Comparison principles, uniqueness and symmetry results of solutions of quasilinear elliptic equations and inequalities

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Abstract

We prove comparison principles, uniqueness, regularity and symmetry results for p-regular distributional solutions of quasilinear very weak elliptic equations of coercive type and to related inequalities. The simplest model examples are

$$-\Delta_p u + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

where $q > p-1 > 0$ and

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

with $q > 0$ and $h \in L^1_{loc}(\mathbb{R}^N)$.

Contents

1 Introduction

Nonlinear elliptic problems of coercive type is still a subject of vital interest in the PDE community. As it is well known, coercive problems have they roots in the classical calculus of variations and precisely in the problems related to the existence of minima for convex functional.

In a celebrated paper [3], Boccardo, Gallouet and Vazquez studied, among other things, the simplest canonical quasilinear problem with non regular data,

$$-\Delta_p u + |u|^{q-1}\, u = h \quad on \quad \mathbb{R}^N, \tag{1}$$

where $q > p-1 > 0$ and $h \in L^1_{loc}(\mathbb{R}^N)$.

An earlier and important contribution to this problem in the case $p = 2$, was obtained by Brezis [4]. Indeed he proved that for any $h \in L^1_{loc}(\mathbb{R}^N)$ the semilinear equation (1) has a unique distributional solution $u \in L^q_{loc}(\mathbb{R}^N)$.

For the general case $p > 1$ existence results have been obtained later in [3]. More precisely, by using a clever approximation procedure, the authors of [3] proved that, if $q > p-1$ and $p > 2 - \dfrac{1}{N}$, then for any $h \in L^1_{loc}(\mathbb{R}^N)$ the equation (1) admits a distributional solution belonging to $X := \{\, u \in W^{1,1}_{loc}(\mathbb{R}^N) \,:\, |\nabla u|^{p-1} \in L^1_{loc}(\mathbb{R}^N),\ |u|^q \in L^1_{loc}(\mathbb{R}^N)\}$.

No general results about uniqueness were claimed in that paper.

In the present work, we shall study the uniqueness problem of solutions of (1) and related qualitative properties. We emphasize that we shall prove the uniqueness of solutions of (1) in the space $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. To this end, first we set up two essential tools which are of independent interest. Namely, the regularity of distributional solutions of (1) in the space $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ and comparison results on $\mathbb{R}^N$ for related inequalities.

Actually, the approach we propose in this article is very general and it applies to a wide family of weakly elliptic operators and to related inequalities. Indeed the same results are obtained for equations involving the mean curvature operator as well as for more degenerate problems of coercive type.

The main results (in their simplest form) proved in this paper are the following.

Theorem 1.1 *Let $1 < p < 2$, $q \geq 1$, $h \in L^1_{loc}(\mathbb{R}^N)$, then the problem*

$$-\Delta_p u + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

has at most one distributional solution $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. Moreover,

$$\inf_{\mathbb{R}^N} h \leq |u|^{q-1} u \leq \sup_{\mathbb{R}^N} h.$$

Theorem 1.2 *Let $q \geq 1$, $h \in L^1_{loc}(\mathbb{R}^N)$ then the problem*

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

has at most one distributional solution $u \in W^{1,1}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. Moreover,

$$\inf_{\mathbb{R}^N} h \leq |u|^{q-1} u \leq \sup_{\mathbb{R}^N} h.$$

The above results are based on the following comparison principle

Theorem 1.3

1. *Let $1 < p < 2$ and $q \geq 1$. Let $u, v \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ such that*

$$\Delta_p v - |v|^{q-1} v \geq \Delta_p u - |u|^{q-1} u \qquad in \quad \mathcal{D}'(\mathbb{R}^N). \tag{2}$$

Then $v \leq u$ a.e. on $\mathbb{R}^N$.

2. *Let $q \geq 1$. Let $u, v \in W^{1,1}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ such that*

$$\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) - |v|^{q-1} v \geq \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - |u|^{q-1} u \qquad in \quad \mathcal{D}'(\mathbb{R}^N). \tag{3}$$

Then $v \leq u$ a.e. on $\mathbb{R}^N$.

The above theorem is a special case of more general comparison principles (cfr. Theorem 4.1 and Theorem 5.4). Indeed, the methods we develop in this article enable us to treat very degenerate weakly elliptic equations as well as problems involving data which are merely distributions. A simple example which well illustrates the situation is contained in the following result.

Theorem 1.4 *Assume either $1 < p < 2$, $q \geq 1$ or $p = 2$, $q > 1$ and let $T \in \mathcal{D}'(\mathbb{R}^N)$ be a distribution. Let a and b two measurable, nonnegative, locally bounded functions satisfying*

$$a + b > 0 \qquad \text{in} \qquad \mathbb{R}^{\mathrm{N}} \tag{4}$$

$$a(x) \leq C\,|x|^r, \qquad b(x) \geq C\,|x|^{-t} \quad \text{for} \quad |x| \geq R_0 > 0, \tag{5}$$

where $r, t \in \mathbb{R}$ and C is a positive constant.
Suppose that either

$$r + t < p \qquad \textit{or} \qquad r + t > p \quad \& \quad \frac{r+t-p}{q-p+1} < \frac{t-N}{q}. \tag{6}$$

Then the problem

$$-\operatorname{div}[a(x)\,|\nabla u|^{p-2}\nabla u] + b(x)|u|^{q-1}\,u = T \quad \textit{on} \quad \mathbb{R}^N \tag{7}$$

can have at most one distributional solution u of class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$.
Moreover, if T is a nonnegative distribution, then $u \geq 0$ a.e. on $\mathbb{R}^N$.

We explicitly note that in the above equation (7) the function a can vanish identically on any measurable subset of $\mathbb{R}^N$, possible the entire space $\mathbb{R}^N$. Also, we observe that (5) is required only outside a fixed ball centered at the origin (whose radius R_0 could be arbitrary large) and that the exponents r and t are real numbers. Hence, equation (7) can be *very* degenerate or singular.

We also emphasize that, when $r + t > p$, the second condition in (6) is essentially necessary in order to obtain the desired conclusion. Indeed, by following [9, Example 3, section 11], for all $N \geq 1$, $1 < p < 2$, $q \geq 1$ (or $p = 2$ and $q > 1$) we can construct nonnegative, continuous functions a and b satisfying (4), (5) and $r + t > p$ with $\frac{r+t-p}{q-p+1} > \frac{t-N}{q}$ such that the equation $-\operatorname{div}[a(x)\,|Du|^{p-2}Du] + b(x)|u|^{q-1}\,u = 0$ admits an explicit positive solution of class $C^\infty(\mathbb{R}^N)$. Since $u \equiv 0$ is also a solution, we see that both uniqueness and the comparison principle do not hold true in this situation.

For a full discussion of this example and other related topics we refer the reader to the Appendix B.

We also point out that the above result is new even in the semilinear case $p = 2$. Indeed, it extends Brezis' uniqueness result [4] in many respects:

1. we replace the standard Laplace operator by a non homogeneous operator with possibly discontinuous and unbounded coefficients (weighted p-Laplacian operator),

2. we allow non homogeneous and possibly discontinuous nonlinear terms,

3. the equation may be very degenerate or singular, as discussed above,

4. we establish uniqueness for any distribution T and not only for distributions in L^1_{loc}.

Another consequence of our methods is the following

Theorem 1.5 (Symmetry) *Assume either* $1 < p < 2$, $q \geq 1$ *or* $p = 2$, $q > 1$ *and let* $h \in L^1_{loc}(\mathbb{R}^N)$. *Let* u *be a distributional solution of class* $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ *of the equation*

$$-\Delta_p u + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N.$$

Then

i) if h *depends only on* $k < N$ *variables (say,* $x_1, ..., x_k$*) then, also* u *depends only on* $x_1, ..., x_k$.

ii) if h *is periodic then, also* u *is periodic.*

iii) if h *is symmetric with respect to an affine hyperplane then, also* u *has the same symmetry.*

iv) if h *is radially symmetric, then also* u *is radially symmetric.*

Theorems 1.4 and 1.5 hold true also for the mean curvature operator. We refer the interested reader to section 5 for this case, as well as for some more general operators.

The situation is more involved when $p > 2$. Some results in this case are presented in Section 6. The following are two simple examples of them

Theorem 1.6 *Let* $q > p - 1 > 1$ *and let* $h \in L^\infty(\mathbb{R}^N)$. *Then there exists exactly one distributional solution of the problem*

$$-\Delta_p u + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

within the class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. *Moreover,*

$$\inf_{\mathbb{R}^N} h \leq |u|^{q-1} u \leq \sup_{\mathbb{R}^N} h$$

and $u \in W^{1,\infty}(\mathbb{R}^N) \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$, *for some* $\beta \in (0, 1)$.

We note that symmetry results (in the spirit of those proved in Theorem 1.5) hold in the context of the above Theorem 1.6.

Theorem 1.7 *Let $p > 2$, $q > 1$. If $h \in L^1_{loc}(\mathbb{R}^N)$ then the problem,*

$$-\Delta_p u + |u|^{q-1} u = h \quad on \quad \mathbb{R}^N,$$

has at most one distributional solution v in the class,

$$\left\{ u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N) : there\ exists\ \theta < \frac{1}{p-2}\ such\ that\ |\nabla u(x)| \leq c\,|x|^\theta\ \ for\ \ |x|\ \ large \right\}.$$

Moreover,

$$\inf_{\mathbb{R}^N} h \leq |v|^{q-1} v \leq \sup_{\mathbb{R}^N} h.$$

Our uniqueness results concern solutions that belong to the class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. Of course, this set is contained in the space X considered in [3]. However we point out that, when dealing with uniqueness results additional regularity is usually required by several authors. See for instance [1]. Indeed, in that paper the authors obtain the existence of solutions of problem (1) belonging to a certain space $\mathcal{T}^{1,p}_0(\mathbb{R}^N)$. The uniqueness result proved in [1] concerns entropy solutions. On the other hand, more regularity on h implies that the solutions belong to the natural class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$, see Theorem 6.14.

Finally, we wish to emphasize that our effort in this paper is to develop a quite general technique. Indeed our approach is essentially based on a careful choice of test functions and on a variant of Kato's inequality (see Theorem 3.4). Several other situations can be studied by our method. For instance, subelliptic equation on Carnot groups. To avoid cumbersome notations, we have decided to present here the results only in the Euclidean framework. However, the interested reader, can be see the Carnot group case in a preliminary version of this paper [8].

The paper is organized as follow. In the next section we describe the setting and the notations. We also give many examples showing the broadness of the considered class of weakly elliptic operators as well as the generality of our approach. In Section 3 we prove some a priori estimates on the solutions of the problems. Sections 4, 5 and 6 are devoted to prove the comparison principles and to derive some of their consequences (uniqueness and symmetry).

In the Appendix A we prove some inequalities that guaranty that an operator is an **M**-p-**C** operator, which plays a central role in some of our results. In the Appendix B we provide and discuss some counterexamples showing the sharpness of our results.

2 Notations, definitions and examples

In this paper Ω will always denote an open subset of $\mathbb{R}^N$, $N \geq 1$. B_R will denote the open ball centered at the origin and of radius $R > 0$ and $A_R := B_{2R} \setminus \overline{B_R}$. By $\mathbf{1}_E$ we shall indicate the characteristic function of the subset $E \subset \mathbb{R}^N$. ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

In what follows we shall assume that $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \operatorname{div}\left(\mathscr{A}(x, u(x), \nabla u(x))\right).$$

Our canonical model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 2.3 below.

Definition 2.1 *Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function. The function $\mathscr{A}$ is called* weakly elliptic *if it generates a weakly elliptic operator L i.e.*

$$\begin{aligned} &\mathscr{A}(x, t, \xi) \cdot \xi \geq 0 \quad \textit{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N, \\ &\mathscr{A}(x, t, 0) = 0 \quad \textit{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R}. \end{aligned} \tag{WE}$$

Let $p \geq 1$, the function $\mathscr{A}$ is called **W**-*p*-**C** *(weakly-p-coercive) (see [2] for $p > 1$), if $\mathscr{A}$ is (WE) and it generates a weakly-p-coercive operator L, i.e. if there exists a constant $k_2 > 0$ such that*

$$(\mathscr{A}(x, t, \xi) \cdot \xi)^{p-1} \geq k_2^p \, |\mathscr{A}(x, t, \xi)|^p \ \text{ for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N. \tag{$\mathbf{W}$-p-$\mathbf{C}$}$$

Let $p > 1$, the function $\mathscr{A}$ is called **S**-*p*-**C** *(strongly-p-coercive) (see [15, 2, 13]), if there exist $k_1, k_2 > 0$ constants such that*

$$(\mathscr{A}(x, t, \xi) \cdot \xi) \geq k_1 \, |\xi|^p \geq k_2^{p'} \, |\mathscr{A}(x, t, \xi)|^{p'} \ \text{ for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N. \tag{$\mathbf{S}$-p-$\mathbf{C}$}$$

Definition 2.2 *Let Ω be an open set of $\mathbb{R}^N$, $T \in \mathcal{D}'(\Omega)$ be a distribution. Let $\mathscr{A}$ be a* weakly elliptic *function and assume $p \geq 1$. By a* p-regular solution *(see [14], Sect. 3.1), of*

$$\operatorname{div}\left(\mathscr{A}(x, u, \nabla u)\right) \geq T \qquad \textit{on} \ \ \Omega, \tag{8}$$

we will understand a function $u \in W^{1,p}_{loc}(\Omega)$ such that $\mathscr{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega)$ and the inequality (8) *is satisfied in the sense of distributions.*

Also, given $S, T \in \mathcal{D}'(\Omega)$, we shall say that u, v are p-regular solutions *of*

$$\operatorname{div}\left(\mathscr{A}(x,v,\nabla v)\right) - S \geq \operatorname{div}\left(\mathscr{A}(x,u,\nabla u)\right) - T \qquad \textit{on} \quad \Omega, \tag{9}$$

if $u, v \in W^{1,p}_{loc}(\Omega)$*,* $\mathscr{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega), \mathscr{A}(\cdot, v, \nabla v) \in L^{p'}_{loc}(\Omega)$ *and the inequality* (9) *is satisfied in the sense of distributions.*

Clearly, when the distribution T in (8) is a member of L^1_{loc}, say $f \in L^1_{loc}(\Omega)$, inequality (8) is equivalent to

$$\forall\, \phi \in \mathscr{C}^1_0(\Omega), \quad \phi \geq 0, \qquad -\int_\Omega \mathscr{A}(x,u,\nabla u)\cdot\nabla\phi \geq \int_\Omega f(x)\phi. \tag{10}$$

Here, and in the sequel, $\mathscr{C}^1_0(\Omega)$ denotes the set of functions of class $\mathscr{C}^1$ with compact support in Ω.

A similar remark applies when u and v are solutions of inequality (9) with S and T distributions such that $S - T$ is a member of L^1_{loc}.

Finally we observe that, when $\mathscr{A}$ is **W**-p-**C**, any distributional solution $u \in W^{1,p}_{loc}(\Omega)$ is automatically p-regular, since $|\mathscr{A}(x,t,\xi)| \leq k_2^{-p}|\xi|^{p-1}$ for each $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

Example 2.3 *1. Let $p > 1$. The p-Laplacian operator*

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$$

is the operator generated by $\mathscr{A}(x,t,\xi) := |\xi|^{p-2}\,\xi$. It is **S**-$p$-**C**.

2. *If $\mathscr{A}$ is of mean curvature type, that is $\mathscr{A}$ can be written as $\mathscr{A}(x,t,\xi) := A(|\xi|)\xi$ with $A : \mathbb{R}_+ \to \mathbb{R}$ a positive bounded continuous function (see [12, 2]), then $\mathscr{A}$ is* **W**-2-**C**.

3. *The mean curvature operator in non parametric form*

$$Tu := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right),$$

is generated by $\mathscr{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$. In this case $\mathscr{A}$ is **W**-p-**C** *with $1 \leq p \leq 2$ and of mean curvature type but it is not* **S**-2-**C**.

4. *Let $m > 1$. The operator*

$$T_m u := \operatorname{div}\left(\frac{|\nabla u|^{m-2}\,\nabla u}{\sqrt{1+|\nabla u|^m}}\right)$$

is **W**-p-**C** *for $m \geq p \geq m/2$ (see [7, Examples 1.3]).*

Definition 2.4 *Let* $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ *be a* weakly elliptic *function.*
We say that $\mathscr{A}$ *is monotone if*

$$(\mathscr{A}(x,u,\xi) - \mathscr{A}(x,v,\eta)) \cdot (\xi - \eta) \geq 0 \qquad \forall\, x \in \mathbb{R}^N,\ \forall\, u, v \in \mathbb{R},\ \forall\, \xi, \eta \in \mathbb{R}^N. \tag{11}$$

Let $p \geq 1$. *We say that* $\mathscr{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *is* **M**-p-**C** *(monotone* p*-coercive) if* $\mathscr{A}$ *is monotone and if there exists* $k_2 > 0$ *such that*

$$((\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta))^{p-1} \geq k_2^p\, |\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)|^p \qquad \forall\, x \in \mathbb{R}^N,\ \forall\, \xi, \eta \in \mathbb{R}^N. \tag{12}$$

Example 2.5 *1. Let* $1 < p \leq 2$ *the function* $\mathscr{A}(\xi) := |\xi|^{p-2}\,\xi$ *is* **M**-p-**C** *(see Appendix A for details).*
2. The mean curvature operator is **M**-p-**C** *with* $1 \leq p \leq 2$ *(see Appendix A).*

In section 5, inspired by the structural conditions introduced in [9], we shall study more general operators and equations. These operators are generated by weakly elliptic functions $\mathscr{A}$ described in the following

Definition 2.6 *Let* $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *be a* monotone *function. We say that* $\mathscr{A}$ *satisfies the* (p,r)-large radii condition *if there exist* $p \geq 1$, $r \in \mathbb{R}$, $R_0 > 0$, $k_2 > 0$ *such that*

$$|x|^r\,((\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta))^{p-1} \geq k_2^p\, |\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)|^p\,. \tag{13}$$

for a.e. $x \in \mathbb{R}^N$ *such that* $|x| > R_0$ *and all* $\xi, \eta \in \mathbb{R}^N$.

Any **M**-p-**C** function $\mathscr{A}$ satisfies the (p,r)*-large radii condition* with $r = 0$ and any $R_0 > 0$. On the other hand, there are weakly elliptic functions $\mathscr{A}$ satisfying definition (2.6) and which are not **M**-p-**C**. Here are some simple examples:

Example 2.7 *1. Let* $r \in \mathbb{R}$, $R_0 > 0$ *and let* a *be a measurable, nonnegative, locally bounded functions satisfying*

$$a(x) \leq C\,|x|^r \qquad \text{for} \quad |x| \geq R_0 \tag{14}$$

for some constant $C > 0$. *Then the* weighted p-Laplacian operator

$$Tu := \operatorname{div}[a(x)\,|\nabla u|^{p-2}\nabla u]$$

and the weighted pseudo p-Laplacian operator

$$Tu := \sum_{j=1}^N \partial_j(a(x)|\partial_j u|^{p-2}\partial_j u)$$

satisfy the (p,r)-large radii condition *with* $1 < p \le 2$.

The weighted mean curvature operator

$$Tu := \operatorname{div}\left(\frac{a(x)\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

satisfy the (p,r)-large radii condition *with* $1 \le p \le 2$.

More generally, it is easily seen that for any $p \ge 1$, *for any function* a *satisfying* (14) *and for any* $\mathscr{A}$ *which is* $\mathbf{M}$-p-$\mathbf{C}$, *the function*

$$\mathscr{A}_a(x,\xi) = a(x)\mathscr{A}(x,\xi)$$

satisfies the (p,r)-large radii condition.

2. **Variational example -** *Assume* $r \in \mathbb{R}$ *and* $R_0 > 0$. *Let* $\mathcal{G} = \mathcal{G}(x,\xi)$ *be* $C^1(\mathbb{R}^N \times \mathbb{R}^N)$ *and convex with respect to* ξ. *The operator*

$$Tu := \operatorname{div}\left[\partial_\xi \mathcal{G}(x,\nabla u)\right]$$

satisfies the $(1,r)$-large radii condition *if*

$$\partial_\xi \mathcal{G}(x,0) = 0, \qquad \forall\, x \in \mathbb{R}^N$$
$$|\partial_\xi \mathcal{G}(x,\eta)| \le C\,|x|^r \qquad \text{for} \quad |x| > R_0, \quad \forall\, \eta \in \mathbb{R}^N,$$

where C *is a positive constant.*

We also observe that for all $x \in \mathbb{R}^N$

$$(\partial_\xi \mathcal{G}(x,\xi) - \partial_\xi \mathcal{G}(x,\eta)) \cdot (\xi - \eta) > 0 \qquad \forall\, \xi,\eta \in \mathbb{R}^N, \quad \xi \ne \eta, \tag{15}$$

whenever $\mathcal{G}$ *is strictly convex with respect to* ξ.

3. *Let* $R_0 > 0$, $p_1 > 1$ *and* $1 < p_2 \le 2$ *with* $p_1 \ne p_2$. *The function*

$$\mathscr{A}_{p_1,p_2}(x,\xi) = |\xi|^{p_1-2}\,\xi\,\mathbf{1}_{\{|x|\le R_0\}} + |\xi|^{p_2-2}\,\xi\,\mathbf{1}_{\{|x|>R_0\}},$$

satisfies the $(p_2,0)$-large radii condition *but it is not* $\mathbf{M}$-p_2-$\mathbf{C}$.

We also observe that for all $x \in \mathbb{R}^N$

$$(\mathscr{A}_{p_1,p_2}(x,\xi) - \mathscr{A}_{p_1,p_2}(x,\eta)) \cdot (\xi - \eta) > 0 \qquad \forall\, \xi,\eta \in \mathbb{R}^N, \quad \xi \ne \eta, \tag{16}$$

and that, if $1 < p_1 < p_2 \le 2$ *and* $u \in W^{1,p_2}_{loc}$, *then* $\mathscr{A}_{p_1,p_2}(\cdot,u) \in L^{p_2'}_{loc}$.

4. **Sum of good operators -** *Let* $\mathscr{A}_1, ..., \mathscr{A}_k$ *be functions satisfying the* $(p, r(k))$-large radii condition *with the same* $p \ge 1$. *Then the sum* $\sum_{j=1}^k \mathscr{A}_j$ *satisfies the* (p,r)-large radii condition *with* $r = \max\{r(1), ..., r(k)\}$ *and* $R_0 = \max\{1, R_0(1), ..., R_0(k)\}$.

Note that, if one of the functions $\mathscr{A}_j$ *satisfies* (16), *then the sum* $\sum_{j=1}^k \mathscr{A}_j$ *also satisfies* (16).

3 A priori estimates

The following results are variations of a result proved in [7].

We shall consider the following inequality,

$$\operatorname{div}(\mathscr{A}(x,\nabla v)) - f \geq \operatorname{div}(\mathscr{A}(x,\nabla u)) - g \qquad on\ \Omega. \tag{17}$$

Theorem 3.1 *Let $\mathscr{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ be $\mathbf{M}$-p-$\mathbf{C}$. Let $f,g\in L^1_{loc}(\Omega)$ and let u,v be p-regular solutions of (17). Set $w:=(v-u)^+$ and let $s>0$. If $(f-g)w\geq 0$ and $w^{s+p-1}\in L^1_{loc}(\Omega)$, then*

$$(f-g)w^s,\quad (\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u))\cdot\nabla w\ w^{s-1}\,\mathbf{1}_{\{w>0\}}\in L^1_{loc}(\Omega). \tag{18}$$

Moreover, for any nonnegative $\phi\in\mathscr{C}^1_0(\Omega)$ we have,

$$\int_\Omega (f-g)w^s\phi + c_1 s\int_{\Omega\cap\{w>0\}}(\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u))\cdot\nabla w\ w^{s-1}\phi \leq c_2 s^{1-p}\int_\Omega w^{s+p-1}\frac{|\nabla\phi|^p}{\phi^{p-1}}, \tag{19}$$

where $c_1 = 1-\frac{p-1}{p}\left(\frac{\epsilon}{k_2}\right)^{\frac{p}{p-1}}>0$, $c_2=\frac{p^p}{p\epsilon^p}$ and $\epsilon>0$ is sufficiently small for $p>1$ and $c_1=1$ and $c_2=1/k_2$ for $p=1$.

Remark 3.2 *i) Notice that from the above result it follows that if u,v are p-regular solutions of (17), then $(f-g)\,w\in L^1_{loc}(\Omega)$.*

ii) If u,v are p-regular solutions of (17) and u is a constant, then Theorem 3.1 still holds even for $\mathbf{W}$-p-$\mathbf{C}$ operators. See the following lemma.

Lemma 3.3 *Let $\mathscr{A}$ be $\mathbf{W}$-p-$\mathbf{C}$. Let $f,g\in L^1_{loc}(\Omega)$ and let v be a p-regular solution of*

$$\operatorname{div}(\mathscr{A}(x,v,\nabla v))\geq f-g \qquad on\ \ \Omega. \tag{20}$$

Let $k>0$ and set $w:=(v-k)^+$ and let $s>0$. If $(f-g)w\geq 0$ and $w^{s+p-1}\in L^1_{loc}(\Omega)$, then

$$(f-g)w^s,\quad \mathscr{A}(x,v,\nabla v)\cdot\nabla w\ w^{s-1}\mathbf{1}_{\{w>0\}}\in L^1_{loc}(\Omega) \tag{21}$$

and for any nonnegative $\phi\in\mathscr{C}^1_0(\Omega)$ we have,

$$\int_\Omega (f-g)w^s\phi + c_1 s\int_{\Omega\cap\{w>0\}}\mathscr{A}(x,v,\nabla v)\cdot\nabla w\ w^{s-1}\phi \leq c_2 s^{1-p}\int_\Omega w^{s+p-1}\frac{|\nabla\phi|^p}{\phi^{p-1}}, \tag{22}$$

where c_1 and c_2 are as in Theorem 3.1.

The above results lie on the following refinement of a result proved in [7, Theorem 2.7].

Theorem 3.4 ([7]) *Let $\mathscr{A}:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ be a monotone function. Let $f,g\in L^1_{loc}(\Omega)$ and let u,v be p-regular solutions of*

$$\operatorname{div}(\mathscr{A}(x,v,\nabla v))-f\geq\operatorname{div}(\mathscr{A}(x,u,\nabla u))-g\qquad\text{on }\Omega.\tag{23}$$

1. *Let $\gamma\in\mathscr{C}^1(\mathbb{R})$ be such that $0\leq\gamma(t),\gamma'(t)\leq M$, then for any nonnegative $\phi\in\mathscr{C}^1_0(\Omega)$ we have,*

$$-\int_\Omega(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u))\cdot\nabla\phi\ \gamma(v-u)\geq\tag{24}$$

$$\geq\int_\Omega\gamma'(v-u)\ (\nabla v-\nabla u)\cdot(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u)\,\phi\tag{25}$$

$$+\int_\Omega\phi\gamma(v-u)(f-g)\quad\text{on }\Omega.\tag{26}$$

Hence

$$\operatorname{div}(\gamma(v-u)(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u)))\geq\gamma(v-u)(f-g)\quad\text{on }\Omega.$$

Moreover[1]

$$\operatorname{div}\big(\operatorname{sign}^+(v-u)(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u))\big)\geq\operatorname{sign}^+(v-u)(f-g)\quad\text{on }\Omega.\tag{27}$$

2. *Let $\gamma\in\mathscr{C}^1(\mathbb{R}_+)$ be such that γ,γ' are bounded, $\gamma\geq 0$, then for any nonnegative $\phi\in\mathscr{C}^1_0(\Omega)$ we have,*

$$-\int_\Omega(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u))\cdot\nabla\phi\ \gamma(w)\geq\tag{28}$$

$$\geq\int_\Omega\gamma'(w)\ \nabla w\cdot(\mathscr{A}(x,v,\nabla v)-\mathscr{A}(x,u,\nabla u)\,\phi\tag{29}$$

$$+\int_\Omega(f-g)\gamma(w)\phi\quad\text{on }\Omega,\tag{30}$$

where $w:=(v-u)^+$.

Proof. It is enough to notice that one can choose any nonnegative function of class $W^{1,p}_{comp}(\Omega)\cap L^\infty(\Omega)$ as test function in (23). The claim then follows by choosing $\gamma(v-u)\phi$ and $\gamma((v-u)^+)\phi$ respectively.

□

Proof of Theorem 3.1. Let $\gamma\in\mathscr{C}^1(\mathbb{R}_+)$ be a bounded nonnegative function with bounded nonnegative first derivative and let $\phi\in\mathscr{C}^1_0(\Omega)$ be a nonnegative test function.

[1]We recall that the function sign^+ is defined as $\operatorname{sign}^+(t):=0$ if $t\leq 0$ and $\operatorname{sign}^+(t):=1$ otherwise.

For simplicity we shall omit the arguments of $\mathscr{A}$. So we shall write $\mathscr{A}_u$ and $\mathscr{A}_v$ instead of $\mathscr{A}(x,\nabla u)$ and $\mathscr{A}(x,\nabla v)$ respectively.

Applying Theorem 3.4, we obtain

$$\begin{aligned}\int_\Omega (f-g)\gamma(w)\phi + \int_\Omega (\mathscr{A}_v-\mathscr{A}_u)\cdot\ \nabla w\ \gamma'(w)\phi &\le -\int_\Omega (\mathscr{A}_v-\mathscr{A}_u)\cdot\nabla\phi\ \gamma(w)\\ &\le \int_\Omega |\mathscr{A}_v-\mathscr{A}_u|\ |\nabla\phi|\,\gamma(w) \qquad (31)\end{aligned}$$

Let $p>1$. From (31) we have

$$\begin{aligned}&\int_\Omega (f-g)\gamma(w)\phi + \int_\Omega (\mathscr{A}_v-\mathscr{A}_u)\cdot\ \nabla w\ \gamma'(w)\phi \le\\ &\le \left(\int_\Omega |\mathscr{A}_v-\mathscr{A}_u|^{p'}\gamma'(w)\phi\right)^{1/p'}\left(\int_\Omega \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla\phi|^p}{\phi^{p-1}}\right)^{1/p}\\ &\le \frac{\epsilon^{p'}}{p'k_2^{p'}}\int_\Omega (\mathscr{A}_v-\mathscr{A}_u)\cdot\nabla w\ \gamma'(w)\phi + \frac{1}{p\epsilon^p}\int_\Omega \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla\phi|^p}{\phi^{p-1}},\end{aligned}$$

where $\epsilon>0$ and all integrals are well defined provided $\frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\in L^1_{loc}(\Omega)$. With a suitable choice of $\epsilon>0$, for any nonnegative $\phi\in\mathscr{C}^1_0(\Omega)$ and $\gamma\in\mathscr{C}^1(\mathbb{R}_+)$ as above such that $\frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\in L^1_{loc}(\Omega)$, it follows that,

$$\int_\Omega (f-g)\gamma(w)\phi + c_1\int_\Omega (\mathscr{A}_v-\mathscr{A}_u)\cdot\nabla w\ \gamma'(w)\phi \le \frac{1}{p\epsilon^p}\int_\Omega \frac{\gamma(w)^p}{\gamma'(w)^{p-1}}\frac{|\nabla\phi|^p}{\phi^{p-1}}. \qquad (32)$$

Now for $s>0$, $1>\delta>0$ and $n\ge 1$, define

$$\gamma_n(t) := \begin{cases}(t+\delta)^s & if\ 0\le t<n-\delta,\\ cn^s-\dfrac{s}{\beta-1}n^{\beta+s-1}(t+\delta)^{1-\beta} & if\ t\ge n-\delta,\end{cases} \qquad (33)$$

where $c:=\frac{\beta-1+s}{\beta-1}$ and $\beta>1$ will be chosen later. Clearly $\gamma_n\in\mathscr{C}^1$,

$$\gamma_n'(t) = \begin{cases}s(t+\delta)^{s-1} & if\ 0\le t<n-\delta,\\ sn^{\beta+s-1}(t+\delta)^{-\beta} & if\ t\ge n-\delta,\end{cases}$$

and γ_n, γ_n' are nonnegative and bounded with $||\gamma_n||_\infty = cn^s$ and $||\gamma_n'||_\infty = sn^{s-1}$. Moreover

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} = \begin{cases}s^{1-p}(t+\delta)^{s+p-1} & for\ t<n-\delta,\\ \theta(t,n) & for\ t\ge n-\delta,\end{cases}$$

where

$$\theta(t,n) := \frac{(cn^s - \frac{s}{\beta-1} n^{\beta+s-1}(t+\delta)^{1-\beta})^p}{(sn^{\beta+s-1}(t+\delta)^{-\beta})^{p-1}} \le (cn^s)^p s^{1-p} n^{-(\beta+s-1)(p-1)} (t+\delta)^{\beta(p-1)}.$$

Choosing $\beta := \frac{s+p-1}{p-1}$ we have $c = p$, and

$$\theta(t,n) \le p^p s^{1-p} n^{sp-(\beta+s-1)(p-1)} (t+\delta)^{s+p-1} = p^p s^{1-p} (t+\delta)^{s+p-1}.$$

Therefore, for $t \ge 0$ we have,

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} \le p^p s^{1-p} (t+\delta)^{s+p-1}.$$

Since by assumption $w^{s+p-1} \in L^1_{loc}(\Omega)$, from (32) with $\gamma = \gamma_n$, it follows that

$$\int_\Omega (f-g)\gamma_n(w)\phi + c_1 \int_\Omega (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla w \ \gamma_n'(w)\phi \le \frac{p^p s^{1-p}}{p\epsilon^p} \int_\Omega (w+\delta)^{s+p-1} \frac{|\nabla\phi|^p}{\phi^{p-1}},$$

hence

$$\int_\Omega (f-g)(\gamma_n(w) - \gamma_n(0))\phi + \int_\Omega (f-g)\delta^s\phi +$$

$$+ c_1 \int_\Omega (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla w \ \gamma_n'(w)\phi \le \frac{p^p s^{1-p}}{p\epsilon^p} \int_\Omega (w+\delta)^{s+p-1} \frac{|\nabla\phi|^p}{\phi^{p-1}}.$$

Now, since $\gamma_n(t) \to (t+\delta)^s$ and $\gamma_n'(t) \to s(t+\delta)^{s-1}$ as $n \to +\infty$, $(f-g)(\gamma_n(w)-\gamma_n(0)) \ge 0$ and $\mathscr{A}$ is monotone (that is $(\mathscr{A}_v - \mathscr{A}_u)\cdot\nabla w \ge 0$), by Fatou's Lemma we obtain

$$\int_\Omega (f-g)\left((w+\delta)^s - \delta^s\right)\phi + c_1 s \int_\Omega (\mathscr{A}_v - \mathscr{A}_u)\cdot\nabla w \ (w+\delta)^{s-1}\phi \le$$

$$c_2 s^{1-p} \int_\Omega (w+\delta)^{s+p-1} \frac{|\nabla\phi|^p}{\phi^{p-1}} - \int_\Omega (f-g)\,\delta^s\phi.$$

By letting $\delta \to 0$ in the above inequality, we complete the proof of the claim in the case $p > 1$. Here we have used Fatou's Lemma for the left hand side and Lebesgue's dominated convergence to treat the right hand side.

Let $p = 1$. From (31) and the fact that $\mathscr{A}_v - \mathscr{A}_u$ is bounded, the estimate (32) holds provided we replace p with 1 and ϵ with k_2. The remaining argument is similar to the case $p > 1$ and we omit it. □

Remark 3.5 *What really matters for the validity of (18) and (21) is the assumption $u^{s+p-1} \in L^1_{loc}(S)$. Here S is the support of $\nabla\phi$. This remark will be useful when dealing with inequalities on unbounded set.*

Lemma 3.6 *Let $p \geq 1$ and let $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be* **M**-p-**C**. *Let $f, g \in L^1_{loc}(\Omega)$ and let u, v are p-regular solutions of (17). Set $w := (v-u)^+$. If $(f-g)w \geq 0$ and $w^q \in L^1_{loc}(\Omega)$ for $q > p-1$, then*

$$(f-g)w^{q-p+1}, \quad ((\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u)) \cdot \nabla w \; w^{q-p}\mathbf{1}_{\{w>0\}} \in L^1_{loc}(\Omega), \tag{34}$$

and for any $\varphi \in \mathscr{C}^1_0(\Omega)$ such that $0 \leq \varphi \leq 1$, we have,

$$\int (f-g)\mathrm{sign}^+(w)\; \varphi^\sigma \leq c_3 \left(\frac{1}{|S|} \int_S w^q \varphi^\sigma \right)^{\frac{p-1}{q}} \left(\frac{1}{|S|} \int_S |\nabla \varphi|^\sigma \right)^{\frac{p}{\sigma}} |S|, \tag{35}$$

where S is the support of $\nabla\varphi$, $c_3 := \frac{\sigma^p}{k_2 s^{p-1}} \left(\frac{c_2}{c_1}\right)^{(p-1)/p}$ with $\sigma \geq \frac{pq}{q-p+1-s}$, $0 < s < \min\{1, q-p+1\}$ and c_1, c_2 as in the above Theorem 3.1.

Proof. The claim (34) follows directly applying Theorem 3.1.

Let $s > 0$ be such that $q \geq s+p-1$. From Theorem 3.1 for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$\int (f-g)w^s\phi + c_1 s \int_{\{w>0\}} (\mathscr{A}_v - \mathscr{A}_u)\cdot \nabla w \; w^{s-1}\phi \leq c_2 s^{1-p} \int w^{s+p-1}\frac{|\nabla\phi|^p}{\phi^{p-1}}, \tag{36}$$

where, as in the proof of Theorem 3.1, we write $\mathscr{A}_v$ and $\mathscr{A}_v$ for $\mathscr{A}(x,\nabla v)$ and $\mathscr{A}(x,\nabla u)$ respectively.

Next, an application of Theorem 3.4 gives (27). That is

$$\mathrm{div}\left(\mathrm{sign}^+(v-u)(\mathscr{A}(x,v,\nabla v) - \mathscr{A}(x,u,\nabla u))\right) \geq \mathrm{sign}^+(v-u)(f-g) \quad on\ \Omega. \tag{37}$$

Next consider the case $p > 1$. Let $0 < s < \min\{1, q-p+1\}$. By definition of p-regular solution and Hölder's inequality with exponent p', taking into account that $\mathscr{A}$ is **M**-p-**C** and from (36) we get,

$$\int \mathrm{sign}^+ w(f-g)\;\phi \leq \int |\mathscr{A}_v - \mathscr{A}_u|\, |\nabla\phi|\, \mathrm{sign}^+ w \tag{38}$$

$$= \int_{\{w>0\}} |\mathscr{A}_v - \mathscr{A}_u|\, w^{\frac{s-1}{p'}} \phi^{\frac{1}{p'}} \;\; |\nabla\phi|\, w^{\frac{1-s}{p'}}\phi^{-\frac{1}{p'}} \tag{39}$$

$$\leq \frac{1}{k_2}\left(\int_{\{w>0\}} (\mathscr{A}_v - \mathscr{A}_u)\cdot\nabla w\, w^{s-1}\phi\right)^{1/p'} \left(\int w^{(1-s)(p-1)}\frac{|\nabla\phi|^p}{\phi^{p-1}}\right)^{1/p} \tag{40}$$

$$\leq \frac{1}{k_2}\left(\frac{c_2}{c_1 s^p}\right)^{1/p'} \left(\int w^{s+p-1}\frac{|\nabla\phi|^p}{\phi^{p-1}}\right)^{1/p'} \left(\int w^{(1-s)(p-1)}\frac{|\nabla\phi|^p}{\phi^{p-1}}\right)^{1/p} \tag{41}$$

Since $q > s+p-1$ and $q > p-1$, applying Hölder inequality to (41) with exponents $\chi := \frac{q}{s+p-1}$ and $y := \frac{q}{(1-s)(p-1)}$, we obtain

$$\int \mathrm{sign}^+ w\;(f-g)\phi \leq c_3' \left(\int w^q\phi\right)^\delta \left(\int \frac{|\nabla\phi|^{p\chi'}}{\phi^{p\chi'-1}}\right)^{\frac{1}{p'\chi'}} \left(\int \frac{|\nabla\phi|^{py'}}{\phi^{py'-1}}\right)^{\frac{1}{py'}}, \tag{42}$$

where

$$\delta := \frac{1}{\chi p'} + \frac{1}{yp} = \frac{p-1}{q}, \qquad c_3' := \left(\frac{c_2}{c_1 s^p}\right)^{1/p'} \frac{1}{k_2}.$$

Next for $\sigma \geq p\chi' > 1$ (and hence $\sigma > py'$ since $p\chi' > py'$) we choose $\phi := \varphi^\sigma$ with $\varphi \in \mathscr{C}_0^1(\Omega)$ such that $0 \leq \varphi \leq 1$. Setting $S := \mathit{support}(\varphi)$, from (42) it follows that

$$\int \operatorname{sign}^+ w\ (f-g)\varphi^\sigma \leq c_3'\sigma^p \left(\int w^q \varphi^\sigma\right)^\delta \left(\frac{1}{|S|}\int_S |\nabla\varphi|^\sigma\right)^{\frac{p}{\sigma}} |S|^{1-\delta}, \tag{43}$$

completing the proof of (35).

Now, we assume that $p = 1$. From (38), with the choice $\phi := \varphi^\sigma$, with $\varphi \in \mathscr{C}_0^1(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\sigma \geq 1$, we have

$$\int \operatorname{sign}^+ w\ (f-g)\varphi^\sigma \leq \frac{\sigma}{k_2}\int_S |\nabla\varphi| \leq \frac{\sigma}{k_2}\left(\frac{1}{|S|}\int_S |\nabla\varphi|^\sigma\right)^{1/\sigma} |S|,$$

which concludes the proof. □

Remark 3.7 *In view of Lemma 3.3 and Theorem 3.4, the same proof gives that the conclusions of Theorem 3.6 hold true for a* **W**-*p*-**C** *function* $\mathscr{A} = \mathscr{A}(x,t,\xi)$*, whenever* u, v *are p-regular solutions of (17) and u is a constant. Here we have used that* $\mathscr{A}(x,t,0) = 0$*, for every x and t (cfr. Definition 2.1).*

Now, by specializing f and g, we study

$$\operatorname{div}(\mathscr{A}(x,\nabla v)) - |v|^{q-1} v \geq \operatorname{div}(\mathscr{A}(x,\nabla u)) - |u|^{q-1} u \qquad \text{on } \Omega. \tag{44}$$

We recall that, when $\mathscr{A}$ is **W**-p-**C**, any distributional solution $u \in W^{1,p}_{loc}(\Omega)$ is automatically p-regular.

Lemma 3.8 *1. Assume* $p > 1$*. Let* $q \geq 1$ *and* $q > p-1$*. Let* $\mathscr{A}$ *be* **M**-*p*-**C***. For any* $\sigma > 0$ *large enough, there exists a constant* $c = c(\sigma, q, p, \mathscr{A}) > 0$ *such that if* $u, v \in W^{1,p}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$ *are distributional solutions of (44) then for any nonnegative* $\varphi \in \mathscr{C}_0^1(\Omega)$ *such that* $||\varphi||_\infty \leq 1$*, we have*

$$\int (|v|^{q-1} v - |u|^{q-1} u)^+ \varphi^\sigma \leq c\left(\frac{1}{|S|}\int_S (v-u)^{+q}\varphi^\sigma\ \right)^{\frac{p-1}{q}} \left(\frac{1}{|S|}\int_S |\nabla\varphi|^\sigma\right)^{\frac{p}{\sigma}} |S| \tag{45}$$

$$\int (|v|^{q-1} v - |u|^{q-1} u)^+ \varphi^\sigma \leq c\ |S|\ \left(\frac{1}{|S|}\int_S |\nabla\varphi|^\sigma\right)^{\frac{1}{\sigma}\frac{pq}{q-p+1}} \tag{46}$$

where $S := support(\varphi)$.

In particular if $B_{2R} \subset\subset \Omega$, then

$$\left(\fint_{B_R} (|v|^{q-1} v - |u|^{q-1} u)^+ \right)^{1/q} \leq c \; R^{-\frac{p}{q-p+1}}. \tag{47}$$

Moreover, for $x \in \Omega$, set $R = dist(x, \partial\Omega)/2$, we have

$$\left(\fint_{B_R(x)} (|v|^{q-1} v - |u|^{q-1} u)^+ \right)^{1/q} \leq c \; dist(x, \partial\Omega)^{-\frac{p}{q-p+1}}. \tag{48}$$

2. *Let $p = 1$. Let $\mathscr{A}$ be $\mathbf{M}$-p-$\mathbf{C}$ and let $q > 0$. For any $\sigma > 0$ large enough, there exists a constant $c = c(\sigma, q, p, \mathscr{A}) > 0$ such that if $u, v \in W^{1,p}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$ are distributional solutions of (44) then for any nonnegative $\varphi \in \mathscr{C}^1_0(\Omega)$ such that $||\varphi||_\infty \leq 1$, we have*

$$\int (|v|^{q-1} v - |u|^{q-1} u)^+ \varphi^\sigma \leq c \left(\frac{1}{|S|} \int_S |\nabla \varphi|^\sigma \right)^{\frac{1}{\sigma}} |S| \tag{49}$$

where $S := support(\varphi)$.

In particular if $B_{2R} \subset\subset \Omega$, then

$$\left(\fint_{B_R} (|v|^{q-1} v - |u|^{q-1} u)^+ \right)^{1/q} \leq c \; R^{-\frac{1}{q}}.$$

Moreover, for $x \in \Omega$, set $R = dist(x, \partial\Omega)/2$, we have

$$\left(\fint_{B_R(x)} (|v|^{q-1} v - |u|^{q-1} u)^+ \right)^{1/q} \leq c \; dist(x, \partial\Omega)^{-\frac{1}{q}}. \tag{50}$$

Proof. Let $p > 1$. From Lemma 3.6 we immediately obtain (45).

Reminding the well known inequality

$$|t|^{q-1} t - |s|^{q-1} s \geq c_q (t-s)^q, \quad for \;\; t > s \qquad (q \geq 1), \tag{51}$$

from (45), we get (46).

In order to obtain the estimate (47) we specialize the test function φ. Indeed, let $\phi \in \mathscr{C}^1_0(\mathbb{R})$ be such that $0 \leq \phi \leq 1$, $\phi(t) = 0$ if $|t| \geq 2$, $\phi(t) = 1$ if $|t| \leq 1$ and $0 < \phi < 1$ if $1 < t < 2$. Next, we define $\phi_R(t) := \phi(t/R)$. The claim follows by choosing $\varphi(x) = \phi_R(|x|)$. Indeed, with this choice we have $|\nabla \varphi| \leq cR^{-1}$ and $|S| = |A_R| = c(N)R^N$, $c(N) > 0$.

The estimate (48) follows by choosing $\varphi(y) = \phi_R(|y - x|)$.

Let $p = 1$. From Lemma 3.6 we immediately obtain (49). In particular we do not need to use (51). The rest of the proof is unchanged and we leave the details to the interested reader. □

Lemma 3.9 *Assume $p \geq 1$ and $\mathscr{A}$ be* **M**-*p*-**C**. *Let $q \geq 1$ and $q > p - 1$. Let $u, v \in W^{1,p}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$ be distributional solutions of (44). Then $(v-u)^+ \in L^r_{loc}(\Omega)$ for any $r < +\infty$.*

In particular, if

$$\operatorname{div}(\mathscr{A}(x, \nabla v)) - |v|^{q-1} v = \operatorname{div}(\mathscr{A}(x, \nabla u)) - |u|^{q-1} u \qquad \text{on } \Omega \tag{52}$$

then $v - u \in L^r_{loc}(\Omega)$

Proof. Set $w := (v-u)^+$. Now since $w \in L^q_{loc}(\Omega)$, and inequality (51) holds we are in position to apply Theorem 3.1, with $s = q - p + 1 > 0$ obtaining $w^{q_1} \in L^1_{loc}(\Omega)$ with $q_1 := 2q - p + 1$. Applying again Theorem 3.1, with $s = q_1 - p + 1$, we get $w^{q_2} \in L^1_{loc}(\Omega)$ with $q_2 := q_1 + q - p + 1 = q + 2(q - p + 1)$. Iterating j times we have that $w^{q_j} \in L^1_{loc}(\Omega)$ with $q_j := q + j(q - p + 1)$. Letting $j \to +\infty$ we have the claim. □

Remark 3.10 *In view of Lemma 3.3, the same proof gives that the conclusions of Lemma 3.9 hold true for a* **W**-*p*-**C** *function $\mathscr{A} = \mathscr{A}(x, t, \xi)$, whenever $u, v \in W^{1,p}_{loc} \cap L^q_{loc}$ are distributional solutions of*

$$\operatorname{div}(\mathscr{A}(x, v, \nabla v)) - |v|^{q-1} v \geq \operatorname{div}(\mathscr{A}(x, u, \nabla u)) - |u|^{q-1} u \qquad \text{on } \Omega \tag{53}$$

and u is a constant. In particular, if the equality sign occurs in (53), *then $v \in L^r_{loc}(\Omega)$ for any $r < +\infty$.*

Remark 3.11 *In the case $u \equiv 0$, inequalities of type (17),(23), (44) and (53), have been widely studied. In this case, results similar to the above ones and to their consequences presented in the subsequent sections, have been obtained in the recent papers [5, 6, 7, 9, 10].*

4 Comparison and Uniqueness

Theorem 4.1 *Assume $p \geq 1$ and $\mathscr{A}$ be* **M**-*p*-**C**. *Let $q \geq 1$ and $q > p - 1$. Let $u, v \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ be distributional solutions of*

$$\operatorname{div}(\mathscr{A}(x, \nabla v)) - |v|^{q-1} v \geq \operatorname{div}(\mathscr{A}(x, \nabla u)) - |u|^{q-1} u \qquad \text{on } \mathbb{R}^N. \tag{54}$$

Then $v \leq u$ a.e. on $\mathbb{R}^N$.

In particular, if

$$\operatorname{div}(\mathscr{A}(x, \nabla v)) - |v|^{q-1} v = \operatorname{div}(\mathscr{A}(x, \nabla u)) - |u|^{q-1} u \qquad \text{on } \mathbb{R}^N \tag{55}$$

then $u = v$ a.e. on $\mathbb{R}^N$.

Proof. Let u, v be distributional solutions of (54) and set $w := (v-u)^+$. From Lemma 3.9 we know that $w \in L^r_{loc}(\mathbb{R}^N)$ for any r, and hence we are in the position to apply Theorem 3.1 with s large enough. Thus, from (51) and (19) we get $w^{q+s} \in L^1_{loc}$ and

$$\int w^{q+s}\phi \leq c(s,q,p) \int w^{s+p-1}\frac{|\nabla\phi|^p}{\phi^{p-1}}.$$

Applying the Hölder inequality with exponent $x := \frac{q+s}{s+p-1} > 1$ we have

$$\int w^{q+s}\phi \leq c(s,q,p) \int \frac{|\nabla\phi|^{px'}}{\phi^{(p-1)x'}}.$$

Replacing ϕ by $\phi^{p'}$ in the latter and, by the same choice of ϕ we made in Lemma 3.8, we have that

$$\int_{B_R} w^{q+s} \leq cR^{N-px'} = cR^{N-p(q+s)/(q-p+1)}.$$

Choosing s large enough and letting $R \to +\infty$, we have that $w \equiv 0$ a.e. that is the claim. By exchanging the roles of u and v, we immediately see that $u = v$ if (55) is in force. □

We then have

Corollary 4.2 *Let p and q as in Theorem 4.1. Let $\mathscr{A}$ be a* **M**-*p*-**C** *function. Then, for any distribution $T \in \mathcal{D}'(\mathbb{R}^N)$, the equation*

$$-\operatorname{div}(\mathscr{A}(x,\nabla u)) + |u|^{q-1}u = T \qquad \text{on } \mathbb{R}^N \tag{56}$$

can have at most one distributional solution of class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$.

Proof. The claim follows by observing that two solutions of (56) are also solutions of (55). □

Remark 4.3 *In view of Lemma 3.3, the same proof gives that the conclusions of Theorem 4.1 hold true for a* **W**-*p*-**C** *function $\mathscr{A} = \mathscr{A}(x,t,\xi)$, whenever $u, v \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ are distributional solutions of (53) and u is a constant. In particular, the equation*

$$-\operatorname{div}(\mathscr{A}(x,u,\nabla u)) + |u|^{q-1}u = C := const. \qquad \text{on } \mathbb{R}^N \tag{57}$$

has exactly one distributional solution in the class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$, the constant function $\operatorname{sign}(C)\,|C|^{1/q}$.

As a further consequence of the above analysis we have

Corollary 4.4 *Let p and q as in Theorem 4.1. Let $\mathscr{A} = \mathscr{A}(x,t,\xi)$ be a* **W**-*p*-**C** *function and assume $h \in L^1_{loc}(\mathbb{R}^N)$. Let $v \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ be a distributional solution of*

$$-\operatorname{div}(\mathscr{A}(x,v,\nabla v)) + |v|^{q-1}\, v = h. \tag{58}$$

Then,

$$\inf_{\mathbb{R}^N} h \le |v|^{q-1}\, v \le \sup_{\mathbb{R}^N} h.$$

In particular, if $h \ge 0$ [resp. ≤ 0], then $v \ge 0$ [resp. ≤ 0] and if $h \in L^\infty(\mathbb{R}^N)$, then $v \in L^\infty(\mathbb{R}^N)$.

Proof. We prove one of the estimates, the other one being similar. If $\sup_{\mathbb{R}^N} h = +\infty$ there is nothing to prove. Let $M := \sup_{\mathbb{R}^N} h < +\infty$. We define $u := \operatorname{sign}(M)\,|M|^{1/q}$. Then

$$\operatorname{div}(\mathscr{A}(x,v,\nabla v)) - |v|^{q-1}\, v = -h \ge -M = 0 - M = \operatorname{div}(\mathscr{A}(x,u,\nabla u)) - |u|^{q-1}\, u$$

that is u, v satisfy (53) with u constant. The conclusion then follows by invoking Remark 4.3. □

5 Completely coercive/monotone equations

In this section we extend the previous results (comparison, uniqueness and regularity) to more general operators and nonlinearities. As a consequence we shall also obtain some new symmetry results.

In the present section we always assume that $\mathscr{A} \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy the (p,r)-*large radii condition* described in Definition (2.6) and we shall consider the following differential inequality,

$$\operatorname{div}(\mathscr{A}(x,\nabla v)) - f \ge \operatorname{div}(\mathscr{A}(x,\nabla u)) - g \qquad \textit{on } \Omega, \tag{59}$$

where Ω is an open set containing $\overline{B_{R_0}}$. Here, and in the sequel, B_{R_0} denotes then open ball centered at the origin and of radius R_0.

We start with two results which are the analogue of Theorem 3.1 and Lemma 3.6 of Section 3.

Theorem 5.1 *Let $p \ge 1$ and $r \in \mathbb{R}$. Let $\mathscr{A}$ satisfy the* (p,r)-large radii condition. *Let $f, g \in L^1_{loc}(\Omega)$ and let u, v be p-regular solutions of (59). Set $w := (v-u)^+$ and let $s > 0$. If $(f-g)w \ge 0$ and $w^{s+p-1} \in L^1_{loc}(\Omega \setminus B_{R_0})$, then*

$$(f-g)w^s, \quad (\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u)) \cdot \nabla w \ w^{s-1} \mathbf{1}_{\{w>0\}} \in L^1_{loc}(\Omega). \tag{60}$$

Moreover, for any nonnegative $\phi \in \mathscr{C}_0^1(\Omega)$ *such that* $\phi \equiv 1$ *on* B_{R_0}*, we have,*

$$\int_\Omega (f-g)w^s\phi + c_1 s\int_{\Omega\cap\{w>0\}}(\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u))\cdot\nabla w\ w^{s-1}\phi \le c_2 s^{1-p}\int_{\Omega\setminus B_{R_0}} w^{s+p-1}\,|x|^r\,\frac{|\nabla\phi|^p}{\phi^{p-1}}, \tag{61}$$

where $c_1 = 1 - \frac{p-1}{p}\left(\frac{\epsilon}{k_2}\right)^{\frac{p}{p-1}} > 0$, $c_2 = \frac{p^p}{p\epsilon^p}$ *and* $\epsilon > 0$ *is sufficiently small for* $p > 1$ *and* $c_1 = 1$ *and* $c_2 = 1/k_2$ *for* $p = 1$.

Lemma 5.2 *Let* $p \ge 1$ *and* $r \in \mathbb{R}$*. Let* $\mathscr{A}$ *satisfy the* (p,r)-large radii condition*. Let* $f, g \in L^1_{loc}(\Omega)$ *and let* u, v *be* p*-regular solutions of (59). Set* $w := (v-u)^+$*. If* $(f-g)w \ge 0$ *and* $w^q \in L^1_{loc}(\Omega \setminus B_{R_0})$ *for* $q > p-1$*, then*

$$(f-g)w^{q-p+1},\ \ ((\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u))\cdot\nabla w\ w^{q-p}\mathbf{1}_{\{w>0\}} \in L^1_{loc}(\Omega), \tag{62}$$

and for any $\varphi \in \mathscr{C}_0^1(\Omega)$ *such that* $0 \le \varphi \le 1$ *and* $\varphi \equiv 1$ *on* B_{R_0}*, we have,*

$$\int (f-g)\mathrm{sign}^+(w)\ \varphi^\sigma \le c_3\left(\frac{1}{|S|}\int_S w^q\varphi^\sigma\right)^{\frac{p-1}{q}}\left(\frac{1}{|S|}\int_S |x|^{\sigma r/p}\,|\nabla\varphi|^\sigma\right)^{\frac{p}{\sigma}}|S|\,, \tag{63}$$

where S *is the support of* $\nabla\varphi$*,* $c_3 := \frac{\sigma^p}{k_2 s^{p-1}}\left(\frac{c_2}{c_1}\right)^{(p-1)/p}$ *with* $\sigma \ge \frac{pq}{q-p+1-s}$*,* $0 < s < \min\{1, q-p+1\}$ *and* c_1, c_2 *as in the above Theorem 5.1.*

Their proofs are similar to those of the above mentioned results and for this reason we omit them.

From now on, we study solutions of

$$\mathrm{div}\,(\mathscr{A}(x,\nabla v)) - f(x,v,\nabla v) \ge \mathrm{div}\,(\mathscr{A}(x,\nabla u)) - g(x,u,\nabla u) \qquad \text{on } \Omega, \tag{64}$$

where $f, g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory functions such that $f(\cdot,u,\nabla u)$, $g(\cdot,u,\nabla u) \in L^1_{loc}(\Omega)$.

We shall assume that f and g satisfy the following (q,t)*-large radii condition*: there exist $q > 0$, $t \in \mathbb{R}$ and $c > 0$ such that

$$(f(x,v,\xi) - g(x,u,\eta))(v-u) \ge 0, \qquad \forall\, x \in \mathbb{R}^N, \quad \forall\, v \ge u, \quad \forall\, \xi,\eta \in \mathbb{R}^N,$$

$$f(x,v,\xi) - g(x,u,\eta) \ge c\,|x|^{-t}\,(v-u)^q, \qquad \text{for a.e. } |x| > R_0, \quad \forall\, v \ge u, \quad \forall\, \xi,\eta \in \mathbb{R}^N. \tag{65}$$

A straightforward application of Theorem 5.1 gives the following regularity result, which is the analogue of Lemma 3.9.

Lemma 5.3 *Let $p \geq 1$, $q \geq 1$ and $q > p-1$. Assume $r, t \in \mathbb{R}$. Let $\mathscr{A}$ satisfy the* (p,r)-large radii condition *and assume that (65) holds. Let u, v be p-regular solutions of (64), then $(v-u)^+ \in L^\sigma_{loc}(\Omega \setminus B_{R_0})$ for any $\sigma < +\infty$.*

We are now in position to prove the comparison principle for general equations satisfying the (p,r)-*large radii condition* as well as the (q,t)-*large radii condition*.

Theorem 5.4 *Let $p \geq 1$, $r \in \mathbb{R}$, $q \geq 1$ and $q > p-1$. Let $\mathscr{A}$ satisfy the* (p,r)-large radii condition *and assume that (65) holds with*

$$r + t < p \qquad or \qquad r + t > p \quad \& \quad \frac{r+t-p}{q-p+1} < \frac{t-N}{q}.$$

Let u, v be p-regular solutions of

$$\operatorname{div}(\mathscr{A}(x, \nabla v)) - f(x, v, \nabla v) \geq \operatorname{div}(\mathscr{A}(x, \nabla u)) - g(x, u, \nabla u) \qquad on\ \mathbb{R}^N. \tag{66}$$

Then $v \leq u$ a.e. on $\mathbb{R}^N \setminus B_{R_0}$ and

$$(f(x, v, \nabla v) - g(x, u, \nabla u))(v-u)^+ \equiv 0, \quad on\ B_{R_0}, \tag{67}$$

$$(\mathscr{A}(x, \nabla v) - \mathscr{A}(x, \nabla u)) \cdot (\nabla v - \nabla u) \equiv 0, \quad on\ B_{R_0}. \tag{68}$$

In particular if

$$\begin{cases} (\mathscr{A}(x, \xi) - \mathscr{A}(x, \eta) \cdot (\xi - \eta) + (f(x, v, \xi) - g(x, u, \eta))(v-u) = 0 \\ \\ \text{implies either } u = v \text{ or } \xi = \eta, \end{cases} \tag{69}$$

then $v \leq u$ a.e. on $\mathbb{R}^N$.

We emphasize that, when $r + t > p$, the second condition in the above theorem is essentially necessary in order to obtain the desired conclusion. This follows from the explicit counterexamples provided in Appendix B.

Proof. We follow the proof of Theorem 4.1. Let u, v be p-regular solutions of (66) and set $w := (v-u)^+$. From Lemma 5.3 we know that $w \in L^\sigma_{loc}(\mathbb{R}^N \setminus B_{R_0})$ for any finite σ, hence we can invoke Theorem 5.1 with s large enough. Thus, from (51), (61) and (65) we get $w^{q+s} \in L^1_{loc}(\mathbb{R}^N \setminus B_{R_0})$ and

$$\int_{\mathbb{R}^N \setminus B_{R_0}} w^{q+s} \, |x|^{-t} \, \phi \leq c(s, q, p) \int w^{s+p-1} \, |x|^r \, \frac{|\nabla \phi|^p}{\phi^{p-1}}.$$

By using Hölder inequality with exponent $\chi := \frac{q+s}{s+p-1}$ we infer

$$\int_{\mathbb{R}^N \setminus B_{R_0}} w^{q+s} |x|^{-t} \phi \leq c \int |x|^{(t+r)\chi' - t} \frac{|\nabla \phi|^{p\chi'}}{\phi^{(p-1)\chi'}}.$$

Thus, by choosing ϕ as in Lemma 3.8, we get

$$\int_{R_0<|x|<R} w^{q+s} |x|^{-t} \leq cR^{\theta} \tag{70}$$

for any $R > 2R_0$, where $\theta := N - t + (r + t - p)\frac{q+s}{q-p+1}$.

Now, we choose s such that $\theta < 0$. When $t+r < p$, this is possible by choosing s large enough. When $r + t > p$ and $\frac{r+t-p}{q-p+1} < \frac{t-N}{q}$, it is immediate to see that we can ensure this by choosing s close to 0. Letting $R \to +\infty$ in (70) we obtain that $w \equiv 0$ in $\mathbb{R}^N \setminus B_{R_0}$. This implies that $v(x) \leq u(x)$ for a.e. on $\{|x| > R_0\}$. Moreover, by using this information in (61) with $s = 1$ we obtain (67) and (68), which in turn, give immediately $v(x) \leq u(x)$ a.e. on $\mathbb{R}^N$, if (69) is in force. □

We are ready to prove some consequences of the above comparison principle.

Corollary 5.5 *Let p, q, r, t and $\mathscr{A}$ as in Theorem 5.4. Assume that f satisfies*

$$(f(x, v, \xi) - f(x, u, \eta))(v - u) \geq 0, \qquad \forall\, x \in \mathbb{R}^N, \quad \forall\, v \geq u, \quad \forall\, \xi, \eta \in \mathbb{R}^N,$$

$$f(x, v, \xi) - f(x, u, \eta) \geq c\, |x|^{-t} (v - u)^q, \qquad \text{for a.e. } |x| > R_0, \quad \forall\, v \geq u, \quad \forall\, \xi, \eta \in \mathbb{R}^N, \tag{71}$$

and

$$\begin{cases} (\mathscr{A}(x, \xi) - \mathscr{A}(x, \eta) \cdot (\xi - \eta) + (f(x, v, \xi) - f(x, u, \eta))(v - u) = 0 \\ \\ \text{implies either } u = v \text{ or } \xi = \eta. \end{cases} \tag{72}$$

Then, for any distribution $T \in \mathcal{D}'(\mathbb{R}^N)$, the equation

$$-\operatorname{div}(\mathscr{A}(x, \nabla v)) + f(x, v, \nabla v) = T \qquad \text{on } \mathbb{R}^N. \tag{73}$$

can have at most one p-regular solution.

Proof. The claim follows by observing that two solutions of (73) are also p-regular solutions of

$$\operatorname{div}(\mathscr{A}(x, \nabla v)) - f(x, v, \nabla v) = \operatorname{div}(\mathscr{A}(x, \nabla u)) - f(x, u, \nabla u) \qquad \text{on } \mathbb{R}^N. \tag{74}$$

The conclusion follows by applying Theorem (5.4) with $f = g$. □

Proof of Theorem 1.4. The considered operator is generated by the function $\mathscr{A} = \mathscr{A}(x,\xi) = a(x)\,|\xi|^{p-2}\,\xi$ (weighted p-Laplacian operator), hence it satisfies the (p,r)-*large radii condition* as observed in Example 2.7. Set $f(x,t,\xi) = g(x,t,\xi) = b(x)\,|t|^{p-1}\,t$. Hence, in view of (5), we see that f satisfies the (q,t)-*large radii condition* (71). Finally we note that (4) implies (72). Next we observe that every distributional solution $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ of the considered equation is automatically $p-$regular, since $a(\cdot)|\nabla u|^{p-2}\nabla u \in L^{p'}_{loc}(\mathbb{R}^N)$ and a is locally bounded. The first conclusion is then a consequence of Corollary 5.5.

The last claim follows in a similar way by applying Theorem 5.4 with $f = g$ and $u = 0$. □

Now we prove a symmetry result. As a corollary, we shall obtain Theorem 1.5 stated in the introduction.

Theorem 5.6 (Symmetry) *Assume either* $1 < p < 2$, $q \geq 1$ *or* $p = 2$, $q > 1$ *and let* $h \in L^1_{loc}(\mathbb{R}^N)$. *Let* a *and* b *two measurable, nonnegative, locally bounded functions satisfying conditions* (4), (5), (6) *of Theorem 1.4.*

Let u *be a distributional solution of class* $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ *of the equation*

$$- \operatorname{div}[a(x)\,|\nabla u|^{p-2}\nabla u] + b(x)|u|^{q-1}\,u = h \quad on \quad \mathbb{R}^N. \tag{75}$$

Then

i) if a, b, h *depend only on* $k < N$ *variables (say,* $x_1, ..., x_k$*) then, also* u *depends only on* $x_1, ..., x_k$.

ii) if a, b, h *are* τ*-periodic then, also* u *is* τ*-periodic.*

iii) if a, b, h *are symmetric with respect to an affine hyperplane then, also* u *has the same symmetry.*

iv) if a, b, h *are radially symmetric, then also* u *is radially symmetric.*

All the above results hold true for the equation

$$- \operatorname{div}\left(\frac{a(x)\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + b(x)|u|^{q-1}\,u = h \quad on \quad \mathbb{R}^N, \tag{76}$$

when $q \geq 1$.

Proof. The proof relies on the uniqueness property established in Theorem 1.4.

i) For any $x = (x_1, ..., x_N) \in \mathbb{R}^N$ we set $\underline{x} := (x_1, ..., x_k)$ and $\overline{x} := (x_{k+1}, ..., x_N)$. Since a, b, h depend only on the first variables $x_1, ..., x_k$, the function $v_{\overline{y}} := u(\underline{x}, \overline{x} + \overline{y})$, is again a solution of the considered equation, and this for every $\overline{y}$. Hence, by Theorem 1.4 we have $v_{\overline{y}} = u$ a.e. on $\mathbb{R}^N$, and this for every $\overline{y}$. The latter implies the desired conclusion.

ii) The function $v_\tau(x) = u(x+\tau)$ is still a solution, hence it must coincide with u. Thus, u is τ-periodic.

iii) Same proof, but using the symmetry with respect to the considered affine hyperplane.

iv) Since a, b, h are symmetric with respect to each hyperplane through the origin, the previous step implies that u has the same symmetry, which is the desired conclusion. □

6 Some results for the case $p > 2$

Notice that the p-Laplacian operator with $p > 2$ is not **M-p-C**. This fact it is easy to see by homogeneity consideration.

In this section we shall require that $p > 2$ and $\mathscr{A} \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a weakly elliptic function such that for all $\xi, \eta \in \mathbb{R}^N \setminus \{0\}$, $x \in \mathbb{R}^N$ satisfies

$$(\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta) \geq k_2 \frac{|\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)|^p}{(|\xi| + |\eta|)^{p(p-2)}}. \tag{77}$$

It is clear that $\mathscr{A}$ is monotone. Also, any distributional solution $u \in W^{1,p}_{loc}(\Omega)$ is automatically p-regular.

Remark 6.1 *The results of this section, with suitable modifications, still hold if we replace the condition (77) with a more general condition having in the left hand side a dependence of x in the same spirit of the large radii condition (13) in Definition 2.6. We leave the details to the interested reader.*

Example 6.2 *Example of function $\mathscr{A}$ satisfying (77) is $\mathscr{A}(x,\xi) = a(x)\,|\xi|^{p-2}\,\xi$ where $a = a(x)$ is a bounded nonnegative function and $p \geq 2$. Indeed, the following inequalities holds*

$$\left|\xi^{p-2}\xi - \eta^{p-2}\eta\right| \leq c_1(|\xi| + |\eta|)^{p-1-\alpha}\,|\xi - \eta|^\alpha\,, \tag{78}$$

$$(\xi^{p-2}\xi - \eta^{p-2}\eta) \cdot (\xi - \eta) \geq c_2(|\xi| + |\eta|)^{p-\beta}\,|\xi - \eta|^\beta\,, \tag{79}$$

with $\beta \geq \max\{p, 2\}$ and $0 \leq \alpha \leq \min\{1, p-1\}$. See [3].

Therefore choosing $\beta = p$ and $\alpha = 1$ in (78) and (79) we have

$$\frac{|\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)|^p}{(|\xi| + |\eta|)^{p(p-2)}} = a(x)^p \frac{|\xi^{p-2}\xi - \eta^{p-2}\eta|^p}{(|\xi| + |\eta|)^{p(p-2)}} \leq a(x)^p c_1^p\,|\xi - \eta|^p$$

$$\leq a(x)^p \frac{c_1^p}{c_2}(\xi^{p-2}\xi - \eta^{p-2}\eta) \cdot (\xi - \eta) = a(x)^{p-1}\frac{c_1^p}{c_2}(\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta).$$

Therefore, (77) is fulfilled with $k_2 = \frac{c_1^p}{c_2}\,||a||_\infty^{p-1}$.

We need of a version of Theorem 3.1 for operator satisfying (77).

Lemma 6.3 *Let $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy (77) with $p > 2$. Let $f, g \in L^1_{loc}(\Omega)$ and let u, v be p-regular solutions of (17). Set $w := (v-u)^+$ and let $s > 0$. If $(f-g)w \geq 0$ and $w^{s+p'-1}\left(|\nabla u| + |\nabla v|\right)^{p'(p-2)} \in L^1_{loc}(\Omega)$, then*

$$(f-g)w^s, \quad (\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u))\cdot \nabla w \ w^{s-1}\mathbf{1}_{\{w>0\}} \in L^1_{loc}(\Omega), \tag{80}$$

and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have,

$$\int_\Omega (f-g)w^s\phi + c_1\int_{\Omega\cap\{w>0\}} (\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u))\cdot \nabla w \ w^{s-1}\phi \tag{81}$$

$$\leq c_2\int_\Omega w^{s+p'-1}\left(|\nabla u| + |\nabla v|\right)^{(p-2)p'}\frac{|\nabla\phi|^{p'}}{\phi^{p'-1}}, \tag{82}$$

where $c_1 = c_1(s,p,k_2), c_2(s,p,k_2) > 0$ are suitable constants independent of u, v and ϕ.

Proof. The proof is analogous to the proof of Theorem 3.1. So we shall sketch it using the same notation. Applying Theorem 3.4 we have (31) which, by using Hölder's inequality, (77) and Young's inequality, yields

$$\int_\Omega (f-g)\gamma(w)\phi + \int_\Omega (\mathscr{A}_v - \mathscr{A}_u)\cdot \ \nabla w \ \gamma'(w)\phi \leq$$

$$\leq \left(\int_\Omega \frac{|\mathscr{A}_v - \mathscr{A}_u|^p}{(|\nabla v| + |\nabla u|)^{(p-2)p}}\gamma'(w)\phi\right)^{1/p}\left(\int_\Omega \frac{\gamma(w)^{p'}}{\gamma'(w)^{p'-1}}\frac{|\nabla\phi|^{p'}}{\phi^{p'-1}}\left(|\nabla v| + |\nabla u|\right)^{(p-2)p'}\right)^{1/p'}$$

$$\leq \frac{\epsilon^p}{pk_2^p}\int_\Omega (\mathscr{A}_v - \mathscr{A}_u)\cdot\nabla w \ \gamma'(w)\phi + \frac{1}{p'\epsilon^{p'}}\int_\Omega \frac{\gamma(w)^{p'}}{\gamma'(w)^{p'-1}}\frac{|\nabla\phi|^{p'}}{\phi^{p'-1}}\ \left(|\nabla v| + |\nabla u|\right)^{(p-2)p'}.$$

Next, constructing a sequence of $\gamma_n(t)$ approximating the function t^s as made in the proof of Theorem 3.1, we conclude the proof. □

Theorem 6.4 *Let $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ $\mathscr{A}$ satisfy (77) with $p > 2$. Let $f, g \in L^1_{loc}(\Omega)$ and let u, v be p-regular solutions of (17). Set $w := (v-u)^+$ and let $s > 0$. If $(f-g)w \geq 0$ and $w^{s(p-1)+1} \in L^1_{loc}(\Omega)$, then*

$$(f-g)w^s, \quad (\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u))\cdot \nabla w \ w^{s-1}\mathbf{1}_{\{w>0\}} \in L^1_{loc}(\Omega) \tag{83}$$

and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have,

$$\int_\Omega (f-g)w^s\phi + c_1\int_{\Omega\cap\{w>0\}} (\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u))\cdot \nabla w \ w^{s-1}\phi \tag{84}$$

$$\leq c_2\left(\int_S w^{s(p-1)+1}\frac{|\nabla\phi|^p}{\phi}\right)^{1/(p-1)}\left(\int_S \left(|\nabla u| + |\nabla v|\right)^p\right)^{(p-2)/(p-1)}, \tag{85}$$

where S is the support of $\nabla\phi$, and $c_1, c_2 > 0$ are suitable constants independent of u, v and ϕ.

In particular if there exist $c > 0, q > 0$ such that

$$(f-g)(v-u)^+ \geq c\left((v-u)^+\right)^q, \tag{86}$$

we have that $w^{q+s} \in L^1_{loc}(\Omega)$. Moreover, $w^{q+1} \in L^1_{loc}(\Omega)$ and if $q > p-1$ we have

$$w^r \in L^1_{loc}(\Omega) \quad for\ any \quad r \in \left]0, \frac{q(p-1)-1}{p-2}\right[. \tag{87}$$

Proof. In order to apply Lemma 6.3 we have to show that $w^{s+p'-1}\left(|\nabla u| + |\nabla v|\right)^{p'(p-2)} \in L^1_{loc}(\Omega)$.

Let $\phi \in \mathscr{C}^1_0(\Omega)$. An application of Hölder's inequality with exponent $z := \frac{p-1}{p-2}$ implies

$$\int_\Omega w^{s+p'-1} \frac{|\nabla\phi|^{p'}}{\phi^{p'-1}} \left(|\nabla u| + |\nabla v|\right)^{(p-2)p'} \leq \left(\int_S w^{s(p-1)+1} \frac{|\nabla\phi|^p}{\phi}\right)^{1/z'} \left(\int_S \left(|\nabla u| + |\nabla v|\right)^p\right)^{1/z}. \tag{88}$$

Since $|\nabla u|, |\nabla v| \in L^p_{loc}(\Omega)$ and $w^{s(p-1)+1} \in L^1_{loc}(\Omega)$ by hypotheses, we obtain the claim. Using (88) in (82) we obtain (85).

Now assuming (86), we have that $w^{q+s} \in L^1_{loc}(\Omega)$.

In order to complete the proof we begin observing that since $w \in L^p_{loc}(\Omega)$, by choosing $s = 1$, we have $u^{q+1} \in L^1_{loc}(\Omega)$.

If $\frac{q(p-1)-1}{p-2} \leq q+1$, there is nothing to prove. Else, if $\frac{q(p-1)-1}{p-2} > q+1$, that is if $q > p-1$, we shall use a bootstrap argument. If $w^r \in L^1_{loc}(\Omega)$, by the first part of the theorem, we have that $w^{h(r)} \in L^1_{loc}(\Omega)$ with $h(r) := q + \frac{r-1}{p-1}$. Therefore, setting $r_0 = q$ and $r_{n+1} := h(r_n)$ we easily verify that the sequence (r_n) is increasing and it converges to $\frac{q(p-1)-1}{p-2}$. This concludes the proof. □

Theorem 6.5 *Let $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy (77) with $p > 2$. Let $u, v \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ be distributional solutions of (54) with $q > 1$. Assume that*

$$\left(\fint_{B_R} |\nabla v|^p\right)^{1/p}, \left(\fint_{B_R} |\nabla u|^p\right)^{1/p} \leq cR^\theta \quad for \quad R\ large, \tag{89}$$

with

$$\theta < \frac{1}{p-2} - \frac{N}{p} \tag{90}$$

Then $v \leq u$.

Proof. Set $w := (v-u)^+$. From (51) we get that (86) is satisfied, and hence (87) and (85) hold. Therefore, fixing $0 < s < \frac{q-1}{p-2}$ we have that

$$\frac{q(p-1)-1}{p-2} > q+s > s(p-1)+1,$$

and hence $w^{s(p-1)+1} \in L^1_{loc}(\mathbb{R}^N)$. From (85) we have

$$\int_{\mathbb{R}^N} w^{q+s}\phi \le c\left(\int_A w^{s(p-1)+1}\frac{|\nabla\phi|^p}{\phi}\right)^{1/(p-1)}\left(\int_A (|\nabla u|+|\nabla v|)^p\right)^{(p-2)/(p-1)}.$$

By Hölder's inequality with exponent $x := \frac{q+s}{s(p-1)+1}$, and choosing $\phi = \phi_R^{p'}$ with ϕ_R as in the proof of Lemma 3.8, we have

$$\begin{aligned}\left(\int_{B_R} w^{q+s}\right)^{1-\frac{1}{x(p-1)}} &\le \left(\int_A \frac{|\nabla\phi|^{px'}}{\phi^{x'}}\right)^{\frac{1}{x'(p-1)}}\left(\int_A (|\nabla u|+|\nabla v|)^p\right)^{(p-2)/(p-1)}\\ &\le cR^{\frac{N-px'}{x'(p-1)}}R^{N\frac{p-2}{p-1}}\left(⨍_A (|\nabla u|+|\nabla v|)^p\right)^{(p-2)/(p-1)}\\ &\le cR^t. \end{aligned} \tag{91}$$

Here the last inequality follows from (89), where

$$t := \frac{N-px'}{x'(p-1)} + N\frac{p-2}{p-1} + \theta p\frac{p-2}{p-1} = \frac{N}{x'(p-1)} + p\frac{p-2}{p-1}\left(\theta + \frac{N}{p} - \frac{1}{p-2}\right).$$

Since

$$\lim_{s\to\frac{q-1}{p-2}^-} x' = \lim_{s\to\frac{q-1}{p-2}^-}\frac{q+s}{q-1-s(p-2)} = +\infty,$$

we can choose s so that x' is large enough and $t<0$. By this choice, letting $R\to+\infty$ in (91), we obtain that $w\equiv 0$. This proves our claim. □

Examples when the growth condition (89) holds are stated in the following results.

Proposition 6.6 *Let $p \ge 1$ and $q > 0$. Let $h \in L^1_{loc}(\mathbb{R}^N)$ and let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ such that $|u|^q \in L^1_{loc}(\mathbb{R}^N)$ be a p-regular solution of*

$$-\operatorname{div}(\mathscr{A}(x,u,\nabla u)) + |u|^{q-1}u = h \quad \text{on} \quad \mathbb{R}^N. \tag{92}$$

1. *Let $\mathscr{A}$ be* weakly elliptic *and assume that $u\,h \in L^1_{loc}(\mathbb{R}^N)$. Then $u \in L^{q+1}_{loc}(\mathbb{R}^N)$.*

2. Let $p>1$, $q>p-1$ and let $\mathscr{A}$ be **S**-*p*-**C**. *Assume that $u\,h\in L^1_{loc}(\mathbb{R}^N)$ and that $h\in L^{1+1/q}_{loc}(\mathbb{R}^N)$, then for any $R>0$ we have*

$$\int_{B_R}|u|^{q+1}+\int_{B_R}|\nabla u|^p\le cR^{N-p\frac{q+1}{q-p+1}}+c\int_{B_{2R}}|h|^{\frac{q+1}{q}}$$

where c is a positive constant depending only on p,q and N.

In particular, if there exist $\sigma\in\mathbb{R}$ and $C>0$ such that

$$\left(⨍_{B_R}|h|^{1+1/q}\right)^{q/(q+1)}\le CR^{\sigma}\quad for\quad R\ large, \tag{93}$$

then

$$\left(⨍_{B_R}|\nabla u|^p\right)^{1/p}\le CR^{\theta}\quad for\quad R\ large, \tag{94}$$

where

$$\theta:=\max\left\{\sigma\frac{q+1}{qp},-\frac{q+1}{q+1-p}\right\}.$$

Proof. Since $|u|^{q-1}u-h$ belongs to $L^1_{loc}(\mathbb{R}^N)$, we can choose any function of class $W^{1,p}_{comp}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ as test function in (92). In particular, the function $T_k(u)\phi$, where T_k is the truncation operator at level $k>0$ and ϕ is any nonnegative function of class $\mathscr{C}^1_0$, is admissible in (92). With this choice of test function we get:

$$\int\mathscr{A}(x,u,\nabla u))\cdot\nabla u\,\mathbf{1}_{|u|<k}\,\phi+\int|u|^{q-1}\,uT_k(u)\phi=-\int\mathscr{A}(x,u,\nabla u)\nabla\phi\,T_k(u)+\int hT_k(u)\phi.$$

The assumption $u\,h\in L^1_{loc}(\mathbb{R}^N)$, the Lebesgue's dominated convergence and the Beppo Levi's monotone convergence lead to

$$\int\mathscr{A}(x,u,\nabla u))\cdot\nabla u\,\phi+\int|u|^{q+1}\,\phi=-\int\mathscr{A}(x,u,\nabla u)\nabla\phi\,u+\int hu\phi. \tag{95}$$

From the latter, we immediately obtain $u\in L^{q+1}_{loc}(\mathbb{R}^N)$, recalling that $\mathscr{A}$ is weakly elliptic, $u\,h\in L^1_{loc}(\mathbb{R}^N)$ and u is p-regular.

Next we turn to the second part of the statement. To this end, we first replace ϕ by ϕ^p in (95). Then, by using the fact that $\mathscr{A}$ is **S**-p-**C** and by Young's inequality we get

$$\int|u|^{q+1}\,\phi^p+c_1\int|\nabla u|^p\,\phi^p\le c_2\int|\nabla\phi|^p\,|u|^p+\int|h|\,|u|\,\phi^p.$$

Next, we assume that $h\in L^{1+1/q}_{loc}(\mathbb{R}^N)$ and we replace ϕ by $\phi^{x'}$ in the latter inequality, where $x:=\frac{q+1}{p}>1$. By Young's inequality with exponents $x=\frac{q+1}{p}$ and $y:=q+1$, it follows that

$$c_3\int|u|^{q+1}\,\phi^{px'}+c_1\int|\nabla u|^p\,\phi^{px'}\le c_4\int|\nabla\phi|^{px'}+c_5\int|h|^{y'}\,\phi^{px'},$$

where $c_1, ..., c_5$ are positive constants depending only on p and q.

Next, by choosing $\phi = \phi_R$ as in Lemma 3.8, from the assumption on h we get

$$\fint_{B_R} |\nabla u|^p \leq c(R^{-px'} + R^{\sigma \frac{q+1}{q}}) \leq cR^{\max\{-px', \sigma \frac{q+1}{q}\}}.$$

□

Remark 6.7 *The assumption $u\,h \in L^1_{loc}(\mathbb{R}^N)$ is obviously satisfied when $u \in L^{q+1}_{loc}(\mathbb{R}^N)$ and $h \in L^{1+1/q}_{loc}(\mathbb{R}^N)$.*

This remark immediately leads to:

Corollary 6.8 *Let $q > p-1 > 1$. Let $\mathscr{A}$ be $\mathbf{S}$-p-$\mathbf{C}$ satisfying (77) and let $h \in L^{1+1/q}_{loc}(\mathbb{R}^N)$ be such that (93) holds for $\sigma \in \mathbb{R}$ and*

$$\max\left\{\sigma\frac{q+1}{qp}, -\frac{q+1}{q+1-p}\right\} < \frac{1}{p-2} - \frac{N}{p}.$$

Then problem

$$-\operatorname{div}(\mathscr{A}(x, \nabla u)) + |u|^{q-1}\, u = h \quad on \quad \mathbb{R}^N \tag{96}$$

has at most one distributional solution in the class $L^{q+1}_{loc}(\mathbb{R}^N) \cap W^{1,p}_{loc}(\mathbb{R}^N)$.

Proof. Under the above assumptions we can can apply Proposition (6.6) to get (94). The desired conclusion is then a consequence of Theorem (6.5). □

Remark 6.9 *The possible p-regular solutions belong to $L^{q+1}_{loc}(\mathbb{R}^N)$ in the following cases.*

1. *$p \geq N$, by Sobolev embedding.*
2. *$p < N$ and $q \leq \frac{N(p-1)+p}{N-p}$, again by Sobolev embedding.*
3. *$h \in L^{p'}_{loc}(\mathbb{R}^N)$, or $h \in L^{q'}_{loc}(\mathbb{R}^N)$ if $q \geq 1$. Indeed, $u\,h \in L^1_{loc}(\mathbb{R}^N)$ and, by Proposition 6.6, it follows that $u \in L^{q+1}_{loc}(\mathbb{R}^N)$.*

Corollary 6.10 *Let $q > p-1 > 1$. Let $\mathscr{A}$ be $\mathbf{S}$-p-$\mathbf{C}$ satisfying (77) and $h \in L^\infty(\mathbb{R}^N)$. If $N = 1$ or*

$$2 < p < \frac{2N}{N-1} \quad and \quad N > 1, \tag{97}$$

then problem (92) has at most one distributional solution of class $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$.

Proof. Since $h \in L^\infty(\mathbb{R}^N)$, the above remark implies that u belongs to $L^{q+1}_{loc}(\mathbb{R}^N)$. Since either $N = 1$ or (97) is in force, the claim follows from Corollary 6.8. □

Corollary 6.11 *Let $h \in L^1_{loc}(\mathbb{R}^N)$, $q > p-1$ and $p > 2$. Then the problem*

$$-\Delta_p u + |u|^{q-1}\, u = h \quad on \quad \mathbb{R}^N$$

has at most one distributional solution $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ such that $\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p < +\infty$.

Proof. It is enough to choose $\theta := -N/p$ in Theorem 6.5. $\square$

Now requiring stronger assumptions on the behavior of the gradient of the solutions, we have the following

Theorem 6.12 *Assume that $\mathscr{A}$ satisfies condition (77) with $p > 2$. Let u, v be p-regular solutions of (54) with $q > 1$.*

Let $\theta < \frac{1}{p-2}$ and assume that there exists $\alpha > \frac{N(p-2)}{1-\theta(p-2)}$ such that

$$\alpha \frac{q(p-1)-1}{p-2} > qp + p', \tag{98}$$

$$\left((v-u)^+\right)^{\alpha \frac{q(p-1)-1}{p(p-2)}} \in L^1(A_R) \quad for \quad R \quad large, \tag{99}$$

$$\left(\fint_{A_R} |\nabla v|^\alpha\right)^{1/\alpha}, \left(\fint_{A_R} |\nabla u|^\alpha\right)^{1/\alpha} \le cR^\theta \quad for \quad R \quad large, \tag{100}$$

where $A_R := B_{2R} \setminus \overline{B_R}$.

Then $v \le u$ a.e. on $\mathbb{R}^N$.

Remark 6.13 *The growth condition on the gradient in the above theorem is essentially sharp, in the sense that if $\theta > \frac{1}{p-2}$, then we can find $q > 1$ such that the problem admits non ordered solutions. See Appendix B for details.*

Proof. Let $w := (v-u)^+$, let $\phi = \phi_R$ as in the proof of Lemma 3.8 and set $s := \frac{\alpha}{p'(p-2)}(q-p'+1) - q$. Note that $s > 0$, since $s = \frac{\alpha}{p'(p-2)}(q-p'+1) - q = \alpha\frac{q(p-1)-1}{p(p-2)} - q > 0$ by (98). Also observe that $s - p' + 1 > 0$ if and only of (98) holds true, hence we see that $x := \frac{q+s}{s-p'+1} > 1$.

By (51) and Lemma 6.3, applying Hölder's inequality to (82) with exponent $x = \frac{q+s}{s-p'+1}$, we have

$$\int w^{q+s}\phi \le c_2 \left(\int_{A_R} w^{q+s}\phi\right)^{1/x} \left(\int_{A_R} (|\nabla u| + |\nabla v|)^{(p-2)p'x'} \frac{|\nabla \phi|^{p'x'}}{\phi^{(p'-1)x'}}\right)^{1/x'}. \tag{101}$$

where all the above integrals are finite thanks to (98), (99), (100) and the fact that $(p-2)p'x' = \alpha$ and $q + s = \frac{\alpha}{p'(p-2)}(q-p'+1)$.

Replacing ϕ by $\phi^{p'}$ in (101), it follows that

$$\int_{B_R} w^{q+s} \le c_2 c(\phi_1) R^{N-p'x'} \fint_{A_R} (|\nabla u| + |\nabla v|)^{(p-2)p'x'}\,. \tag{102}$$

Next, from (102) and (100), we obtain

$$\int_{B_R} w^{q+s} \le cR^{\gamma} \qquad where \;\; \gamma = N - p'x' + \theta(p-2)p'x'. \tag{103}$$

Finally, we observe that since

$$\gamma = N - \alpha\frac{1}{p-2} + \theta\alpha < N - (\frac{1}{p-2} - \theta)\frac{N(p-2)}{1-\theta(p-2)} = 0,$$

by letting $R \to +\infty$ in (103), the claim follows. □

We are now in position to prove Theorem 1.6 stated in the introduction. Its proof is based on the above Theorem 6.12 and on the following existence result (of independent interest).

Theorem 6.14 *Let $N \ge 1$, $q > p-1$, $p > 2 - \frac{1}{N}$. For every $h \in L^{1+1/q}_{loc}(\mathbb{R}^N)$ there exists $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ solution of*

$$-\Delta_p u + |u|^{q-1} u = h \qquad in \quad \mathcal{D}'(\mathbb{R}^N).$$

Proof. When $p > N$ this follows from Theorem 2 of Boccardo, Gallouet and Vazquez [3]. Thus, we bound ourselves to the case $p \le N$. It is enough to prove that, when $h \in L^{1+1/q}_{loc}(\mathbb{R}^N)$, the solution u constructed by Boccardo, Gallouet and Vazquez in [3, Theorem 1], actually belongs to $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. To this end, we recall that in [3] (cfr.(3.1)-(3.3) therein) the distributional solution u is obtained (up to an extraction of a subsequence) as weak limit in $W^{1,\theta}_{loc}(\mathbb{R}^N)$, for some $\theta \ge 1$, of the sequence (u_n) defined as follows:

for every $n \ge 1$, $u_n \in W^{1,p}_0(B_n)$ is the unique solution of

$$-\Delta_p u_n + |u|_n^{q-1} u_n = h_n \qquad in \quad \mathcal{D}'(B_n), \tag{104}$$

where $h_n := \inf(|h|, n)\mathrm{sign}(h) = T_n(h)$ (here we make no distinction between u_n defined on B_n and its canonical extension by zero outside the ball B_n).

In particular we have (cfr. (3.2) of [3]), for all $v \in W^{1,p}_0(B_n) \cap L^\infty(B_n)$:

$$\int_{B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v + \int_{B_n} |u_n|^{q-1} u_n v = \int_{B_n} h_n v. \tag{105}$$

To conclude we prove that (u_n) is bounded in $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. In order to demonstrate the latter claim we follow the proof of our Proposition 6.6. We fix $R > 0$ and plug into (105) the test function $v = T_k(u_n)\varphi_R$, where $k > 0, n > R+2$ and φ_R is a standard nonnegative cut-off function which is equal to 1 on B_R and zero outside B_{R+1}. With this choice of v and with the same arguments used in Proposition 6.6, we immediately get:

$$c_3 \int_{B_n} |u_n|^{q+1} \varphi_R^{px'} + c_1 \int_{B_n} |\nabla u_n|^p \, \varphi_R^{px'} \le c_4 \int_{B_n} |\nabla \varphi_R|^{px'} + c_5 \int_{B_n} |h_n|^{1+\frac{1}{q}} \varphi_R^{px'}, \qquad (106)$$

where x' is the conjugate exponent of $x := \frac{q+1}{p} > 1$ and $c_1, ..., c_5$ are positive constants depending only on p and q.

Using the assumption $h \in L^{1+1/q}_{loc}(\mathbb{R}^N)$ and the definition of h_n in (106) we obtain: for all $n > R+2$

$$c_3 \int_{B_n} |u_n|^{q+1} \varphi_R^{px'} + c_1 \int_{B_n} |\nabla u_n|^p \, \varphi_R^{px'} \le c_4 \int_{B_n} |\nabla \varphi_R|^{px'} + c_5 \int_{B_n} |h|^{1+\frac{1}{q}} \varphi_R^{px'}, \qquad (107)$$

hence, for all $n > R+2$

$$c_3 \int_{B_R} |u_n|^{q+1} \varphi_R^{px'} + c_1 \int_{B_R} |\nabla u_n|^p \, \varphi_R^{px'} \le c_4 \int_{B_{R+1}} |\nabla \varphi_R|^{px'} + c_5 \int_{B_{R+1}} |h|^{1+\frac{1}{q}}, \qquad (108)$$

which clearly yields the desired bound in $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$.

□

Proof of Theorem 1.6. Since h belongs to $L^\infty(\mathbb{R}^N)$, Theorem 6.14 gives the existence of a distributional solution in $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$. By Corollary 4.4 we have that every solution u of the considered equation belongs to $L^\infty(\mathbb{R}^N)$ and therefore $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a distributional solution of $-\Delta_p u = \tilde{h} \in L^\infty(\mathbb{R}^N)$. Thus, we can apply the classical results of [11] and [16] to get $|\nabla u| \in L^\infty(\mathbb{R}^N)$ and $u \in C^{1,\beta}_{loc}(\mathbb{R}^N)$ for some $\beta \in (0,1)$. The claim then follows by applying Theorem 6.12 with $\theta = 0$ and α large enough. □

As a final remark we note that the knowledge of a pointwise estimate on the gradient of the solutions on an exterior domain enable us to use Theorem 6.12. More precisely, an estimate of the type

$$|\nabla u(x)| \le c\,|x|^\theta \qquad for \quad |x| \quad large,$$

where $\theta < 1/(p-2)$, immediately leads to Theorem 1.7.

Proof of Theorem 1.7. The growth assumption on ∇u implies that (100) holds for any $\alpha > 0$. Since $|\nabla u|$ is locally bounded outside a large ball centered at the origin, the Sobolev embedding theorem implies that $u \in L^r(A_R)$ for any $r > 1$ and for R large. Therefore by choosing α large enough the claim follows from Theorem 6.12. □

A Inequalities and M-p-C Operators

Here, we shall prove some fundamental elementary inequalities that we use throughout the paper. Very likely these inequalities are well known, nevertheless for completeness we shall include their proof here.

In what follows we shall assume that $\mathscr{A}$ has the form

$$\mathscr{A}(x,\xi) = A(|\xi|)\xi,$$

where $\mathscr{A}\colon \mathbb{R}_+ \to \mathbb{R}$. We set $\phi(t) := A(t)t$.

Theorem A.1 *Let A be nonincreasing and bounded function such that*

$$\phi(0) = 0, \quad \phi(t) > 0 \text{ for } t > 0, \phi \textit{ is nondecreasing.} \tag{109}$$

Then $\mathscr{A}$ is **M**-p-**C** *with $p = 2$.*

Theorem A.2 *Let $1 < p \leq 2$. Let ϕ be increasing, concave function satisfying (109) and such that there exist positive constants $c_p, c_\phi > 0$ such that*

$$\phi(t) \leq c_p t^{p-1} \tag{110}$$

and

$$\phi'(s)s \leq c_\phi \phi(s). \tag{111}$$

Then $\mathscr{A}$ is **M**-p-**C**.

Remark A.3 *We notice that 110 is necessary condition for $\mathscr{A}$ to be an* **M**-p-**C** *operator. Indeed, if $\mathscr{A}$ is* **M**-p-**C**, *by taking $\eta = 0$, then it follows that $\mathscr{A}$ is* **W**-p-**C**, *and (110) holds by Hölder inequality.*

We set

$$I := (\mathscr{A}(\xi) - \mathscr{A}(\eta)) \cdot (\xi - \eta), \qquad J := |\mathscr{A}(\xi) - \mathscr{A}(\eta)|\,.$$

Our goal is to prove that there exists a constant $c > 0$ such that $I^{p-1} \geq cJ^p$.

We set $t := |\xi|$, $s := |\eta|$ and let θ be such that $\xi \cdot \eta = \theta\,|\xi|\,|\eta| = \theta ts$. Hence $\theta \in [-1.1]$, $t, s > 0$. Moreover by symmetry we can assume that $s \geq t$.

We rewrite I and J as

$$\begin{aligned} I &= A(|\xi|)\,|\xi|^2 + A(|\eta|)\,|\eta|^2 - A(|\xi|)\xi\cdot\eta - A(|\eta|)\xi\cdot\eta \\ &= \phi(t)t + \phi(s)s - \phi(t)s\theta - \phi(s)t\theta, \\ J^2 &= \phi^2(t) + \phi^2(s) - 2\phi(t)\phi(s)\theta. \end{aligned}$$

Remark A.4 *From (109) we deduce that: if $I = 0$ then $\phi(t) = \phi(s)$, $\theta = 1$, and $J = 0$. Indeed, assuming $s \geq t$*

$$I = \phi(t)(t - s\theta) + \phi(s)(s - t\theta) \geq \phi(t)(t - s\theta) + \phi(t)(s - t\theta) = \phi(t)(1 - \theta)(t + s) \geq 0$$

Therefore, if $I = 0$, then $\theta = 1$ or $\phi(t) = 0$. If $\phi(t) = 0$ then $t = 0$ and hence (since $I = 0$) also $s = 0$. If $\theta = 1$, then we have $0 = I = (\phi(s) - \phi(t))(s - t)$ and hence the claim follows.

We notice that if ϕ is increasing, then $I = 0$ implies also that $t = s$.

Therefore, in order to prove that $\mathscr{A}$ is **M**-*p*-**C**, *we restrict ourselves to the case $s > t > 0$.*

Proof of Theorem A.1. Set

$$I_1 := \frac{I}{\phi(t)t} = 1 + \frac{\phi(s)s}{\phi(t)t} - \theta\frac{\phi(s)}{\phi(t)} - \theta\frac{s}{t}$$

$$J_1 := \frac{J^2}{\phi^2(t)} = 1 + \frac{\phi^2(s)}{\phi^2(t)} - 2\theta\frac{\phi(s)}{\phi(t)}$$

We have

$$\begin{aligned} I_1 - J_1 &= \frac{\phi(s)s}{\phi(t)t} - \frac{\phi^2(s)}{\phi^2(t)} + \theta\frac{\phi(s)}{\phi(t)} - \theta\frac{s}{t} = \frac{\phi(s)}{\phi(t)}\left(\frac{s}{t} - \frac{\phi(s)}{\phi(t)}\right) - \theta\left(\frac{s}{t} - \frac{\phi(s)}{\phi(t)}\right) \\ &= \left(\frac{s}{t} - \frac{\phi(s)}{\phi(t)}\right)\left(\frac{\phi(s)}{\phi(t)} - \theta\right) = \frac{s}{t}\left(1 - \frac{A(s)}{A(t)}\right)\left(\frac{\phi(s)}{\phi(t)} - \theta\right). \end{aligned}$$

Since $s > t > 0$, ϕ is nondecreasing and A is nonincreasing it follows that $I_1 - J_1 \geq 0$. Therefore

$$I = \phi(t)tI_1 \geq \phi(t)tJ_1 = \frac{t}{\phi(t)}J^2 = \frac{1}{A(t)}J^2 \geq \frac{1}{||A||_\infty}J^2 = cJ^2,$$

that is the claim. □

Proof of Theorem A.2. Our goal is to show that $I^{p-1}/J^p \geq const > 0$.

We have

$$\frac{I^{p-1}}{J^p} = \frac{s^{p-1}}{\phi(s)}\,\frac{(\frac{I}{\phi(s)s})^{p-1}}{\left(\frac{J^2}{\phi^2(s)}\right)^{p/2}} = \frac{s^{p-1}}{\phi(s)}\,\frac{\left(1 + \frac{\phi(t)t}{\phi(s)s} - \theta\frac{\phi(t)}{\phi(s)} - \theta\frac{t}{s}\right)^{p-1}}{\left(1 + \frac{\phi^2(t)}{\phi^2(s)} - 2\theta\frac{\phi(t)}{\phi(s)}\right)^{p/2}} \geq cF(t, s, \theta),$$

where

$$F(t, s, \theta) := \frac{\left(1 + \frac{\phi(t)t}{\phi(s)s} - \theta\frac{\phi(t)}{\phi(s)} - \theta\frac{t}{s}\right)^{p-1}}{\left(1 + \frac{\phi^2(t)}{\phi^2(s)} - 2\theta\frac{\phi(t)}{\phi(s)}\right)^{p/2}}.$$

In order to prove the claim it is enough to show that F is uniformly positive for $\theta \in [-1,1]$ and $s > t > 0$. Since ϕ is nondecreasing, setting

$$\alpha := \frac{\phi(t)}{\phi(s)}, \quad z := \frac{t}{s},$$

it is enough to prove that

$$G(\alpha, z, \theta) := \frac{(1 + \alpha z - \alpha\theta - z\theta)^{p-1}}{(1 + \alpha^2 - 2\theta\alpha)^{p/2}}$$

is uniformly positive for $(\alpha, z, \theta) \in D := [0,1] \times [0,1] \times [-1,1] \setminus \{1,1,1\}$. The function G is well defined in D. Indeed the denominator vanishes if and only if $J = 0$ that is $\phi(t) = \phi(s)$ and $\theta = 1$ that is $\alpha = z = \theta = 1$. On the other hand the numerator of G vanishes if $I = 0$, that is if $\alpha = z = \theta = 1$. Therefore, G is strictly positive on D.

Moreover taking into account that ϕ is concave we have

$$\frac{\phi(s) - \phi(t)}{s - t} \leq \phi'(s),$$

which, together with (111), yields

$$\frac{\phi(s) - \phi(t)}{s - t} \, \frac{s}{\phi(s)} \leq \phi'(s) \frac{s}{\phi(s)} \leq c_\phi.$$

That is

$$1 - \alpha \leq c_\phi (1 - z). \tag{112}$$

Now the claim will follows by proving that

$$\liminf_{\alpha, z, \theta \to 1} G(\alpha, z, \theta) > 0. \tag{113}$$

Here the lim inf is computed for $(\alpha, z, \theta) \in D$ and under the constrained (112). Introducing

$$H(a, b, e) := \frac{(ab + 2e - ae - be)^{p-1}}{(a^2 + 2e - 2ae)^{p/2}}$$

and setting $a := 1 - \alpha$, $b := 1 - z$, $e := 1 - \theta$, (113) is equivalent to

$$\liminf_{a, b, e \to 0^+} H(a, b, e) > 0$$

with $a \leq c_\phi b$.

We shall argue by contradiction. Let a_n, b_n, e_n be three infinitesimal sequences such that $H(a_n, b_n, e_n) \to 0$. Since $a_n e_n$ and $b_n e_n$ are infinitesimal sequences of order greater then e_n, we have that

$$0 = \lim_n H(a_n, b_n, e_n) = \lim_n \frac{(a_n b_n + 2e_n)^{p-1}}{(a_n^2 + 2e_n)^{p/2}}.$$

Taking into account that $a_n \leq c_\phi b_n$, we have

$$0 = \lim_n \frac{(a_n b_n + 2e_n)^{p-1}}{(a_n^2 + 2e_n)^{p/2}} \geq \liminf_n \frac{(a_n^2/c_\phi + 2e_n)^{p-1}}{(a_n^2 + 2e_n)^{p/2}} > 0,$$

because $a_n^2/c_\phi + 2e_n$ and $a_n^2 + 2e_n$ are infinitesimal of the same order and $p \leq 2$. This contradiction concludes the proof. □

Remark A.5 *If $\mathscr{A}$ has the form*

$$\mathscr{A}(x, \xi) = a(x) A(|\xi|)\xi,$$

then the above Theorems A.1 and A.2 hold provided $a \in L^\infty(\mathbb{R}^N)$ and it is positive a.e..

B Counterexamples

In this section we provide counterexamples to the uniqueness property as well as to the comparison principles, as claimed in the introduction.

The next example is taken from [9, Example 3, section 11]. It shows that, for all $N \geq 1$, $1 < p \leq 2$, $q > 0$ with

$$q > p - 1, \qquad r + t > p, \qquad \nu = \frac{r + t - p}{q - p + 1} > \frac{t - N}{q} \tag{114}$$

the function

$$u(x) = (1 + |x|^2)^{\nu/2} \tag{115}$$

is a smooth positive solution of the weighted p-Laplacian equation

$$-\operatorname{div}[a(x)\, |\nabla u|^{p-2} \nabla u] + b(x)|u|^{q-1}\, u = 0 \quad on \quad \mathbb{R}^N \tag{116}$$

where

$$a(x) = C(x)^{2-p-r} |x|^r, \qquad b(x) = \nu^{p-1} \left\{ N + (\nu q + N - t)|x|^2 \right\} (1 + |x|^2)^{-1-t/2}, \tag{117}$$

with $C(x) = |x|/\sqrt{1 + |x|^2}$.

b is smooth and positive, while a is a nonnegative continuous function, since $p \leq 2$. Also the large radii condition (5) is satisfied for any $R_0 \geq 1$.

Since $u \equiv 0$ is also a solution, we see that both uniqueness and the comparison principle do not hold true in this situation, even for the homogeneous problem (116) of p-Laplace type with regular coefficients.

Following [9, Example 3, section 11], one can construct counterexamples also for the weighted mean-curvature operator. We leave the details to the interested reader.

Next we show that the restriction $q > p-1 > 0$ is necessary to obtain the uniqueness property and the comparison principles of Section 4.

1. *Case :* $q = p-1$. For any $p > 1$ we set $\alpha = (\frac{1}{p-1})^{\frac{1}{p}}$. For every $N \geq 1$ and every $t \in \mathbb{R}$, the function $u_t(x) = u_t(x_1, ..., x_N) = e^{\alpha(x_1 - t)}$ is a smooth solution of the equation
$$- \operatorname{div}[|\nabla u|^{p-2} \nabla u] + |u|^{q-1} u = 0 \quad on \quad \mathbb{R}^N. \tag{118}$$
It is also clear that the function $v_t(x) = u_t(-x_1, ..., x_N)$ solves the equation (118).

2. *Case :* $0 < q < p-1$. For any $p > 1$ and any q such that $0 < q < p-1$ we set $\gamma = \frac{p}{p-1-q} > 1$, $\omega = \gamma^{p-1}(p-1)(\gamma - 1) > \gamma^{p-1}$ and $\lambda = \omega^{-\frac{1}{p-1-q}}$. For every $N \geq 1$ and every $t \in \mathbb{R}$, the function
$$u_t(x) = u_t(x_1, ..., x_N) := \begin{cases} \lambda(x_1 - t)^{\gamma} & if \quad x_1 > t, \\ \\ 0 & if \quad x_1 \leq t, \end{cases} \tag{119}$$
is of class $C^1(\mathbb{R}^N)$ and such that $|\nabla u_t|^{p-2} \nabla u_t \in C^1(\mathbb{R}^N)$. An immediate calculation yields that u_t is a solution of (118), as well as the function $v_t(x) = u_t(-x_1, ..., x_N)$.

   Finally we observe that the latter example also shows that the restriction $\theta < \frac{1}{p-2}$ in Theorem (6.12) is essentially necessary. Indeed, for every $p > 2$ and every $\theta > \frac{1}{p-2}$, we can find $q \in (1, p-1)$, $\alpha > \frac{N(p-2)}{1-\theta(p-2)}$ and solutions u and v of (118) such that (98), (99) and (100) are satisfied but the functions u and v are not ordered.

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