac{u^k_{i+1} - u^k_i}{\varepsilon}$$

If $0 < i < N_\varepsilon - 1$, then by the optimality conditions in Proposition 2 we have

$$\begin{aligned} v^{k+1}_i - v^k_i &= \frac{1}{\varepsilon}\left(u^{k+1}_{i+1} - u^{k+1}_i - u^k_{i+1} + u^k_i\right) \\ &= \frac{1}{\varepsilon}(u^{k+1}_{i+1} - u^k_{i+1}) - \frac{1}{\varepsilon}(u^{k+1}_i - u^k_i) \\ &= \frac{\tau}{\varepsilon^2}\left[\varphi'_\varepsilon\left(\frac{u^{k+1}_{i+2} - u^{k+1}_{i+1}}{\varepsilon}\right) + \varphi'_\varepsilon\left(\frac{u^{k+1}_i - u^{k+1}_{i-1}}{\varepsilon}\right) - 2\varphi'_\varepsilon\left(\frac{u^{k+1}_{i+1} - u^{k+1}_i}{\varepsilon}\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} (v^{k+1}_i - v^k_i) &+ 2\frac{\tau}{\varepsilon^2}\varphi'_\varepsilon\left(\frac{u^{k+1}_{i+1} - u^{k+1}_i}{\varepsilon}\right) \\ &= \frac{\tau}{\varepsilon^2}\left[\varphi'_\varepsilon\left(\frac{u^{k+1}_{i+2} - u^{k+1}_{i+1}}{\varepsilon}\right) + \varphi'_\varepsilon\left(\frac{u^{k+1}_i - u^{k+1}_{i-1}}{\varepsilon}\right)\right], \end{aligned} \tag{34}$$

and

$$(v^{k+1}_i - v^k_i) + 2\frac{\tau}{\varepsilon^2}\varphi'_\varepsilon\left(\frac{u^{k+1}_{i+1} - u^{k+1}_i}{\varepsilon}\right) \le 2\frac{\tau}{\varepsilon^2}\max\varphi'_\varepsilon.$$

We introduce the function

$$g(z) = 2\frac{\tau}{\varepsilon^2}\varphi'_\varepsilon(z).$$

Then, the previous inequality can be re-written in the form

$$(v^{k+1}_i - v^k_i) + g\big(v^{k+1}_i\big) \le \max g. \tag{35}$$

Note that in case $i = 0$ or $i = N_\varepsilon - 1$, only one of the two terms on the right-hand side of equation (34) remains. Since $\max\varphi'_\varepsilon$ is positive, estimate (35) still holds unchanged for $i = 0$ and $i = N_\varepsilon - 1$.

Lemma 4.2. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $L < 1$. Let $(a_k)_{k\ge 0}$ be a sequence of real numbers, and let $C \in \mathbb{R}$ be such that*

$$a_{k+1} - a_k + g(a_{k+1}) \le g(C) \qquad \text{for every } k.$$

Then

$$a_0 \le C \quad\Rightarrow\quad \big(a_k \le C \quad \text{for every } k\big).$$

Proof. If we set $\tilde{a}_k = a_k - C$ and $\tilde{g}(z) = g(C+z) - g(C)$ then we can argue with $C = 0$ and $g(0) = 0$. Therefore, for every k

$$a_{k+1} - a_k \leq -g(a_{k+1}) \leq L|a_{k+1}|.$$

The inequality $a_k \leq 0$ now yields

$$a_{k+1} \leq L|a_{k+1}|,$$

hence $a_{k+1} \leq 0$ if $L < 1$. □

We would like to apply the previous lemma with $C = z_0/\sqrt{\varepsilon}$; i.e., with the maximizer of g. The Lipschitz constant of g involves the second derivative $\varphi''_\varepsilon(z) = \psi''(\sqrt{\varepsilon}z)$. Now recall the boundedness of $\big(F_\varepsilon(u^0_\varepsilon)\big)_\varepsilon$ (see assumption $(B2)$), hence the uniform boundedness of $F_\varepsilon(u^k_{\varepsilon,\tau})$ with respect to ε, τ and k by Proposition 1. Thus, if $\zeta_0 \in (-1, 0)$ is such that $\psi(\zeta_0) > M$, then (see Lemma 2.2)

$$\frac{u^k_{j+1} - u^k_j}{\varepsilon} > \frac{\zeta_0}{\sqrt{\varepsilon}} \quad \text{for every } k \in \mathbb{N} \text{ and } j = 0, \dots, N_\varepsilon - 1.$$

Therefore, the relevant domain for the function g in (35) is $[\zeta_0/\sqrt{\varepsilon}, +\infty)$. Hence, we meet the requirement that the Lipschitz constant of g is less than 1 if

$$2\frac{\tau}{\varepsilon^2} \max_{[\zeta_0,+\infty)} \psi'' < 1. \tag{36}$$

The application of Lemma 4.2 to the sequence (v_k) now gives the following result.

Proposition 4. *If condition* (36) *holds, then*

$$I^+_\varepsilon(u^{k+1}_{\varepsilon,\tau}) \subseteq I^+_\varepsilon(u^k_{\varepsilon,\tau})$$

for every $k \geq 0$.

By the estimate of Lemma 2.3 we have $u^0_\varepsilon \in SBV(0,1)$, and we can define m points $x^\varepsilon_1 \leq x^\varepsilon_2 \leq \ldots \leq x^\varepsilon_m$ (not necessarily distinct, and with m independent of ε), such that for every $\varepsilon > 0$ we have

$$I^+_\varepsilon(u^0_\varepsilon) \subseteq \{x^\varepsilon_j : \ j = 1, \dots, m\}.$$

Therefore, up to a subsequence, we can assume that

$$\lim_{\varepsilon \to 0} x^\varepsilon_j = x_j \qquad \text{for every } j = 1, \dots, m. \tag{37}$$

Denote by S this set of limit points.

Since, for every $t \geq 0$, each jump point of $u(\cdot, t)$ is the limit of a sequence of jump points of the piecewise-linear functions $\hat{v}_n(\cdot, t)$ (see Theorem 3.1), if (36) holds then, by Proposition 4

$$S\big(u(\cdot, t)\big) \subseteq S \qquad \text{for every } t \geq 0.$$

Taking into account this condition and Theorem 4.1, we characterize the limit motion as the heat equation with Neumann boundary conditions on $(0,1) \setminus S(u^0)$; that is, the same as the minimizing movement of the Mumford-Shah energy as described in [7] Section 8.3. This characterization is valid until the first collision time, for which $\#S\big(u(\cdot, t^+)\big) < \#S\big(u(\cdot, t^-)\big) = \#(S(u^0))$.

Corollary 1 (commutativity of minimizing movements)**.** *Let* u^0_ε *be an initial datum satisfying assumptions* (B1) *and* (B2) *at the end of Section* 2*. Let* u_ε *be a minimizing movement of the functional* F_ε *in* (8)*. Then* u_ε *converges in* $L^\infty\big((0,T); L^2(0,1)\big)$ *as* $\varepsilon \to 0$ *to a minimizing movement of the Mumford-Shah functional.*

Proof. As in the proof of Theorem 3.1, $u_\varepsilon\colon [0,+\infty) \to L^2(0,1)$ satisfies a Hölder continuity estimate which is uniform with respect to ε. Hence, up to a subsequence, $u_\varepsilon\colon [0,1]\times[0,+\infty) \to \mathbb{R}$ converges in $L^\infty\big((0,T);L^2(0,1)\big)$ to a function u. By a diagonal argument we deduce that such u is the minimizing movement along F_ε with a suitable time scale $\tau=\tau(\varepsilon)$ (see also Theorem 8.1(ii) in [7]). Since we may choose $\tau(\varepsilon) << \varepsilon^2$, we can apply Theorem 4.1 to deduce that u is a solution of the heat equation (hence it is a minimizing movement for the Mumford-Shah energy). □

4.3. **Initial data with controlled Dirichlet energy.** We conclude with a simple result which gives a sufficient condition for the uniqueness of the evolution *without assuming condition* (36) on ε and τ. First, we state a remark on the discontinuity lines of the evolution function u.

Remark 6. Let u be as in Theorem 4.1, and let $T>0$. Then the distributional derivative of $u(x,\cdot)$ is in $L^2(0,T)$ for a.e. $x\in(0,1)$; thus, $u(x,\cdot)$ is in $H^1(0,T)$, hence continuous for a.e. $x\in(0,1)$. This implies that the projection onto the x-axis of the jump set S_u of u as a function of both x and t has zero Lebesgue measure. In other words, u can only present discontinuity lines which are parallel to the t-axis.

Proposition 5. *Let u^0 be a piecewise C^1-function on $(0,1)$ satisfying the condition*

$$\frac{1}{2}\int_0^1 (u^0_x)^2\,\mathrm{d}x < \frac{1}{\psi''(0)}\,. \tag{38}$$

Let $\varepsilon=\varepsilon_n$ and $\tau=\tau_n$ be positive infinitesimal sequences, and let u^0_ε be the discretization of u^0. Let u be defined as in Theorem 4.1 with respect to the initial datum u^0_ε. Then there exists $\sigma>0$ such that

$$S\big(u(\cdot,t)\big) = S(u^0)$$

for $t\in(0,\sigma)$.

The value σ actually represents the first 'collision time', in which at least one jump of $u(\cdot,t)$ disappears. As a consequence of this result, under assumption (38) we have the uniqueness, up to σ, of the evolution u from the initial datum u^0, since u is uniquely characterized as the solution of the heat equation, with Neumann boundary conditions, between two consecutive jump points.

Proof. From the definition of $u_{\varepsilon,\tau}$ and Proposition 1 we get

$$F_\varepsilon\big(u_{\varepsilon,\tau}(\cdot,t)\big) \le F_\varepsilon(u^0_\varepsilon)$$

(where $\varepsilon=\varepsilon_n$ and $\tau=\tau_n$). Since $u_{\varepsilon,\tau}(\cdot,t)\to u(\cdot,t)$ in $L^2(0,1)$ for every $t\ge 0$, and (F_ε) Γ-converges to the Mumford-Shah functional, we have

$$MS(u(\cdot,t)) := \frac{1}{2}\psi''(0)\int_0^1 |u_x(x,t)|^2\,\mathrm{d}x + \#S(u(\cdot,t)) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u^0_\varepsilon)\,.$$

Since u^0_ε is the discretization of a piecewise-C^1 function, it turns out that

$$F_\varepsilon(u^0_\varepsilon) \to MS(u^0)\,,$$

so that

$$MS(u(\cdot,t)) \le MS(u^0). \tag{39}$$

Let us now show that there exists $\sigma>0$ such that

$$\#S(u^0) \le \#S\big(u(\cdot,t)\big) \qquad \text{for any } t\in(0,\sigma). \tag{40}$$

Let $S(u^0) = \{x_1, \dots, x_s\}$ and let V_i be a neighbourhood of x_i such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Let

$$A_i = \{t \geq 0 : \; S\big(u(\cdot, t)\big) \cap V_i \neq \emptyset\}.$$

Inequality (40) is satisfied if each A_i contains a right neighbourhood of $t = 0$. Otherwise, there exists i and a sequence $t_n \to 0$ such that $S\big(u(\cdot, t_n)\big) \cap V_i = \emptyset$ for every $n \in \mathbb{N}$. In this case we get a contradiction since $u(\cdot, t_n)$ would be a precompact family of functions in H^1 on a full neighbourhood of x_i, while the limit u^0 has a discontinuity in x_i.

Now, (39) and (40) yield that

$$\begin{aligned} 0 \leq \#S(u(\cdot,t)) - \#S(u^0) &\leq \frac{1}{2}\psi''(0)\Big[\int_0^1 |u_x^0(x)|^2 \,\mathrm{d}x - \int_0^1 |u_x(x,t)|^2 \,\mathrm{d}x\Big] \\ &\leq \frac{1}{2}\psi''(0)\int_0^1 |u_x^0(x)|^2 \,\mathrm{d}x\,. \end{aligned} \tag{41}$$

By (38) the last term is less than 1, which implies that $\#S(u(\cdot,t)) = \#S(u^0)$. By Remark 6 ($u$ can only present discontinuity lines which are parallel to the t-axis) we conclude that $S\big(u(\cdot,t)\big) = S(u^0)$ in a right neighbourhood of $t = 0$. □

Remark 7. The first line of inequality (41) clearly suggests the search for an improvement of the previous result aiming to show the convergence, as $t \to 0$, of $\int_0^1 |u_x(x,t)|^2 \,\mathrm{d}x$ to $\int_0^1 |u_x^0(x)|^2 \,\mathrm{d}x$. Note that, formally, it is easy to prove the monotonicity of the L^2-norm of the space derivative, since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |u_x(x,t)|^2 \,\mathrm{d}x &= \int_0^1 2u_x(x,t)u_{xt}(x,t)\,\mathrm{d}x \\ &= -2\int_0^1 (u_x)_x u_t \,\mathrm{d}x = -2\psi''(0)\int_0^1 (u_t)^2 \,\mathrm{d}x \leq 0\,, \end{aligned}$$

where we have used the vanishing of the derivatives on the boundary of $(0,1)$ and the fact that u satisfies (32). The continuity in $t = 0$, which is standard within the classical geometrical framework of the heat equation on a rectangle, here requires to manage the subtle problem of the unknown positions of the discontinuity lines of u. We could make use of the classical results if we could prove that the following condition is satisfied:

> there exists $\sigma > 0$ such that $S_u \cap \big((0,1)\times(0,T)\big)$ consists of a finite number of segments $\{x_i\} \times (0,\sigma)$, with $i = 1, \dots, s$ and $0 = x_0 < x_1 < \dots < x_s < x_{s+1} = 1$.

In this case assumption (38) can be removed.

Acknowledgments. The first author gratefully acknowledges the hospitality of the Mathematical Institute in Oxford and financial support of the EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1).

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