A VARIATIONAL MODEL FOR THE QUASI-STATIC GROWTH OF FRACTIONAL DIMENSIONAL BRITTLE FRACTURES

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Abstract. We propose a variational model for the irreversible quasi-static evolution of brittle fractures having fractional Hausdorff dimension in the setting of two-dimensional antiplane and plane elasticity. The evolution along such irregular crack paths can be obtained as Γ-limit of evolutions along one-dimensional cracks when the fracture toughness tends to zero.

Keywords: variational models, energy minimization, von Koch curve, crack propagation, quasi-static evolution, brittle fractures.

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Contents

1. Introduction

We consider a model for the evolution of cracks in brittle materials which contain extremely fragile parts that allow the fracture to develop along highly irregular paths. We settle the problem in the framework of quasi-static evolutions and of Griffith's theory. The first refers to the fact that loads are assumed to vary so slowly in time to let the body be in equilibrium at any instant, so that we can neglect any inertial and viscous effect. According to Griffith's theory the crack advance is the result of the competition between the elastic energy released in the process of crack opening and the energy spent to create new crack. In

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an isotropic homogeneous material this energy is usually seen as proportional to the newly created crack surface, and the proportionality constant is the material fracture toughness.

Variational models for the crack growth based on this idea were first introduced by Francfort and Marigo in [24]. The evolution is typically obtained through an approximation procedure based on time discretization in which the approximating solutions solve suitable incremental minimum problems. The first complete mathematical analysis of a continuous-time formulation of such a model in the case of two-dimensional antiplane linear elasticity was given in [19] under the assumption that the cracks are compact connected sets of finite length. This result has then been extended to plane elasticity [13], to a larger functional space [25], to finite elasticity [17, 18]. An important feature of these models is that the path followed by the crack during its evolution is not prescribed, it is instead the result of energy minimization.

The involved energy functional can be written as

$$\mathcal{W}(u,K)+\mathcal{S}(u,K)\,,$$

where $\mathcal{W}(u,K)$ and $\mathcal{S}(u,K)$ represent the bulk elastic energy and the dissipated surface energy, respectively, associated to a displacement u and a crack K.

For simplicity we now focus on the antiplane shear case in the framework of linear elasticity. The reference configuration is then represented by an open bounded subset Ω of $\mathbb{R}^2$, and for a brittle isotropic solid the elastic energy and the dissipated energy are of the form

$$\mathcal{W}(u,K)=\int_{\Omega\setminus K}|\nabla u|^2dx \qquad \text{and} \qquad \mathcal{S}(u,K)=\int_K \kappa(x)d\mathcal{H}^1(x)\,,$$

respectively.

So far Griffith's model has mainly been studied assuming that the material fracture toughness κ is bounded both from above and from below:

$$0<\beta_1\le\kappa(x)\le\beta_2<+\infty \tag{1.1}$$

at every point x of the body. By (1.1), $\mathcal{S}(u,K)$ amounts to consider as admissible cracks only sets of finite one-dimensional Hausdorff measure. In [19] the admissible cracks are compact sets having an *a priori* bounded number of connected components and finite length, and the displacements are Sobolev functions out of the crack, while in [25, 17] the displacements belong to suitable spaces of SBV-type and the cracks are rectifiable sets related to the jump sets of the displacements.

In order to validate Griffith's model in a wider range of possibilities, one should be able to treat cases in which (1.1) is violated. In the context of homogenization, the extremal case when the material toughness is infinite in some parts of the material was investigated, e.g., in [20, 4].

In our work, instead, we are interested in the case when the material has extremely fragile parts, so that the bound from below in (1.1) is not guaranteed anymore. The crack tends to develop in the most fragile zone, since it is energetically convenient. The low toughness coefficient allows the crack to grow quite a lot in length, without paying so much in terms of dissipated energy; the consequence is a very irregular crack, concentrated in the fragile zone. In the limit as $\kappa(x)$ vanishes in some part of the body, the crack is no longer one-dimensional as in [19]: its dimension might increase at values strictly higher than 1, and we are led to consider surface dissipation energies of the form

$$\tilde{\mathcal{S}}(u,K)=\int_K\tilde\kappa(x)\,d\mathcal{H}^d(x)\,, \tag{1.2}$$

where K is a d-dimensional curve and $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. It is worth to notice that, in agreement with Griffith's principle, the dissipated energy $\tilde{\mathcal{S}}(u,K)$ is still proportional to the number of molecular bonds which are broken to get the fracture. By a Γ-convergence result, we show how evolutions with dissipated energy of the type (1.2) can be seen as the limit regime of quasi-static evolutions in materials whose brittleness increases in some parts. This also shows that the SBV approach for the Griffith's model is not omnicomprehensive, since jump sets of $SBV/GSBV$ functions are always 1-rectifiable.

Different materials, like glass and ceramics, present highly irregular crack surfaces, as reported in several experimental papers, see, e.g., [6, 34]; the fracture shows roughness characteristics suggesting that the appropriate model for it might be given by a fractional Hausdorff dimensional set rather than by a "smooth" surface. Furthermore in the analysis of real cracks different scales seem to play a role and patterns of various dimension emerge [8]. Theoretical aspects of fracture mechanics in this framework have been developed by, e.g., [5, 7, 12, 36], among many others.

In this paper, starting from the original formulation in [19], we enlarge the class of admissible cracks to include curves of fractional Hausdorff dimension, as for instance the well-known von Koch curve. The curves we consider need not have the same Hausdorff dimension; they may also intersect each other, provided that the dimension of the intersection is strictly less than the dimension of any of the involved curves (see Subsection 2.2 for the precise definition of the class $\mathcal{C}$ of admissible cracks). The price to pay is that, at least at this stage, these curves along which the fracture develops are assigned.

In the energetic framework for rate-independent processes introduced by Mielke, see, e.g. [30], we prove the existence of a quasi-static evolution in this class of fractures with fractional dimension; more precisely, we show that there exists an irreversible crack evolution satisfying at each instant global stability and energy balance. To our knowledge, the present work is the first attempt to extend the variational approach to fracture evolution in order to encompass fractional dimensional cracks.

The paper is organized as follows: in Section 2 we describe the setting of the problem and recall some definitions and preliminary results, while in Section 3 we define the irreversible quasi-static evolution and state the main result of the paper, Theorem 3.3. Based on a careful study of the geometrical, topological and metric properties of our class $\mathcal{C}$ of admissible cracks carried out in Section 4, we are able to prove in Section 5 the existence of a quasi-static evolution (Theorem 3.3). In Section 6 we explain how our model represents a limit case when the lower bound in (1.1) is violated. In Section 7 we discuss the extension of quasi-static fracture evolutions with cracks of fractional dimension to the nonlinear and linearized cases. Finally, the Appendix contains the construction of a "good" parametrization for the von Koch curve.

2. Setting of the problem

In this section we introduce the class of admissible fractional dimensional cracks and the precise functional setting for the displacements.

2.1. **Reference configuration.** Let us fix a bounded connected open subset Ω of $\mathbb{R}^2$ with Lipschitz boundary. It will represent the reference configuration of a brittle elastic body in the antiplane shear case. We also fix a relatively open (nonempty) subset $\partial_D\Omega$ of $\partial\Omega$, on which we will impose a Dirichlet boundary condition. We set $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$; on it a homogeneous Neumann boundary condition will be assumed (in a weak sense).

2.2. **Admissible cracks.** We consider as admissible cracks compact subsets of curves of non-integer Hausdorff dimension having an *a priori* bounded number of connected components. First, let us recall that, for every $d > 0$, the d-dimensional Hausdorff measure $\mathcal{H}^d$ is defined as

$$\mathcal{H}^d(A) = m(d) \sup_{\delta>0} \inf \Big\{ \sum_{i\in I} (\operatorname{diam} A_i)^d : A_i \text{ are measurable sets, } A \subset \cup_i A_i, \operatorname{diam} A_i \le \delta \Big\},$$

where $m(d) = 2^{-d}\Gamma(\frac{1}{2})^d/\Gamma(\frac{d}{2}+1)$, with Γ denoting here the Euler function.

The curves we have in mind are of the following type: given $d \in (1,2)$, let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous curve such that for some constants $c, L > 0$ it holds

$$\frac{1}{c}|a-b|^{1/d} \le |\gamma(a) - \gamma(b)| \le c|a-b|^{1/d} \tag{2.1}$$

and

$$\mathcal{H}^d(\gamma(a,b)) = L(b-a) \tag{2.2}$$

for any $0 \le a < b \le 1$.

If $\mathcal{K} := \gamma([0,1])$, then $0 < \mathcal{H}^d(\mathcal{K}) < +\infty$.

As an explicit example of a set $\mathcal{K}$ of the above form, in the Appendix we will construct a natural parametrization for the von Koch curve, for which $d = \log 4/\log 3$.

Remark 2.1. By (2.1), the function $\gamma : [0,1] \to \mathcal{K}$ is invertible with continuous inverse. Hence, if K is a compact connected subset of $\mathcal{K}$ there exist $a, b \in [0,1]$ such that $K = \gamma([a,b])$.

We fix a finite number of sets $\mathcal{K}_1, \ldots, \mathcal{K}_M$ contained in Ω with the property that, for each $m \in \{1, \ldots, M\}$, there exists $d_m \in [1,2[$ such that $\mathcal{K}_m$ is parametrized by a continuous function $\gamma_m : [0,1] \to \mathcal{K}_m$ satisfying (2.1) with $d = d_m$ and some positive constants c_m, L_m, and

$$\mathcal{H}^{d_m}(\gamma_m([a,b])) = L_m(b-a) \qquad \forall a, b \in [0,1]. \tag{2.3}$$

Moreover we assume that

$$\dim(\mathcal{K}_{m_1} \cap \mathcal{K}_{m_2}) < \min\{\dim(\mathcal{K}_{m_1}), \dim(\mathcal{K}_{m_2})\} \qquad \forall m_1 \neq m_2. \tag{2.4}$$

The class $\mathcal{C}_p$ of *admissible cracks* is

$$\mathcal{C}_p := \Big\{K \subset \bigcup_{m=1}^{M} \mathcal{K}_m : K \text{ nonempty compact set with at most } p \text{ connected components}\Big\}.$$

Note that each connected component of an admissible crack K may contain "pieces" of different Hausdorff dimension.

On this class we will consider the convergence with respect to the Hausdorff distance. Recall that given any two compact subsets $K_1, K_2 \subset \Omega$, the *Hausdorff distance* between them is defined as

$$\mathrm{dist}_H(K_1, K_2) := \max\left\{\sup_{x \in K_1} \mathrm{dist}(x, K_2), \sup_{y \in K_2} \mathrm{dist}(y, K_1)\right\},$$

with the convention that $\mathrm{dist}(x, \emptyset) = \mathrm{diam}\,\Omega$ and $\sup \emptyset = 0$.

For simplicity of notation in the following discussions, we define the set function

$$\mathcal{L}(K) := \mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \ldots + \mathcal{H}^{d_M}(K \cap \mathcal{K}_M). \tag{2.5}$$

Notice that, by (2.4),

$$\mathcal{H}^{d_m}(K \cap \mathcal{K}_m) = \mathcal{H}^{d_m}\Big(K \setminus \bigcup_{n \neq m} \mathcal{K}_n\Big)$$

for any subset K and $m = 1, \ldots, M$.

2.3. **Admissible displacements.** In the antiplane shear case the body undergoes a deformation of the form

$$(x_1, x_2, x_3) \in \Omega \times \mathbb{R} \mapsto (x_1, x_2, x_3 + u(x_1, x_2))$$

so that we are led to consider only the out-of-plane component of the displacement, the scalar function $u : \Omega \to \mathbb{R}$. In this situation, if on $\partial_D\Omega$ we impose a bounded displacement $g \in H^1(\Omega) \cap L^\infty(\Omega)$, by a truncation argument we may deduce that minimizers of the elastic energy $\mathcal{W}(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 dx$ belong to the Sobolev space $H^1(\Omega \setminus K)$. However, in our setting the cracks $K \in \mathcal{C}_p$ are so irregular that even if they do not disconnect the domain, the H^1 regularity of the boundary datum is not necessarily inherited by the admissible displacements. Therefore we will consider $g \in H^1(\Omega)$ (not necessarily bounded) and we will use for the displacements the *Deny-Lions space* introduced in [21], defined, for any open set $A \subset \mathbb{R}^2$, by

$$L^{1,2}(A) := \{u \in L^2_{loc}(A) : \nabla u \in L^2(A;\mathbb{R}^2)\}.$$

Notice that if A is an open set with Lipschitz boundary then $L^{1,2}(A) = H^1(A)$ (see [29, Corollary 1.1.11]); moreover the set

$$\{\nabla u : u \in L^{1,2}(A)\}$$

is closed in $L^2(A;\mathbb{R}^2)$ (see [29, Section 1.1.13]).

To give a precise mathematical meaning to the fact that the boundary value of the displacement is imposed, we need to use fine properties of functions in the Deny-Lions space related to the notion of capacity, for which we refer to [22, 27, 37]. Let us only recall that if B is a bounded open set in $\mathbb{R}^2$, the *capacity* of an arbitrary subset E of B is defined as

$$\mathrm{cap}(E,B) := \inf_{u \in \mathcal{U}^B_E} \int_B |\nabla u|^2\,dx\,,$$

where $\mathcal{U}^B_E$ is the set of all functions $u \in H^1_0(B)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

In the sequel we shall use the expression *quasi-everywhere on* E, abbreviated as *q.e. on* E, to indicate that a property holds on a set E except a subset of capacity zero, while we shall use the abbreviation *a.e. on* E when referring to the Lebesgue measure.

We remind also that any function $u \in L^{1,2}(A)$ admits a quasi-continuous representative $\tilde{u}$ (cf, e.g., [22, 27, 37]) that can be extended up to the Lipschitz part $\partial_L A$ of the boundary of A; moreover, if $u_n \to u$ strongly in $H^1(A)$, then a subsequence of $(\tilde{u}_n)$ converges to $\tilde{u}$ q.e. in $A \cup \partial_L A$. We shall always identify each function $u \in L^{1,2}(A)$ with its quasi-continuous representative $\tilde{u}$.

Throughout the paper, given a function $u \in L^{1,2}(\Omega \setminus K)$ for some K of null $\mathcal{L}^2$ measure, we always extend ∇u to Ω by setting $\nabla u = 0$ a.e. on K. We stress that, however, ∇u is the distributional gradient of u only in $\Omega \setminus K$ and, in general, it does not coincide in Ω with the gradient of an extension of u.

We denote by $(\cdot|\cdot)$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega;\mathbb{R}^2)$.

3. Irreversible quasi-static evolution

For every compact set $K \in \mathcal{C}_p$ and every $g \in H^1(\Omega)$ we consider the minimum elastic energy of the unfractured part of the body, given by

$$E(g,K) := \min_{v \in \mathcal{V}(g,K)} \int_{\Omega\setminus K} |\nabla v|^2 dx\,, \tag{3.1}$$

where

$$\mathcal{V}(g,K) := \{v \in L^{1,2}(\Omega \setminus K) : v = g \quad \text{q.e. on } \partial_D\Omega\ \}. \tag{3.2}$$

According to Griffith's theory, the dissipation energy is proportional to the "length" of the crack, *i.e.* to the number of broken atomic bonds; in our setting it is given by the functional $\mathcal{L}$ defined in (2.5). Consequently, the total energy of the system is

$$\mathcal{E}(g,K) := E(g,K) + \mathcal{L}(K)\,. \tag{3.3}$$

Remark 3.1. The minimum problem (3.1) admits a solution $u \in \mathcal{V}(g,K)$. Indeed, by standard arguments on the minimization of quadratic forms it is easy to see that u is a solution of (3.1) if and only if it solves the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K \\ u = g & \text{on } \partial_D \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } K \cup \partial_N \Omega \,. \end{cases} \tag{3.4}$$

Due to the irregularity of K it is clear that the Neumann boundary condition cannot be satisfied in the classical sense. By a solution of (3.4) we mean a function u which satisfies the following conditions:

$$\begin{cases} u \in L^{1,2}(\Omega \setminus K)\,, \quad u = g \quad \text{q.e. on } \partial_D \Omega\,, \\ (\nabla u | \nabla z) = 0 \quad \forall z \in L^{1,2}(\Omega \setminus K)\,,\; z = 0 \quad \text{q.e. on } \partial_D \Omega\,. \end{cases} \tag{3.5}$$

The existence of a solution is assured by the Lax-Milgram lemma. We underline that uniqueness is guaranteed only in the connected components of $\Omega \setminus K$ whose boundary intersects $\partial_D \Omega$; in the connected components for which this is not the case, the solution can be any arbitrary constant, therefore uniqueness is lost. However, ∇u is always unique.
Moreover, the map $g \mapsto \nabla u$ is linear from $H^1(\Omega)$ into $L^2(\Omega \setminus K; \mathbb{R}^2)$ and satisfies the estimate

$$\int_{\Omega \setminus K} |\nabla u|^2 \, dx \le \int_{\Omega} |\nabla g|^2 \, dx \,.$$

Given a time-dependent boundary displacement $t \mapsto g(t)$, we consider quasi-static evolutions of global minimizers, for which an irreversibility condition and an energy balance condition hold.

Definition 3.2. Given $T > 0$ and $g \in AC([0,T]; H^1(\Omega))$, we say that a map $K : [0,T] \to \mathcal{C}_p$ is an *irreversible quasi-static evolution* on $[0,T]$ with imposed boundary condition g if it satisfies the following conditions:

(I) irreversibility: $K(s) \subseteq K(t)$ for $0 \le s \le t \le T$,
(GS) global stability: for every $t \in [0,T]$

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K)$$

for every $K \in \mathcal{C}_p\,,\; K \supseteq K(t)$,
(EB) energy balance: for every s, t with $0 \le s < t \le T$

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(s), K(s)) + 2 \int_s^t (\nabla u(\tau) | \nabla \dot{g}(\tau)) d\tau \,,$$

where $u(\tau)$ is a solution of the minimum problem (3.1) which defines $E(g(\tau), K(\tau))$.

This derivative-free form of the problem is an energetic formulation in the sense of Mielke [30], in which the irreversibility condition can be enclosed in the description of the process by means of the so-called dissipation distance.
We now state the main result of the paper.

Theorem 3.3. *Let $T > 0$ and $g \in AC([0,T]; H^1(\Omega))$. Let $p \ge 1$ and $K_0 \in \mathcal{C}_p$. Then there exists an irreversible quasi-static evolution $K \colon [0,T] \to \mathcal{C}_p$ such that $K_0 \subseteq K(0)$ and*

$$\mathcal{E}(g(0), K(0)) \le \mathcal{E}(g(0), K) \tag{3.6}$$

for every $K \in \mathcal{C}_p$ with $K \supseteq K_0$.

4. Properties of sets in $\mathcal{C}_p$ and lower semicontinuity of $\mathcal{L}$

In this section we prove some geometrical, topological and metric properties for the class $\mathcal{C}_p$, the lower semicontinuity of the functional $\mathcal{L}$, and an approximation result for sets in $\mathcal{C}_p$ that will play an important role in the proof of the global minimality conditions (GS) and (3.6). We begin by showing the (sequential) compactness of the class $\mathcal{C}_p$.

Proposition 4.1. *If (K_n) is a sequence in $\mathcal{C}_p$, then there exists a subsequence which converges to a set $K \in \mathcal{C}_p$ in the Hausdorff distance.*

Proof. By Blaschke's Selection Theorem (see, e.g., [3]), there exists a subsequence converging to a nonempty compact set K. As all K_n are contained in the union $\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_M$, also the limit K is. By a simple contradiction argument one proves that the number of connected components of K is at most p. □

We now establish some results on the lower semicontinuity of the Hausdorff measures $\mathcal{H}^d$ (and of the functional $\mathcal{L}$) with respect to the Hausdorff convergence in $\mathcal{C}_p$.

Proposition 4.2. *Let (K_n) be a sequence of closed connected nonempty subsets of $\mathcal{K}_1$ converging to K in the Hausdorff metric. Then for every open set $U \subset \Omega$ it holds*

$$\mathcal{H}^{d_1}(K \cap U) \leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \cap U). \tag{4.1}$$

Proof. The set $\mathcal{K}_1 \cap U$ is made of at most countable many connected components $\hat{\mathcal{K}}^i$, of the form

$$\hat{\mathcal{K}}^i = \gamma_1(I^i)$$

with $I^i \subset [0,1]$ an interval, and $I^i \cap I^j = \emptyset$ for $i \neq j$.

Let $K_n = \gamma_1([a_n, b_n])$ and $K = \gamma_1([a,b])$. Then $K_n \cap \hat{\mathcal{K}}^i = \gamma_1([a_n,b_n] \cap I^i)$ and $K \cap \hat{\mathcal{K}}^i = \gamma_1([a,b] \cap I^i)$. By the Hausdorff convergence and (2.1) (with $d = d_1$) we have $a_n \to a$ and $b_n \to b$.

For every $i \in \mathbb{N}$, by (2.3) it holds

$$\mathcal{H}^{d_1}(K \cap \hat{\mathcal{K}}^i) = \lim_{n\to+\infty} \mathcal{H}^{d_1}(K_n \cap \hat{\mathcal{K}}^i).$$

Therefore for every $N \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{H}^{d_1}(K \cap \bigcup_{i=1}^N \hat{\mathcal{K}}^i) &= \sum_{i=1}^N \mathcal{H}^{d_1}(K \cap \hat{\mathcal{K}}^i) = \sum_{i=1}^N \lim_{n\to+\infty} \mathcal{H}^{d_1}(K_n \cap \hat{\mathcal{K}}^i) \\
&= \lim_{n\to+\infty} \mathcal{H}^{d_1}(K_n \cap \bigcup_{i=1}^N \hat{\mathcal{K}}^i) \\
&\leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \cap U).
\end{aligned}$$

As $N \to \infty$, we obtain (4.1). □

Proposition 4.3. *Let (K_n) be a sequence in $\mathcal{C}_1$ converging to K in the Hausdorff metric. Then*

$$\mathcal{L}(K) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n).$$

Proof. For simplicity, we consider the case $M = 2$. We have to prove that

$$\mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2) \leq \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2)\right).$$

If either $K_n \subset \mathcal{K}_1$ for all n large enough, or $K_n \subset \mathcal{K}_2$, the result follows by Proposition 4.2 with $U = \Omega$.

Assume now that $K_n \setminus \mathcal{K}_1 \neq \emptyset \neq K_n \setminus \mathcal{K}_2$ for all n large. We first prove that

$$\mathcal{H}^{d_1}(K \setminus \mathcal{K}_2) \leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2). \tag{4.2}$$

For every $\varepsilon > 0$, consider the open set

$$U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{K}_2) < \varepsilon\}.$$

Let V be an open set with $V \subset\subset \mathbb{R}^2 \setminus \overline{U}_\varepsilon$, and define $\delta := \operatorname{dist}(V, \partial U_\varepsilon)$. We claim that the number of connected components C of $K_n \setminus U_\varepsilon$ that intersect V is uniformly bounded with respect to n. Indeed, if $C \cap \partial U_\varepsilon \neq \emptyset$, then by (2.1) and the fact that $C \subset \mathcal{K}_1$ it is

$$\frac{L_1}{c_1}\delta^{d_1} \leq \mathcal{H}^{d_1}(C) \leq L_1.$$

Hence the number of these connected components is at most c_1/δ^{d_1}. If $C \cap \partial U_\varepsilon = \emptyset$, then $C \subset \mathcal{K}_1 \setminus \mathcal{K}_2$ and it is a connected component of K_n, so that actually $C = K_n$.

Let $F_n^1, \ldots, F_n^{N_n}$ be the connected components of $K_n \setminus U_\varepsilon$ which intersect V. Up to subsequences, we can assume that $N_n = N \leq 1 + c_1/\delta^{d_1}$ for every n and $F_n^i \to F^i$ in the Hausdorff metric, for $i = 1, \ldots, N$. Notice that

$$K \cap V \subset F^1 \cup \ldots \cup F^N.$$

Indeed, if $x \in K \cap V$ there exists $x_n \in K_n$ converging to x. For n large enough, $x_n \in V$, so that $x_n \in F_n^{i_n}$ for some $i \in \{1, \ldots, N\}$. Therefore, there exists i such that $i_n = i$ for infinitely many n, hence $x \in F^i$.

By the fact that F_n^i and F^i verify the hypotheses in Proposition 4.2 and the curves F_n^i are pairwise disjoint, we have

$$\begin{aligned}
\mathcal{H}^{d_1}(K \cap V) &\leq \sum_{i=1}^N \mathcal{H}^{d_1}(F^i) \leq \sum_{i=1}^N \liminf_{n\to+\infty} \mathcal{H}^{d_1}(F_n^i) \\
&\leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(F_n^1 \cup \ldots \cup F_n^N) \leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus U_\varepsilon) \\
&\leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2).
\end{aligned}$$

As $V \nearrow \mathbb{R}^2 \setminus \mathcal{K}_2$, we obtain (4.2).

Of course, in an analogous way we can prove that

$$\mathcal{H}^{d_2}(K \setminus \mathcal{K}_1) \leq \liminf_{n\to+\infty} \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1).$$

Being $\mathcal{H}^{d_j}(\mathcal{K}_1 \cap \mathcal{K}_2) = 0$ for $j = 1, 2$ by (2.4), we can conclude that

$$\begin{aligned}
\mathcal{H}^{d_1}(K \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2) &= \mathcal{H}^{d_1}(K \setminus \mathcal{K}_2) + \mathcal{H}^{d_2}(K \setminus \mathcal{K}_1) \\
&\leq \liminf_{n\to+\infty} \mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2) + \liminf_{n\to+\infty} \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1) \\
&\leq \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \setminus \mathcal{K}_2) + \mathcal{H}^{d_2}(K_n \setminus \mathcal{K}_1)\right) \\
&= \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2)\right).
\end{aligned}$$

The general case can be proved similarly. □

Corollary 4.4. *Let (K_n) be a sequence in $\mathcal{C}_p$ that converges in the Hausdorff metric to a set K, and let $U \subset \Omega$ be an open set. Then*

$$\mathcal{L}(K \cap U) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n \cap U). \tag{4.3}$$

Proof. For simplicity we consider the case when $M = 2$ and the sets K_n are connected. We have to show that for every open set $U \subset \Omega$ it holds

$$\mathcal{H}^{d_1}(K \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2 \cap U) \leq \liminf_{n\to+\infty} (\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2 \cap U)). \tag{4.4}$$

Consider V_1 and V_2 open sets, such that $V_1 \subset\subset V_2 \subset\subset U$. Arguing as in the proof of Proposition 4.3, the number of connected components $F_n^1, \ldots, F_n^{N_n}$ of $K_n \cap \overline{V_2}$ which intersect V_1 is uniformly bounded. As before, we can assume that $N_n = N$ and

$$K \cap V_1 \subset F^1 \cup \ldots \cup F^N$$

where F^i is the limit of F_n^i in the Hausdorff metric, for $i = 1, \ldots, N$. Observe that the sequences (F_n^i) satisfy the hypotheses of Proposition 4.3. Then we have

$$\begin{aligned}
\mathcal{H}^{d_1}(K \cap \mathcal{K}_1 \cap V_1) + \mathcal{H}^{d_2}(K \cap \mathcal{K}_2 \cap V_1) &\leq \sum_{i=1}^{N} \mathcal{H}^{d_1}(F^i \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(F^i \cap \mathcal{K}_2) \\
&\leq \sum_{i=1}^{N} \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(F_n^i \cap \mathcal{K}_1) + \mathcal{H}^{d_2}(F_n^i \cap \mathcal{K}_2)\right) \\
&\leq \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}\Big(\bigcup_{i=1}^{N} F_n^i \cap \mathcal{K}_1\Big) + \mathcal{H}^{d_2}\Big(\bigcup_{i=1}^{N} F_n^i \cap \mathcal{K}_2\Big)\right) \\
&\leq \liminf_{n\to+\infty} \left(\mathcal{H}^{d_1}(K_n \cap \mathcal{K}_1 \cap U) + \mathcal{H}^{d_2}(K_n \cap \mathcal{K}_2 \cap U)\right) .
\end{aligned}$$

As $V_1 \nearrow U$, we obtain (4.4). ☐

Corollary 4.5. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric. Let (H_n) be a sequence of compact sets converging to H in the Hausdorff metric. Then*

$$\mathcal{L}(K \setminus H) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus H_n) .$$

Proof. For every $\varepsilon > 0$, let $U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, H) < \varepsilon\}$. Since, for n large, $H_n \subset U_\varepsilon$, it is $K_n \setminus \overline{U}_\varepsilon \subset K_n \setminus H_n$. By Corollary 4.4 with $U = \mathbb{R}^2 \setminus \overline{U}_\varepsilon$, we have

$$\mathcal{L}(K \setminus \overline{U}_\varepsilon) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus \overline{U}_\varepsilon) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n \setminus H_n).$$

The thesis follows letting $\varepsilon \to 0$. ☐

We need to establish a connection between some topological and measure properties of elements in $\mathcal{C}_p$, which will be useful in the proof of Theorem 5.1 on the continuity of minimizers of (3.1) as K varies in $\mathcal{C}_p$.

Lemma 4.6. *Let $K \in \mathcal{C}_1$ with $\mathcal{L}(K) = 0$. Then $K = \{x\}$.*

Proof. For simplicity, assume $M = 2$. If $K \subset \mathcal{K}_1$ or $K \subset \mathcal{K}_2$ the conclusion follows from Remark 2.1 and (2.3).

Assume now that

$$K \setminus \mathcal{K}_1 \neq \emptyset \neq K \setminus \mathcal{K}_2 \tag{4.5}$$

and let

$$U_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{K}_2) < \varepsilon\} .$$

Notice that there exists $\overline{\varepsilon} > 0$ such that $K \setminus U_{\overline{\varepsilon}} \neq \emptyset$. Indeed, otherwise $K \subset \bigcap_{\varepsilon>0} U_\varepsilon = \mathcal{K}_2$ which contradicts $K \setminus \mathcal{K}_2 \neq \emptyset$. As $K \cap \mathcal{K}_2 \neq \emptyset$ we have also $K \cap U_{\overline{\varepsilon}/2} \neq \emptyset$. Since K is connected we deduce that there exists a connected subset C of K which intersects both $\partial U_{\overline{\varepsilon}}$ and $\partial U_{\overline{\varepsilon}/2}$ (otherwise K would have at least two connected components). Then $C \subset \mathcal{K}_1$ and $\operatorname{diam} C > \overline{\varepsilon}/2$. By (2.3) we have $\mathcal{H}^{d_1}(C) > 0$, in contradiction with $\mathcal{L}(K) = 0$. This shows that (4.5) cannot happen, therefore either $K \setminus \mathcal{K}_1 = \emptyset$ or $K \setminus \mathcal{K}_2 = \emptyset$, which is the situation considered at the beginning of the proof. ☐

Lemma 4.7. *For every $l > 0$ there exists a constant $C_l > 0$ such that, if $K \in \mathcal{C}_1$ with $\operatorname{diam} K > l$, then $\mathcal{L}(K) > C_l$.*

Proof. By contradiction, assume that there exists $l > 0$ such that, for every $n \in \mathbb{N}$, there exists $K_n \in \mathcal{C}_1$, with $\operatorname{diam} K_n > l$ and $\mathcal{L}(K_n) \leq 1/n$.

Up to subsequences, by Proposition 4.1 we can assume that (K_n) converges to a set $K \in \mathcal{C}_1$ in the Hausdorff metric. By the lower semicontinuity of $\mathcal{L}$ (Proposition 4.3), we have

$$\mathcal{L}(K) \leq \liminf_{n\to+\infty} \mathcal{L}(K_n) = 0\,.$$

Then Lemma 4.6 implies that K is a singleton: $K = \{z\}$.

On the other hand, since $\operatorname{diam} K_n > l$ there exists $x_n, y_n \in K_n$ with

$$|x_n - y_n| > l\,. \tag{4.6}$$

By Hausdorff convergence it is $x_n, y_n \to z$, which is clearly a contradiction to (4.6). □

In [32, §2.2] the following definition is given.

Definition 4.8. A closed set $A \subset \mathbb{R}^2$ is locally connected if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any two points $x, y \in A$ with $|x - y| < \delta$ we can find a continuum (*i.e.* compact connected set) B with $x, y \in B \subset A$, $\operatorname{diam} B < \varepsilon$.

Lemma 4.9. *If $K \in \mathcal{C}_p$ then K is locally connected.*

Proof. We follow the proof of [14, Lemma 1]. It is enough to prove the result for a single connected component of K, since we can choose δ in Definition 4.8 smaller than the distance between two connected components. Assume by contradiction that K is not locally connected; hence there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exist $x_n\,, y_n \in K$ with $|x_n - y_n| < \frac{1}{n}$ with the property that any continuum $B \subset K$ connecting x_n to y_n must have $\operatorname{diam} B > \varepsilon$. Note that such an ε is necessarily less than $\operatorname{diam} K$. Up to subsequences, we may assume that $\lim_n x_n = \lim_n y_n = z \in K$, $x_n \in \mathcal{K}_{m_1}$, and $y_n \in \mathcal{K}_{m_2}$. Then $z \in \mathcal{K}_{m_1} \cap \mathcal{K}_{m_2}$.

For n large enough $x_n, y_n \in B(z, \frac{\varepsilon}{2})$. Let $\widetilde{X}_n$ be the connected component of $K \cap \overline{B(z, \frac{\varepsilon}{2})}$ that contains x_n and $\widetilde{Y}_n$ the one containing y_n. Then $\widetilde{X}_n \cap \widetilde{Y}_n = \emptyset$ (otherwise $\widetilde{X}_n \cup \widetilde{Y}_n$ would be a continuum connecting x_n and y_n of diameter less than ε), therefore either $z \notin \widetilde{X}_n$ or $z \notin \widetilde{Y}_n$. Assume $z \notin \widetilde{X}_n$ for infinetely many indices n. As K is connected and $\operatorname{diam} K > \varepsilon$, $\widetilde{X}_n \cap \partial B(z, \frac{\varepsilon}{2}) \neq \emptyset$. Since $x_n \to z$, for n large enough $\widetilde{X}_n \cap B(z, \frac{\varepsilon}{4}) \neq \emptyset$. Thus $\operatorname{diam} \widetilde{X}_n > \varepsilon/4$ and by Lemma 4.7, we have $\mathcal{L}(\widetilde{X}_n) > C_\varepsilon > 0$ for every n. Since, except for a finite number, the sets $\widetilde{X}_n$ are pairwise disjoint we deduce that $\mathcal{L}(K) = +\infty$, which is impossible since $K \in \mathcal{C}_p$. □

The following approximation results for sets in $\mathcal{C}_p$ are in the spirit of [19, Lemmas 3.5-3.8]. In case their proof is only slightly different, we remark the differences and refer to [19] for the core of it.

Lemma 4.10. *Let $p, q \geq 1$. Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, and let $K \in \mathcal{C}_q$ be such that $H \subset K$. Then there exists a sequence (K_n) in $\mathcal{C}_q$ such that it converges to K in the Hausdorff metric, $H_n \subset K_n$ and $\mathcal{L}(K_n \setminus H_n) \to \mathcal{L}(K \setminus H)$.*

Its proof is a direct consequence of Lemma 4.14 below, for which we need some preliminaries.

Lemma 4.11. *Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, with $H \in \mathcal{C}_1$. Then there exist a sequence $(\widehat{H}_n)$ in $\mathcal{C}_1$ such that $H_n \subset \widehat{H}_n$, $\widehat{H}_n \to H$ in the Hausdorff metric and $\mathcal{L}(\widehat{H}_n \setminus H_n) \to 0$.*

Proof. Without loss of generality, we may assume that all the sets H_n have exactly $q \leq p$ connected components $H^1_n, \dots H^q_n$ with H^i_n converging to $\widetilde{H}^i$ in the Hausdorff metric, for $i = 1, \dots, q$, with $\widetilde{H}^i \in \mathcal{C}_1$; of course, $H = \widetilde{H}^1 \cup \ldots \cup \widetilde{H}^q$.

Being H connected, there exists a finite set of indices $(\sigma_i)_{1\leq i\leq l}$ such that $\{\sigma_1,\ldots,\sigma_l\} = \{1,\ldots,q\}$ and $\widetilde{H}^{\sigma_i}\cap\widetilde{H}^{\sigma_{i+1}}\neq\emptyset$ for every $i=1,\ldots,l-1$. Fixed a point $x^i\in\widetilde{H}^{\sigma_i}\cap\widetilde{H}^{\sigma_{i+1}}$ for every $i=1,\ldots,l-1$, consider $x_n^i\in H_n^{\sigma_i}$ and $y_n^i\in H_n^{\sigma_{i+1}}$ with $x_n^i, y_n^i\to x^i$ as $n\to+\infty$.

Fix $i\in\{1,\ldots,l\}$. For every $m=1,\ldots,M$, let

$$I_m := \{n\in\mathbb{N}:\ x_n^i\in\mathcal{K}_m\}.$$

For m with I_m infinite, it is $x^i\in\mathcal{K}_m$. For such indices m and for every $n\in I_m$ consider the arc $X_n^i\subset\mathcal{K}_m$ connecting x_n^i and x^i. Then, by (2.2) and (2.1), we have that $\mathcal{H}^{d_m}(X_n^i)\leq c_m|x_n^i-x^i|^{d_m}$, with c_m independent of i and n. Hence $\mathcal{H}^{d_m}(X_n^i)\to 0$ as $n\to+\infty$.

Similarly, defined J_m for the points y_n^i, we choose the sets Y_n^i. Finally we set

$$\widehat{H}_n := H_n\cup\bigcup_{i=1}^{l-1}X_n^i\cup\bigcup_{i=1}^{l-1}Y_n^i\,.$$

By Lemma 4.7 we obtain that X_n^i and Y_n^i converge to $\{x^i\}$ in the Hausdorff metric, so that $\widehat{H}_n\to H$; in addition $\mathcal{L}(\widehat{H}^n\setminus H_n)\to 0$. Finally, being

$$\widehat{H}_n = H_n^{\sigma_1}\cup X_n^1\cup Y_n^1\cup H_n^{\sigma_2}\cup\ldots\cup H_n^{\sigma_{l-1}}\cup X_n^{l-1}\cup Y_n^{l-1}\cup H_n^{\sigma_l}\,,$$

the sets $\widehat{H}^n$ are connected and contained in $\bigcup_{m=1,\ldots,M}\mathcal{K}_m$, *i.e* $\widehat{H}_n\in\mathcal{C}_1$. □

Lemma 4.12. *If C is a connected subset of $\mathcal{K}_1\cup\ldots\cup\mathcal{K}_M$, then $\mathcal{L}(\overline{C})=\mathcal{L}(C)$.*

Proof. For simplicity, we assume $M=2$. If $C\subset\mathcal{K}_1$, then by Remark 2.1 $C=\gamma_1(I)$, where $I\subset[0,1]$ is an interval of the form (a,b), $[a,b)$, $(a,b]$ or $[a,b]$. By (2.2), the thesis follows. The case $C\subset\mathcal{K}_2$ is analogous.

For every $\varepsilon>0$ let $U_\varepsilon := \{x\in\Omega:\ \mathrm{dist}(x,\mathcal{K}_2)<\varepsilon\}$. Arguing as in the proof of Proposition 4.3, the number of connected components F of $C\setminus\mathcal{K}_2$ such that $\overline{F}\cap\mathcal{K}_2\neq\emptyset\neq F\setminus U_\varepsilon$ is finite, say N_ε. Note that $\overline{C}\setminus U_\varepsilon\subset\bigcup_{i=1}^{N_\varepsilon}\overline{F}_i$. In addition, by construction $F_i\subset\mathcal{K}_1$, $F_i\cap F_j=\emptyset$ for $i\neq j$, and $\mathcal{H}^{d_1}(\overline{F}_i)=\mathcal{H}^{d_1}(F_i)$ by the previous part. Then we have

$$\mathcal{H}^{d_1}(\overline{C}\setminus U_\varepsilon)\leq\sum_{i=1}^{N_\varepsilon}\mathcal{H}^{d_1}(\overline{F}_i)=\sum_{i=1}^{N_\varepsilon}\mathcal{H}^{d_1}(F_i)\leq\mathcal{H}^{d_1}(C\setminus\mathcal{K}_2)\,.$$

As $\varepsilon\to0$, we obtain $\mathcal{H}^{d_1}(\overline{C}\setminus\mathcal{K}_2)\leq\mathcal{H}^{d_1}(C\setminus\mathcal{K}_2)$; hence the equality holds.

Similarly, we have $\mathcal{H}^{d_2}(\overline{C}\setminus\mathcal{K}_1)\leq\mathcal{H}^{d_2}(C\setminus\mathcal{K}_1)$. Recalling the definition (2.5) of $\mathcal{L}$, and (2.4), the thesis follows. □

Lemma 4.13. *Let $K\in\mathcal{C}_1$ and $H\subset K$ be a compact set with $p\geq2$ connected components $H^1,\ldots,H^p$. Then there exists a family of indices $(\sigma_j)_{0\leq j\leq l}$, with $\{\sigma_0,\ldots,\sigma_l\}=\{1,\ldots,p\}$, and a family $(\Gamma_j)_{0\leq j\leq l}$ of connected components of $K\setminus H$, such that $\overline{\Gamma}_j$ connects $H^{\sigma_{j-1}}$ with H^{σ_j} for $1\leq j\leq l$.*

Proof. It is enough to argue as in [19, Lemma 3.7], noticing that: by Lemma 4.9 the set K is locally connected; by Lemma 4.12 it is $\mathcal{L}(\overline{C}_n)=\mathcal{L}(C_n)$, where C_n are defined in the cited result. □

Lemma 4.14. *Let (H_n) be a sequence in $\mathcal{C}_p$ converging to H in the Hausdorff metric, and let $K\in\mathcal{C}_1$ be such that $H\subset K$. Then there exists a sequence (K_n) in $\mathcal{C}_1$ such that (K_n) converges to K in the Hausdorff metric, $H_n\subset K_n$ and $\mathcal{L}(K_n\setminus H_n)\to\mathcal{L}(K\setminus H)$.*

Proof. Following the lines of [19, Lemma 3.8], apply Lemma 4.13, Lemma 4.11, Lemma 4.12 and Corollary 4.5 instead of Lemma 3.7, Lemma 3.6, Proposition 2.5 and Corollary 3.4 in [19], respectively. In the construction of the sets corresponding to X_n^j, Y_n^j and Z_n^i in [19], it is enough to argue as in Lemma 4.11. □

5. Proof of the main result

In this section we prove the existence of a quasi-static evolution for cracks in $\mathcal{C}_p$, satisfying the global minimality condition and the energy balance (Theorem 3.3), by the usual time discretization procedure. We therefore follow the steps of [19].

We shall need the following result on the convergence of the minimum points of problems (3.1) corresponding to converging sequences in $\mathcal{C}_p$.

Theorem 5.1. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff distance, and let (g_n) be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$E(g_n, K_n) = \min_{v \in \mathcal{V}(g_n, K_n)} \int_{\Omega \setminus K_n} |\nabla v|^2 \, dx \,,$$

and let u be a solution of the minimum problem (3.1)

$$E(g, K) = \min_{v \in \mathcal{V}(g, K)} \int_{\Omega \setminus K} |\nabla v|^2 \, dx \,,$$

where $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ are defined by (3.2)*. Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$.*

Proof. The proof can be done in the same manner as for [19, Theorem 5.1], as long as we check that the key facts therein are satisfied. The first one lies in the application of [19, Theorem 4.3], for which the set K needs to be locally connected; in our case this is assured by Lemma 4.9.

The second important step is the following: given any $x \in \overline{\Omega}$, an open rectangle V containing x and an open set $U \subset\subset V$, we need to bound uniformly the number of connected components of $\overline{V} \cap K_n$ which meet U. We can argue in the following way. Let $l = \operatorname{dist}(U, \partial V)$ and let C be a connected component of $\overline{V} \cap K_n$ which meets U. If $C \cap \partial V \neq \emptyset$, then $\operatorname{diam} C \geq l$ and by Lemma 4.7 there exists a constant C_l such that $\mathcal{L}(C) \geq C_l$. Being $\mathcal{L}(K_n) \leq \lambda$, the number of those connected components is smaller than λ / C_l. If $C \cap \partial V = \emptyset$, then C is a connected component of K_n, and there are at most p of them.

Having established these two key issues, the proof carries on as in the cited result, based on the construction of a harmonic conjugate for u. ☐

Given $\delta > 0$, we denote by N_δ the largest integer such that $\delta N_\delta \leq T$; for $0 \leq i \leq N_\delta$, let $t_i^\delta := i\delta$ and $g_i^\delta := g(t_i^\delta)$. The sets K_i^δ are defined inductively as a solution to the following minimization problem

$$\min_K \left\{ \mathcal{E}(g_i^\delta, K) : K \in \mathcal{C}_p \,, \; K \supseteq K_{i-1}^\delta \right\} , \tag{5.1}$$

where we set $K_{-1}^\delta := K_0$.

Lemma 5.2. *There exists a solution of the minimum problem* (5.1).

Proof. Assume by induction that $K_{i-1}^\delta \in \mathcal{C}_p$. Consider a minimizing sequence (K_n) of problem (5.1). By Proposition 4.1, we may assume that (up to a subsequence) (K_n) converges in the Hausdorff distance to some compact set $K \in \mathcal{C}_p$ which contains K_{i-1}^δ. For every n let u_n be a solution of the minimum problem (3.1) which defines $E(g_i^\delta, K_n)$.

By Theorem 5.1 the sequence (∇u_n) converges strongly in $L^2(\Omega; \mathbb{R}^2)$ to ∇u, where u is a solution of the minimum problem (3.1) which defines $E(g_i^\delta, K)$. As by Corollary 4.4

$$\mathcal{L}(K) \leq \liminf_n \mathcal{L}(K_n) \,,$$

we conclude that $\mathcal{E}(g_i^\delta, K) \leq \liminf_n \mathcal{E}(g_i^\delta, K_n)$. Since (K_n) is a minimizing sequence, this proves that K is a solution of the minimum problem (5.1). ☐

We define now the piecewise constant functions g_δ, K_δ, and u_δ on $[0,T]$ by setting $g_\delta(t) := g_i^\delta = g(t_i^\delta)$, $K_\delta(t) := K_i^\delta$, and $u_\delta(t) := u_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$, where u_i^δ is a solution of the minimum problem (3.1) which defines $E(g_i^\delta, K_i^\delta)$.

Lemma 5.3. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that*

$$\|\nabla u_j^\delta\|^2 + \mathcal{L}(K_j^\delta) \le \|\nabla u_i^\delta\|^2 + \mathcal{L}(K_i^\delta) + 2\int_{t_i^\delta}^{t_j^\delta} (\nabla u_\delta(t)|\nabla \dot g(t))\,dt + \rho(\delta) \tag{5.2}$$

for $0 \le i < j \le N_\delta$.

Proof. Let us fix an integer r with $i \le r < j$. From the absolute continuity of g we have

$$g_{r+1}^\delta - g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \dot g(t)\,dt\,,$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$. This implies that

$$\nabla g_{r+1}^\delta - \nabla g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \nabla \dot g(t)\,dt\,, \tag{5.3}$$

where the integral is a Bochner integral for functions with values in $L^2(\Omega;\mathbb{R}^2)$.

As $u_r^\delta + g_{r+1}^\delta - g_r^\delta \in L^{1,2}(\Omega\backslash K_r^\delta)$ and $u_r^\delta + g_{r+1}^\delta - g_r^\delta = g_{r+1}^\delta$ q.e. on $\partial_D\Omega\backslash K_r^\delta$, we have

$$\mathcal{E}(g_{r+1}^\delta, K_r^\delta) \le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{L}(K_r^\delta)\,. \tag{5.4}$$

By the minimality of u_{r+1}^δ and by (5.1) it is

$$\|\nabla u_{r+1}^\delta\|^2 + \mathcal{L}(K_{r+1}^\delta) = \mathcal{E}(g_{r+1}^\delta, K_{r+1}^\delta) \le \mathcal{E}(g_{r+1}^\delta, K_r^\delta)\,. \tag{5.5}$$

From (5.3), (5.4), and (5.5) we obtain

$$\begin{aligned}
&\|\nabla u_{r+1}^\delta\|^2 + \mathcal{L}(K_{r+1}^\delta) \le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{L}(K_r^\delta) \le \\
&\quad \le \|\nabla u_r^\delta\|^2 + \mathcal{L}(K_r^\delta) + 2\int_{t_r^\delta}^{t_{r+1}^\delta} (\nabla u_r^\delta|\nabla \dot g(t))\,dt + \Big(\int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\,dt\Big)^2 \le \\
&\quad \le \|\nabla u_r^\delta\|^2 + \mathcal{L}(K_r^\delta) + 2\int_{t_r^\delta}^{t_{r+1}^\delta} (\nabla u_\delta(t)|\nabla \dot g(t))\,dt + \sigma(\delta)\int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\,dt\,,
\end{aligned}$$

where

$$\sigma(\delta) := \max_{0\le r<N_\delta} \int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\,dt \ \longrightarrow\ 0$$

by the absolute continuity of the integral. Iterating this inequality for $i \le r < j$ we get (5.2) with $\rho(\delta) := \sigma(\delta)\int_0^T \|\nabla \dot g(t)\|\,dt$. ☐

Lemma 5.4. *There exists a constant λ, depending only on g and K_0, such that*

$$\|\nabla u_i^\delta\| \le \lambda \quad \textit{and} \quad \sum_{m=1}^M \mathcal{H}^{d_m}(K_i^\delta \cap \mathcal{K}_m) \le \lambda \tag{5.6}$$

for every $\delta > 0$ and for every $0 \le i \le N_\delta$.

Proof. As g_i^δ is admissible for the problem (3.1) which defines $E(g_i^\delta, K_i^\delta)$, by the minimality of u_i^δ we have $\|\nabla u_i^\delta\| \le \|\nabla g_i^\delta\|$, hence $\|\nabla u_\delta(t)\| \le \|\nabla g_\delta(t)\|$ for every $t \in [0,T]$. As $t \mapsto g(t)$ is absolutely continuous with values in $H^1(\Omega)$ the function $t \mapsto \|\nabla \dot g(t)\|$ is integrable on $[0,T]$ and there exists a constant $C > 0$ such that $\|\nabla g(t)\| \le C$ for every $t \in [0,T]$. This implies (5.6).

The latter inequality follows now from Lemma 5.3 and from the inequality $\|\nabla u_0^\delta\|^2 + \mathcal{L}(K_0^\delta) \le \|\nabla g(0)\|^2 + \mathcal{L}(K(0))$, which is an obvious consequence of (5.1) for $i = 0$. ☐

At this point we have all the elements to obtain a continuous-time evolution as limit of discrete-time ones when the time step δ vanishes.

By Helly's Theorem (see, e.g., [19]), there exists a subsequence of K_δ, not relabelled, and an increasing function $K : [0,T] \to \mathcal{C}_p$ such that

$$K_\delta(t) \to K(t)$$

in the Hausdorff metric for every $t \in [0,T]$.

In the rest of this section, when we write $\delta \to 0$, we always refer to the sequence given above by Helly's Theorem.

For every $t \in [0,T]$ let $u(t)$ be a solution of the minimum problem (3.1) which defines $E(g(t), K(t))$. Then

$$\mathcal{E}(g(t), K(t)) = \|\nabla u(t)\|^2 + \mathcal{L}(K(t)).$$

Lemma 5.5. *For every $t \in [0,T]$ we have $\nabla u_\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$.*

Proof. As $u_\delta(t)$ is a solution of the minimum problem (3.1) which defines $E(g_\delta(t), K_\delta(t))$, and $g_\delta(t) \to g(t)$ strongly in $H^1(\Omega)$, the conclusion follows from Theorem 5.1. □

Lemma 5.6. *For every $t \in [0,T]$ we have*

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K) \quad \forall K \in \mathcal{C}_p, \ K \supset K(t). \tag{5.7}$$

Moreover

$$\mathcal{E}(g(0), K(0)) \le \mathcal{E}(g(0), K) \quad \forall K \in \mathcal{C}_p, \ K \supset K_0. \tag{5.8}$$

Proof. Fix $t \in [0,T]$. By construction, $K(t)$ is the limit of the sequence $(K_\delta(t))$ in the Hausdorff metric as δ vanishes. Fix $K \in \mathcal{C}_p$ with $K \supset K(t)$. Applying Lemma 4.10 we find a sequence (K_δ) in $\mathcal{C}_p$ with $K_\delta \supset K_\delta(t)$, such that $K_\delta \to K$ in the Hausdorff metric and $\mathcal{L}(K_\delta \backslash K_\delta(t)) \to \mathcal{L}(K \backslash K(t))$.

Consider the minimizers v_δ and v of the elastic energies corresponding to $E(g_\delta(t), K_\delta)$ and $E(g(t), K)$, respectively. By Theorem 5.1 we have that $\nabla v_\delta \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$. By the choice of $K_\delta(t)$ as minimizers of (5.1), it is $\mathcal{E}(g_\delta(t), K_\delta(t)) \le \mathcal{E}(g_\delta(t), K_\delta)$, which implies $\|\nabla u_\delta(t)\|^2 \le \|\nabla v_\delta\|^2 + \mathcal{L}(K_\delta \backslash K_\delta(t))$. By Lemma 5.5 and the properties of the sequence (K_δ), we obtain $\|\nabla u(t)\|^2 \le \|\nabla v\|^2 + \mathcal{L}(K \backslash K(t))$. To get (5.7) it is now enough to add $\mathcal{L}(K(t))$ to both sides of the last inequality.

The proof for (5.8) is similar, exploiting the minimality of $K_\delta(0)$ in (5.1) with respect to all sets $K \in \mathcal{C}_p$ containing K_0, and applying Corollary 4.5 for the functional $\mathcal{L}$. □

The previous lemma proves the global minimality conditions (GS) and (3.6).

Finally, after a technical result, we will deal with the energy balance (EB), the only missing property in Theorem 3.3.

Lemma 5.7. *For every $K \in \mathcal{C}_p$ the function $\mathcal{E}(\cdot, K) : H^1(\Omega) \to \mathbb{R}$ is of class C^1, and for every $g, h \in H^1(\Omega)$ it is*

$$\partial_g \mathcal{E}(g, K)[h] = 2(\nabla u(g, K) | \nabla h), \tag{5.9}$$

where $u(g, K)$ is the solution to the minimum problem (3.1).

Proof. Being K fixed, for simplicity of notation we write $u_g := u(g, K)$. By linearity, for every $\eta \in \mathbb{R}$ it is $u_{g+\eta h} = u_g + \eta u_h$ a.e. in Ω. Then

$$\begin{aligned}\mathcal{E}(g + \eta h, K) - \mathcal{E}(g, K) &= \|\nabla u_{g+\eta h}\|^2 - \|\nabla u_g\|^2 \\ &= 2\eta\, (\nabla u_g | \nabla u_h) + \eta^2 \|\nabla u_h\|^2 = 2\eta\, (\nabla u_g | \nabla h) + \eta^2 \|\nabla u_h\|^2,\end{aligned}$$

where the last equality is obtained by (3.5) with $z = u_h - h$, since $u_h - h \in L^{1,2}(\Omega \setminus K)$ and $u_h - h = 0$ q.e. on $\partial_D \Omega$. Dividing by $\eta \neq 0$ and letting η vanish, we get (5.9). Finally, the C^1-regularity is consequence of the continuity of the map $g \mapsto \nabla u(g, K)$ (see Theorem 5.1). □

Lemma 5.8. *For every* $s,\, t$ *with* $0 \le s < t \le T$

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(s), K(s)) + 2 \int_s^t (\nabla u(\tau) | \nabla \dot{g}(\tau)) d\tau \,. \tag{5.10}$$

Proof. The strategy is to show that the map $t \mapsto \mathcal{E}(g(t), K(t))$ is absolutely continuous on $[0,T]$, with pointwise derivative $2(\nabla u(t)|\nabla \dot{g}(t))$ for a.e. $t \in [0,T]$.

Let us fix s, t with $0 \le s < t \le T$, and $\delta > 0$. Applying Lemma 5.3 we obtain

$$\|\nabla u_\delta(t)\|^2 + \mathcal{L}(K_\delta(t) \backslash K_\delta(s)) \le \|\nabla u_\delta(s)\|^2 + 2 \int_{s_\delta}^{t_\delta} (\nabla u_\delta(\tau) | \nabla \dot{g}(\tau)) \, d\tau + \rho(\delta) \,, \tag{5.11}$$

with $\rho(\delta)$ converging to zero as $\delta \to 0$, where s_δ, t_δ are the discrete times such that $s_\delta \le s < s_\delta + \delta$, $t_\delta \le t < t_\delta + \delta$. For every $\tau \in [0,T]$ we have, by Lemma 5.5, that $\nabla u_\delta(\tau) \to \nabla u(\tau)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ as $\delta \to 0$, and, by Lemma 5.4, that $\|\nabla u_\delta(\tau)\| \le \lambda$. Moreover, by Corollary 4.5 we get

$$\mathcal{L}(K(t) \backslash K(s)) \le \liminf_{\delta \to 0} \; \mathcal{L}(K_\delta(t) \backslash K_\delta(s)) \,,$$

so that, passing to the limit in (5.11) as $\delta \to 0$, we obtain

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(s), K(s)) + 2 \int_s^t (\nabla u(\tau) | \nabla \dot{g}(\tau)) d\tau \,. \tag{5.12}$$

To prove the opposite inequality note that, by the global stability (GS) of Definition 3.2 we have $\mathcal{E}(g(s), K(s)) \le \mathcal{E}(g(s), K(t))$, and by Lemma 5.7

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2 \int_s^t (\nabla u(\tau, t) | \nabla \dot{g}(\tau)) \, d\tau \,,$$

where $u(\tau, t)$ is a solution of the minimum problem (3.1) which defines $E(g(\tau), K(t))$. Therefore

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \ge 2 \int_s^t (\nabla u(\tau, t) | \nabla \dot{g}(\tau)) \, d\tau \,. \tag{5.13}$$

Since for $s \le \tau \le t$ the uniform bounds $\|\nabla u(\tau)\| \le \|\nabla g(\tau)\| \le C$ and $\|\nabla u(\tau, t)\| \le \|\nabla g(\tau)\| \le C$ hold, from (5.12) and (5.13) we obtain

$$\left| \mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \right| \le 2\, C \int_s^t \|\nabla \dot{g}(\tau)\| \, d\tau \,,$$

which proves the absolute continuity of the map $t \mapsto \mathcal{E}(g(t), K(t))$.

Observe that by Theorem 5.1 $\nabla u(\tau, t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ as $\tau \to t$. Dividing now both (5.12) and (5.13) by $t - s$ and letting $s \to t-$, we get

$$\lim_{s \to t-} \frac{\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s))}{t - s} = 2\, (\nabla u(t) | \nabla \dot{g}(t))$$

for a.e. $t \in [0,T]$, and thus the proof is concluded. □

6. Fractional dimensional crack evolution as limit of one-dimensional ones

In this section we show that the energy functional considered in the previous sections arises as a natural extension of the Griffith setting; indeed, it can be obtained as Γ-limit of energies involving small toughness coefficients and the $\mathcal{H}^1$-measure restricted to polygonal approximations of the curves with fractional Hausdorff dimension. We illustrate this idea in the case of a single curve $\mathcal{K}$.

Let $\mathcal{K}$ be a curve of the form $\mathcal{K} = \gamma([0,1])$ with γ satisfying (2.1) and (2.2), and $d \in (1,2)$. For $n \in \mathbb{N}$ we construct a sequence of polygonal approximations $\mathcal{K}^n$ in the following way: define $\gamma_n : [0,1] \to \mathbb{R}^2$ as

$$\gamma_n(s) := \gamma(i/n) + (ns - i)(\gamma((i+1)/n) - \gamma(i/n))$$

for $i/n \le s < (i+1)/n$ and $i = 0, \dots, n-1$, and set $\mathcal{K}^n := \gamma_n([0,1])$. By (2.1), it is

$$\mathcal{K}^n \to \mathcal{K} \tag{6.1}$$

in the Hausdorff metric, as $n \to +\infty$.

We define the "toughness coefficients"

$$\kappa_n^i = \frac{L}{n\,|\gamma((i+1)/n) - \gamma(i/n)|}\,,$$

for $i = 0, \dots, n-1$, where $L = \mathcal{H}^d(\mathcal{K})$, and set $\kappa_n(x) = \kappa_n^i$ if $x \in \gamma_n([i/n, (i+1)/n))$. Finally, we introduce the set-function

$$\mathcal{L}_n(K) := \int_{K \cap \mathcal{K}^n} \kappa_n(x)\, d\mathcal{H}^1(x)\,.$$

Lemma 6.1. *Let (K_n) be a sequence of compact connected sets such that $K_n \subset \mathcal{K}^n$ for every n. Assume that (K_n) converges to K in the Hausdorff metric. Then K is a compact connected set, contained in $\mathcal{K}$, and*

$$\mathcal{L}_n(K_n) \to \mathcal{H}^d(K)\,.$$

Proof. The set K is compact, connected and contained in $\mathcal{K}$ by properties of the Hausdorff convergence (and (6.1)). For every n, it is $K_n = \gamma_n([a_n, b_n])$ for some $a_n, b_n \in [0,1]$, and $K = \gamma([a,b])$ for $a, b \in [0,1]$. It is not difficult to verify that $a_n \to a$ and $b_n \to b$. Set $i_n, j_n \in \{0, \dots, 1/n\}$ such that $ni_n \le a_n < n(i_n+1)$ and $nj_n \le b_n < n(j_n+1)$, we have

$$\mathcal{L}_n(K_n) = L(j_n - (i_n+1)) + \kappa_n^{i_n}\big|\gamma(n(i_n+1)) - \gamma(a_n)\big| + \kappa_n^{j_n}\big|\gamma(b_n) - \gamma(nj_n)\big|\,,$$

which converges to $L(b-a)$ as $n \to +\infty$. Being $\mathcal{H}^d(K) = L(b-a)$ by (2.2), the lemma is proved. □

On the other hand, given a compact connected set $K \subset \mathcal{K}$, there exists a sequence K_n of compact connected sets such that $K_n \subset \mathcal{K}^n$, $K_n \to K$ in the Hausdorff distance and

$$\mathcal{L}_n(K_n) \to \mathcal{H}^d(K)\,. \tag{6.2}$$

Indeed, being $K = \gamma([a,b])$, it is enough to take

$$K_n := \gamma_n([a,b])\,. \tag{6.3}$$

Then Lemma 6.1 provides (6.2).

Remark 6.2. The length of the approximating polygonals K_n in the previous lemma tends to infinity:

$$\begin{aligned}\mathcal{H}^1(K_n) &\ge \sum_{h=i_n+1}^{j_n} |\gamma(h/n) - \gamma((h+1)/n)| \\ &\ge c^{-1} \sum_{h=i_n+1}^{j_n} (1/n)^{1/d} = c^{-1}\frac{L}{b-a} n^{1-1/d} + o(1) \to +\infty\,.\end{aligned}$$

Conversely, the toughness coefficients κ_n vanish, so that the lower bound in (1.1) is violated: indeed

$$\sup_i \kappa_n^i = \sup_i \frac{L}{n\,|\gamma((i+1)/n) - \gamma(i/n)|} \le c\, n^{-(1-1/d)} \to 0$$

as $n \to +\infty$, being $d > 1$.

We consider the functionals

$$F(u,g,K) := \begin{cases} \displaystyle\int_{\Omega \setminus K} |\nabla u|^2\, dx + \mathcal{H}^d(K) & \text{if } K \subset \mathcal{K}, g \in H^1(\Omega) \text{ and } u \in \mathcal{V}(g,K) \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F_n(u,g,K) := \begin{cases} \displaystyle\int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{L}_n(K) & \text{if } K\subset \mathcal{K}^n, g\in H^1(\Omega) \text{ and } u\in\mathcal{V}(g,K) \\ +\infty & \text{otherwise}, \end{cases}$$

where $\mathcal{V}(g,K)$ is defined in (3.2) for $K\subset\mathcal{K}$ and similarly when $K\subset\mathcal{K}^n$. The two functionals are related in the following way.

Theorem 6.3. *Let (K_n) be a sequence of compact sets with at most p connected components and $K_n\subset\mathcal{K}^n$, and assume it converges to K in the Hausdorff metric. Let (g_n) be a sequence converging to g in $H^1(\Omega)$. Then $F_n(\cdot,g_n,K_n)$ Γ-converges to $F(\cdot,g,K)$ with respect to the weak convergence in L^2 of the gradients.*

The proof of the above theorem will be a consequence of the result below, proved in [16, Theorem 6.3], and that we rewrite for the ease of the reader. Similar results, concerning Dirichlet and Neumann boundary data, were proved, e.g., in [35] and [11, 10, 14], respectively.

Theorem 6.4. *Let (g_n) be a sequence in $H^1(\Omega)$ converging to $g\in H^1(\Omega)$, and let (K_n) be a sequence of compact subsets of $\overline{\Omega}$ converging to K in the Hausdorff metric. Assume that $|K_n|$ converges to $|K|$ and that K_n have a uniformly bounded number of connected components. Then the space*

$$H_n := \big\{\nabla u\, 1_{\Omega\setminus K_n} :\ u\in L^{1,2}(\Omega\setminus K_n),\ u=g_n \ \text{ on } \partial_D\Omega\big\}$$

converges to

$$H := \big\{\nabla u\, 1_{\Omega\setminus K} :\ u\in L^{1,2}(\Omega\setminus K),\ u=g \ \text{ on } \partial_D\Omega\big\}$$

in the sense of Mosco [31], i.e. *the following two conditions hold:*

- (M_1) *for every $u\in L^{1,2}(\Omega\setminus K)$ with $u=g$ on $\partial_D\Omega$ there exists a sequence $u_n\in L^{1,2}(\Omega\setminus K_n)$ with $u=g_n$ on $\partial_D\Omega$, such that $\nabla u_n 1_{\Omega\setminus K_n}$ converges strongly to $\nabla u\, 1_{\Omega\setminus K}$ in $L^2(\Omega;\mathbb{R}^2)$;*
- (M_2) *if (h_n) is a sequence of indices that tends to $+\infty$, and (u_n) is a sequence such that $u_n\in L^{1,2}(\Omega\setminus K_{h_n})$ with $u_n=g_{h_n}$ on $\partial_D\Omega$ for every n and $\nabla u_n 1_{\Omega\setminus K_{h_n}}$ converges weakly in $L^2(\Omega;\mathbb{R}^2)$ to ψ, then there exists a function $u\in L^{1,2}(\Omega\setminus K)$ with $u=g$ on $\partial_D\Omega$ and $\psi=\nabla u\, 1_{\Omega\setminus K}$.*

Proof of Theorem 6.3. Let us observe immediately that the hypotheses on the K_n and K in Theorem 6.4 are satisfied: indeed the K_n have at most p connected components and, since $\mathcal{L}_n(K_n)<\infty$ and $\mathcal{H}^d(K)<\infty$, it is $|K_n|=|K|=0$. Below we apply Theorem 6.4 with $H_n=\{\nabla u:\ u\in\mathcal{V}(g_n,K_n)\}$ and $H=\{\nabla u:\ u\in\mathcal{V}(g,K)\}$.

$\Gamma-\liminf$ *inequality.* Let $u\in\mathcal{V}(g,K)$ and let (u_n) be a sequence such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$. We may assume that

$$F_n(u_n,g_n,K_n)\le C \tag{6.4}$$

for some $C>0$ for every n (otherwise the $\Gamma-\liminf$ inequality is trivially satisfied); hence $u_n\in\mathcal{V}(g_n,K_n)$ for every n. By Lemma 6.1 it is

$$\mathcal{H}^d(K)=\lim_{n\to+\infty}\mathcal{L}_n(K_n).$$

Since

$$\int_{\Omega\setminus K}|\nabla u|^2\,dx \le \liminf_{n\to+\infty}\int_{\Omega\setminus K_n}|\nabla u_n|^2\,dx,$$

we get

$$F(u,g,K)\le \liminf_{n\to+\infty} F_n(u_n,g_n,K_n).$$

$\Gamma-\limsup$ *inequality.* Consider a function $u \in \mathcal{V}(g,K)$ and the sequence $u_n \in \mathcal{V}(g_n,K_n)$ provided by (M_1) in Theorem 6.4. Then ∇u_n converges to ∇u strongly in $L^2(\Omega;\mathbb{R}^2)$ and

$$F(u,g,K) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{H}^d(K)$$
$$= \lim_{n\to+\infty} \int_{\Omega\setminus K_n} |\nabla u_n|^2\,dx + \mathcal{L}_n(K_n) = \lim_{n\to+\infty} F_n(u_n,g_n,K_n)\,.$$

□

At this point we want to prove that the evolutions described in Theorem 3.3 are indeed limits of irreversible quasi-static crack evolutions $t\mapsto (u_n(t),K_n(t))$ (of global minimizers) whose crack set $K_n(t)$ is 1-dimensional and contained in $\mathcal{K}^n$, with fracture dissipation energy given by

$$\mathcal{L}_n(K_n(t)) = \int_{K_n(t)} \kappa_n(x)\,d\mathcal{H}^1(x)\,.$$

In analogy to Sections 2 and 3, we define the set

$$\mathcal{C}_p^n := \Big\{K \subset \mathcal{K}^n :\ K \text{ nonempty compact set with at most } p \text{ connected components}\Big\}\,,$$

and the energy functional

$$\mathcal{E}_n(g,K) := \min_{u\in\mathcal{V}(g,K)} \int_{\Omega\setminus K} |\nabla u|^2 dx + \mathcal{L}_n(K)\,.$$

The results in [19] (in particular [19, Theorem 7.1]) guarantee the existence of irreversible quasi-static crack evolutions $t\mapsto (u_n(t),K_n(t))$ for the total energy $\mathcal{E}_n$,with the constraint $K_n(t)\subset\mathcal{K}^n$, $K_n(t)$ having at most p connected components (with p prescribed *a priori*), and satisfying conditions analogous to those in Theorem 3.3. More precisely, for every n, given $K_n^0\subset\mathcal{K}^n$ and $g\in AC([0,T];H^1(\Omega))$, there exists an evolution $t\in[0,T]\mapsto K_n(t)\subset\mathcal{K}^n$ fulfilling the following conditions:

(I_n) $K_n^0 \subseteq K_n(\tau)\subseteq K_n(t)$ for $0\le\tau\le t\le T$;

(GS_n) $\mathcal{E}_n(g(0),K_n(0)) \le \mathcal{E}_n(g(0),K) \quad \forall\, K\in\mathcal{C}_p^n,\ K\supseteq K_n^0,$ and for $0\le t\le T$

$$\mathcal{E}_n(g(t),K_n(t)) \le \mathcal{E}_n(g(t),K) \quad \forall\, K\in\mathcal{C}_p^n,\ K\supseteq K_n(t)\,;$$

(EB_n) for every s, t with $0\le s<t\le T$

$$\mathcal{E}_n(g(t),K_n(t)) = \mathcal{E}_n(g(s),K_n(s)) + 2\int_s^t (\nabla u_n(\tau)|\nabla\dot g(\tau))d\tau\,,$$

where $u_n(t)$ is the unique solution of the minimum problem defining $\mathcal{E}_n(g(t),K_n(t))$.

Theorem 6.5. *For every $n\in\mathbb{N}$, let $t\to K_n(t)$ be an irreversible quasi-static evolution satisfying* (I_n) - (GS_n) - (EB_n) *and such that $K_n(t)\subset\mathcal{K}^n$ for every $t\in[0,T]$. Then there exists a subsequence, not relabelled, and an evolution $t\mapsto K(t)$, such that it satisfies the conditions in Theorem 3.3 and $K_n(t)$ converges to $K(t)$ in the Hausdorff metric for every $t\in[0,T]$.*

Proof. Monotonicity of the maps $t\mapsto K_n(t)$ due to (I_n), and Helly's theorem [19, Theorem 6.3], guarantee the existence of a subsequence (not relabelled) and of an increasing set-function $t\mapsto K(t)$ such that, for every $t\in[0,T]$, $K_n(t)$ converges to $K(t)$ in the Hausdorff metric. Since $(\mathcal{K}^n)$ converges to $\mathcal{K}$ in the Hausdorff metric and $K_n(t)\subset\mathcal{K}^n$, it is $K(t)\subset\mathcal{K}$. Moreover $K(t)$ has at most p connected components, so that $K(t)\in\mathcal{C}_p$ for every t. Hence condition (I) in Theorem 3.3 is satisfied.

We have to check the global unilateral minimality conditions (3.6) and (GS) at any instant t. Fix $t\in[0,T]$ and $K\in\mathcal{C}_p$ with $K\supset K(t)$ for $t>0$, and with $K\supset K_0$ if $t=0$.

We claim that there exists a sequence (K_n) converging to K in the Hausdorff metric and such that, for every n, K_n has at most p connected components and $K_n(t) \subset K_n \subset \mathcal{K}^n$.

By the minimality of $K_n(t)$, corresponding to (GS$_n$), we have

$$\mathcal{E}_n(g(t), K_n(t)) \leq \mathcal{E}_n(g(t), K_n)\,,$$

where K_n is the sequence provided by the claim above. By Theorem 6.3 and the properties of Γ-convergence (the functionals $F_n(\cdot, g(t), K_n)$ and $F_n(\cdot, g(t), K_n(t))$ are asymptotically sequentially coercive; see [15, Chapter 7]) we get the convergence of the minima:

$$\mathcal{E}_n(g(t), K_n) = \min_{u \in \mathcal{V}(g(t), K_n)} F_n(u, g(t), K_n) \to \mathcal{E}(g(t), K) = \min_{u \in \mathcal{V}(g(t), K)} F(u, g(t), K)$$

and, analogously,

$$\mathcal{E}_n(g(t), K_n(t)) \to \mathcal{E}(g(t), K(t))\,. \tag{6.5}$$

The three relations above prove conditions (GS) and (3.6).

The conservation of the energy (EB) follows by (EB$_n$) and (6.5).

Proof of the claim.

We now illustrate how to construct the sets K_n; the main issue is to fulfil the condition on the maximum number of connected components. Let $t \in [0, T]$ be fixed. Assume that

$$K(t) = \gamma([a_1, b_1]) \cup \ldots \cup \gamma([a_q, b_q])$$

for some $q \leq p$, with $b_i < a_{i+1}$ for $i = 1, \ldots, q-1$. Without loss of generality, we can assume that the sets $K_n(t)$ have r connected components for every n, more precisely they are of the form

$$K_n(t) = \gamma_n([a_1^n, b_1^n]) \cup \ldots \cup \gamma_n([a_r^n, b_r^n])$$

with $b_j^n < a_{j+1}^n$ for $j = 1, \ldots, r-1$.

In general, $r \geq q$. If $r > q$ we want to substitute the set $K_n(t)$ with a set $\widetilde{K}_n(t)$ having exactly q connected components, containing $K_n(t)$ and still converging to $K(t)$ in the Hausdorff metric. The construction can be done in the following way. We firstly observe that

$$\gamma_n([a_{i_n}^n, b_{i_n}^n] \cup \ldots \cup [a_{h_n}^n, b_{h_n}^n]) \to \gamma([a_i, b_i])$$

in the Hausdorff metric if and only if

$$a_{i_n}^n \to a_i \qquad b_{h_n}^n \to b_i \qquad a_l^n - b_{l-1}^n \to 0$$

for $l = i_n + 1, \ldots, h_n$.

Let $\eta > 0$ be such that $a_{i+1} - b_i > 3\eta$ for all $i = 1, \ldots, q-1$. Set $\alpha_1^n := a_1^n$ and $\beta_1^n := b_j^n$ with the index j satisfying

$$b_j^n < a_2 - \eta \leq a_{j+1}^n\,,$$

and $\beta_q^n = b_r^n$. For $i = 2, \ldots, q-1$ we define the intervals $[\alpha_i^n, \beta_i^n] = [a_j^n, b_h^n]$, where the indices j, h are such that

$$b_{j-1}^n < a_i - \eta \leq a_j^n < b_h^n \leq b_i + \eta < a_{h+1}^n\,.$$

Set

$$\widetilde{K}_n(t) := \gamma_n([\alpha_1^n, \beta_1^n]) \cup \ldots \cup \gamma_n([\alpha_q^n, \beta_q^n])\,.$$

By construction, $K_n(t) \subset \widetilde{K}_n(t) \subset \mathcal{K}^n$ and $\widetilde{K}_n(t)$ has q connected components; by the previous observation, $\widetilde{K}_n(t)$ converges to $K(t)$ in the Hausdorff metric.

Let $K \in \mathcal{C}_p$ with $K \supset K(t)$. It is of the form

$$K = \gamma([c_1, d_1]) \cup \ldots \cup \gamma([c_s, d_s])$$

for some $s \leq p$. Notice that, by inclusion, every interval $[a_i, b_i]$ is contained in an interval $[c_j, d_j]$. It is not difficult to verify that the set

$$K_n := \gamma_n([c_1, d_1]) \cup \ldots \cup \gamma_n([c_s, d_s]) \cup \widetilde{K}_n(t)$$

fulfils the requests of the claim: it has the same number of connected components as K (hence less then p), contains $K_n(t)$, is a subset of $\mathcal{K}^n$, and converges to K in the Hausdorff metric. □

The result above is consistent with the justification of the model, as discussed in the introduction, when the lower bound in (1.1) is violated (see Remark 6.2). Indeed, where the material becomes more and more fragile, the $\mathcal{H}^1$ measure of the crack is no longer appropriate for the dissipative term, and it is necessary to introduce fractional Hausdorff measures in order to take into account the increased roughness of the fracture in the fragile area.

7. The linearized and nonlinear cases

The results of the previous sections, which for simplicity have been proved in the antiplane linear setting, can be extended to more general frameworks, in particular to the vectorial 2-dimensional setting, corresponding to the mode I and mode II fracture models, both in the nonlinear and linearized case.

7.1. **Nonlinear elasticity.** Our setting can be extended to the case of hyperelastic materials, under suitable assumptions on the nonlinear energy density that guarantee the existence of global minimizers. We consider both the antiplane and the plane case. We briefly discuss the main steps.

The bulk energy for a deformation v of the unfractured part of the body $\Omega \setminus K$ is given by the functional

$$\int_{\Omega\setminus K} W(x, \nabla v(x))\, dx\,,$$

where $W : \Omega\times\mathbb{R}^{N\times 2} \to \mathbb{R}$ is a given energy density, dependent on the material. Here $N = 1$ in the antiplane case, with v describing the out-of-plane vertical deformation; $N = 2$ if v describes the in-plane deformation.

We assume W to satisfy the following properties:

- W is a Carathéodory function;
- for every $x \in \Omega$ the function $\xi \mapsto W(x, \xi)$ is C^1 and *quasiconvex*, *i.e.* for every $\xi \in \mathbb{R}^{N\times 2}$ and for every $\phi \in C^1_c(\Omega; \mathbb{R}^N)$
$$\frac{1}{|\Omega|}\int_{\Omega} W(x, \xi + \nabla\phi(y))\, dy \geq W(\xi)\,;$$
- for some constants $a_0, a_1 > 0$ and a non-negative function $b \in L^1(\Omega)$ it is
$$a_0|\xi|^2 \leq W(x, \xi) \leq a_1|\xi|^2 + b(x) \tag{7.1}$$
for every $x \in \Omega$ and $\xi \in \mathbb{R}^{N\times 2}$.

Note that for $N = 1$ quasiconvexity and convexity coincide.

Similarly to (3.2), for every $g \in H^1(\Omega; \mathbb{R}^N)$ and $K \in \mathcal{C}_p$ define the set

$$\mathcal{V}_N(g, K) := \{w \in L^{1,2}(\Omega\setminus K; \mathbb{R}^N) : w = g \quad \text{q.e. on } \partial_D\Omega\}$$

and consider the functional

$$\mathcal{W}(g, K, v) := \begin{cases} \displaystyle\int_{\Omega\setminus K} W(x, \nabla v(x))\, dx & \text{if } v \in \mathcal{V}_N(g, K) \\ +\infty & \text{otherwise}\,. \end{cases}$$

Proposition 7.1. *Let (g_n) be a sequence converging to g in $H^1(\Omega; \mathbb{R}^N)$. Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric. Let $v_n \in \mathcal{V}(g_n, K_n)$ be such that (∇v_n) converges to ψ weakly in $L^2(\Omega; \mathbb{R}^{N\times 2})$. Then $\psi = \nabla v$ for some $v \in \mathcal{V}(g, K)$, and*

$$\mathcal{W}(g, K, v) \leq \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n)\,. \tag{7.2}$$

Proof. The existence of $v \in \mathcal{V}(g,K)$ with $\psi = \nabla v$ is consequence of Theorem 6.4 (when $N=2$, hence $v_n(x) = (v_n^1(x), v_n^2(x))$, it is enough to apply it to each component v_n^1, v_n^2).

Consider a subsequence (v_{n_m}) of (v_n) such that

$$\liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n) = \lim_{m\to+\infty} \mathcal{W}(g_{n_m}, K_{n_m}, v_{n_m}).$$

Consider a Lipschitz open set $\omega \subset\subset (\Omega \setminus K) \cup \partial_D\Omega$ with $\mathcal{H}^1(\partial\omega \cap \partial_D\Omega) > 0$. By Hausdorff convergence, $K_n \cap \omega = \emptyset$ for n sufficiently large. As ω has a Lipschitz boundary, $v_n \in H^1(\omega;\mathbb{R}^N)$ for every n. By Rellich theorem and the convergence in $H^1(\Omega;\mathbb{R}^N)$ of (g_n) to g, there exists a subsequence (not relabelled) of (v_{n_m}) that converges to v strongly in $L^2(\omega;\mathbb{R}^N)$. Therefore (v_{n_m}) converges to v weakly in $H^1(\omega;\mathbb{R}^N)$ and we can apply the semicontinuity result [1, Theorem II.4] to obtain

$$\begin{aligned}
\int_\omega W(x, \nabla v(x))\, dx &\le \liminf_{m\to+\infty} \int_\omega W(x, \nabla v_{n_m}(x))\, dx \\
&\le \liminf_{m\to+\infty} \int_{\Omega\setminus K_{n_m}} W(x, \nabla v_{n_m}(x))\, dx \\
&= \lim_{m\to+\infty} \mathcal{W}(g_{n_m}, K_{n_m}, v_{n_m}) = \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n)\,,
\end{aligned}$$

where the last inequality is due to the fact that $W \ge 0$ and $\omega \subset \Omega \setminus K_{n_m}$ for m large. As $\omega \nearrow \Omega \setminus K$ we obtain

$$\mathcal{W}(g, K, v) \le \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n)\,.$$

□

Corollary 7.2. *For every g, K, the minimum problem*

$$\min_{w\in\mathcal{V}_N(g,K)} \mathcal{W}(g, K, w) \tag{7.3}$$

has a solution.

The following result is the counterpart of Theorem 5.1 in the nonlinear setting.

Proposition 7.3. *Let (K_n) be a sequence in $\mathcal{C}_p$ converging to K in the Hausdorff metric, and let (g_n) be a sequence converging to g in $H^1(\Omega;\mathbb{R}^N)$. For every n let $v_n \in \mathcal{V}_N(g_n, K_n)$ be a minimizer of $\mathcal{W}(g_n, K_n, \cdot)$, and assume that*

$$\sup_n \mathcal{W}(g_n, K_n, v_n) < +\infty\,. \tag{7.4}$$

Then, up to subsequences, ∇v_n converges to ∇v weakly in $L^2(\Omega;\mathbb{R}^{N\times 2})$, with $v \in \mathcal{V}_N(g,K)$ which minimizes $\mathcal{W}(g,K,\cdot)$.

Proof. By (7.4) and (7.1), it results that $\sup_n \|\nabla v_n\| < +\infty$. Hence, up to subsequences, (∇v_n) converges to a function ψ weakly in $L^2(\Omega;\mathbb{R}^{N\times 2})$. Theorem 6.4 guarantees the existence of a function $v \in \mathcal{V}_N(g,K)$ with $\nabla v = \psi$ (as before, when $N=2$, *i.e.* $v_n(x) = (v_n^1(x), v_n^2(x))$, it is enough to apply it to each component v_n^1, v_n^2).

It remains to show that v minimizes $\mathcal{W}(g,K,\cdot)$ in $\mathcal{V}_N(g,K)$. Let $w \in \mathcal{V}_N(g,K)$; by (M_1) in Theorem 6.4, there exists a sequence (w_n) with $w_n \in \mathcal{V}_N(g_n, K_n)$ and ∇w_n converging to ∇w strongly in $L^2(\Omega;\mathbb{R}^{N\times 2})$. Up to subsequences, we can assume that $\nabla w_n(x) \to \nabla w(x)$ for a.e. $x \in \Omega$, so that $W(x, \nabla w_n(x)) \to W(x, \nabla w(x))$ for a.e. $x \in \Omega$; by the growth assumption (7.1) and the Generalized Dominated Convergence Theorem, we obtain

$$\int_\Omega W(x, \nabla w_n(x))\, dx \to \int_\Omega W(x, \nabla w(x))\, dx\,.$$

Finally, by the lower semicontinuity result in Proposition 7.1 and by the minimality of the v_n it follows

$$\mathcal{W}(g,K,v) \le \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, v_n) \le \liminf_{n\to+\infty} \mathcal{W}(g_n, K_n, w_n) = \mathcal{W}(g, K, w)\,,$$

which proves that v is a minimizer of $\mathcal{W}(g,K,\cdot)$ in $\mathcal{V}_N(g,K)$. □

For $g\in H^1(\Omega;\mathbb{R}^N)$ and $K\in\mathcal{C}_p$ we define

$$\mathcal{E}_{nl}(g,K):=\inf_{w\in\mathcal{V}_N(g,K)}\mathcal{W}(g,K,w)+\mathcal{L}(K)\,.$$

At this point, considering Proposition 7.1, Proposition 7.3 and the lower semicontinuity of the functional $\mathcal{L}$ (see Corollary 4.4), in order to show the existence of a quasi-static crack evolution in the context of nonlinear elasticity it is sufficient to argue as for Theorem 3.3. In other words, we can prove the following result:

Theorem 7.4. *Let $T>0$ and $g\in AC([0,T];H^1(\Omega;\mathbb{R}^N))$. Let $p\geq 1$ and $K_0\in\mathcal{C}_p$. Then there exists a function $K\colon[0,T]\to\mathcal{C}_p$ such that*

(I$_{nl}$) $K_0\subseteq K(s)\subseteq K(t)$ *for* $0\leq s\leq t\leq T$,

(GS$_{nl}$) *for every* $0\leq t\leq T$

$$\mathcal{E}_{nl}(g(t),K(t))\leq\mathcal{E}_{nl}(g(t),K)$$

for all $K\in\mathcal{C}_p$ *with* $K\supseteq K(t)$;
moreover, $\mathcal{E}_{nl}(g(0),K(0))\leq\mathcal{E}_{nl}(g(0),K)$ *for all* $K\in\mathcal{C}_p$ *with* $K\supseteq K_0$,

(EB$_{nl}$) *for every* s, t *with* $0\leq s<t\leq T$

$$\mathcal{E}_{nl}(g(t),K(t))=\mathcal{E}_{nl}(g(s),K(s))+\int_s^t(D_\xi W(x,\nabla v(\tau))|\nabla\dot g(\tau))d\tau\,,$$

where $v(\tau)$ is a solution of the minimum problem (7.3) *with $g(\tau)$ and $K(\tau)$.*

7.2. **Linearized elasticity.** The extension of our model of crack growth to the linearized case cannot be done in a straightforward way by means of Korn's inequality: indeed, due to the irregularity of the crack sets, it cannot be applied. Instead, the key role is played by the approximation result proved by Chambolle [13, Theorem 1] (see also [9]), which can be used similarly to Theorem 5.1 in the proof of existence of minimizers for the energy $\mathcal{E}_{sym}$ introduced below. Roughly speaking, [13, Theorem 1] states that if $\mathbb{R}^2\setminus\Omega$ has a finite number of connected components then $H^1(\Omega;\mathbb{R}^2)$ is dense in $\{u\in L^2_{loc}(\Omega;\mathbb{R}^2):e(u)\in L^2(\Omega;\mathbb{R}^{2\times2}_{sym})\}$. Here

$$e(u):=\frac{\nabla u+(\nabla u)^T}{2}$$

is the symmetrized gradient of u, and $\mathbb{R}^{2\times2}_{sym}$ is the space of 2×2 symmetric matrices.

Let A be a positive definite quadratic form on the space of symmetric matrices, *i.e.*, $A\xi:\xi\geq C|\xi|^2$ for every $\xi\in\mathbb{R}^{2\times2}_{sym}$, where ":" denotes the scalar product between matrices, and $C>0$. Combining together [19, Theorem 7.1], [13, Theorem 1] and Theorem 3.3, we can state that Theorem 3.3 holds true for the energy

$$\mathcal{E}_{sym}(g,K):=\min_{v\in\mathcal{V}_{sym}(g,K)}\int_{\Omega\setminus K}Ae(v):e(v)\,dx+\sum_{m=1}^M\mathcal{H}^{d_m}(K\cap\mathcal{K}_m)\,,\tag{7.5}$$

where

$$\mathcal{V}_{sym}(g,K):=\{v\in L^2_{loc}(\Omega\setminus K;\mathbb{R}^2):e(v)\in L^2(\Omega\setminus K;\mathbb{R}^{2\times2}_{sym}),\ v=g\quad\text{q.e. on }\partial_D\Omega\}\,.$$

Indeed, the approximation theorem [13, Theorem 1], together with the metric and topological properties shown in Section 4 and used to extend the results in [19], can be applied in order to prove the lower semicontinuity of $\mathcal{E}_{sym}(\cdot,\cdot)$ with respect to the convergence of functions g_n to g in $H^1(\Omega)$ and of sets $K_n\in\mathcal{C}_p$ to K in the Hausdorff metric, and to construct appropriate recovery sequences in order to obtain (GS) and (3.6) in Theorem 3.3 with $\mathcal{E}_{sym}$ instead of $\mathcal{E}$, and $e(u),e(g)$ instead of $\nabla u,\nabla g$ in the condition (EB).

8. Appendix

The von Koch curve, denoted in this subsection by $\mathcal{K}$, represents a significative example for the class of admissible fractal cracks considered in this paper. Therefore, let us describe now the constructive iterative process that defines this self-similar fractal starting from the segment $[0,1]\times\{0\}\subset\mathbb{R}^2$, and provides a parametrization which satisfies (2.1) and (2.2).

With reference to Figure 1, for $i=1,\dots,4$ let $S_i:\mathbb{R}^2\to\mathbb{R}^2$ be the unique similitude that maps the segment $[0,1]\times\{0\}\subset\mathbb{R}^2$ into the segment l_1^i (with length $1/3$) and has positive determinant. It results (see for example [28]) that the von Koch curve is the unique compact set $\mathcal{K}$ such that

$$\mathcal{K}=\bigcup_{i=1}^{4}S_i(\mathcal{K}).$$

We now construct iteratively a parametrization for the von Koch curve.

Let $\gamma_0:[0,1]\to\mathbb{R}^2$ be such that $\gamma_0([0,s])=[0,s]\times\{0\}$.

Let $\gamma_1:[0,1]\to\mathbb{R}^2$ be a continuous parametrization of the set $\tilde{\mathcal{K}}_1$ as in Figure 1, such that $\gamma_1(0)=0\in\mathbb{R}^2$ and $\mathcal{H}^1(\gamma_1([0,s]))=\frac{4}{3}s$. It results that $\gamma_1([(i-1)/4,i/4])=l_1^i$ for $i=1,\dots,4$.

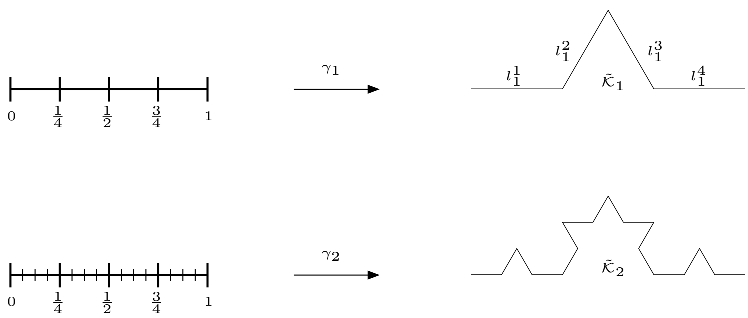

Figure 1. The first and second iterations in the construction of the natural parametrization γ of the von Koch curve.

Iteratively construct the set $\tilde{\mathcal{K}}_2=\bigcup_{i=1,\dots,4}S_i(\tilde{\mathcal{K}}_1)$ and its continuous parametrization $\gamma_2:[0,1]\to\mathbb{R}^2$ such that $\gamma_2(0)=0\in\mathbb{R}^2$, $\mathcal{H}^1(\gamma_2([0,s]))=\left(\frac{4}{3}\right)^2 s$ and $\gamma_2([(i-1)/4^2,i/4^2])=l_2^i$ for $i=1,\dots,4^2$.

It results that for any $n\in\mathbb{N}$ it is

$$\|\gamma_n-\gamma_{n+1}\|_\infty=\frac{1}{3^{n+1}}\frac{\sqrt{3}}{2}$$

and, as consequence, for any $n,j\in\mathbb{N}$ we have

$$\|\gamma_n-\gamma_{n+j}\|_\infty\le\frac{1}{3^n}\frac{3\sqrt{3}}{4}.$$

Therefore the sequence γ_n is a Cauchy sequence in $(C([0,1];\mathbb{R}^2),\|\cdot\|_\infty)$, and there exists a continuous function $\gamma:[0,1]\to\mathbb{R}^2$ such that

$$\gamma_n\to\gamma \tag{8.1}$$

uniformly on $[0,1]$.

The sequence of compact sets $\tilde{\mathcal{K}}_n$ converges in the Hausdorff metric to the von Koch curve $\mathcal{K}$. This fact, together with the uniform convergence (8.1), implies that $\gamma([0,1])=\mathcal{K}$.

It can be proved that $\mathcal{K}$ has Hausdorff dimension

$$d := \frac{\log 4}{\log 3}$$

and $0 < \mathcal{H}^d(\mathcal{K}) < +\infty$.

The map γ we just obtained corresponds to the one that in [33] is called *natural parametrization.* The following result shows that γ fulfils (2.1) and (2.2).

Proposition 8.1. *There exists a constant $c > 0$ such that for any $a, b \in [0,1]$ the natural parametrization γ satisfies*

$$\frac{1}{c}|a-b|^{1/d} \le |\gamma(a) - \gamma(b)| \le c|a-b|^{1/d} \tag{8.2}$$

and, for $a < b$,

$$\mathcal{H}^d(\gamma(a,b)) = (b-a)\mathcal{H}^d(\mathcal{K})\,.$$

Proof. The first statement is proved in [33, Theorem 1].

Concerning the second fact, firstly note that, by construction, the von Koch curve $\mathcal{K}$ and the parametrization γ have the following self-similarity property: for every $n \in \mathbb{N}$ and $j = 1, \dots, 4^n - 1$ there exists an affine isometry $\Phi_n^j : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\Phi_n^j\left(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\right) = \gamma\Big(0, \frac{1}{4^n}\Big)\,.$$

For any $s, h \in [0,1]$ let $i_n^s, i_n^h \in \{1, \dots, 4^n\}$ be such that

$$\frac{i_n^s}{4^n} \le s < \frac{i_n^s + 1}{4^n} \quad \text{and} \quad \frac{i_n^h}{4^n} \le h < \frac{i_n^h + 1}{4^n}\,.$$

For n sufficiently large (so that $i_n^h \ge 2$) it is

$$(s, s+h) = (s, (i_n^s+1)/4^n) \cup [(i_n^s+1)/4^n, (i_n^s+i_n^h)/4^n] \cup ((i_n^s+i_n^h)/4^n, s+h)\,.$$

Then, being the Φ_n^j Lipschitz continuous maps with Lipschitz constant equal to 1, we have

$$\begin{aligned}
\mathcal{H}^d(\gamma(s,s+h)) =& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\Big) \\
&+ \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\Phi_n^j\Big(\gamma\Big(\frac{j}{4^n}, \frac{j+1}{4^n}\Big)\Big)\Big) \\
&+ \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + \sum_{j=i_n^s+1}^{i_n^s+i_n^h-1} \mathcal{H}^d\Big(\gamma\Big(0, \frac{1}{4^n}\Big)\Big) + \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big) \\
=& \mathcal{H}^d\Big(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\Big) + (i_n^h - 2)\mathcal{H}^d\Big(\gamma\Big(0, \frac{1}{4^n}\Big)\Big) + \mathcal{H}^d\Big(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\Big).
\end{aligned}$$

Since γ is $(1/d)$-Hölder continuous by (8.2), it holds that

$$\mathcal{H}^d\left(\gamma\Big(s, \frac{i_n^s+1}{4^n}\Big)\right) \le C(d)\left(\frac{i_n^s+1}{4^n} - s\right) \le C(d)\frac{1}{4^n} \to 0$$

and

$$\mathcal{H}^d\left(\gamma\Big(\frac{i_n^s+i_n^h}{4^n}, s+h\Big)\right) \le C(d)\left(s+h - \frac{i_{i_n^s} + N_n^h}{4^n}\right) \le 2C(d)\frac{1}{4^n} \to 0$$

as $n \to +\infty$, with $C(d)$ independent of t and h. Hence we obtain

$$\begin{aligned}\mathcal{H}^d(\gamma(s,s+h)) &= \lim_{n\to+\infty}(i_n^h-2)\mathcal{H}^d\left(\gamma\Big(0,\frac{1}{4^n}\Big)\right)\\ &= \lim_{n\to+\infty}(1_n^h-2)\frac{1}{4^n}\mathcal{H}^d\left(\gamma(0,1)\right) = h\mathcal{H}^d(\mathcal{K}),\end{aligned}$$

where, in the second equality, we used the self-similiarity property of $\mathcal{K}$, that is, $\mathcal{K} = \gamma([0,1])$ contains exactly 4^n distinct copies of $\gamma([0,1/4^n])$.

Consider now $0 \leq a < b \leq 1$. Set $s = a$ and $h = b - a$ in the above argument, the thesis follows. □

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