On Limits of Variational Problems. The case of a Non-Coercive Functional*

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Typical convergence theorems for values functions and solutions of (parametric families of) optimization problems based on Γ-convergence of the corresponding functionals usually rely on equi-coercivity assumptions. Without them the connection between the Γ-limit of the functionals and values and/or solutions of the problems may be completely broken. The question to be discussed is whether it is possible, even in the absence of a coercivity-type assumption, to find limiting optimization problems (parametrized in a similar way and determined by functionals which may differ from the Γ-limits of the functionals of the sequence) such that the value functions and solutions of the problems of the sequence converge in a certain sense to those of the limiting problems. A positive answer to the question is given to a class of variational problems (containing optimal control problems with linear dynamics).

1 Introduction.

In the paper we shall consider the simplest variational problem associated with integral functionals of the form

$$I(u) = \begin{cases} \displaystyle\int_0^1 f(t, u(t))\, dt, & \text{if the integral makes sense,} \\ \infty, & \text{otherwise,} \end{cases} \tag{1}$$

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namely the problem of the following type:

$$\text{minimize} \quad I(u) \quad \text{s.t.} \quad \int_0^1 u(t)\,dt = x, \tag{2}$$

where $f(t,u):\ [0,1]\times\mathbb{R}^N \to (-\infty,\infty]$ is an integrand of which we shall basically require only that $f(t,u(t))$ be measurable for any measurable $u(t)$.

The question to be discussed is what kind of a limit can be associated with sequences of such problems. There is an extensive literature related to the question, even for more general classes of functionals and variational problems, and concerned with the problems of duality, relaxation, integral representation and variational convergence of (1)-like and related functionals on spaces of functions and measures (e.g. [2, 3, 4, 5, 8, 9, 12, 13, 14]).

A complete account of relaxation theory for integral functionals in spaces of R^N-valued Radon measures with the weak*-topology is presented in the monograph of Buttazzo [4] in which the analysis is mainly measure-theory oriented. However, in the context of the variational problem (2), many basic ingredients of the theory were developed twenty years earlier by Ioffe and Tihomirov [8] (see also [9]) who used a different approach based on convex duality.

The main relaxation theorem given in [4] (Theorem 3.3.1) basically says that if we consider functions $u(t)$ as densities of R^N-valued measures on the segment then the relaxed functional with respect to the weak*-topology of the space of measures can be written in the form

$$I_{relaxed}(\nu) = \int_0^1 \varphi(t, \frac{d\nu}{dt}(t))\,dt + \int_0^1 h(t, \frac{d\nu_s}{d|\nu_s|}(t))\,d|\nu_s|$$

where φ is a normal convex integrand satisfying $\varphi(t,x) \le f(t,x)$ and $h(t,x)$ is a lower semi-continuous function which, as a function of x, coincides with the recession function of $\varphi(t,\cdot)$. Let's recall that the recession function of (a convex function) φ is defined by $\varphi^\infty(x) = \lim_{t\to\infty} t^{-1}\varphi(\overline{x}+tx)$, where $\overline{x}$ is an arbitrary element of $\operatorname{dom}\varphi$.

In [8, 9] a slightly different functional (with $f^{**}(1,x)$, the second Fenchel conjugate of f with respect to the second argument, was considered on a smaller set $\mathcal{K}$ of measures with singular parts consisting of no more than $N+1$ jumps (that is on a subset of SBV in modern terminology). It was proved that the corresponding problem has a solution belonging to $\mathcal{K}$ under a mild coercivity condition and the arguments in the proof also lead to the conclusion that this special solution also solves the relaxed problem (with the same value of the functional).

Γ-limits of (1)-like functionals were studied in [2, 5] under the assumption that the functionals of the sequence are equi-coercive. It was shown that the Γ-limit, if exists, has the form

$$J(\nu) = \int_0^1 \varphi(t, \frac{d\nu}{d\mu}(t))\,d\mu + \int_0^1 h(t, \frac{d\nu_s}{d|\nu_s|}(t))\,d|\nu_s| \tag{3}$$

(with μ being a positive finite measure, φ a normal convex integrand and $h(t,\cdot)$ the recession function of $\varphi(t,\cdot)$ for μ-almost all t), and the solution and the value of a suitably defined variational problem for the limiting functional can be obtained as limits of approximate solutions and values of the problems associated with the functionals of the sequence.

This is no longer the case in the absence of the coercivity condition (although weaker forms of the above described convergence result were established for certain classes of non-coercive problems): for it may happen that the Γ-limit of the sequence of integral functionals gives no information about the limiting behavior of the associated variational problems (which is of course a well known phenomenon: see e.g. [6], Example 7.5). Consider for instance the following example.

Example 1 Let

$$f_n(t,u) = \begin{cases} e^{u/n}, & \text{if } \dfrac{2k}{n} \le t < \dfrac{2k+1}{n},\ k = 0,1,\ldots, \le \dfrac{n-1}{2}, \\ e^{-u/n}, & \text{if } \dfrac{2k-1}{n} \le t < \dfrac{2k}{n},\ k = 1,\ldots, \le \dfrac{n}{2}. \end{cases}$$

If u_n is a sequence of functions converging weakly* in $\mathcal{M}$, then $\int |u_n(t)|\,dt$ are uniformly bounded, hence for any n the measure of the set $\{t :\ |u_n(t)| \ge r\}$ goes to zero as $r \to \infty$. If now $|u_n(t)| \le r$ on a certain set $E \subset [0,1]$, then

$$\int_E f_n(t,u_n(t))\,dt \ge e^{-r/n}\mathrm{meas}(E)$$

(where "meas" stands for the Lebesgue measure) from which it follows immediately that

$$\liminf_{n\to\infty} \int_0^1 f_n(t,u_n(t))\,dt \ge 1.$$

On the other hand, it is equally easy to see that for any bounded u

$$\lim_{n\to\infty} \int_0^1 f_n(t,u(t))\,dt = 1.$$

This means that the Γ-limit of our sequence with respect to the weak topology in L^1 is the functional identically equal to 1.

Meanwhile, it is quite clear that in the problems

$$\text{minimize} \quad \int_0^1 f_n(t,u(t))\,dt,\ s.t.\ \int_0^1 u(t)\,dt = x$$

the value function is identical zero (as a function of x) for all n.

The main result of this paper shows, however, that even in the absence of coercivity there is a functional of the same form as in (3) such that for a certain subsequence

(a) the liminf inequality of the Γ-convergence holds for this functionals and elements of the subsequence;

(b) a weaker form of the limsup inequality also holds;

(c) the value functions of problems (1), (2) for the functionals of the subsequence Γ-converge to the lower closure of the value function of a corresponding problem for the limit functional (which, being a convex function, coincides with its closure at all relatively interior points).

The quoted result of Bouchitté (for the equi-coercive case) is an easy consequence of the main theorem as well as the existence of a solution of the limit problem belonging to the mentioned set $\mathcal{K}$.

2 Statements of the main results.

So let $f_n(t,u)$, $n = 1,2,...$ be a sequence of extended-real-valued integrands on $[0,1]\times\mathbb{R}^N$, that is to say, a sequence of functions defined on $[0,1]\times\mathbb{R}^N$ and taking values in $(-\infty,\infty]$. Assume that all integrands satisfy the standard measurability requirements mentioned in the introduction: *$f_n(t,u(t))$ is measurable whenever $u(t)$ is measurable.* Set

$$I_n(u) = \begin{cases} \displaystyle\int_0^1 f_n(t,u(t))\,dt, & \text{if the integral exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Throughout the paper we use the following notation:

L_1^N – the space of $\mathbb{R}^N$-valued summable (w.r.t. the Lebesgue measure) function on $[0,1]$;

$L_1^N(\mu)$, where μ is a Radon measure on $[0,1]$ – the space of $\mathbb{R}^N$-valued μ-integrable functions on $[0,1]$;

$\mathcal{M}^N$ – the space of finite $\mathbb{R}^N$-valued measures on $[0,1]$;

f^* the Young-Fenchel conjugate of f, and f^{**} its biconjugate;

$s(w,P) = \sup\limits_{p\in P} p\cdot w$ – the support function of $P\subset\mathbb{R}^N$;

$\mathcal{A}$ – the collection of open subsets of $[0,1]$;

It has to be emphasized that by an open subset of [0,1] we always mean a *relatively* open set, that is an intersection of an open subset of $\mathbb{R}$ with $[0,1]$. In the same way we understand the expression "open interval in $[0,1]$", in particular such are all intervals $[0,\beta)$ and $(\alpha,1]$. Moreover, we shall never be interested in intervals $(0,\alpha)$ and $(\beta,1)$.

Remark 1 It may be more convenient to speak about arbitrary open subsets of $\mathbb{R}$, including intervals (α,β) with $\alpha<0$ and/or $\beta>1$. Of course in such cases we identify every such set with its intersection with $[0,1]$. This convention can be implemented formally if we agree to consider all functions to be defined on the whole $\mathbb{R} \times \mathbb{R}^N$ and equal to zero at points (t, x) with $t \notin [0, 1]$.

The following two assumptions will be adopted:

($\mathbf{A}_1$) *there exist $u_0 \in L_1^N$ and $\rho \in L_1$ such that for any n*

$$f_n\big(t, u_0(t)\big) \le \rho(t) \quad \text{a.e.};$$

($\mathbf{A}_2$) *for any $n = 1, 2, \ldots$ there exist (q_n) bounded in $\mathbb{R}^N$ and (ρ_n) bounded in L_1 such that*

$$f_n(t, u) \ge q_n \cdot u - \rho_n(t) \ \ \forall u \in \mathbb{R}^N, \ \forall n \in \mathbb{N}, \text{ a.e. } t \in [0, 1].$$

In the theorems below we deal with the following four objects

- a positive finite measure μ on $[0, 1]$;
- a μ-measurable normal convex integrand $\varphi(t, u)$ on $[0, 1] \times \mathbb{R}^N$;
- a lower semi-continuous set-valued mapping $P(t)$ from $[0, 1]$ into $\mathbb{R}^N$ with nonempty, convex and closed values;
- the functional $J(\nu)$ given by (3) with $h(t, x) = s(x, P(t))$.

For every n we denote by $V_n(x)$ the value function of the problem (2) (with I_n instead of I) and by $V(x)$ the value function of the corresponding problem for J:

$$\text{minimize} \quad J(\nu) \quad \text{s.t.} \quad \int_0^1 d\nu = x. \tag{4}$$

As well known, V_n are convex functions, therefore any reasonable limit of V_n, whenever it exists, is also a convex function.

Along with V_n and V we consider value functions $V_n(\Delta, \cdot)$ and $V(\Delta, \cdot)$ of the problems

$$\text{minimize} \quad \int_\alpha^\beta f_n(t, u(t))\, dt, \quad \text{s.t.} \quad \int_\alpha^\beta u(t)\, dt = x$$

and

$$\text{minimize} \quad J_\Delta(\nu) = \int_\Delta \varphi(t, \frac{d\nu}{d\mu})\, d\mu + \int_\Delta h(t, \frac{d\nu}{d|\nu_s|})\, d|\nu_s|, \quad \text{s.t.} \quad \int_\Delta d\nu = x,$$

where Δ is an open subinterval in $[0, 1]$. By an *open subinterval* of $[0, 1]$ we always mean either (α, β) with $0 < \alpha < \beta < 1)$ or $[0, \beta)$ with $\beta < 1$ or $(\alpha, 1]$ with $\alpha > 0$ or finally, $[0, 1]$ itself.

The notation $\int_\alpha^\beta$ refers to integration over the *closed* interval. In case we wish to emphasize that we integrate over the open subinterval Δ we use the symbol $\int_\Delta$.

Given a partition π of $[0,1]$ by points $0 < \alpha_1 < \ldots < \alpha_k < 1$, we say that an open interval Δ *belongs to* π if either $\Delta = [0, \alpha_i)$, or $\Delta = (\alpha_i, 1]$ or $\Delta = (\alpha_i, \alpha_j)$ $(j > i)$. The interval $[0,1]$ belongs to every partition by definition. Adding $\alpha_0 = 0$ and $a_{k+1} = 1$ we define *diameter* of the partition as the maximal distance between two adjacent points (including 0 and 1):

$$\mathrm{diam}(\pi) = \max_{0 \le i \le k} (\alpha_{i+1} - \alpha_i)$$

An interval Δ belonging to the partition and bounded by two adjacent points of the partition is called *minimal.* The collection of all minimal intervals of π will be denoted $\min(\pi)$. We say that a sequence (π_k) of partitions is *decreasing* if every interval belonging to π_k belongs also to π_{k+1}.

Finally, given a positive measure μ on $[0,1]$, we say that a collection $\mathcal{D}$ of intervals is μ-dense if for any interval $\Delta \subset [0,1]$ and any $\varepsilon > 0$ there is a $\Delta' \in \mathcal{D}$ contained in Δ and such that both the length and the μ-measure of Δ' are smaller than the length and the μ-measure of Δ by less than ε.

Theorem 1 *Under* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *there are a probability measure* μ *on* $[0,1]$, *a normal convex integrand* $\varphi(t,x)$ *on* $[0,1] \times \mathbb{R}^N$, *a lower semi-continuous multifunction* $P(t)$ *from* $[0,1]$ *into* $\mathbb{R}^N$ *with nonempty closed and convex values (connected to* h *by* $h(t,w) = s(w, P(t))$*) and a subsequence* $\{n_j\}$ *of indices such that*

(a) the value functions $V_{n_j}(\Delta, \cdot)$ Γ*-converge to* $V^{**}(\Delta, \cdot)$ *for every interval* Δ *of a* μ*-dense set of intervals including* $[0,1]$ *itself;*

(b) if a sequence $(u_j) \subset L_1^N$ *converges to a measure* ν *in the weak*-topology of* $\mathcal{M}^N$, *then*

$$\liminf_{j \to \infty} I_{n_j}(u_j) \ge J(\nu);$$

(c) for any $\nu \in \mathcal{M}^N$ *there is a decreasing sequence* (π_k) *of partitions of* $[0,1]$ *by points which are not atoms of either* μ *or* ν *and with diameters going to zero and a sequence* $(u_j) \subset L_1^N$ *such that*

$$\limsup_{j \to \infty} I_{n_j}(u_j) \le J(\nu) \tag{5}$$

and for any interval Δ *belonging to one of the partitions* π_k *the value functions* $V_{n_j}(\Delta, \cdot)$ Γ*-converge to* $V^{**}(\Delta, \cdot)$ *and*

$$\lim_{j \to \infty} \int_\Delta u_j(t)\, dt \;\to \int_\Delta d\nu.$$

Remark 2 The measure μ is not unique. On the other hand, as soon as μ is chosen, the integrand φ is determined uniquely. The construction described in the proof of the theorem leads to the *minimal* integrand among those which determine the same lower closure of the value function. As to the set-valued mapping $P(t)$, it is completely and uniquely defined by the chosen subsequence of integrals and therefore does not depend on μ.

We also note that, according to the theorem, jumps of measures ν for which $J(\nu) < \infty$ may occur either at atoms of μ or at those t at which $P(t)$ does not coincide with the whole of $\mathbb{R}^N$.

We next introduce the subset $\mathcal{K}^N \subset \mathcal{M}^N$ of all $\nu \in \mathcal{M}^N$ of the form

$$\nu(E) = \int_E u(t)\, d\mu + \sum_{i=1}^{N} \xi_i \delta_{\tau_i}(E),$$

where $u \in L_1^N(\mu)$, $\xi_i \in \mathbb{R}^N$, $\tau_i \in [0,1]$ and δ_{τ_i} stands for the unit mass at τ_i. Thus $\mathcal{K}^N$ is the collection of measures whose μ-singular part consist of at most N jumps.

Theorem 2 *Assume* ($\mathbf{A}_1$) *and the following "mild coercivity condition":*

($\mathbf{A}_1'$) *there are $q_0 \in \mathbb{R}^N$, $r > 0$ and (ρ_n) bounded in L_1 such that the inequality*

$$f_n(t,u) \geq q \cdot u - \rho_n(t)$$

holds almost everywhere for all n and all q of the r-ball around q_0.

Then

(a) *the conclusion of Theorem 1 holds with the limit functional J being the Γ-limit of I_{n_j} (in the weak*-topology of $\mathcal{M}^N$),*

(b) *for every $x \in \mathrm{ri}(\mathrm{dom}\, V)$ the problem (4) has a solution belonging to $\mathcal{K}^N$.*

3 Preliminaries

1. Γ-**convergence** [6]. Let X be a topological space, and let (f_n) be a sequence of extended-real-valued function on X. The functions

$$(\Gamma - \liminf\ f_n)(x) = \inf_{(x_n)\to x} \liminf_{n\to\infty} f_n(x_n)$$

and

$$(\Gamma - \limsup\ f_n)(x) = \inf_{(x_n)\to x} \limsup_{n\to\infty} f_n(x_n)$$

are called the (*sequential*) Γ-*lower* and Γ-*upper limits* of (f_n). If both functions coincide and $f(x)$ is the common value of the limits, then it is said that f_n (*sequentially*) Γ-*converges* to f and f is called the (*sequential*) Γ-*limit* of (f_n).

The following proposition gives a convenient characterization of sequential Γ-limits.

Proposition 1 *The sequence* (f_n) *(sequentially)* Γ*-converges to* f *if and only if for any* x

(a) $\liminf_{n\to\infty} f_n(x_n) \geq f(x)$ *whenever* $x_n \to x$*;*

(b) there is a sequence (x_n) *converging to* x *such that* $\limsup_{n\to\infty} f_n(x_n) \leq f(x)$.

Under additional assumptions on the space X, namely if X satisfies the first axiom of countability (e.g. $X = \mathbb{R}^N$), or if X is dual of a separable Banach space (e.g. $X = \mathcal{M}^N$) and the sequence of functionals is weak* equi-coercive, the sequential Γ-limits coincide with a more general topological kind of Γ-limits for an account of those we refer to the book of Dal Maso [6]. This implies in particular that the Γ-limit, if exists, is always a lower semi-continuous function. Moreover, if all elements of the sequence are convex, then so is the Γ-limit.

2. $\overline{\Gamma}$**-convergence**. Let $\mu(E)$ be an extended-real-valued function on $\mathcal{A}$. The *inner regular envelope* of μ is the function on $\mathcal{A}$ defined as follows:

$$\mu_-(E) = \sup\{\mu(G) : \ G \in \mathcal{A}, \ G \ll E\},$$

where $G \ll E$ means that the closure of G is compact and contained in E. The function μ is a *measure* on $\mathcal{A}$ if there is a Borel measure coinciding with μ on $\mathcal{A}$.

The central role in the proof of Theorem 1 will be played by extended-real-valued functions on $\mathbb{R}^N \times \mathcal{A}$ which are non-negative, convex with respect to the first variable and measures with respect to the second variable. For brevity we shall call them *non-negative convex measures* on $\mathbb{R}^N \times \mathcal{A}$.

Given a sequence (F_n) of such functions, we say that the sequence $\overline{\Gamma}$*-converges* to F if for any p the functions $F'(p,E) = \Gamma - \liminf F_n(\cdot,E)(p)$ and $F''(p,E) = \Gamma - \limsup F_n(\cdot,E)(p)$ have a common inner regular envelope coinciding with $F(p,E)$.

Proposition 2 ([6], Theorems 16.9, 18.5) *Let* (S_n) *be a sequence of non-negative measures on* $\mathbb{R}^N \times \mathcal{A}$*. Then a subsequence of* (S_n) $\overline{\Gamma}$*-converges to a function* S *which is also a non-negative convex measure and lower-semicontinuous with respect to the first variable.*

$\overline{\Gamma}$-convergence of S_n does not imply Γ-convergence of $S_n(\cdot,E))$ for all $E \in \mathcal{A}$. It is known, however, that for a vast majority of the sets the implication is valid. Let (E_t) be a family of elements of $\mathcal{A}$ with the parameter t running through an open interval, say (α,β). It is called a *chain* if it is increasing or decreasing in the sense that $E_t \ll E_\tau$ either whenever $t < \tau$ or whenever $t > \tau$. We shall call a family $\mathcal{D}$ of open intervals in $[0,1]$ *rich* if for any chain $(\Delta_t) \subset \mathcal{A}$, all Δ_t except at most countably many of them belong to $\mathcal{D}$.

Proposition 3 (cf. [6], Proposition 16.4) *Let* (S_n) *be a sequence of non-negative convex measures on* $\mathbb{R}^N \times \mathcal{A}$ $\overline{\Gamma}$*-converging to* S*. Then the sequence* $(S_n(\cdot,\Delta))$ Γ*-converges to* $S(\cdot,\Delta)$ *for all* Δ *of a rich family of open intervals in* $[0,1]$.

3. **Unbounded set functions**. Let μ be an increasing function on $\mathcal{A}$ with values in $[0,\infty]$. We say that a point $x \in \mathbb{R}^N$ is *μ-extraordinary* if $\mu(U) = \infty$ for any open neighborhood U of x. A point x which is not μ-extraordinary will be called *μ-ordinary.* We denote by $\mathrm{Od}(\mu)$ the collection of all μ-ordinary points.

Clearly, $\mathrm{Od}(\mu)$ is an open (maybe empty) subset of $\mathbb{R}^N$ and the set of μ-extraordinary points is closed. If $E \ll \mathrm{Od}(\mu)$, $E \in \mathcal{A}$, then $\mu(E) < \infty$ by compactness. It follows that μ is locally finite on $\mathrm{Od}(\mu)$.

Proposition 4 *Let μ be a measure on $\mathcal{A}$ with values in $[0,\infty]$. Then there is a probability measure λ such that μ is locally absolutely continuous with respect to λ on $\mathrm{Od}(\mu)$. In other words, there exists a function $\varphi(\cdot) \in L^1_{\mathrm{loc}}(\mathrm{Od}(\mu),\lambda)$ such that $\mu(E) = \int_E \varphi(t)d\lambda$ whenever $E \in \mathcal{A}$ and $\mu(E) < \infty$.*

Indeed, the proposition is trivial if $\mu(\mathrm{Od}(\mu)) = 0$. Otherwise $\mathrm{Od}(\mu)$ is a union of sets $E_i \in \mathcal{A}$ with $0 < \mu(E_i) < \infty$. Set

$$\lambda(E) = \sum_{i=1}^{\infty} 2^{-i} \frac{\mu(E \cap U_i)}{|\mu|(U_i)}.$$

4. **Normal integrands and conjugates of value functions**. Let μ be a Borel measure on $[0,1]$, and let $f(t,x)$ be an extended-real-valued function on $[0,1] \times \mathbb{R}^N$ (an integrand). It is called *normal integrand* if (a) as a function of x it is lower semicontinuous for μ-almost all t and (b) the epigraph of f is a Borel subset of $[0,1] \times \mathbb{R}^N$ up to a set whose projection onto $[0,1]$ has μ-measure zero. In this paper we consider only positive measures and non-negative integrands, so we shall assume this in what follows. If in addition to being a normal integrand, f is convex in x for μ-almost every t, then f is called a *normal convex integrand.*

Let f be a normal integrand. Then $f(t,x(t))$ is μ-measurable whenever $x(t)$ is μ-measurable and the Young-Fenchel conjugate of f with respect to the second variable,

$$f^*(t,p) = \sup_x (p \cdot x - f(t,x))$$

is a normal convex integrand.

Assume now that the integrand f has only the property mentioned in the introduction, that is that $f(t,x(t))$ is μ-measurable whenever $x(t)$ is μ-measurable. It is well known that in this case the value function $V(x)$ of the problem (2) (with $d\mu$ instead of dt in the integrals) is convex. If, in addition, f is a normal integrand, then

$$V^*(p) = \sup_x (p \cdot x - V(x)) = \int_0^1 f^*(t,p)\, d\mu.$$

The following proposition (which is an adaptation of the virtual measurability theorem of [7]) shows that the same is true under weaker assumptions on the integrand.

Proposition 5 *Suppose that μ is a finite positive measure and $f(t,x(t))$ is μ-measurable whenever $x(t)$ is μ-measurable. Then there is a normal integrand $\varphi(t,x)$ such that for any measurable $x(t)$ the inequality $\varphi(t,x(t)) \le f(t,x(t))$ holds μ-almost everywhere and there is a sequence $(x_n(t))$ converging to $x(t)$ in measure and such that* $\liminf_{n\to\infty} f(t,x_n(t)) \le \varphi(t,x(t))$ *μ-almost everywhere.*

PROOF. To prove the proposition we have to consider the space X of all pairs $(x(t),\alpha(t))$ where $x(t)$ and $\alpha(t)$ are (equivalence classes of) μ-measurable functions with values in $\mathbb{R}^N$ and $\mathbb{R}$ respectively with the topology of convergence in measure. As μ is finite, this is a metrizable topology and the corresponding metric space is separable. Considering now the subset of X consisting of all pairs satisfying $\alpha(t) \ge f(t,x(t))$, it is possible to choose a dense countable collection of elements of this set. Taking a representative pair in each of the chosen element, we shall get a countable family $\mathcal{F} = \{(x_n(t),\alpha_n(t))\}$ of pairs of everywhere defined μ-measurable functions. Then it is easy to check that an integrand φ with the desired properties is defined by

$$\varphi(t,x) = \inf\{\liminf_{n\to\infty} \alpha_n(t) \;:\; (x_n(t),\alpha_n(t)) \in \mathcal{F} \text{ and } x_n(t) \to x \text{ in } \mathbb{R}^N\},$$

that is for any t the epigraph of $\varphi(t,\cdot)$ coincides with the closure of the values of $(x_n(\cdot),\alpha_n(\cdot))$ at t.

It is clear that $\varphi^*(t,p) = f^*(t,p)$ almost everywhere for every p and that the lower closure of $V(x)$ is not smaller than the value function of the corresponding problem for φ.

4 Constructions of μ, $\varphi(t,x)$ and $P(t)$.

At the first step of the proofs of the theorems we shall construct the three main objects that enter the statements: the probability measure μ, the normal convex integrand $\varphi(t,x)$ and the set-valued mapping $P(t)$. Thanks to $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, we only need to prove the theorems for the case when

$$u_0(t) \equiv 0, \quad f_n(t,0) \le 0, \quad q_n = 0, \tag{6}$$

(that is when every f_n is bounded from below by a summable function). Indeed, otherwise we could consider the functions

$$\tilde{f}_n(t,u) = f_n(t,u_0(t)+u) - q_n \cdot u - \rho(t)$$

clearly having this properties.

Suppose that Theorem 1 is valid for $\tilde{f}_n$, and let $\tilde{\mu}$, $\tilde{\varphi}$ and $\tilde{P}$ be the corresponding measure, convex integrand and set-valued mapping. We can assume, of course, that the limit relations in (a), (b) and (c) are satisfied for the entire sequence, not just for a subsequence, and that q_n converge to some $q \in \mathbb{R}^N$. Set

$$\varphi(t,u) = \tilde{\varphi}(t, u - u_0(t)) + q \cdot (u - u_0(t)) + \rho(t);\ P(t) = \tilde{P}(t) + q;\ \mu = \tilde{\mu}.$$

It is not a difficult matter to verify that the conditions (a), (b) and (c) of Theorem 1 are satisfied for f_n, φ, μ and P if they hold for $\tilde{f}_n$, $\tilde{\varphi}$, $\tilde{\mu}$ and $\tilde{P}$.

If (6) holds, then for every n and every p

$$f_n^*(t,p) \geq 0, \quad f_n^*(t,0) \leq \rho_n(t) \quad \text{a.e.}. \tag{7}$$

Define the following functions on $\mathbb{R}^N \times \mathcal{A}$:

$$S_n(p,E) = \int_E f_n^*(t,p)\,dt.$$

As immediately follows from the definition and (7), these functions are non-negative convex measures on $\mathbb{R}^N \times \mathcal{A}$ which are lower semi-continuous with respect to the first variable and satisfy $S_n(0,E) < \infty$ for all $p \in \mathbb{R}^N$ and all $E \in \mathcal{A}$;

By Proposition 2 a subsequence of these functions $\overline{\Gamma}$-converges to a certain function $S(p,E)$ satisfying the same properties. In what follows we assume for notational simplicity that the sequence (S_n) itself $\overline{\Gamma}$-converges to S.

Let $\mathcal{E}$ be the collection of open sub-intervals of $[0,1]$ with rational end-points. For any $E \in \mathcal{E}$ choose a dense countable subset of $\operatorname{dom} S(\cdot,E)$, and let Π stands for the union of all such subsets. As $\mathcal{E}$ is a countable set, so is Π.

Clearly, any open subset of $[0,1]$ can be obtained as the union of elements of $\mathcal{E}$. As every $S(p,\cdot)$ is inner regular and increasing, it follows that the intersection of Π with $\operatorname{dom} S(\cdot,E)$ is dense in the latter for any $E \in \mathcal{A}$.

1. **Construction of** $P(t)$. First we define, for any t, the set $P_0(t) \subset \mathbb{R}^N$ as the collection of all p such that t is not an extraordinary point of $S(p,\cdot)$. The set $P(t)$ is defined as the closure of $P_0(t)$.

Proposition 6 *The set-valued mappings P_0 and P are convex-valued and lower semi-continuous (in the sense that, whenever $t_k \to t$, $p \in P(t)$, there are $p_k \to p$ such that $p_k \in P(t_k)$), and Π is dense in $P_0(t)$ (hence in $P(t)$) for any t.*

PROOF. It is enough to prove the proposition for P_0. By definition, $p \in P_0(t)$ if and only if there is a neighborhood U of t such that $p \in \operatorname{dom} S(\cdot,U)$. Thus $p \in P_0(\tau)$ means that $p \in P(t)$ for all t of a neighborhood of τ and, consequently, P_0 is lower semi-continuous. Furthermore, as $S(p,\cdot)$ is increasing, for any two $p_1,\ p_2 \in P(t)$ there is a common $U \in \mathcal{A}$ such that both p_1 and p_2 belong to the domain of $S(\cdot,U)$. Every point of the line segment joining p_1 and p_2 also belongs to the domain as S is convex in the vector argument. Hence every point of the segment also belong to

$P(t)$. Finally, it was already explained that Π is dense in the domain of $S(\cdot, E)$ for every $E \in \mathcal{A}$, and this immediately implies that the intersection of Π with every $P(t)$ is dense in the latter.

2. **Construction of μ.** By Proposition 4, for any $p \in \mathbb{R}^N$ there is a probability measure λ_p such that $S(p, \cdot)$ is locally absolutely continuous with respect to λ_p on $\mathrm{Od}(S(p, \cdot))$. Let $p_1, p_2, \ldots$ be an ordering of elements of Π. Set

$$\mu(E) = (1/2) \sum_{i=1}^{\infty} 2^{-i}(\lambda_{p_i}(E) + \mathrm{meas}(E))$$

(where $\mathrm{meas}(E)$ stands for the Lebesgue measure of E). Clearly, μ is a probability measure and the Lebesgue measure is absolutely continuous with respect to μ. We claim that every $S(p, \cdot)$ is locally absolutely continuous with respect to μ on $\mathrm{Od}(S(p, \cdot))$. To see this, take an open $E \subset \mathrm{Od}(S(p, \cdot))$ with $S(p, E) < \infty$. Then $p \in \mathrm{dom}\, S(\cdot, E)$. Let $q_1, q_2, \ldots$ be the intersection of $\mathrm{dom}\, S(\cdot, E)$ with Π. Then $E \subset \mathrm{Od}(S(q_i, \cdot))$ for all q_i and all $S(q_i, \cdot)$ are absolutely continuous with respect to μ on E.

Consider first the case when $p \in \mathrm{ri}(\mathrm{dom}\, S(\cdot, E))$. Then we can find $k \leq N + 1$ points $q_{i_1}, \ldots, q_{i_k}$ such that p belongs to the convex hull of these points. In this case $S(p, E) \leq \max_j S(q_{i_j}, E)$ and therefore $S(p, \cdot)$ is also absolutely continuous with respect to μ on E.

Now let p be an arbitrary element of $\mathrm{dom}\, S(\cdot, E)$, and let $p_0 \in \mathrm{ri}(\mathrm{dom}\, S(\cdot, E))$. Then for any $\alpha \in (0, 1)$

$$p_\alpha = \alpha p + (1 - \alpha) p_0 \in \mathrm{ri}(\mathrm{dom}\, S(\cdot, E)).$$

We have for $0 < \alpha < \alpha'$ and for any $G \subset E$, $G \in \mathcal{A}$

$$S(p_\alpha, G) \leq (\alpha/\alpha') S(p_{\alpha'}, G) + (1 - (\alpha/\alpha')) S(p_0, G),$$

that is

$$\alpha^{-1}(S(p_\alpha, G) - S(p_0, G)) \leq (\alpha')^{-1}(S(p_{\alpha'}, G) - S(p_0, G))$$

and, as $S(\cdot, G)$ is convex and l.s.c., $S(p_\alpha, G) \to S(p, G)$ as $\alpha \to 1$. Thus the functions

$$F_\alpha(\cdot) = \alpha^{-1}(S(p_\alpha, \cdot) - S(p_0, \cdot))$$

are finite measures on E increasingly converging to F_1 as $\alpha \to 1$. But all F_α for $0 < \alpha < 1$ are absolutely continuous w.r.t. μ, hence by the theorem of Beppo–Levi so is F_1. This proves the claim.

3. **Construction of $\varphi(t, u)$.** We shall complete the construction in three steps. Denote $\Delta(t, \varepsilon) = (t - \varepsilon, t + \varepsilon) \cap [0, 1]$ and set

$$g(t, p) = \begin{cases} \limsup\limits_{\varepsilon \to 0} \dfrac{S(p, \Delta(t, \varepsilon))}{\mu(\Delta(t, \varepsilon))}, & \text{if } t \in \mathrm{Od}(S(p, \cdot)), \\ \infty, & \text{otherwise.} \end{cases}$$

The following properties of g are immediate from the corresponding properties of S:

- g is a non-negative function and $\int_0^1 g(t,0)\,d\mu < \infty$;
- $g(t,\cdot)$ is convex for all t (as an upper limit of convex functions);
- for every p, $g(\cdot,p)$ is μ-measurable and the limsup in its definition is μ-almost everywhere the real limit (by the theorem of Radon-Nikodym as $S(p,\cdot)$ is locally absolutely continuous with respect to μ on $\mathrm{Od}(S(p,\cdot))$);
- for any open E we have $S(p,E) = \int_E g(t,p)\,d\mu$.

Next we define another function $\psi(t,p)$:

$$\psi(t,p) = \inf\{\sum_{i=1}^{N+1} \alpha_i g(t,p_i) :\ p_i \in \Pi,\ \alpha_i \geq 0,\ \sum \alpha_i = 1, \sum \alpha_i p_i = p\}.$$

This is also a non-negative function, convex with respect to the second argument and μ-measurable with respect to the first. It is also clear that $\psi(t,p) \geq g(t,p)$ everywhere and $\psi(t,p) \leq g(t,p)$ for all t if $p \in \Pi$. Therefore $S(p,E) \leq \int_E \psi(t,p)\,d\mu$ for all open E and all p and $S(p,E) = \int_E \psi(t,p)\,d\mu$ if $p \in \mathrm{ri}(\mathrm{dom}\, S(\cdot,E))$.

Finally, we define φ as the Young-Fenchel conjugate of $\psi(t,\cdot)$:

$$\varphi(t,x) = \sup_p (p \cdot x - \psi(t,p)).$$

It is clear from the definition that

$$\varphi(t,x) = \sup_{p\in\Pi} (p \cdot x - g(t,p))$$

from which it immediately follows that φ is a normal convex integrand (as so is every function $(t,x) \mapsto p \cdot x - g(t,p)$).

4. **Some properties of φ and P.** We conclude this section by proving two more propositions explaining connections between the just constructed objects.

First we observe that, as immediately follows from (7),

$$\varphi^*(t,p) \geq 0,\ \forall p, \quad \text{and} \quad \int_0^1 \varphi^*(t,0)\,d\mu < \infty.$$

Proposition 7 *$P(t)$ coincides with the closure of* $\mathrm{dom}\,\varphi^*(t,\cdot)$ *μ-almost everywhere.*

PROOF. By definition $g(t,p) < \infty$ only if $t \in \mathrm{Od}(S(p,\cdot))$, and the latter is also a necessary and sufficient condition for $p \in P_0(t)$. Thus $\mathrm{dom}\, g(t,\cdot) \subset P_0(t)$ for all t. Furthermore, as $\psi(t,p)$ is nowhere smaller than $g(t,p)$ and coincides with $g(t,p)$ on the relative interior of its domain, the closures of $\mathrm{dom}\,\psi(t,\cdot)$ and $\mathrm{dom}\, g(t,\cdot)$ coincide

μ-almost everywhere and are subsets of $P(t)$. This proves that $\operatorname{dom}\varphi^*(t,\cdot) \subset P(t)$ as $\varphi^*(t,\cdot) = \psi^{**}(t,\cdot)$ and therefore its domain is contained in the closure of $\operatorname{dom}\psi(t,\cdot)$.

On the other hand, if $p \in \Pi$, then $p \in \operatorname{dom}\psi(t,\cdot)$ μ-almost everywhere on $\mathrm{Od}(S(p,\cdot))$. As Π is a countable set, it follows that for μ-almost every t we have $p \in \operatorname{dom}\psi(t,\cdot)$, provided $t \in \mathrm{Od}(S(p,\cdot))$ and $p \in \Pi$. By Proposition 6, Π is dense in $P(t)$, whence the opposite inclusion.

Proposition 8 *For any $p \in \mathbb{R}^N$ and any $E \in \mathcal{A}$*

$$S(p,E) = \int_E \varphi^*(t,p)\,d\mu;$$

for any continuous $\mathbb{R}^N$-valued function $p(t)$ on $[0,1]$ and any interval Δ

$$J^*_\Delta(p(\cdot)) = \int_\Delta \varphi^*(t,p(t))\,d\mu.$$

PROOF. For $p \in \mathrm{ri}(\operatorname{dom} S(\cdot,E))$, the first equality has been already established as in this case $\varphi^*(t,p) = \psi(t,p)$. For all p it now follows from the fact that both parts of the equality are convex, lower semi-continuous and $\int_E \varphi^*(t,p)\,d\mu \ge S(p,E)$ as $\int_E \psi(t,p)\,d\mu \ge S(p,E)$.

We have furthermore

$$J^*_\Delta(p(\cdot)) = \sup_{\nu\in\mathcal{M}^N} \Big(\int_\Delta p(t)\,d\nu - \int_\Delta \varphi(t,\frac{d\nu}{d\mu})\,d\mu - \int_\Delta s(\frac{d\nu_s}{d|\nu_s|},P(t))\,d|\nu_s|\Big).$$

If $J^*_\Delta(p(\cdot)) < \infty$, then $\int p(t)\,d\nu \le \int s(\frac{d\nu}{d|\nu|},P(t))\,d|\nu|$ for any μ-singular measure ν for otherwise the supremum would be equal to infinity due to homogeneity of $s(\cdot,P(t))$. As we can change independently the absolutely continuous and the singular part of the measure when calculating the extremum, in follows that

$$J^*_\Delta(p(\cdot)) = \sup_{\substack{\nu\in\mathcal{M}^N\\ \nu\ll\mu}} \Big(\int_\Delta p(t)\,d\nu - \int_\Delta \varphi(t,\frac{d\nu}{d\mu})\,d\mu\Big) = \int_\Delta \varphi^*(t,p(t))\,d\mu.$$

On the other hand, if $J^*_\Delta(p(\cdot)) = \infty$ then $\int \varphi^*(t,p(t))\,d\mu$ also must be equal to infinity for otherwise we should conclude that $p(\tau) \notin P(\tau)$ for at least one τ of the support of ν_s in which case $p(t) \notin P(t)$ for all t of a neighborhood of τ as $p(t)$ is continuous and $P(t)$ is lower semi-continuous. By Proposition 7 it follows that $p(t) \notin \operatorname{dom}\varphi^*(t,\cdot)$ on a set of positive μ-measure and therefore $\int \varphi^*(t,p(t))\,d\mu = \infty$.

5 Proofs of the theorems.

1. **Proof of Part (a) of Theorem 1**. We agreed in the previous section that the entire sequence (S_n) $\overline{\Gamma}$-converges to S and $S_n(\cdot,[0,1])$ Γ-converge to $S(\cdot,[0,1])$.

Denote by A_0 the collection of $\tau \in (0,1)$ such that

– τ is not an atom of μ;

– $S_n(\cdot,\Delta)$ Γ-converge to $S(\cdot,\Delta)$ if either $\Delta = [0,\tau)$ or $\Delta = (\tau,1]$.

By Proposition 3 the set $B_0 = (0,1)\backslash A_0$ is at most countable. Take a $\tau \in A_0$ and a $E \in \mathcal{A}$ and set

$$E^-(\tau) = E \cap [0,\tau),\ E^+(\tau) = E \cap (\tau,1],\ G^-(\tau) = E \cap [0,\tau],\ G^+ = E \cap [\tau,1].$$

As follows from Proposition 8 $S(p,G)$ is well defined for all Borel sets G and, since τ is not an atom of μ, we have

$$S(p,E^-(\tau)) = S(p,G^-(\tau)), \quad S(p,E^+(\tau)) = S(p,G^+(\tau)). \tag{8}$$

Denote by $\mathcal{A}^-$ the collection of all relatively open subsets of $[0,\tau]$ and by $\mathcal{A}^+$ the collection of all relatively open subsets of $[\tau,1]$ and by $S^{\pm}(p,\cdot)$ the restrictions of $S(p,\cdot)$ to $\mathcal{A}^{\pm}$. Then the above equations say that the restrictions of S_n to $\mathbb{R}^N \times \mathcal{A}^{\pm}$ Γ-converge to $S^{\pm}$.

Applying Proposition 3 to $[0,\tau]$ and $[\tau,1]$ in the same way as it has been done with $[0,1]$ in the beginning of the proof, we can find two sets $A_0^-(\tau) \subset (0,\tau)$ and $A_0^+(\tau) \subset (\tau,1)$ with at most countable complements to $(0,\tau)$ and $(\tau,1)$ respectively such that (cf. (8)) $S_n(\cdot,\Delta)$ Γ-converge to $S(\cdot,\Delta)$ whenever Δ is either (α,τ) with $\alpha \in A_0^-(\tau)$ or (τ,β) with $\beta \in A_0^+(\tau)$.

To summarize, we conclude that $S_n(\cdot,\Delta)$ Γ-converge to $S(\cdot,\Delta)$ for any interval Δ with one end point $\tau \in A_0$ and the other end point being either 0 or 1 (in which case as usual we speak about $[0,\tau)$ or $(\tau,1]$) or belonging to $A_0(\tau) = A_0^-(\tau) \cup A_0^+(\tau)$. Clearly, this collection of intervals is μ-dense in $[0,1]$.

As well known ([11]) for any $p(\cdot) \in L_\infty^N$

$$I_n^*(p(\cdot)) = \int_0^1 f_n^*(t,p(t))\,dt.$$

On the other hand, $V(\cdot,\Delta)$ is the inverse image of J_Δ under the linear weak*-continuous mapping $T:\ \mathcal{M}^N \mapsto \mathbb{R}^N$ defined by $T(\nu) = \int_\alpha^\beta d\nu$. The adjoint mapping associates with every $p \in \mathbb{R}^N$ the function identically equal to p. Therefore, according to the standard rules of convex analysis (e.g. [9], §3.4, Theorem 3), we get

$$V^*(p,\Delta) = J_\Delta^*(T^*(p)) = \int_\alpha^\beta \varphi^*(t,p)\,d\mu = S(p,\Delta)$$

and this completes the proof of the first statement of the theorem because Γ-convergence of convex functions on $\mathbb{R}^N$ implies Γ-convergence of their conjugates (see [10] or [1] Theorem 3.7) and convex functions Γ-converge if and only if their second conjugates Γ-converge (which is immediate from the definitions).

2. **Proof of Part (b) of Theorem 1**. First we observe that J is lower semicontinuous in the weak*-topology. This follows e.g. from Theorem 3.4.1 of [4] as by Propositions 6 and 7

$$\varphi(t,u) = \sup_{p\in\Pi}(p\cdot u - \varphi^*(t,p)).$$

Let $\nu \in \mathcal{M}^N$, and let the sequence $(u_n) \subset L_1^N$ converge to ν in the weak* topology. We also fix an $\varepsilon > 0$. As J is l.s.c., there is a continuous function $p(t)$ on $[0,1]$ such that

$$\begin{aligned} J(\nu) + J^*(p(\cdot)) &< \int_0^1 p(t)\,d\nu + \varepsilon, \quad \text{if } J(\nu) < \infty, \\ \int_0^1 p(t)\,d\nu - J^*(p(\cdot)) &> \frac{1}{\varepsilon}, \qquad\quad \text{if } J(\nu) = \infty. \end{aligned} \tag{9}$$

In both cases $p(\cdot) \in \operatorname{dom} J^*$, hence (by Proposition 8) $\int_0^1 \varphi^*(t,p(t))\,d\mu < \infty$.

Lemma 1 *There is a piecewise constant function $q(t)$ on $[0,1]$ such that*

(a) $|\int_0^1 (q(t) - p(t))\,d\nu| < \varepsilon$ *and* $|\int_0^1 (\varphi^*(t,p(t)) - \varphi^*(t,q(t))\,d\mu| < \varepsilon;$

(b) if $0 \le \tau_1 < \ldots < \tau_k \le 1$ are points of discontinuity of $q(t)$, then $\mu(\{\tau_i\}) = \nu(\{\tau_i\}) = 0$ for all $i = 1,\ldots,k$ and $S_n(\cdot,(\tau_i,\tau_{i+1}))$ Γ-converge to $S(\cdot,(\tau_i,\tau_{i+1}))$ for all $i = 0,\ldots,k$ whenever $\tau_0 < 0$ and $\tau_{k+1} > 1$.

We shall prove the lemma in the next section and now, assuming that it has been already proved, continue the proof of the theorem. Set $\Delta_i = (\tau_i,\tau_{i+1})$, and let q_i be the value of q on Δ_i (to avoid confusion we can extend $q(t)$ beyond $[0,1]$ by setting $q_0 = q(0)$ and $q_k = q(1)$). As $S_n(\cdot,\Delta_i)$ Γ-converge to $S(\cdot,\Delta_i)$, we can for any i find a sequence $(q_{in}) \subset \mathbb{R}^N$ converging to q_i and such that $S_n(q_{in},\Delta_i) \to S(q_i,\Delta_i)$. Define $q_n(t)$ by setting it equal to q_{in} on Δ_i and choose $n(\varepsilon)$ so big that for $n \ge n(\varepsilon)$

$$|S_n(q_{in},\Delta_i) - S(q_i,\Delta_i)| < \frac{\varepsilon}{k+2}.$$

We have

$$\begin{aligned} I_n^*(q_n) = \int_0^1 f_n^*(t,q_n(t))\,dt &= \sum_i S_n(q_{in},\Delta_i) \le \sum_i S(q_i,\Delta_i) + \varepsilon \\ &= \int_0^1 \varphi^*(t,q(t))\,d\mu + \varepsilon \\ &\le \int_0^1 \varphi^*(t,p(t))\,d\mu + 2\varepsilon = J^*(p) + 2\varepsilon. \end{aligned} \tag{10}$$

On the other hand, as follows from part (b) of the lemma,

$$\int_0^1 q(t)\cdot u_n(t)\,dt = \sum_i \int_{\Delta_i} q_i\cdot u_n(t)\,dt \;\to\; \sum_i \int_{\Delta_i} q_i\,d\nu = \int_0^1 q(t)\,d\nu$$

and also,

$$|\int_0^1 (q_n(t) - q(t)) \cdot u_n(t)\, dt| \le \|u_n\|_1 \max_i \|q_{in} - q_i\| \ \to\ 0$$

as (u_n) is bounded in L_1^N, so that (by the property (a) of the lemma) we can choose $n(\varepsilon)$ to make sure that

$$|\int_0^1 q(t) u_n(t)\, dt - \int_0^1 p(t)\, d\nu| < 2\varepsilon. \tag{11}$$

Combining (9),(10) and (11) we get for $n \ge n(\varepsilon)$

$$\begin{aligned} I_n(u_n) &\ge \int_0^1 q_n(t) u_n(t)\, dt - \int_0^1 f_n^*(t, q_n(t))\, dt, \\ &\ge \int_0^1 p(t)\, d\nu - J^*(p) - 4\varepsilon \\ &\ge \begin{cases} J(\nu) - 5\varepsilon, & \text{if } J(\nu) < \infty, \\ 1/\varepsilon, & \text{if } J(\nu) = \infty. \end{cases} \end{aligned}$$

As ε is an arbitrary positive number, we get from here that

$$\liminf_{n\to\infty} I_n(u_n) \ge J(\nu)$$

as claimed.

3. **Proof of Part (c) of Theorem 1**. Suppose we are given a $\nu \in \mathcal{M}^N$. Let $A_1 = \{\tau_1, \tau_2, ...\}$ be a dense countable subset of A_0 (see the proof of part (a) of the theorem) such that none of τ_i is an atom of ν, and let $B_0(\tau) = (0,1)\backslash A_0(\tau)$ for any $\tau \in A_0$. Let B_1 stands for the union of the set of atoms of ν with $\bigcup_i B_0(\tau_i)$. Then B_1 is at most countable. Set $A_2 = (0,1)\backslash B_1$. Then $A_2 \cap A_1 = \emptyset$. Let finally $\mathcal{D}$ stands for the collection of intervals which either have one end point in A_1 and the other in A_2 or have the form $[0,\tau)$ or $(\tau, 1]$ with $\tau \in A_1$. According to what has been established in the proof of part (a), $S_n(\cdot, \Delta)$ Γ-converge to $S(\cdot, \Delta)$ whenever $\Delta \in \mathcal{D}$.

Now a suitable sequence (π_k) of partitions can be constructed as follows. Choose a $\tau_i \in A_1$ such that $\min\{\tau_i, 1-\tau_i\} > 1/3$, and let π_1 be the partition into two intervals $[0,\tau)$ and $(\tau, 1]$. Then the two intervals of the partition belong to $\mathcal{D}$ and the diameter of the partition is not greater than $2/3$. Suppose now that we have already defined partitions up to π_s with the diameters of every π_k, $k = 1, ..., s$ not greater than $(2/3)2^{-(k-1)}$ and such that for each k every $\Delta \in \min(\pi_k)$ belongs to $\mathcal{D}$.

The partition π_{s+1} is defined as follows. Let $\Delta \in \min(\pi_s)$, that is the end points of Δ are adjacent points of the partition. By the assumption $\Delta \in \mathcal{D}$. This means that one of the end points of the interval belongs to A_1. Suppose, to be certain, that this is the right end point, call it β. Let α be the left end point. As both A_1 and A_2 are dense in $(0,1)$, we can find $\tau' \in A_1$ and $\tau'' \in A_2$ such that $\alpha < \tau' < \tau'' < \beta$

and the lengths of the three intervals into which τ' and τ'' break Δ is smaller than the half of the length of Δ, that is smaller than $(1/2)\mathrm{diam}(\pi_s)$. It is also clear that each of the three intervals belongs to $\mathcal{D}$. Having done this with every $\Delta \in \min(\pi_s)$ we shall have a new partition (π_{s+1}) into intervals belonging to $\mathcal{D}$ with diameter not greater than $(1/2)\mathrm{diam}(\pi_s)$ and such that every interval belonging to π_s belongs also to π_{s+1}.

This completes the construction. We observe that every Δ belonging to one of π_k is the union of several adjacent intervals belonging to $\min(\pi_k)$ and since the latter all belong to $\mathcal{D}$, the same is true for Δ. Then $S_n(\cdot,\Delta)$ Γ-converge to $S(\cdot,\Delta)$ and, according to the proof of part (a), $V_n(\Delta,\cdot)$ Γ-converge to $V^{**}(\Delta,\cdot)$.

Fix a k and take an interval $\Delta \in \min(\pi_k)$. As $V_n(\Delta,\cdot)$ Γ-converge to $V^{**}(\Delta,\cdot)$, we can find a sequence (x_n^Δ) converging to $x^\Delta = \int_\Delta d\nu$ and such that $V_n(\Delta, x_n^\Delta)$ converge to $V^{**}(\Delta, x^\Delta)$. On the other hand, for any n we can find a $u_n^\Delta(t)$ such that

$$\int_\Delta u_n^\Delta(t)\,dt = x_n^\Delta$$

and

$$\int_\Delta f_n(t, u_n^\Delta(t))\,dt \le V_n(\Delta, x_n^\Delta) + \frac{1}{n}.$$

Having done this for any $\Delta \in \min(\pi_k)$, define $u_{kn}(t)$ by $u_{kn}(t) = u_n^\Delta(t)$ if $t \in \Delta$. Then of course

$$\int_\Delta u_{kn}(t)\,dt \to x^\Delta = \int_\Delta d\nu$$

as $n \to \infty$, for any interval Δ belonging to π_k and

$$\begin{aligned}\limsup_{n\to\infty} I_n(u_{kn}) &= \limsup_{n\to\infty} \sum_{\Delta\in\min(\pi_k)} \int_\Delta f_n(t, u_n^\Delta(t))\,dt \\ &\le \lim_{n\to\infty} \sum_{\Delta\in\min(\pi_k)} V_n(\Delta, x_n^\Delta) = \sum_{\Delta\in\min(\pi_k)} V^{**}(\Delta, x^\Delta) \ = J(\nu).\end{aligned} \tag{12}$$

We therefore can find an $n(k)$ such that for all $n \ge n(k)$ we have

$$\sum_{\Delta\in\min(\pi_k)} \left|\int_\Delta u_{kn}(t)\,dt - \int_\Delta d\nu\right| < \frac{1}{k}; \quad I_n(u_{kn}) \le J(\nu) + \frac{1}{k}. \tag{13}$$

It is possible to assume without loss of generality that $(n(k))$ is a strictly increasing sequence of integers.

As π_k is a decreasing sequence of partitions, every interval belonging to π_k belongs also to all subsequent partitions. So defining $u_n(t)$ by

$$u_n = u_{kn} \quad \text{if} \quad n(k) \le n < n(k+1),$$

we complete the proof of (c) and of Theorem 1.

4. **Proof of part (a) of Theorem 2.** As in the proof of Theorem 1, it is enough to consider the case when (6) holds, now with $q_n \equiv q_0 = 0$, in which case by $(\mathbf{A}_1')$

$$f_n(t,u) \geq r\|u\| - \rho_n(t). \tag{14}$$

We can also deal with the entire sequence. In view of the part (b) of Theorem 1, we only have to show that in this case for any $\nu \in \mathcal{M}^N$ there is sequence $u_n \in L_1^N$ converging in the weak*-topology of $\mathcal{M}^N$ to ν and such that

$$\limsup_{n\to\infty} I_n(u_n) \leq J(\nu). \tag{15}$$

We assume below that $J(\nu) < \infty$, otherwise any weak*-converging sequence will satisfy (15).

Let (π_k) and (u_n) be the sequences of partitions and summable functions constructed in the proof of part (c) of Theorem 1. We shall show that u_n is the desired sequence. To this end, we only have to check that (u_n) actually weak* converges to ν as the "limsup" inequality (15) follows from (13). But this is an easy consequence of (14) and the fact that the diameters of π_k go to zero as $k \to \infty$. Indeed, as follows from (13), (14)

$$J(\nu) \geq \limsup_{n\to\infty} I_n(u_n) \geq \limsup_{n\to\infty}(r\int_0^1 \|u_n(t)\|\,dt - \int_0^1 \rho_n(t)\,dt).$$

This means that the L_1-norms of u_n are uniformly bounded by some constant R.

Let now $p(t)$ be a continuous function on $[0,1]$ with values in $\mathbb{R}^N$. Fix an $\varepsilon > 0$ and choose k so big that the oscillation of the function on every interval of length $\operatorname{diam}(\pi_k)$ or less is not greater than ε. Take an $n > k$ and for any $\Delta \in \min(\pi_k)$, let a_Δ be, say, the value of $p(\cdot)$ at the middle point of Δ. Then, as the μ-measures and ν-measures of the points defining π_k are all equal to zero and in view of (13)

$$\begin{aligned}
|\int_0^1 p(t)\,d\nu - \int_0^1 p(t)\cdot u_n(t)\,dt| &= |\sum_{\Delta\in\min(\pi_k)} (\int_\Delta p(t)\,d\nu - \int_\Delta p(t)\cdot u_n(t)\,dt)| \\
&\leq |\sum_{\Delta\in\min(\pi_k)} a_\Delta \cdot (\int_\Delta d\nu - \int_\Delta u_n(t)\,dt)| + \varepsilon(\int_0^1 d|\nu| + R) \\
&\leq k^{-1}\|p(\cdot)\| + \varepsilon(\int_0^1 d|\nu| + R).
\end{aligned}$$

It follows that $\int_0^1 p(t)\cdot u_n(t)\,dt \to \int_0^1 p(t)\,d\nu$ for any continuous p (since $k(n) = \max\{k:\ n(k) \leq n\} \to \infty$ as $n \to \infty$).

5. **Proof of part (b) of Theorem 2.** If $\overline{x} \in \operatorname{ri}(\operatorname{dom} V)$, then $\partial V(\overline{x}) \neq \emptyset$, that is to say, there exists a $\overline{p} \in \mathbb{R}^N$ such that $\overline{x} \in \partial V^*(\overline{p}) = \partial S(\overline{p})$ (let's agree, from now on, that $S(p)$ denote $S(p,[0,1])$) .

The theorem will be proved if we manage to show that there are a μ-measurable mapping $\overline{u}(t)$, $t_1, ..., t_k \in [0,1]$ and $\overline{v}_1, ..., \overline{v}_k \in \mathbb{R}^N$, $(k \le N)$ such that

$$\overline{p} \cdot \overline{u}(t) = \varphi(t, \overline{u}(t)) + \varphi^*(t, \overline{p}) \quad \mu - \text{almost everywhere,}$$

$$\overline{v}_i \in \mathbf{N}(P(t_i), \overline{p}) \ i = 1, ..., k,$$

(where $\mathbf{N}(Q, p)$ stands for the normal cone to Q at p), and

$$\int_0^1 \overline{u}(t)\, d\mu + \sum_{i=1}^k \overline{v}_i = \overline{x}. \tag{16}$$

Indeed, in this case the measure $\overline{\nu}$ defined by

$$\overline{\nu}(E) = \int_E \overline{u}(t)\, d\mu + \sum_{t_i \in E} \overline{v}_i$$

is a solution to the problem since, for any other measure ν with $\int d\nu = \overline{x}$ and density $u(t)$ w.r.t. μ, we have

$$\begin{aligned} J(\nu) &= \int_0^1 \varphi(t, u(t))\, d\mu + \int_0^1 h(t, \frac{d\nu_s}{d|\nu_s|})\, d|\nu_s| \\ &\ge \int_0^1 (\overline{p} \cdot u(t) - \varphi^*(t, \overline{p}))\, d\mu + \overline{p} \cdot \int_0^1 d\nu_s \\ &= \overline{p} \cdot \overline{x} - \int_0^1 \varphi^*(t, \overline{p})\, d\mu \\ &= \int_0^1 (\overline{p} \cdot \overline{u}(t) - \varphi^*(t, \overline{p}))\, d\mu + \sum_i \overline{p} \cdot \overline{v}_i \\ &= \int_0^1 \varphi(t, \overline{u}(t))\, d\mu + \sum_i h(t_i, \overline{v}_i) = J(\overline{\nu}) \end{aligned}$$

where the first inequality comes from the definition of h and the fact that $\overline{x} \in \partial S(\overline{p})$ implies $\overline{p} \in \bigcap_t P(t)$.

As follows from (14) and $(\mathbf{A}_1)$,

$$p \cdot u_0(t) - \rho(t) \le f_n^*(t, p) \le \rho_n(t), \quad \text{a.e.}$$

if $\|p\| \le r$, that is (setting $x_0 = \int u_0(t)\, dt$, $R = \int \rho(t)\, dt$ and $R_1 = \sup_n \int \rho_n(t)\, dt$)

$$p \cdot x_0 - R \le S_n(p) \le R_1, \quad \text{if } \|p\| \le r.$$

Since S_n Γ-converge to S, the inequality remains valid if we replace S_n by S. The latter implies that

$$0 \in \operatorname{int}(\operatorname{dom} S) \ne \emptyset.$$

The proof now can be completed by a reference to Theorem 5 of [9], § 9.3. But the situation considered there is somewhat different, so we give the necessary details.

By Theorem 4 of [9], § 8.3, $\overline{x} \in \partial S(\overline{p})$ means that

$$\overline{x} \in \int_0^1 \partial\varphi^*(t,\overline{p})\, d\mu + \mathbf{N}(\operatorname{dom} S(\cdot),\overline{p}).$$

In other words, there are a μ-measurable $\overline{u}(\cdot)$ and a $\overline{v} \in \mathbf{N}(\operatorname{dom} S(\cdot),\overline{p})$ such that

$$\overline{u}(t) \in \partial\varphi^*(t,\overline{p})\ \mu\text{– a.e.};\quad \overline{x} = \int_0^1 \overline{u}(t)\, d\mu + \overline{v}.$$

What remains is to verify that every non-zero element v of $\mathbf{N}(\operatorname{dom} S(\cdot),\overline{p})$ (if exists) can be decomposed into a sum of at most N vectors v_i, each belonging to the normal cone to $P(t_i)$ at $\overline{p}$ for some t_i.

If $0 \neq v \in \mathbf{N}(\operatorname{dom} S(\cdot),\overline{p}))$, then $\overline{p} \notin \operatorname{int}(\operatorname{dom} S(\cdot))$, hence $\overline{p} \neq 0$. As the ball of radius r around zero belongs to $\operatorname{dom} S(\cdot)$, we have

$$v \in \mathbf{N}(\operatorname{dom} S(\cdot),\overline{p}) \ \Rightarrow\ \overline{p}\cdot v \geq r\|v\| \tag{17}$$

With no loss of generality we may assume that $\overline{p}\cdot v = 1$.

The closure of $\operatorname{dom} S(\cdot)$ coincides with $P = \bigcap_t P(t)$ (this is immediate from Propositions 7 and 8 combined with the fact that $P(t)$ is lower semi-continuous). Therefore the normal cones to P and to $\operatorname{dom} S(\cdot)$ at every point of $\operatorname{dom} S(\cdot)$ coincide. Set

$$C = \{w \in \mathbb{R}^N :\ w \in \mathbf{N}(P,\overline{p}),\ \overline{p}\cdot w = 1\};$$
$$C(t) = \{w \in \mathbb{R}^N :\ w \in \mathbf{N}(P(t),\overline{p}),\ \overline{p}\cdot w = 1\}.$$

Clearly $C(t) \subset C$ for all t (as $P \subset P(t)$) and the closure of the convex hull of $\bigcup_t C(t)$ coincides with C (as P is the intersection of $P(t)$). On the other hand, the set valued mapping $t \mapsto C(t)$ is compact-valued (by (17)) and upper semi-continuous (as $P(t)$ is lower semi-continuous). Therefore $\bigcup_t C(t)$ is a compact set and so is its convex hull. It remains to apply the Carathéodory theorem and take into account that C is a subset of an affine manifold of dimension $N-1$. This completes the proof of the theorem.

6 Proof of Lemma 1.

To begin with, let us agree to denote by $B_t(x,r)$ the ball of radius r around x in the affine hull of $P(t)$ (that is the intersection of the affine hull of $P(t)$ with the ball of radius r around x). We also agree, to avoid confusion, that the piecewise constant functions we shall consider are continuous from the right and also at 1. Finally, we set $\bar{\mu} = \mu + |\nu|$.

1. Choose a $\delta > 0$ so small that

$$\int_E \|p(t)\|\, d|\nu| < \frac{\varepsilon}{4}; \qquad \int_E \varphi^*(t, p(t))\, d\mu < \frac{\varepsilon}{4}; \qquad \int_E \varphi^*(t, 0)\, d\mu < \frac{\varepsilon}{4}$$

whenever $\mu(E) < \delta$.

Let $t_1, t_2, \ldots$ be points of discontinuity (atoms) of $\bar{\mu}$; set $\bar{\mu}(\{t_l\}) = \bar{\mu}_l$. Choose s so big that

$$\sum_{l>s} \bar{\mu}_l < \delta \tag{18}$$

(or s equal to the number of atoms if the latter is finite). Take an $\eta > 0$ so small that $|t_l - t_k| > \eta$ for $1 \le l \ne k \le s$ and

$$\bar{\mu}(\bigcup_{l\le s}([t_l - \eta, t_l) \cup (t_l, t_l + \eta])) < \delta. \tag{19}$$

We can also assume without loss of generality that among the points $t_l \pm \eta$ there are no atoms of $\bar{\mu}$.

2. Set

$$U = [0,1] \setminus \bigcup_{l=1}^{s} [t_l - \eta, t_l + \eta].$$

Then U is an open subset of $[0,1]$. Set further

$$U_i = \{t \in U : \dim P(t) \ge i\}.$$

Every U_i is also an open subset of $[0,1]$ (as $P(t)$ is a lower semi-continuous set-valued mapping). We can choose recursively, starting with $i = N$ down, disjoint open sets $V_N,\ V_{N-1}, \ldots, V_0$ such that

$$V_i \subset U_i \setminus \bigcup_{j=i+1}^{N} V_j,$$

every V_i is a finite union of open intervals and

$$\bar{\mu}(V_i) > \bar{\mu}\big(U_i \setminus \bigcup_{j=i+1}^{N} V_j\big) - \frac{\delta}{2(N+1)}.$$

It follows that

$$\bar{\mu}(U_i \setminus \bigcup_{j=i}^{N} V_j) < \frac{\delta}{2(N+1)}, \qquad i = 0, 1, \ldots, N,$$

so that

$$\bar{\mu}\{t \in V_i : \dim P(t) > i\} = \bar{\mu}(V_i \cap U_{i+1}) < \frac{\delta}{2(N+1)}$$

and therefore, setting $Z_i = \{t \in V_i :\ \dim P(t) = i\}$, we get

$$\bar{\mu}(Z_i) > \bar{\mu}\big(U_i \backslash \bigcup_{j=i+1}^{N} V_j\big) - \frac{\delta}{N+1}.$$

Hence

$$\bar{\mu}(\bigcup_i Z_i) > \sum_i \bar{\mu}\big(U_i \backslash \bigcup_{j=i+1}^{N} V_j\big) - \delta \geq \sum_i \bar{\mu}(U_i \backslash U_{i+1}) - \delta = \bar{\mu}(U) - \delta. \tag{20}$$

3. Given a convex set P with $\dim P \geq 1$, let $\rho(P)$ be the upper bound of radii of balls in the affine hull of P which are contained in P. If $\dim P = 0$ (that is P and the affine hull of P is a singleton), we set $\rho(P) = \infty$, as in any other case when P is itself an affine manifold. By definition $\rho(P) > 0$ for any convex set. It is clear furthermore that $\rho(P(t))$ is lower semi-continuous on every subset of $[0,1]$ on which $\dim P(t)$ is constant.

For any $i = 0, ..., N$ choose a closed set $Q_i \subset Z_i$ with

$$\bar{\mu}(Q_i) > \bar{\mu}(Z_i) - \frac{\delta}{N+1}. \tag{21}$$

Set $T = Q_0 \cup ... \cup Q_N \cup \{t_1, ..., t_s\}$. This is a compact set and therefore $\rho(P(t))$ attains minimum on T which is a positive number. Fix a positive

$$\rho < \min\{\rho(P(t)) :\ t \in T\}. \tag{22}$$

4. Take a $\tau \in T$. By (22) there is a $q \in \mathbb{R}^N$ such that $B_\tau(q, \rho) \subset P_0(\tau)$ (as the relative interior of $P(\tau)$ is contained in $P_0(\tau)$). Moreover it follows from (22) that a ball of a slightly bigger radius than ρ (and of the same dimension as the dimension of $\mathrm{aff} P(t)$) is contained in every $P(t)$ with $t \in T$. Therefore we can assume that there is a polyhedron containing $B_\tau(q, \rho)$ and contained in $P_0(\tau)$. It follows that there is a $\sigma = \sigma_\tau > 0$ such that the polyhedron, and hence $B_\tau(q, \rho)$, is contained in $P(t)$ for $|t - \tau| < \sigma$ and, moreover, that $\xi_\tau(t) = \sup\{\varphi^*(t, p) :\ p \in B_\tau(q, \rho)\}$ is summable on $\Delta_\tau = (\tau - \sigma, \tau + \sigma)$.

We can choose σ so small that $\Delta_\tau \subset V_i$, if $\tau \in Q_i$ or $\sigma < \eta$ if τ is one of t_l, $l = 1, ..., s$. It is also an easy matter to guarantee that $\tau \pm \sigma$ are not among the atoms of $\bar{\mu}$.

Applying the above procedure to every point of T we get a collection of open intervals covering T, hence there is a finite sub-collection still covering T. Let Δ'_{ij}, $j = 1, ..., k_i$ be those interval of the sub-collection that cover Q_i. These intervals are contained in V_i and therefore do not contain points of other V_k as well the atoms t_l. In general the intervals Δ'_{ij} for the given i are not disjoint. We replace them by smaller disjoint intervals whose union coincides with the union of Δ'_{ij} up to finitely many points of zero $\bar{\mu}$-measure.

To this end (assuming that none of Δ'_{ij} belongs to another, otherwise we could simply drop the smaller one) we set

$$\Delta_{i1} = \Delta'_{i1},\ \Delta_{i2} = \Delta'_{i2} \backslash \overline{\Delta_{i1}},\ \dots\ , \Delta_{ij} = \Delta'_{ij} \backslash \overline{(\Delta_{i1} \cup \dots \cup \Delta_{i,j-1} \dots)}.$$

Clearly the only points of the union of Δ'_{ij} which may not belong to the new intervals are end points of the former which are finitely many and have, by construction, zero $\bar{\mu}$-measure.

As a result of application of this procedure to every $i = 0, \dots, N$ we shall have a finite collection of disjoint intervals covering T up to finitely many points of zero $\bar{\mu}$-measure, and associated with each of the intervals, triples $(\tau, q, \xi(t))$ such that $\xi(t)$ is summable on the corresponding interval and for every t in the interval $|\varphi^*(t,p)| \leq \xi(t)$ for all $p \in B_\tau(q, \rho)$.

Denote by $W_i,\ i = 0, \dots, N$ the union of Δ_{ij} and by W_{N+1} the union of the intervals containing $t_l,\ l = 1, \dots, s$. Finally, we define a piecewise constant mapping $\tilde{q}(t)$ by setting it equal to the q associated with the interval to which t belongs, if $t \in W = \bigcup W_i$ and equal to zero if $t \notin \overline{W}$. Likewise we define a function $\tilde{\xi}(t)$ by setting it equal to $\xi(t)$ for the $\xi(\cdot)$ associated with the interval to which t belongs if $t \in W$, and to $\varphi^*(t, 0)$ if $t \notin \overline{W}$.

5. Denote by $\overline{\lambda}$ the minimal length of the intervals constructed in the preceding step of the proof. Take a small $\alpha > 0$ and choose a positive $\lambda < \overline{\lambda}$ such that

$$\|p(t) - p(t')\| < \alpha\rho, \quad \text{if } |t - t'| < \lambda. \tag{23}$$

With such a λ fixed, choose a partition of $[0, 1]$ into intervals of length not greater than λ by points of $(0, 1)$ which are not atoms of $\bar{\mu}$ and consider a new collection of intervals consisting of intersections of intervals of the partition with the intervals defined in the preceding step (that is with Δ_{ij} and the intervals containing $t_l,\ l = 1, \dots, s$). Clearly, the union of newly defined intervals can differ from W by at most finite number of points of zero $\bar{\mu}$-measure.

Let $0 < \tau_1 < \dots < \tau_m < 1$ be the end points of the new intervals so that any $\Delta_k = (\tau_k, \tau_{k+1})$ as well as $\Delta_0 = [0, \tau_1)$ and $\Delta_m = (\tau_k, 1]$ is contained either in W or in the complement of its closure. We denote by I^+ the collection of indices $k = 0, 1, \dots, m$ corresponding to intervals of the first group (contained in W).

6. Next we define a piecewise constant mapping $q_\alpha(t)$ as follows: if $t \in \Delta_k$, then

(a) $q_\alpha(t) = \alpha\tilde{q}(t) + (1 - \alpha)p(\tilde{t}_k)$ if $\Delta_k \cap T \neq \emptyset$, where $\tilde{t}_k$ is an arbitrary element of $\Delta_k \cap Q_i$ if the intersection is nonempty, or $\tilde{t}_k = t_l$ if $t_l \in \Delta_k$ for some $l = 1, \dots, s$ (clearly only one of the possibilities can take place as any Δ_k meeting some Q_i is contained in V_i and any Δ_k containing some t_l does not meet U, hence any of Q_i);

(b) $q_\alpha(t) = 0$ if $\Delta_k \cap T = \emptyset$ (in particular if $k \notin I^+$).

7. To complete the proof we need estimates for

$$\int_0^1 \|p(t) - q_\alpha(t)\|\, d\nu \quad \text{and} \quad \int_0^1 (\varphi^*(t,p(t)) - \varphi^*(t,q_\alpha(t)))\, d\mu.$$

Set $Q = \bigcup Q_i$. By (19), (20) and (21) we have

$$\bar{\mu}(T) < \bar{\mu}(Q) - 3\delta. \tag{24}$$

Let $t \in \Delta_k$. If $\Delta_k \cap T \neq \emptyset$, then by (23)

$$\|p(t) - q_\alpha(t)\| \le \alpha\|p(t) - \tilde{q}(t)\| + (1-\alpha)\|p(t) - p(\tilde{t}_k)\| \le \alpha(\rho + \|p(t) - \tilde{q}(t)\|). \tag{25}$$

If $\Delta_k \cap T = \emptyset$, then $\|p(t) - q_\alpha(t)\| = \|p(t)\|$, so by the choice of δ, (24) and (25)

$$\int_0^1 \|p(t) - q_\alpha(t)\|\, d\nu < \alpha \int_T (\rho + \|p(t) - \tilde{q}(t)\|)\, d\bar{\mu} + \int_{[0,1]\setminus T} \|p(t)\|\, d\bar{\mu} \le K\alpha + \frac{3\varepsilon}{4},$$

where we set $K = \int_0^1 (\rho + \|p(t) - \tilde{q}(t)\|)\, d\bar{\mu}$. Thus we shall have the desired estimate

$$\int_0^1 \|p(t) - q_\alpha(t)\|\, d\nu < \varepsilon \tag{26}$$

if $4\alpha < K\varepsilon$.

Let us estimate the second integral. If Δ_k meets some Q_i, then Δ_k is contained in a certain Δ_{ij} and there is a $\tau \in Q_i$ such that $\dim P(\tau) = i$ and $B_\tau(q,\rho) \subset P_0(t)$ for all $t \in \Delta_{ij}$, where q is the value of $\tilde{q}(\cdot)$ on Δ_k. Let t be an arbitrary point of $\Delta_k \cap Q_i$. Then $\dim P(t) = i$. As $P(t)$ contains $B_\tau(q,\rho)$, the affine hulls of $P(\tau)$ and $P(t)$ coincide. It follows that there is a $q_t \in P(t)$ such that $\|q_t - \tilde{q}(t)\| < \rho$ and $q_\alpha(t) = \alpha q_t + (1-\alpha)p(t)$. Indeed, the equality means that $\alpha\tilde{q}(t) + (1-\alpha)p(\tilde{t}_k) = \alpha q_t + (1-\alpha)p(t)$, that is

$$\|q - q_t\| = \frac{1-\alpha}{\alpha}\|p(\tilde{t}_k) - p(t)\| \le (1-\alpha)\rho < \rho$$

(as $|\tilde{t}_k - t| < \lambda$), that is $q_t \in B_t(q,\rho) = B_\tau(q,\rho)$. It follows that for every $t \in \Delta_k \cap Q_i$ we have

$$\varphi^*(t, q_\alpha(t)) \le \alpha\varphi^*(t,q_t) + (1-\alpha)\varphi^*(t,p(t)) \le \alpha\tilde{\xi}(t) + (1-\alpha)\varphi^*(t,p(t)). \tag{27}$$

It is clear that the same inequality is valid for $t = t_l$, $l = 1, \dots, s$ (in which case $q_t = q_\alpha(t)$). Thus

$$\begin{aligned}
\int_0^1 \varphi^*(t, q_\alpha(t))\, d\mu &= \int_T \varphi^*(t, q_\alpha(t))\, d\mu + \int_{[0,1]\setminus T} \varphi^*(t,0)\, d\mu \\
&\le \alpha \int_T \tilde{\xi}(t)\, d\mu + (1-\alpha)\int_T \varphi^*(t,p(t))\, d\mu + \int_{[0,1]\setminus T} \varphi^*(t,0)\, d\mu \\
&\le \int_T \varphi^*(t,p(t))\, d\mu + \alpha\int_0^1 \tilde{\xi}(t)\, d\mu + \int_{[0,1]\setminus T} \varphi^*(t,0)\, d\mu.
\end{aligned}$$

Again, as $\mu([0,1]\backslash T) < 3\delta$, we have

$$\int_0^1 \varphi^*(t,0)\,d\mu < \frac{3\varepsilon}{4},$$

by the choice of δ, so the inequality

$$|\int_0^1 (\varphi^*(t,p(t)) - \varphi^*(t,q_\alpha(t)))\,d\mu| < \varepsilon$$

will be satisfied if $4\alpha < (\int_0^1 \tilde{\xi}(t)\,d\mu)^{-1}\varepsilon$.

Let's then choose an α to make sure that the above inequality holds. Then $q_\alpha(t)$ is a piecewise constant function taking values $q_0,\dots,q_k,\dots,q_m$ on intervals $[0,\tau_1),\dots,(\tau_k,\tau_{k+1}),\dots,(\tau_m,1]$ respectively. Set (for sufficiently small λ)

$$q^\lambda(t) = \begin{cases} q_0, & \text{if } t \in [0,\tau_1-\lambda), \\ \dots & \dots \\ q_k, & \text{if } t \in [\tau_k + (-1)^k\lambda, \tau_{k+1} + (-1)^{k+1}\lambda), \\ \dots & \dots \\ q_m, & \text{if } t \in [\tau_m + (-1)^m\lambda, 1]. \end{cases}$$

As none of τ_k is an atom of μ, it follows that

$$\lim_{\lambda\to 0}\int_0^1 \varphi^*(t,q^\lambda(t))\,d\mu = \int_0^1 \varphi^*(t,q_\alpha(t)))\,d\mu.$$

It remains to observe that for all λ with a possible exception of a countable set the points $\tau_k + (-1)^k\lambda$ are not atoms of $\bar{\mu}$ and (by Proposition 3) $S_n(\cdot,\Delta)$ Γ-converge to $S(\cdot,\Delta)$ whenever $\Delta = [0,\tau_1-\lambda)$ or $\Delta = (\tau_k + (-1)^k\lambda, \tau_{k+1} + (-1)^{k+1})$, or $\Delta = (\tau_m + (-1)^m\lambda, 1]$. So it is possible to find a λ satisfying all these properties and such that

$$|\int_0^1 (\varphi^*(t,p(t)) - \varphi^*(t,q^\lambda(t)))\,d\mu| < \varepsilon.$$

Setting $q(t) = q^\lambda(t)$ with the chosen λ, we conclude the proof.

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