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Multiscale analysis for nonlinear variational problems arising from discrete systems

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Contents

Introduction

The object of this thesis is the study of high density discrete systems as variational limit of low density discrete energies indexed by the number of nodes of the system itself. In this context the term discreteness should be understood rather broadly as inferring to different scales from crystal lattice to grain structure while the low-to-high density limit refers to the discrete-to-continuum passage in the energetic description of the system. Within this framework we focus our attention on *central lattice systems*; i.e., systems where the reference positions of the interacting points lie on a prescribed lattice, whose parameters change as the number of points changes and where all the interactions are pair interactions. In more precise terms, we consider an open set $\Omega \subset \mathbb{R}^N$ and take as reference lattice $Z_\varepsilon = \varepsilon\mathbb{Z}^N \cap \Omega$. The general form of a pair-potential energy in a central system is

$$E_\varepsilon(u) = \sum_{i,j \in Z_\varepsilon} \psi^\varepsilon_{ij}(u(i), u(j)), \tag{0.0.1}$$

where $u : Z_\varepsilon \to \mathbb{R}^d$. The analysis of energies of the form (0.0.1) has been performed under various hypotheses on ψ^ε_{ij}. The first natural assumption is the invariance under translations (in the target space); that is,

$$\psi^\varepsilon_{ij}(u, v) = g^\varepsilon_{ij}(u - v).$$

Furthermore, an important class of pair potentials is that of homogeneous interactions (i.e., invariant under translations in the reference space); this can be expressed as

$$\psi^\varepsilon_{ij}(u, v) = g^\varepsilon_{(i-j)/\varepsilon}(u, v).$$

If both conditions are satisfied, then the energies E_ε above may be rewritten in the form

$$E_\varepsilon(u) = \sum_{k \in Z^n} \sum_{i,j \in Z_\varepsilon, i-j=\varepsilon k} \varepsilon^n f^\varepsilon_k\Big(\frac{u(i) - u(j)}{\varepsilon}\Big),$$

where $f^\varepsilon_k(\xi) = \varepsilon^{-n} g^\varepsilon_k(\varepsilon\xi)$. In this way we can highlight the dependence of the potentials on the (discrete) difference quotients of the function u. Upon identifying each function u with its piecewise-constant interpolation, we can consider E_ε as defined on (a subset of) $L^p(\Omega;\mathbb{R}^d)$, and hence consider the Γ-limit as ε tends to zero with respect to the L^p-topology. Under some coerciveness conditions the computation of the Γ-limit will give a continuous approximate description of the behaviour of minimum problems involving the energies E_ε for ε small (see Chapter 1, and [13] for a quick introduction to the theory of Γ-convergence).

In Chapter 1, looking for a microscopical theoretical justification of theories in Continuum Mechanics and having in mind those concerning with hyperelastic materials, we face the problem of finding the widest class of discrete systems with an energy of local type as continuum counterpart, defined on the Sobolev space $W^{1,p}(\Omega,\mathbb{R}^d)$ with $p>1$. In particular, even in a general space dependent case, if we make the following assumptions on the functions f_k^ε:

(i) (coerciveness on nearest neighbors) there exit $c_1, c_2 > 0$ such that for all $(x,z)\in\Omega\times\mathbb{R}^d$ and $i\in\{1,\dots,n\}$

$$c_1|z|^p - c_2 \le f_\varepsilon^{e_i}(x,z)$$

(ii) (decay of long-range interactions) for all $(x,z)\in\Omega\times\mathbb{R}^d$, and $k\in\mathbb{Z}^N$

$$f_\varepsilon^k(x,z) \le c_k^\varepsilon(1+|z|^p), \tag{0.0.2}$$

where c_k^ε satisfy

$$\text{(H1):}\quad \limsup_{\varepsilon\to 0^+}\sum_{k\in\mathbb{Z}^N} c_k^\varepsilon < +\infty;$$

$$\text{(H2):}\quad \text{for all } \delta>0\ M_\delta>0 \text{ exists such that } \limsup_{\varepsilon\to 0^+}\sum_{|k|>M_\delta} c_k^\varepsilon < \delta,$$

then we can state a compactness theorem asserting that, the energies E_ε, defined by

$$E_\varepsilon(u) = \sum_{k\in\mathbb{Z}^N}\sum_{i\in R_\varepsilon^k}\varepsilon^n f_\varepsilon^k\left(i, \frac{u(i+\varepsilon k)-u(i)}{\varepsilon|k|}\right),$$

where $R_\varepsilon^k := \{i\in Z_\varepsilon :\ i+\varepsilon k\in Z_\varepsilon\}$ are such that, for every sequence (ε_j) of positive real numbers converging to 0, there exists a subsequence (ε_{j_k}) and a Carathéodory function $f:\Omega\times\mathbb{R}^{d\times N}$ satisfying

$$c(\|M\|^p-1)\le f(x,M)\le C(\|M\|^p+1),$$

with $0<c<C$, such that $(E_{\varepsilon_{j_k}}(\cdot))$ Γ-converges with respect to the $L^p(\Omega)$-topology to the functional $F: L^p(\Omega)\to[0,+\infty]$ defined as

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)\,dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise.}\end{cases} \tag{0.0.3}$$

Note that here the growth hypothesis of superlinear type on nearest neighbors translates into a boundedness condition on the gradient of proper piecewise-affine interpolations and it ensures that the limit is defined in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$, while the decay assumption as $k \to +\infty$ allows to asymptotically neglect very long-range interactions. Observe that these hypotheses are natural for our purposes in the sense that, if the first is lifted then the limit may be defined on sets of functions with bounded total variation where a different analytical approach is needed (see Braides and Gelli [18], [20], [19]), while, if the second is removed, Braides has proven in [12] that the limit may exhibit a non local behavior.

In the special case of periodic structure, when the energies are defined by a scaling process, i.e. when

$$f_k^\varepsilon(x,z) = f_k\Big(\frac{x}{\varepsilon}, z\Big), \tag{0.0.4}$$

then the limit energy density $\varphi(M) = f(x,M)$ is independent of x and of the subsequence, and is characterized by the *asymptotic homogenization formula*

$$\varphi(M) = \lim_{T\to+\infty} \frac{1}{T^N} \min\left\{\mathcal{F}_T(u),\ u|_{\partial Q_T} = Mi\right\}, \tag{0.0.5}$$

where $Q_T = (0,T)^N$,

$$\mathcal{F}_T(u) = \sum_{k\in\mathbb{Z}^N} \sum_{i\in R_1^k(Q_T)} \psi_k\left(\frac{u(i+k)-u(i)}{|k|}\right)$$

and $u|_{\partial Q_T} = Mi$ means that "near the boundary" of Q_T the function u is the discrete interpolation of the affine function Mx (see Section1.2 for details). In the one-dimensional case this formula was first derived in [21] and it is the discrete analog of the nonlinear asymptotic formula for the homogenization of nonlinear energies of the form $G_\varepsilon(u) = \int_\Omega g(x/\varepsilon, Du)\,dx$, that reads

$$\varphi(M) = \lim_{T\to+\infty} \frac{1}{T^N} \inf\big\{\mathcal{G}_T(u) : u = Mx \text{ on } \partial Q_T\big\},$$

where now

$$\mathcal{G}_T(u) = \int_{Q_T} g(y, Du)\,dy$$

(see [16] for exact statements and hypotheses on g).

In Chapter 1 we also examine formula (0.0.5) in some special cases. First, if all f_k are convex then a periodicity cell problem formula holds and, apart from a possible lower-order boundary contribution, the solution in (0.0.5) is simply $u_i = Mi$. In this case the Γ-limit coincides with the pointwise limit and this means that the continuous counterparts of these discrete systems are obtained by simply substituting difference quotients with directional derivatives. Next, if only *nearest-neighbor interactions* are present then it reduces to

$$\varphi(M) = \sum_{i=1}^{n} f_i^{**}(Me_i),$$

where $f_i = f_{e_i}$ and f^{**} denotes the lower semicontinuous and convex envelope of f. Note that convexity is not a necessary condition for lower semicontinuity at the discrete level: this convexification operation should be interpreted as an effect due to oscillations at a 'mesoscopic scale' (i.e., much larger than the 'microscopic scale' ε but still vanishing as $\varepsilon \to 0$). Moreover the previous formula highlights that this mesoscopic phenomenon acts in every direction without mixing them, thus in some sense we can say that the limit energy density is obtained relaxing the energies due to interactions in every coordinate direction independently and then summing over them. This observation allows us, in the last section of Chapter 1, to build an example of vector-valued discrete interaction energies whose continuous counterpart has an energy density which is a quasiconvex not convex function.

An issue of interest when dealing with continuum limits of discrete systems is the problem of the validity or failure of the *Cauchy-Born rule*; i.e. whether to a 'macroscopic' gradient there corresponds at the 'microscopic' scale a 'regular' arrangement of lattice displacements. For energies of the form (0.0.4) this can be translated into the study of the asymptotic behavior of minimizers for the problems defining $\varphi(M)$; in particular we say that the strict Cauchy-Born rule holds at M if minimizers converge to the affine state $u_i = Mi$ and that the weak Cauchy-Born rule holds at M if minimizers tend to a periodic perturbation of Mi. In Chapter 2, we consider the simple one dimensional case of next-to-nearest-neighbor discrete systems. The energy of these systems can be written as

$$E_\varepsilon(u) = \sum_{i,j\in Z_\varepsilon, i-j=\varepsilon} \varepsilon f_1\Big(\frac{u(i)-u(j)}{\varepsilon}\Big) + \sum_{i,j\in Z_\varepsilon, i-j=2\varepsilon} \varepsilon f_2\Big(\frac{u(i)-u(j)}{\varepsilon}\Big),$$

and here $\varphi = f_0^{**}$, where

$$f_0^{**}(z) = f_2(2z) + \frac{1}{2}\min\Big\{f_1(z_1) + f_1(z_2) : z_1 + z_2 = 2z\Big\}. \tag{0.0.6}$$

The second term, obtained by minimization, is due to oscillations at the microscopic scale: nearest neighbors rearrange in order to minimize their interaction coupled with that between second neighbors (see [13] for a simple treatment of these one-dimensional problems).

If, for the sake of simplicity, we suppose that the minimum problem in (0.0.6) has a unique solution, upon changing z_1 into z_2, then we can read the microscopic behavior as follows:

(i) first case: f_0 *is convex at* z (i.e., $f_0(z) = \varphi(z)$). We have the two cases

(a) $f(z) = f_1(z) + f_2(z)$; in this case $z = z_1 = z_2$ minimizes the formula giving φ; hence, the strict Cauchy-Born rule applies;

(b) $f(z) < f_1(z) + f_2(z)$; in this case we have a 2-periodic ground state with 'slopes' z_1 and z_2, and the weak Cauchy-Born rule applies;

(ii) second case: f_0 *is not convex at* z (i.e., $f_0(z) > \varphi(z)$). In this case the Cauchy-Born rule is violated, non uniform states may be preferred as minimizers and phase transitions may occur. Thus surface energies must be taken into account in order to provide a better description of the limit. To this end a higher order analysis of the energy is necessary. Thanks to the notion of development by Γ-convergence introduced by Anzellotti and Baldo in [7], analyzing proper scaling of the energies E_ε, we show that, even if globally the Cauchy-Born rule is violated, minimizers are fine mixtures of states satisfying a weak Cauchy-Born rule, which in this sense holds locally (in each phase).

Another issue of interest in presence of phase transitions is whether the effect of the boundary conditions on the system is or not confined in a neighborhood of the boundary of the domain. In Chapter two a sensitive dependence of the phase transition energy on the boundary conditions is observed for next-to-nearest-neighbors systems. Moreover the appearance of boundary layers contribution to the energy is discussed. Here again the advantage of passing from the macroscopic to the more refined microscopic scale is evident. In fact the description of this phenomenon is possible studying not only the behavior of averaged fields as the natural high density counterpart of the microscopic ground states, but also the formation of microscopical patterns. Finally, using the notion of equivalence by Γ-convergence introduced by Braides and Truskinowsky in [22], we infer that (under some technical assumptions) these discrete systems are equivalent to the perturbation of a non-convex energy on the continuum, of the form

$$\int_\Omega \psi(u')\,dt + \varepsilon^2 C \int_\Omega |u''|^2\,dt,$$

thus recovering a well-known formulation of the gradient theory of phase transitions. This result shows that a surface term (generated by the second gradient) penalizes high oscillations between states locally satisfying some Cauchy-Born rule.

A further problem addressed in the thesis is the analysis of a new type of continuum limits involving energies of the form

$$F_\varepsilon(u) = \sup_{k\in Z^n}\ \sup_{i,j\in Z_\varepsilon, i-j=\varepsilon k} f_k^\varepsilon\Big(\frac{u(i)-u(j)}{\varepsilon}\Big). \tag{0.0.7}$$

This seems to be a complex problem in its generality; in this work the one dimensional case only is treated. Such energies are the discrete analog of L^∞-energies of the gradient, that have been recently widely studied in the framework of spaces of Lipschitz functions (see [6], [8]). In the generality of the conditions that we require on f_k^ε, the continuum limit of (0.0.7) will take a new form that can be interpreted as the natural SBV version of the L^∞-energies of the gradient. This type of energies have been only partially studied in the literature and for this reason much part of Chapter 3 is devoted to the analysis of semicontinuity and relaxation results for energies defined on $SBV(\Omega)$ of the form

$$F(u) = \max\Big\{\sup_{t\in\Omega} f(u'(t)), \sup_{t\in S(u)\cap\Omega} g([u](t))\Big\}. \tag{0.0.8}$$

In particular we show that necessary and sufficient conditions for the lower semicontinuity of F are of two types:
(i) *structure conditions* on f and g. Namely, that f be level convex and g be *sub-maximal*; i.e. that

$$g(a+b) \leq \max\{g(a), g(b)\};$$

(ii) *compatibility conditions* between the growth of g at 0 and of f at infinity:

$$\lim_{z\to 0+} g(z) = \lim_{z\to +\infty} f(z).$$

Moreover, in order to study the structure of solution of minimum problems of the type

$$m(d) = \min\{F(u) : u(0) = 0,\ u(1) = d\}, \tag{0.0.9}$$

we prove a relaxation theorem showing that the L^1-lower semicontinuous envelope of such F is a functional of the same form with f and g substituted by the suitably defined level-convex and sub-maximal envelopes, respectively. By plotting the 'stress-strain' curve relating the bulk gradient of the solutions of problem (0.0.9) to the boundary datum we highlight a 'multiple cracking' phenomenon analogous to that observed for non-subadditive free-discontinuity integral energies (see [11] for a survey on this argument).

In the last section of the chapter we come back to the original problem proving a first approximation result via Γ-convergence of energies of the form (0.0.8) by one dimensional discrete systems with energies of the type (0.0.7) under the assumption that $k = 1$.

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Chapter 1

An integral representation result for continuum limits of energies of discrete systems

The energetic description of the asymptotic behavior of lattice systems when the mesh size tends to zero turns out to be useful both as a microscopical theoretical justification of theories in Continuum Mechanics and as a powerful means thanks to which a great number of microscopical phenomena can be read in the macroscopical setting. In this chapter we describe variational limits of discrete lattice systems in a vectorial and non convex setting when general "atomic" interaction energies are taken into account, that lead to continuum "elastic" theories described by bulk integral energies. We will limit our analysis to square lattices, but more general geometries, e.g. hexagonal lattices, can be easily included in this framework by a change of variables (see for instance [20] Examples 5.1 and 5.2 for details). In mathematical terms, given a fixed open set $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, we consider energies defined on functions $u : \alpha \in \varepsilon\mathbb{Z}^N \cap\ \Omega \mapsto u(\alpha) \in \mathbb{R}^d$, of the general form

$$F_\varepsilon(u) = \sum_{\substack{\alpha,\beta\in\varepsilon\mathbb{Z}^N \\ [\alpha,\beta]\subset\Omega}} g_\varepsilon(\alpha, \beta, u(\alpha) - u(\beta)),$$

In the case $N = d = 3$ we can picture the lattice $\varepsilon\mathbb{Z}^N \cap\ \Omega$ as the reference configuration of a set of interacting material points (see fig. 1). Here u is the field mapping the reference configuration into the deformed one, thus the total stored energy $F_\varepsilon(u)$ is obtained, according to the classical theory of cristalline structures in "hyperelastic" regime, by the superposition of the energy densities

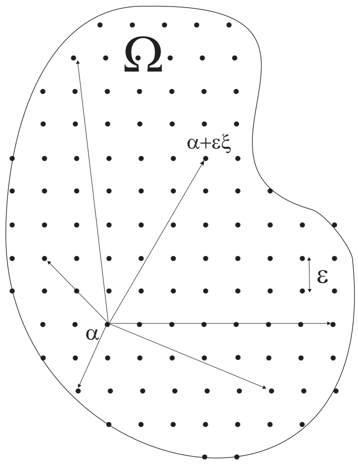

Figure 1: interactions on the lattice $\varepsilon\mathbb{Z}^N$

$g_\varepsilon(\alpha, \beta, u(\alpha) - u(\beta))$ weighing the pairwise interaction between points in the positions α and β in the reference configuration lattice. Note that the only assumption we make is that g_ε depends on the displacement field in α and β through the differences $u(\alpha) - u(\beta)$. This condition arises naturally in many situations as for example in frame indifferent models.

It is usually more convenient to group the energy densities as

$$F_\varepsilon(u) = \sum_{\xi \in \mathbb{Z}^N} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} g_\varepsilon(\alpha, \alpha + \varepsilon\xi, u(\alpha + \varepsilon\xi) - u(\alpha)),$$

where $R_\varepsilon^\xi(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^N : \ [\alpha, \alpha + \varepsilon\xi] \subset \Omega\}$. Setting

$$f_\varepsilon^\xi(\alpha, \zeta) = \varepsilon^{-N} g_\varepsilon(\alpha, \alpha + \varepsilon\xi, \varepsilon|\xi|\zeta)$$

we can rewrite

$$F_\varepsilon(u) = \sum_{\xi \in \mathbb{Z}^N} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^N f_\varepsilon^\xi \left(\alpha, \frac{u(\alpha + \varepsilon\xi) - u(\alpha)}{\varepsilon|\xi|} \right), \tag{1.0.1}$$

thus highlighting the dependence of the energy on discrete difference quotients in the direction ξ.

In this chapter we provide a characterization of all the possible variational limits, as the mesh size ε tends to zero, of a very general class of energies of the form

(1.0.1). Upon identifying u with a function constant on each cell of the lattice $\varepsilon\mathbb{Z}^N$, we can make the asymptotic analysis precise thanks to the notions and the methods of De Giorgi's Γ-convergence (see [30], [13], [29]). On the functions $f_\varepsilon^\xi(\alpha,\cdot)$ we make assumptions of two types: a growth hypothesis of superlinear type on nearest neighbors (see 1.2.2) that ensures that the limit is finite only on $W^{1,p}(\Omega;\mathbb{R}^d)$, and a decay assumption as $\xi \to +\infty$ (see (1.2.3), (H1), (H2)) that allows to neglect very long-range interactions. Under these conditions, a compactness theorem holds asserting that, up to passing to a subsequence, the energies F_ε have a Γ-limit energy F defined on the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$ and taking the form

$$F(u) = \int_\Omega f(x, Du)\; dx,$$

(see Theorem 1.2.1). A similar compactness result for quadratic interactions in planar networks has been observed by Vogelius [46] (see also Piatnitski and Remy [42]).

Note that the decay assumption on the density energies f_ε^ξ as $|\xi| \to +\infty$ guarantees that the non locality of our discrete functionals disappears in the limit. If this hypothesis is lifted then we may have non local Γ-limits (see [12]). On the other hand, if growth conditions are removed, the limit may be defined on sets of functions with bounded variation where a different analytical approach is needed (see [45], [15], [26], [3], [12], [18], [20]).

To perform our analysis, we develop the discrete analogue of a localization argument used, for example, in the context of homogenization theory for multiple integrals which allows us to regard our energies and their Γ-limits as functionals defined on pairs function-set and then to prove that all the hypotheses of an integral representation theorem are fulfilled. In order to treat minimum problems with boundary data, we also derive a compactness theorem in the case that our functionals are subject to Dirichlet boundary conditions (see (1.2.30) and Theorem 1.2.10).

An interesting special case is when the arrangement of the "material points" presents a periodic feature; *i.e.,* in terms of f_ε, we have

$$f_\varepsilon^\xi(\cdot, z) = f^\xi\left(\frac{\cdot}{\varepsilon}, z\right) \qquad f^\xi(\cdot, z) \quad Q_k\text{-periodic}$$

where $Q_k = (0,k)^N$. By adapting the integral homogenization arguments to our discrete setting, we prove that the whole family F_ε Γ-converges to a limit energy of the form

$$F(u) = \int_\Omega f_{hom}(Du)\; dx.$$

Note that in this setting we also include, when $k = 1$, the situation when $f^\xi(\alpha, z)$ is independent of α. If not only nearest neighbor interactions are present, the formula for f_{hom} highlights a multiple-scale effect also in this case (see [13]). An interesting example showing the effect of nonlinearities of "geometrical" origin is contained in a work by Friesecke and Theil [34], where an interpretation in terms of the Cauchy-Born rule is given.

Here f_{hom} is given by the following homogenization formula

$$f_{hom}(M) = \lim_{h\to+\infty} \frac{1}{h^N} \min\{\mathcal{F}_h(u),\ u|_{\partial Q_h} = M\alpha\} \tag{1.0.2}$$

where

$$\mathcal{F}_h(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in R^\xi(Q_h)} f^\xi\left(\alpha, \frac{u(\alpha+\xi)-u(\alpha)}{|\xi|}\right)$$

and $u|_{\partial Q_h} = M\alpha$ means that "near" the boundary of Q_h the function u is the discrete interpolation of the affine function Mx (for more precise definitions see (1.2.29) and Theorem 1.3.1). This formula generalizes that obtained in [21] in a one dimensional-scalar setting.

In general (1.0.2) cannot be simplified to a cell problem formula and gives rise to a quasiconvex function even for simple interactions. Indeed, in Section 7 we provide an example of quasiconvex f_{hom} drawing inspiration from Šverák's construction of a quasiconvex function which is not polyconvex (see [44]).

In Sections 5 and 6 we study some important cases when the formula for f_{hom} can be simplified. For convex interactions a periodicity cell problem formula holds: if f^ξ is a convex function in the second variable for all $\xi \in \mathbb{Z}^N$, then (1.0.2) can be written as

$$f_{hom}(M) = \frac{1}{k^N} \min\{\mathcal{F}(u),\ u\ Q_k\text{-periodic}\}$$

where

$$\mathcal{F}(u) = \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha\in\{0,1,\dots,k-1\}^N} f^\xi\left(\alpha, \frac{u(\alpha+\xi)-u(\alpha)}{|\xi|} + M\cdot\frac{\xi}{|\xi|}\right),$$

(see Theorem 1.4.1). An analogous result for discrete quadratic forms has been obtained by Piatnitski and Remy [42]. Our result has been used by Braides and Francfort [17] as a step for the derivation of optimal bounds for composite conducting networks in the particular case of quadratic interactions (see Remarks 1.2.2 and 1.4.2).

If we consider only interactions along independent directions a reduction to the 1-dimensional case occurs: if $k = 1$, that is f^ξ does not depend on α, and

$$f^\xi \equiv 0 \quad \forall\xi \in \mathbb{Z}^N : \ \xi \neq je_i,\ i \in \{1,2,\dots,N\},\ j \in \mathbb{N}, \tag{1.0.3}$$

where $\{e_1, e_2, \dots, e_N\}$ is the standard orthonormal base in $\mathbb{R}^N$, then

$$f_{hom}(M) = \sum_{i=1}^{N} (\tilde{f}_i)(M^i)$$

being $(\tilde{f}_i)$ convex functions defined by a 1-dimensional homogenization formula and M^i the i-th column of M (see Theorem 1.5.3). Note that here a superposition principle holds, in the sense that the limit energy is obtained by relaxing the energies due to the interactions in every coordinate direction independently and then summing over them.

From the results obtained in the 1-dimensional setting in [21] (see Theorems 1.5.1, 1.5.2), we deduce that the limit energy density f_{hom} can be rewritten by a non-asymptotic formula if only nearest and next-to-nearest neighbor interactions along the coordinate directions are considered (see Remark 1.5.5). In particular, in the case of only nearest neighbor interactions, the only effect of the passage from the discrete setting to the continuum is a separate convexification process in the coordinate directions.

1.1 Notation and Preliminaries

We denote by $\{e_1, e_2, \ldots, e_N\}$ the standard basis in $\mathbb{R}^N$, by $|\cdot|$ the usual euclidean norm and by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^N$. We denote by $\mathcal{M}^{d\times N}$ and $\mathcal{M}^{d\times d}_{sym}$ the space of $d\times N$ matrices and symmetric $d\times d$ matrices, respectively. For $P \in \mathcal{M}^{d\times N}$, $Q \in \mathcal{M}^{N\times l}$, $P \cdot Q$ denotes the standard row by column product. For $x,\ y \in \mathbb{R}^N$, $[x, y]$ denotes the segment between x and y. If Ω is a bounded open subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω while $\mathcal{A}_0(\Omega)$ denotes the family of all open subsets of Ω whose closure is a compact subset of Ω. If $B \subset \mathbb{R}^N$ is a Borel set, we will denote by $|B|$ its Lebesgue measure. We use standard notation for L^p and Sobolev spaces.

We also recall the standard notation for slicing arguments (see [13]). Let $\xi \in S^{N-1}$, and let $\Pi_\xi = \{y \in \mathbb{R}^N : \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbb{R}^N$ we define $E^\xi = \{y \ s.t. \ \exists t \in \mathbb{R} : y + t\xi \in E\}$ and $E^\xi_y = \{t \in \mathbb{R} : y + t\xi \in E\}$. Moreover, if $u : E \to \mathbb{R}$ we set $u_{\xi,y} : E^\xi_y \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$.

We also introduce a useful notation for difference quotient along any direction. Fix $\xi \in \mathbb{R}^N$; for $\varepsilon > 0$ and for every $u : \mathbb{R}^N \to \mathbb{R}^d$ we define

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon|\xi|}.$$

1.1.1 Necessary conditions for weak lower semicontinuity

Definition 1.1.1 *We say that a function $f : M^{d\times N} \to \mathbb{R}$ is quasiconvex if f is continuous, and for all $A \in M^{d\times N}$ and for every bounded open subset E of $\mathbb{R}^N$*

$$|E| f(A) \leq \int_E f(A + D\varphi)\, dx$$

for every $\varphi \in C^\infty_0(E; \mathbb{R}^d)$.

Definition 1.1.2 *We say that a function $f : M^{d\times N} \to \mathbb{R}$ is $W^{1,p}$-quasiconvex if, for all $A \in M^{d\times N}$, there exists a bounded open subset E of $\mathbb{R}^N$ such that*

$$|E|f(A) \leq \int_E f(A + D\varphi(x))\ dx$$

for every $\varphi \in W_0^{1,p}(E;\mathbb{R}^d)$; that is equivalently, such that

$$f(A) = \min\left\{ \frac{1}{|E|}\int_E f(A + D\varphi(x))\ dx\ \ : \varphi \in W_0^{1,p}(E;\mathbb{R}^d) \right\}.$$

Remark 1.1.3 *Note that if $1 \leq p < \infty$, and f satisfies the growth condition from above*

$$0 \leq f(A) \leq c(1 + |A|^p)$$

for all $A \in M^{d\times N}$, then f is quasiconvex if and only if f is $W^{1,p}$-quasiconvex.

Theorem 1.1.4 *Let $1 \leq p < \infty$. If the integral functional*

$$F(u) = \int_\Omega f(Du(x))\ dx$$

is weak lower semicontinuous on $W^{1,p}(\Omega)$, then f is $W^{1,p}$-quasiconvex.

1.1.2 Γ-convergence

We recall the notion of Γ-convergence in $L^p(\Omega)$ (see [30],[29],[13]). A sequence of functionals $F_j : L^p(\Omega) \to [0, +\infty]$ is said to Γ-converge to a functional $F : L^p(\Omega) \to [0, +\infty]$ at $u \in L^p(\Omega)$ as $j \to +\infty$, and we write $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if the following two conditions hold:

(i) (lower semicontinuity inequality) for all sequences (u_j) converging to u in $L^p(\Omega)$ we have that $F(u) \leq \liminf_j F_j(u_j)$;

(ii) (existence of a recovery sequence) there exists a sequence (u_j) converging to u in $L^p(\Omega)$ such that $F(u) = \lim_j F_j(u_j)$.

We say that F_j Γ-converges to F if $F(u) = \Gamma\text{-}\lim_j F_j(u)$ at all points $u \in L^p(\Omega)$ and that F is the Γ-limit of F_j. The main reason for the introduction of this convergence is the following fundamental theorem.

Theorem 1.1.5 *Let $F = \Gamma\text{-}\lim_j F_j$, and let a compact set $K \subset L^p(\Omega)$ exist such that $\inf_{L^p(\Omega)} F_j = \inf_K F_j$ for all j. Then*

$$\exists \min_{L^p(\Omega)} F = \lim_j \inf_{L^p(\Omega)} F_j.$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j)$=$\lim_j \inf_{L^p(\Omega)} F_j$ then its limit is a minimum point for F.

If (F_ε) is a family of functionals indexed by $\varepsilon > 0$ then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_j F_{\varepsilon_j}$ for all (ε_j) converging to 0. If we define the *lower and upper* Γ*-limits* by

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon \to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$
$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, then F_ε Γ-converges to F as $\varepsilon \to 0^+$ if and only if $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous (see [29] Proposition 6.8).

Theorem 1.1.6 (Compactness) *Let $(F_\varepsilon) : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be a family of functionals. Suppose that for every sequence (ε_k) of positive real numbers converging to 0 and for every $u \in W^{1,p}(\Omega)$*

$$F''(U) = \Gamma(L^p)\text{-}\liminf_k F_{\varepsilon_k}(u, U)$$

define an inner regular increasing set function. Then for every sequence (ε_j) of positive real numbers converging to 0 there exists a subsequence (ε_{j_k}) such that the Γ-limit

$$F(u, U) = \Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u, U)$$

exists for all $U \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega)$.

Before stating the next theorem about the Γ-convergence of a family of real quadratic forms, we recall the following pure algebraic proposition which provides a useful characterization for quadratic forms.

Proposition 1.1.7 *Let $F : X \to [0, \infty]$ be an arbitrary function. If*

(i) $F(0) = 0$,

(ii) $F(tx) \leq t^2 F(x)$ *for every* $x \in X$ *and for every* $t > 0$,

(iii) $F(x + y) + F(x - y) \leq 2F(x) + 2F(y)$ *for every* $x, y \in X$,

then F is a quadratic form. Conversely, if F is a quadratic form, then $(i), (ii), (iii)$ are satisfied, and, in addition,

(iv) $F(tx) = t^2 F(x)$ *for every* $x \in X$ *and for every* $t \in \mathbb{R}$ *with* $t \neq 0$,

(v) $F(x + y) + F(x - y) = 2F(x) + 2F(y)$ *for every* $x, y \in X$.

Theorem 1.1.8 *Suppose that (F_ε) Γ-converges to a function F, and that, for every ε, F_ε is a non negative quadratic form. Then F is a non negative quadratic form.*

1.1.3 Integral representation on Sobolev spaces

In this section we recall an integral representation result on Sobolev spaces for functionals defined on pairs function-sets (see [24]).

Theorem 1.1.9 *Let $1 \leq p < \infty$ and let $F : W^{1,p}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be a functional satisfying the following conditions:*

(i) (locality) *F is local, i.e. $F(u, A) = F(v, A)$ if $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$;*

(ii) (measure property) *for all $u \in W^{1,p}(\Omega)$ the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;*

(iii) (growth condition) *there exists $c > 0$ and $a \in L^1(\Omega)$ such that*

$$F(u, A) \leq c \int_A (a(x) + |Du|^p) \, dx$$

for all $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$;

(iv) (translation invariance in u) *$F(u + z, A) = F(u, A)$ for all $z \in \mathbb{R}^d$, $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$;*

(v) (lower semicontinuity) *for all $A \in \mathcal{A}(\Omega)$ $F(\cdot, A)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega)$.*

Then there exists a Carathéodory function $f : \Omega \times \mathbb{M}^{d\times N} \to [0, +\infty)$ satisfying the growth condition

$$0 \leq f(x, M) \leq c(a(x) + |M|^p)$$

for all $x \in \Omega$ and $M \in M^{d\times N}$, such that

$$F(u, A) = \int_A f(x, Du(x)) \, dx$$

for all $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$.
If in addition it holds

(vi) (translation invariance in x)

$$F(Mx, B(y, \varrho)) = F(Mx, B(z, \varrho))$$

for all $M \in M^{d\times N}$, $y, z \in \Omega$, and $\varrho > 0$ such that $B(y, \varrho) \cup B(z, \varrho) \subset \Omega$, then f does not depend on x.

Theorem 1.1.10 (Integral representation of homogeneous functionals) *Let $1 \leq p < \infty$ and let $F : W^{1,p}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$. There exists a quasiconvex function $f : M^{d\times N} \to [0, +\infty)$ satisfying*

$$0 \leq f(x, A) \leq c(a(x) + |A|^p) \quad \forall A \in M^{d\times N}$$

such that the functional F can be represented by

$$F(u, U) = \int_U f(Du(x))\, dx$$

if and only if conditions $(i) - (v)$ of Theorem 1.1.9 hold and in addition

(vi) (translation invariance in x)

$$F(Ax, B(y, \varrho)) = F(Ax, B(z, \varrho))$$

for all $A \in M^{d\times N}$, $y, z \in \Omega$, and $\varrho > 0$ such that $B(y, \varrho) \cup B(z, \varrho) \subset \Omega$.

1.2 Compactness and integral representation

In this section we define the class of discrete energies we are going to consider in the rest of the chapter and we prove a general compactness theorem, asserting that any sequence of energies in this class has a subsequence whose Γ-limit F is an integral functional.

In what follows Ω will denote a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary. We consider the family of functionals $F_\varepsilon : L^p(\Omega) \to [0, +\infty]$ defined as

$$F_\varepsilon(u) = \begin{cases} \sum\limits_{\xi\in\mathbb{Z}^N} \sum\limits_{\alpha\in R_\varepsilon^\xi(\Omega)} \varepsilon^N f_\varepsilon^\xi\left(\alpha, D_\varepsilon^\xi u(\alpha)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2.1}$$

where for any $\xi \in \mathbb{Z}^N$ and $\varepsilon > 0$

$$R_\varepsilon^\xi(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^N : \ [\alpha, \alpha + \varepsilon\xi] \subset \Omega\},$$

$$\mathcal{A}_\varepsilon(\Omega) := \{u : \mathbb{R}^N \to \mathbb{R}^d \ : \ u \text{ constant on } \alpha + [0, \varepsilon)^N \text{ for any } \alpha \in \varepsilon\mathbb{Z}^N \cap \Omega\},$$

and $f_\varepsilon^\xi : (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d \to [0, +\infty)$ is a given function. On f_ε^ξ we make the following assumptions:

$$f_\varepsilon^{e_i}(\alpha, z) \geq c_1(|z|^p - 1) \quad \forall (\alpha, z) \in (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d,\ i \in \{1, \dots, N\} \tag{1.2.2}$$

$$f_\varepsilon^\xi(\alpha, z) \leq C_\varepsilon^\xi(|z|^p + 1) \quad \forall (\alpha, z) \in (\varepsilon\mathbb{Z}^N \cap \Omega) \times \mathbb{R}^d,\ \xi \in \mathbb{Z}^N, \tag{1.2.3}$$

where $c_1 > 0$ and $\{C_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfies:

$$\limsup_{\varepsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi < +\infty; \tag{H1}$$

$$\forall \delta > 0 \quad \exists\, M_\delta > 0 : \quad \limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta. \tag{H2}$$

The main result of this section is stated in the following theorem.

Theorem 1.2.1 (**compactness**) *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (1.2.2), (1.2.3) and let* (H1)-(H2) *hold. Then for every sequence (ε_j) of positive real numbers converging to* 0*, there exists a subsequence (ε_{j_k}) and a Carathéodory function quasiconvex in the second variable $f : \Omega \times \mathbb{R}^{d\times N}$ satisfying*

$$c(|M|^p - 1) \le f(x, M) \le C(|M|^p + 1),$$

with $0 < c < C$, such that $(F_{\varepsilon_{j_k}}(\cdot))$ Γ-converges with respect to the $L^p(\Omega)$-topology to the functional $F : L^p(\Omega) \to [0, +\infty]$ defined as

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.4}$$

Remark 1.2.2 (**quadratic forms**) Under the hypotheses of Theorem 1.2.1, if in addition for any $\xi \in \mathbb{Z}^N$ and $\varepsilon > 0$ $f_\varepsilon^\xi(\alpha, \cdot)$ is a positive quadratic form on $\mathbb{R}^d$, that is

$$f_\varepsilon^\xi(\alpha, z) = \langle A_\varepsilon^\xi(\alpha) z, z\rangle \quad A_\varepsilon^\xi(\alpha) \in \mathcal{M}^{d\times d}_{sym}$$

then, by the properties of Γ-convergence (see [29]), the limit energy density $f(x, \cdot)$ is a quadratic form on $\mathcal{M}^{d\times N}$, that is

$$f(x, M) = A(x)\,(M, M)\,, \quad A(x) \in T_2\mathcal{M}^{d\times N} \tag{1.2.5}$$

where $T_2\mathcal{M}^{d\times N}$ is the vectorial space of all two times covariant tensors on $\mathcal{M}^{d\times N}$.

To prove Theorem 1.2.1 we use a localization technique, which is a standard argument dealing with limits of integral functionals (see for example [16] in the context of homogenization theory). We stress the fact that here this analysis becomes more difficult to perform because of the non locality of our discrete energies.

The first step is to define a "localized" version of our energies: given an open set A we isolate the contributions due to interactions within A as follows. For $u \in \mathcal{A}_\varepsilon(\Omega)$, $A \in \mathcal{A}(\Omega)$ and $\xi \in \mathbb{Z}^N$, set

$$\mathcal{F}_\varepsilon^\xi(u, A) := \sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^N f_\varepsilon^\xi\left(\alpha, D_\varepsilon^\xi u(\alpha)\right), \tag{1.2.6}$$

where

$$R^{\xi}_{\varepsilon}(A) := \{\alpha \in \varepsilon\mathbb{Z}^N : \ [\alpha, \alpha + \varepsilon\xi] \subset A\}.$$

The function $\mathcal{F}^{\xi}_{\varepsilon}$ represents the energy due to the interactions within A along the direction ξ. Then the local version of the functional in (1.2.1) is given by

$$F_{\varepsilon}(u, A) = \begin{cases} \sum_{\xi \in \mathbb{Z}^N} \mathcal{F}^{\xi}_{\varepsilon}(u, A) & \text{if } u \in \mathcal{A}_{\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.7}$$

We will prove also the following result.

Theorem 1.2.3 (**local compactness**) *Let $\{f^{\xi}_{\varepsilon}\}_{\varepsilon,\xi}$ satisfy (1.2.2), (1.2.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to* 0, *let (ε_{j_k}) and f be as in Theorem 1.2.1. Then for any $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$ there holds*

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) = \int_A f(x, \nabla u)\, dx.$$

We will derive the proof of Theorems 1.2.1 and 1.2.3 as a direct consequence of some propositions and lemmas which are fundamental steps to show that our limit functionals satisfy all the hypotheses of the Representation Theorem 1.1.9.

In the next two propositions we show that, thanks to hypotheses (1.2.2) and (1.2.3), the Γ-$\liminf$ and the Γ-$\limsup$ of F_{ε} are finite only on $W^{1,p}(\Omega;\mathbb{R}^d)$ and satisfy standard p-growth conditions.

Proposition 1.2.4 *Let $\{f^{e_i}_{\varepsilon}\}_{\varepsilon,i}$ satisfy (1.2.2). If $u \in L^p(\Omega)$ is such that $F'(u, A) < +\infty$, then $u \in W^{1,p}(A;\mathbb{R}^d)$ and*

$$F'(u, A) \geq c\ \Big(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} - |A| \Big) \tag{1.2.8}$$

for some positive constant c independent on u and A.

Proof. Let $\varepsilon_n \to 0^+$ and let u_n converge to u in $L^p(\Omega)$ and be such that $\liminf_n F_{\varepsilon_n}(u_n, A) < +\infty$. By the growth condition (1.2.2) we get

$$F_{\varepsilon_n}(u_n, A) \geq c_1 \sum_{i=1}^{N} \sum_{\alpha \in R^{e_i}_{\varepsilon_n}(A)} \varepsilon_n^N |D^{e_i}_{\varepsilon_n} u_n(\alpha)|^p - c_1 N |A|.$$

For any $i \in \{1, \ldots, N\}$, consider the sequence of piecewise-affine functions (v^i_n) defined as follows

$$v^i_n(x) := u_n(\alpha) + D^{e_i}_{\varepsilon_n} u_n(\alpha)(x_i - \alpha_i) \quad x \in \big(\alpha + [0, \varepsilon_n)^N\big) \cap \Omega,\ \alpha \in R^{e_i}_{\varepsilon_n}(A).$$

For any $\eta > 0$, set

$$A_{\eta} := \{x \in A : dist(x, A^c) > \eta\}.$$

Then, fixed $\eta > 0$, it is easy to check that $v_n^i \to u$ in $L^p(A_\eta;\mathbb{R}^d)$ for every $i \in \{1,\dots,N\}$. Moreover, set $\frac{\partial v_n^i}{\partial x_i}(x)$ the absolutely continuous part of the Radon measure $D_{x_i} v_n^i$, since $\frac{\partial v_n^i}{\partial x_i}(x) = D_{\varepsilon_n}^{e_i} u_n(\alpha)$ for $x \in \alpha + [0,\varepsilon_n)^N$, we get

$$F_{\varepsilon_n}(u_n, A) \quad \geq \quad c_1 \sum_{i=1}^{N} \int_{A_\eta} \left| \frac{\partial v_n^i}{\partial x_i}(x) \right|^p \, dx - c_1 N|A|. \tag{1.2.9}$$

We apply now a standard slicing argument. By Fubini's Theorem and Fatou's Lemma for any i we get

$$\liminf_n \int_{(A_\eta)} \left| \frac{\partial v_n^i}{\partial x_i}(x) \right|^p \geq \int_{(A_\eta)^{e_i}} \liminf_n \int_{(A_\eta)_y^{e_i}} \left| (v_n^i)'_{e_i,y}(t) \right|^p \, dt \, d\mathcal{H}^{N-1}(y).$$

Since, up to passing to a subsequence, we may assume that, for $\mathcal{H}^{N-1}$-a.e. $y \in (A_\eta)^{e_i}$ $(v_n^i)_{e_i,y} \to u_{e_i,y}$ in $L^p\left((A_\eta)_y^{e_i};\mathbb{R}^d\right)$, we deduce that $u_{e_i,y} \in W^{1,p}\left((A_\eta)_y^{e_i};\mathbb{R}^d\right)$ for $\mathcal{H}^{N-1}$-a.e. $y \in (A_\eta)^{e_i}$ and

$$\liminf_n \int_{(A_\eta)} \left| \frac{\partial v_n^i}{\partial x_i}(x) \right|^p \geq \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left| u'_{e_i,y}(t) \right|^p \, dt \, d\mathcal{H}^{N-1}(y).$$

Then, by (1.2.9), we have

$$\liminf_n F_{\varepsilon_n}(u_n, A) \geq c_1 \sum_{i=1}^{N} \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left| u'_{e_i,y}(t) \right|^p \, dt \, d\mathcal{H}^{N-1}(y) - c_1 N|A|.$$

Since, in particular, the previous inequality implies that

$$\sum_{i=1}^{N} \int_{(A_\eta)^{e_i}} \int_{(A_\eta)_y^{e_i}} \left| u'_{e_i,y}(t) \right|^p \, dt \, d\mathcal{H}^{N-1}(y) < +\infty,$$

thanks to the characterization of $W^{1,p}$ by slicing, we obtain that $u \in W^{1,p}\left(A_\eta;\mathbb{R}^d\right)$ and

$$\begin{aligned} \liminf_n F_{\varepsilon_n}(u_n, A) \quad &\geq \quad c_1 \sum_{i=1}^{N} \int_{A_\eta} \left| \frac{\partial u}{\partial x_i}(x) \right|^p \, dx - c_1 N|A| \\ &\geq \quad c \left(\int_{A_\eta} \|\nabla u(x)\|^p \, dx - |A| \right). \end{aligned}$$

Letting $\eta \to 0^+$, we get the conclusion. $\square$

Proposition 1.2.5 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (1.2.3) and let* (H1) *hold. Then for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ there holds*

$$F''(u, A) \leq C \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A| \right) \tag{1.2.10}$$

for some positive constant C independent on u and A.

Proof. We first show that inequality (1.2.5) holds for u smooth and then we recover the proof for any $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ by using a density argument.

Let $u \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^d)$ and consider the family $(u_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega)$ defined as

$$u_\varepsilon(\alpha) := u(\alpha), \quad \alpha \in \varepsilon\mathbb{Z}^N.$$

Then $u_\varepsilon \to u$ in $L^p(\Omega)$ as $\varepsilon \to 0^+$. Moreover, for any $\alpha \in \varepsilon\mathbb{Z}^N$, we have

$$D_\varepsilon^\xi u_\varepsilon(\alpha) = \frac{1}{|\xi|}\int_0^1 \nabla u(\alpha+\varepsilon\xi s)\xi\, ds,$$

so that, by Jensen's inequality, we get

$$\begin{aligned}
|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p &= \frac{1}{|\xi|^p}\left|\int_0^1 \nabla u(\alpha+\varepsilon\xi s)\xi\, ds\right|^p \\
&\leq \frac{1}{|\xi|^p}\int_0^1 |\nabla u(\alpha+\varepsilon\xi s)\xi|^p\ ds \leq \int_0^1 |\nabla u(\alpha+\varepsilon\xi s)|^p\ ds.
\end{aligned}$$

By the regularity hypothesis on u and by Fubini's Theorem, we easily obtain the following inequalities

$$\begin{aligned}
&\varepsilon^N \int_0^1 |\nabla u(\alpha+\varepsilon\xi s)|^p\ ds = \int_{\alpha+[0,\varepsilon)^N}\int_0^1 |\nabla u(\alpha+\varepsilon\xi s)|^p\ ds\, dx \\
&\leq \int_{\alpha+[0,\varepsilon)^N}\int_0^1 |\nabla u(x+\varepsilon\xi s)|^p\ ds\, dx + c(u)\int_{\alpha+[0,\varepsilon)^N}\int_0^1 |x-\alpha|^p\, ds\, dx \\
&\leq \int_0^1 \int_{\alpha+s\varepsilon\xi+[0,\varepsilon)^N} |\nabla u(x)|^p\ dx\, ds + c(u)\varepsilon^p\varepsilon^N,
\end{aligned}$$

where by $c(u)$ we denote a constant depending only on u. By (1.2.3) and the last inequality, we then have

$$\begin{aligned}
F_\varepsilon(u_\varepsilon, A) &\leq \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(A)} \int_0^1\int_{\alpha+s\varepsilon\xi+[0,\varepsilon)^N} |\nabla u(x)|^p\ dx\, ds \\
&\quad + (1+c(u)\varepsilon^p)\sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^N \\
&\leq \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi \left(\int_{A^\varepsilon} |\nabla u(x)|^p\, dx + (1+c(u)\varepsilon^p)\,|A^\varepsilon|\right),
\end{aligned}$$

where

$$A^\varepsilon := A + [0,\varepsilon)^N.$$

Eventually, letting $\varepsilon \to 0^+$, by (H1) we get

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) \leq C\left(\int_A |\nabla u(x)|^p\, dx + |A|\right)$$

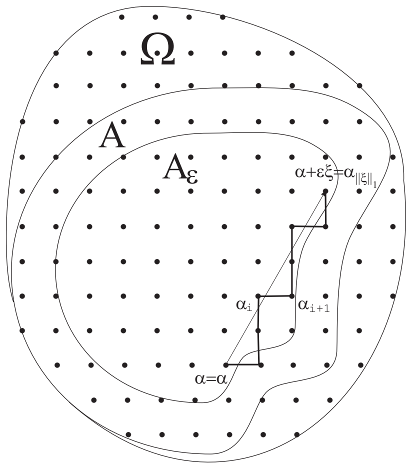

Figure 2

and the conclusion follows by the definition of F''. Now let $u \in W^{1,p}(\Omega)$ and let $(u_n) \subset C_c^\infty(\mathbb{R}^N;\mathbb{R}^d)$ converge to u in the $W^{1,p}(\Omega)$-topology. Then, by the lower-semicontinuity of F'', we obtain

$$\begin{aligned} F''(u,A) &\leq \liminf_n F''(u_n,A) \leq \lim_n C\left(\|\nabla u_n\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A|\right) \\ &= C\left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N})} + |A|\right). \end{aligned}$$

□

The next technical lemma asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions.

Lemma 1.2.6 *Let $A \in \mathcal{A}(\Omega)$ and set $A_\varepsilon := \{x \in A : \ dist(x,\partial A) > 2\sqrt{N}\varepsilon\}$. Then for any $\xi \in \mathbb{Z}^N$ and $u \in \mathcal{A}_\varepsilon(\Omega)$ there holds*

$$\sum_{\alpha \in R^\xi_\varepsilon(A_\varepsilon)} |D^\xi_\varepsilon u(\alpha)|^p \leq C \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_\varepsilon(A)} |D^{e_i}_\varepsilon u(\alpha)|^p. \tag{1.2.11}$$

Proof. Let us fix some notations : for $\xi \in \mathbb{Z}^N$ and $\alpha \in \varepsilon\mathbb{Z}^N$, set

$$I^\xi_\varepsilon(\alpha) := \left\{\beta \in \varepsilon\mathbb{Z}^N : \ \left(\beta + [-\varepsilon,\varepsilon]^N\right) \cap [\alpha, \alpha+\varepsilon\xi] \neq \emptyset\right\};$$

moreover we will denote by $\|\cdot\|_1$ the norm on $\mathbb{R}^N$ defined as

$$\|\xi\|_1 := \sum_{i=1}^N |\xi_i|, \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N.$$

Let $\alpha \in R_\varepsilon^\xi(A_\varepsilon)$ and consider $\{\alpha_h\}_{h=1}^{\|\xi\|_1} \subset I_\varepsilon^\xi(\alpha)$ such that

$$\alpha_{\|\xi\|_1} = \alpha + \varepsilon\xi, \quad \alpha_1 = \alpha, \quad \alpha_h = \alpha_{h-1} + \varepsilon e_{i(h)},$$

for some $i(h) \in \{1, \dots, N\}$ (see fig. 2). Then, since

$$D_\varepsilon^\xi u(\alpha) = \frac{1}{|\xi|} \sum_{h=1}^{\|\xi\|_1} D_\varepsilon^{e_{i(h)}} u(\alpha_h)$$

by Jensen's inequality, we get

$$\begin{aligned} |D_\varepsilon^\xi u(\alpha)|^p &= \left(\frac{\|\xi\|_1}{|\xi|}\right)^p \left| \frac{1}{\|\xi\|_1} \sum_{h=1}^{\|\xi\|_1} D_\varepsilon^{e_{i(h)}} u(\alpha_h) \right|^p \\ &\leq \left(\frac{\|\xi\|_1}{|\xi|}\right)^p \frac{1}{\|\xi\|_1} \sum_{h=1}^{\|\xi\|_1} |D_\varepsilon^{e_{i(h)}} u(\alpha_h)|^p. \end{aligned}$$

Since for any $h = 1, \dots, N$, $\alpha_h \in R_\varepsilon^{e_{i(h)}}(A)$ and all the norms are equivalent in a finite dimensional space, we infer that

$$\sum_{\alpha \in R_\varepsilon^\xi(A_\varepsilon)} |D_\varepsilon^\xi u(\alpha)|^p \leq C \sum_{i=1}^N \sum_{\beta \in R_\varepsilon^{e_i}(A)} \frac{\gamma_\varepsilon^\xi(\beta)}{\|\xi\|_1} |D_\varepsilon^{e_i} u(\beta)|^p,$$

where

$$\gamma_\varepsilon^\xi(\beta) := \#\{\alpha \in R_\varepsilon^\xi(A_\varepsilon) : \ \beta \in I_\varepsilon^\xi(\alpha)\}.$$

Hence, the proof is complete if we show that $\gamma_\varepsilon^\xi(\beta) \leq C|\xi|$. To this aim, notice that

$$\{\alpha \in R_\varepsilon^\xi(A_\varepsilon) : \ \beta \in I_\varepsilon^\xi(\alpha)\} \subseteq \varepsilon\mathbb{Z}^N \cap Q_\varepsilon^\xi(\beta),$$

where

$$Q_\varepsilon^\xi(\beta) := \{x \in \mathbb{R}^N : x = y + t\xi, \ y \in \beta + [-\varepsilon, \varepsilon]^n, \ t \in [-\varepsilon, \varepsilon]^N\}.$$

Thus, we infer that

$$\gamma_\varepsilon^\xi(\beta) \leq C \frac{|Q_\varepsilon^\xi(\beta)|}{\varepsilon^N}.$$

Now we use a slicing argument to provide an estimate of $\left|Q_\varepsilon^\xi(\beta)\right|$. By Fubini's Theorem, we get

$$\begin{aligned}|Q_\varepsilon^\xi(\beta)| &= \int_{\left(Q_\varepsilon^\xi(\beta)\right)^\xi} \mathcal{H}^1\left(Q_\varepsilon^\xi(\beta)\right)_y^\xi \, d\mathcal{H}^{N-1}(y)\\ &\leq \mathcal{H}^{N-1}\left((Q_\varepsilon^\xi(\beta))^\xi\right) 2(\sqrt{N}+|\xi|)\varepsilon \leq c(N)|\xi|\varepsilon^N,\end{aligned}$$

where the last inequality holds, since for any $\xi \in \mathbb{Z}^N$

$$\mathcal{H}^{N-1}\left((Q_\varepsilon^\xi(\beta))^\xi\right) \leq c(N)\varepsilon^{N-1}.$$

□

In the next two propositions we establish the subadditivity and the inner regularity of the set function $F''(u,\cdot)$. To this end we use a careful modification of De Giorgi's cut-off functions argument, which appears frequently in the proof of the integral representation of Γ-limits of integral functionals (see [16],[29]). We underline that the non locality of our energies requires a deeper analysis in which a key role is played by hypothesis (H2), which allows us to show that very long range interactions do not lead to non local terms in the limit.

Proposition 1.2.7 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (1.2.2),(1.2.3) and let* (H1)-(H2) *hold. Let $A, B \in \mathcal{A}(\Omega)$ and let $A', B' \in \mathcal{A}(\Omega)$ be such that $A' \subset\subset A$ and $B' \subset\subset B$. Then for any $u \in W^{1,p}(\Omega)$,*

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, B).$$

Proof. Without loss of generality, we may suppose $F''(u,A)$ and $F''(u,B)$ finite. Let $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ both converge to u in $L^p(\Omega)$ and be such that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u,A), \quad \limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, B) = F''(u,B).$$

By (1.2.2) and Lemma 1.2.6, we infer that

$$\sup_{\xi\in\mathbb{Z}^N}\sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi(A_\varepsilon)} \varepsilon^N |D_\varepsilon^\xi u_\varepsilon(\alpha)|^p < +\infty, \tag{1.2.12}$$

$$\sup_{\xi\in\mathbb{Z}^N}\sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi(B_\varepsilon)} \varepsilon^N |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p < +\infty, \tag{1.2.13}$$

where A_ε and B_ε are defined as in Lemma 1.2.6. Moreover, since (u_ε) and (v_ε) converge to u in the $L^p(\Omega)$-topology, we have

$$\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap\Omega'} \varepsilon^N\left(|u_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha)|^p\right) \leq \|u_\varepsilon\|^p_{L^p(\Omega)} + \|v_\varepsilon\|^p_{L^p(\Omega)} \leq C < +\infty, \tag{1.2.14}$$

$$\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap\Omega'} \varepsilon^N \left(|u_\varepsilon(\alpha)-v_\varepsilon(\alpha)|^p\right) \le \|u_\varepsilon - v_\varepsilon\|^p_{L^p(\Omega)} \to 0^+ \tag{1.2.15}$$

for any $\Omega' \subset\subset \Omega$. Set

$$d := \mathrm{dist}\,(A', A^c)$$

and for any $i \in \{1,\dots,N\}$ define

$$A_i := \{x \in A :\ \mathrm{dist}(x, A') < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \le 2\frac{N}{d}$. Then for any $i \in \{1,\dots,N\}$ consider the family of functions $w^i_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ still converging to u in $L^p(\Omega)$ defined as

$$w^i_\varepsilon(\alpha) := \varphi_i(\alpha)u_\varepsilon(\alpha) + (1-\varphi_i(\alpha))\, v_\varepsilon(\alpha).$$

Note that, for any $\xi \in \mathbb{Z}^N$, we have

$$\begin{aligned} D^\xi_\varepsilon w^i_\varepsilon(\alpha) = \varphi_i(\alpha+\varepsilon\xi) D^\xi_\varepsilon u_\varepsilon(\alpha) \quad &+ \quad (1-\varphi_i(\alpha+\varepsilon\xi))\, D^\xi_\varepsilon v_\varepsilon(\alpha) \\ &+ (u_\varepsilon(\alpha)-v_\varepsilon(\alpha))\, D^\xi_\varepsilon \varphi(\alpha). \end{aligned} \tag{1.2.16}$$

Fix $i \in \{1, 2, \dots, N-3\}$. Given $\xi \in \mathbb{Z}^N$ and $\alpha \in R^\xi_\varepsilon(A' \cup B')$, then either $\alpha \in R^\xi_\varepsilon(A_i)$, or $\alpha \in R^\xi_\varepsilon(\overline{A}^c_{i+1} \cap B')$, or

$$[\alpha, \alpha+\varepsilon\xi] \cap \left(\overline{A}_{i+1} \setminus A_i\right) \cap B' \ne \emptyset.$$

Then, if we set

$$\left(\overline{A}_{i+1} \setminus A_i\right)^{\varepsilon,\xi} := \{x = y + t\xi,\ |t| \le \varepsilon,\ y \in \overline{A}_{i+1} \setminus A_i\},$$

$$S^{\varepsilon,\xi}_i := \left(\overline{A}_{i+1} \setminus A_i\right)^{\varepsilon,\xi} \cap (A' \cup B'),$$

we get

$$R^\xi_\varepsilon(A' \cup B') \subseteq R^\xi_\varepsilon(A_i) \cup R^\xi_\varepsilon(B' \setminus \overline{A}_{i+1}) \cup R^\xi_\varepsilon\left(S^{\varepsilon,\xi}_i\right)$$

(see fig. 3). Thus, since $D^\xi_\varepsilon w^i_\varepsilon(\alpha) = D^\xi_\varepsilon u_\varepsilon(\alpha)$ if $\alpha \in R^\xi_\varepsilon(A_i)$ and $D^\xi_\varepsilon w^i_\varepsilon(\alpha) = D^\xi_\varepsilon v_\varepsilon(\alpha)$ if $\alpha \in R^\xi_\varepsilon(\overline{A}^c_{i+1} \cap B')$, we get by (1.2.3) and (1.2.16)

$$\begin{aligned} \mathcal{F}^\xi_\varepsilon(\, w^i_\varepsilon, A' \cup B') &\le \mathcal{F}^\xi_\varepsilon(u_\varepsilon, A) + \mathcal{F}^\xi_\varepsilon(v_\varepsilon, B) \\ &+ \ C\, C^\xi_\varepsilon \sum_{\alpha\in R^\xi_\varepsilon(S^{\varepsilon,\xi}_i)} \varepsilon^N \left(|D^\xi_\varepsilon u_\varepsilon(\alpha)|^p + |D^\xi_\varepsilon v_\varepsilon(\alpha)|^p + N^p\, |u_\varepsilon(\alpha)-v_\varepsilon(\alpha)|^p + 1\right). \end{aligned} \tag{1.2.17}$$

If $\varepsilon|\xi| \le \frac{d}{2N}$, then

$$S^{\varepsilon,\xi}_i \subseteq \left(A_{N-1} \setminus \overline{A'}\right) \cap B' =: S_N \subset\subset A \cap B. \tag{1.2.18}$$

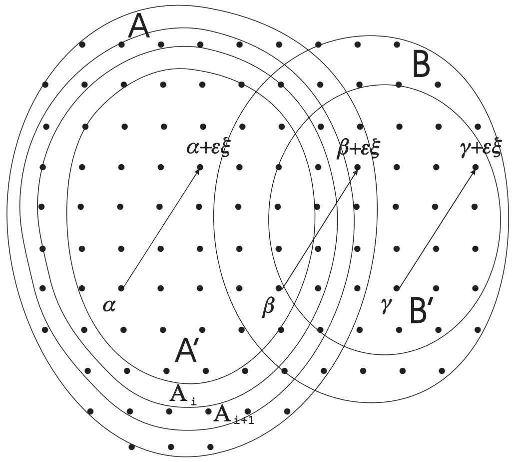

Figure 3 : $\alpha \in R_\varepsilon^\xi(A_i), \beta \in R_\varepsilon^\xi(S_i^{\varepsilon,\xi}), \gamma \in R_\varepsilon^\xi(B' \setminus A_{i+1}^c)$

If $\varepsilon|\xi| \geq \frac{d}{2N}$, then

$$\frac{1}{\varepsilon^p|\xi|^p} \leq \frac{2^p N^p}{d^p},$$

and so

$$|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p \leq CN^p \left(|u_\varepsilon(\alpha)|^p + |u_\varepsilon(\alpha+\varepsilon\xi)|^p\right),$$

and the same inequality holds for v_ε. Thus, in this case we get by (1.2.17)

$$\begin{aligned} &\mathcal{F}_\varepsilon^\xi\left(w_\varepsilon^i, A' \cup B'\right) \leq \mathcal{F}_\varepsilon^\xi(u_\varepsilon, A) + \mathcal{F}_\varepsilon^\xi(v_\varepsilon, B) \\ &+ \; CN^p \, C_\varepsilon^\xi \sum_{\alpha \in R_\varepsilon^\xi(A' \cup B')} \varepsilon^N \left(|u_\varepsilon(\alpha)|^p + |u_\varepsilon(\alpha+\varepsilon\xi)|^p + |v_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha+\varepsilon\xi)|^p + 1\right). \end{aligned} \tag{1.2.19}$$

Let $M_\delta > 0$ be such that $\limsup\limits_{\varepsilon\to 0^+} \sum\limits_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta$. Then, by (1.2.17), (1.2.18) and (1.2.19), summing over $\xi \in \mathbb{Z}^N$,for ε small enough we get

$$\begin{aligned} &F_\varepsilon\left(w_\varepsilon^i, A' \cup B'\right) \leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\ &\quad + C \sum_{|\xi|\leq M_\delta} C_\varepsilon^\xi \sum_{\alpha \in R_\varepsilon^\xi(S_i^{\varepsilon,\xi})} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\,|u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right) \\ &\quad + C \sum_{M_\delta<|\xi|\leq \frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha \in R_\varepsilon^\xi(S_N)} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\,|u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right) \\ &\quad + CN^p \sum_{|\xi|>\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha \in \varepsilon\mathbb{Z}^N \cap A' \cup B'} \varepsilon^N \left(|u_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha)|^p + 1\right). \end{aligned}$$

Note that, for ε small enough and $|\xi| \leq M_\delta$ we have that $R^\xi_\varepsilon(S^{\varepsilon,\xi}_i) \cap R^\xi_\varepsilon(S^{\varepsilon,\xi}_j) \neq \emptyset$ if and only if $|i-j| = 1$, and $\bigcup_{i=1}^{N-3} R^\xi_\varepsilon(S^{\varepsilon,\xi}_i) \subseteq R^\xi_\varepsilon(A_\varepsilon \cap B_\varepsilon)$. Thus, summing over $i \in \{1, 2, \ldots, N-3\}$, averaging, and taking into account (1.2.12), (1.2.13), (1.2.14) and (1.2.15), we get

$$
\begin{aligned}
\frac{1}{N-3}\sum_{i=1}^{N-3} F_\varepsilon(w^i_\varepsilon, A' \cup B') \leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \qquad (1.2.20)\\
+\frac{C}{N-3}\,(1 + N^p O(\varepsilon)) + C(\delta + O(\varepsilon))(1 + N^p O(\varepsilon))\\
+C\,(\delta + O(\varepsilon))(N^p)
\end{aligned}
$$

For any $\varepsilon > 0$ there exists $i(\varepsilon) \in \{1, \ldots, N-3\}$ such that

$$
F_\varepsilon(w^{i(\varepsilon)}_\varepsilon, A' \cup B') \leq \frac{1}{N-3}\sum_{i=1}^{N-3} F_\varepsilon(w^i_\varepsilon, A' \cup B'). \qquad (1.2.21)
$$

Then, since $w^{i(\varepsilon)}_\varepsilon$ still converges to u in $L^p(\Omega)$, by (1.2.20) and (1.2.21), letting $\varepsilon \to 0^+$, we get

$$
F''(u, A' \cup B') \leq F''(u, A) + F''(u, B) + \frac{C}{N-3} + C\delta\,(1 + N^p)\,.
$$

Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain the thesis. □

Proposition 1.2.8 *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (1.2.2),(1.2.3) and let* (H1)-(H2) *hold. Then for any $u \in W^{1,p}(\Omega)$ and for any $A \in \mathcal{A}(\Omega)$, there holds*

$$
\sup_{A' \subset\subset A} F''(u, A') = F''(u, A).
$$

Proof. Since $F''(u, \cdot)$ is an increasing set function, it suffices to prove that

$$
\sup_{A' \subset\subset A} F''(u, A') \geq F''(u, A).
$$

To do this, we apply the same argument of the proof of Proposition 1.2.7. Given $\delta > 0$, there exists $A'' \subset\subset A$ such that

$$
|A \setminus \overline{A''}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A''})} \leq \delta.
$$

Let $\tilde{\Omega} \supset\supset \Omega$ and let $\tilde{u} \in W^{1,p}(\tilde{\Omega}; \mathbb{R}^d)$ an extension of u. By reasoning as in the proof of Proposition 1.2.5, we may find $v_\varepsilon \in \mathcal{A}_\varepsilon(\tilde{\Omega})$ such that v_ε converges to $\tilde{u}$ in $L^p(\tilde{\Omega}; \mathbb{R}^d)$ and

$$
\limsup_{\varepsilon \to 0^+} F_\varepsilon(v_\varepsilon, A \setminus \overline{A''}) \leq C\left(|A \setminus \overline{A''}| + \|\nabla u\|^p_{L^p(A \setminus \overline{A''})}\right) \leq C\delta. \qquad (1.2.22)
$$

We remark that this extension on $\tilde{\Omega}$ is just a technical tool to exploit an analogue of inequality (1.2.14) and obtain a control of the interactions near the boundary of Ω. Let $A' \in \mathcal{A}(\Omega)$ be such that $A'' \subset\subset A' \subset\subset A$ and let $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ converge to u in $L^p(\Omega)$, with

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A') = F''(u, A').$$

Set

$$d := \operatorname{dist}(A'', A'^c)$$

and for any $i \in \{1, \ldots, N\}$ define

$$A_i := \{x \in A : \ \operatorname{dist}(x, A') < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \le 2\frac{N}{d}$. Then for any $i \in \{1, \ldots, N\}$ consider the family of functions $w_\varepsilon^i \in \mathcal{A}_\varepsilon(\Omega)$ still converging to u in $L^p(\Omega)$ defined as

$$w_\varepsilon^i(\alpha) := \varphi_i(\alpha)u_\varepsilon(\alpha) + (1 - \varphi_i(\alpha))\, v_\varepsilon(\alpha).$$

Now we can set

$$S_i^{\varepsilon,\xi} := \left(\overline{A}_{i+1} \setminus A_i\right)^{\varepsilon,\xi} \cap A,$$

so that

$$R_\varepsilon^\xi(A) \subseteq R_\varepsilon^\xi(A_i) \cup R_\varepsilon^\xi(A \setminus \overline{A}_{i+1}) \cup R_\varepsilon^\xi\left(S_i^{\varepsilon,\xi}\right).$$

Let $\delta > 0$ and let $M_\delta > 0$ be such that $\limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta$. Then, by reasoning as in the proof of Proposition 1.2.7,for ε small enough, we get

$$\begin{aligned}
&F_\varepsilon\left(w_\varepsilon^i, A\right) \le F_\varepsilon(u_\varepsilon, A') + F_\varepsilon(v_\varepsilon, A \setminus \overline{A''}) \\
&\quad + C \sum_{|\xi|\le M_\delta} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_i^{\varepsilon,\xi})} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\, |u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right) \\
&\quad + C \sum_{M_\delta<|\xi|\le\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \sum_{\alpha\in R_\varepsilon^\xi(S_N)} \varepsilon^N \left(|D_\varepsilon^\xi u_\varepsilon(\alpha)|^p + |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p + N^p\, |u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p + 1\right) \\
&\quad + CN^p \sum_{|\xi|>\frac{d}{2N\varepsilon}} C_\varepsilon^\xi \left(\|u_\varepsilon\|^p_{L^p(\Omega)} + \|v_\varepsilon\|^p_{L^p(\tilde{\Omega};\mathbb{R}^d)} + 1\right).
\end{aligned}$$

Since u_ε and v_ε satisfy (1.2.12), (1.2.13), (1.2.14) and (1.2.15) with A_ε replaced by A'_ε and B_ε by $(A \setminus \overline{A''_\varepsilon})$ then we can choose $i(\varepsilon) \in \{1, \ldots, N-3\}$ such that

$$\begin{aligned}
&F_\varepsilon(w_\varepsilon^{i(\varepsilon)}, A) \le \frac{1}{N-3} \sum_{i=1}^{N-3} F_\varepsilon(w_\varepsilon^i, A) \le \\
&F_\varepsilon(u_\varepsilon, A') + C\delta + \frac{C}{N-3}\,(1 + N^p O(\varepsilon)) + \\
&C(\delta + O(\varepsilon))\,(1 + N^p O(\varepsilon)) + CN^p(\delta + O(\varepsilon)).
\end{aligned} \tag{1.2.23}$$

Then, since $w_\varepsilon^{i(\varepsilon)}$ still converges to u in $L^p(\Omega)$, by (1.2.23), letting $\varepsilon \to 0^+$, we get

$$F''(u,A) \le \sup_{A' \subset\subset A} F''(u,A') + C(\frac{1}{N-3} + \delta + \delta N^p).$$

Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain the thesis. □

The following proposition asserts that $F''(\cdot,\cdot)$ satisfies hypothesis *(i)* of Theorem 1.1.9. The argument we use for the proof is still the same exploited in the last two propositions.

Proposition 1.2.9 *Let* $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ *satisfy (1.2.2),(1.2.3) and let* (H1)-(H2) *hold. Then for any* $A \in \mathcal{A}(\Omega)$ *and for any* $u,\ v \in W^{1,p}(\Omega)$*, such that* $u = v$ *a.e. there holds*

$$F''(u,A) = F''(v,A).$$

Proof. Thanks to Proposition 1.2.8, we may assume that $A \in \mathcal{A}_0(\Omega)$. We first prove

$$F''(u,A) \ge F''(v,A). \tag{1.2.24}$$

Once more we apply the argument used in the previous proposition. Given $\delta > 0$, there exists $A_\delta \subset\subset A$ such that

$$|A \setminus \overline{A_\delta}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A_\delta})} \le \delta.$$

Let $v_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ and $u_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ be such that

$$v_\varepsilon \to v \text{ in } L^p(\Omega;\mathbb{R}^d) \tag{1.2.25}$$

$$u_\varepsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^d) \tag{1.2.26}$$

and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u,A),$$

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, A\setminus\overline{A_\delta}) = F''(v, A\setminus\overline{A_\delta}) \le C\left(|A\setminus\overline{A_\delta}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A_\delta})}\right) \le C\delta \tag{1.2.27}$$

Set

$$d := \operatorname{dist}(A_\delta, A^c)$$

and for any $i \in \{1,\dots,N\}$ define

$$A_i := \{x \in A:\ \operatorname{dist}(x, A_\delta) < i\frac{d}{N}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \le 2\frac{N}{d}$. Then for any $i \in \{1,\dots,N\}$ consider the family of functions $w_\varepsilon^i \in \mathcal{A}_\varepsilon(\Omega)$ converging to v in $L^p(\Omega)$ defined as

$$w_\varepsilon^i(\alpha) := \varphi_i(\alpha)u_\varepsilon(\alpha) + (1-\varphi_i(\alpha))\,v_\varepsilon(\alpha).$$

Then, following the same steps as in the proof of Propositions 1.2.7 and 1.2.8, we can choose $i(\varepsilon) \in \{1, \ldots, N-3\}$ such that

$$\begin{aligned} F_\varepsilon(w_\varepsilon^{i(\varepsilon)}, A) &\leq \frac{1}{N-3} \sum_{i=1}^{N-3} F_\varepsilon(w_\varepsilon^i, A) \qquad (1.2.28) \\ &\leq F_\varepsilon(u_\varepsilon, A) + C\delta + \frac{C}{N-3}\,(1 + N^p O(\varepsilon)) \\ &+ C(\delta + O(\varepsilon))\,(1 + N^p O(\varepsilon)) + C(\delta + O(\varepsilon))N^p. \end{aligned}$$

Then, since $w_\varepsilon^{i(\varepsilon)}$ still converges to v in $L^p(\Omega)$, by (1.2.28), letting $\varepsilon \to 0^+$, we get

$$F''(v, A) \leq F''(u, A) + C\left(\frac{1}{N-3} + \delta + \delta N^p\right).$$

Eventually, letting first $\delta \to 0^+$ and then $N \to +\infty$, we obtain 1.2.24. Reversing the roles of u and v we obtain the thesis. $\square$

Proof of Theorems 1.2.1 and 1.2.3. By the compactness property of the Γ-convergence and by Proposition 1.2.8, there exists a subsequence (ε_{j_k}) such that, for any $(u, A) \in W^{1,p}(\Omega) \times \mathcal{A}(\Omega)$, there holds

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) := F(u, A)$$

(see [16] Theorem 10.3). Moreover, by Proposition 1.2.4,

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u) = +\infty$$

for $u \in L^p(\Omega) \setminus W^{1,p}(\Omega)$. So far, it suffices to check that, for every $(u, A) \in W^{1,p}(\Omega) \times \mathcal{A}(\Omega)$, $F(u, A)$ satisfies all the hypotheses of Theorem 1.1.9. In fact, it can be easily seen that the superadditivity property of $F_\varepsilon(u, \cdot)$ is conserved in the limit. Thus, as an easy consequence of Propositions 1.2.5, 1.2.7, 1.2.8, 1.2.9 and thanks to De Giorgi - Letta Criterion (see [16]), hypotheses (i), (ii), (iii) hold true. Moreover, as $F_\varepsilon(u, A)$ depends on u only through its difference quotients, hypothesis (iv) is satisfied and finally, by the lower semicontinuity property of Γ-limit, also hypothesis (v) is fulfilled. $\square$

1.2.1 Convergence of minimum problems

In order to treat minimum problems with boundary data, we also derive a compactness theorem in the case that our functionals are subject to Dirichlet boundary conditions.

Given $\varphi \in Lip\ (\mathbb{R}^N)$ and $l \in \mathbb{N}$, set, for any $\varepsilon > 0$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{A}^l_{\varepsilon,\varphi}(A) := \{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N) : u(\alpha) = \varphi(\alpha) \text{ if } \big(\alpha + [-l\varepsilon, l\varepsilon]^N\big) \cap A^c \neq \emptyset\big)\}. \quad (1.2.29)$$

Then define $F_\varepsilon^{\varphi,l} : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as

$$F_\varepsilon^{\varphi,l}(u, A) = \begin{cases} F_\varepsilon(u, A) & \text{if } u \in \mathcal{A}^l_{\varepsilon,\varphi}(A) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.30}$$

By simplicity of notation we set $\mathcal{A}_{\varepsilon,\varphi}(A) := \mathcal{A}^1_{\varepsilon,\varphi}(A)$ and $F_\varepsilon^\varphi := F_\varepsilon^{\varphi,1}$.

Theorem 1.2.10 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (1.2.2), (1.2.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to 0, let (ε_{j_k}) and f be as in Theorem 1.2.1. For any $\varphi \in Lip\ (\mathbb{R}^N)$, let $F^\varphi : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be defined as*

$$F^\varphi(u, A) = \begin{cases} \displaystyle\int_A f(x, \nabla u)\, dx & \textit{if } u - \varphi \in W_0^{1,p}(A; \mathbb{R}^d) \\ \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, for any $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary and $l \in \mathbb{N}$, $(F^{\varphi,l}_{\varepsilon_{j_k}}(\cdot, A))$ Γ-converges with respect to the $L^p(\Omega)$-topology to the functional $F^\varphi(\cdot, A)$.

Proof. For the sake of simplicity we prove the Theorem with $l = 1$, the proof being the same in the other cases. Let us first prove the Γ-liminf inequality. Let (u_k) be a sequence of functions belonging to $\mathcal{A}_{\varepsilon_{j_k},\varphi}(A)$ converging to u in the L^p topology such that

$$\liminf_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) = \lim_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) < +\infty.$$

Then, from (1.2.2), we get in particular that

$$\sup_k \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_{\varepsilon_{j_k}}(A)} \varepsilon^N_{j_k} |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty. \tag{1.2.31}$$

Thanks to the boundary conditions on u_k it is easy to deduce that

$$\sup_k \sum_{i=1}^N \sum_{\alpha \in R^{e_i}_{\varepsilon_{j_k}}(\Omega)} \varepsilon^N |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty.$$

Then, as in the proof of Proposition 1.2.4, we can prove that $u \in W^{1,p}(\Omega)$ and, since (u_k) converge to φ in $L^p(\Omega \setminus A; \mathbb{R}^d)$, we get that $u - \varphi \in W_0^{1,p}(A; \mathbb{R}^d)$. By Theorem 1.2.3 one has

$$\liminf_k F^\varphi_{\varepsilon_{j_k}}(u_k, A) = \liminf_k F_{\varepsilon_{j_k}}(u_k, A) \geq F^\varphi(u, A).$$

To prove the Γ-limsup inequality, let us first consider $u \in W^{1,p}(\Omega)$ such that supp $(u-\varphi) \subset\subset A$. Let $u_k \in \mathcal{A}_{\varepsilon_{j_k}}(\Omega)$, be such that (u_k) converges to u in $L^p(\Omega)$ and

$$\limsup_k F_{\varepsilon_{j_k}}(u_k, A) = F^{\varphi}(u, A).$$

Then, by reasoning as in the proof of Proposition 1.2.8, given $\delta > 0$, we can find suitable cut-off functions ϕ_k with supp $(u-\varphi) \subset\subset$ supp $\phi_k \subset\subset A$ such that, set

$$v_k(\alpha) := \phi_k(\alpha)u_k(\alpha) + (1-\phi_k(\alpha))\varphi(\alpha),$$

then (v_k) still converges to u in $L^p(\Omega)$, $v_k \in \mathcal{A}_{\varepsilon_{j_k},\varphi}(\Omega)$ for k large enough and

$$\limsup_k F_{\varepsilon_{j_k}}(v_k, A) \leq \limsup F_{\varepsilon_{j_k}}(u_k, A) + \delta.$$

Thus, thanks to the definition of Γ-limsup we have

$$\Gamma\text{-limsup} F^{\varphi}_{\varepsilon_{j_k}}(u, A) \leq F^{\varphi}(u, A) + \delta.$$

By the arbitrariness of δ, we obtain the required inequality. In the general case the thesis follows by a density argument, thanks to the lower semicontinuity of Γ-limsup and to the continuity of F with respect to the strong convergence in $W^{1,p}(\Omega)$. □

As a consequence of the previous theorem we derive the following result about the convergence of minimum problems with boundary data.

Corollary 1.2.11 *Under the hypotheses of Theorem 1.2.10 we get that, for any $\varphi \in Lip\ (\mathbb{R}^N)$, $l \in \mathbb{N}$ and $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary*

$$\lim_k \inf\{F_{\varepsilon_{j_k}}(u, A) :\ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}\} = \min\{F(u, A) :\ u - \varphi \in W^{1,p}_0(A; \mathbb{R}^d)\}.$$

Moreover, if (u_k) is a converging sequence such that

$$\lim_k F_{\varepsilon_{j_k}}(u_k, A) = \lim_k \inf\{F_{\varepsilon_{j_k}}(u, A) :\ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}\},$$

then its limit is a minimizer for $\min\{F(u, A) :\ u - \varphi \in W^{1,p}_0(A; \mathbb{R}^d)\}$.

Proof. Let (u_k) be a sequence such that $F_{\varepsilon_{j_k}}(u_k, A) < +\infty$. Then, by (1.2.2) and by the boundary conditions on u_k, it is easy to show that

$$\sup_n \sum_{i=1}^{N} \sum_{\alpha \in \varepsilon_n \mathbb{Z}^N \cap K} \varepsilon^N |D^{e_i}_{\varepsilon_{j_k}} u_k(\alpha)|^p < +\infty,$$

for any compact set K of $\mathbb{R}^N$. By virtue of this property, up to passing to a continuous extension of u_k vanishing outside a bounded open set containing Ω, we get

$$\lim_{|h|\to 0} \sup_k \|\tau_h u_k - u_k\|_{L^p(\mathbb{R}^N;\mathbb{R}^d)} = 0,$$

where we have set

$$(\tau_h u)(x) := u(x+h),\ x \in \mathbb{R}^N,\ h \in \mathbb{R}^N.$$

Then, by Frechét-Kolmogorov Theorem, there exists a subsequence (u_{k_n}) converging in $L^p(\Omega)$ to a function $u \in L^p(\Omega)$. Arguing as in the previous proof it is easy to show that $u - \varphi \in W_0^{1,p}(\Omega)$. The thesis follows thanks to Theorem 1.2.10 and Theorem 1.1.5. □

We can also derive the analogue of Theorem 1.2.10 and Corollary 1.2.11 about the convergence of minimum problems with periodic conditions.

Let $\mathcal{Q}(\Omega)$ be the family of all open N-cubes contained in Ω. For any $\varepsilon > 0$, $r > 0$, $Q = (x_0, x_0 + r)^N \in \mathcal{Q}(\Omega)$ and $\varphi \in Lip\ (\mathbb{R}^N)$, set

$$r_\varepsilon = \varepsilon\left(\left[\frac{r}{\varepsilon}\right] - 2\right),$$

$$\mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) = \left\{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N) :\ u - \hat{\varphi}\ \ r_\varepsilon - \text{periodic}\right\},$$

where $\hat{\varphi} \in \mathcal{A}_\varepsilon(\mathbb{R}^N),\ \hat{\varphi}(\alpha) = \varphi(\alpha)$ for any $\alpha \in \varepsilon\mathbb{Z}^N$. Then define $F_\varepsilon^{\varphi,\#} : L^p(\Omega) \times \mathcal{Q}(\Omega) \to [0, +\infty]$ as

$$F_\varepsilon^{\varphi,\#}(u, Q) = \begin{cases} F_\varepsilon(u, Q) & \text{if } u \in \mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.32}$$

Theorem 1.2.12 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (1.2.2), (1.2.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to* 0, *let (ε_{j_k}) and f be as in Theorem 1.2.1. Then, for any $\varphi \in Lip\ (\mathbb{R}^N)$, let $F^{\#} : L^p(\Omega) \times \mathcal{Q}(\Omega) \to [0, +\infty]$ be defined as*

$$F^{\varphi,\#}(u, Q) = \begin{cases} \displaystyle\int_Q f(x, \nabla u)\, dx & \text{if } u \in W^{1,p}_{\#}(Q; \mathbb{R}^d) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for any $Q \in \mathcal{Q}(\Omega)$, $(F^{\varphi,\#}_{\varepsilon_{j_k}}(\cdot, Q))$ Γ-converges with respect to the $L^p(\Omega)$-topology to the functional $F^{\varphi,\#}(u, Q)$.

Proof. To prove the Γ-liminf inequality, let (u_k) be a sequence of functions belonging to $\mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)$ converging to u in the L^p topology such that

$$\liminf_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k, Q) = \lim_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k, Q) < +\infty.$$

Then, arguing as in the proof of Theorem 1.2.10 and observing that $r_\varepsilon \to r$, we can conclude that $u - \varphi \in W^{1,p}_{\#}(Q; \mathbb{R}^d)$ and

$$\liminf_k F^{\varphi,\#}_{\varepsilon_{j_k}}(u_k, Q) \geq F^{\varphi,\#}(u, Q).$$

By a density argument it suffices to prove the Γ-limsup inequality for u such that $u-\varphi \in W^{1,\infty}_{\#}(Q';\mathbb{R}^d)$ for any open N-cube Q' such that $(x_0+\delta, x_0+r-\delta) \subseteq Q' \subseteq Q$ for some $\delta > 0$. Note that, for such a u, $\mathcal{A}_{\varepsilon_{j_k},u} \subseteq \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}$ for k large enough. Then the existence of a recovery sequence is assured by Theorem 1.2.10. $\square$

As a consequence of the previous theorem, by reasoning as in the proof of Corollary 1.2.11 one can prove the following result.

Corollary 1.2.13 *Under the hypotheses of Theorem 1.2.12 we get that, for any* $\varphi \in Lip\ (\mathbb{R}^N)$ *and* $Q \in \mathcal{Q}(\Omega)$

$$\lim_k \inf\{F_{\varepsilon_{j_k}}(u,Q) :\ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)\} = \min\{F(u,Q) :\ u-\varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}.$$

Moreover, if (u_k) *is a converging sequence such that*

$$\lim_k F_{\varepsilon_{j_k}}(u_k,Q) = \lim_k \inf\{F_{\varepsilon_{j_k}}(u,Q) :\ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}\},$$

then its limit is a minimizer for $\min\{F(u,Q) :\ u-\varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}$.

1.3 Homogenization

In this section we will show that if the functions f^{ξ}_{ε} are obtained by rescaling by ε functions f^{ξ} periodic in the space variable, then a Γ-convergence result holds true. This models the case when the arrangement of the "material points" presents a periodic feature (see fig. 4).

Let $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ be given and set

$$\mathcal{R}_{\mathbf{k}} := (0,k_1) \times \cdots \times (0,k_N).$$

For any $\xi \in \mathbb{Z}^N$, let $f^{\xi} : \mathbb{Z}^N \times \mathbb{R}^d \to [0,+\infty)$ be such that $f^{\xi}(\cdot,z)$ is $\mathcal{R}_{\mathbf{k}}$-periodic for any $z \in \mathbb{R}^d$. Then we consider f^{ξ}_{ε} of the following form

$$f^{\xi}_{\varepsilon}(\alpha,z) := f^{\xi}\left(\frac{\alpha}{\varepsilon}, z\right). \tag{1.3.1}$$

In this case, the growth conditions (1.2.2) and (1.2.3) and hypotheses (H1) and (H2) can be rewritten as follows:

$$f^{e_i}(\alpha,z) \geq c_1(|z|^p - 1), \quad \forall i \in \{1,\ldots,N\}, \tag{1.3.2}$$

$$f^{\xi}(\alpha,z) \leq C^{\xi}(|z|^p + 1), \tag{1.3.3}$$

where

$$\sum_{\xi \in \mathbb{Z}^N} C^{\xi} < +\infty. \tag{$H3$}$$

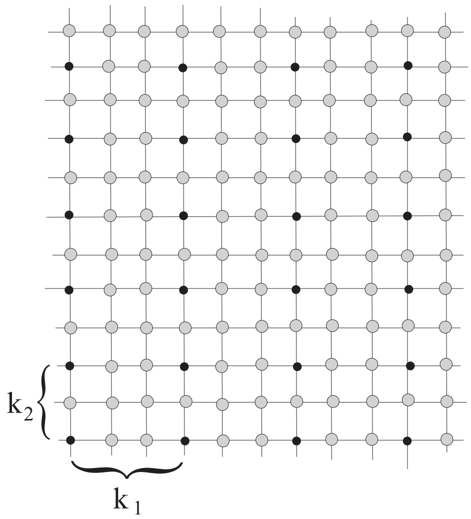

Figure 4: example of periodic structure

In the sequel we will use the following notation: for any $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ define

$$[x]_{\mathbf{k}} := \left(\left[\frac{x_1}{k_1} \right] k_1, \dots, \left[\frac{x_N}{k_N} \right] k_N \right).$$

Moreover, for any $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$, $l \in \mathbb{N}$ and $M \in \mathcal{M}^{d \times N}$ we denote by $\mathcal{A}^l_{\varepsilon,M}(A)$ the set defined in formula (1.2.29) with $\varphi(x) = Mx$. By simplicity of notation, we set $\mathcal{A}^1_{\varepsilon,M}(A) := \mathcal{A}_{\varepsilon,M}(A)$. Finally for every $r > 0$ we set $Q_r := (0, r)^N$.

The following theorem is the main result of this section and its proof is obtained by adapting a homogenization argument to the discrete setting. We remark that a central role is played by Theorems 1.2.1 and 1.2.3 and by the convergence of minimum problems with boundary data stated in Corollary 1.2.11. Moreover, we recall that the following result has been already proven in [21] in the one-dimensional case, where a more straightforward proof is possible.

Theorem 1.3.1 *Let* $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ *satisfy (1.3.1)-(1.3.3) and let* (H3) *hold. Then,* (F_ε) Γ*-converges with respect to the* $L^p(\Omega)$*-topology to the functional* $F : L^p(\Omega) \to [0, +\infty]$ *defined as*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\, dx & \textit{if } u \in W^{1,p}(\Omega) \\ +\infty & \textit{otherwise,} \end{cases} \tag{1.3.4}$$

where $f_{hom} : \mathcal{M}^{d\times N} \to [0,+\infty)$ *is given by the following homogenization formula*

$$f_{hom}(M) := \lim_{h\to+\infty} \frac{1}{h^N} \min\Big\{ \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_h) \Big\} \quad (1.3.5)$$

Proof. Let (ε_n) be a sequence of positive numbers converging to 0. Then, by Theorems 1.2.1 and 1.2.3, we can extract a subsequence (not relabelled) such that (F_{ε_n}) Γ-converges to a functional F defined as in (1.2.4) and such that, for any $u \in W^{1,p}(\Omega)$, $A \in \mathcal{A}(\Omega)$

$$\Gamma\text{-}\lim_n F_{\varepsilon_n}(u, A) = \int_A f(x, \nabla u)\, dx.$$

The theorem is proved if we show that f does not depend on the space variable x and $f \equiv f_{hom}$. To prove the first claim, by Theorem 1.1.9, it suffices to show that, set

$$F(u, A) = \int_A f(x, \nabla u)\, dx,$$

then

$$F(Mx, B(y, \rho)) = F(Mx, B(z, \rho))$$

for all $M \in \mathcal{M}^{d\times N}$, $y, z \in \Omega$ and $\rho > 0$ such that $B(y,\rho) \cup B(z,\rho) \subset \Omega$. We will prove that

$$F(Mx, B(y, \rho)) \le F(Mx, B(z, \rho)),$$

the proof of the opposite inequality being analogous. By the inner regularity of $F(Mx, \cdot)$, given by Proposition 1.2.8, it suffices to show that for any $\rho' < \rho$ we get

$$F(Mx, B(y, \rho')) \le F(Mx, B(z, \rho)). \quad (1.3.6)$$

Let, then, $v_n \in \mathcal{A}_{\varepsilon_n}(\Omega)$ be such that (v_n) converges to Mx in $L^p(\Omega)$ and

$$\lim_n F_{\varepsilon_n}(v_n, B(z, \rho)) = F(Mx, B(z, \rho)). \quad (1.3.7)$$

For $n \in \mathbb{N}$, define $u_n \in \mathcal{A}_{\varepsilon_n}(\Omega)$ as

$$u_n(\alpha) := \begin{cases} v_n\left(\alpha - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}}\right) + \varepsilon_n M \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}} & \text{if } \alpha \in \varepsilon_n\mathbb{Z}^N \cap B(y, \rho') \\ M\alpha & \text{otherwise.} \end{cases}$$

Then it is easy to verify that (u_n) converges to Mx in $L^p(\Omega)$. Moreover for n large enough

$$R^\xi_{\varepsilon_n}(B(y, \rho')) - \varepsilon_n \left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}} \subseteq R^\xi_{\varepsilon_n}(B(z, \rho)).$$

Thus, since, by the periodicity hypothesis, $f^\xi\left(\alpha-\varepsilon_n\left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}},z\right)=f^\xi(\alpha,z)$ and $D^\xi_\varepsilon u_n(\alpha)=D^\xi_\varepsilon v_n\left(\alpha-\varepsilon_n\left[\frac{y-z}{\varepsilon_n}\right]_{\mathbf{k}}\right)$, we get for n large enough

$$F_{\varepsilon_n}(u_n,B(y,\rho'))\le F_{\varepsilon_n}(v_n,B(z,\rho)).$$

Eventually, by (1.3.7), we obtain

$$\begin{aligned}F(Mx,B(y,\rho')) &\le \liminf_{n\to+\infty}F_{\varepsilon_n}(u_n,B(y,\rho'))\\ &\le \lim_{n\to+\infty}F_{\varepsilon_n}(v_n,B(z,\rho))=F(Mx,B(z,\rho)).\end{aligned}$$

In order to prove that $f\equiv f_{hom}$, first note that, by the lower semicontinuity of F in $W^{1,p}(\Omega)$, f is quasiconvex, so that, by the p-growth properties of f, for any $A\in\mathcal{A}(\Omega)$ with Lipschitz boundary and for any $M\in\mathcal{M}^{d\times N}$ there holds

$$\begin{aligned}f(M) &= \frac{1}{|A|}\min\left\{\int_A f(\nabla u)\,dx:\quad u-Mx\in W^{1,p}_0(A;\mathbb{R}^d)\right\}\\ &= \frac{1}{|A|}\min\left\{F(u,A):\quad u-Mx\in W^{1,p}_0(A;\mathbb{R}^d)\right\}\\ &= \frac{1}{|A|}\liminf_n\left\{F_{\varepsilon_n}(u,A):\quad u\in\mathcal{A}_{\varepsilon_n,M}(A)\right\},\end{aligned}$$

where the last equality follows by Corollary 1.2.11. In particular, if $x_0\in\Omega$ and $r>0$ are such that $Q_r(x_0):=(x_0,x_0+r)^N\subseteq\Omega$, then

$$f(M)=\lim_n\frac{1}{r^N}\inf\{F_{\varepsilon_n}(u,Q_r(x_0)):\ u\in\mathcal{A}_{\varepsilon_n,M}(Q_r(x_0))\}.$$

Without loss of generality, we may suppose $x_0=0$. If we set

$$T_n:=\left[\frac{r}{\varepsilon_n}\right]+1,$$

then it is easy to show that $\mathcal{A}_{\varepsilon_n,M}(Q_r)=\mathcal{A}_{\varepsilon_n,M}(Q_{\varepsilon_nT_n})$ and that for $\xi\in\mathbb{Z}^N$ $R^\xi_{\varepsilon_n}(Q_r)=R^\xi_{\varepsilon_n}(Q_{\varepsilon_nT_n})$. Thus

$$f(M)=\lim_n\frac{1}{r^N}\inf\{F_{\varepsilon_n}(u,Q(0,\varepsilon_nT_n)):\ u\in\mathcal{A}_{\varepsilon_n,M}(Q_{\varepsilon_nT_n})\}.$$

Eventually, through the change of variable

$$\beta=\frac{\alpha}{\varepsilon},\quad v(\beta)=\frac{1}{\varepsilon}u(\varepsilon\beta),\tag{1.3.8}$$

we get

$$\begin{aligned}f(M) &= \lim_n\left(\frac{\varepsilon_n}{r}\right)^N\inf\Big\{\sum_{\xi\in\mathbb{Z}^N}\ \sum_{\beta\in R^\xi_1(Q_{T_n})}f^\xi(\beta,D^\xi_1v(\beta)),\quad v\in\mathcal{A}_{1,M}(Q_{T_n})\Big\}\\ &= \lim_n\frac{1}{T_n^N}\inf\Big\{\sum_{\xi\in\mathbb{Z}^N}\ \sum_{\beta\in R^\xi_1(Q_{T_n})}f^\xi(\beta,D^\xi_1v(\beta)),\quad v\in\mathcal{A}_{1,M}(Q_{T_n})\Big\},\end{aligned}$$

where the last equality holds since

$$\lim_n T_n \frac{\varepsilon_n}{r} = 1.$$

Then the thesis will follow by the next proposition.

Proposition 1.3.2 *Let f^ξ satisfy (1.3.2),(1.3.3) and (H3) for any $\xi \in \mathbb{Z}^N$. Then the limit*

$$\lim_{h\to+\infty} \frac{1}{h^N} \inf\Big\{ \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}_{1,M}\left(Q_h\right)\Big\}$$

exists for all $M \in \mathcal{M}^{d\times N}$.

Proof. Let $M \in \mathcal{M}^{d\times N}$ be fixed and set

$$F_1(v, A) := \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in R_1^\xi(A)} f^\xi(\beta, D_1^\xi v(\beta)),$$

$$f_h(M) := \frac{1}{h^N} \inf\left\{F_1(v, Q_h), \quad v \in \mathcal{A}_{1,M}\left(Q_h\right)\right\}.$$

Moreover for any $R > 0$, set

$$F_1^R(v, A) := \sum_{|\xi|\leq R} \sum_{\beta\in R_1^\xi(A)} f^\xi(\beta, D_1^\xi v(\beta)),$$

$$f_h^R(M) := \frac{1}{h^N} \inf\left\{F_1^R(v, Q_h), \quad v \in \mathcal{A}_{1,M}\left(Q_h\right)\right\}.$$

We prove that

$$\lim_{R\to+\infty} \sup_h |f_h^R(M) - f_h(M)| = 0. \tag{1.3.9}$$

To this end, since $f_h^R(M) \leq f_h(M)$ for any $h \in \mathbb{N}$ and $R > 0$, it suffices to prove that for any $\delta > 0$, there exist $R_\delta > 0$ such that

$$f_h(M) \leq f_h^R(M) + \delta, \quad \forall R > R_\delta\ , h \in \mathbb{N}$$

Fix $\delta > 0$ and let $v_h^R \in \mathcal{A}_{1,M}(Q_h)$ be such that

$$\frac{1}{h^N} F_h^R(v_h^R, Q_h) \leq f_h^R(M) + \frac{1}{R}. \tag{1.3.10}$$

By testing the minimum problem defining $f_h^R(M)$ with $v(\alpha) = M\alpha$, we get, by (1.3.3) and (H3), that

$$f_h^R(M) \leq \frac{1}{h^N} F_h^R(M\alpha, Q_h) \leq C|M|^p.$$

Thus, by (1.3.10) and (1.3.2), we obtain that

$$\sup_{h,R} \frac{1}{h^N} \sum_{i=1}^N \sum_{\beta \in R_1^{e_i}(Q_h)} |D_1^{e_i} v_h^R(\beta)|^p < +\infty.$$

Then, by arguing as in the proof of Lemma 1.2.6 and thanks to the particular geometry of the sets Q_h, we deduce that

$$\sup_{h,R} \frac{1}{h^N} \sup_{\xi \in \mathbb{Z}^N} \sum_{\beta \in R_1^{\xi}(Q_h)} |D_1^{\xi} v_h^R(\beta)|^p < +\infty.$$

Eventually, we have

$$\begin{aligned} f_h(M) \quad &\leq \quad \frac{1}{h^N} F_h(v_h^R, Q_h) \leq \frac{1}{h^N} F_h^R(v_h^R, Q_h) + \frac{1}{h^N} \sum_{|\xi|>R} C^{\xi} \sum_{\beta \in R_1^{\xi}(Q_h)} |D_1^{\xi} v_h^R(\beta)|^p \\ &\leq f_h^R(M) + \frac{1}{R} + C \sum_{|\xi|>R} C^{\xi}. \end{aligned}$$

Thus, it suffices to choose $R_\delta > 0$ such that for $R > R_\delta$

$$\frac{1}{R} + C \sum_{|\xi|>R} C^{\xi} \leq \delta.$$

So far, in order to prove the thesis, it suffices to show that for any $R > 0$ there exists the limit

$$\lim_h f_h^R(M).$$

Set

$$f_h^{R,R}(M) := \frac{1}{h^N} \inf \left\{ F_1^R(v, Q_h), \quad v \in \mathcal{A}_{1,M}^{[R]}(Q_h) \right\}.$$

Using backward the scaling argument exploited in the proof of the previous proposition and thanks to Theorem 1.2.10 and Corollary 1.2.11, one can show that, for any subsequence $(h_n) \subset \mathbb{N}$ it is possible to extract a further subsequence (not relabelled) such that

$$\lim_n f_{h_n}^R(M) = \lim_n f_{h_n}^{R,R}(M). \tag{1.3.11}$$

Thus, to complete the proof, it is sufficient to prove that there exists the limit

$$\lim_h f_h^{R,R}(M).$$

Let $h \in \mathbb{N}$ and let $v_h \in \mathcal{A}_{1,M}^{[R]}(Q_h)$ be such that

$$\frac{1}{h^N} F_1^R(v_h, Q_h) \leq f_h^{R,R}(M) + \frac{1}{h}.$$

For any $k > h$ define a function $u_k \in \mathcal{A}^{[R]}_{1,M}(Q_k)$ as follows

$$u_k(\alpha) = \begin{cases} v_h(\alpha - h\mathbf{i}) + hM\mathbf{i} & \text{if } \alpha \in h\mathbf{i} + Q_h,\ \mathbf{i} \in \big\{0, \dots, \big[\frac{k}{h}\big] - 1\big\}^N \\ M\alpha & \text{otherwise} \end{cases}$$

Note that for any $\xi \in \mathbb{Z}^N$, $|\xi| \le R$ we have

$$\begin{aligned} R_1^{\xi}(Q_k) \quad \subseteq \quad & \Big(\bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} R_1^{\xi}(h\mathbf{i} + Q_h) \Big) \cup R_1^{\xi}\Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h)\Big) \\ & \cup \Big(\bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} \big(h\mathbf{i} + \big(\{0,\dots,h+R\}^N \setminus \{0,\dots,h-R\}^N\big)\big)\Big). \end{aligned}$$

Moreover $D_1^{\xi} u_k(\alpha) = M\frac{\xi}{|\xi|}$ if $\alpha \in R_1^{\xi}\Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h)\Big)$ or $\alpha \in \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} \big(h\mathbf{i} + \big(\{0,\dots,h+R\}^N \setminus \{0,\dots,h-R\}^N\big)\big)$, and

$$\#\Big(R_1^{\xi}\Big(Q_k \setminus \bigcup_{\mathbf{i} \in \{0,\dots,[\frac{k}{h}]-1\}^N} (h\mathbf{i} + Q_h)\Big)\Big) \le k^N - \left[\frac{k}{h}\right]^N h^N,$$

$$\#\left(\{0,\dots,h+R\}^N \setminus \{0,\dots,h-R\}^N\right) \le (h+R)^N - (h-R)^N.$$

Then, by (1.3.3) and (H3), we get

$$\begin{aligned} f_k^{R,R}(M) \quad \le \quad & \frac{1}{k^N} F_1^R(u_k, Q_k) \le \left[\frac{k}{h}\right]^N \frac{1}{k^N} F_1^R(v_h, Q_h) \\ & + C|M|^P \frac{1}{k^N}\left(k^N - \left[\frac{k}{h}\right]^N h^N + \left[\frac{k}{h}\right]^N \left((h+R)^N - (h-R)^N\right)\right) \\ & \le \left[\frac{k}{h}\right]^N \frac{h^N}{k^N}\left(f_h^{R,R}(M) + \frac{1}{h}\right) \\ & + C|M|^P \frac{1}{k^N}\left(k^N - \left[\frac{k}{h}\right]^N h^N + \left[\frac{k}{h}\right]^N \left((h+R)^N - (h-R)^N\right)\right). \end{aligned}$$

By letting k tend to $+\infty$, we then get

$$\limsup_k f_k^{R,R}(M) \le f_h^{R,R}(M) + \frac{1}{h} + C|M|^P \frac{1}{h^N}\left((h+R)^N - (h-R)^N\right)$$

Eventually, letting h tendo to $+\infty$, we obtain

$$\limsup_k f_k^{R,R}(M) \le \liminf_h f_h^{R,R}(M),$$

that is the conclusion. □

Remark 1.3.3 In formula (1.3.5) we can replace $\mathcal{A}_{1,M}(Q_h)$ by $\mathcal{A}^l_{1,M}(Q_h)$ for any fixed $l \in \mathbb{N}$, the proof being exactly the same.

Remark 1.3.4 The function f_{hom} in Theorem 1.3.1 also satisfies

$$f_{hom}(M) = \lim_{h\to+\infty} \frac{1}{h^N} \inf \Big\{ \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}(Q_{h-2}) \Big\}, \tag{1.3.12}$$

where, for every $k \in \mathbb{R}$,

$$\mathcal{A}_{1,\#}(Q_k) := \{v \in \mathcal{A}_1(\mathbb{R}^N) : \ v \ k\text{-periodic}\}.$$

This characterization can be proved by arguing as in the proof of Theorem 1.3.1 and Proposition 1.3.2 taking into account Corollary 1.2.13 and recalling that, since f_{hom} is quasiconvex, there holds

$$\begin{aligned} f_{hom}(M) &= \frac{1}{r^N}\min\left\{\int_{Q_r} f_{hom}(M+\nabla\psi)\,dx : \psi \in W^{1,p}_\#(Q_r;\mathbb{R}^d)\right\} \\ &= \frac{1}{r^N}\min\left\{F(M\alpha+\psi, Q_r) : \psi \in W^{1,p}_\#(Q_r;\mathbb{R}^d)\right\}. \end{aligned}$$

As a consequence of Theorem 1.2.10, Corollary 1.2.11 and Theorem 1.3.1 we immediately derive the following result about Γ-convergence and convergence of minimum problems for homogeneous functionals subject to Dirichlet boundary conditions.

Theorem 1.3.5 *For any $\varphi \in Lip\ (\mathbb{R}^N)$ and $l \in \mathbb{N}$ let $F_\varepsilon^{\varphi,l}$ be defined by (1.2.30) and let $F^\varphi : L^p(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ be defined as*

$$F^\varphi(u,A) = \begin{cases} \displaystyle\int_A f_{hom}(\nabla u)\,dx & \text{if } u-\varphi \in W_0^{1,p}(A;\mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3.13}$$

Under the hypotheses of Theorem 1.3.1, $F_\varepsilon^\varphi(\cdot,A)$ Γ-converges with respect to the $L^p(\Omega)$-topology to $F^\varphi(\cdot,A)$ for any $A\in\mathcal{A}$.

Corollary 1.3.6 *Under the hypotheses of Theorem 1.3.5, for any $\varphi \in Lip\ (\mathbb{R}^N)$, $l\in\mathbb{N}$ and $A \in \mathcal{A}(\Omega)$*

$$\lim_{\varepsilon\to0}\inf\{F_\varepsilon(u,A) : \ u \in \mathcal{A}^l_{\varepsilon,\varphi}\} = \min\{F(u,A) : \ u-\varphi \in W_0^{1,p}(A;\mathbb{R}^d)\}.$$

Moreover, for any (ε_j) converging to zero as j tends to infinity, if (u_j) is a converging sequence such that

$$\lim_j F_{\varepsilon_j}(u_j,A) = \lim_j \inf\{F_{\varepsilon_j}(u,A) : \ u\in\mathcal{A}^l_{\varepsilon_j,\varphi}\},$$

then its limit is a minimizer for $\min\{F(u,A) : \ u-\varphi\in W_0^{1,p}(A;\mathbb{R}^d)\}$.

An analogous result about the convergence of minimum problems with periodic conditions follows by Theorem 1.2.12 and Corollary 1.2.13.

1.4 The convex case: a cell problem formula

In this section we will see that in the convex case the function f_{hom} can be rewritten by a single periodic minimization problem on the periodic cell $\mathcal{R}_{\mathbf{k}}$. Set

$$\hat{k} := \prod_{i=1}^{N} k_i,$$

$$I_{\mathbf{k}} := \prod_{i=1}^{N} \{0, \dots, k_i - 1\}$$

and

$$\mathcal{A}_{1,\#}(\mathcal{R}_{\mathbf{k}}) := \{u \in \mathcal{A}_1(\mathbb{R}^N) \ : \ u \text{ is } \mathcal{R}_{\mathbf{k}} - \text{periodic}\}$$

Theorem 1.4.1 *Let $(f_\varepsilon^\xi)_{\varepsilon,\xi}$ satisfies all the assumptions of Theorem 1.3.1 and in addition let $f_\varepsilon^\xi(\alpha,\cdot)$ be convex for all $\alpha \in \varepsilon\mathbb{Z}^N$, $\varepsilon > 0$ and $\xi \in \mathbb{Z}^N$. Then the conclusion of Theorem 1.3.1 holds with f_{hom} satisfying*

$$f_{hom}(M) = \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi \in \mathbb{Z}^N} \sum_{\beta \in I_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_{\mathbf{k}}\right)\Big\},$$

for all $M \in \mathcal{M}^{d\times N}$.

Proof. Set

$$\overline{f}(M) := \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi \in \mathbb{Z}^N} \sum_{\beta \in I_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_{\mathbf{k}}\right)\Big\}.$$

We first prove that

$$f_{hom}(M) \leq \overline{f}(M). \tag{1.4.1}$$

With fixed $\delta > 0$, let $v \in \mathcal{A}_{1,\#}\left(\mathcal{R}_k\right)$ be such that

$$\frac{1}{\hat{k}} \sum_{\xi \in \mathbb{Z}^N} \sum_{\beta \in I_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \leq \overline{f}(M) + \delta.$$

For $n \in \mathbb{N}$, since in particular $v \in \mathcal{A}_{1,\#}(Q_{n\hat{k}})$, we get

$$\begin{aligned} f^{\#}_{n\hat{k}+2}(M) &\leq \sum_{\xi \in \mathbb{Z}^N} \sum_{\beta \in R_1^\xi(Q_{n\hat{k}})} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \\ &\leq n^N \hat{k}^{N-1} \sum_{\xi \in \mathbb{Z}^N} \sum_{\beta \in I_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \end{aligned}$$

where the last inequality follows by the periodicity of v, $(f(\cdot,z))$ and by the fact that $Q_{n\hat{k}}$ is union of $n^N\hat{k}^{N-1}$ periodicity cells. Eventually, by Remark 1.3.12, we get

$$\begin{aligned} f_{hom}(M) &\leq \limsup_n \frac{1}{(n\hat{k}+2)}^N f^{\#}_{n\hat{k}+2}(M) \\ &\leq \frac{1}{\hat{k}} \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \leq \overline{f}(M) + \delta, \end{aligned}$$

and inequality (1.4.1) follows, by letting δ tend to 0. Let us prove that

$$f_{hom}(M) \geq \overline{f}(M).$$

For any $R > 0$, set

$$f^R_{hom}(M) := \lim_{h\to+\infty} \frac{1}{h^N} \inf\Big\{ \sum_{|\xi|\leq R} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}^{[R]}_{1,M}(Q_h) \Big\},$$

$$\overline{f}^R(M) := \frac{1}{\hat{k}} \inf\Big\{ \sum_{|\xi|\leq R} \sum_{\beta\in I_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \quad v \in \mathcal{A}_{1,\#}(\mathcal{R}_{\mathbf{k}}) \Big\}.$$

By (1.3.9) and (1.3.11)we easily derive that

$$\lim_{R\to+\infty} f^R_{hom}(M) = f_{hom}(M).$$

Analogously one can prove that

$$\lim_{R\to+\infty} \overline{f}^R(M) = \overline{f}(M).$$

Thus it suffices to prove that for any $R > 0$

$$f^R_{hom}(M) \geq \overline{f}^R(M). \tag{1.4.2}$$

For $n \in \mathbb{N}$, let $u \in \mathcal{A}^{[R]}_{1,M}(Q_{n\hat{k}})$ and let $v \in \mathcal{A}_{1,\#}(Q_{n\hat{k}})$ be such that

$$v(\alpha) = u(\alpha) - M\alpha, \quad \forall \alpha \in Q_{n\hat{k}}.$$

Moreover set

$$I^n_{\mathbf{k}} := \prod_{i=1}^N \{0, \ldots, n\prod_{j\neq i} k_j - 1\}.$$

Then, we get

$$\frac{1}{(n\hat{k})^N} \sum_{|\xi|\leq R} \sum_{\beta\in R_1^\xi(Q_{n\hat{k}})} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right)$$

$$= \frac{1}{(n\hat{k})^N} \sum_{|\xi|\le R} \sum_{\beta\in\{0,\dots,n\hat{k}\}^N} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) - O(\frac{1}{n})$$

$$= \frac{1}{\hat{k}} \sum_{|\xi|\le R} \sum_{\beta\in I_k} \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta + \sum_{i=1}^N \gamma_i k_i e_i)\right) - O(\frac{1}{n})$$

$$\ge \frac{1}{\hat{k}} \sum_{|\xi|\le R} \sum_{\beta\in I_k} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} D_1^\xi v(\beta + \sum_{i=1}^N \gamma_i k_i e_i)\right) - O(\frac{1}{n}),$$

where in the last inequality we have used the convexity hypothesis on f^ξ. Eventually, set

$$v_n(\beta) := \frac{1}{\hat{k}^{N-1}n^N} \sum_{\gamma\in I^n_{\mathbf{k}}} v(\beta + \sum_{i=1}^N \gamma_i k_i e_i).$$

It is easy to show that $v_n \in \mathcal{A}_{1,\#}(\mathcal{R}_k)$ and so, by the previous inequality, we get

$$\begin{aligned}
&\frac{1}{(n\hat{k})^N} \sum_{|\xi|\le R} \sum_{\beta\in R_1^\xi(Q_{n\hat{k}})} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right) \\
&\ge \frac{1}{\hat{k}} \sum_{|\xi|\le R} \sum_{\beta\in I_k} f^\xi \left(\beta, M\frac{\xi}{|\xi|} + D_1^\xi v_n(\beta)\right) - O(\frac{1}{n}) \\
&\ge \overline{f}^R(M) - O(\frac{1}{n}).
\end{aligned}$$

Passing to the inf with respect to $u \in \mathcal{A}^R_{1,\#}(Q_{n\hat{k}})$, we get

$$f^{R,R}_{n\hat{k}+2}(M) \ge \overline{f}^R(M) - O(\frac{1}{n}),$$

and then, letting n tend to $+\infty$, we obtain (1.4.2). □

Remark 1.4.2 (**quadratic forms**) Under the hypotheses of Theorem 1.4.1, if in addition for any $\xi \in \mathbb{Z}^N$ $f^\xi(\alpha,\cdot)$ is a positive quadratic form on $\mathbb{R}^d$, that is

$$f^\xi(\alpha, z) = \langle A^\xi(\alpha)z, z\rangle \quad A^\xi(\alpha) \in \mathcal{M}^{d\times d}_{sym},$$

then, thanks to Remark 1.2.2, the limit energy density $f_{hom}(\cdot)$ is a homogeneous quadratic form on $\mathcal{M}^{d\times N}$ and formula (1.2.5) becomes

$$\begin{aligned}
f_{hom}(M) &= A_{hom}(M,M) \\
&= \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi\in\mathbb{Z}^N} \sum_{\beta\in I_{\mathbf{k}}} \langle A^\xi(\beta)\cdot \left(M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \left(M\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right)\rangle, \\
&\qquad v \in \mathcal{A}_{1,\#}(\mathcal{R}_{\mathbf{k}}) \Big\}
\end{aligned}$$

with $A_{hom} \in T_2(\mathcal{M}^{d\times N})$.
If $N = d = 1$ and only nearest-neighbor interactions are taken into account, that is

$$f^\xi \equiv 0 \text{ if } \xi \neq e_1, \qquad f^{e_1}(\alpha, z) = a(\alpha)z^2,$$

with $a : \mathbf{z}^N \to (0, +\infty)$ k-periodic, the previous minimum problem can be easily solved (see [19])giving the analogue in the discrete setting of a well known homogenization result for integral functionals (see [16]). In fact, in this case

$$A_{hom} = \frac{1}{k}\left(\sum_{\beta=0}^{k-1} \frac{1}{a(\beta)}\right)^{-1}$$

is the harmonic mean of $a(\cdot)$.

Remark 1.4.3 Note that if $\mathcal{R}_{\mathbf{k}} = (0, 1)^N$, that is f^ξ does not depend on the space variable α, in Theorem 1.4.1 we obtain

$$f_{hom}(M) = \sum_{\xi\in\mathbb{Z}^N} f^\xi\left(M\frac{\xi}{|\xi|}\right).$$

1.5 Interactions along independent directions

In this section we first recall some results proven in the 1-dimensional setting in [21], where a non-asymptotic formula defining the limit energy density f_{hom} is provided when only nearest and next-to-nearest neighbor interactions are considered.

Then in Theorem 1.5.3 we will show that, if only interactions along the coordinate directions are taken into account, the N-dimensional problem can be reduced to a 1-dimensional one.

The following two theorems have been proven in [21] in the case $d = 1$. Their proof in the case $d > 1$ is the same.

Theorem 1.5.1 (**nearest-neighbor interactions**) *Let $\Omega = (0, l) \subset \mathbb{R}$ and let $F_\varepsilon : L^p(\Omega) \to [0, +\infty)$ be defined as*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\sum_{i=1}^{l-2} \varepsilon f\left(\frac{u(\varepsilon(i+1)) - u(\varepsilon i)}{\varepsilon}\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

with $f : \mathbb{R}^d \to [0, +\infty)$ satisfying $f(z) \geq C(|z|^p - 1)$. Then the conclusions of Theorem 1.3.1 hold with

$$f_{hom}(z) = f^{**}(z).$$

Theorem 1.5.2 (**next-to-nearest neighbor interactions**) *Let* $\Omega = (0,l) \subset \mathbb{R}$ *and let* $F_\varepsilon : L^p(\Omega) \to [0,+\infty)$ *be defined as*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\sum_{i=1}^{l-2} \varepsilon f^1\left(\frac{u(\varepsilon(i+1)) - u(\varepsilon i)}{\varepsilon}\right) + \sum_{i=1}^{l-3} \varepsilon f^2\left(\frac{u(\varepsilon(i+2)) - u(\varepsilon i)}{2\varepsilon}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise,} \end{cases}$$

with $f^1, f^2 : \mathbb{R}^d \to [0,+\infty)$ *satisfying* $f^1(z) \geq C(|z|^p - 1)$. *Then the conclusions of Theorem 1.3.1 hold with*

$$f_{hom}(z) = \tilde{f}^{**}(z),$$

where $\tilde{f}(z) = f^2(z) + \frac{1}{2}\inf\{f^1(z_1) + f^1(z_2),\ z_1 + z_2 = 2z\}$.

Back to the general N-dimensional setting, we consider now energies of the form

$$F_\varepsilon(u) = \begin{cases} \sum_{i=1}^{N} \mathcal{F}_\varepsilon^i(u,\Omega) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.5.1}$$

where, for any $i \in \{1,\dots,N\}$, $\mathcal{F}_\varepsilon^i : \mathcal{A}_\varepsilon(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ is defined as

$$\mathcal{F}_\varepsilon^i(u,A) := \sum_{k=1}^{+\infty} \sum_{\alpha \in R_\varepsilon^{ke_i}(A)} \varepsilon^N f_i^k(D_\varepsilon^{ke_i} u(\alpha)) \tag{1.5.2}$$

with $f_i^k : \mathbb{R}^d \to [0,+\infty)$ satisfying

$$f_i^1(z) \geq c(|z|^p - 1), \qquad f_i^k(z) \leq C_i^k(|z|^p + 1)$$

and

$$\sum_{i=1}^{N}\sum_{k=1}^{+\infty} C_i^k < +\infty.$$

This is a particular case of the model considered in Section 4, with $f^\xi \equiv 0$ if $\xi \neq ke_i$, $f^{ke_i}(0,z) = f_i^k(z)$, $i \in \{1,\dots,N\}$, $k \in \mathbb{N}$, and $\mathcal{R}_{\mathbf{k}} = (0,1)^N$.

The following theorem shows that, in this case, the homogenization formula defining f_{hom} can be rewritten as a sum of N one-dimensional homogenization formulas.

Theorem 1.5.3 *Let* F_ε *be defined by (1.5.2). Then the* Γ*-convergence result stated in Theorem 1.3.1 holds with* f_{hom} *satisfying*

$$f_{hom}(M) = \sum_{i=1}^{N} \tilde{f}_i(M^i) \tag{1.5.3}$$

for any $M = (M^1, \dots, M^N) \in \mathcal{M}^{d\times N}$, *where* $\tilde{f}_i : \mathbb{R}^d \to \mathbb{R}$, $i \in \{1, \dots, N\}$, *is defined by the following* 1*-dimensional homogenization formula*

$$\tilde{f}_i(z) := \lim_{h\to+\infty} \frac{1}{h} \inf \left\{ \sum_{k=1}^{+\infty} \sum_{j=1}^{h-k-1} f_i^k \left(\frac{v(j+k) - v(j)}{k} \right),\ v \in \mathcal{A}_{1,z}((0,h)) \right\}$$

Proof. We first prove that

$$f_{hom}(M) \geq \sum_{i=1}^{N} \tilde{f}_i(M^i).$$

To do this, by the definition of $f_{hom}(M)$, it suffices to show that for any $i \in \{1, \dots, N\}$, $u \in \mathcal{A}_{1,M}(Q_h)$ we have

$$\frac{1}{h^N} \mathcal{F}_1^i(u, Q_h) \geq \tilde{f}_i(M^i) + O(h). \tag{1.5.4}$$

We use a slicing argument. For $i \in \{1, \dots, N\}$, set

$$m_h^i(z) := \frac{1}{h} \inf \left\{ \sum_{k=1}^{+\infty} \sum_{j=1}^{h-k-1} f_i^k \left(\frac{v(j+k) - v(j)}{k} \right),\ v \in \mathcal{A}_{1,z}((0,h)) \right\}.$$

By simplicity of notation, we prove (1.5.4) for $i = 1$. Given $u \in \mathcal{A}_{1,M}(Q_h)$, we may write

$$\mathcal{F}_1^1(u, Q_h) = \sum_{\beta\in\{1,\dots,h-1\}^{N-1}} \sum_{k=1}^{+\infty} \sum_{j=1}^{h-k-1} f_1^k \left(\frac{u(j+k,\beta) - u(j,\beta)}{k} \right) \tag{1.5.5}$$

Since for any $\beta \in \{1, \dots, h-1\}^{N-1}$ the function $v(j) := u(j, \beta) - \tilde{M}\beta$ belongs to $\mathcal{A}_{1,M^1}(0, h)$, where $\tilde{M} := \left(M^2, \dots, M^N\right)$, from (1.5.5) we get

$$\frac{1}{h^N} \mathcal{F}_1^1(u, Q_h) \geq \frac{1}{h^{N-1}} \# \left(\{1, \dots, h-1\}^{N-1} \right) m_h^1(M^1) \geq m_h^1(M^1).$$

We then easily infer inequality (1.5.4).

We now prove that

$$f_{hom}(M) \leq \sum_{i=1}^{N} \tilde{f}_i(M^i). \tag{1.5.6}$$

With fixed $\eta > 0$, for any $i \in \{1, \dots N\}$ let $v_h^i \in \mathcal{A}^2_{1,M^i}(0, h)$ be such that

$$\frac{1}{h} \sum_{k=1}^{+\infty} \sum_{j=1}^{h-k-1} f_i^k \left(\frac{v_h^i(j+k) - v_h^i(j)}{k} \right) \leq m_h^i(M^i) + \eta, \tag{1.5.7}$$

and set

$$u_h(\alpha) := \sum_{i=1}^{N} v_h^i(\alpha_i), \quad \alpha = (\alpha_1, \dots, \alpha_N).$$

Note that $u_h \in M\alpha + \mathcal{A}_{1,\#}(Q_{h-2})$. Moreover by the analogue of (1.5.5) applied to $\mathcal{F}_1^i(u_, Q_h)$ for any $i \in \{1, \dots, N\}$ and by (1.5.7), we easily deduce that

$$\frac{1}{h^N} \sum_{i=1}^{N} \mathcal{F}_1(u_h, Q_h) \leq \sum_{i=1}^{N} m_h^i(M^i) + N\eta.$$

Eventually, by the characterization of f_{hom} given by formula (1.3.12), letting first h tend to $+\infty$ and then η tend to 0, we get (1.5.6). □

Remark 1.5.4 Note that formula (1.5.3) highlights that a superposition principle holds, in the sense that the limit energy is obtained by relaxing the energies due to the interactions in every coordinate direction independently and then summing over them.

Remark 1.5.5 (a)(*nearest-neighbors*) By Theorem 1.5.1, if $f_i^k = 0$ for all $k \neq 1$, then formula (1.5.3) can be rewritten as

$$f_{hom}(M) = \sum_{i=1}^{N} (f_i^1)^{**}(M^i);$$

(b)(*next-to-nearest neighbors*) By Theorem 1.5.2, if $f_i^k = 0$ for all $k \neq 1, 2$, then formula (1.5.3) can be rewritten as

$$f_{hom}(M) = \sum_{i=1}^{N} (\tilde{f}_i)^{**}(M^i),$$

with

$$\tilde{f}_i(z) = f_i^2(z) + \frac{1}{2} \inf\{f_i^1(z_1) + f_i^1(z_2),\ z_1 + z_2 = 2z\}.$$

1.6 Quasiconvexity of the limit energy density

In the following we provide an example of vector-valued discrete interaction energies defined in the plane whose continuous counterpart has an energy density which is a quasiconvex (not polyconvex) function. Our example draws inspiration from Šverák's construction of a quasiconvex function which is not polyconvex (see [44]). Let $N = d = 2$, $p > 1$ and define $f_i : \mathbb{R}^2 \to [0, +\infty)$, $i = 1, 2, 3$, as

$$f_i(z) = \begin{cases} 1 + |z|^p & \text{if } z \neq \pm\frac{\xi_i}{|\xi_i|} \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi_1 = e_1$, $\xi_2 = e_2$, $\xi_3 = e_1 + e_2$. Let F_ε be defined as

$$F_\varepsilon(u) = \sum_{i=1}^{3} \sum_{\alpha \in R_\varepsilon^{\xi_i}} \varepsilon^2 f_i\left(D_\varepsilon^{\xi_i} u(\alpha)\right),$$

then the conclusions of Theorems 1.3.1, 1.3.5 and Corollary 1.3.6 hold with f_{hom} given by

$$f_{hom}(M) = \lim_{h\to+\infty} \frac{1}{h^N} \min\Big\{ \sum_{i=1}^{3} \sum_{\beta \in R_1^{\xi_i}(Q_h)} f_i(D_1^{\xi_i} v(\beta)), \quad v \in \mathcal{A}_{1,M}(Q_h) \Big\}$$

Theorem 1.6.1 *f_{hom} is not convex.*

Proof. By testing the minimum problem defining f_{hom} with the identity function and its opposite, we immediately obtain that

$$f_{hom}(I) = f_{hom}(-I) = 0$$

where I is the identity matrix in $M^{2\times 2}$. The claim is proven if we show that $f_{hom}(0) > 0$. We argue by contradiction. Without loss of generality we may assume that Theorem 1.3.5 hold with $A = Q_1$. Were $f_{hom}(0)$ zero, there should exist a sequence $u_n \in \mathcal{A}_{\varepsilon_n,0}(Q_1)$ such that $u_n \to 0$ in $L^p(Q_1;\mathbb{R}^2)$ and

$$\lim_n F_{\varepsilon_n}(u_n) = 0. \tag{1.6.1}$$

Set

$$\begin{aligned} T^+ &:= \{(x_1,x_2) \in \mathbb{R}^2 : \ 0 \le x_1 \le 1, \quad x_1 \le x_2 \le 1\} \\ T^- &:= \{(x_1,x_2) \in \mathbb{R}^2 : \ 0 \le x_1 \le 1, \quad 0 \le x_2 \le x_1\}, \end{aligned}$$

we consider the family of piecewise affine functions $v_n : Q_1 \to \mathbb{R}^2$ defined as follows

$$v_n(x) = \begin{cases} u_n(\alpha) + D_{\varepsilon_n}^{e_1} u_n(\alpha)(x_1 - \alpha_1) \\ +D_{\varepsilon_n}^{e_2} u_n(\alpha + \varepsilon_n e_1)(x_2 - \alpha_2) & \text{if } x \in \alpha + \varepsilon_n T^- \\ \\ u_n(\alpha) + D_{\varepsilon_n}^{e_1} u_n(\alpha + \varepsilon_n e_2)(x_1 - \alpha_1) \\ +D_{\varepsilon_n}^{e_2} u_n(\alpha)(x_2 - \alpha_2) & \text{if } x \in \alpha + \varepsilon_n T^+. \end{cases}$$

Note that $v_n|_{\partial Q_1} = 0$. Moreover it is easy to check that

$$F_{\varepsilon_n}(u_n) = \int_{Q_1} \tilde{f}(\nabla v_n)\, dx. \tag{1.6.2}$$

where $\tilde{f} : M^{2\times 2} \to [0,+\infty)$ is defined as

$$\tilde{f}(\zeta) := f_1(\zeta_1) + f_2(\zeta_2) + f_3\left(\frac{\zeta_1 + \zeta_2}{\sqrt{2}}\right) \quad \zeta = (\zeta_1, \zeta_2) \in M^{2\times 2}.$$

In particular, by (1.6.1)

$$\lim_n \int_{Q_1} \tilde{f}(\nabla v_n)\ dx = 0 \tag{1.6.3}$$

Since we have

$$\tilde{f}(\zeta) \geq c(|\zeta_{11} - \zeta_{22}|^p + |\zeta_{12} + \zeta_{21}|^p),$$

by (1.6.1) and (1.6.2) we obtain

$$\lim_n \int_{Q_1} (|\nabla_1 v_n^1 - \nabla_2 v_n^2|^p + |\nabla_1 v_n^2 + \nabla_2 v_n^1|^p)\ dx = 0. \tag{1.6.4}$$

Since

$$\Delta v_n^1 = \operatorname{div}(\nabla_1 v_n^1 - \nabla_2 v_n^2, \nabla_1 v_n^2 + \nabla_2 v_n^1)$$

$$\Delta v_n^2 = \operatorname{div}(\nabla_1 v_n^2 + \nabla_2 v_n^1, -\nabla_1 v_n^1 + \nabla_2 v_n^2),$$

using the L^p estimates for the Laplace operator (see [40]) we obtain that

$$\begin{aligned} \|\nabla v_n^i\|^p_{L^p(Q_1;\mathbb{R}^2)} &\leq \|\Delta v_n^i\|^p_{W^{-1,p}(Q_1;\mathbb{R}^2)} \\ &\leq \int_{Q_1} (|\nabla_1 v_n^1 - \nabla_2 v_n^2|^p + |\nabla_1 v_n^2 + \nabla_2 v_n^1|^p)\ dx \end{aligned}$$

for $i = 1, 2$. Then, by (1.6.4) and the previous estimates, ∇v_n converges to 0 strongly in $L^p(Q_1; M^{2\times 2})$, so that

$$\lim_n \int_{Q_1} \tilde{f}(\nabla v_n)\ dx = \tilde{f}(0)\ |Q_1| > 0.$$

Hence we reach a contradiction. □

Remark 1.6.2 In the particular case $1 < p < 2$, thanks to the growth hypotheses on f_i, f_{hom} is a quasiconvex not polyconvex function (see [16], Remark 6.9).

Chapter 2

Surface energies for discrete systems

As we have seen, nonconvex interactions in lattice systems lead to a number of interesting phenomena that can be translated into a variety of energies within their limit continuum description as the lattice size tends to zero. In general, these effects may be due to different superposed causes. When only nearest-neighbour interactions are taken into account, a scaling effect for nonconvex energy densities with non-faster-than-linear growth at infinity (such as Lennard-Jones potentials or the 'weak membrane' energies considered by Blake and Zisserman in Image Processing) show the appearance of a competing surface term besides a convex bulk integral. In this way one can derive the Mumford Shah functional of Computer Vision as the limit of finite-difference schemes [26], explain Griffith's theory of Fracture as a phase transition with one 'well' at infinity [45], or give a microscopical interpretation of softening phenomena [15]. In the one-dimensional case a complete description can be given highlighting in addition oscillations and micro-cracking (see [20] and also Del Piero and Truskinowsky [32] for a Mechanical insight).

For energies with 'superlinear' growth (we remind that these growth conditions are expressed in terms of the scaled difference quotients), in the previous Chapter (see also [1]), we have shown that, upon some natural decay conditions on the energy densities ϕ_ε, the Γ-limit as $\varepsilon \to 0$ of an arbitrary system of interactions

$$\sum_{i,j \in \varepsilon\mathbb{Z}^N \cap \Omega} \varepsilon^N \phi_\varepsilon\Big(\frac{i-j}{\varepsilon}, \frac{u_j - u_i}{\varepsilon}\Big)$$

(Ω a bounded open subset of $\mathbb{R}^n$) always exists (upon passing to subsequences) and is expressed as an integral functional

$$\int_\Omega \varphi(Du)\, dx.$$

The simplest case is when only nearest-neighbour interactions are present, in which case the function φ is computed via a convexification process. When not only nearest-neighbour interactions are taken into account, in contrast, the description of the limit problem turns out more complex involving in general some 'homogenization' process (see [21], [1]). It is worth noting that the necessity of such a complex description arises also for simple linear spring models where the nonlinearity is of a more 'geometrical' origin (see [34]). Even in the simple one-dimensional case of next-to-nearest-neighbour interaction the limit bulk energy density is described by a formula of 'convolution type' that highlights a non-trivial balance between first and second neighbours (see [41], [13]). Additional phenomena arise in the case when the range of interaction does not vanish with the lattice size, in which case a complex non-local interaction can take place (see [12]).

In this Chapter we provide a higher-order description of one-dimensional next-to-nearest-neighbour systems of the form

$$\sum_{i,i+1\in\varepsilon\mathbb{Z}\cap\Omega} \varepsilon\psi_1\Big(\frac{u^{i+1}-u^i}{\varepsilon}\Big) + \sum_{i,i+2\in\varepsilon\mathbb{Z}\cap\Omega} \varepsilon\psi_2\Big(\frac{u^{i+2}-u^i}{2\varepsilon}\Big)$$

using the terminology of developments by Γ-convergence (introduced in Anzellotti and Baldo [7]) and equivalence of variational theories (in the spirit of Braides and Truskinovsky [22]). In this one-dimensional case the integrand φ is given as the convex envelope of an effective energy ψ described by an explicit convolution-type formula describing oscillations at the lattice level

$$\psi(z) = \psi_2(z) + \frac{1}{2}\min\{\psi_1(z_1) + \psi_1(z_2) : z_1 + z_2 = 2z\},$$

that allows an easier description of the phenomena. Besides the possibility of oscillatory solutions on the microscopic scale, we show some additional features: first, the appearance of a boundary-layer contribution on the boundary due to the asymmetry of the boundary interactions. This type of boundary contribution has been studied by Charlotte and Truskinovsky [28] in terms of local minima for quadratic interactions and here is described in energetic terms in the general case. The formula for the boundary contribution is quite general, and can be also formulated for higher-dimensional problems, where additional difficult technical issues arise (see *e.g.* the recent work by Theil [**?**]).

A second feature is the appearance of a phase-transition surface energy, that is due to the non convexity of the zero-order energy density ψ that forces the appearance of phase transitions and the appearance of internal boundary layers due to the presence of next-to-nearest neighbour interactions. By showing an equivalent family of continuum energies we highlight that second neighbours play the same role as the higher-order gradients in the gradient theory of phase transitions. It is worth noting that, under some assumptions on the geometry of displacements, by combining this result with the description of Lennard-Jones systems by Pagano and Paroni [41] we obtain a variational asymptotic theory with first and second gradients that qualitatively differs from that obtained as a pointwise limit (see *e.g.* Blanc, Le Bris and Lions [9]).

A third issue is that 'macroscopic' transitions must be coupled to 'microscopic' ones; *i.e.*, even if the limit deformation is an affine function $u(t) = zt$ the corresponding microscopic deformations may be forced to have oscillations corresponding to a minimizing pair (z_1, z_2) with $z_1 = (u^{i+1} - u^i)/\varepsilon$ for i odd mixed with oscillations corresponding to the same pair, but with $z_1 = (u^{i+1} - u^i)/\varepsilon$ for i even, thus introducing an 'anti-phase' boundary that may not be detected by the macroscopic averaged field u. This justifies a necessarily more complex description of the limit in terms of a vector variable $\mathbf{u} = (u_1, u_2)$ that separately describes 'even' and 'odd' oscillations. If we integrate out microscopic patters the limit theory takes a non-local form where the internal surface terms are influenced by the boundary and also between themselves. Note that anti-phase boundaries necessarily arise under some boundary conditions. It must be remarked that the use of the new vector variable u brings more information than the description by Young measures (see Paroni [**?**]), by which the interaction of micro-oscillations with phase transitions cannot be detected.

Finally, an additional fourth feature appears in the description of Lennard-Jones type microscopic interactions, where the higher-order Γ-limit gives a fracture term. The microscopic pattern influence the value of the fracture energy through the appearance of boundary layers on the two sides of the fracture. Note that these fracture boundary layer may compete with those forced by boundary conditions; as a consequence, for example, for Lennard-Jones interactions we obtain that fracture at the boundary is energetically favoured, in contrast with the nearest-neighbour case when fracture may appear anywhere in the sample.

2.1 Setting of the problem. Notation and preliminaries

We will consider one-dimensional next-to-nearest neighbour interactions on (a portion of) a lattice $\lambda_n\mathbb{Z}$ of the form $E_n(u) : \mathcal{B}_n(0, L) \to [0, +\infty)$ given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_1\Big(\frac{u^{i+1} - u^i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi_2\Big(\frac{u^{i+2} - u^i}{2\lambda_n}\Big),$$

where ψ_1, ψ_2 are Borel functions bounded from below (this condition can be relaxed). Here and in the following we set $\lambda_n = \frac{L}{n}$ and $\mathcal{B}_n(0, L) = \mathcal{B}_{\lambda_n}(0, L) = \{u : \mathbb{R} \to \mathbb{R} : \ u \in C(\mathbb{R}),\ u(t) \text{ is affine for } t \in (i, i+1)\lambda_n \ \forall\ i \in \{0, 1, \ldots, n-1\}\}$. We will also consider problems with fixed boundary data. To this end, given $l \in \mathbb{R}$ we define $E_n^l(u) : \mathcal{B}_n(0, L) \to [0, +\infty]$ as

$$E_n^l(u) = \begin{cases} E_n(u) & \text{if } u(0) = 0,\ u(L) = l \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1.1}$$

We also define the *effective* (*zero-order*) *energy density* of the system ψ_0 by

$$\psi_0(\alpha) = \psi_2(\alpha) + \frac{1}{2}\inf\{\psi_1(z_1) + \psi_1(z_2) \ : \ z_1 + z_2 = 2\alpha\} \tag{2.1.2}$$

obtained by 'minimizing out' the nearest-neighbour interactions (see [21], [41], [13]).

For any $\alpha \in \mathbb{R}$, we define the set of (microscopic) *minimal states* of the effective energy density $\mathbb{M}^\alpha$ as the set of all the pairs optimizing the minimum problem for $\psi_0(\alpha)$; *i.e.*

$$\mathbb{M}^\alpha := \{(z_1, z_2) \in \mathbb{R}^2 : z_1 + z_2 = 2\alpha,\ \psi_0(\alpha) = \psi_2(\alpha) + \frac{1}{2}\left(\psi_1(z_1) + \psi_1(z_2)\right)\}.$$

The case $\#\mathbb{M}^\alpha = 1$, so that $\mathbb{M}^\alpha = \{(\alpha,\alpha)\}$ and $\psi_0^{**}(\alpha) = \psi_1(\alpha)+\psi_2(\alpha)$, is usually referred to as the *strict Cauchy-Born hypothesis*, while the case $\#\mathbb{M}^\alpha = 2$ as *the local Cauchy-Born hypothesis*. In this case $\mathbb{M}^\alpha = \{(z_1^\alpha, z_2^\alpha), (z_2^\alpha, z_1^\alpha)\}$ with $z_2^\alpha \neq z_1^\alpha$.

For $l \in \mathbb{R}$, if $\psi_0^{**}(\frac{l}{L}) < \psi_0(\frac{l}{L})$, then ψ_0^{**} coincides with an affine function on a neighborhood of $(\frac{l}{L})$. We denote by r such affine function and let $N(l)$ be the number of α_i such that $\psi_0(\alpha_i) = \psi_0^{**}(\alpha_i) = r(\alpha_i)$. In the following we will make the assumption that $\#N(l) < +\infty$. We also define the set $\mathbb{M}_l$ as follows:

$$\mathbb{M}_l = \begin{cases} \emptyset & \text{if} \quad \psi_0^{**}(\frac{l}{L}) = +\infty \\ \mathbb{M}^{\frac{l}{L}} & \text{if} \quad \psi_0^{**}(\frac{l}{L}) = \psi_0(\frac{l}{L}) \\ \bigcup\limits_{i=1}^{N(l)} \mathbb{M}^{\alpha_i} & \text{if} \quad \psi_0^{**}(\frac{l}{L}) < \psi_0(\frac{l}{L}). \end{cases}$$

Let $\mathbf{z}^\alpha = (z_1^\alpha, z_2^\alpha) \in \mathbb{M}^\alpha$; we define the *minimal energy configurations* $u_{\mathbf{Z}^\alpha} : \mathbb{Z} \to \mathbb{R}$ and $\overline{u}_{\mathbf{Z}^\alpha} : \mathbb{Z} \to \mathbb{R}$ by

$$u_{\mathbf{Z}^\alpha}(i) = \left[\frac{i}{2}\right] z_2^\alpha + \left(i - \left[\frac{i}{2}\right]\right) z_1^\alpha, \quad \overline{u}_{\mathbf{Z}^\alpha}(i) = u_{\mathbf{Z}^\alpha}(i+1) - z_1^\alpha$$

and $u_{\mathbf{Z}^\alpha,n} : \lambda_n\mathbb{Z} \to \mathbb{R}$, $\overline{u}_{\mathbf{Z}^\alpha,n} : \lambda_n\mathbb{Z} \to \mathbb{R}$ by

$$u_{\mathbf{Z}^\alpha,n}(x_i^n) = u_{\mathbf{Z}^\alpha}(i)\lambda_n, \quad \overline{u}_{\mathbf{Z}^\alpha,n}(x_i^n) = \overline{u}_{\mathbf{Z}^\alpha}(i)\lambda_n. \tag{2.1.3}$$

Note that the gradient of (the piecewise affine interpolation corresponding to) $u_{\mathbf{Z}^\alpha}$ takes the values z_1^α, z_2^α on intervals $(i, i+1)$ with i even/odd respectively while the converse holds for $\overline{u}_{\mathbf{Z}^\alpha}$ and that the piecewise affine interpolations of both $u_{\mathbf{Z}^\alpha,n}$ and $\overline{u}_{\mathbf{Z}^\alpha,n}$ converge uniformly to αt.

2.1.1 Even and odd interpolation

In order to describe the fine behaviour of discrete minimizers we will separately consider even and odd indices. In order to separately track the limits of the corresponding interpolations, given $u : \mathcal{B}_n(0, L) \to \mathbb{R}$ we define the *even interpolator*

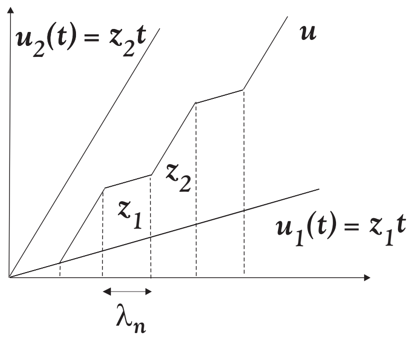

Figure 2.1: Interpolator functions for minimal energy configuration.

function $u_1 : \mathcal{B}_n(0, L) \to \mathbb{R}$ and the *odd interpolator function* $u_2 : \mathcal{B}_n(0, L) \to \mathbb{R}$ as follows:

$$u_1^0 = 0, \quad u_1^{i+1} - u_1^i = \begin{cases} u^{i+1} - u^i & \text{i is even} \\ u^i - u^{i-1} & \text{i is odd} \end{cases}$$

$$\text{if n is even,} \quad u_2^0 = 0 \quad u_2^{i+1} - u_2^i = \begin{cases} u^{i+1} - u^i & \text{i is odd} \\ u^{i+2} - u^{i+1} & \text{i is even} \end{cases}$$

$$\text{if n is odd,} \quad u_2^0 = 0 \quad u_2^{i+1} - u_2^i = \begin{cases} u^{i+1} - u^i & \text{i is odd} \\ u^{i+2} - u^{i+1} & \text{i is even, } i < n-1 \\ u^i - u^{i-1} & i = n-1. \end{cases}$$

We will say that a sequence of functions u_n belonging to $\mathcal{B}_n(0, L)$ converges to $\mathbf{u} = (u_1, u_2)$ in L^p $(1 \leq p \leq \infty)$ if $\mathbf{u}_n = (u_{n,1}, u_{n,2})$ converges to $\mathbf{u} = (u_1, u_2)$ in L^p. Note that this convergence implies that (but is not equivalent to) $u_n \to \frac{1}{2}(u_1 + u_2)$ in the usual sense in L^p. Moreover, for any functional space $\mathcal{B}$, we will write $\mathbf{u} \in \mathcal{B}$ meaning that $u_1, u_2 \in \mathcal{B}$.

With this notation the *minimal energy configurations* $u_{\mathbf{Z}^\alpha}$ and $\overline{u}_{\mathbf{Z}^\alpha}$ can be respectively identified with

$$\mathbf{u}_{\mathbf{Z}^\alpha}(i) = (z_1^\alpha i, z_2^\alpha i), \quad \overline{\mathbf{u}}_{\mathbf{Z}^\alpha}(i) = (z_2^\alpha i, z_1^\alpha i)$$

(see Fig. 2.1).

2.1.2 Crease and boundary-layer energies

We will show that (proper scalings of) the energies E_n give rise to phase-transition energies with interfacial energy and boundary terms. The quantification of these energies will be done by optimizing boundary and transition layers on the lattice level on the whole lattice (or only its positive part in the case of boundary layers) with minimal configurations as conditions at infinity. To this end we introduce the energy densities below. Note that the energies do not depend only on $\mathbf{z}$, but we have to take into account also a possible 'shift' since it may occur that it is energetically convenient not to match the minimal configuration exactly but its translation by a constant (which gives the same bulk contribution). Note that such a fixed translation is lost in the passage to the continuum.

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$, $\phi,\ l \in \mathbb{R}$. The *right-hand side boundary layer energy* of $\mathbf{z}$ with shift ϕ is

$$\begin{aligned} B_+(\mathbf{z}, \phi) = \inf_{N \in \mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^1 - u^0) + \sum_{i \geq 0} \Big(\psi_2\Big(\tfrac{u^{i+2} - u^i}{2}\Big) \\ & + \tfrac{1}{2}\Big(\psi_1\Big(u^{i+2} - u^{i+1}\Big) + \psi_1\Big(u^{i+1} - u^i\Big)\Big) - \psi_0\Big(\tfrac{z_1 + z_2}{2}\Big)\Big) \ : \\ & u : \mathbb{N} \to \mathbb{R}, u(0) = 0, u^i = u^i_{\mathbf{z}} - \phi \text{ if } i \geq N \Big\}. \end{aligned}$$

The *left-hand side boundary layer energy* of $\mathbf{z}$ with shift ϕ is

$$\begin{aligned} B_-(\mathbf{z}, \phi) = \inf_{N \in \mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^{-1} - u^0) + \sum_{i \leq 0} \Big(\psi_2\Big(\tfrac{u^{i} - u^{i-2}}{2}\Big) \\ & + \tfrac{1}{2}\Big(\psi_1\Big(u^{i} - u^{i-1}\Big) + \psi_1\Big(u^{i-1} - u^{i-2}\Big)\Big) - \psi_0\Big(\tfrac{z_1 + z_2}{2}\Big)\Big) \ : \\ & u : -\mathbb{N} \to \mathbb{R}, u(0) = 0, u^i = u^i_{\mathbf{z}} + \phi \text{ if } i \leq -N \Big\}. \end{aligned}$$

Let $\mathbf{z}' = (z_1', z_2') \in \mathbb{R}^2$. The (crease) *transition energy* between $\mathbf{z}$ and $\mathbf{z}'$ with shift ϕ is

$$\begin{aligned} C(\mathbf{z}, \mathbf{z}', \phi) = \inf_{N \in \mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^0 - u^{-1}) + \sum_{i \leq -1} \Big(\psi_2\Big(\tfrac{u^{i+2} - u^i}{2}\Big) \\ & + \tfrac{1}{2}\Big(\psi_1\Big(u^{i+2} - u^{i+1}\Big) + \psi_1\Big(u^{i+1} - u^i\Big)\Big) - \psi_0\Big(\tfrac{z_1 + z_2}{2}\Big)\Big) \\ & + \ \tfrac{1}{2}\psi_1(u^1 - u^0) + \sum_{i \geq 0} \Big(\psi_2\Big(\tfrac{u^{i+2} - u^i}{2}\Big) \\ & + \tfrac{1}{2}\Big(\psi_1\Big(u^{i+2} - u^{i+1}\Big) + \psi_1\Big(u^{i+1} - u^i\Big)\Big) - \psi_0\Big(\tfrac{z_1' + z_2'}{2}\Big)\Big) \ : \\ & \quad u : \mathbb{Z} \to \mathbb{R}, \quad u^i = u^i_{\mathbf{z}} + \phi_1 \text{ if } i \leq -N \\ & \quad u^i = u^i_{\mathbf{z}'} + \phi_2 \text{ if } i \geq N, \quad \phi = \phi_1 - \phi_2 \Big\}. \end{aligned}$$

Remark 2.1.1 Note that $B_+(\mathbf{z}, \phi) = B_-(\overline{\mathbf{z}}, -\phi)$. In the case of affine minimal-energy configurations, which is to say when $\#\mathbb{M}^z = \#\mathbb{M}^{z'} = 1$, we often have a simpler description of the limit. We introduce a slightly different notation for this case. If $\mathbf{z} = (z, z)$ and $\mathbf{z}' = (z', z')$ we set

$$B_\pm(z, \phi) = B_\pm(\mathbf{z}, \phi) \quad \text{and} \quad C(z, z', \phi) = C(\mathbf{z}, \mathbf{z}', \phi).$$

Remark 2.1.2 If ψ_1, ψ_2 are such that $\psi_0 \in C^1(\mathbb{R})$, then it can be easily shown that, for all $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^2$, B_+, B_- and C are shift independent, thus it is possible to rewrite the previous energies as follows

$$
\begin{aligned}
B_+(\mathbf{z}) = \inf_{N\in\mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^1-u^0) + \sum_{i\geq 0}\Big(\psi_2\Big(\tfrac{u^{i+2}-u^i}{2}\Big) \\
& +\tfrac{1}{2}\Big(\psi_1\Big(u^{i+2}-u^{i+1}\Big)+\psi_1\Big(u^{i+1}-u^i\Big)\Big)-\psi_0\Big(\tfrac{z_1+z_2}{2}\Big)\Big) \ : \\
& u:\mathbb{N}\to\mathbb{R}, \\
& u(0)=0,\ (u_1^{i+1}-u_1^i)=z_1,\ (u_2^{i+1}-u_2^i)=z_2 \text{ if } i\geq N\Big\}
\end{aligned}
$$

$$
\begin{aligned}
B_-(\mathbf{z}) = \inf_{N\in\mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^{-1}-u^0) + \sum_{i\leq 0}\Big(\psi_2\Big(\tfrac{u^{i}-u^{i-2}}{2}\Big) \\
& +\tfrac{1}{2}\Big(\psi_1\Big(u^{i}-u^{i-1}\Big)+\psi_1\Big(u^{i-1}-u^{i-2}\Big)\Big)-\psi_0\Big(\tfrac{z_1+z_2}{2}\Big)\Big) \ : \\
& u:-\mathbb{N}\to\mathbb{R}, \\
& u(0)=0,\ (u_1^{i+1}-u_1^i)=z_1,\ (u_2^{i+1}-u_2^i)=z_2 \text{ if } i\leq -N\Big\}
\end{aligned}
$$

$$
\begin{aligned}
C(\mathbf{z},\mathbf{z}') = \inf_{N\in\mathbb{N}} \min \Big\{ & \tfrac{1}{2}\psi_1(u^0-u^{-1}) + \sum_{i\leq -1}\Big(\psi_2\Big(\tfrac{u^{i+2}-u^i}{2}\Big) \\
& +\tfrac{1}{2}\Big(\psi_1\Big(u^{i+2}-u^{i+1}\Big)+\psi_1\Big(u^{i+1}-u^i\Big)\Big)-\psi_0\Big(\tfrac{z_1+z_2}{2}\Big)\Big) \\
& +\ \tfrac{1}{2}\psi_1(u^1-u^0) + \sum_{i\geq 0}\Big(\psi_2\Big(\tfrac{u^{i+2}-u^i}{2}\Big) \\
& +\tfrac{1}{2}\Big(\psi_1\Big(u^{i+2}-u^{i+1}\Big)+\psi_1\Big(u^{i+1}-u^i\Big)\Big)-\psi_0\Big(\tfrac{z_1'+z_2'}{2}\Big)\Big) \ : \\
& \ u:\mathbb{Z}\to\mathbb{R}, \\
& \ (u_1^{i+1}-u_1^i)=z_1,\ (u_2^{i+1}-u_2^i)=z_2 \text{ if } i\leq -N \\
& \ (u_1^{i+1}-u_1^i)=z_1',\ (u_2^{i+1}-u_2^i)=z_2' \text{ if } i\geq N\Big\}.
\end{aligned}
$$

Remark 2.1.3 By the previous two remarks we observe that, in the case $\#\mathbb{M}^z = 1$, if ψ_1, ψ_2 are such that $\psi_0 \in C^1(\mathbb{R})$, then $B_+(z) = B_-(z)$. Moreover, by the formula defining $B_\pm$ one easily gets that, set $\mathbf{z} = (z, z)$, if z is a minimum point for ψ_1, then $B_\pm(z) = \frac{1}{2}\min\psi_1$.

2.1.3 Development by Γ-convergence

As already pointed out, in our asymptotic description we will make use of proper scaling of the energies E_n. The following theorem will make precise our analysis.

Theorem 2.1.4 *Let $F_\varepsilon : X \to \overline{\mathbb{R}}$ be a family of d-equi-coercive functions and let $F^0 = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$. Let $m_\varepsilon = \inf F_\varepsilon$ and $m^0 = \min F^0$. Suppose that for some $\alpha > 0$ there exists the Γ-limit*

$$
F^\alpha = \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon - m^0}{\varepsilon^\alpha},
$$

and that the sequence $F_\varepsilon^\alpha = \frac{F_\varepsilon - m^0}{\varepsilon^\alpha}$ is d'-equi-coercive for a metric d' which is not weaker than d. Define $m^\alpha = \min F^\alpha$ and suppose that $m^\alpha \neq +\infty$; then we have *that*

$$m_\varepsilon = m^0 + \varepsilon^\alpha m^\alpha + o(\varepsilon)^\alpha$$

and from all sequences (x_ε) *such that* $F_\varepsilon(x_\varepsilon) - m_\varepsilon = o(\varepsilon)^\alpha$ *(in partcular this holds for minimizers) it is possible to extract a subsequence converging in* d' *to a point* x *which minimizes both* F^0 *and* F^α*.*

2.2 Γ-convergence for superlinear growth densities

2.2.1 Zero-order Γ-limit

In this section we give a description of the (zero-order) Γ-limit of the sequence E_n showing that the result by Braides, Gelli and Sigalotti [21] can be extended to Dirichlet and periodic boundary conditions. We have to take some extra care in that we do not assume that our potentials are everywhere finite. For the sake of simplicity, and without losing in generality, we will suppose $\psi_1,\ \psi_2$ to be non negative.

Theorem 2.2.1 (Zero order Γ-limit - Dirichlet boundary data) *Let* $\psi_1, \psi_2 : \mathbb{R} \to [0, +\infty]$ *be Borel functions such that the following hypotheses hold:*

[A1] *dom* $\psi_1 =$ *dom* ψ_2 *is an interval of* $\mathbb{R}$*,*

[A2] ψ_1 *and* ψ_2 *are lower semicontinuous on their domain,*

[A3] $\lim_{|z|\to+\infty} \frac{\psi_1(z)}{|z|} = +\infty.$

Then the Γ*-limit of* E^l_n *with respect to the* L^1*-topology is given by*

$$E^l(u) = \begin{cases} \displaystyle\int_0^L \psi_0^{**}(u'(t))dt & u \in W^{1,1}(0,L),\ u(0)=0,\ u(L)=l \\ +\infty & \textit{otherwise} \end{cases}$$

on $L^1(0, L)$*.*

Remark 2.2.2 It is possible to weaken hypothesis [A1] supposing that dom $\psi_1 = \bigcup_i A_i$ where A_i is an interval of the real line. In this case some extra condition is needed. For example, one can suppose that, if $z \in$ dom ψ_0 and $z_1,\ z_2$ are optimal for z in the sense of (2.1.2), then $z_1,\ z_2 \in A_i$ and that dom ψ_2 contains the convex hull of dom ψ_1.

Proof of Theorem 2.2.1. In the following we suppose that $L = 1$. Let $u_n \to u$ in $L^1(0, 1)$ and be such that $\sup_n E^l_n(u_n) < +\infty$, then, up to subsequences $u_n \rightharpoonup u$ weakly in $W^{1,1}(0, 1)$ and $u(0) = 0$, $u(1) = l$. Moreover

$$\liminf_n E^l_n(u_n) \geq \int_{(0,1)} \psi_0^{**}(u'(t))\, dt.$$

To prove the Γ-$\limsup$ inequality we consider two cases:

$$(a)\ l \text{ is an internal point of dom } \psi_0, \qquad (b)\ l \in \partial\text{dom } \psi_0.$$

In case (a) we use a density argument. Let u be such that $E^l(u) < +\infty$. Then $u'(t) \in \mathrm{dom}\psi_0$ for a.e. $t \in (0,1)$. Without loss of generality we may suppose that $\mathrm{dom}\psi_0 = [0, +\infty)$ or $\mathrm{dom}\psi_0 = (0, +\infty)$. If $u' \geq c > 0$ the density argument is easy as it is possible to construct a sequence of piecewise affine functions (u_n) such that $u_n' \geq \frac{c}{2} > 0$, $u_n \to u$ in $W^{1,1}(0,1)$ and $\lim_n \int_0^1 \psi_0^{**}(u_n') = \int_0^1 \psi_0^{**}(u')$. If otherwise $\inf u' = 0$, then $|\{t : \ u'(t) > l\}| \neq 0$ and $\eta > 0$ exists such that $|\{t : \ l + \eta < u'(t) < \frac{1}{\eta}\}| > 0$. Let $u_T \in W^{1,\infty}(0,1)$ be such that $u_T(0) = 0$ and $u_T' = u' \vee T$, and let

$$v_T(t) = u_T(t) + c_T \int_0^t \chi_{\{t:\ u' \in (l+\eta, \frac{1}{\eta})\}} \text{ where } c_T = \frac{l - u_T(1)}{|\{t : \ l + \eta < u' < \frac{1}{\eta}\}|}.$$

Observe that $\lim_{T \to 0^+} c_T = 0$. We have that $v_T \in W^{1,\infty}(0,1)$, $v_T(0) = 0$, $v_T(1) = l$ and that, for $T \to 0^+$, $v_T \to u$ in $W^{1,1}(0,1)$. By lower semicontinuity we have

$$\liminf_{T \to a} \int_{(0,1)} \psi_0^{**}(v_T'(t))\, dt \geq \int_{(0,1)} \psi_0^{**}(u'(t))\, dt.$$

Moreover

$$\begin{aligned} \int_{(0,1)} \psi_0^{**}(v_T')\, dt = & \int_{\{t:\ u' \leq T\}} \psi_0^{**}(T)\, dt + \int_{\{t:\ T < u' \leq l+\eta\}} \psi_0^{**}(u')\, dt \\ & + \int_{\{t:\ l+\eta \leq u' \leq \frac{1}{\eta}\}} \psi_0^{**}(u' + c_T)\, dt + \int_{\{t:\ u' > \frac{1}{\eta}\}} \psi_0^{**}(u')\, dt. \end{aligned} \tag{2.2.1}$$

Observe that, thanks to the uniform continuity of ψ_0^{**} on compact sets, we have

$$\lim_{T \to 0^+} \int_{\{t:\ l+\eta \leq u' \leq \frac{1}{\eta}\}} \psi_0^{**}(u' + c_T)\, dt = \int_{\{t:\ l+\eta \leq u' \leq \frac{1}{\eta}\}} \psi_0^{**}(u')\, dt. \tag{2.2.2}$$

To pass to the limit in the equality (2.2.1) we need to consider the following two cases:

$$\text{(i)} \lim_{T \to 0^+} \psi_0^{**}(T) = +\infty, \quad \text{(ii)} \lim_{T \to 0^+} \psi_0^{**}(T) < +\infty.$$

In case (i), for T small enough we have that $\psi_0^{**}(T) \leq \psi_0^{**}(u'(t))$ for a.e. t such that $u'(t) \leq T$, hence

$$\int_{\{t:\ u' \leq T\}} \psi_0^{**}(T)\, dt \leq \int_{\{t:\ u' \leq T\}} \psi_0^{**}(u')\, dt.$$

In case (ii) we have that ψ_0^{**} is uniformly continuous in $[0, \frac{1}{\eta}]$. Hence, passing to the limit as $T \to 0^+$ in (2.2.1), thanks to (2.2.2) we finally have that

$$\limsup_{T\to 0^+} \int\limits_{(0,1)} \psi_0^{**}(v_T'(t))\, dt \le \int\limits_{(0,1)} \psi_0^{**}(u'(t))\, dt. \tag{2.2.3}$$

Thanks to (2.2.3) and a density argument it suffices to prove the Γ- lim sup inequality for $u(t)$ piecewise affine. For the sake of simplicity we prove it for $u(t) = zt$ with z such that $\psi_0^{**}(z) = \psi_0(z)$ as the general case can be easily obtained by a convexity argument. Thanks to hypothesis [A2] there exist $\mathbf{z} = (z_1, z_2) \in \mathbb{M}^l$. Setting $u_n = u_{\mathbf{z},n}$ as in (2.1.3), then $u_n \to u$ in $L^1(0,1)$ and $\lim_n E_n(u_n) = \int\limits_{(0,1)} \psi_0(u')\, dt$. Defining

$$v_n(t) = \begin{cases} u_n(t) & \text{if } t \in [0, 1-\lambda_n] \\ u_n(1-\lambda_n) + \dfrac{(l - u_n(1-\lambda_n)}{\lambda_n}(t - 1 + \lambda_n) & \text{if } t \in [1-\lambda_n, 1] \end{cases}$$

then $v_n(0) = 0$, $v_n(1) = z$ and it holds

$$E_n(u_n) - E_n(v_n) \le \lambda_n |\psi_2(z) - \psi_2(\frac{z+z_2}{2})| + \lambda_n |\psi_1(z_1) - \psi_1(z)|.$$

Thanks to hypotheses [A1] we have that

$$\lim_n (E_n(u_n) - E_n(v_n)) = 0,$$

thus proving that v_n is the recovery sequence for our problem.

In case (b), observing that $\psi_0^{**}(l) = \psi_0(l)$ and that $u(t) = lt$, the proof is easily obtained as the boundary condition is automatically satisfied for $u_n = u_{\mathbf{z},n}$ when $z_1 = z_2 = l$. □

Remark 2.2.3 Observe that in the previous proof the construction of v_n is simplified in the case of *even* lattices.

We now state the analog result in the periodic case. Set

$$\mathcal{B}_n(\mathbb{R}) := \{u : \mathbb{R} \to \mathbb{R} : \; u \in C(\mathbb{R}), \; u(t) \text{ is affine for } t \in (i, i+1)\lambda_n \; \text{ for all } \; i \in \mathbb{Z}\},$$

let $E_n^{\#,l}(u) : \mathcal{B}_n(\mathbb{R}) \to [0, +\infty]$ be defined as

$$E_n^{\#,l}(u) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \lambda_n \psi_1\Big(\frac{u^{i+1} - u^i}{\lambda_n}\Big) + \sum_{i=0}^{n-1} \lambda_n \psi_2\Big(\frac{u^{i+2} - u^i}{2\lambda_n}\Big) & \text{if } u \in \mathcal{B}_n^{\#,l}(0, L) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_n^{\#,l}(0, L) := \{u \in \mathcal{B}_n(\mathbb{R}) \; : u((i+n)\lambda_n) = u(i\lambda_n) + l\}$ (*i.e.*, $u(t)$ are $L-$periodic perturbation of the affine function $\frac{l}{L}t$).

Theorem 2.2.4 (Zero order Γ-limit - Periodic boundary data) *In the hypotheses of Theorem* 2.2.1 *the* Γ*-limit of* $E_n^{\#,l}$ *with respect to the* $L^1_{\rm loc}$*-topology is given by*

$$E^{\#,l}(u) = \begin{cases} \displaystyle\int_0^L \psi_0^{**}(u'(t))\,dt & \text{if } u \in W^{1,1}_{\#,l}(0,L) \\ +\infty & \text{otherwise} \end{cases}$$

on $L^1_{\rm loc}(\mathbb{R})$*, where* $W^{1,1}_{\#,l}(0,L) := \{u \in W^{1,1}_{\rm loc}(\mathbb{R}) : \ u(t) - lt \text{ is } L \text{ periodic}\}$.

Proof. Supposing that $L = 1$, let $u_n \to u$ in $L^1_{\rm loc}(\mathbb{R})$ such that $\sup_n E_n^{\#,l}(u_n) < +\infty$. Then, up to subsequences, $u_n \rightharpoonup u$ weakly in $W^{1,1}_{\rm loc}(\mathbb{R})$ and a.e. in every compact set of $\mathbb{R}$. Thus

$$u(x+1) - u(x) = \lim_n (u_n(x+1) - u_n(x)) = \lim_n (u_n([x]+1) - u_n([x])) = l$$

and then $u \in W^{1,1}_{\#,l}(0,1)$. The thesis is easily obtained by arguing as in the proof of Theorem 2.2.1. □

2.2.2 First-order Γ-limit

In this section we compute the first-order Γ-limit of E_n under periodic or Dirichlet type boundary conditions, and show the appearance of phase transitions and boundary terms in the limit energy. Interfacial energies will appear in the case when ψ_0 is a non-convex function. Our model case is when ψ_0 is a double-well potential with two minimum points (in particular there is only one 'interval of non-convexity'), and each minimum point z possesses only one (in the trivial case (z,z)) or two (*i.e.*, (z_1,z_2) and (z_2,z_1) with $z_1 \neq z_2$) minimal-energy configuration. We nevertheless treat a more general case, for which some hypotheses (that are always satisfied except for 'degenerate' cases) must be made clear as follows:

[H1](*discreteness of the energy states*)

$$\#(\{x \in \mathbb{R} : \psi_0(x) = \psi_0^{**}(x)\} \cap \{x \in \mathbb{R} : \psi_0 \text{ is affine}\}) < +\infty.$$

This condition is necessary in order to deal with a finite number of accessible energy states;

[H2] (*finiteness of minimal energy configurations*) for every $\alpha \in \mathbb{R}$ such that $\psi_0(\alpha) = \psi_0^{**}(\alpha)$

$$\#\mathbb{M}^\alpha < +\infty;$$

[H3] (*compatibility of minimal energy configurations*) for every $\alpha, \ \beta \in \mathbb{R}$ with $\alpha \neq \beta$, such that $\psi_0(\alpha) = \psi_0^{**}(\alpha)$ and $\psi_0(\beta) = \psi_0^{**}(\beta)$ and for every $\mathbf{z}^\alpha = (z_1^\alpha, z_2^\alpha) \in \mathbb{M}^\alpha$ and $\mathbf{z}^\beta = (z_1^\beta, z_2^\beta) \in \mathbb{M}^\beta$ it holds

$$z_i^\alpha \neq z_j^\beta \quad i,j \in \{1,2\}.$$

This condition is necessary in order to have a non-zero surface energy for the transition from α to β;

[H4] (*continuity and growth conditions*) $\psi_1,\ \psi_2\ : \mathbb{R} \to [0, +\infty]$ are Lipschitz functions such that

$$\psi_1(z) \geq mz + q$$

for some $m, q \in \mathbb{R}$ with $m \neq 0$ (note that m is not required to be positive) and that l is such that

$$\lim_{|z| \to +\infty} \psi_0(z) - pz = +\infty \qquad \text{for all } p \in \partial\psi_0^{**}(l);$$

[H5] (*non-degeneracy for the boundary datum*) l is such that

$$\lim_{|z| \to +\infty} \psi_1(z) - pz = +\infty \qquad \text{for all } p \in \partial\psi_0^{**}(l);$$

[H6] (*finiteness of the intervals of non-convexity*) l is such that $N(l) < +\infty$ ($N(l)$ defined as in (2.1.3)).

The following compactness result states that functions u_n such that $E_n^l(u_n) = \min E^l + O(\lambda_n)$ locally have microscopic oscillations close to minimal-energy configurations belonging to $\mathbb{M}_l$, except for a finite number of interactions that concentrate on a finite set S in the limit.

Proposition 2.2.5 (Compactness - Dirichlet boundary data) *Suppose that hypotheses* [H1]–[H6] *hold. If* $\{u_n\}$ *is a sequence of functions such that*

$$\sup_n E_n^{1,l}(u_n) = \sup_n \frac{E_n^l(u_n) - \min E^l}{\lambda_n} < +\infty \tag{2.2.4}$$

then there exists a set $S \subset (0, L)$ *with* $\#S < +\infty$ *such that, up to subsequences,* u_n *converges to* $\mathbf{u} = (u_1, u_2)$ *in* $W^{1,\infty}_{\text{loc}}((0, L) \setminus S)$ *where* u_1, u_2 *are piecewise-affine functions and* $u_1(L) + u_2(L) = 2l$. *Moreover* $\mathbf{u}'(t) \in \mathbb{M}_l$ *for a.e.* $t \in (0, L)$ *and* $S(\mathbf{u}') = S(u_1') \cup S(u_2') \subseteq S$.

Proof. For simplicity of notation we can suppose that $L = 1$ and that n is even, the proof being analogous in the general case.

Set

$$\tilde{\psi}_1(z) = \psi_1(z) - \frac{m}{2} z,$$

(m as in $[H4]$) we have that

$$\tilde{\psi}_1(z) \geq c(|z| - 1) \tag{2.2.5}$$

for some $c > 0$. Thus we have

$$+\infty > E_n(u_n) \quad \geq \quad \sum_{i=0}^{n-1} \lambda_n \psi_1\left(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big) + \frac{m}{2}\sum_{i=0}^{n-1}(u_n^{i+1}-u_n^i)\\
&= \sum_{i=0}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big) + \frac{m}{2}l.
\end{aligned}
$$

Then, by the definition of even and odd interpolations, we have

$$
\begin{aligned}
+\infty &> 2\sum_{i=0,\ even}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big) + 2\sum_{i=0,\ odd}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big)\\
&= \sum_{i=0}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_{1,n}^{i+1}-u_{1,n}^i}{\lambda_n}\Big) + \sum_{i=0}^{n-1} \lambda_n \tilde{\psi}_1\Big(\frac{u_{2,n}^{i+1}-u_{2,n}^i}{\lambda_n}\Big)\\
&= \int_0^1 \tilde{\psi}_1(u'_{1,n}(t))\,dt + \int_0^1 \tilde{\psi}_1(u'_{2,n}(t))\,dt.
\end{aligned}
$$

Thanks to (2.2.5) we get, for $h \in \{1,2\}$

$$
\int_0^1 |u'_{h,n}(t)|\,dt < +\infty. \tag{2.2.6}
$$

Let $\{J_j\},\ \{K_j\}$ be two families of intervals of the real line where ψ_0^{**} is, respectively, a straight line or a strictly convex function, satisfying

$$
\begin{gathered}
\partial\psi_0^{**}(x) \neq \partial\psi_0^{**}(y) \quad \text{for all } x \in J_j,\ y \in J_{j+1}\\
K_j,\ K_{j+1} \text{ are not contiguous,}
\end{gathered}
$$

where we have denoted by $\partial\psi_0^{**}(x)$ the sub-differential of ψ_0^{**} in x. Note that, by the growth conditions on ψ_0, J_j is a bounded interval. Suppose that $l \in J_j$ for some j and that $p(l) \in \partial\psi_0^{**}(l)$. We define $r_j(x) = p(l)(x-l) + \psi_0^{**}(l)$, the straight line such that $\psi_0^{**}(x) = r_j(x)$ for all $x \in J_j$. Since $\min E^l = \psi_0^{**}(l)$, by (2.2.4) we get

$$
\begin{aligned}
C \geq E_n^{1,l}(u_n) &= \sum_{i=0}^{n-2}\Big(\psi_2\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)\\
&\quad +\frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big)\Big) - \psi_0^{**}(l)\Big)\\
&\quad +\frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{n}-u_n^{n-1}}{\lambda_n}\Big) + \psi_1\Big(\frac{u_n^1-u_n^0}{\lambda_n}\Big)\Big) - \psi_0^{**}(l)\\
&= \sum_{i=0}^{n-2}\mathcal{E}_n^i(u_n) + \frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{n}-u_n^{n-1}}{\lambda_n}\Big)\\
&\quad +\psi_1\Big(\frac{u_n^1-u_n^0}{\lambda_n}\Big) - r_j\Big(\frac{u_n^{n}-u_n^{n-1}}{\lambda_n}\Big) - r_j\Big(\frac{u_n^1-u_n^0}{\lambda_n}\Big)\Big),
\end{aligned}
$$

where we have set

$$\mathcal{E}_n^i(u_n) = \psi_2\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)+\frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big)\Big)-r_j\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big).$$

Thanks to the continuity of $\psi_1(z)$ and $r_j(z)$ and to hypothesis [H5], we have that $\psi_1(z)-r_j(z)$ has a finite minimum since $\lim_{|z|\to+\infty}(\psi_1(z)-r_j(z)) = +\infty$. It follows that

$$\sum_{i=0}^{n-2}\mathcal{E}_n^i(u_n) \le C. \tag{2.2.7}$$

We infer that, for every $\eta > 0$, if we define $I_n(\eta) := \{i \in \{0,1,\dots,n-2\} : \mathcal{E}_n^i(u_n) > \eta\}$, then

$$\sup_n \#I_n(\eta) \le C(\eta) < +\infty.$$

Let $i \notin I_n(\eta)$; then by (2.2.7)

$$0\le\psi_2\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)+\frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n}\Big)\Big)-\psi_0\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)\le\eta$$

$$0\le\psi_0\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)-r_j\Big(\frac{u_n^{i+2}-u_n^i}{2\lambda_n}\Big)\le\eta\ .$$

Let $\varepsilon = \varepsilon(\eta)$ be defined so that if

$$\begin{aligned}
&0 \le \psi_0(z) - r_j(z) \le \eta,\\
&0 \le \psi_2(z) + \frac{1}{2}\Big(\psi_1(z_1)+\psi_1(z_2)\Big) - \psi_0(z) \le \eta \quad \text{with } z_1+z_2 = 2z,
\end{aligned}$$

then

$$\mathrm{dist}\,((z_1,z_2),\mathbb{M}_l) \le \varepsilon(\eta).$$

Chosen $\eta > 0$ such that

$$2\varepsilon(\eta) < \min\{|\mathbf{z}'-\mathbf{z}''|,\ \mathbf{z}',\mathbf{z}''\in\mathbb{M}_l\},$$

we deduce that, if $i-1,\ i \notin I_n(\eta)$ then there exists $\mathbf{z} \in \mathbb{M}_l$ such that

$$\Big|\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n},\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big)-\mathbf{z}\Big|\le\varepsilon$$

and

$$\Big|\Big(\frac{u_n^{i}-u_n^{i-1}}{\lambda_n},\frac{u_n^{i+1}-u_n^{i}}{\lambda_n}\Big)-\overline{\mathbf{z}}\Big|\le\varepsilon.$$

Hence, there exists a finite number of indices $0 = i_1 < i_2 < \ldots < i_{N_n} = n$ such that for all $k = 1,2,\dots,N_n$ there exists $\mathbf{z}_k^n = (z_{1,k}^n, z_{2,k}^n) \in \mathbb{M}_l$ such that for all $i \in \{i_{k-1}+1, i_{k-1}+2,\dots,i_k-1\}$ we have

$$\Big|\Big(\frac{u_n^{i+1}-u_n^i}{\lambda_n},\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big)-\mathbf{z}_k^n\Big|\le\varepsilon.$$

Then, by the definitions of even and odd interpolations it can be easily seen that

$$\begin{aligned}
&\Big|\frac{u_{1,n}^{i+1}-u_{1,n}^{i}}{\lambda_n}-z_{1,k}^{n}\Big|\le\varepsilon \quad i\in\{i_{k-1}+2,i_{k-1}+3,\dots,i_k-1\},\\
&\Big|\frac{u_{2,n}^{i+1}-u_{2,n}^{i}}{\lambda_n}-z_{2,k}^{n}\Big|\le\varepsilon \quad i\in\{i_{k-1}+1,i_{k-1}+2,\dots,i_k-2\}.
\end{aligned}\tag{2.2.8}$$

Let $\{j_1,j_2,\dots,j_{M_n}\}$ be the maximal subset of $0=i_1<i_2<\ldots<i_{N_n}=n$ defined by the requirement that if $\mathbf{z}_{j_k}^n\in\mathbb{M}^\beta$ then $\mathbf{z}_{j_k+1}^n\in\mathbb{M}^\gamma$ with $\beta\neq\gamma$ and $\mathbb{M}^\beta,\mathbb{M}^\gamma\subset\mathbb{M}_l$. Thanks to (2.2.4) there exists $C(\eta)>0$ such that $E_n^1(u_n)\ge C(\eta)M_n$. Thus, up to further subsequences, we can suppose that $M_n=M$, $\mathbf{z}_{j_k}^n=\mathbf{z}_k=(z_{1,k},z_{2,k})$ and that $x_{j_k}^n\to x_k$. Fix δ, set $S=\bigcup\limits_k x_k$ and $S_\delta=\bigcup\limits_k(x_k-\delta,x_k+\delta)$. Then, by (2.2.8) we get

$$\sup_{t\in(0,1)\setminus S_\delta}|u_{s,n}'(t)-z_{s,k}|\le\varepsilon \quad s\in\{1,2\}.$$

The previous estimates, together with (2.2.6) ensure that $\mathbf{u}_n$ is an equicontinuous and equibounded sequence in $(0,1)\setminus S_\delta$. Thus, thanks to the arbitrariness of δ, up to passing to a further subsequence (not relabelled), $\mathbf{u}_n$ converges in $W^{1,\infty}_{\rm loc}((0,1)\setminus S)$ to a function $\mathbf{u}$ such that $\mathbf{u}'(t)\in\mathbb{M}_l$ a.e. $t\in(0,1)$. Moreover $S(\mathbf{u}')\subseteq S$.

To prove that u_1 and u_2 are piecewise affine functions, we need to prove that they are continuous. Suppose by contradiction that $S(u_1)\neq\emptyset$. Then, for n large enough,

$$\text{for all } M\in\mathbb{N} \text{ there exits } i:\ \Big|\frac{u_{1,n}^{i+1}-u_{1,n}^{i}}{\lambda_n}\Big|>M.\tag{2.2.9}$$

Then, by (2.2.4), we have that, for some j

$$\begin{aligned}
C \ \ge\ &\sum_{i=0,\ even}^{n-2}\Bigg(\psi_2\Big(\frac{u_n^{i+2}-u_n^{i}}{2\lambda_n}\Big)+\frac12\Big(\psi_1\Big(\frac{u_n^{i+2}-u_n^{i+1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_n^{i+1}-u_n^{i}}{\lambda_n}\Big)\Big)\\
&-r_j\Big(\frac{u_n^{i+2}-u_n^{i}}{2\lambda_n}\Big)\Bigg).
\end{aligned}$$

Then, for i even, by the definition of even and odd interpolations, we get

$$C\ge\frac12\Big(\psi_1\Big(\frac{u_{2,n}^{i+2}-u_{2,n}^{i+1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_{1,n}^{i+1}-u_{1,n}^{i}}{\lambda_n}\Big)\Big)-r_j\Big(\frac{u_{2,n}^{i+2}-u_{2,n}^{i+1}+u_{1,n}^{i+1}-u_{1,n}^{i}}{2\lambda_n}\Big).$$

Since $r_j(z)=pz+q$ with $p\in\partial\psi_0^{**}(l)$, the previous estimate gives

$$C\ge\frac12\Big(\psi_1\Big(\frac{u_{2,n}^{i+2}-u_{2,n}^{i+1}}{\lambda_n}\Big)-p\Big(\frac{u_{2,n}^{i+2}-u_{2,n}^{i+1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_{1,n}^{i+1}-u_{1,n}^{i}}{\lambda_n}\Big)-p\Big(\frac{u_{1,n}^{i+1}-u_{1,n}^{i}}{\lambda_n}\Big)\Big).$$

By the previous inequality, we get the contradiction thanks to (2.2.9) and to hypothesis [H5].

The same argument can be exploited also in the case $l\in K_j$ for some j. $\square$

Remark 2.2.6 (Boundary terms blow-up) Observe that, if hypothesis [H5] is dropped, it is possible to produce an example of ψ_1 and ψ_2 and a sequence (u_n) equibounded in energy such that

$$\lim_n \frac{1}{2}\Big(\psi_1\Big(\frac{u_n^n-u_n^{n-1}}{\lambda_n}\Big)+\psi_1\Big(\frac{u_n^1-u_n^0}{\lambda_n}\Big)-r_j\Big(\frac{u_n^n-u_n^{n-1}}{\lambda_n}\Big)-r_j\Big(\frac{u_n^1-u_n^0}{\lambda_n}\Big)\Big)=-\infty \tag{2.2.10}$$

preventing us from deducing inequality (2.2.7). In fact if

$$\psi_1(z)=|z|-1, \qquad \psi_2(z)=\begin{cases}-2z-6 & z\in(-\infty,-1],\\ 5z+1 & z\in(-1,0],\\ -z+1 & z\in(0,1],\\ 4z-4 & z\in(1,+\infty),\end{cases}$$

we have that

$$\psi_0^{**}(z)=\begin{cases}-3z-7 & z\in(-\infty,-1],\\ 2z-2 & z\in(-1,+\infty).\end{cases}$$

For $l=1$ we have that $\partial\psi_0^{**}(l)=\{2\}$ and that

$$\lim_{z\to\infty}\psi_1(z)-2z=-\infty,$$

thus not fulfilling hypothesis [H5]. The sequence

$$u_n^0=0, \qquad u_n^{i+1}-u_n^i=\begin{cases}\sqrt{\lambda_n} & i=0,\\ \lambda_n & i=1,2,\dots,n-2,\\ 2\lambda_n-\sqrt{\lambda_n} & i=n-1,\end{cases}$$

satisfies (2.2.10) and is such that $E_n^{1,l}(u_n)=0$.

The first Γ-limit is given in terms of the variables $\mathbf{u}$ (giving microscopic oscillations) and s (the shift). It describes transitions between different phases through the term C and with the boundary through the terms $B_\pm$. The final form of the limit is obtained by optimizing in the shift term, taking care of the compatibility restrictions due to the boundary conditions. Note the difference in the limit boundary conditions in the even and odd cases.

Theorem 2.2.7 (First order Γ-limit - Dirichlet boundary data) *Suppose that hypotheses* [H1]–[H6] *hold and let* $E_n^{1,l}:\mathcal{B}_n(0,L)\to[0,+\infty]$ *be defined by*

$$E_n^{1,l}(u)=\frac{E_n^l(u)-\min E^l}{\lambda_n}.$$

We then have:

(Case n even) $E_n^{1,l}$ Γ-*converges with respect to the* L^∞-*topology to*

$$E_{even}^{1,l}(\mathbf{u}) = \inf\Big\{E_{even}^{1,l}(\mathbf{u}, s): \ s : S(\mathbf{u}') \cup \{0, L\} \to \mathbb{R}, \sum_{t\in S(\mathbf{u}')\cup\{0,L\}} s(t) = l\Big\}$$

where

$$E_{even}^{1,l}(\mathbf{u}, s) = \begin{cases} \sum\limits_{t\in S(\mathbf{u}')} C(\mathbf{u}'(t-), \mathbf{u}'(t+), s(t)) + B_+(\mathbf{u}'(0), s(0)) + B_-(\mathbf{u}'(L), s(L)), \\ \qquad\qquad \text{if } \mathbf{u}' \in PC(0, L),\ \mathbf{u}' \in \mathbb{M}_l,\ u_1(L) + u_2(L) = 2l \\ +\infty \qquad\qquad \text{otherwise} \end{cases}$$

on $W^{1,\infty}(0, L)$.

(Case n odd) $E_n^{1,l}$ Γ-*converges with respect to the* L^∞-*topology to*

$$E_{odd}^{1,l}(\mathbf{u}) = \inf\Big\{E_{odd}^{1,l}(\mathbf{u}, s): \ s : S(\mathbf{u}') \cup \{0, L\} \to \mathbb{R}, \sum_{t\in S(\mathbf{u}')\cup\{0,L\}} s(t) = l\Big\}$$

where

$$E_{odd}^{1,l}(\mathbf{u}, s) = \begin{cases} \sum\limits_{t\in S_{\mathbf{u}'}} C(\mathbf{u}'(t-), \mathbf{u}'(t+), s(t)) + B_+(\mathbf{u}'(0), s(0)) + B_-(\overline{\mathbf{u}'(L)}, s(L)) \\ \qquad\qquad \text{if } \mathbf{u}' \in PC(0, L),\ \mathbf{u}' \in \mathbb{M}_l,\ u_1(L) + u_2(L) = 2l \\ +\infty \qquad\qquad \text{otherwise} \end{cases}$$

on $W^{1,\infty}(0, L)$.

PROOF. The general case being dealt with similarly, in the following we will suppose that n is even, $L = 1$ and, using the notation of the previous proof, that $l \in J_j$ for some j.

Γ-liminf inequality. Let $u_n \to \mathbf{u}$ in $L^\infty(0, 1)$ be such that $E_n^{1,l}(u_n) < +\infty$. Then, thanks to Proposition 2.2.5 there exist $M \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_M \in \mathbb{M}_l$ and $0 = x_0 < x_1 < \ldots < x_M = 1$ such that

$$u_n'(t) \to \mathbf{z}^{\alpha_j} \quad t \in (x_{j-1}, x_j) \quad j \in \{1, 2, \ldots, M\}. \tag{2.2.11}$$

For $i \in \{0, 1, \ldots, M\}$, let $\{k_n^i\}_n$ be a sequence of indices such that $k_n^0 = 0$ and

$$\lim_n \Big(k_n^i - \sum_{j=1}^{i} \frac{x_j - x_{j-1}}{\lambda_n}\Big) = 0, \tag{2.2.12}$$

and let $\{h_n^i\}_n$ be a sequence of indices such that

$$\lim_n \lambda_n h_n^i = \frac{x_i - x_{i-1}}{2}.$$

Since ψ_0^{**} is affine in J_j, we have that

$$n\psi_0^{**}(l) \quad = \quad n\psi_0^{**}\Big(\sum_{j=1}^{M} \alpha_j(x_j - x_{j-1})\Big) = \sum_{j=1}^{M} n(x_j - x_{j-1})\psi_0^{**}(\alpha_j)$$

$$
\begin{aligned}
&= \sum_{j=1}^{M} \sum_{i=k_n^{j-1}}^{k_n^j-1} n\frac{(x_j - x_{j-1})}{(k_n^j - k_n^{j-1})}\psi_0(\alpha_j) \\
&= \sum_{j=1}^{M-1} \sum_{i=k_n^{j-1}}^{k_n^j-1} \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \sum_{i=k_n^{M-1}}^{n-2} \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \psi_0(\alpha_M) + R_n,
\end{aligned}
$$

with

$$
\begin{aligned}
R_n \;=\; & \sum_{j=1}^{M-1} \sum_{i=k_n^{j-1}}^{k_n^j-1} \Big(\Big(n\frac{(x_j - x_{j-1})}{(k_n^j - k_n^{j-1})}\Big)\psi_0(\alpha_j) - \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big)\Big) \\
& + \sum_{i=k_n^{M-1}}^{n-2} \Big(n\frac{(x_M - x_{M-1})}{(k_n^M - k_n^{M-1})}\Big)\Big(\psi_0(\alpha_M) - \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big)\Big) \\
& + \Big(n\frac{(x_M - x_{M-1})}{(k_n^M - k_n^{M-1})} - 1\Big)\psi_0(\alpha_M).
\end{aligned}
$$

Thanks to Proposition 2.2.5, (2.2.12) and to the continuity of ψ_0 we have that $R_n \to 0$. To get the Γ-liminf inequality it is useful to rewrite the energy as follows:

$$
\begin{aligned}
E_n^1(u_n) = \; & \sum_{i=0}^{n-2} \psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \sum_{i=0}^{n-1} \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) \\
& - \sum_{j=1}^{M-1} \sum_{i=k_n^{j-1}}^{k_n^j-1} \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) - \sum_{i=k_n^{M-1}}^{n-2} \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) - \psi_0(\alpha_M) - R_n \\
= \; & E_n^1(u_n, h_n^1) + \sum_{j=1}^{M-1} E_n^1(u_n, h_n^j, h_n^{j+1}) + E_n^1(u_n, h_n^M) - R_n \qquad (2.2.13)
\end{aligned}
$$

where we have set

$$
\begin{aligned}
E_n^1(u_n, h_n^1) &= \sum_{i=0}^{h_n^1-1} \Big(\psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) - \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big)\Big), \\
E_n^1(u_n, h_n^j, h_n^{j+1}) &= \sum_{i=h_n^j}^{h_n^{j+1}-1} \Big(\psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) - \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big)\Big), \\
E_n^1(u_n, h_n^M) &= \sum_{i=h_n^M}^{n-2} \Big(\psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) - \psi_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big)\Big) \\
& \quad + \psi_1\Big(\frac{u_n^n - u_n^{n-1}}{\lambda_n}\Big) - \psi_0(\alpha_M).
\end{aligned}
$$

As the general case can be obtained by slightly modifying the definition of $\tilde{\mathbf{u}}_n$, in the sequel we will suppose that $h^j_n, k^j_n, h^j_n - k^j_n, h^{j+1}_n - k^j_n$ are even. Defining

$$\tilde{\mathbf{u}}^i_n = \begin{cases} \dfrac{\mathbf{u}^i_n}{\lambda_n} & \text{if } i \in \{0, 1, \dots, h^1_n\} \\ \mathbf{u}_{\mathbf{Z}^{\alpha_1}}(i) - \mathbf{u}_{\mathbf{Z}^{\alpha_1}}(h^1_n) + \dfrac{\mathbf{u}^{h^1_n}_n}{\lambda_n} & \text{if } i \geq h^1_n, \end{cases}$$

by the continuity of ψ_1 and ψ_2, we can find a suitable continuous function $\omega(\varepsilon)$: $\mathbb{R} \to \mathbb{R}$, $\omega(0) = 0$ such that, as $\tilde{\mathbf{u}}^i_n$ is a test function for the minimum problem defining $B_+(\mathbf{u}'(0), \phi(0))$, for any $\varepsilon > 0$, we have, for n large enough,

$$\begin{aligned} E^1_n(u_n, h^1_n) &= \frac{1}{2}\psi_1(\tilde{u}^1_n - \tilde{u}^0_n) + \sum_{i \geq 0} \Big(\psi_2\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big) \\ &\quad + \frac{1}{2}\Big(\psi_1\Big(\tilde{u}^{i+2}_n - \tilde{u}^{i+1}_n\Big) + \psi_1\Big(\tilde{u}^{i+1}_n - \tilde{u}^i_n\Big)\Big) - \psi_0\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big)\Big) + \omega(\varepsilon) \\ &\geq B_+(\mathbf{u}'(0), \phi(0)) + \omega(\varepsilon), \end{aligned} \tag{2.2.14}$$

where

$$\phi(0) = u_{\mathbf{Z}^{\alpha_1}}(h^1_n) - \frac{u^{h^1_n}_n}{\lambda_n}.$$

Exploiting the same argument, for $j \in \{1, 2, \dots, M-1\}$, we can define

$$\tilde{\mathbf{u}}^i_n = \begin{cases} \mathbf{u}_{\mathbf{Z}^{\alpha_j}}(i) - \mathbf{u}_{\mathbf{Z}^{\alpha_j}}(h^j_n - k^j_n) + \dfrac{\mathbf{u}^{h^j_n}_n}{\lambda_n} & \text{if } i \leq h^j_n - k^j_n, \\ \dfrac{\mathbf{u}^{i+k^j_n}_n}{\lambda_n} & \text{if } h^j_n - k^j_n \leq i \leq h^{j+1}_n - k^j_n, \\ \mathbf{u}_{\mathbf{Z}^{\alpha_{j+1}}}(i) - \mathbf{u}_{\mathbf{Z}^{\alpha_{j+1}}}(h^{j+1}_n - k^j_n) + \dfrac{\mathbf{u}^{h^{j+1}_n}_n}{\lambda_n} & \text{if } i \geq h^{j+1}_n - k^j_n\ , \end{cases}$$

and we have that

$$\begin{aligned} E^1_n(u_n, h^j_n, h^{j+1}_n) &= \frac{1}{2}\psi_1(\tilde{u}^0_n - \tilde{u}^{-1}_n) + \sum_{i \leq -1} \Big(\psi_2\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big) \\ &\quad + \frac{1}{2}\Big(\psi_1\Big(\tilde{u}^{i+2}_n - \tilde{u}^{i+1}_n + \psi_1\Big(\tilde{u}^{i+1}_n - \tilde{u}^i_n\Big)\Big) - \psi_0\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big)\Big) \\ &\quad + \frac{1}{2}\psi_1(\tilde{u}^1_n - \tilde{u}^0_n) + \sum_{i \geq 0} \Big(\psi_2\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big) \\ &\quad + \frac{1}{2}\Big(\psi_1\Big(\tilde{u}^{i+2}_n - \tilde{u}^{i+1}_n\Big) + \psi_1\Big(\tilde{u}^{i+1}_n - \tilde{u}^i_n\Big)\Big) - \psi_0\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big)\Big) \\ &\quad + \omega(\varepsilon) \\ &\geq C(\mathbf{u}'(x_j-), \mathbf{u}'(x_j+), \phi(x_j)) + \omega(\varepsilon), \end{aligned} \tag{2.2.15}$$

with

$$\phi(x_j) = \frac{u_n^{h_n^j}}{\lambda_n} - u_{\mathbf{Z}^{\alpha_j}}(h_n^j - k_n^j) + u_{\mathbf{Z}^{\alpha_{j+1}}}(h_n^{j+1} - k_n^j) - \frac{u_n^{h_n^{j+1}}}{\lambda_n}.$$

Finally, with

$$\tilde{\mathbf{u}}_n^i = \begin{cases} \mathbf{u}_{\mathbf{Z}^{\alpha_M}}(i) - \mathbf{u}_{\mathbf{Z}^{\alpha_M}}(h_n^M - n) + \dfrac{\mathbf{u}_n^{h_n^M}}{\lambda_n} & \text{if } i \leq h_n^M - n, \\ \dfrac{\mathbf{u}_n^{i+n}}{\lambda_n} - \dfrac{l}{\lambda_n} & h_n^M - n \leq i \leq 0, \end{cases}$$

we obtain

$$\begin{aligned} E_n^1(u_n, h_n^M) &= \frac{1}{2}\psi_1(\tilde{u}_n^0 - \tilde{u}_n^{-1}) + \sum_{i\leq 0}\Big(\psi_2\Big(\frac{\tilde{u}_n^i - \tilde{u}_n^{i-2}}{2}\Big) \\ &\quad + \frac{1}{2}\Big(\psi_1\Big(\tilde{u}_n^i - \tilde{u}_n^{i-1}\Big) + \Big(\tilde{u}_n^{i-1} - \tilde{u}_n^{i-2}\Big)\Big) - \psi_0\Big(\frac{\tilde{u}_n^{i+2} - \tilde{u}_n^i}{2}\Big)\Big) + \omega(\varepsilon) \\ &\geq B_-(\mathbf{u}'(1), \phi(1)) + \omega(\varepsilon), \end{aligned} \tag{2.2.16}$$

where

$$\phi(1) = \frac{u_n^{h_n^M}}{\lambda_n} - u_{\mathbf{Z}^{\alpha_M}}(h_n^M - n).$$

Since we have

$$\sum_{t\in S(\mathbf{u}')\cup\{0,1\}} \phi(t) = l, \tag{2.2.17}$$

we obtain that, thanks to (2.2.13), (2.2.14), (2.2.15) and (2.2.16),

$$\begin{aligned} E_n^{1,l}(u_n) &\geq B_+(\mathbf{u}'(0), \phi(0)) \\ &\quad + \sum_{t\in S(\mathbf{u}')} C(\mathbf{u}'(t-), \mathbf{u}'(t+), \phi(t)) + B_-(\mathbf{u}'(1), \phi(1)) + c\omega(\varepsilon) - R_n \\ &\geq \inf\Big\{E_{even}^{1,l}(u, s) : \ s : S(\mathbf{u}') \cup \{0,1\} \to \mathbb{R}, \sum_{t\in S(\mathbf{u}')\cup\{0,1\}} s(t) = l\Big\} + c\,\omega(\varepsilon) - R_n. \end{aligned}$$

Thus, by the arbitrariness of ε, we get

$$\liminf_n E_n^{1,l}(u_n) \geq \inf\Big\{E_{even}^{1,l}(u, s) : \ s : S(\mathbf{u}') \cup \{0,1\} \to \mathbb{R}, \sum_{t\in S(\mathbf{u}')\cup\{0,1\}} s(t) = l\Big\}. \tag{2.2.18}$$

Γ-limsup inequality. Let $\mathbf{u}$ be such that $E_{even}^{1,l}(\mathbf{u}) < +\infty$. Then there exist $M \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_M \in \mathbb{M}_l$ and $0 = x_0 < x_1 < \ldots < x_M = 1$ such that $\#S(\mathbf{u}') = M - 1$ and

$$\mathbf{u}'(t) = \mathbf{z}^{\alpha_j} \quad t \in (x_{j-1}, x_j) \quad j \in \{1, 2, \ldots, M\}. \tag{2.2.19}$$

Thanks to the boundary conditions on $\mathbf{u}$, we have that $\sum_{i=1}^{M-1}(x_{i+1}-x_i)=1$. For $\varepsilon>0$ let $\varphi: S(\mathbf{u}')\cup\{0,1\}\to\mathbb{R}$ be such that

$$\begin{aligned}
&\sum_{t\in S(\mathbf{u}')\cup\{0,1\}}\varphi(t)=l,\\
&\sum_{t\in S(\mathbf{u}')} C(\mathbf{u}'(t-),\mathbf{u}'(t+),\varphi(t))+B_+(\mathbf{u}'(0),\varphi(0))+B_-(\mathbf{u}'(1),\varphi(1))\\
&\qquad\le E^{1,l}_{even}(\mathbf{u})+\varepsilon.
\end{aligned}\tag{2.2.20}$$

Fix $\eta>0$. For $j\in\{1,2,\dots,M-1\}$ let $\mathbf{v}_1=(v_{1,1},v_{1,2})$, $\mathbf{v}_{j,j+1}=(v_{j,j+1,1},v_{j,j+1,2})$ and $\mathbf{v}_M=(v_{M,1},v_{M,2})$ be such that

$$\begin{aligned}
&v_1^0=0,\qquad v_1^i=u^i_{\mathbf{Z}^{\alpha_1}}-\varphi(0)\quad\text{for } i\ge N,\\
&v^i_{j,j+1}=\begin{cases} u^i_{\mathbf{Z}^{\alpha_j}}+\phi_1^{j,j+1} & \text{for } i\le -N,\\ u^i_{\mathbf{Z}^{\alpha_{j+1}}}+\phi_2^{j,j+1} & \text{for } i\ge N,\end{cases}\\
&v_M^0=0,\qquad v_M^i=u^i_{\mathbf{Z}^{\alpha_M}}-\varphi(1)\quad\text{for } i\le -N,
\end{aligned}$$

where

$$\phi_1^{j,j+1}=-\sum_{k=0}^{j}\varphi(x_k),\qquad \phi_2^{j,j+1}=-\sum_{k=0}^{j+1}\varphi(x_k)$$

and

$$\begin{aligned}
&\frac12\psi_1(v_1^1-v_1^0)+\sum_{i\ge0}\Big(\psi_2\Big(\frac{v_1^{i+2}-v_1^i}{2}\Big)\\
&\quad+\frac12\Big(\psi_1\Big(v_1^{i+2}-v_1^{i+1}\Big)+\psi_1\Big(v_1^{i+1}-v_1^i\Big)\Big)-\psi_0(\alpha_1)\Big)\\
&\le B_+(\mathbf{u}'(0),\varphi(0))+\eta,
\end{aligned}$$

$$\begin{aligned}
&\frac12\psi_1(v^0_{j,j+1}-v^{-1}_{j,j+1})+\sum_{i\le-1}\Big(\psi_2\Big(\frac{v^{i+2}_{j,j+1}-v^i_{j,j+1}}{2}\Big)\\
&\quad+\frac12\Big(\psi_1\Big(v^{i+2}_{j,j+1}-v^{i+1}_{j,j+1}\Big)+\psi_1\Big(v^{i+1}_{j,j+1}-v^i_{j,j+1}\Big)\Big)-\psi_0\Big(\frac{z_1^{\alpha_j}+z_2^{\alpha_j}}{2}\Big)\Big)\\
&\quad+\frac12\psi_1(v^1_{j,j+1}-v^0_{j,j+1})+\sum_{i\ge0}\Big(\psi_2\Big(\frac{v^{i+2}_{j,j+1}-v^i_{j,j+1}}{2}\Big)\\
&\quad+\frac12\Big(\psi_1\Big(v^{i+1}_{j,j+1}-v^i_{j,j+1}\Big)+\psi_1\Big(v^{i+1}_{j,j+1}-v^i_{j,j+1}\Big)\Big)-\psi_0\Big(\frac{z_1^{\alpha_{j+1}}+z_2^{\alpha_{j+1}}}{2}\Big)\Big)\\
&\le C(\mathbf{u}'(x_j-),\mathbf{u}'(x_j+),\varphi(x_j))+\eta,
\end{aligned}$$

$$\frac{1}{2}\psi_1(v_M^1 - v_M^0) + \sum_{i\le 0}\Big(\psi_2\Big(\frac{v_M^i - v_M^{i-2}}{2}\Big) + \frac{1}{2}\Big(\psi_1\Big(v_M^i - v_M^{i-1}\Big) + \psi_1\Big(v_M^{i-1} - v_M^{i-2}\Big)\Big) - \psi_0(\alpha_M)\Big) \le B_-(\mathbf{u}'(1), \varphi(1)) + \eta.$$

Consider the sequence of functions $(\mathbf{u}_n)$ defined as follows

$$\mathbf{u}_n^i = \begin{cases} \lambda_n \mathbf{v}_1^i & \text{if } 0 \le i \le [x_1 n] - N, \\ \lambda_n \mathbf{v}_{j,j+1}^{i-[x_j n]} + \lambda_n D_j & \text{if } [x_j n] - N \le i \le [x_{j+1}n] - N \\ & j \in \{1, 2, \dots, M-1\}, \\ \lambda_n \mathbf{v}_M^{i-n} + \lambda_n D_M & \text{if } n - N \le i \le n-1, \\ 0 & \text{if } i = n, \end{cases}$$

where

$$\begin{aligned} D_1 &= -u_{\mathbf{Z}^{\alpha_1}}(-N) + u_{\mathbf{Z}^{\alpha_1}}([x_1 n] - N), \\ D_j &= -\sum_{k=1}^{j-1} u_{\mathbf{Z}^{\alpha_k}}(-N) + \sum_{k=1}^{j} u_{\mathbf{Z}^{\alpha_k}}([x_k n] - \sum_{h=1}^{j-1}[x_h n] - N) \quad j \in \{2, 3, \dots, M\}. \end{aligned}$$

Then $\mathbf{u}_n \to \mathbf{u}$ in L^∞ and

$$\begin{aligned} E(u_n) \le\ & B_+(\mathbf{u}'(0), \varphi(0)) + \sum_{j=1}^{M-1} C(\mathbf{u}'(x_j-), \mathbf{u}'(x_j+), \varphi(x_j)) + B_-(\mathbf{u}'(1), \varphi(1)) \\ & + \tilde{R}_n + c\eta, \end{aligned} \tag{2.2.21}$$

where

$$\begin{aligned} \tilde{R}_n =\ & \psi_2\Big(\frac{v_M^0 - v_M^{-2}}{2}\Big) + \frac{1}{2}\Big(\psi_1(v_M^0 - v_M^{-1}) + \psi_1(v_M^{-1} - v_M^{-2})\Big) \\ & -\psi_0\Big(\frac{v_M^0 - v_M^{-2}}{2}\Big) - \psi_2\Big(\frac{u_n^n - u_n^{n-2}}{2\lambda_n}\Big) - \frac{1}{2}\Big(\psi_1(\frac{u_n^n - u_n^{n-1}}{\lambda_n}) \\ & +\psi_1(\frac{u_n^{n-1} - u_n^{n-2}}{\lambda_n})\Big) + \psi_0\Big(\frac{u_n^n - u_n^{n-2}}{2\lambda_n}\Big) \end{aligned}$$

and, by the continuity of ψ_1, ψ_2 and ψ_0, $\tilde{R}_n \to 0$. Thanks to (2.2.20) and (2.2.21) we have that

$$\limsup_n E_n(u_n) \le E_{even}^{1,l}(\mathbf{u}) + c\eta + \varepsilon.$$

We obtain the thesis thanks to the arbitrariness of η and ε. □

Remark 2.2.8 In the case that ψ_1, ψ_2 are such that $\psi_0 \in C^1(\mathbb{R})$ then, thanks to Remark 2.1.2, the first order Γ-limit has no shift minimization formula:

(Case n even)

$$E^{1,l}_{even}(\mathbf{u}) = \begin{cases} \sum\limits_{t\in S(\mathbf{u}')} C(\mathbf{u}'(t-), \mathbf{u}'(t+)) + B_+(\mathbf{u}'(0)) + B_-(\mathbf{u}'(L)), \\ \qquad\qquad \text{if } \mathbf{u}' \in PC(0,L),\ \mathbf{u}' \in \mathbb{M}_l,\ u_1(L) + u_2(L) = 2l, \\ +\infty \qquad\qquad \text{otherwise,} \end{cases}$$

(Case n odd)

$$E^{1,l}_{odd}(\mathbf{u}) = \begin{cases} \sum\limits_{t\in S_{\mathbf{u}'}} C(\mathbf{u}'(t-), \mathbf{u}'(t+)) + B_+(\mathbf{u}'(0)) + B_-(\overline{\mathbf{u}'(L)}), \\ \qquad\qquad \text{if } \mathbf{u}' \in PC(0,L),\ \mathbf{u}' \in \mathbb{M}_l,\ u_1(L) + u_2(L) = 2l, \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

Remark 2.2.9 In the case that

$$\#\mathbb{M}^\alpha = 1 \quad \text{ for all } \alpha \in \mathbb{R} \text{ such that } \psi_0(\alpha) = \psi_0^{**}(\alpha),$$

then, by Remark 2.1.1, the first order Γ-limit does not depend on the parity of the lattice, or, in formula,

$$E^{1,l}(u) = \inf\Big\{E^{1,l}(u,s) : \ s : S(u') \cup \{0,L\} \to \mathbb{R}, \sum_{t\in S(u')\cup\{0,L\}} s(t) = l\Big\}$$

where

$$E^{1,l}(u,s) = \begin{cases} \sum\limits_{t\in S(u')} C(u'(t-), u'(t+), s(t)) + B_+(u'(0), s(0)) + B_-(u'(L), s(L)), \\ \qquad\qquad \text{if } u' \in PC(0,L),\ (u',u') \in \mathbb{M}_l,\ u(0) = 0,\ u(L) = l, \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

The Γ-limit in the periodic case is similar to that with Dirichlet boundary conditions, except for the absence of boundary terms. Note that in the case of odd interactions non-uniform minimal-energy configurations are not admissible test functions, and hence phase transitions may be forced by the periodicity constraints.

Theorem 2.2.10 (First-order Γ-limit - Periodic boundary data) *Suppose that hypotheses* [H1]–[H4] *and* [H6] *hold and let* $E^{\#,l}_{1,n} : \mathcal{B}_n(0,L) \to [0,+\infty]$ *be defined by*

$$E^{\#,l}_{1,n}(u) = \frac{E^{\#,l}_n(u) - \min E^{\#,l}}{\lambda_n}. \tag{2.2.22}$$

We then have:

(Case n even) $E^{\#,l}_{1,n}$ *Γ-converges with respect to the $L^\infty_{\rm loc}$-topology to*

$$E^{\#,l}_1(\mathbf{u}) = \begin{cases} \sum\limits_{t\in S(\mathbf{u}')\cap[0,L)} C(\mathbf{u}'(t-), \mathbf{u}'(t+)), & \text{if } \mathbf{u}' \in PC_{\rm loc}(\mathbb{R}),\ \mathbf{u}' \in \mathbb{M}_l, \\ & u(t) - lt \text{ is } L\text{-periodic}, \\ +\infty & \text{otherwise} \end{cases}$$

on $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ *and* $u_1'(0+) = u_1'(L+)$ *and* $u_2'(0+) = u_2'(L+)$.

(Case n odd) *The same results hold but* $u_1'(0+) = u_2'(L+)$ *and* $u_2'(0+) = u_1'(L+)$.

Proof. Since the Γ-liminf and the Γ-limsup inequalities are easily deducible from the proof of Theorem 2.2.7, we will only prove the compactness result.

In the following, without loss of generality, we suppose $L = 1$, n even and $l \in J_j$ for some j. Moreover, with the same notation of the previous proposition, we define r_j to be the straight line such that $\psi_0^{**}(x) = r_j(x)$ for all $x \in J_j$. Let $u_n \to \mathbf{u}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ be such that $\sup_n E^{\#,l}_{1,n}(u_n) < +\infty$. By the definition of $E^{\#,l}_{n,1}$, we have that u_n is such that $\sup_n E^{\#,l}_n(u_n) < +\infty$, and then, as in Theorem 2.2.4,

$$\frac{u_1(t) + u_2(t)}{2} - lt = u(t) - lt \text{ is 1-periodic.}$$

Thanks to the periodicity assumption, we have that

$$\frac{u_n^{n+1} - u_n^n}{\lambda_n} = \frac{u_n^1 - u_n^0}{\lambda_n}$$

and then

$$\begin{aligned} +\infty > E^{\#,l}_{n,1}(u_n) &= \sum_{i=0}^{n-1}\Big(\psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) + \psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) - \psi_0^{**}(l)\Big) \\ &= \sum_{i=0}^{n-1}\Big(\psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) \\ &\quad + \frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{i+2} - u_n^{i+1}}{\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big)\Big) - \psi_0^{**}(l)\Big) \\ &= \sum_{i=0}^{n-1}\mathcal{E}_n^i(u_n) \end{aligned} \tag{2.2.23}$$

where

$$\mathcal{E}_n^i(u_n) = \psi_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + \frac{1}{2}\Big(\psi_1\Big(\frac{u_n^{i+2} - u_n^{i+1}}{\lambda_n}\Big) + \psi_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big)\Big) - r_j\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big).$$

Thanks to (2.2.23) we can deduce, as in the proof of Proposition 2.2.5, that there exists $S \subset (0,1]$ with $\#(S) < +\infty$ such that, up to subsequences, $u_n \to \mathbf{u}$ in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \setminus (S + k))$, $k \in \mathbb{Z}$ and that $\mathbf{u}' \in \mathbb{M}_l$. By the definition of even and odd interpolations, thanks to the periodicity hypothesis, we have

$$\begin{aligned} \frac{u_{1,n}^1 - u_{1,n}^0}{\lambda_n} &= \frac{u_n^1 - u_n^0}{\lambda_n} = \frac{u_n^{n+1} - u_n^n}{\lambda_n} = \frac{u_{1,n}^{n+1} - u_{1,n}^n}{\lambda_n}, \\ \frac{u_{2,n}^1 - u_{2,n}^0}{\lambda_n} &= \frac{u_n^2 - u_n^1}{\lambda_n} = \frac{u_n^{n+2} - u_n^{n+1}}{\lambda_n} = \frac{u_{2,n}^{n+1} - u_{2,n}^n}{\lambda_n}. \end{aligned}$$

Passing to the limit in the previous expressions we get that $u_1'(0+) = u_1'(1+)$ and $u_2'(0+) = u_2'(1+)$. $\square$

Remark 2.2.11 We observe that, in contrast with Theorem 2.2.7, here, the absence of boundary layer terms in the limit, allowed us to skip hypothesis [H5] to obtain inequality (2.2.23).

Remark 2.2.12 Note that in the case n is odd, if $\mathbf{u}' \equiv (z_1, z_2)$ in $(0, L)$ with $z_1 \neq z_2$, then $kL \in S(\mathbf{u}')$ for all $k \in \mathbb{Z}$.

2.3 Γ-convergence for Lennard-Jones type densities

In this section we deal with the zero- and first-order Γ-limit, under periodic and Dirichlet boundary conditions, of energies H_n of the form

$$H_n(u) = \sum_{i=0}^{n-1} \lambda_n J_1\Big(\frac{u^{i+1} - u^i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n J_2\Big(\frac{u^{i+2} - u^i}{2\lambda_n}\Big),$$

where J_1 and J_2 are Lennard-Jones type potentials. Our model case being the standard $(6, 12)$ Lennard-Jones potential, we will treat more general energy densities. With the same notation of the previous section we define $H_n^{\#,l}(u) : \mathcal{B}_n(\mathbb{R}) \to [0, +\infty]$ as

$$H_n^{\#,l}(u) = \begin{cases} H_n(u) & \text{if } u \in \mathcal{B}_n^{\#,l}(0, L) \\ +\infty & \text{otherwise} \end{cases}$$

and $H_n^l(u) : \mathcal{B}_n(0, L) \to [0, +\infty]$ as

$$H_n^l(u) = \begin{cases} H_n(u) & \text{if } u(0) = 0,\ u(L) = l \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3.1}$$

We also set

$$\begin{aligned} J_0(z) &= J_2(z) + \frac{1}{2} \inf\{J_1(z_1) + J_1(z_2) :\ z_1 + z_2 = 2z\}, \\ BV^{\#,l}(0, L) &= \{u \in BV_{\rm loc}(\mathbb{R}) :\ u(t) - lt \text{ is } L \text{ periodic}\}, \\ BV^l(0, L) &= \{u \in BV(0, L) :\ u(0+) = 0,\ u(L-) = l\}, \end{aligned}$$

and use the analogous notation for SBV spaces.

Adapting the proof of Theorem 3.2 in [**?**] it is possible to prove the following two theorems which are the analogue of Theorem 2.2.4 and Theorem 2.2.1.

Theorem 2.3.1 (Zero order Γ-limit - Periodic boundary data) *Let $\psi_j : \mathbb{R} \to (-\infty, +\infty]$ be Borel functions bounded below. Suppose that there exists a convex function $\Psi : \mathbb{R} \to [0, +\infty]$ such that*

$$\lim_{z \to -\infty} \frac{\Psi(z)}{|z|} = +\infty$$

and there exist constants $c_1,\ c_2 > 0$ such that

$$c_1(\Psi(z) - 1) \leq J_j(z) \leq c_2 \max\{\Psi(z), |z|\} \quad \text{for all } z \in \mathbb{R}, \quad j = 1, 2,$$

then the Γ-limit of $H_n^{\#,l}$ with respect to the L^1_{loc}-topology is given by

$$H^{\#,l}(u) = \begin{cases} \displaystyle\int_0^L J_0^{**}(u'(t))\, dt & \text{if } u \in BV^{\#,l}(0, L),\ D^s u > 0 \\ +\infty & \text{otherwise} \end{cases}$$

on $L^1_{\mathrm{loc}}(\mathbb{R})$, where $D^s u$ denotes the singular part of the measure Du with respect to the Lebesgue measure.

Theorem 2.3.2 (Zero order Γ-limit - Dirichlet boundary data) *Let $\psi_j : \mathbb{R} \to (-\infty, +\infty]$ be Borel functions bounded below satisfying the same conditions as in the previous theorem; then the Γ-limit of H_n^l with respect to the L^1_{loc}-topology is given by*

$$H^l(u) = \begin{cases} \displaystyle\int_0^L J_0^{**}(u'(t))\, dt & \text{if } u \in BV^l(0, L),\ D_s u > 0 \\ +\infty & \text{otherwise} \end{cases}$$

on $L^1(0, L)$.

In the same spirit of Section 2.2.2, we now deal with the problem of computing the first order $\Gamma-$limit of H_n in order to describe boundary layer phenomena in the continuum limit. The following set of hypotheses makes clear what kind of Lennard-Jones type potentials we will consider in this case:

$[H1]_{LJ}$(*discreteness of the energy states*)

$$\#(\{x \in \mathbb{R} : J_0(x) = J_0^{**}(x)\} \cap \{x \in \mathbb{R} : J_0 \text{ is affine}\}) < +\infty,$$

$[H2]_{LJ}$ (*finiteness of minimal energy configurations*) for every $\alpha \in \mathbb{R}$ such that $J_0(\alpha) = J_0^{**}(\alpha)$

$$\#\mathbb{M}^\alpha < +\infty,$$

$[H3]_{LJ}$ (*compatibility of minimal energy configurations*) for every $\alpha,\ \beta \in \mathbb{R}$ with $\alpha \neq \beta$, such that $J_0(\alpha) = J_0^{**}(\alpha)$ and $J_0(\beta) = J_0^{**}(\beta)$ and for every $\mathbf{z}^\alpha = (z_1^\alpha, z_2^\alpha) \in \mathbb{M}^\alpha$ and $\mathbf{z}^\beta = (z_1^\beta, z_2^\beta) \in \mathbb{M}^\beta$ it holds

$$z_i^\alpha \neq z_j^\beta \quad i, j \in \{1, 2\},$$

$[H4]_{LJ}$ (*continuity and growth conditions*) $J_1,\ J_2\ \ : \mathbb{R} \to (-\infty, +\infty]$ are sufficiently smooth functions bounded below such that $J_0 \in C^1(\mathbb{R})$ and there exists a convex function $\Psi : \mathbb{R} \to [0, +\infty]$ and constants $c_1,\ c_2 > 0$ such that

$$\lim_{z\to-\infty} \frac{\Psi(z)}{|z|} = +\infty,$$

and

$$c_1(\Psi(z) - 1) \le J_j(z) \le c_2 \max\{\Psi(z), |z|\} \quad \text{for all } z \in \mathbb{R}, \quad j = 1, 2;$$

$[H5]_{LJ}$ (*structure of* $J_1,\ J_2$ *and* J_0) the following limits exist:

$$\lim_{z\to+\infty} J_1(z) = J_1(+\infty), \quad \lim_{z\to+\infty} J_2(z) = J_2(+\infty) \quad \lim_{z\to+\infty} J_0(z) = J_0(+\infty),$$

$J_0(z) = \min J_0$ if and only if $z = \gamma$, and $J_0(+\infty) > J_0(\gamma)$;

$[H6]_{LJ}$ (*finiteness of the intervals of non-convexity*) l is such that $N(l) < +\infty$ ($N(l)$ defined as in (2.1.3)).

The following compactness result will be used in proving Theorem 2.3.4. It describes functions with $H_n^{\#,l}(u_n) = \min H^{\#,l} + O(\lambda_n)$, stating that below the threshold γ they behave as in the Sobolev case and develop no discontinuity. Above the threshold they may develop a finite number of discontinuities, behaving otherwise as in the Sobolev case with periodic condition corresponding to γ.

Proposition 2.3.3 (Compactness - Periodic boundary data) *Suppose that hypotheses $[H1]_{LJ}$–$[H6]_{LJ}$ hold. If $\{u_n\}$ is a sequence of functions such that*

$$\sup_n H_{1,n}^{\#,l}(u_n) = \sup_n \frac{H_n^{\#,l}(u_n) - \min H^{\#,l}}{\lambda_n} < +\infty \tag{2.3.2}$$

and there exists $t \in [0, L)$ such that $\sup_n |u_n(t)| < +\infty$, then, up to subsequences, $u_n \to \mathbf{u}$ strongly in $L^1_{\rm loc}(\mathbb{R})$ where $\mathbf{u} \in SBV^{\#,l}(0, L)$ is such that

(i) $\#(S(\mathbf{u}) \cap [0, L)) < +\infty$. *In particular*

  (a) *if $l \le \gamma$ then $S(\mathbf{u}) = \emptyset$,*

  (b) *if $l > \gamma$ then $0 < \#(S(\mathbf{u}) \cap [0, L)) < +\infty$,*

(ii) $[\mathbf{u}_s(t)] > 0 \quad s = 1, 2 \quad$ *for all $t \in S(\mathbf{u})$,*

(iii) $\#(S(\mathbf{u}') \cap [0, L)) < +\infty$,

(iv) $\mathbf{u}'(t) \in \mathbb{M}_l$ *a.e. $t \in (0, L)$. In particular*

  (a) *if $l \le \gamma$ then $\mathbf{u}'(t) \in \mathbb{M}^{\frac{l}{L}}$,*

  (b) *if $l > \gamma$ then $\mathbf{u}'(t) \in \mathbb{M}^{\gamma}$.*

Proof. To fix the ideas let us suppose that $u_n(0) = 0$ and that $L = 1$. Let us observe that (2.3.2) implies that

$$\sup_n H_n^{\#,l}(u_n) \leq C < +\infty. \tag{2.3.3}$$

With the notation so far used, let us set $\mathbf{u}_n = (u_{n,1}, u_{n,2})$. Since if $u_{n,1} = u_{n,2}$ it is possible to prove that $u_n \to u$ strongly in $L^1_{loc}(\mathbb{R})$ and that $u \in SBV^{\#,l}(0, L)$ repeating the same proof of Theorem 3.7 in [18] first and then using Theorem 3.1 in [18], we only sketch this part of the proof. For all $n \in \mathbb{N}$, let $T_n \in \mathbb{R}$ be such that $\lim_n T_n = +\infty$, $\lim_n \lambda_n T_n = 0$ and set $I_n := \{i \in \{0, 1, \dots, n-1\} : \ |u_n^{i+1} - u_n^i| > \lambda_n T_n\}$. Let w_n be defined as

$$w_n(t) = \begin{cases} 0 & \text{if } t = 0 \\ u_n(t) & \text{if } t \in (i, i+1)\lambda_n,\ i \notin I_n \\ u_n(i\lambda_n) & \text{if } t \in (i, i+1)\lambda_n,\ i \in I_n \end{cases}$$

and let $v_n(t)$ be an extension of $w_n(t)$ given by the following formula $v_n(t + k) = w_n(t) + kl$ for all $k \in \mathbb{Z}$. By (2.3.3), thanks to the growth hypotheses, by arguing as in [18] (Theorem 3.7), we have that $\|v_n\|_{BV_{loc}(\mathbb{R})} \leq C$. Then, up to subsequences not relabelled, $v_n \to u$ strongly in $L^1_{loc}(\mathbb{R})$. The same holds true for u_n since, by construction, for all compact sets $K \subset \mathbb{R}$, $\lim_n \int_K |u_n(t) - v_n(t)|\, dt = 0$. Set

$$\mathcal{H}_n^1(u_n) = \sum_{i=0}^{n-2} \Big(J_0\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) - J_0(\gamma)\Big), \tag{2.3.4}$$

by (2.3.2) we have that $\sup_n \mathcal{H}_n^1(u_n) \leq C < +\infty$. Let us set $\mathbf{v}_n^i = \mathbf{u}_n^i - i\lambda_n \mathbf{z}^\gamma$ and $\tilde{J}_0(z) = J_0(z + \gamma) - J_0(\gamma)$. We have that the sequence of functionals $\mathcal{H}_n^1$ satisfies all the hypotheses of Theorem 3.1 in [18] which implies in particular that $u \in SBV^{\#,l}(0, L)$. In the general case, when $u_{n,1}(t) \neq u_{n,2}(t)$ for some $t \in [0, 1)$, the previous argument need to be modified in order to prove the convergence of even and odd interpolator functions independently. In this case, observing that $H_n^{\#,l}(u_n) = \frac{1}{2} \sum_{s=1}^{2} \mathcal{E}_{n,s}^{\#,l}(u_{n,s})$, where

$$\mathcal{E}_{n,s}^{\#,l}(u_{n,s}) = \sum_{i=0}^{n-1} \lambda_n J_1\Big(\frac{u_{n,s}^{i+1} - u_{n,s}^i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n J_2\Big(\frac{u_{n,s}^{i+2} - u_{n,s}^i}{2\lambda_n}\Big),$$

we get

$$\sup_n \mathcal{E}_{n,s}^{\#,l}(u_{n,s}) \leq C < +\infty. \tag{2.3.5}$$

Thus the convergence to $\mathbf{u} \in SBV^{\#,l}(0, L)$ can be now easily proved using the argument we have exploited before independently for $s = 1, 2$. By (2.3.5) and Theorem 2.3.1 we also get *(ii)*. The proof of (iii) and (iv) can be obtained arguing as in the proof of Theorem 2.2.7. Let us prove (ii) in case (a). If $l < \gamma$ then, thanks to the hypothesis $[H5]_{LJ}$ on J_0, we have that for all $p \in \partial J_0^{**}(l)$ $\lim_{|z|\to+\infty} J_0(z) - pz = +\infty$ and the claim follows again arguing as in the proof of Theorem 2.2.10. If $l = \gamma$, by the boundary conditions and (iv), $u(t) = \frac{u_1(t)+u_2(t)}{2} = \gamma t$ *a.e.* $t \in (0,1)$, thus $S(u) \cap [0,1) = \emptyset$. This, together with (i), for $s = 1, 2$, implies that $S(u_s) \cap [0,1) = \emptyset$ and then the claim follows by the definition of $S(\mathbf{u})$.

Let us prove (ii) in the case (b). Arguing as in the proof of Theorem 2.2.7, set

$$\mathcal{H}_n^1(u_{s,n}) = \sum_{i=0}^{n-2} \Big(J_0\Big(\frac{u_{s,n}^{i+2} - u_{s,n}^i}{2\lambda_n}\Big) - J_0(\gamma)\Big), \tag{2.3.6}$$

by (2.3.2) we have that $\sup_n \mathcal{H}_n^1(u_{s,n}) \le C < +\infty$. Let us set $\mathbf{v}_n^i = \mathbf{u}_n^i - i\lambda_n \mathbf{z}^\gamma$ and $\tilde{J}_0(z-\gamma) = J_0(z) - J_0(\gamma)$. Observing that $\mathbb{M}_l = \mathbb{M}^\gamma$, $\mathbf{v}_n \to \mathbf{u} - t\mathbf{z}^\gamma$ strongly in $L^1_{\rm loc}(\mathbb{R})$. Thus, by Theorem 3.1 in [20], for $s = 1, 2$ we have

$$\begin{aligned} C \ge \liminf_n \mathcal{H}_n^1(u_{s,n}) &= \liminf_n \sum_{i=0}^{n-2} \tilde{J}_0\Big(\frac{v_{s,n}^{i+2} - v_{s,n}^i}{2\lambda_n}\Big) \\ &\ge \int_{(0,1)} F(u_s'(t))\, dt + \sum_{t\in S(u_s)\cap(0,1)} G([u_s](t), \end{aligned} \tag{2.3.7}$$

where

$$F(z) = \begin{cases} 0 & \text{if } z = \gamma \\ +\infty & \text{otherwise} \end{cases}$$

and

$$G(w) = \begin{cases} J_0(+\infty) - J_0(\gamma) & \text{if } w > 0 \\ 0 & \text{if } w = 0 \\ +\infty & \text{if } w < 0. \end{cases}$$

By (2.3.7) and hypothesis $[H5]_{LJ}$ we finally get that

$$\#S(\mathbf{u}) = \# \bigcup_{s=1}^{2} S(u_s) \le C < +\infty.$$

The Γ-limit described below takes into account both phase transitions and discontinuities. Note that the energy of a discontinuity takes into account boundary layers on both sides of the jump. For simplicity of notation we define

$$SBV_c^{\#,l}(0,L) = \{\mathbf{u} \in SBV^{\#,l}(0,L) : \text{ (i)-(iv) of Proposition 2.3.3 hold}\}. \tag{2.3.8}$$

Theorem 2.3.4 (First order Γ-limit - Periodic boundary data) *Suppose that hypotheses $[H1]_{LJ} - [H6]_{LJ}$ hold and let $H^{\#,l}_{1,n} : \mathcal{B}_n(0,L) \to [0,+\infty]$ be defined by*

$$H^{\#,l}_{1,n}(u) = \frac{H^{\#,l}_n(u) - \min H^{\#,l}}{\lambda_n}. \tag{2.3.9}$$

We then have:

(Case n even)

(i) *if* $l \leq \gamma$

$H^{\#,l}_{1,n}$ Γ*-converges with respect to the* $L^\infty_{\rm loc}$*-topology to*

$$H^{\#,l}_1(\mathbf{u}) = \begin{cases} \sum\limits_{t \in S(\mathbf{u}') \cap [0,L)} C(\mathbf{u}'(t-), \mathbf{u}'(t+)), & \text{if } \mathbf{u} \in SBV^{\#,l}_c(0,L), \\ +\infty & \text{otherwise} \end{cases}$$

on $W^{1,\infty}_{\rm loc}(\mathbb{R})$*, where* $SBV^{\#,l}_c(0,L)$ *is defined in* (2.3.8)*, and* $u_1'(0+) = u_1'(L+)$ *and* $u_2'(0+) = u_2'(L+)$.

(ii) *if* $l > \gamma$

$H^{\#,l}_{1,n}$ Γ*-converges with respect to the* $L^1_{\rm loc}$*-topology to*

$$H^{\#,l}_1(\mathbf{u}) = \begin{cases} \sum\limits_{t \in S(\mathbf{u}') \setminus S(\mathbf{u}) \cap [0,L)} C(\mathbf{u}'(t-), \mathbf{u}'(t+)) + \sum\limits_{t \in S(\mathbf{u}) \cap [0,L)} B_J(\mathbf{u}'(t-), \mathbf{u}'(t+)), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathbf{u} \in SBV^{\#,l}_c(0,L), \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases}$$

on $L^1_{\rm loc}(\mathbb{R})$ *where*

$$B_J(\boldsymbol{z}, \boldsymbol{z}') = B_+(\boldsymbol{z}) + B_-(\boldsymbol{z}') - 2J_0(\gamma) + 2J_2(+\infty) + J_1(+\infty)$$

and $u_1'(0+) = u_1'(L+)$ *and* $u_2'(0+) = u_2'(L+)$.

(Case n odd) *The same results hold but* $u_1'(0+) = u_2'(L+)$ *and* $u_2'(0+) = u_1'(L+)$.

Proof. Since the proof is similar to that of Theorem 2.2.10, we only highlight the main differences in the case $l > \gamma$ proving the Γ-liminf inequality for the second term in the energy.

In the following we will suppose that $L = 1$ and n is even. Let $u_n \to \mathbf{u}$ in $L^1_{\rm loc}(\mathbb{R})$ be such that $\sup_n H^{\#,l}_{1,n}(u_n) < +\infty$. Then, thanks to Proposition 2.3.3 and to the translation invariance of the energies, without loss of generality, we can further suppose that

$$\mathbf{u}(0) = 0, \quad \mathbf{u}(1) = l, \qquad S(\mathbf{u}) \cap [0,1) = S(\mathbf{u}') \cap [0,1) = \{\bar{t}\}.$$

Let $\mathbf{z}^\gamma_1,\ \mathbf{z}^\gamma_2 \in \mathbb{M}^\gamma$ be such that

$$\mathbf{u}'(\bar{t}-) = \mathbf{z}^\gamma_1, \quad \mathbf{u}'(\bar{t}+) = \mathbf{z}^\gamma_2$$

and let $\{h_n\}_n$ be a sequence of indices such that,

$$\lambda_n h_n \leq \bar{t} \quad \text{and} \quad \lim_n \lambda_n h_n = \bar{t}. \tag{2.3.10}$$

It is convenient to rewrite the energy as follows

$$\begin{aligned} H^{\#,l}_{1,n}(u_n) &= \mathcal{H}_n(u_n, h_n-) + \mathcal{H}_n(u_n, h_n+) + J_2\Big(\frac{u_n^{h_n+1} - u_n^{h_n-1}}{2\lambda_n}\Big) \\ &+ J_2\Big(\frac{u_n^{h_n+2} - u_n^{h_n}}{2\lambda_n}\Big) + J_1\Big(\frac{u_n^{h_n+1} - u_n^{h_n}}{\lambda_n}\Big) - 2J_0(\gamma), \end{aligned} \tag{2.3.11}$$

where

$$\begin{aligned} \mathcal{H}_n(u_n, h_n-) &= \sum_{i=0}^{h_n-2} \Big(J_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + J_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) - J_0(\gamma)\Big) \\ &\quad + J_1\Big(\frac{u_n^{h_n} - u_n^{h_n-1}}{\lambda_n}\Big) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_n(u_n, h_n+) &= \sum_{i=h_n+1}^{n-2} \Big(J_2\Big(\frac{u_n^{i+2} - u_n^i}{2\lambda_n}\Big) + J_1\Big(\frac{u_n^{i+1} - u_n^i}{\lambda_n}\Big) - J_0(\gamma)\Big) \\ &\quad + J_1\Big(\frac{u_n^{n} - u_n^{n-1}}{\lambda_n}\Big) - J_0(\gamma). \end{aligned}$$

Defining

$$\tilde{\mathbf{u}}_n^i = \begin{cases} \dfrac{\mathbf{u}_n^{i+h_n+1}}{\lambda_n} & \text{if } 0 \leq i \leq n - h_n - 1 \\ \dfrac{\mathbf{u}_n^{n}}{\lambda_n} + u_{\mathbf{z}_2^\gamma}(i) - u_{\mathbf{z}_2^\gamma}(n - h_n - 1) & \text{if } i \geq n - h_n - 1, \end{cases}$$

by the continuity of J_1 and J_2, we can find a suitable continuous function $\omega(\varepsilon) : \mathbb{R} \to \mathbb{R}$, $\omega(0) = 0$ such that,for all $\varepsilon > 0$, as $\tilde{\mathbf{u}}_n^i$ is a test function for the minimum problem defining $B_+(\mathbf{z}_2^\gamma)$, for n large enough we have

$$\begin{aligned} \mathcal{H}_n(u_n, h_n+) &= \frac{1}{2} J_1(\tilde{u}_n^1 - \tilde{u}_n^0) + \sum_{i\geq 0} \Big(J_2\Big(\frac{\tilde{u}_n^{i+2} - \tilde{u}_n^i}{2}\Big) \\ &\quad + \frac{1}{2}\Big(J_1\Big(\tilde{u}_n^{i+2} - \tilde{u}_n^{i+1}\Big) + J_1\Big(\tilde{u}_n^{i+1} - \tilde{u}_n^{i}\Big)\Big) - J_0(\gamma)\Big) + \omega(\varepsilon) \\ &\geq B_+(\mathbf{z}_2^\gamma) + \omega(\varepsilon). \end{aligned} \tag{2.3.12}$$

Analogously, defining

$$\tilde{\mathbf{u}}_n^i = \begin{cases} \dfrac{\mathbf{u}_n^{0}}{\lambda_n} + u_{\mathbf{z}_1^\gamma}(i) - u_{\mathbf{z}_1^\gamma}(-h_n) & \text{if } i \leq -h_n, \\ \dfrac{\mathbf{u}_n^{i+h_n}}{\lambda_n} & \text{if } -h_n \leq i \leq 0, \end{cases}$$

we have,

$$\begin{aligned}\mathcal{H}_n(u_n,h_n-) &= \frac{1}{2}J_1(\tilde{u}_n^0-\tilde{u}_n^{-1})+\sum_{i\leq 0}\Big(J_2\Big(\frac{\tilde{u}_n^i-\tilde{u}_n^{i-2}}{2}\Big)\\ &\quad+\frac{1}{2}\Big(J_1\Big(\tilde{u}_n^i-\tilde{u}_n^{i-1}\Big)+\Big(\tilde{u}_n^{i-1}-\tilde{u}_n^{i-2}\Big)\Big)-J_0(\gamma)\Big)+\omega(\varepsilon)\\ &\geq B_-(\mathbf{z}_1^\gamma)+\omega(\varepsilon).\end{aligned} \tag{2.3.13}$$

Thanks to inequalities (2.3.12), (2.3.13) and formula (2.3.11), we get

$$\begin{aligned}H_{1,n}^{\#,l}(u_n) &\geq J_2\Big(\frac{u_n^{h_n+1}-u_n^{h_n-1}}{2\lambda_n}\Big)+J_2\Big(\frac{u_n^{h_n+2}-u_n^{h_n}}{2\lambda_n}\Big)+J_1\Big(\frac{u_n^{h_n+1}-u_n^{h_n}}{\lambda_n}\Big)\\ &\quad-2J_0(\gamma)+B_-(\mathbf{z}_1^\gamma)+B_+(\mathbf{z}_2^\gamma)+c\,\omega(\varepsilon).\end{aligned}$$

By (2.3.10), the definition of $\bar{t}$ and hypothesis $[H5]_{LJ}$, we have

$$\begin{aligned}\liminf_n H_{1,n}^{\#,l}(u_n) &\geq 2J_2(+\infty)+J_1(+\infty)-2J_0(\gamma)+B_-(\mathbf{z}_1^\gamma)+B_+(\mathbf{z}_2^\gamma)+c\omega(\varepsilon)\\ &= B_J(\mathbf{u}'(\bar{t}-),\mathbf{u}'(\bar{t}+))+c\,\omega(\varepsilon).\end{aligned}$$

The claim follows by the arbitrariness of ε.

Slightly modifying the construction made in the proof of Γ-limsup inequality in Theorem 2.2.7, it can be proven that this bound is optimal. □

We find useful to set

$$\tilde{\mathbf{u}}(t)=\begin{cases}\mathbf{u}(0+) & \text{if } t=0\\ \mathbf{u}(t) & \text{if } t\in(0,L)\\ \mathbf{u}(L-) & \text{if } t=L.\end{cases} \tag{2.3.14}$$

The proof of the following result can be straightly derived by that of Proposition 2.3.3.

Proposition 2.3.5 (Compactness - Dirichlet boundary data) *Suppose that hypotheses $[H1]_{LJ}$–$[H6]_{LJ}$ hold. If $\{u_n\}$ is a sequence of functions such that*

$$\sup_n H_{1,n}^l(u_n)=\sup_n\frac{H_n^l(u_n)-\min H^l}{\lambda_n}<+\infty, \tag{2.3.15}$$

then, up to subsequences, $u_n\to\mathbf{u}$ strongly in $L^1_{\mathrm{loc}}(0,L)$ where $\mathbf{u}\in SBV^l(0,L)$ is such that

(i) *$\#S(\tilde{\mathbf{u}})<+\infty$ ($\tilde{\mathbf{u}}$ defined in (2.3.14)). In particular*

  (a) *if $l\leq\gamma$ then $S(\tilde{\mathbf{u}})=\emptyset$,*

  (b) *if $l>\gamma$ then $0<\#(S(\tilde{\mathbf{u}}))<+\infty$,*

(ii) $[\tilde{\mathbf{u}}_s(t)] > 0 \quad s = 1, 2 \qquad$ *for all* $t \in S(\mathbf{u})$,

(iii) $\#(S(\mathbf{u}')) < +\infty$,

(iv) $\mathbf{u}'(t) \in \mathbb{M}_l$ *a.e.* $t \in (0, L)$. *In particular*

  (a) *if* $l \le \gamma$ *then* $\mathbf{u}'(t) \in \mathbb{M}^{\frac{l}{L}}$,

  (b) *if* $l > \gamma$ *then* $\mathbf{u}'(t) \in \mathbb{M}^{\gamma}$.

The Γ-limit for Dirichlet boundary conditions takes the form below, where boundary-layer effects at the boundary are taken into account. For simplicity of notation we define

$$SBV^l_c(0, L) = \{u \in SBV^l(0, L) : \text{ conditions (i)-(iv) of Proposition 2.3.5 hold}\}. \tag{2.3.16}$$

Theorem 2.3.6 (First order Γ-limit - Dirichlet boundary data) *Suppose that hypotheses* $[H1]_{LJ}$*–*$[H6]_{LJ}$ *hold and let* $H^l_{1,n} : \mathcal{B}_n(0, L) \to [0, +\infty]$ *be defined by*

$$H^l_{1,n}(u) = \frac{H^l_n(u) - \min H^l}{\lambda_n}. \tag{2.3.17}$$

We then have:

(i) *if* $l \le \gamma$
$H^l_{1,n}$ Γ-converges with respect to the L^∞-topology to

$$H^l_1(\mathbf{u}) = \begin{cases} \sum\limits_{t \in S(\mathbf{u}')} C(\mathbf{u}'(t-), \mathbf{u}'(t+)), & \text{if } \mathbf{u} \in SBV^l_c(0, L), \\ +\infty & \text{otherwise} \end{cases}$$

on $W^{1,\infty}(0, L)$*, with* $SBV^l_c(0, L)$ *defined in* (2.3.16).

(ii) *if* $l > \gamma$ *and* $\#\mathbb{M}^\gamma = 1$
$H^l_{1,n}$ Γ-converges with respect to the L^1_{loc}-topology to

$$H^l_1(u) = \begin{cases} C(\gamma, \gamma)\#(S(u') \setminus S(u)) + B_{IJ}\#S(u) + B_{BJ}\#S(\tilde{u}) + 2B(\gamma) \\ \qquad\qquad\qquad \text{if } u \in SBV^l_c(0, L) \\ +\infty \qquad\qquad \text{otherwise} \end{cases}$$

on $L^1_{\mathrm{loc}}(\mathbb{R})$*, where*

$$B_{BJ} = J_1(+\infty) + J_2(+\infty) - J_0(\gamma)$$

is the boundary layer energy for a jump at the boundary of the domain, and

$$B_{IJ} = 2B(\gamma) - 2J_0(\gamma) + 2J_2(+\infty) + J_1(+\infty)$$

is the boundary layer energy for a jump at an internal point of the domain.

Remark 2.3.7 Note that, compared to the periodic case, we have further restricted our analysis to the case $\#\mathbb{M}^\gamma = 1$ when $l > \gamma$. In the general case a dependence on the parity of the lattice would appear in the limit as in Theorem 2.2.7.

Proof of Theorem 2.3.6. Since the proof is similar to that of Theorem 2.2.10, we only highlight the main differences in the case $l > \gamma$ proving the Γ-liminf inequality for the last term in the energy. In what follows we will suppose $L = 1$ and n even. Let $u_n \to \mathbf{u}$ in $L^1_{\rm loc}(0, L)$ be such that $\sup_n H^l_{1,n}(u_n) < +\infty$. Moreover, for simplicity, suppose that

$$S(\mathbf{u}) = \{0\}. \tag{2.3.18}$$

By the compactness result of Proposition 2.3.5, we have that $\mathbf{u}'(t) = \mathbf{z}^\gamma \in \mathbb{M}^\gamma$ for a.e. $t \in (0, L)$. Let $\{h_n\}_n$ be a sequence of indices such that $\lim_n \lambda_n h_n = \frac{1}{2}$. It is convenient to rewrite the energy as follows:

$$\begin{aligned} H^l_{1,n}(u_n) &= \mathcal{H}_n(u_n, 0+) + \mathcal{H}_n(u_n, 1-) + J_1\Big(\frac{u^1_n - u^0_n}{\lambda_n}\Big) \\ &\quad + J_2\Big(\frac{u^2_n - u^0_n}{2\lambda_n}\Big) - J_0(\gamma), \end{aligned} \tag{2.3.19}$$

where

$$\mathcal{H}_n(u_n, 0+) = \sum_{i=1}^{h_n} \Big(J_2\Big(\frac{u^{i+2}_n - u^i_n}{2\lambda_n}\Big) + J_1\Big(\frac{u^{i+1}_n - u^i_n}{\lambda_n}\Big) - J_0(\gamma)\Big),$$

$$\begin{aligned} \mathcal{H}_n(u_n, 1-) &= \sum_{i=h_n+1}^{n-2} \Big(J_2\Big(\frac{u^{i+2}_n - u^i_n}{2\lambda_n}\Big) + J_1\Big(\frac{u^{i+1}_n - u^i_n}{\lambda_n}\Big) - J_0(\gamma)\Big) \\ &\quad + J_1\Big(\frac{u^n_n - u^{n-1}_n}{\lambda_n}\Big) - J_0(\gamma). \end{aligned}$$

Defining

$$\tilde{\mathbf{u}}^i_n = \begin{cases} \dfrac{\mathbf{u}^{i+1}_n}{\lambda_n} & \text{if } 0 \le i \le h_n - 1 \\ \dfrac{\mathbf{u}^{h_n}_n}{\lambda_n} + u_{\mathbf{z}^\gamma}(i) - u_{\mathbf{z}^\gamma}(h_n) & \text{if } i \ge h_n - 1, \end{cases}$$

by the continuity of J_1 and J_2, we can find a suitable continuous function $\omega(\varepsilon)$: $\mathbb{R} \to \mathbb{R}$, $\omega(0) = 0$ such that,for all $\varepsilon > 0$, as $\tilde{\mathbf{u}}^i_n$ is a test function for the minimum problem defining $B(\gamma)$, for n large enough we have

$$\begin{aligned} \mathcal{H}_n(u_n, 0+) &= \frac{1}{2} J_1(\tilde{u}^1_n - \tilde{u}^0_n) + \sum_{i \ge 0} \Big(J_2\Big(\frac{\tilde{u}^{i+2}_n - \tilde{u}^i_n}{2}\Big) \\ &\quad + \frac{1}{2}\Big(J_1\Big(\tilde{u}^{i+2}_n - \tilde{u}^{i+1}_n\Big) + J_1\Big(\tilde{u}^{i+1}_n - \tilde{u}^i_n\Big)\Big) - J_0(\gamma)\Big) + \omega(\varepsilon) \\ &\ge B_+(\mathbf{z}^\gamma) + \omega(\varepsilon). \end{aligned} \tag{2.3.20}$$

Analogously, defining

$$\tilde{\mathbf{u}}_n^i = \begin{cases} \dfrac{\mathbf{u}_n^{h_n}}{\lambda_n} + u_{\mathbf{Z}^\gamma}(i) - u_{\mathbf{Z}^\gamma}(h_n - n) & \text{if } i \le h_n - n, \\ \dfrac{\mathbf{u}_n^{i+n}}{\lambda_n} - \dfrac{l}{n} & \text{if } h_n - n \le i \le 0, \end{cases}$$

we have,

$$\begin{aligned} \mathcal{H}_n(u_n, 1-) &= \frac{1}{2} J_1(\tilde{u}_n^0 - \tilde{u}_n^{-1}) + \sum_{i \le 0} \Big(J_2\Big(\frac{\tilde{u}_n^i - \tilde{u}_n^{i-2}}{2}\Big) \\ &\quad + \frac{1}{2}\Big(J_1\Big(\tilde{u}_n^i - \tilde{u}_n^{i-1}\Big) + \Big(\tilde{u}_n^{i-1} - \tilde{u}_n^{i-2}\Big)\Big) - J_0(\gamma)\Big) + \omega(\varepsilon) \\ &\ge B(\gamma) + \omega(\varepsilon). \end{aligned} \tag{2.3.21}$$

By (2.3.18) and hypothesis $[H5]_{LJ}$, we have

$$\begin{aligned} \liminf_n H_{1,n}^l(u_n) &\ge J_1(+\infty) + J_2(+\infty) - J_0(\gamma) + 2B(\gamma) + c\omega(\varepsilon) \\ &= B_{BJ} + 2B(\gamma) + c\omega(\varepsilon). \end{aligned}$$

The Γ-liminf inequality follows by the arbitrariness of ε. □

In the following two examples we consider the case of standard *Lennard-Jones* and *Morse* potentials pointing out some interesting features about phase transition energies in these cases.

Example 2.3.8 Let us consider the Lennard-Jones case:

$$J_1(z) = \begin{cases} +\infty & \text{if } z \le 0 \\ \dfrac{k_1}{z^{12}} - \dfrac{k_2}{z^6} & \text{if } z > 0, \end{cases} \qquad J_2(z) = J_1(2z)$$

for some $k_1, k_2 > 0$. Set $z_{\min} = (2k_1/k_2)^{\frac{1}{6}}$ the minimum point of J_1 and γ the minimum point of J_0, it can be proven that

$$J_0^{**}(z) = \begin{cases} J_0(z) & \text{if } 0 < z \le \gamma := \left(\frac{1+2^{-12}}{1+2^{-6}}\right)^{\frac{1}{6}} z_{\min} \\ J_0(\gamma) & \text{otherwise.} \end{cases}$$

Hence no mesoscopic phase transition energies come into play because $N(l) = 1$ being

$$\mathbb{M}_l = \begin{cases} \emptyset & \text{if } l/L \le 0 \\ \mathbb{M}^{\frac{l}{L}} & \text{if } 0 < l/L < \gamma \\ \mathbb{M}^\gamma & \text{otherwise.} \end{cases}$$

It is also possible to show that neither microscopic phase transition energies appear as $\#\mathbb{M}_l \le 1$.

Example 2.3.9 Let us consider the Morse case:

$$J_1(z) = k_1\Big(1 - e^{-k_2(z - z_{\min})}\Big)^2, \quad J_2(z) = J(2z)$$

for some $k_1, k_2 > 0$. Set γ the minimum point of J_0, it can be proven that

$$J_0^{**}(z) = \begin{cases} J_0(z) & \text{if } z \le \gamma < z_{\min} \\ J_0(\gamma) & \text{otherwise.} \end{cases}$$

This again gives that no mesoscopic phase transition energy appear in the first order Γ-limit as $N(l) = 1$ being

$$\mathbb{M}_l = \begin{cases} \mathbb{M}^{\frac{l}{L}} & \text{if } l/L < \gamma \\ \mathbb{M}^{\gamma} & \text{otherwise.} \end{cases}$$

We now give an example of Lennard-Jones type potentials leading to mesoscopic phase transition terms in the limit.

Example 2.3.10 Let

$$\begin{aligned} J_1(z) &= (z - z_m)^2 \wedge t\chi_{(z_m, +\infty)}(z) \\ J_2(z) &= J_1(z/k) \end{aligned}$$

for some $t < z_m^2$ and $k > \frac{z_m + \sqrt{t/2}}{z_m - \sqrt{t/2}}$. Then (see Fig. 2.2)

$$J_0(z) = \begin{cases} (z - z_m)^2 + (\frac{1}{k}z - z_m)^2 & \text{if } z \le z_m + \sqrt{t/2} \\ (\frac{1}{k}z - z_m)^2 + \frac{1}{2}\,t & \text{if } z_m + \sqrt{t/2} < z \le k(z_m + \sqrt{t}) \\ \frac{3}{2}t & \text{if } z > k(z_m + \sqrt{t}) \end{cases}$$

and

$$J_0^{**}(z) = \begin{cases} J_0(z) & \text{if } z \le z_m + \sqrt{\frac{t}{2(1+k^2)}} \\ a(z - z_m - \sqrt{\frac{t(1+k^2)}{2}}) + b & \text{if } z_m + \sqrt{\frac{t}{2(1+k^2)}} < z \le z_m + \sqrt{\frac{t(1+k^2)}{2}} \\ J_0(z) & \text{if } z_m + \sqrt{\frac{t(1+k^2)}{2}} < z \le kz_m \\ \frac{t}{2} & \text{if } z > kz_m, \end{cases}$$

where

$$a = \frac{2}{k^2}\Big(z_m(1-k) + \sqrt{\frac{t(1+k^2)}{2}}\Big), \quad b = \frac{1}{k^2}\Big(z_m(1-k) + \sqrt{\frac{t(1+k^2)}{2}}\Big)^2 + \frac{t}{2}.$$

In this case we have that

$$\mathbb{M}_l = \begin{cases} \mathbb{M}^{\frac{l}{L}} & \text{if } l/L \le \alpha \\ \mathbb{M}^{\alpha} \cup \mathbb{M}^{\beta} & \text{if } \alpha < l/L \le \beta \\ \mathbb{M}^{\frac{l}{L}} & \text{if } \beta < l/L \le \gamma \\ \mathbb{M}^{\gamma} & \text{otherwise} \end{cases}$$

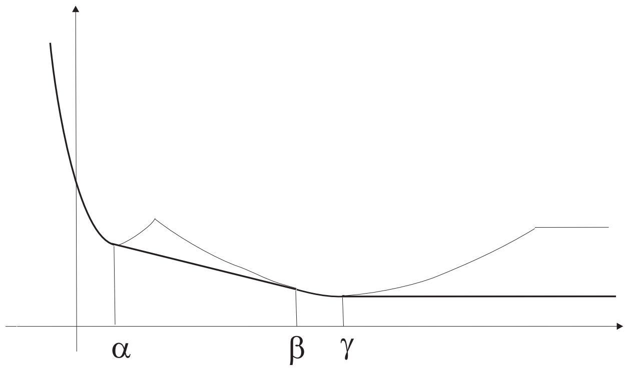

Figure 2.2: J_0 and J_0^{**} (bold line) in example 2.3.10

where

$$\alpha = z_m + \sqrt{\frac{t}{2(1+k^2)}}, \quad \beta = z_m + \sqrt{\frac{t(1+k^2)}{2}}, \quad \gamma = kz_m.$$

A mesoscopic phase transition energy will occur in the limit being $N(l) = 2$ for $\alpha \le \frac{l}{L} \le \beta$.

2.4 Minimum Problems

In this section we describe the structure of the minima for the first order discrete energies we studied in Section 2.2 and 2.3 in some special cases. In particular we will focus on the periodic case for superlinear growth densities and on the Dirichlet case for Lennard-Jones densities.

2.4.1 Superlinear-growth densities

The next theorem deals with the convergence of minimizer for first-order discrete energies of the form (2.2.22) in two special cases. For the sake of simplicity and without losing in generality we can set $L = 1$.

Theorem 2.4.1 *Suppose that hypotheses* [H1]–[H4] *and* [H6] *hold and suppose that* ψ_0 *is such that*

$$\mathbb{M}_l = \mathbb{M}^\alpha \cup \mathbb{M}^\beta \ \textit{if} \ \ l \in (\alpha, \beta).$$

Then the minimizers $(\mathbf{u}_n)$ *of* $\min\{E^{\#,l}_{1,n}(u)\}$*, for* n *even and* $l \in (\alpha,\beta)$*, converge, up to subsequences, to one of the functions:*

(i) *if* $\mathbb{M}^\alpha = \{(\alpha,\alpha)\},\ \mathbb{M}^\beta = \{(\beta,\beta)\}$*, then* $\mathbf{u} = (u,u)$*,*

$$u(t) = \alpha\chi_I(t) + \beta\chi_{(0,1)\setminus I}(t)$$

where $I \subset (0,1)$ *is an interval such that* $|I|\alpha + (1-|I|)\beta = l$*. Moreover* $E_1^{\#,l}(u) = 2C(\alpha,\beta)$*.*

(ii) *if* $\mathbb{M}^\alpha = \{(\alpha,\alpha)\},\ \mathbb{M}^\beta = \{(\beta_1,\beta_2),(\beta_2,\beta_1)\}$*, then* $\mathbf{u} = (u_1,u_2)$*,*

$$\begin{array}{l} u_1(t) = \alpha\chi_I(t) + \beta_1\chi_{(0,1)\setminus I}(t) \\ u_2(t) = \alpha\chi_I(t) + \beta_2\chi_{(0,1)\setminus I}(t) \end{array}$$

where $I \subset (0,1)$ *is an interval such that* $|I|\alpha + (1-|I|)\beta = l$*. Moreover* $E_1^{\#,l}(u) = 2C((\alpha,\alpha),(\beta_1,\beta_2))$*.*

Proof. The claim follows thanks to Theorem 2.2.10 applying the minima convergence result in Γ-convergence problem (see [13] and [29]) and observing that we have

$$E_1^{\#,l}(u) \geq 2C((\alpha,\alpha),(\beta,\beta)),$$

$$E_1^{\#,l}(u) \geq C((\alpha,\alpha),\mathbf{z}^\beta) + C((\alpha,\alpha),\mathbf{z}_1^\beta) \quad \text{for all } \mathbf{z}^\beta,\ \mathbf{z}_1^\beta \in \mathbb{M}^\beta.$$

in case (i) and (ii), respectively. □

2.4.2 A graphic reduction method

In what follows we describe a graphic reduction method which can be useful to treat cases more complicated than those seen in the previous theorem. Let $l \in (\alpha,\beta)$. We introduce some terminology: the plane (z_1,z_2) is said to be the *micro-phase plane (m-p plane)*. A point $\mathbf{w} = (w_1,w_2)$ in the m-p plane is said to be a *micro-configuration (m-c)* if $\mathbf{w} \in \mathbb{M}_l$. An arrow in the m-p plane connecting two m-cs, starting from a m-c $(\overline{z}_1,\overline{z}_2)$ and pointing to a m-c $(\tilde{z}_1,\tilde{z}_2)$ is said to be a *phase-transition (p-t)* and is indicated by $(\overline{z}_1,\overline{z}_2) \to (\tilde{z}_1,\tilde{z}_2)$.

Definition 2.4.2 *Two p-ts* $(z_1,z_2) \to (z_1',z_2'),\ (w_1,w_2) \to (w_1',w_2')$ *are said to be* connected *if* $(z_1,z_2) \equiv (w_1',w_2')$ *or if* $(z_1',z_2') \equiv (w_1,w_2)$*. A set of connected p-ts is said to be a* loop *if every m-c is starting and ending point for two p-ts. A loop is said to be of* length $n \in \mathbb{N}$ *(or an n-loop) if it is built connecting n p-ts.*

Definition 2.4.3 *A real function F defined on the cartesian product of two m-p planes is called an* energy*.*

Let F be a given energy. The energy of a phase transition $(\overline{z}_1,\overline{z}_2) \to (\tilde{z}_1,\tilde{z}_2)$ *is* $F((\overline{z}_1,\overline{z}_2),(\tilde{z}_1,\tilde{z}_2))$*. The energy of a sets of p-ts is the sum of the energies of all p-ts. Two sets of p-ts are said to be (energetically)* equivalent *if they have the same energy. An n-loop is said to be* reducible *if it is equivalent to another set of p-ts containing an m-loop with* $m < n$*.*

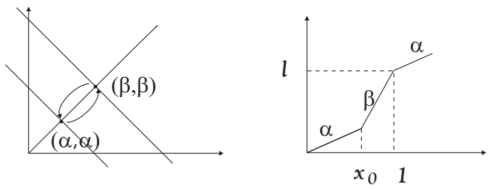

Figure 2.3: 2-loop and minimizing configuration in Example 2.4.4

We are interested in solving the minimum problem for the Γ-limit of energies of the type (2.2.22) in the same hypotheses of the previous theorem where, following the definitions above, $F(\mathbf{z}_1, \mathbf{z}_2) = C(\mathbf{z}_1, \mathbf{z}_2)$. Observe that, by the compactness result obtained in the previous section and the definition of transition energy $C(\cdot, \cdot)$, we know that

(R1) $\mathbf{u}' \in \mathbb{M}_l = \mathbb{M}^\alpha \cup \mathbb{M}^\beta$,

(R2) $\mathbf{u}'$ is 1-periodic,

(R3) $C(\mathbf{z}^\alpha, \mathbf{z}^\beta) = C(\overline{\mathbf{z}}^\alpha, \overline{\mathbf{z}}^\beta) = C(\overline{\mathbf{z}}^\beta, \overline{\mathbf{z}}^\alpha) = C(\mathbf{z}^\beta, \mathbf{z}^\alpha)$,

(R4) $C(\cdot, \cdot) > 0$.

Thanks to $(R1)$ and the definition of $\mathbb{M}^\alpha$ and $\mathbb{M}^\beta$, we know that the m-cs we have to consider in the m-p plane, where we are going to plot our minimal configurations, are those laying on the straight lines

$$z_1 + z_2 = 2\alpha, \qquad z_1 + z_2 = 2\beta.$$

Moreover, by $(R2)$, we know that the allowed p-ts form a loop. By $(R3)$ two p-ts symmetric with respect to $z_1 = z_2$ as well as two p-ts with starting and ending points exchanged are equivalent. We will describe this graphic method with three examples. The first two are cases (i) and (ii) in Theorem (2.4.1).

Example 2.4.4 Let $\mathbb{M}^\alpha = \{(\alpha, \alpha)\}$ and $\mathbb{M}^\beta = \{(\beta, \beta)\}$. In this case only one 2-loop is possible. Thus there is only one minimizing configuration (see Fig. 2.3).

Example 2.4.5 Let $\mathbb{M}^\alpha = \{(\alpha, \alpha)\}$ and $\mathbb{M}^\beta = \{(\beta_1, \beta_2), (\beta_2, \beta_1)\}$. In this case two equivalent 2-loops can be built (see Fig.2.4). Moreover a 3-loop can be built, but it can be reduced to a 2-loop as shown in Fig. 2.5, thus the minimum configuration has two transitions and the associated fields $u = \frac{u_1+u_2}{2}$, u_1 and u_2 look like those in Fig. 2.6.

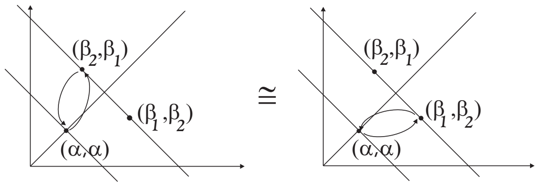

Figure 2.4: equivalent 2-loops (Example 2.4.5)

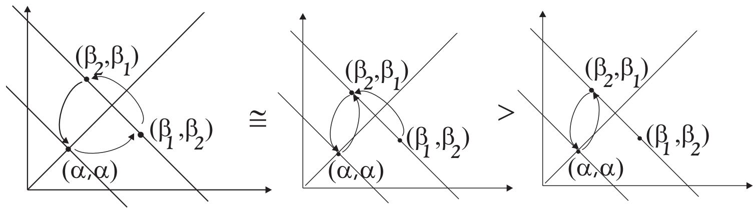

Figure 2.5: reduction of a 3-loop (Example 2.4.5)

Example 2.4.6 Let $\mathbb{M}^\alpha = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$ and $\mathbb{M}^\beta = \{(\beta_1, \beta_2), (\beta_2, \beta_1)\}$. In this case two pairs of equivalent 2-loops and two of 3-loops can be built. Moreover three 4-loops can be built but each of them can be reduced to a 2-loop, thus the minimum configuration has two or three transitions. To say which loop minimizes the energy we have to compare the minimum 2-loop energy

$$m_2 \equiv \min\{2C((\alpha_1, \alpha_2), (\beta_1, \beta_2)), 2C((\alpha_1, \alpha_2), (\beta_2, \beta_1))\},$$

with the minimum 3-loop energy

$$\begin{aligned} m_3 \quad \equiv \quad & \min\{C((\alpha_1, \alpha_2), (\beta_1, \beta_2)) + C((\beta_1, \beta_2), (\beta_2, \beta_1)) + C((\beta_2, \beta_1), (\alpha_1, \alpha_2)), \\ & \quad C((\alpha_1, \alpha_2), (\beta_1, \beta_2)) + C((\beta_1, \beta_2), (\alpha_2, \alpha_1)) + C((\alpha_2, \alpha_1), (\alpha_1, \alpha_2))\}. \end{aligned}$$

Three cases can occur. If $m_2 < m_3$ a 2-loop configuration is minimal and the corresponding minimizing fields are shown in Fig.2.7. If $m_2 > m_3$ a 3-loop configuration minimizes the energy and the corresponding fields are shown in Fig 2.8. If $m_2 = m_3$ the 2-loop and 3-loop configurations are equienergetic.

The following is an example of interaction energies ψ_1, ψ_2 leading to a ψ_0 satisfying the hypotheses of Theorem 2.4.1 in cases (i) and (ii).

Example 2.4.7 Consider $\psi_1 = (z+1)^2 \wedge (z-1)^2$. It is possible to compute explicitly $\phi(z) \equiv \min\{\psi_1(z_1) + \psi_1(z_2) : \ z_1 + z_2 = 2z\}$ obtaining $\phi(z) = (z+1)^2 \wedge (z-1)^2 \wedge z^2$ and in particular

$$2\phi(z) = \begin{cases} \psi_1(z-1) + \psi_1(z+1) & z \in (-\frac{1}{2}, \frac{1}{2}), \\ 2\psi_1(z) & \text{otherwise} \end{cases}.$$

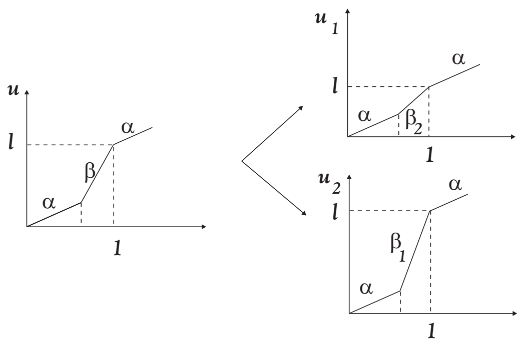

Figure 2.6: minimizing fields (Example 2.4.5)

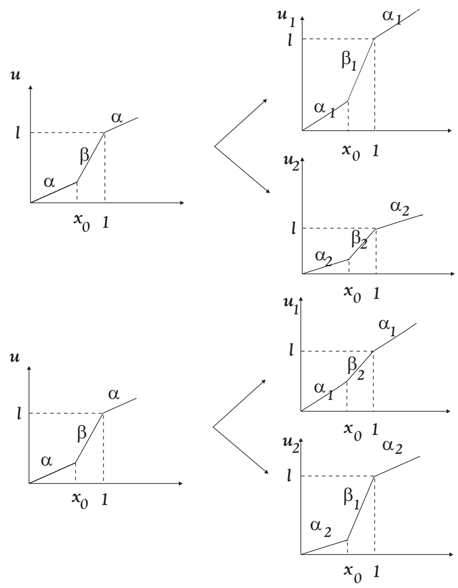

Figure 2.7: fields in the 2-loop configuration (Example 2.4.6)

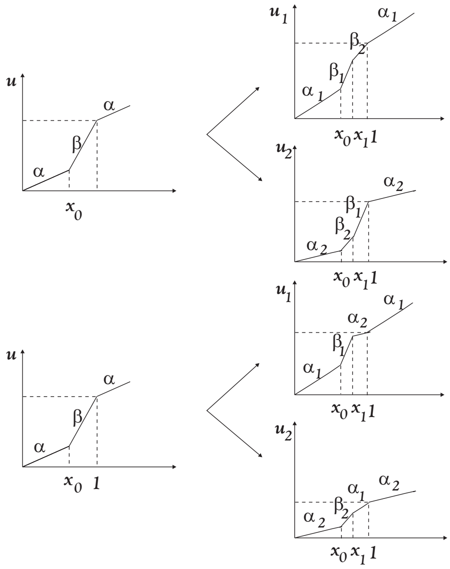

Figure 2.8: fields in the 3-loop configuration: $x_0 \in (0, \overline{x})$ with $\alpha\overline{x} + \beta(1-\overline{x}) = l$, $x_1 \in (x_0, x_0 + 1 - \overline{x})$ (Example 2.4.6)

i) If $\psi_2 = -\frac{z^2}{2}$, computing explicitly ψ_0^{**} (see Fig. 2.9) one gets that, for $l \in (-1, 1)$, $\mathbb{M}_l = \mathbb{M}^{-1} \cup \mathbb{M}^1$ where $\#\mathbb{M}^{-1} = \#\mathbb{M}^1 = 1$.

ii) If $\psi_2 = z^2$, computing explicitly ψ_0^{**} one gets that, for $l \in (-\frac{3}{4}, -\frac{1}{4})$, $\mathbb{M}_l = \mathbb{M}^{-\frac{3}{4}} \cup \mathbb{M}^{-\frac{1}{4}}$ and, for $l \in (\frac{1}{4}, \frac{3}{4})$, $\mathbb{M}_l = \mathbb{M}^{\frac{1}{4}} \cup \mathbb{M}^{\frac{3}{4}}$ where $\#\mathbb{M}^{-\frac{3}{4}} = \#\mathbb{M}^{\frac{3}{4}} = 1$ while $\#\mathbb{M}^{-\frac{1}{4}} = \#\mathbb{M}^{\frac{1}{4}} = 2$.

We end this section by giving an example of potentials leading to the energetic description we showed in Example 2.4.6.

Example 2.4.8 Consider $\psi_1 = (z+2)^2 \wedge z^2 \wedge (z-2)^2$ and $\psi_2 = (z+1)^2 \wedge ((z-1)^2 + 1)$. Again it is possible to compute $\phi(z) = (z+2)^2 \wedge (z+1)^2 \wedge (z-1)^2 \wedge (z-2)^2 \wedge z^2$. In particular

$$2\phi(z) = \begin{cases} 2\psi_1(z) & z \in (-\infty, -\frac{3}{2}) \\ \psi_1(z-1) + \psi_1(z+1) & z \in (-\frac{3}{2}, -\frac{1}{2}) \\ \psi_1(z-2) + \psi_1(z+2) & z \in (-\frac{1}{2}, \frac{1}{2}) \\ \psi_1(z-1) + \psi_1(z+1) & z \in (\frac{1}{2}, \frac{3}{2}) \\ 2\psi_1(z) & z \in (\frac{3}{2}, +\infty) \end{cases}$$

and, computing ψ_0^{**}, it can be seen that for $l \in (-\frac{7}{8}, \frac{9}{8})$ we have that $\mathbb{M}_l = \mathbb{M}^{-\frac{7}{8}} \cup \mathbb{M}^{\frac{9}{8}}$ with $\#\mathbb{M}^{-\frac{7}{8}} = \#\mathbb{M}^{\frac{9}{8}} = 2$. Observe that, to construct an example like

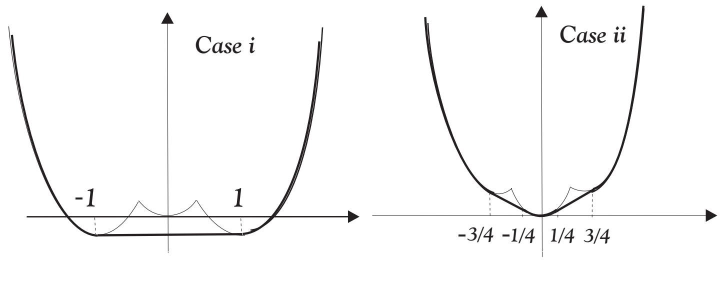

Figure 2.9: Example 2.4.7: ψ_0 and ψ_0^{**}.

this, it is not possible to substitute the second order asymmetric interaction we used with an even one, otherwise hypothesis [H3] would not be satisfied.

2.4.3 Lennard-Jones densities

Since the analysis of minimum problems for the scaled Lennard-Jones type energies of the form (2.3.1) when $l \leq \gamma$ does not present new features with respect to the superlinear case, we will focus on minimum problems with $l > \gamma$ when a contribution due to the crack appears in the limit. Although a more general description like the one we have provided in the previous section is possible, in order to give a simplified analysis of the phase transition phenomena for standard Lennard-Jones NNN energies, we restrict to the case $J_1(z) = J_2(2z) = J(z)$ with $\min J < J(+\infty)$ and $\#\mathbb{M}^\gamma = 1$.

For the sake of simplicity and without losing in generality we can set $L = 1$.

Theorem 2.4.9 (Localization of fracture) *Suppose that hypotheses $[H1]_{LJ}$–$[H6]_{LJ}$ hold and suppose that $J_1(z) = J_2(2z) = J(z)$ is such that*

$$\min J < J(+\infty), \qquad \#M^\gamma = 1. \tag{2.4.22}$$

Then the minimizers $(\mathbf{u}_n)$ of $\min\{H^l_{1,n}(u)\}$, for $l > \gamma$, converge, up to subsequences, to one of the functions:

$$u_1(t) = \gamma t, \qquad u_2(t) = \gamma t + (l - \gamma)$$

Moreover $H^l_1(u) = 3J(+\infty) - J_0(\gamma)$.

Remark 2.4.10 Note that the previous result asserts that, at first order in λ_n, the fracture of the ground state can be localized at the boundary of the domain.

Proof of Theorem 2.4.9. Thanks to Remark 2.1.3 we get

$$B_\pm(\gamma) = \frac{1}{2} J(\gamma).$$

Since by Theorem 2.3.6 we have that $\mathbf{u}'(t) = u'(t) = \gamma$ a.e. $t \in (0, L)$, we have that

$$B_{IJ} = J(\gamma) - 2J_0(\gamma) + 3J(+\infty), \qquad B_{BJ} = 2J(+\infty) - J_0(\gamma).$$

The claim follows applying the minima convergence result in Γ-convergence problems and observing that, since

$$H_1^l(u) \geq B_{IJ}(\gamma)\#S(u) + B_{BJ}\#S(\tilde{u}) + 2B(\gamma).$$

one has that $\#S(u) \leq 1$ and $\#S(\tilde{u}) \leq 1$. It remains to compare the energy $H_1^l(u)$ in the following four cases:

$$\begin{array}{ll} \text{(a) } \#S(u) = 0 \quad \#S(\tilde{u}) = 1, & \text{(b) } \#S(u) = 1 \quad \#S(\tilde{u}) = 0, \\ \text{(c) } \#S(u) = 0 \quad \#S(\tilde{u}) = 0, & \text{(d) } \#S(u) = 1 \quad \#S(\tilde{u}) = 1. \end{array}$$

By the boundary conditions, (c) must be rejected. By the positiveness of our energies (d) has an energy greater than (a) and (b). If we are in the case (a), then the only two minimizers are u_1 and u_2 while in the case (b) all the possible minimizers are functions of the type $\overline{u}(t) = \gamma t + (l - \gamma)\chi(\overline{t})$ where $\overline{t} \in (0, L)$. We have that

$$H_1^l(u_1) = H_1^l(u_2) = 3J(+\infty) - J_0(\gamma), \qquad H_1^l(\overline{u}) = 5J(+\infty) - 2J_0(\gamma),$$

and the claim follows observing that, by the definition of J_0 and thanks to hypothesis (2.4.22) $J_0(\gamma) \leq J(\gamma) + \min J < J(\gamma) + J(+\infty)$. □

2.5 Equivalence by Γ-convergence

In this section we give an interpretation of the results of Section 2.2 by linking them with the gradient theory of phase transitions. We show that in a sense discrete energies with next-to-nearest neighbour interactions act as singular perturbation of non-convex energies with higher-order gradients. In order to give a rigorous meaning to this statement we will use the notion of *equivalence* by Γ-convergence (see [22]).

Definition 2.5.1 Let $\mathcal{L}$ be a set of parameters and for $l \in \mathcal{L}$ let $F_\varepsilon^l(u)$ and $G_\varepsilon^l(u)$ be parameterized families of functionals. We say that F_ε^l and G_ε^l are *equivalent to first order* along the sequence ε_n if

(i) for all $l \in \mathcal{L}$ $\quad \Gamma\text{-}\lim_{n\to\infty} F_{\varepsilon_n}^l(u) = \Gamma\text{-}\lim_{n\to\infty} G_{\varepsilon_n}^l(u) =: F_0^l(u)$

(ii) for all $l \in \mathcal{L}$ $\quad \Gamma\text{-}\lim_{n\to\infty} \dfrac{F_{\varepsilon_n}^l(u) - \min F_0^l(u)}{\varepsilon_n} = \Gamma\text{-}\lim_{n\to\infty} \dfrac{G_{\varepsilon_n}^l(u) - \min F_0^l(u)}{\varepsilon_n}.$

With a slight abuse we use the same notation if F^l_ε and G^l_ε are defined only for $\varepsilon = \varepsilon_n$. In the following, after setting $\varepsilon_n = \lambda_n$, $F^l_{\varepsilon_n}(u) = E^{\#,l}_n(u)$, and

$$G^l_{\varepsilon_n}(u) = G^{\#,l}_n(u) = \begin{cases} \displaystyle\int_{(0,L)} \tilde{\psi}_0(u')\,dt + \lambda_n^2 \int_{(0,L)} |u''|^2\,dt & u \in W^{2,2}_{\rm loc}(\mathbb{R}), \\ & u - lt \text{ is } L\text{-periodic} \\ +\infty & \text{otherwise on } L^1(0,L), \end{cases}$$

we prove the following equivalence result.

Theorem 2.5.2 (Γ-equivalence - Periodic boundary data) *Let $\tilde{\psi}_0 : \mathbb{R} \to \mathbb{R}$ be a Borel function such that*

(i) $\displaystyle\lim_{|z|\to+\infty} \frac{\tilde{\psi}_0(z)}{|z|} = +\infty,$

(ii) $(\tilde{\psi}_0)^{**} = \psi_0^{**}$.

(iii) $\{z \in \mathbb{R} :\ \tilde{\psi}_0(z) = (\tilde{\psi}_0)^{**}(z)\} = \{z \in \mathbb{R} :\ \psi_0(z) = \psi_0^{**}(z)\}$

(iv) $\tilde{\psi}_0(z^i + z) - (\tilde{\psi}_0)^{**}(z^i + z) = O(z^\alpha),\ \alpha > 1$ *for all z^i such that* $\tilde{\psi}_0(z^i) = (\tilde{\psi}_0)^{**}(z^i)$

If $\{z : \psi_0^{**}$ is affine$\} = \bigcup_{i=1}^{N} [\alpha_i, \beta_i]$ disjoint intervals, suppose that

$$\#\mathbb{M}^{\alpha_j} = \#\mathbb{M}^{\beta_j} = 1 \text{ and} \tag{2.5.23}$$

$$2\int_{\alpha_j}^{\beta_j} \sqrt{\tilde{\psi}_0(s) - \psi_0^{**}(s)}\ ds = C(\alpha_j, \beta_j) \text{ for some } j \in \{1, 2, \ldots, N(l)\}, \tag{2.5.24}$$

with $N(l) < +\infty$, then $F^l_{\varepsilon_n}$ and $G^l_{\varepsilon_n}$ are equivalent up to the first order for $l \in [\alpha_j, \beta_j]$.

Remark 2.5.3 In the special case that hypotheses (i) and (iv) are satisfied by ψ_0, it is possible to restate the previous result asserting that $F^l_{\varepsilon_n}$ is equivalent up to the first order, for $l \in [\alpha_j, \beta_j]$, to the following family of functionals

$$H^l_\varepsilon(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi_0(u')\,dt + k\varepsilon^2 \int_{(0,L)} |u''|^2\,dt & u \in W^{2,2}_{\rm loc}(\mathbb{R}), \\ & u - lt \text{ is } L\text{-periodic} \\ +\infty & \text{otherwise on } L^1(0,L), \end{cases}$$

where

$$k = \left(\frac{C(\alpha_j, \beta_j)}{\int_{\alpha_j}^{\beta_j} \sqrt{\psi_0(s)}\ ds} \right)^2,$$

along the sequence $\varepsilon_n = \lambda_n$.

Proof of Theorem 2.5.2

Zero-order equivalence. In what follows we set $L = 1$. By Theorem 2.2.4 we need to prove that

$$\Gamma\text{-}\lim_n G_n^{\#,l}(u) = E^{\#,l}(u) = \begin{cases} \int_0^1 \psi_0^{**}(u'(t))\,dt & \text{if } u \in W^{1,1}_{\#,l}(0,1) \\ +\infty & \text{otherwise on } L^1_{\rm loc}(\mathbb{R}). \end{cases}$$

First observe that, thanks to hypothesis (i) and the definition of $G_n^{\#,l}$, we have that, as in Theorem 2.2.4, the limit is finite only on $W^{1,1}_{\#,l}(0,1)$. Moreover, as

$$G_n^{\#,l}(u) \geq \int_0^1 \tilde{\psi}_0(u'(t))\,dt \geq \int_0^1 (\tilde{\psi}_0)^{**}(u'(t))\,dt,$$

then

$$\Gamma\text{-}\liminf_n G_n^{\#,l}(u) \geq \int_0^1 (\tilde{\psi}_0)^{**}(u'(t))\,dt.$$

By an easy density argument it suffices to obtain the $\Gamma\text{-}\limsup$ inequality for $u \in C^2(\mathbb{R})$ such that $u(t) - lt$ is 1-periodic. In this case we have, from the definition of $\Gamma\text{-}\limsup$, taking the pointwise limit of $G_n^{\#,l}(u)$ and passing to its lower semicontinuous envelope with respect to the strong L^1 convergence,

$$\Gamma\text{-}\limsup_n G_n^{\#,l}(u) \leq \int_0^1 (\tilde{\psi}_0)^{**}(u'(t))\,dt.$$

First-order equivalence. Set

$$\begin{aligned} G_{1,n}^{\#,l}(u) &:= \frac{G_n^{\#,l}(u) - \min E^{\#,l}}{\lambda_n} \\ &= \begin{cases} \dfrac{1}{\lambda_n}\displaystyle\int_0^1 (\tilde{\psi}_0(u') - (\tilde{\psi}_0)^{**}(l))\,dt + \lambda_n \int_0^1 |u''|^2\,dt & \text{if } u \in W^{2,2}_{\rm loc}(\mathbb{R}), \\ & u(t) - lt \text{ is 1-periodic,} \\ +\infty & \text{otherwise on } L^1_{\rm loc}(\mathbb{R}). \end{cases} \end{aligned}$$

Thanks to Theorem 2.2.10 and hypothesis (2.5.23), we need to prove that

$$\Gamma\text{-}\lim_n G_{1,n}^{\#,l}(u) = E_1^{\#,l}(u) = \begin{cases} \displaystyle\sum_{t \in S(\mathbf{u}') \cap (0,1]} C(u'(t-), u'(t+)), & \text{if } \mathbf{u}' \in PC_{\rm loc}(\mathbb{R}), \\ & u' \in \mathbb{M}_l,\ u(t) - lt \text{ is 1-periodic,} \\ +\infty & \text{otherwise on } W^{1,\infty}_{\rm loc}(\mathbb{R}). \end{cases}$$

Compactness. Let $u_n \to u$ in $W^{1,1}_{\rm loc}(\mathbb{R})$ be such that

$$\sup_n G_{1,n}^{\#,l}(u_n) \leq c. \tag{2.5.25}$$

As in the proof of the zero order equivalence, we have that $u(t) - lt$ is a 1-periodic function. Without loss of generality, we may suppose that $(\tilde{\psi}_0)^{**}$ is a straight line in a neighborhood of l and we write $(\tilde{\psi}_0)^{**}(z) = r(z)$ (the case $(\tilde{\psi}_0)^{**}$ is strictly convex can be proved in the same way). Since

$$\int_0^1 (\tilde{\psi}_0)^{**}(u_n'(t))\, dt \geq (\tilde{\psi}_0)^{**}(l) \quad \text{and} \quad \tilde{\psi}_0 \geq (\tilde{\psi}_0)^{**},$$

we have that

$$G_{1,n}^{\#,l}(u_n) \geq \frac{1}{\lambda_n}\int_0^1 ((\tilde{\psi}_0)^{**}(u_n') - (\tilde{\psi}_0)^{**}(l))\, dt = \frac{1}{\lambda_n}\int_0^1 ((\tilde{\psi}_0)^{**}(u_n') - r(u_n'))\, dt$$

and

$$G_{1,n}^{\#,l}(u_n) \geq \frac{1}{\lambda_n}\int_0^1 (\tilde{\psi}_0(u_n') - (\tilde{\psi}_0)^{**}(u_n'))\, dt = \frac{1}{\lambda_n}\int_0^1 W(u_n')\, dt$$

where we have set for short $W(z) = \tilde{\psi}_0(z) - (\tilde{\psi}_0)^{**}(z)$. By (2.5.25) we get that , for all $\eta > 0$

$$\lim_n |\{t : (\tilde{\psi}_0)^{**}(u_n'(t)) - r(u_n'(t)) > \eta\} \cap \{t : W(u_n'(t)) > \eta\}| = 0.$$

Since, thanks to hypothesis (iii), we have that $\{z \in \mathbb{R} : \tilde{\psi}_0(z) = (\tilde{\psi}_0)^{**}(z) = r(z)\} = \mathbb{M}_l$, we get that, up to subsequences, $u_n' \to z$ a.e. where $z \in \mathbb{M}_l$.

Let us prove that $u' \in PC_{\mathrm{loc}}(\mathbb{R})$. By the 1-periodicity of u it suffices to consider K a compact set of $(0,1]$ and prove that $u' \in PC(K)$. Without loss of generality we can suppose that $K = [a,b]$ and that $t_1, t_2, \ldots, t_M \in S(u') \cap [a,b]$. For $i = 1, 2, \ldots, M$ we can find $a_i^{\pm} \in [a,b]$ such that

$$a < a_i^- < t_i < a_i^+ < a_{i+1}^- < b \tag{2.5.26}$$

and that there exist the limits

$$\lim_n u_n'(a_i^{\pm}) = u'(a_i^{\pm}) \in \mathbb{M}_l \quad \text{with} \quad u'(a_i^+) \neq u'(a_i^-). \tag{2.5.27}$$

It holds

$$\begin{aligned} G_{1,n}^{\#,l}(u_n) &\geq \frac{1}{\lambda_n}\int_a^b W(u_n')\, dt + \lambda_n \int_a^b |u_n''|^2\, dt \\ &\geq \sum_{i=1}^M \Big(\frac{1}{\lambda_n}\int_{a_i^-}^{a_i^+} W(u_n') + \lambda_n \int_{a_i^-}^{a_i^+} |u_n''|^2\Big), \end{aligned}$$

and, by Young's inequality,

$$G_{1,n}^{\#,l}(u_n) \geq \sum_{i=1}^M 2\int_{a_i^-}^{a_i^+} \sqrt{W(u_n')}|u_n''|\, dt \geq 2\sum_{i=1}^M \Big|\int_{a_i^-}^{a_i^+} \sqrt{W(u_n')}|u_n''|\, dt\Big|$$

$$\geq \quad 2\sum_{i=1}^{M}\Big|\int_{u_n'(a_i^-)}^{u_n'(a_i^+)} \sqrt{W(s)}\; ds\Big| = \sum_{i=1}^{M} C(u_n'(a_i^-), u_n'(a_i^+)).$$

Since we have that

$$\begin{aligned}\liminf_{n\to\infty} G_{1,n}^{\#,l}(u_n) &\geq \sum_{i=1}^{M} \liminf_{n\to\infty} C(u_n'(a_i^-), u_n'(a_i^+)) \geq \sum_{i=1}^{M} C(u'(a_i^-), u'(a_i^+)) \\ &\geq M \min\{C(u'(a_i^-), u'(a_i^+)),\ i \in \{1,2,\dots,M\}\} \geq cM,\end{aligned}$$

thanks to (2.5.25) we get that $u' \in PC([a,b])$.

Γ-liminf inequality. Thanks to the compactness result we have just proven, we can infer that there exist $0 < t_1 < t_2 < \dots < t_N \leq 1$ such that

$$S(u') = \{t \in \mathbb{R} :\ t + q = t_i,\ q \in \mathbb{Z},\ i = 1,2,\dots,N\}.$$

With an abuse of notation we can choose again $a_i^{\pm}$ such that (2.5.26) and (2.5.27) hold true with $a = 0$, $b = 1$ and $M = N$. Then, by the periodicity of u, we get

$$\begin{aligned}G_{1,n}^{\#,l}(u_n) &\geq \int_0^{a_1^-} \sqrt{W(u_n')}|u_n''|\, dt \\ &\quad + \sum_{i=1}^{N-1} \int_{a_i^-}^{a_{i+1}^-} \sqrt{W(u_n')}|u_n''|\, dt + \int_{a_N^-}^{1} \sqrt{W(u_n')}|u_n''|\, dt \\ &\geq \sum_{i=1}^{N-1} \int_{a_i^-}^{a_{i+1}^-} \sqrt{W(u_n')}|u_n''|\, dt + \int_{a_N^-}^{1+a_1^-} \sqrt{W(u_n')}|u_n''|\, dt.\end{aligned}$$

Passing to the $\liminf$ for $n \to +\infty$ in the previous inequality we have

$$\begin{aligned}\liminf_{n} G_{1,n}^{\#,l}(u_n) &\geq \sum_{i=1}^{N-1} \liminf_{n\to+\infty} \int_{a_i^-}^{a_{i+1}^-} \sqrt{W(u_n')}|u_n''|\, dt \\ &\quad + \liminf_{n\to+\infty} \int_{a_N^-}^{1+a_1^-} \sqrt{W(u_n')}|u_n''|\, dt \\ &= \sum_{t\in S(u')\cap(0,1]} C(u'(t-), u'(t+)).\end{aligned}$$

Γ-limsup inequality. We now construct a recovery sequence (u_n) for the Γ-limsup in a periodicity cell. Fix $l \in [\alpha_j, \beta_j]$, let u be such that $E_1^{\#,l}(u) < +\infty$. Supposing, without loss of generality, that $\alpha_j = 0,\ \beta_j = 1$, since the limit energy depends only on u', our approximation construction modifies u' only in a small neighborhood of $S(u')$ and is invariant under translation, it is not restrictive to suppose that u' is the 1-periodic extension of $\chi_{(a,b)}$ where $(a,b) \subseteq [0,1]$ and that $0 < a < b \leq 1$. Following the well known construction of the recovery sequence in the Modica-Mortola problem [38] (see also [11]) it is possible to find $v_n \to u$ in $W^{1,1}(0,1)$ such that

$$\lim_{n\to+\infty} \frac{1}{\lambda_n} \int_0^1 W(v_n')\,dt + \lambda_n \int_0^1 |v_n''|^2\,dt = \sum_{t\in S(u')\cap(0,1]} C(u'(t-),u'(t+)) \quad (2.5.28)$$

From now on we will call (v_n) the Modica-Mortola recovery sequence for u. In the following we will modify the sequence (v_n) to obtain our recovery sequence (u_n) which has to satisfy (2.5.28) but also the condition

$$u_n(t) - lt \text{ is 1-periodic}$$

which can be rephrased as

$$\int_0^1 u_n'(t)\,dt = \int_0^1 u'(t)\,dt. \quad (2.5.29)$$

Since v_n' modifies u' only in $(a-\lambda_n, a+\lambda_n)\cup(b-\lambda_n, b+\lambda_n)$ we can define u_n' to be

$$u_n'(t) = \begin{cases} t+1-a-\lambda_n & \text{if } t\in(a+\lambda_n, a+\lambda_n+k_n) \\ -t+1+b+\lambda_n & \text{if } t\in(b-\lambda_n-k_n, b-\lambda_n) \\ 1+k_n & \text{if } t\in(a+\lambda_n+k_n, b-\lambda_n-k_n) \\ v_n'(t) & \text{otherwise} \end{cases}$$

where k_n has to be chosen such that (2.5.29) holds. Since

$$\int_0^1 u_n'\,dt = \int_0^1 v_n'\,dt + k_n^2 + k_n(b-a-2\lambda_n-2k_n),$$

setting

$$\alpha_n := \int_0^1 (u_n' - v_n')\,dt, \quad (2.5.30)$$

the equation for k_n becomes

$$k_n^2 - k_n(b-a-2\lambda_n) + \alpha_n = 0$$

and it can be chosen to be

$$k_n = \Big(\frac{b-a-2\lambda_n}{2}\Big)\Big(1-\sqrt{1-\frac{4\alpha_n}{(b-a-2\lambda_n)^2}}\Big) = O(\alpha_n).$$

By hypothesis (iv) it holds true that

$$G_{1,n}^{\#,l}(u_n) \le \int_0^1 \frac{W(v_n')}{\lambda_n} + \lambda_n|v_n''|^2\,dt + \int_{a+\lambda_n}^{b-\lambda_n} \frac{W(u_n')}{\lambda_n} + \lambda_n|u_n''|^2\,dt$$

$$
\begin{aligned}
&\leq \int_0^1 \frac{W(v_n')}{\lambda_n} + \lambda_n |v_n''|^2 \, dt + \int\limits_{a+\lambda_n}^{a+\lambda_n+k_n} \frac{W(t+1-a-\lambda_n)}{\lambda_n} \, dt \\
&\quad + \int\limits_{b-\lambda_n-k_n}^{b-\lambda_n} \frac{W(-t+1+b+\lambda_n)}{\lambda_n} \, dt + 2\lambda_n k_n + \frac{(b-a-2\lambda_n-2k_n)}{\lambda_n} W(1+k_n) \\
&\leq \int_0^1 \frac{W(v_n')}{\lambda_n} + \lambda_n |v_n''|^2 \, dt + \frac{2}{\lambda_n} k_n \sup\{W(s) : \ s \in (1, 1+k_n)\} \\
&\quad +2\lambda_n k_n + \frac{c}{\lambda_n} O(k_n^\alpha).
\end{aligned}
$$

Passing to the lim sup in the previous inequality, by (2.5.28), observing that

$$
\alpha_n = \int_0^1 u' - v_n' \, dt \leq C\lambda_n,
$$

we conclude the proof. □

Theorem 2.5.4 (Γ-equivalence - Dirichlet boundary data) *In the same hypotheses of Theorem* 2.5.2, *if* $\varepsilon_n = \lambda_n$,

$$
G^l_{\varepsilon_n}(u) = G^l_n(u) = \begin{cases} \displaystyle\int_{(0,L)} \Big(\tilde{\psi}_0(u') + \lambda_n^2 |u''|^2\Big) \, dt + \lambda_n \Big(\overline{B}_+(u'(0)) + \overline{B}_-(u'(L))\Big) \\ \qquad\qquad\qquad \text{if } u \in W^{2,2}(0,L), \ u(0) = 0, \ u(L) = l, \\ +\infty \qquad\qquad\qquad \text{otherwise on } L^1(0,L), \end{cases}
$$

where

$$
\begin{aligned}
&\overline{B}_+(z) : \mathbb{R} \to \mathbb{R} \text{ is such that } \inf_z \{C(w,z) + \overline{B}_+(z)\} = B_+(w), \\
&\overline{B}_-(z) : \mathbb{R} \to \mathbb{R} \text{ is such that } \inf_z \{C(w,z) + \overline{B}_-(z)\} = B_-(w),
\end{aligned} \tag{2.5.31}
$$

and $F^l_{\varepsilon_n}(u) = E^l_n(u)$, *then* $F^l_{\varepsilon_n}$ *and* $G^l_{\varepsilon_n}$ *are equivalent up to the first order for* $l \in [\alpha_j, \beta_j]$.

Proof. Suppose that $L = 1$. As the proof is analogous to that of the previous theorem, we only point out the main differences in the construction of the recovery sequence for the first order equivalence. As before, let $\alpha_j = 0$, $\beta_j = 1$ and let u be such that $u'(t) = \chi_{(a,b)}(t)$ with $(a,b) \subseteq (0,1)$. In the following we set

$$
\begin{aligned}
G^l_{1,n}(u) \ &:= \ \frac{G^l_n(u) - \min E^l}{\lambda_n} \\
&= \begin{cases} \displaystyle\int_0^L \Big(\frac{\tilde{\psi}_0(u') - (\tilde{\psi}_0)^{**}(u')}{\lambda_n} + \lambda_n |u''|^2 \Big) \, dt + \overline{B}_+(u'(0)) + \overline{B}_-(u'(L)) \\ \qquad\qquad\qquad \text{if } u \in W^{2,2}(0,L), \ u(0) = 0, \ u(1) = l, \\ +\infty \qquad\qquad\qquad \text{otherwise on } L^1(0,1). \end{cases}
\end{aligned}
$$

Fix $\varepsilon > 0$. Thanks to (2.5.31) there exist $\overline{z}, \tilde{z}$ such that

$$
\begin{array}{l}
C(\overline{z}, u'(0+)) + \overline{B}_+(\overline{z}) \leq B_+(u'(0)) + \varepsilon \quad \text{and} \\
C(\tilde{z}, u'(1-)) + \overline{B}_-(\tilde{z}) \leq B_-(u'(1-)) + \varepsilon.
\end{array}
$$

Fix $\eta > 0$, let $\tilde{u}$ be such that

$$
\tilde{u}'(t) = \begin{cases} \overline{z} & \text{if } t \in (-\eta, 0) \\ u'(t) & \text{if } t \in (0,1) \\ \tilde{z} & \text{if } t \in (1, \eta), \end{cases}
$$

and let $(\tilde{v}_n)$ be the Modica-Mortola recovery sequence for $\tilde{u}$. It holds that $\tilde{v}_n \to u$ in $W^{1,1}(0,1)$ and that

$$
\begin{aligned}
\lim_n & \int_0^1 \Big(\frac{\tilde{\psi}_0(\tilde{v}_n') - (\tilde{\psi}_0)^{**}(\tilde{v}_n')}{\lambda_n} + \lambda_n |\tilde{v}_n''|^2 \Big)\, dt \\
&= \sum_{t \in S(u') \cap (0,1)} C(u'(t-), u'(t+)) + C(\overline{z}, u'(0+)) + C(\tilde{z}, u'(1-)). \quad (2.5.32)
\end{aligned}
$$

As done in the proof of the previous theorem, we can modify $(\tilde{v}_n)$ in order to construct our recovery sequence (u_n) which fulfils the boundary conditions $u_n(0) = 0,\ u_n(1) = l$. We have that

$$
G^l_{1,n}(u_n) = \int_0^1 \Big(\frac{\tilde{\psi}_0(\tilde{v}_n') - (\tilde{\psi}_0)^{**}(\tilde{v}_n')}{\lambda_n} + \lambda_n |\tilde{v}_n''|^2 \Big)\, dt + \overline{B}_+(\overline{z}) + \overline{B}(\tilde{z}).
$$

Passing to the $\limsup_n$ in the previous expression, thanks to (2.5.32), we get that

$$
\begin{aligned}
\limsup_n G^l_{1,n}(u_n) &= \sum_{t \in S(u') \cap (0,1)} C(u'(t-), u'(t+)) \\
&\quad + C(\overline{z}, u'(0+)) + \overline{B}_+(\overline{z}) + C(\tilde{z}, u'(1-)) + \overline{B}_-(\tilde{z}) \\
&\leq E^l_1(u) + 2\varepsilon.
\end{aligned}
$$

The claim follows by the arbitrariness of ε. $\square$

Chapter 3

L^{∞} energies on discontinuous functions and their approximation by discrete systems

Many interesting phenomena of a variational nature present a complex dependence on boundary conditions or forcing terms, highlighting the relevant effect of non-convex energies. Among the many examples, in the framework of the so-called free-discontinuity problems we point out variational theories of softening and fracture (see e.g. [25] and [33]) and models in Computer Vision (see e.g. [39]). The study of problems involving such types of energies has been fruitfully addressed in the framework of the direct methods of the Calculus of Variations by characterizing some classes of non-convex functionals that are lower semicontinuous on spaces of discontinuous functions; namely, BV spaces or Ambrosio and De Giorgi's SBV spaces (see [4]). In one space dimension we can think of SBV functions as piecewise-Sobolev functions; if we denote by $S(u)$ the set of discontinuity points of an SBV function u, then the typical shape of such functionals is

$$\int_{\Omega} f(u')\,dt + \sum_{S(u)\cap\Omega} g([u]), \tag{3.0.1}$$

where $[u](t) = u(t+) - u(t-)$ is the jump of u across the point $t \in S(u)$ and u' is the derivative of u, that is defined almost everywhere on Ω. In order for such energies (or more precisely their extensions on BV) to be lower-semicontinuous in an appropriate topology, the necessary and sufficient conditions are of two types: (i) *structure conditions* on f and g (namely, that f be convex and g be subadditive; i.e. that $g(a+b) \le g(a) + g(b)$); (ii) *compatibility conditions* between the growth of g at 0 and of f at infinity; i.e.,

$$\lim_{z\to\pm\infty}\frac{f(z)}{z}=\lim_{z\to 0\pm}\frac{g(z)}{z}.$$

Note that these two conditions do not imply that the energy (3.0.1) be convex, and in particular they allow for complex non-monotone behaviour with respect to the boundary data. We refer, for example, to [15] and [45] for an analysis of the behaviour of (local) minima with Dirichlet boundary conditions and their interpretation in terms of softening and fracture. Note that similar behaviours can be obtained with 'asymptotically equivalent' energies of different shapes involving scale parameters (see [11] for a survey on this subject).

Recently, other types of functionals of a different nature that allow for a non-convex behaviour have been studied by considering energies of the sup norm. The simplest type of such energies is defined on $W^{1,\infty}(\Omega)$ and takes the form

$$\sup_{x\in\Omega} f(\nabla u(x)) \tag{3.0.2}$$

(by sup we mean the essential supremum). This type of functionals with $f(z)=|z|$ arises naturally, for example, when looking for the best Lipschitz extension of a function defined on $\partial\Omega$ (see e.g. [6, 8]). Necessary and sufficient conditions for the lower semicontinuity of functionals of this type have been studied in [5, 8], where it is shown that a necessary condition is that the sub-level sets of f be convex or, equivalently, that f satisfy

$$f(tx+(1-t)y)\le\max\{f(x),f(y)\} \tag{3.0.3}$$

for all $t\in[0,1]$. This property is called the *level convexity* or *quasi-convexity* of f. Note that level convexity is implied by convexity. In one space dimension this property is equivalent to being either monotone or decreasing-increasing; in particular this class of f contains many more functions than just convex functions.

An interesting remark, observed in [35] for convex f and subsequently proved in [27] for f super-linear at infinity, is that the functional in (3.0.2) is the Γ-limit of the power-law integral energies (for simplicity f is considered positive)

$$F_p(u)=\Big(\int_\Omega (f(\nabla u(x)))^p\,dx\Big)^{1/p}, \tag{3.0.4}$$

and hence it can be used to deduce 'approximate variational principles' for F_p for p large. This observation extends to more complex integrands and in itself is a justification of the study of the energies in (3.0.2). As an example of an application to dielectric breakdown following this approach we refer to [35].

Here we analyze the structure of minimum problems of the sup norm for discontinuous functions, in the simple case of one space dimension. The goal of the chapter is to show that some energies of the sup norm are meaningful also in BV and SBV spaces, that necessary and sufficient conditions for lower semicontinuity can be easily described and compared with the corresponding condition for integral functionals, and that the solutions of general minimum problems can be described by relaxation showing interesting new behaviours. In the last section of the chapter we will also make a first step in the study of this kind of energies in the framework of discrete systems proving an approximation result via Γ-convergence.

We first extend the definition in (3.0.2) to functions in SBV in parallel with (3.0.1), by considering

$$F(u) = \max\Big\{\sup_{t\in\Omega} f(u'(t)), \sup_{t\in S(u)\cap\Omega} g([u](t))\Big\}. \tag{3.0.5}$$

A technical issue must be mentioned at this point: since the lower-semicontinuity properties of L^∞ energies are invariant under composition of the energy densities with bijective increasing functions, it is not restrictive to impose a one-sided growth condition on f (namely, that $f(z) \to +\infty$ as $z \to -\infty$), that automatically also implies that $g(z) = +\infty$ for $z < 0$. Our model energy densities f are then Lennard-Jones type potentials (the case of monotone f being less interesting).

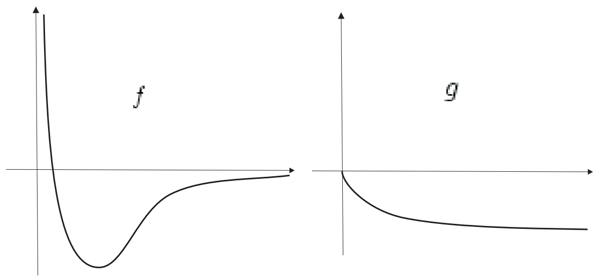

Figure 3.1: The functions f and g

In parallel with the theory for the energies in (3.0.1), we then show that necessary and sufficient conditions for the lower semicontinuity of F are again of two types:
(i) *structure conditions* on f and g. Namely, that f be level convex and g be *sub-maximal*; i.e. that

$$g(a + b) \leq \max\{g(a), g(b)\}; \tag{3.0.6}$$

(ii) *compatibility conditions* between the growth of g at 0 and of f at infinity:

$$\lim_{z\to 0+} g(z) = \lim_{z\to+\infty} f(z). \tag{3.0.7}$$

It must be remarked that while level convexity is implied by convexity, the new condition of 'sub-maximality' of g implies its sub-additivity. A simple class of sub-maximal g are decreasing functions, but much more complex shapes can be exhibited. In the case of f of Lennard-Jones type and g decreasing as in Figure 1 we can plot as an example the minimal energy $m(d)$ of the problems

$$m(d) = \min\{F(u) : u(0) = 0,\ u(1) = d\}, \tag{3.0.8}$$

obtaining a graph as in Figure 2.

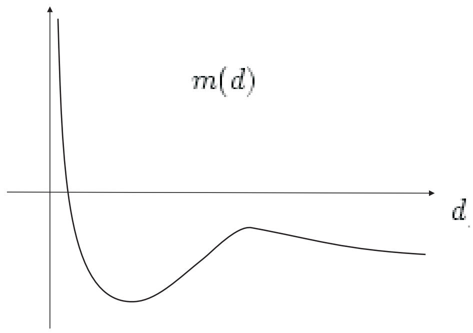

Figure 3.2: Non-monotone dependence on boundary conditions

In order to study the structure of solutions of minimum problems for general non-lower semicontinuous F, we prove a relaxation theorem showing that the L^1-lower semicontinuous envelope of such F is a functional of the same form with f and g substituted by the suitably defined level-convex and sub-maximal envelopes, respectively. A simple formula is obtained when g is a level-convex function itself, in which case the sub-maximal envelope is simply $\inf_k g(x/k)$.

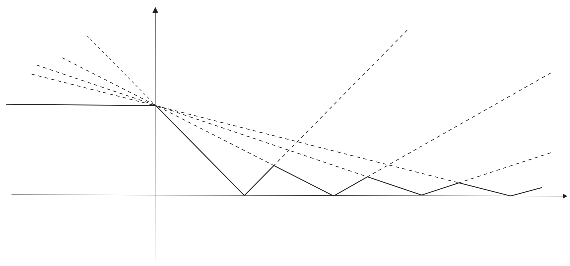

Figure 3.3: A non-trivial sub-maximal envelope

By plotting the 'stress-strain' curve relating the bulk gradient of the solutions to the boundary datum we highlight a 'multiple cracking' phenomenon analogous to that observed for non-subadditive free-discontinuity integral energies (see the concluding section).

In the last section of the chapter we will also provide a discrete approximation for energies of the form (3.0.5).

3.1 Preliminaries

The integer part of t will be denoted by $[t]$. We will sometimes write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $\bigvee_i a_i = \sup\{a_i\}$, and so on. If μ is a measure and f a real function μ-$\sup f$ denotes the essential supremum of the values of f with respect to the measure μ. With a little abuse of notation, we simply write $\sup f$ if μ is the Lebesgue measure. We write l.s.c. as a shorthand for 'lower semicontinuous'.

We first recall the notion of level convexity (also referred to as *quasiconvexity* – not to be confused with Morrey's quasiconvexity – in part of the literature). A Borel function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *level convex* if for any $t \in \overline{\mathbb{R}}$ the sub-level set $\{x \in \mathbb{R}^n : \ f(x) \leq t\}$ is convex. We remark that level convexity can be equivalently stated by requiring the condition:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq f(x_1) \vee f(x_2) \quad \forall \lambda \in (0,1),\ x_1, x_2 \in \mathbb{R}^n.$$

In the one-dimensional case $n = 1$, this condition is equivalent to f being either monotone or decreasing in $(-\infty, t_0]$ and increasing in $(t_0, +\infty)$ for some $t_0 \in \mathbb{R}$ (we label this situation as a decreasing/increasing f, for short). Moreover we recall the analogue of Jensen's inequality for level convex functions (see [43]).

Theorem 3.1.1 *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a lower-semicontinuous and level-convex function and let μ be a probability measure on $\mathbb{R}$. Then for every function $u \in L^1_\mu(\mathbb{R})$*

$$f\Big(\int_{\mathbb{R}} u \ d\mu\Big) \leq \mu\text{-}\sup(f \circ u).$$

In what follows $I = (a, b)$ is a bounded open interval in $\mathbb{R}$. Let $F : L^\infty(I) \to \overline{\mathbb{R}}$ be defined as follows:

$$F(u) = \sup_{t \in I} f(u(t)). \tag{3.1.9}$$

The following theorems, concerning the lower-semicontinuity and relaxation of L^∞-functionals on $L^\infty(I)$, have been proved in [43].

Theorem 3.1.2 *F is L^∞ weakly*-lower semicontinuous if and only if f is lower semicontinuous and level convex.*

Theorem 3.1.3 *Let F be as in* (3.1.9)*. Then the lower-semicontinuous envelope of F with respect to the weak*-topology of $L^\infty(I)$ is given by*

$$\overline{F}(u) = \sup_{t \in I} f^{\mathrm{lc}}(u(t))$$

where $f^{\mathrm{lc}}(z) = \sup\{g(z) : \ g$ is lower semicontinuous and level convex, $g \leq f\}$ is the lower-semicontinuous and level-convex envelope of f.

We finally recall an extension of L^∞ energies on gradients in a BV setting (see Section 3.3 for definitions).

Remark 3.1.4 Let f be a lower-semicontinuous level-convex function. Upon changing $f(z)$ into $f(-z)$, we can suppose that

$$\lim_{z\to-\infty} f(z) = \sup f.$$

Let $K \in \overline{\mathbb{R}}$, and let $F_\infty : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ be defined as

$$F_\infty(u) = \begin{cases} \sup\limits_{t\in I} f(u'(t)) & \text{if } u \in W^{1,1}_{\rm loc}(I) \\ \max\{\sup\limits_{t\in I} f(u'(t)), C_\infty\} & \text{if } u \in BV^+_{\rm loc}(I) \setminus W^{1,1}_{\rm loc}(I) \\ K & \text{otherwise.} \end{cases} \tag{3.1.10}$$

Thanks to the results in [36], F_∞ is l.s.c. with respect to $L^1_{\rm loc}(I)$ convergence if and only if $K = \sup f$ and $C_\infty = \lim\limits_{z\to+\infty} f(z)$.

3.2 Sub-maximality

In this section we define the notion of sub-maximality which is related to the lower semicontinuity of L^∞ energies defined on piecewise-constant functions.

We say that a function $u : (a,b) \to \mathbb{R}$ is *piecewise constant* on (a,b) if there exist points $a = t_0 < t_1 < \ldots < t_N < t_{N+1} = b$ such that

$$u(t) \text{ is constant a.e. on } (t_{i-1}, t_i) \text{ for all } i = 1, 2, \ldots, N+1. \tag{3.2.11}$$

We define $S(u)$ as the minimal set $\{t_1, t_2, \ldots, t_N\} \subset (a,b)$ such that (3.2.11) holds. The subspace of $L^\infty(a,b)$ of all such u is denoted by $PC(a,b)$.

In this section, we will study functionals $H(u) : PC(I) \to \overline{\mathbb{R}}$ of the form

$$H(u) = \sup_{t\in S(u)} g([u](t)), \tag{3.2.12}$$

where $g : \mathbb{R}\setminus\{0\} \to \overline{\mathbb{R}}$.

We say that a function $g : \mathbb{R}\setminus\{0\} \to \overline{\mathbb{R}}$ is *sub-maximal* if the following inequality holds true for all $x_1, x_2 \in \mathbb{R}\setminus\{0\}$ such that $x_1 \neq -x_2$:

$$g(x_1 + x_2) \le \max\{g(x_1), g(x_2)\}. \tag{3.2.13}$$

We will also consider functions $g : (0,+\infty) \to \overline{\mathbb{R}}$ or $g : (-\infty, 0) \to \overline{\mathbb{R}}$ for which the definition extends likewise.

In the sequel we sometimes prefer to consider g as defined on the whole $\mathbb{R}$, even though its value in 0 is never taken into account, by setting $g(0) = \inf g$. Note that g is sub-maximal and lower semicontinuous in $\mathbb{R}\setminus\{0\}$ if and only if such its extension is sub-maximal and lower semicontinuous.

The following theorem holds true.

Theorem 3.2.1 (**Semicontinuity**) *H is lower semicontinuous with respect to the $L^1(I)$-convergence if and only if g is lower semicontinuous and sub-maximal.*

PROOF. First suppose H to be semicontinuous. To prove the lower semicontinuity of g, let (w_n) be a sequence of real numbers converging to w. Let $u_h(t)$ be defined as

$$u_h(t) = \begin{cases} z & \text{if } t \le t_0 \\ z + w_h & \text{if } t > t_0. \end{cases}$$

As u_h converges to

$$u(t) = \begin{cases} z & \text{if } t \le t_0 \\ z + w & \text{if } t > t_0 \end{cases}$$

in $L^\infty(I)$, we get

$$g(w) \le \liminf_{h \to +\infty} g(w_h). \tag{3.2.14}$$

Consider now u_h be defined as

$$u_h(t) = \begin{cases} z & \text{if } t \le t_0 \\ z + w_1 & \text{if } t_0 < t \le t_0 + \frac{1}{h} \\ z + w_1 + w_2 & \text{if } t > t_0 + \frac{1}{h}. \end{cases}$$

As u_h converges to

$$u(t) = \begin{cases} z & \text{if } t \le t_0 \\ z + w_1 + w_2 & \text{if } t > t_0 \end{cases}$$

in $L^1(I)$, we obtain the sub-maximality of g.

Conversely suppose g to be lower semicontinuous and sub-maximal. Let (u_h) be a sequence of functions such that $u_h \to u$ in $L^1_{\mathrm{loc}}(I)$. Up to subsequences we can suppose that $u_h \to u$ a.e.. As $u \in PC(I)$, there exists $\varepsilon > 0$ such that $\varepsilon < \inf\{|t-s| \ : \ t, s \in S(u),\ t \neq s\}$. Fix $t \in S(u)$, we can suppose that

$$u_h(t \pm \varepsilon) \to u(t \pm \varepsilon) = u(t\pm)$$

and that, for all h, $(t \pm \varepsilon) \notin S(u_h)$. Thanks to the hypotheses on g, we have

$$g([u](t)) \le \liminf_{h \to +\infty} \sup_{s \in S(u_h) \cap (t-\varepsilon, t+\varepsilon)} g([u_h](s)) \le \liminf_{h \to +\infty} \sup_{t \in S(u_h)} g([u_h](t)).$$

Thanks to the arbitrariness of $t \in S(u)$ we finally get

$$\sup_{t \in S(u)} g([u](t)) \le \liminf_{h \to +\infty} H(u_h)$$

as desired. $\square$

Remark 3.2.2 (i) If $g : (0, +\infty) \to \mathbb{R}$ is decreasing then it is sub-maximal. Analogously, if $g : (-\infty, 0) \to \mathbb{R}$ is increasing then it is sub-maximal.

(ii) If $g : (0, +\infty) \to \mathbb{R}$ ($g : (-\infty, 0) \to \mathbb{R}$) is sub-maximal, a sub-maximal extension $\tilde{g}$ of g defined on $\mathbb{R} \setminus \{0\}$ is given by

$$\tilde{g}(w) = \begin{cases} C & \text{if } w < 0 \ (w > 0) \\ g(w) & \text{if } w > 0 (w < 0), \end{cases}$$

where $C \geq \sup_{(0,+\infty)} g$.

Example 3.2.3 The function $g(w) = \dfrac{|\sin(w)|}{w}$ is sub-maximal on $(0, +\infty)$. More generally, if $f : (0, +\infty) \to [0, +\infty)$ is sub-additive and $h : (0, +\infty) \to (0, +\infty)$ is super-additive, then the function $g := \dfrac{f}{h}$ is sub-maximal. In fact, given $w_1, \ w_2 > 0$ there holds

$$\begin{aligned} g(w_1 + w_2) &\leq \frac{1}{h(w_1 + w_2)} (f(w_1) + f(w_2)) \\ &= \frac{h(w_1)}{h(w_1 + w_2)} \frac{f(w_1)}{h(w_1)} + \frac{h(w_2)}{h(w_1 + w_2)} \frac{f(w_2)}{h(w_2)} \\ &\leq \left(\frac{h(w_1)}{h(w_1 + w_2)} + \frac{h(w_2)}{h(w_1 + w_2)} \right) \Big(g(w_1) \vee g(w_2) \Big) \\ &\leq g(w_1) \vee g(w_2). \end{aligned}$$

Remark 3.2.4 (properties of sub-maximal functions) Let g be sub-maximal.

(i) If g is positive then g is sub-additive;

(ii) If $k \in \mathbb{N}$ then $g(kw) \leq g(w)$ for all $w \in \mathbb{R}$;

(iii) $g + c$ is sub-maximal (c a constant).

Furthermore, if (g_i) is a family of sub-maximal functions then $g = \sup_i g_i$ is sub-maximal.

We introduce the *lower-semicontinuous and sub-maximal envelope* of g as

$$g^{\mathrm{sm}}(w) = \sup\{f(w) : f \text{ lower semicontinuous and sub-maximal, } f \leq g\}.$$

Following the analogous argument for sub-additive functions (see e.g. [13]) we easily check that

$$g^{\mathrm{sm}}(w) = \inf\Big\{ \liminf_{j \to +\infty} \bigvee_{i=1}^{N_j} g(w_j^i) \ : \sum_{i=1}^{N_j} w_j^i \to w \Big\}. \tag{3.2.15}$$

Theorem 3.2.5 (relaxation) *Let $H(u) : PC(I) \to \mathbb{R}$ be the functional defined in* (3.2.12). *Then the lower-semicontinuous envelope of H with respect to the $L^1(I)$-topology is given on $PC(I)$ by the functional G defined as*

$$G(u) = \sup_{t \in S(u)} g^{\mathrm{sm}}([u](t)). \tag{3.2.16}$$

PROOF. Since g^{sm} is l.s.c., sub-maximal and $g^{\mathrm{sm}} \le g$, thanks to Theorem 3.2.1 $G(u) \le \overline{F}(u)$. We now prove that $\overline{F}(u) \le G(u)$. For simplicity we can suppose that

$$u(t) = \begin{cases} \alpha & \text{if } t \in (a, t_0) \\ \alpha + w & \text{if } t \in [t_0, b). \end{cases}$$

Then, for all $\varepsilon > 0$, thanks to (3.2.15) there exist w_j^i and N_j such that

$$g^{\mathrm{sm}}(w) \ge \bigvee_{i=1}^{N_j} g(w_j^i) - \varepsilon \qquad \text{and} \qquad \Big|\sum_{i=1}^{N_j} w_j^i - w\Big| < \frac{1}{j}.$$

Let $M_j \in \mathbb{N}$ be such that $\sup_i \left(\dfrac{N_j^2 |w_j^i|}{M_j} \right) < \dfrac{1}{j}$ and let u_j be defined as follows

$$u_j(t) = \begin{cases} \alpha & a < t \le t_0 \\ \alpha + \sum\limits_{i=1}^{k} w_j^i & t_0 + \frac{k-1}{M_j} < t \le t_0 + \frac{k}{M_j} \\ & k = 1, 2, \ldots, N_j - 1 \\ \alpha + \sum\limits_{i=1}^{N_j} w_j^i & t_0 + \frac{N_j - 1}{M_j} < t < b. \end{cases}$$

We have that $u_j \to u$ in $L^1(I)$ and that

$$F(u_j) = \bigvee_{i=1}^{N_j} g(w_j^i) \le g^{\mathrm{sm}}(w) + \varepsilon = G(u) + \varepsilon.$$

Letting $j \to +\infty$ we get

$$F(u) \le G(u) + \varepsilon$$

and the conclusion follows by the arbitrariness of ε. □

The following simple formula for the sub-maximal envelope of a level-convex function holds.

Proposition 3.2.6 *If g is lower semicontinuous, level convex and continuous in zero then*

$$g^{\mathrm{sm}}(w) = \bigwedge_{j=1}^{+\infty} g\Big(\frac{w}{j}\Big). \tag{3.2.17}$$

PROOF. Set $\phi(w) = \sup\{f(w) : \ f \text{ sub-maximal}, \ f \le g\}$. It can be easily checked that $\phi(w) = \inf\Big\{\bigvee\limits_{i=1}^{N} g(w_j) \ : \sum\limits_{j=1}^{N} w_j = w\Big\}$. Note that $g^{\mathrm{sm}} \le \phi \le g$. We will prove (3.2.6) in two steps.

Step 1. We want to show that $\phi(w) = \bigwedge_{j=1}^{+\infty} g(\frac{w}{j})$. For all j, choosing $w_j = \frac{w}{j}$ in the minimum problem defining $\phi(w)$ we get that $\phi(w) \leq g(\frac{w}{j})$ and so

$$\phi(w) \leq \bigwedge_{j=1}^{+\infty} g(\frac{w}{j}).$$

To prove the opposite inequality, thanks to the level convexity of g, we have that $g(\sum_{i=1}^{j} \frac{w_i}{j}) \leq \bigvee_{i=1}^{j} g(w_i)$, hence

$$\phi(w) \geq \inf\Big\{g(\sum_{i=1}^{j} \frac{w_i}{j}),\ \sum_{i=1}^{j} w_i = w\Big\} = \bigwedge_{j=1}^{+\infty} g(\frac{w}{j}).$$

Step 2. To prove (3.2.6) ot suffices to prove that $\phi(w)$ is l.s.c. Let $w_n \to w$. Fixed $\varepsilon > 0$, thanks to the continuity of g in zero, there exists $k \in \mathbb{N}$ such that

$$\bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) > \bigwedge_{j=k+1}^{+\infty} g(\frac{w}{j}) - \varepsilon \quad \forall n \in \mathbb{N}. \tag{3.2.18}$$

Then, thanks to the l.s.c. of g and to (3.2.18), we have

$$\begin{aligned} \liminf_n \phi(w_n) &= \liminf_n \bigwedge_{j=1}^{k} g(\frac{w_n}{j}) \wedge \liminf_n \bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) \\ &= \bigwedge_{j=1}^{k} \liminf_n g(\frac{w_n}{j}) \wedge \liminf_n \bigwedge_{j=k+1}^{+\infty} g(\frac{w_n}{j}) \\ &\geq \bigwedge_{j=1}^{k} g(\frac{w}{j}) \wedge \bigwedge_{j=k+1}^{+\infty} g(\frac{w}{j}) - \varepsilon = \phi(w) - \varepsilon. \end{aligned}$$

Step 2 is completed by letting ε tend to zero. □

Remark 3.2.7 If $g : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $g(w) = +\infty$ for all $w < 0$, the previous proposition still holds if the continuity hypothesis is replaced by the continuity from the right in zero.

Example 3.2.8 Let $g : (0, +\infty) \to \mathbb{R}$ be of quadratic type and such that $g(0+) = 0$, that is

$$g(w) = b(w^2 - 2ww_0)$$

for some positive constants b and w_0, and set

$$K := \min g = g(w_0) = -bw_0^2.$$

By Proposition 3.2.6 and Remark 3.2.7, we have that the lower semicontinuous and sub-maximal envelope of g on $(0,+\infty)$ is given by formula (3.2.17). Note that $g^{\mathrm{sm}}(kw_0)=K$ for any $k\in\mathbb{N}$. Moreover a simple calculation shows that

$$g^{\mathrm{sm}}(w)=g(w/k) \text{ if } w\in[w_{k-1},w_k]$$

where $w_k:=2w_0k(k+1)/(2k+1)$ are relative maximum points for g^{sm}. Note that $kw_0<w_k<(k+1)w_0$ and $g^{\mathrm{sm}}(w_k)$ converge decreasingly to K as $k\to\infty$.

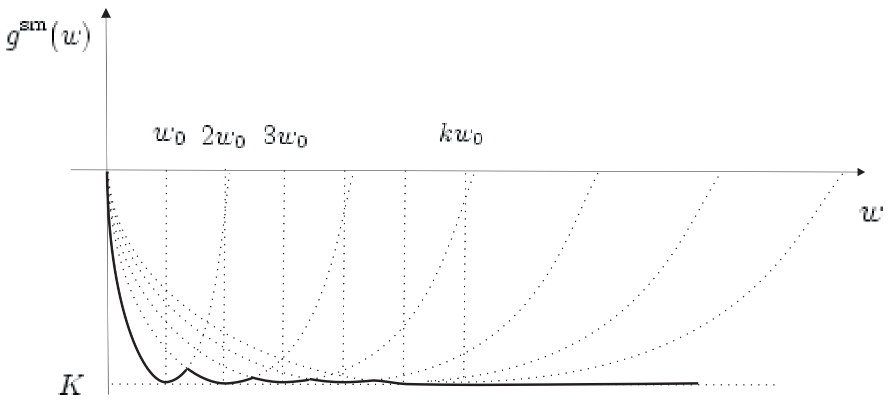

Figure 3.4: the sub-maximal envelope of g in Example 3.2.8

Remark 3.2.9 Let $g:\mathbb{R}\to\overline{\mathbb{R}}$. Then

$$\inf g(w)\le g^{\mathrm{sm}}(w)\le \limsup_{w\to0^+} g(w)\quad \forall w>0,$$

$$\inf g(w)\le g^{\mathrm{sm}}(w)\le \limsup_{w\to0^-} g(w)\quad \forall w<0.$$

The lower bound inequality is trivial. We prove the upper bound for $w>0$ only, the proof in the other case being analogous. For any $\varepsilon\in(0,w)$ we can write

$$\begin{aligned} g^{\mathrm{sm}}(w) &= g^{\mathrm{sm}}\Big(\varepsilon\Big[\frac{w}{\varepsilon}\Big]+\Big(w-\varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big)\Big)\le g^{\mathrm{sm}}\Big(\varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big)\vee g^{\mathrm{sm}}\Big(w-\varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big)\\ &\le g^{\mathrm{sm}}(\varepsilon)\vee g^{\mathrm{sm}}v\Big(w-\varepsilon\Big[\frac{w}{\varepsilon}\Big]\Big)\le \sup_{(0,\varepsilon]} g(w), \end{aligned}$$

and we conclude by letting ε go to zero.

3.3 Semicontinuity and relaxation

In this section we study lower-semicontinuity properties for functionals whose natural framework is that of functions with bounded variation.

Let $I \subset \mathbb{R}$ be a bounded open set. We recall that a function $u \in L^1(I)$ is a function of bounded variation if its distributional derivative Du is a measure on I. In this case u is approximately differentiable almost everywhere and its a.e. derivative is denoted by u'; moreover the set of approximate discontinuity points (or jump set) of u, denoted by $S(u)$, is at most countable and at each point $t \in S(u)$ there exist the right-hand and left-hand approximate limits, denoted by $u(t\pm)$. With a slight abuse of notation we will also denote $[u](t) = u(t+) - u(t-)$ the *jump* of u at t. With this notation, the distributional derivative Du admits the decomposition

$$Du = u'\mathcal{L}^1 + D^s u = u'\mathcal{L}^1 + \sum_{t \in S(u)} (u(t+) - u(t-))\delta_t + D^c u,$$

where $\mathcal{L}^1$ stands for the one-dimensional Lebesgue measure, δ_t is the Dirac delta at t, and $D^c u$, the *Cantor part* of Du, is a non-atomic measure which is orthogonal to the Lebesgue measure. The notation $D^s u$ stands for the *singular part* of the measure Du (with respect to the Lebesgue measure). The space of functions of bounded variation on I is denoted by $BV(I)$.

The general (homogeneous and translation-invariant) local functional on $BV(I)$ is of the form

$$\int_I f(u')\,dt + \sum_{t \in S(u)} g([u]) + \int_I h\Big(\frac{D^c u}{|D^c u|}\Big) d|D^c u|. \tag{3.3.19}$$

Lower-semicontinuity properties for F are translated into the convexity of f, the subadditivity of g and a compatibility condition between the three energy densities (see e.g. [10] for details).

A function $u \in BV(I)$ is said to be a *special function of bounded variation* if $D^c u$ is the null measure. We will write $u \in SBV(I)$. In particular, piecewise-Sobolev functions belong to $SBV(I)$. Integral functionals on $SBV(I)$ take the form (3.3.19) without the integral depending on the Cantor part, or, equivalently, can be see as particular functionals on $BV(I)$ with $h(0) = 0$ and $h(z) = +\infty$ otherwise.

We refer to [4] for a thorough introduction to (special) functions of bounded variation; a quick overview is contained in [11].

In this section we study lower-semicontinuity properties of L^∞-functionals whose natural domain is $BV_{\rm loc}(I)$. In parallel with (3.3.19) these energies take the form

$$F(u) = \max\Big\{\sup_{t \in I} f(u'(t)),\ \sup_{t \in S(u) \cap I} g([u](t)),\ |D^c u|\text{-}\sup_{t \in I} h\Big(\frac{D^c u}{|D^c u|}(t)\Big)\Big\}, \tag{3.3.20}$$

with $f : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \setminus \{0\} \to \overline{\mathbb{R}}$ and $h : \{-1, 1\} \to \overline{\mathbb{R}}$ Borel functions. Note that we denote by $|D^c u|$- sup the essential supremum with respect to the measure

$|D^c u|$. A particular case is when we take $h(z) = +\infty$ if $z \neq 0$, in which case the domain of F is $SBV(I)$. We can also equivalently consider F as defined on $L^1_{\rm loc}(I)$, by setting

$$F(u) = \begin{cases} \max\Big\{\sup\limits_{t\in I} f(u'),\ \sup\limits_{t\in S(u)\cap I} g([u]),\ |D^c u|\text{-}\sup\limits_{t\in I} h\Big(\frac{D^c u}{|D^c u|}\Big)\Big\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in BV_{\rm loc}(I) \\ \\ K \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} \end{cases} \tag{3.3.21}$$

with $K \in \overline{\mathbb{R}}$ a constant (in particular we can take $K = +\infty$, in which case the functional defined in $L^1_{\rm loc}(I)$ is equivalent to that defined on $BV_{\rm loc}(I)$).

3.3.1 Lower semicontinuity. Statements of the results

We first analyze necessary and sufficient conditions for F as above to be lower semicontinuous with respect to the $L^1(I)$-convergence.

Theorem 3.3.1 (necessary conditions) *Let $F : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ be defined in* (3.3.21). *Then necessary conditions for F to be lower semicontinuous with respect to the $L^1_{\rm loc}(I)$-convergence are that*

(a) *f is lower semicontinuous and level convex;*

(b) *$g \vee \inf f$ is lower semicontinuous and sub-maximal;*

and that the following compatibility conditions *hold:*

(c) $h(\pm 1) \vee \inf f = \lim\limits_{z\to\pm\infty} f(z) = \lim\limits_{w\to 0\pm} g(w) \vee \inf f = \sup\limits_{\pm w>0} g(w) \vee \inf f$*;*

(d) $K = \sup f$*;*

(e) *either*

$$g(w) \vee \inf f = K \qquad \textit{for all } w < 0 \tag{3.3.22}$$

or $g(w) \vee \inf f = K$ for all $w > 0$.

Remark 3.3.2 (necessary conditions under one-sided growth conditions) *Note that, due to its level-convexity, f is either monotone or decreasing/increasing. In both cases, the two limits $f(\pm\infty) := \lim_{z\to\pm\infty} f(z)$ exist and it is not restrictive to suppose that $f(-\infty) = \sup f$. Necessary conditions can be further simplified if the two limits are not equal, for example if*

$$\lim_{z\to-\infty} f(z) > \lim_{z\to+\infty} f(z). \tag{3.3.23}$$

We will assume that this condition holds whenever the limits are different; if the converse inequality holds, clearly the correct statements must be obtained by changing signs.

Then, if $\inf f < \sup f$*, the necessary conditions (e) leads to*

$$g(w) = h(-1) = K = \lim_{z\to-\infty} f(z) = \sup f \qquad \textit{for all } w < 0, \tag{3.3.24}$$

and the compatibility conditions become

$$h(1) \vee \inf f = \lim_{z \to +\infty} f(z) = \lim_{w \to 0^+} g(w) \vee \inf f = \sup_{w>0} g(w) \vee \inf f. \qquad (3.3.25)$$

The next proposition shows that (3.3.24) is a sufficient condition for coerciveness in $BV_{\rm loc}(I)$.

Proposition 3.3.3 (coerciveness) *Let* (3.3.24) *hold. Then from every bounded sequence* (u_j) *in* $L^1(I)$ *such that* $\sup_j F(u_j) < K$ *we can extract a subsequence weakly* converging in* $BV_{\rm loc}(I)$.

Upon changing sign to variables, we have that (a), (b), (3.3.24) and (3.3.25) are necessary for combined lower-semicontinuity and coerciveness of F. The following theorem shows that they are also sufficient.

Theorem 3.3.4 (sufficient conditions) *Let conditions* (a) *and* (b) *of Theorem* 3.3.1 *and conditions* (3.3.24) *and* (3.3.25) *of Remark* 3.3.2 *be satisfied. Then* $F : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ *defined in* (3.3.21) *is lower semicontinuous with respect to the* $L^1_{\rm loc}(I)$-*convergence.*

Note that in Theorem 3.3.4 we do not require that (3.3.23) holds. Moreover, clearly, conditions (3.3.24) and (3.3.25) can be replaced by symmetric conditions changing z into $-z$, etc.

We summarize the results above in the following theorem.

Theorem 3.3.5 (necessary and sufficient conditions) *Conditions* (a)–(e) *of Theorem* 3.3.1 *are necessary and sufficient for the lower semicontinuity of* F *with respect to the* $L^1_{\rm loc}(I)$-*convergence. Moreover, such* F *is coercive with respect to the weak** BV-*convergence on the set* $\{u : F(u) < K\}$.

Before entering into the details of the proof of the results above, we note in the following remark that it is not restrictive to require that $K = +\infty$ and that one-sided growth conditions be imposed on f and g, due to the necessary conditions.

Remark 3.3.6 (modification of growth conditions by composition) We preliminarily note that by definition the value $F(u)$ does not change if we substitute g and h by $g \vee \inf f$ and $h \vee \inf f$, respectively, or, equivalently it is not restrictive to suppose that $g \geq \inf f$ and $h \geq \inf f$.

If F is lower semicontinuous then by Theorem 3.3.1 it is not restrictive to suppose that

$$\inf f \leq g \leq \sup f, \qquad \inf f \leq h \leq \sup f, \qquad K = \sup f = f(-\infty), \qquad (3.3.26)$$

and that f is level convex, in particular the limits $f(\pm\infty)$ exist. Let now $H : [\inf f, \sup f] \to \overline{\mathbb{R}}$ be a strictly-increasing function, and consider the functional

$$G(u) = \begin{cases} \max\Big\{\sup_{t\in I} H(f(u')), \sup_{t\in S(u)\cap I} H(g([u])), \ |D^c u|\text{-}\sup_{t\in I} H\Big(h\Big(\frac{D^c u}{|D^c u|}\Big)\Big)\Big\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in BV_{\rm loc}(I) \\ \\ H(K) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{3.3.27}$$

Then $G(u) = H(F(u))$ so that G is lower semicontinuous with respect to the $L^1_{\rm loc}$-convergence if and only if such is F. Note moreover that all notions involved in the statement of the results above are invariant with respect to the composition with H.

We are now free to choose H such that $H(\sup f) = +\infty$ (and, if needed, $H(\inf f) = 0$), so that by (3.3.26) it is not restrictive to suppose that a 'one-sided growth condition' on f holds:

$$\lim_{z\to-\infty} f(z) = +\infty. \tag{3.3.28}$$

Note that condition (3.3.24) translates into

$$g(w) = +\infty \quad \text{if } w < 0, \quad h(-1) = +\infty. \tag{3.3.29}$$

In conclusion, if (3.3.24) holds, it is not restrictive to assume that F is of the type

$$F(u)= \begin{cases} \sup_{t\in I} f(u'(t)) & \text{if } u \in W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} f(u'(t)), \sup_{t\in S(u)\cap I} g([u](t))\} & \text{if } u \in SBV^+_{\rm loc}(I) \setminus W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} f(u'(t)), \sup_{t\in S(u)\cap I} g([u](t)), C_\infty\} & \text{if } u \in BV^+_{\rm loc}(I) \setminus SBV^+_{\rm loc}(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3.30}$$

where $C_\infty \in \overline{\mathbb{R}}$ and

$$\begin{aligned} &BV^+_{\rm loc}(I) = \{u \in BV_{\rm loc}(I) : \ D_s u \text{ is a positive measure}\}, \\ &SBV^+_{\rm loc}(I) = SBV_{\rm loc}(I) \cap BV^+_{\rm loc}(I). \end{aligned}$$

Note that the condition for a function $u \in SBV_{\rm loc}(I)$ to belong to $SBV^+_{\rm loc}(I)$ is simply that all jumps be positive; i.e. $u(t+) > u(t-)$ on $S(u)$.

3.3.2 Lower semicontinuity. Proofs of the results

We now prove the results stated in the previous section.

Proof of Theorem 3.3.1. Without loss of generality we assume that $g \geq \inf f$ and $h \geq \inf f$ (see Remark 3.3.6).

(a) Suppose that F is l.s.c. with respect to the $L^1_{\rm loc}(I)$ convergence. Since, in particular, the restriction of F to $W^{1,\infty}(I)$ is l.s.c. with respect to the $L^\infty(I)$-convergence, thanks to [43] f is l.s.c. and level convex.

(b) We now check that g is lower semicontinuous. Let (w_n) be a sequence of real numbers converging to w. For any $z \in \mathbb{R}$ let $u_n(t)$ be defined as

$$u_n(t) = \begin{cases} zt & \text{if } t \leq t_0 \\ zt + w_n & \text{if } t > t_0. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} zt & \text{if } t \leq t_0 \\ zt + w & \text{if } t > t_0 \end{cases}$$

in $L^\infty(I)$, we get

$$f(z) \vee g(w) \leq f(z) \vee \liminf_{n \to +\infty} g(w_n). \tag{3.3.31}$$

Let (z_k) be a sequence such that $f(z_k) \to \inf f$, then passing to the limit as $k \to +\infty$ in (3.3.31) with $z = z_k$ we get the conclusion. For the sake of simplicity, from now on we suppose that there exists z_0 such that $f(z_0) = \inf f$. In order to prove that g is sub-maximal, take $w_1,\ w_2$ and let u_n be defined as

$$u_n(t) = \begin{cases} z_0 t & \text{if } t \leq t_0 \\ z_0 t + w_1 & \text{if } t_0 < t \leq t_0 + \frac{1}{n} \\ z_0 t + w_1 + w_2 & \text{if } t > t_0 + \frac{1}{n}. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} z_0 t & \text{if } t \leq t_0 \\ zt + w_1 + w_2 & \text{if } t > t_0 \end{cases}$$

in $L^1(I)$, we obtain the sub-maximality of g.

(c) We only deal the case of positive variables. We first prove that $\lim_{z \to +\infty} f(z) = \lim_{w \to 0^+} g(w) = \sup_{w>0} g$. Consider, for $z > z_0$, $u(t) = zt$ and let (w_n) be a decreasing sequence of positive real numbers converging to 0 such that $\lim_n g(w_n) = \liminf\limits_{w \to 0^+} g(w)$. Let u_n be defined as

$$u_n(t) = \begin{cases} z_0 t + j w_n & \text{if } a + j(\frac{b-a}{k_n}) < t < a + (j+1)(\frac{b-a}{k_n}) \\ & \text{with } j \in \{1, 2, \ldots, [k_n] - 1\} \\ & \\ z_0 t + [k_n] w_n & \text{if } a + [k_n](\frac{b-a}{k_n}) < t < b \end{cases}$$

where $k_n = (z - z_0)(\frac{b-a}{w_n})$. With this approximation we obtain that $f(z) \leq \liminf\limits_{w \to 0^+} g(w)$. This inequality gives

$$\lim_{z \to +\infty} f(z) \leq \liminf_{w \to 0^+} g(w). \tag{3.3.32}$$

Consider now, for all $w > 0$, u_n defined as

$$u_n(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ nwt + (z_0 - nw)t_0 & \text{if } t_0 \le t < t_0 + \frac{1}{n} \\ z_0 t + w + z_0 t_0 & \text{if } t_0 + \frac{1}{n} \le t < b. \end{cases}$$

As u_n converges to

$$u(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ z_0 t + w & \text{if } t_0 \le t < b \end{cases}$$

in $L^1(I)$ we have that

$$g(w) \le \liminf_{z \to +\infty} f(z). \tag{3.3.33}$$

Letting $w \to 0^+$, from inequalities (3.3.32) and (3.3.33) we get

$$\limsup_{w \to 0^+} g(w) \le \sup_{w > 0} g(w) \le \lim_{z \to +\infty} f(z) \le \liminf_{w \to 0^+} g(w). \tag{3.3.34}$$

Let now $\gamma : [0,1] \to [0,1]$ be the Cantor-Vitali function and let $\gamma_n : [0,1] \to [0,1]$ be its standard piecewise affine approximations with maximum slope $z_n = (\frac{3}{2})^n$. Set $u(t) = \gamma(t) + z_0 t$ we have that $u_n(t) = \gamma_n(t) + z_0 t$ converges to u and that

$$\lim_{n \to +\infty} f(z_n) = \lim_{z \to +\infty} f(z) \ge h(1). \tag{3.3.35}$$

To get the opposite inequality, for all $w > 0$ consider the sequence

$$u_n(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ w\gamma(n(t - t_0)) + z_0 t & \text{if } t_0 \le t < t_0 + \frac{1}{n} \\ z_0 t + w & \text{if } t_0 + \frac{1}{n} \le t < b. \end{cases}$$

Then u_n converges in L^1 to

$$u(t) = \begin{cases} z_0 t & \text{if } a < t < t_0 \\ w + z_0 t_0 & \text{if } t_0 \le t < b \end{cases}$$

and so

$$h(1) \ge g(w) \quad \forall w > 0.$$

Passing to the limsup as $w \to 0^+$ in the right-hand side of the previous estimate, thanks to (3.3.35) and to (3.3.34) we conclude the proof.

(d) Take $u \in L^1(I) \setminus BV_{\text{loc}}(I)$, and let $u_j \to u$ in $L^1(I)$ with $u_j \in W^{1,\infty}(I)$. We then obtain $K = F(u) \le \liminf_j F(u_j) \le \sup f$. Conversely, let $z \in \mathbb{R}$, $I = (0,1)$, and let

$$u_j(t) = \begin{cases} \frac{1}{j} u(jt) & \text{if } t \in (0, 1/j), \\ zt & \text{otherwise} \end{cases}$$

(u as above). We then have $u_j \to zt$ so that $f(z) = F(zt) \leq \liminf_j F(u_j) = K$. By the arbitrariness of z we conclude the proof.

(e) Suppose (e) is not satisfied; i.e., $w_1 > 0$ and $w_2 < 0$ exist such that $\max\{g(w_1), g(w_2)\} < K$ (and $\inf f < \sup f = K$ by (d)). Now, note that from the sub-maximality of g we have

$$g(k_1 w_1 + k_2 w_2) \leq C := \max\{g(w_1), g(w_2)\} < K \qquad \text{for all } k_1, k_2 \in \mathbb{N}. \quad (3.3.36)$$

If w_1 and w_2 are linearly independent in $\mathbb{Z}$, then $\mathbb{N}w_1 + \mathbb{N}w_2$ is dense in $\mathbb{R}$. Since by (b) g is also lower semicontinuous, from (3.3.36) we deduce that $\sup g \leq C < K$, that contradicts condition (c) (taking (d) into account).

If otherwise w_1 and w_2 are linearly dependent in $\mathbb{Z}$ then let $\pm w_0 \in \mathbb{N}w_1 + \mathbb{N}w_2$, let

$$u_h(t) = z_0 t + w_0 \sum_{j=1}^{h} (-1)^j \chi_{(0,1/j)},$$

and u the limit of u_h. Note that $u \notin BV_{\mathrm{loc}}(-1,1)$, so that (taking $I = (-1,1)$) we obtain $K = F(u) \leq \liminf_h F(u_h) = \max\{g(w_0), g(-w_0)\} \vee \inf f$, that contradicts (3.3.36). □

We now prove the remaining results under the simplifying non-restrictive hypotheses derived from Remark 3.3.6.

Proof of Proposition 3.3.3. Consider a bounded sequence (u_n) in $L^1(I)$ with $F(u_n)$ equibounded. The sequence then belongs to $BV^+_{\mathrm{loc}}(I)$ and by (3.3.28) there exists $\overline{z}$ such that $u_n' \geq \overline{z}$ a.e. for all n. Upon extracting a subsequence, for each fixed $\eta > 0$ we can find $a_\eta \in (a, a+\eta)$ and $b_\eta \in (b-\eta, b)$ such that $(u_n(a_\eta))$ and $(u_n(b_\eta))$ converge. We then have

$$\begin{aligned} |Du_n|(a_\eta, b_\eta) &= D^s u_n(a_\eta, b_\eta) + \int_{a_\eta}^{b_\eta} |u_n'|\, dt \\ &\leq u_n(b_\eta) - u_n(a_\eta) + 2(b-a)|\overline{z}|, \end{aligned}$$

so that the sequence (u_n) is bounded in $BV(a_\eta, b_\eta)$ and hence we can extract from it a further converging subsequence. From this argument, we easily obtain the thesis. □

Proof of Theorem 3.3.4. Suppose now that f is lower semicontinuous and level convex, g is lower semicontinuous and sub-maximal and (3.3.25) holds true. Let (u_h) be a sequence of functions such that $u_h \to u$ in $L^1_{\mathrm{loc}}(I)$. Up to subsequences we can suppose that also $u_h \to u$ a.e. and that there exists

$$\lim_{h \to +\infty} F(u_h) < +\infty.$$

Thanks to (3.3.28), $u_h \in BV^+_{\mathrm{loc}}(I)$ and there exists $k > 0$ such that

$$u_h'(t) \geq -k \text{ a.e. } t \in I, \ \forall h \in \mathbb{N}.$$

If we set $v_h(t) = u_h(t) + kt$, then (v_h) is a sequence of non decreasing functions in $BV^+_{\text{loc}}(I)$ converging in L^1_{loc} to $v(t) = u(t) + kt$. It implies that u belongs to $BV^+_{\text{loc}}(I)$. Note that the lower-semicontinuity inequality along a sequence (u_h) can be easily checked in the following three cases:

1. $u_h \in BV^+_{\text{loc}}(I) \setminus SBV^+_{\text{loc}}(I) \cup W^{1,1}_{\text{loc}}(I)$,
2. $\liminf\limits_{h\to+\infty} \inf\limits_{t\in S(u_h)} ([u_h](t)) = 0$,
3. $\limsup\limits_{h\to+\infty} \sup_{t\in I} f(u'_h(t)) = +\infty$.

In fact, thanks to hypothesis (3.3.25), $F(u) \le F_\infty(u)$ and $F(u) = F_\infty(u)$ $\forall u \in W^{1,1}_{\text{loc}}(I) \cup \big(BV^+_{\text{loc}}(I) \setminus SBV^+_{\text{loc}}(I)\big)$. Then, by Remark 3.1.4, it holds

$$\lim_{h\to+\infty} F(u_h) = \lim_{h\to+\infty} F_\infty(u_h) \ge F_\infty(u) \ge F(u).$$

It remains to prove the lower semicontinuity when none of the above cases is satisfied; i.e. when $(u_h) \in SBV^+_{\text{loc}}(I) \setminus W^{1,1}_{\text{loc}}(I)$ and is such that

$$\exists k > 0,\ \delta > 0 \qquad \text{such that} \quad |u'_h(t)| \le k \quad \forall h \in \mathbb{N}, \quad \text{for a.e. } t \in I \tag{3.3.37}$$

$$\inf_{t\in S(u_h)} [u_h](t) \ge \delta. \tag{3.3.38}$$

Let $\eta > 0$ be such that $a + \eta,\ b - \eta \in (a, b) \setminus \bigcup_h S(u_h)$, $a + \eta < b - \eta$ and $u_h(a+\eta) \to u(a+\eta) < +\infty$, $u_h(b-\eta) \to u(b-\eta) < +\infty$. Set $I_\eta := (a+\eta, b-\eta)$. By (3.3.38) we can suppose that $\#(S(u_h) \cap I_\eta) < +\infty$ and that it is independent of h. If $\#(S(u_h) \cap I_\eta) = 0$ then the sequence is weakly* converging in $W^{1,\infty}_{\text{loc}}(I)$ and the lower semicontinuity follows by the results in Sobolev spaces (see e.g. [36] and [43]). Hence, we can suppose that $S(u_h) = \{t^0_h, \dots, t^N_h\}$, with $t^{j-1}_h < t^j_h$, and that $t^j_h \to t^j$ as $h \to +\infty$. Let $S = \{t^0, \dots, t^N\}$, and for each $\delta > 0$ let $S_\delta = \{t \in I\ :\ \inf_j |t - t^j| \le \delta\}$. As before, $u_h \rightharpoonup u$ weakly* in $W^{1,\infty}(I_\eta \setminus S_\delta)$ so that

$$\liminf_{h\to+\infty} F(u_h) \ge \liminf_{h\to+\infty} \sup_{t\in I_\eta\setminus S_\delta} f(u'_h(t)) \ge \sup_{t\in I_\eta\setminus S_\delta} f(u'(t)).$$

and, as η and δ go to zero, we have

$$\liminf_{h\to+\infty} F(u_h) \ge \sup_{t\in I} f(u'(t)). \tag{3.3.39}$$

If $S \cap I = \emptyset$, the proof is completed . Suppose otherwise that there exists $t \in S \cap I$ and let $j = j_0, j_0 + 1, \dots, j_0 + M$ be the indices such that $t = t^j$. As $u_h \to u$ uniformly on $L^\infty(I_\eta \setminus S_\delta)$ for all $\delta > 0$ and u_h are equi-uniformly continuous on each interval (t^{j-1}_h, t^j_h), we get that $u_h(t^{j_0}_h-) \to u(t-)$, $u_h(t^{j_0+M}_h+) \to u(t+)$ and that $(u_h(t^{j-1}_h+) - u_h(t^j_h-)) \to 0$ $\forall j \in \{j_0+1, j_0+2, \dots, j_0+M\}$. Thus we have

$$\sum_{j=j_0}^{j_0+M} (u_h(t^j_h+) - u_h(t^j_h-)) \to (u(t+) - u(t-)).$$

By the sub-maximality and lower-semicontinuity of g we obtain

$$\begin{aligned} g(u(t+)-u(t-)) &\leq \liminf_{h\to+\infty} g\Big(\sum_{j=j_0}^{j_0+M}(u_h(t_h^j+)-u_h(t_h^j-))\Big) \\ &\leq \liminf_{h} \sup_{j\in\{j_0,j_0+1,\ldots,j_0+M\}} g(u_h(t_h^j+)-u_h(t_h^j-)) \\ &\leq \liminf_{h\to+\infty} F(u_h). \end{aligned}$$

Thanks to the arbitrariness of $t \in S\cap I_\eta$, when $\eta \to 0$, as $S(u) \subset S \cap I$ we have

$$\sup_{t\in S(u)} g(u(t+)-u(t-)) \leq \liminf_{h\to+\infty} F(u_h). \tag{3.3.40}$$

The thesis follows from the inequalities in (3.3.39) and in (3.3.40). □

Remark 3.3.7 Following the outline of the previous proof, one can show that, were F of the type (3.3.21), then the lower semicontinuity of F implies that the following three conditions are equivalent

(a) $\lim_{z\to-\infty} f(z) = +\infty$, (b) $\lim_{w\to 0^-} g(w) = +\infty$, (c) $h(-1) = +\infty$.

Proof of Theorem 3.3.5. The proof immediately follows from Theorems 3.3.1 and 3.3.4 since it is not restrictive to suppose that the hypotheses of Theorem 3.3.4 hold. □

3.3.3 Relaxation

We now prove a relaxation theorem under the simplifying hypotheses derived from Remark 3.3.6. For a general introduction to relaxation we refer to [24]. In what follows, for short, we set $f(+\infty) := \lim_{z\to+\infty} f(z)$ and $\underline{g}(0^+) := \liminf_{w\to0^+} g(w)$.

Theorem 3.3.8 (Relaxation) *Let f be such that condition* (3.3.28) *holds and let $F : L^1_{\rm loc}(I) \to [0,+\infty]$ be the functional defined in* (3.3.30)*. Then the lower-semicontinuous envelope of F with respect to the $L^1_{\rm loc}(I)$-topology is given by the functional $G : L^1_{\rm loc}(I) \to \overline{\mathbb{R}}$ defined as*

$$G(u) = \begin{cases} \sup_{t\in I} \hat f(u'(t)) & \text{if } u \in W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} \hat f(u'(t)), \sup_{t\in S(u)\cap I} \hat g([u](t))\} & \text{if } u \in SBV^+_{\rm loc}(I)\setminus W^{1,1}_{\rm loc}(I) \\ \max\{\sup_{t\in I} f(u'(t)), \hat C_\infty\} & \text{if } u \in BV^+_{\rm loc}(I)\setminus SBV^+_{\rm loc}(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3.41}$$

where

$$\hat{f}(z) = f^{\mathrm{lc}}(z) \wedge \tilde{f}(z) \wedge C_\infty \ \textit{with}\ \tilde{f}(z) = \inf\{f^{\mathrm{lc}}(y),\ y \le z\} \vee g^{\mathrm{sm}}(0^+),$$
$$\hat{g}(w) = g^{\mathrm{sm}}(w) \wedge f^{\mathrm{lc}}(+\infty) \wedge g^{\mathrm{sm}}(0^+) \wedge C_\infty,$$
$$\hat{C}_\infty = C_\infty \wedge f^{\mathrm{lc}}(+\infty) \wedge g^{\mathrm{sm}}(0^+).$$

PROOF. By Theorems 3.1.3 and 3.2.5 the functional H defined by substituting f with f^{lc} and g with g^{sm} in formula (3.3.30) is such that $\overline{H} = \overline{F}$. Thus we may suppose f and g to be l.s.c., and level-convex and sub-maximal, respectively, (see Sections 2 and 3). We want to prove that $\overline{F}(u) = G(u)$. Since $G(u)$ is such that $G(u) \le F(u)$ by definition, and it satisfies all the hypotheses of the lower semicontinuity Theorem 3.3.5, we have that $\overline{F}(u) \ge G(u)$. Now we prove that $\overline{F}(u) \le G(u)$ in the case f is decreasing-increasing, the proof in the other case being analogous. The following two cases are possible:

1. $G(u) < f(+\infty) \wedge \underline{g}(0^+) \wedge C_\infty$,
2. $G(u) = f(+\infty) \wedge \underline{g}(0^+) \wedge C_\infty$.

In the first case there is nothing to prove since $G(u) = F(u)$. The second possibility can be further subdivided into three sub-cases:

2a. $G(u) = f(+\infty)$, 2b. $G(u) = \underline{g}(0^+)$, 2c. $G(u) = C_\infty$.

Case 2a. In this case, set

$$\tilde{z} = \min\{z :\ f(z) \le f(+\infty)\},$$

we have that $u'(t) \ge \tilde{z}$ for a.e. $t \in I$. Consider the function $v(t) = u(t) - \tilde{z}t$, and let (v_n) be a sequence of increasing piecewise-constant functions converging to v in $L^1_{\mathrm{loc}}(I)$. Fixed n, suppose for simplicity, the construction being analogous in the general case, that $S(v_n) = \{t_0\}$. Let $v_{n,j}$ be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } t \in (a, t_0) \cup (t_0 + \frac{1}{j}, b) \\ v_n(t_0-) + j[v_n](t_0)(t - t_0) & \text{if } t \in [t_0, t_0 + \frac{1}{j}] \end{cases}$$

Using a diagonalization argument there exists $n(j)$ such that $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ satisfies $u'_{n(j),j} \ge \tilde{z}$ and $D^s(u_{n(j),j}) = 0$. Moreover, it converges to u in $L^\infty(I)$ and

$$\lim_{j\to+\infty} F(u_{n(j),j}) = \lim_{j\to+\infty} \sup_{t\in I} f(u'_{n(j),j}(t)) = f(+\infty) = G(u).$$

Case 2b. In this case, denoted $\tilde{z} := \min\{z :\ f(z) \le \underline{g}(0+)\}$ we have $u'(t) \ge \tilde{z}$ for a.e. $t \in I$. Consider the function $v(t) = u(t) - \tilde{z}t$, and let v_n be a sequence of increasing piecewise-constant functions converging to v in $L^1_{\mathrm{loc}}(I)$. With fixed n, suppose for simplicity that $S(v_n) = \{t_0\}$, the construction being analogous in the general case. Let (w_j) be a sequence such that $\lim_j g(w_j) = \liminf_{w\to 0^+} g(w) = \underline{g}(0^+)$ and let $v_{n,j}$ be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } a < t \le t_0 \\ v_n(t_0-) + hw_j & \text{if } t_0 + (h-1)M_j < t \le t_0 + hM_j \\ & \quad \text{and } h \in \left\{1, 2, \dots \left[\frac{[v_n](t_0)}{w_j}\right]\right\} \\ v_n(t_0-) + \left[\frac{[v_n](t_0)}{w_j}\right] w_j & \text{if } \left[\frac{[v_n](t_0)}{w_j}\right] M_j < t < b \end{cases}$$

where $M_j := \left[\frac{w_j}{jw}\right]$. Using a diagonalization argument we obtain that the sequence $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ converges to u with $u'_{n(j),j} \ge \tilde{z}$, $D^c(u_{n(j),j}) = 0$ and is such that,

$$\begin{aligned} \lim_{j\to+\infty} F(u_{n(j),j}) &= \max\{\sup_{t\in I} f(u'_{n(j),j}(t)), \sup_{t\in S(u_{n(j),j})\cap I} g([u_{n(j),j}](t))\} \\ &= \underline{g}(0^+) = G(u). \end{aligned}$$

Case 2c. Set $\tilde{z} := \min\{z : \ f(z) \le C_\infty\}$. We have that $u'(t) \ge \tilde{z}$ for a.e. $t \in I$. We can construct v, v_n and $v_{n,j}$ as before. In particular, supposing for simplicity that, with n fixed, $S(v_n) = \{t_0\}$ and denoting by $\gamma : [0,1] \to [0,1]$ the Cantor-Vitali function, $v_{n,j}$ will be defined as follows

$$v_{n,j}(t) = \begin{cases} v_n(t) & \text{if } t \in (a, t_0) \cup (t_0 + \frac{1}{j}, b) \\ v_n(t_0-) + [v_n](t_0)\gamma(j(t-t_0)) & \text{if } t_0 \le t \le t_0 + \frac{1}{j}. \end{cases}$$

Thanks to a diagonalization argument, the sequence $u_{n(j),j}(t) = v_{n(j),j}(t) + \tilde{z}t$ converges to u with $u'_{n(j),j} \ge \tilde{z}$, $D^j(u_{n(j),j}) = 0$ and is such that

$$\lim_{j\to+\infty} F(u_{n(j),j}) = \max\{\sup_{t\in I} f(u'_{n(j),j}(t)), C_\infty\} = C_\infty = G(u),$$

as desired. □

Example 3.3.9 Let f and g satisfy the hypotheses of Theorem 3.3.8 and let $C_\infty = +\infty$. Suppose in addition that f be continuous and level convex, decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$, and g be continuous and decreasing on $(0, +\infty)$. Then we simply have

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \le 0 \\ f(z) \wedge (g(0^+) \vee \inf f) & \text{if } z > 0 \end{cases}, \ \hat{g}(w) = g(w) \wedge f(+\infty), \ \hat{C}_\infty = f(+\infty) \wedge g(0^+).$$

3.4 Minimum Problems

Let F be an L^∞-functional defined as in (3.3.30) with f satisfying (3.3.28). In what follows, for simplicity, we suppose that $g \geq \inf f$, $C_\infty \geq \inf f$ and $I = (0,1)$. We will consider minimum problems with Dirichlet boundary conditions

$$m(d) = \inf\Big\{F(u) : u(0) = 0,\ u(1) = d\}.$$

As usual in problems on BV spaces, it is convenient to 'relax' the boundary conditions. In place of the definition above, for all $d \geq 0$ set

$$m(d) := \inf\{F(u) \vee \tilde{g}(u(0+)) \vee \tilde{g}(d - u(1-))\}$$

where $\tilde{g} : \mathbb{R} \to \overline{\mathbb{R}}$ is an extension of g such that $\tilde{g}(0) = \inf f$ and $\tilde{g}(w) = +\infty$ for all $w < 0$. We will show *a posteriori* that the two definitions of $m(d)$ are the same.

Remark 3.4.1 If F is lower semicontinuous, then it can be easily checked that, for any $d \geq 0$, the functional F^d defined as

$$F^d(u) = F(u) \vee \tilde{g}(u(0+)) \vee \tilde{g}(d - u(1-))$$

is also lower semicontinuous (the functional F^d can be viewed e.g. as obtained by extending each function u to the affine functions $z_0 t$ for $t \leq 0$ and $d + z_0(t-1)$ for $t \geq 1$, where $f(z_0) = \min f$). Moreover F^d is coercive in $L^1(I)$. In fact, let $u_n \in BV^+_{\mathrm{loc}}(I)$ be such that $\sup_n F(u_n) < +\infty$. Then, for any n, $u_n(0+) \geq 0$, $u_n(1-) \leq d$ and there exists $k > 0$ such that $\inf_{t\in I} u_n'(t) \geq -k$. Then

$$|Du_n|(0,1) \leq Du_n(0,1) + 2k = u_n(1-) - u_n(0+) + 2k \leq d + 2k.$$

Since the previous inequality implies that u_n is bounded in $L^\infty(I)$, we get that (u_n) is relatively compact in $L^1(I)$. If F is lower semicontinuous then it follows that $m(d)$ is actually a minimum.

Proposition 3.4.2 *If F is lower semicontinuos, then for any $d > 0$, we have*

$$m(d) = f(d) \wedge \inf_{z<d}\{f(z) \vee g(d-z)\}, \tag{3.4.42}$$

and, for any $\bar{t} \in I$ the function

$$u^d(t) := \begin{cases} dt & \text{if } m(d) = f(d) \\ z_d t + (d - z_d)\chi_{[\bar{t},+\infty)}(t) & \text{if } m(d) = f(z_d) \vee g(d - z_d) \end{cases} \tag{3.4.43}$$

is a minimizer for $m(d)$.

Proof. We can restrict to $SBV^+(I)$ in the definition of $m(d)$. In fact, let $u \in BV^+(I)$ be such that $u(0) = 0$ and $u(1) = d$ and set

$$v(t) := u(t) - D^c u(0,t) + D^c u(I)\chi_{[\bar{t},+\infty)}(t).$$

Then $v \in SBV^+(I)$, $v(0) = 0$, $v(1) = d$ and, by (3.3.25), $F(v) \leq F(u)$.

Moreover, we may assume that $\#S(u) \leq 1$. In fact, if $u \in SBV^+(I)$ and $S(u) = \{t_i : i \in \mathbb{N}\}$, $z_i := [u](t_i)$, the function

$$v(t) := u(t) - \sum_{i=1}^{+\infty} z_i \chi_{[t_i,+\infty)}(t) + \sum_{i=1}^{+\infty} z_i \chi_{[\bar{t},+\infty)}(t)$$

satisfies the same boundary condition as u, $S(v) = \{\bar{t}\}$, and, by the sub-maximality of g, $F(v) \leq F(u)$.

Eventually, we may suppose $u'(t)$ constant. In fact, if $u \in SBV^+(I)$ and $z := \int_0^1 u'(t)\, dt$, the function

$$v(t) = zt + (d - z)\chi_{[\bar{t},+\infty)}(t)$$

satisfies the same boundary condition of u and $F(v) \leq F(u)$, since, by the level-convexity of f, we get

$$f(z) = f\Big(\int_0^1 u'(t)\, dt\Big) \leq \sup_{t \in I} f(u'(t)),$$

and it is easy to get the conclusion. $\square$

In the following examples we give a detailed description of $m(d)$, under particular choices of f and g.

Example 3.4.3 Let f be a Lennard-Jones type potential; that is, a continuous decreasing-increasing function such that $f(z) = +\infty$ if $z \leq 0$, $\lim_{z\to 0+} f(z) = +\infty$ and $\lim_{z\to+\infty} f(z) = 0$. In particular f is a level-convex function. Set

$$f(\hat{z}) = 0, \qquad f(z_m) = m := \min_{\mathbb{R}} f.$$

Let $g : (0, +\infty) \to \mathbb{R}$ be any decreasing and continuous function such that $g(0+) = 0$ and

$$K := \lim_{z\to+\infty} g(z) > m.$$

In particular g is sub-maximal. By Theorem 3.3.5, the functional F, defined as in (3.3.30), with this choice of f and g and with $C_\infty = 0$ is lower semicontinuous. We study $m(d)$ in this case. Set

$$H(z, d) = f(z) \vee g(d - z), \quad h(d) = \min_{0<z<d} H(z, d),$$

and let $z_1 < z_2$ be such that

$$f^{-1}(K) = \{z_1, z_2\}.$$

Note that, for $z \in [z_1, z_2]$, $f(z) \leq K$ and so $H(z, d) = g(d - z)$.

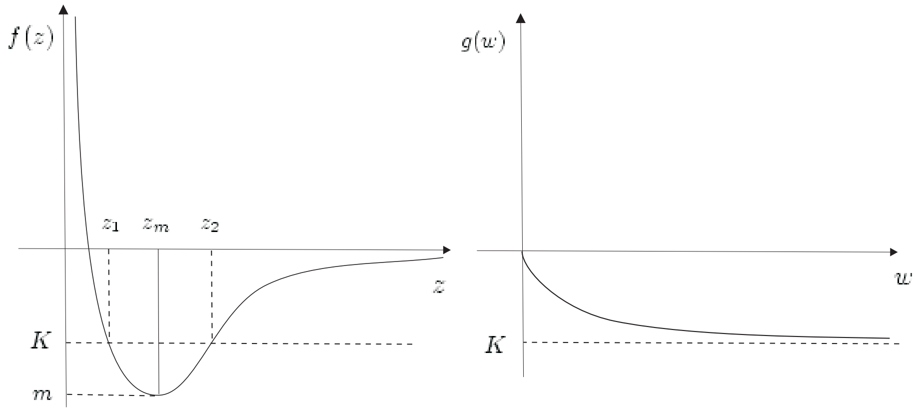

Figure 3.5: f and g in Example 3.4.3

We have that $m(d) = f(d)$ for $d \leq z_2$. In fact, if $d \leq z_m$ then f is decreasing and so $f(d) \leq h(d)$. If $z_m < d \leq z_2$ then $f(d) \leq \min g \leq h(d)$ and so $m(d) = f(d)$.

The function $z_0 : (\hat{z}, +\infty) \to (\hat{z}, z_1]$ defined as

$$z_0(d) = \min\{z > \hat{z} : f(z) = g(d - z)\}$$

is continuous increasing and such that

$$h(d) = f(z_0(d)) = g(d - z_0(d)), \qquad \text{and} \qquad \lim_{d \to +\infty} z_0(d) = z_1.$$

In particular, h is decreasing in $(\hat{z}, +\infty)$ and $\lim_{d \to +\infty} h(d) = K$. Then there exists a unique $d_0 > z_2$ such that $h(d_0) = f(d_0)$ and

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, d_0] \\ f(z_0(d)) & \text{if } d \in (d_0, +\infty). \end{cases}$$

Moreover, a minimizer for $m(d)$ is given by the following function

$$u^d(t) = \begin{cases} dt & \text{if } d \in (0, d_0] \\ z_0(d)t + (d - z_0(d))\chi_{[\bar{t}, +\infty)}(t) & \text{if } d \in (d_0, +\infty). \end{cases}$$

Example 3.4.4 Let f be as in Example 3.4.3 and let g be as in Example 3.2.8 with $K > m$. We study $m(d)$ with this choice of f and g and with $C_\infty = 0$. By Theorem 3.3.8 and Proposition 3.4.2, we get that

$$m(d) = f(d) \wedge \min_{0<z<d} \{f(z) \vee g^{\mathrm{sm}}(d - z)\}.$$

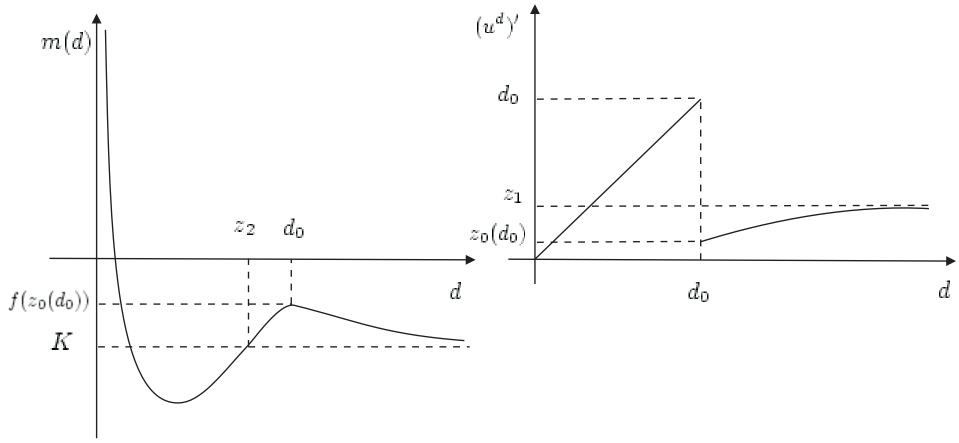

Figure 3.6: $m(d)$ and $(u^d)'$ in Example 3.4.3

Set

$$H(z,d) = f(z) \vee g^{\mathrm{sm}}(d-z), \quad h(d) = \min_{0<z<d} H(z,d),$$

and let $z_1 < z_2$ be such that

$$f^{-1}(K) = \{z_1, z_2\}.$$

As in Example 3.4.3, we easily infer that $m(d) = f(d)$ for $d \le z_2$. For $d > z_2$ we distinguish the two cases $w_0 \le z_2 - z_1$ and $w_0 > z_2 - z_1$.

Case 1: $w_0 \le z_2 - z_1$. In this case, if $d \in [z_1 + kw_0, z_2 + kw_0]$, $k \in \mathbb{N}$, then $m(d) = H(d - kw_0, d) = g^{\mathrm{sm}}(kw_0) = K$. Thus, we have

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, z_2] \\ K & \text{if } d \in (z_2, +\infty) \end{cases}$$

and a minimizer for $m(d)$ is given by the following function

$$u^d(t) = \begin{cases} dt & \text{if } d \in (0, z_2] \\ (d - kw_0)t + w_0 \sum_{i=1}^{k} \chi_{[t_i,+\infty)}(t) & \\ \qquad \text{if } d \in (z_2 + (k-1)w_0, z_2 + kw_0],\ k \in \mathbb{N}, & \end{cases}$$

where $(t_i)_{i\in\mathbb{N}}$ is any increasing sequence of points in $(0,1)$.

Case 2: $w_0 > z_2 - z_1$. Analogously to Example 3.4.3, the function $z_0 : (\hat{z}, w_0 + z_1] \to (\hat{z}, z_1]$ defined as

$$z_0(d) = \min\{z > \hat{z} : f(z) = g(d-z)\}$$

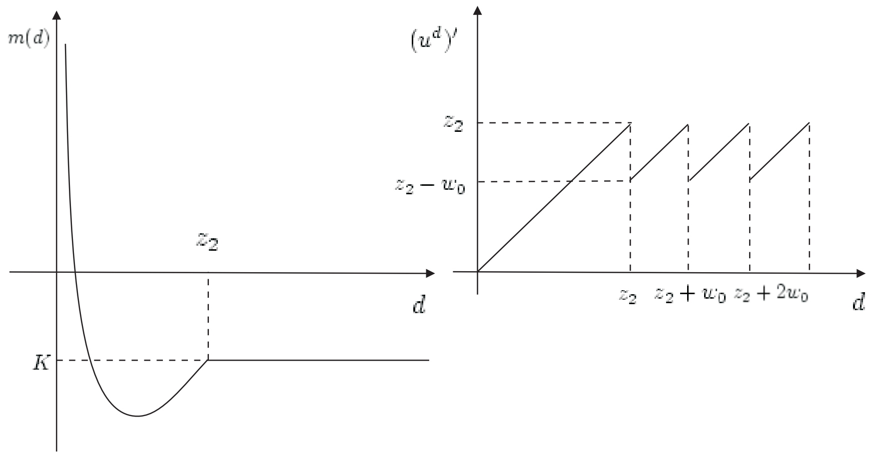

Figure 3.7: $m(d)$ and $(u^d)'$ in Example 3.4.4 - Case 1

is continuous, increasing, and such that

$$h(d) = f(z_0(d)) = g(d - z_0(d)) \quad \text{and} \quad z_0(w_0 + z_1) = z_1.$$

In particular, h is continuous and decreasing in $(\hat{z}, w_0 + z_1]$ and $h(w_0 + z_1) = K$. Then there exists a unique $d_0 \in (z_2, w_0 + z_1]$ such that $h(d_0) = f(d_0)$ and

$$m(d) = \begin{cases} f(d) & \text{if } d \in (0, d_0] \\ f(z_0(d)) & \text{if } d \in (d_0, w_0 + z_1]. \end{cases}$$

If $k \in \mathbb{N}$ and $d \in [kw_0 + z_1, kw_0 + z_2]$, then $m(d) = h(d) = H(d - kw_0, d) = g^{\text{sm}}(kw_0) = K$.

If $d \in [kw_0 + z_2, (k+1)w_0 + z_1]$, then it can be easily shown that there exist two increasing and continuous functions $z_k^1 : [kw_0 + z_2, (k+1)w_0 + z_1] \to [\hat{z}, z_1]$, $z_k^2 : [kw_0 + z_2, (k+1)w_0 + z_1] \to [z_2, w_0 + z_1]$ such that

$$z_k^1((k+1)w_0 + z_1) = z_1, \quad z_k^2(kw_0 + z_2) = z_2,$$

$$f(z_k^1(d)) = g^{\text{sm}}(d - z_k^1(d)) = g\big(\frac{d - z_k^1(d)}{k+1}\big),$$

$$f(z_k^2(d)) = g^{\text{sm}}(d - z_k^2(d)) = g\big(\frac{d - z_k^2(d)}{k}\big),$$

and

$$m(d) = f(z_k^1(d)) \wedge f(z_k^2(d)).$$

Set

$$h_k^1(d) := f(z_k^1(d)), \quad h_k^2(d) := f(z_k^2(d)).$$

Since h_k^1 is continuous and decreasing, h_k^2 is continuous and increasing, and

$$\begin{aligned} &h_k^1((k+1)w_0+z_1)=K<h_k^2((k+1)w_0+z_1),\\ &h_k^2(kw_0+z_2)=K<h_k^1(kw_0+z_2), \end{aligned}$$

there exist a unique $d_k \in (kw_0+z_2,(k+1)w_0+z_1)$ such that $h_k^1(d_k)=h_k^2(d_k)$ and

$$m(d)=\begin{cases} h_k^2(d) & \text{if } d\in[kw_0+z_2,d_k]\\ h_k^1(d) & \text{if } d\in(d_k,(k+1)w_0+z_1]. \end{cases}$$

Note that $m(d_k)\searrow K$ as $k\to+\infty$.

Eventually, we can write

$$m(d)=\begin{cases} f(d) & \text{if } d\in(0,d_0]\\ f(z_0(d)) & \text{if } d\in(d_0,w_0+z_1]\\ K & \text{if } d\in(kw_0+z_1,kw_0+z_2],\ k\in\mathbb{N}\\ f(z_k^2(d)) & \text{if } d\in(kw_0+z_2,d_k],\ k\in\mathbb{N}\\ f(z_k^1(d)) & \text{if } d\in(d_k,(k+1)w_0+z_1],\ k\in\mathbb{N}, \end{cases}$$

and a minimizer for $m(d)$ is given by the following function

$$u^d(t)=\begin{cases} dt & \text{if } d\in(0,d_0]\\ z_0(d)t+(d-z_0(d))\chi_{[\bar t,+\infty)}(t) & \text{if } d\in(d_0,w_0+z_1]\\ (d-kw_0)t+w_0\sum_{i=1}^k\chi_{[t_i,+\infty)}(t) & \text{if } d\in(kw_0+z_1,kw_0+z_2],\ k\in\mathbb{N}\\ z_k^2(d)t+\sum_{i=1}^k\big(\frac{d-z_k^2(d)}{k}\big)\chi_{[t_i,+\infty)}(t) & \text{if } d\in(kw_0+z_2,d_k],\ k\in\mathbb{N}\\ z_k^2(d)t+\sum_{i=1}^{k+1}\big(\frac{d-z_k^1(d)}{k+1}\big)\chi_{[t_i,+\infty)}(t) & \text{if } d\in(d_k,(k+1)w_0+z_1],\ k\in\mathbb{N}, \end{cases}$$

where $(t_i)_{i\in\mathbb{N}}$ is any increasing sequence of points in $(0,1)$.

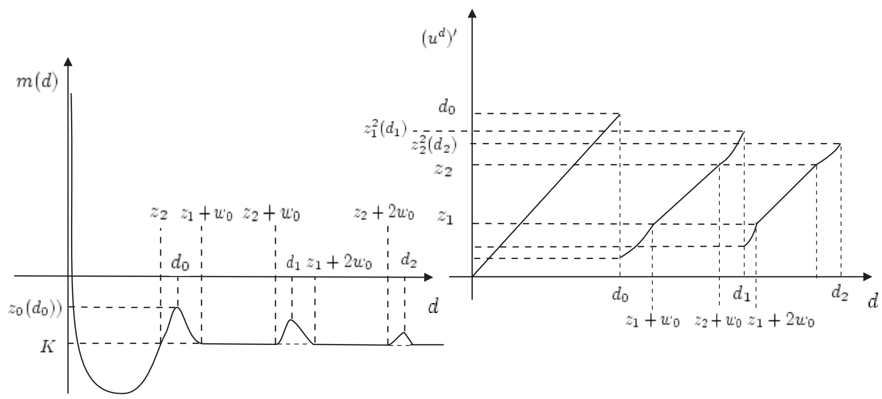

Figure 3.8: $m(d)$ and $(u^d)'$ in Example 3.4.4 - Case 2

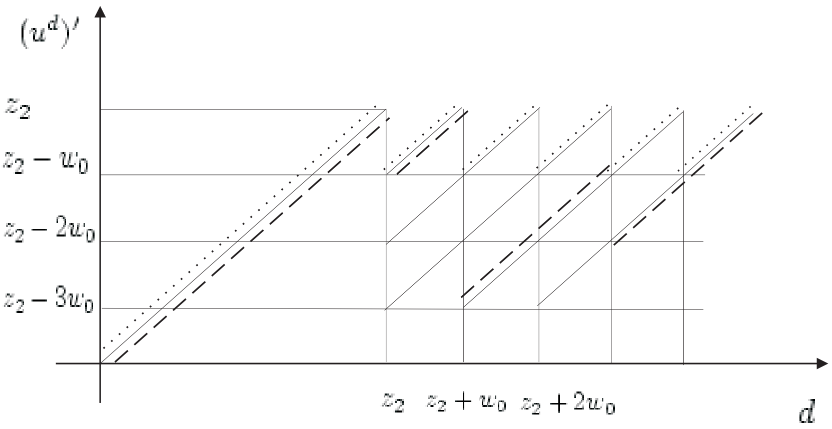

Figure 3.9: Non uniqueness: the dashed and the pointed $(u^d)'$ are equi-energetic.

Remark 3.4.5 Note that, in general, many equi-energetic states are possible. This feature is highlighted in Fig. 9 where a non unique multiple crack behavior is shown in the particular case $[\frac{z_2 - z_1}{w_0}] = 3$.

3.5 Approximation of L^∞ energies

In this section we will provide a discrete approximation of the energies treated in this chapter. For simplicity let us consider energies of the form

$$F(u) = \begin{cases} \sup\limits_{t \in I} f(u'(t)) & \text{if } u \in W^{1,1}_{\text{loc}}(0,1) \\ \max\{ \sup\limits_{t\in(0,1)} f(u'(t)), \sup\limits_{t \in S(u)\cap(0,1)} g([u](t))\} & \text{if } u \in SBV^+_{\text{loc}}(0,1) \setminus W^{1,1}_{\text{loc}}(0,1) \\ +\infty & \text{otherwise on } L^1_{\text{loc}}(0,1), \end{cases}$$

under the particular choice of f and g we have done in Example 3.4.3. We will approximate F via Γ-convergence by energies F_n of the type

$$F_n(u) = \sup_{i \in \{0,1,\ldots,n-1\}} \psi_n\Big(\frac{u^{i+1} - u^i}{\lambda_n}\Big).$$

Let us define the space of the SBV interpolations $\tilde{u}$ for $u : \{0,1,\ldots,n\} \to \mathbb{R}$ and denote it by $\mathcal{D}_n(0,1)$. We will say that $\tilde{u}$ is an SBV interpolation for u and we will write $\tilde{u} \in \mathcal{D}_n(0,1)$ if for all $t \in [i, i+1]\lambda_n$, $i \in \{0, 1, \dots, n-1\}$:

$$\begin{cases} \tilde{u}'(t) = \dfrac{u^{i+1} - u^i}{\lambda_n} & \text{if } \dfrac{u^{i+1} - u^i}{\lambda_n} \le M_n, \\ \tilde{u}'(t) = z_m, \; [\tilde{u}]((i+1)\lambda_n) = u^{i+1} - u^i - z_m\lambda_n & \text{if } \dfrac{u^{i+1} - u^i}{\lambda_n} > M_n. \end{cases}$$

where the sequence of numbers (M_n) is such that $M_n \to +\infty$, $\lambda_n M_n \to 0$. In the following, we will identify u with $\tilde{u}$. We can define the sequence of functionals F_n in $L^1(0,1)$ and rewrite them as follows:

$$F_n(u) = \begin{cases} \sup\{\sup\limits_{t\in(0,1)} f_n(u'(t)), \sup\limits_{t\in S(u)\cap(0,1)} g_n([u](t))\} & u \in \mathcal{D}_n(0,1), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.5.1}$$

where we have set:

$$\psi_n(u'(t)) =: f_n(u'(t)), \qquad \psi_n(\frac{[u](t)}{\lambda_n} + z_m) =: g_n([u](t)).$$

Theorem 3.5.1 *Let F_n be the sequence of functionals defined in (3.5.1) and let f_n and g_n be such that:*

$$f_n(z) = \begin{cases} f(z) & \text{if } z \le M_n, \\ +\infty & \text{if } z > M_n \end{cases}$$

and

$$g_n(w) = \begin{cases} g(w) & \text{if } w \ge \lambda_n(M_n - z_m), \\ +\infty & \text{if } w < \lambda_n(M_n - z_m). \end{cases}$$

Then the Γ-limit of the family of functionals (F_n) with respect to the $L^1_{\rm loc}(0,1)$ topology on $SBV_{\rm loc}(0,1)$ is the functional $F : SBV_{\rm loc}(0,1) \to \mathbb{R}$ defined as

$$F(u) = \max\{\sup_{t\in(0,1)} f(u'(t)), \sup_{t\in S(u)\cap(0,1)} g([u](t))\}$$

Proof. We first prove the Γ-liminf inequality. By the definition of f_n and g_n we have that

$$F_n(u) \ge F(u)$$

and then

$$\Gamma\text{-}\liminf_n F_n(u) \ge F(u).$$

To prove the Γ-limsup inequality we argue by density. We first prove it for $u \in SBV_{\rm loc}(0,1)$ such that

$$\|u'\|_\infty \le C < +\infty$$

$$[u](t) \geq c > 0 \quad \forall\, t \in S(u) \cap (0,1). \tag{3.5.2}$$

In this case we have, from the definition of $\Gamma\text{-}\limsup$, taking the pointwise limit of $F_n(u)$, that

$$\Gamma\text{-}\limsup_n F_n(u) \leq F(u).$$

To conclude the proof we observe that, for all $u \in SBV_{\text{loc}}(0,1)$ it is possible to construct a sequence (u_n) of functions belonging to $SBV_{\text{loc}}(0,1)$, satisfying

$$\|u_n'\|_\infty \leq n \qquad [u_n](t) \geq \frac{1}{n}\ \forall t \in S(u_n) \cap (0,1)$$

and such that $u_n \to u$ in the strong $L^1(0,1)$ topology and $\lim_n F(u_n) = F(u)$. To this end, let $a \in (0,1)$ be such that $u(a) < +\infty$. Let (v_n) be the sequence satisfying the following requirements:

$$v_n(a) = u(a) \qquad v_n' = u' \vee n,$$

then

$$u_n(t) = v_n(t) - \sum_{s \in S(u):[u](s) < \frac{1}{n}} [u](s)\chi_{(s,+\infty)}(t).$$

Thus, by the lower semicontinuity of $F''(u) := \Gamma\text{-}\limsup_n F_n(u)$, we have that

$$F''(u) \leq \liminf_n F''(u_n) \leq \lim_n F(u_n) = F(u).$$

This inequality, together with (3.5.2), proves the thesis. □

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