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A class of Phase Transition problems with the Line Tension effect

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Giampiero Palatucci

Contents

Introduction

In this thesis we study a class of problems concerning the analysis of liquid-liquid phase transitions, from a variational point of view. In the literature, there are many variants of functionals of the Calculus of Variation, describing phase transitions phenomena.

We now give a brief overview of the *iter* that brings us through the choice of this class of problems.

In the classical theory of phase transitions, two-phase systems are modeled as follows: the container is represented by a bounded regular domain Ω in $\mathbb{R}^3$; every configuration of the fluid is described by a mass density u on Ω which takes only two values, α and β, corresponding to the phases $A := \{u = \alpha\}$ and $B := \{u = \beta\} = \Omega \setminus A$. The singular set of u (the set of discontinuity points of u) is the interface between the two phases, that we denote by Su.

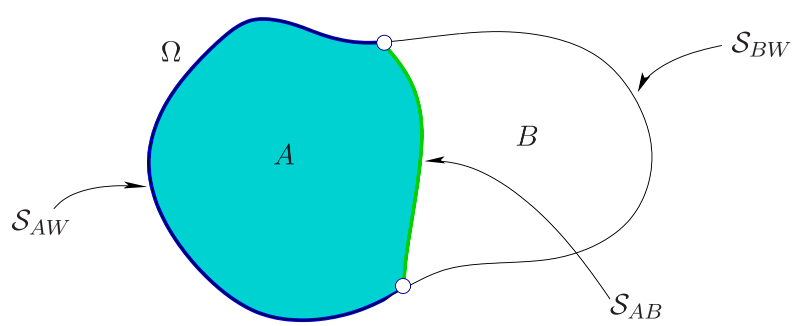

Figure 0.1: A two-phase system.

The space of admissible configurations is given by all $u : \Omega \to \{\alpha, \beta\}$ satisfying some volume constraint. The energy is located on the interface $\mathcal{S}_{AB} := Su$ which separates the two phases and on the contact surfaces $\mathcal{S}_{AW} := \partial A \cap \partial\Omega$ and $\mathcal{S}_{BW} := \partial B \cap \partial\Omega$ between the wall of the container and the phases A and B (see Fig. 0.1).

Hence, the equilibrium configurations are assumed to minimize the capillary energy

$$\mathcal{E}_0(A) := \sigma\mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW}\mathcal{H}^2(\mathcal{S}_{AW}) + \sigma_{BW}\mathcal{H}^2(\mathcal{S}_{BW}), \tag{0.1}$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. The positive constants σ, σ_{AW}, σ_{BW} in (0.1) are referred to as **surface tensions**.

In the late 50's, Cahn and Hilliard [24] proposed an alternative way to study two-phase fluids. They followed the continuum mechanics approach by Gibbs and assumed that the transition is not given by a separating interface, but is a continuous phenomenon occurring in a thin layer which, on a macroscopic level, is identified with the interface. In this region, a fine mixture of the two-phases fluid is allowed.
Hence, a configuration of the system is described by a mass density u which varies continuously from the value α to the value β, under a suitable volume constraint.

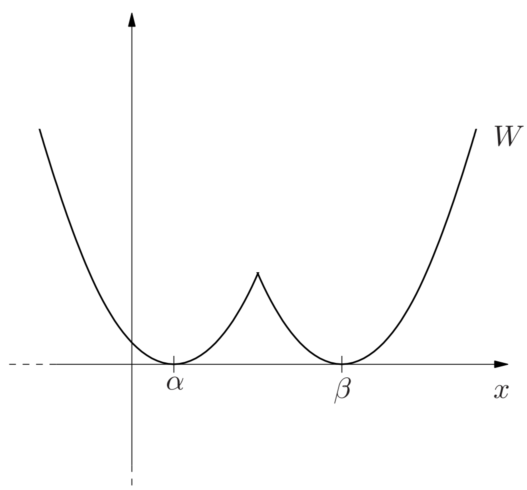

Figure 0.2: A double-well type potential W.

Neglecting the interactions between the fluid and the wall of the container, the energy in the Cahn-Hilliard model associated to u is the sum of a bulk term $\int_\Omega W(u)dx$, where W is a so-called "double-well potential" (a continuous positive function which vanishes only at α and β; see Fig. 0.2), and a singular perturbation $\varepsilon^2 \int_\Omega |Du|^2 dx$ which penalizes the spatial non-homogeneity of the fluid:

$$\varepsilon^2 \int_\Omega |Du|^2 dx + \int_\Omega W(u)dx, \tag{0.2}$$

where ε is a small parameter, giving the characteristic length of the thickness of the interface. This model is also known as "diffuse interface model" for phase transitions. Since the length ε is smaller than the size of the container, it is natural to study the equilibrium of the fluid in an asymptotic way, as ε goes to 0 (see Section 2.2).

A connection between the classical sharp interface model and the diffuse interface model, without taking into account the interactions with the wall, was established by Modica only in 1987 ([48]), by means of De Giorgi's notion of Γ-convergence (see Section 1.1). Modica proved that a suitable rescaling of the energy (0.2) Γ-converges to the surface energy functional $u \mapsto \sigma\mathcal{H}^2(Su)$; he was also able to prove that the minimizers arrange themselves in order to minimize the area of the separation interface. This result was conjectured by De Giorgi at the end of the 70's. It is important to remark that a great contribution to the work of Modica was already given by Modica himself and Mortola in [50], where a suitable scaling of energy (0.2) was proposed as a first interesting example of Γ-convergence. Since then, several results were given which extend the "Modica-Mortola" convergence result in different directions.

It is worth noting that the extension of (0.2) to a super-quadratic version; i.e., an energy with the perturbation of the form $\varepsilon^p \int_\Omega |Du|^p dx$ ($p > 2$), is an immediate consequence of the result by Modica (see Section 2.3). In [54] Owen and Sternberg treated the same problem of Modica, in a more general setting. They considered a wider class of quadratic perturbations that may give rise to anisotropic limits; i.e., qualitatively the limit is very similar to what one gets using the simplest perturbation $\varepsilon^2 \int_\Omega |Du|^2 dx$, but the surface tension may depend on the orientation of the interface (for more general anisotropic limits see also Barroso and Fonseca [16] and Bouchitté [19]).

Another variation of the Cahn-Hilliard functional arises as scalings of the free energy of a continuum limit of spin systems on lattices, or Ising systems. It is obtained by replacing the Dirichlet energy $\varepsilon^2 \int_\Omega |Du|^2 dx$ by suitable scalings of a non-local interaction

$$\iint_{\Omega\times\Omega} J_\varepsilon(x'-x)(u(x')-u(x))^2 dx' dx,$$

where $J_\varepsilon(y) := \varepsilon^{-N} J(y/\varepsilon)$, with J positive interaction potential in $L^1(\mathbb{R}^N)$. Also in this case the qualitative behavior of the functional is similar to the Modica-Mortola functional and the limit is possibly anisotropic (see Alberti and Bellettini in [5] and [6]).

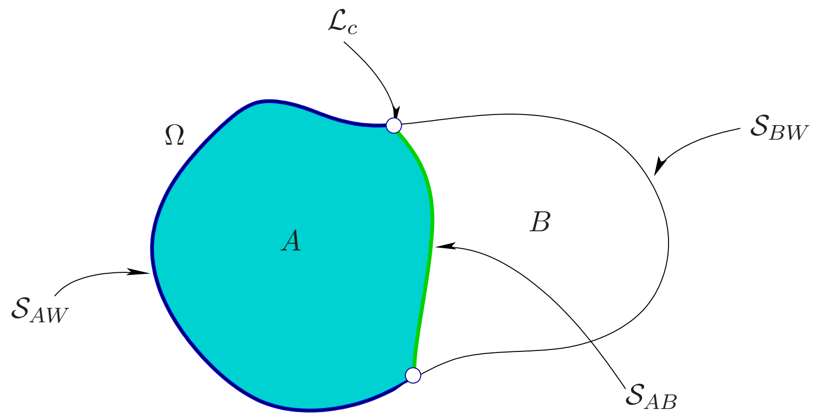

Figure 0.3: The line tension effect.

An extension of the classical model for two-phase fluids is obtained by adding to the energy (0.1) a **line tension** energy with density τ concentrated along the line $\mathcal{L}_c \equiv \partial \mathcal{S}_{AW}$, where $\mathcal{S}_{AB}$ meets the wall of the container; i.e., along the "contact line" (see Fig. 0.3). In this model, the capillary energy becomes

$$\mathcal{E}(A) := \sigma \mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW} \mathcal{H}^2(\mathcal{S}_{AW}) + \sigma_{BW} \mathcal{H}^2(\mathcal{S}_{BW}) + \tau \mathcal{H}^1(\mathcal{L}_c). \tag{0.3}$$

At the end of the 80's, Modica provided a partial rigorous connection between this classical model and the diffuse interface model. In [49], Modica added to (0.2) a boundary contribution of the form $\lambda \int_{\partial\Omega} g(Tu) d\mathcal{H}^2$, where Tu denotes the trace of u on $\partial\Omega$, λ does not depend on ε, and g is a positive continuous function:

$$E^g_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon} \int_\Omega W(u) dx + \lambda \int_{\partial\Omega} g(Tu) d\mathcal{H}^{N-1}. \tag{0.4}$$

Confirming a conjecture of Gurtin [43], Modica was able to prove that a sequence of minimizers (u_ε) for the energy (0.4) is pre-compact in $L^1(\Omega)$, each limit point u takes only the values α and β, and the corresponding phase $A := \{u = \alpha\}$ is a solution of the "liquid-drop" problem[1] associated to the energy (0.1).

[1] The problem of minimizing (0.1) is called "liquid-drop" problem and the existence of a solution is ensured by the "wetting condition" $\sigma \geq |\sigma_{AW} - \sigma_{BW}|$ (see Section 2.1 and 2.4).

In the 90's, Alberti, Bouchitté and Seppecher ([9]) proved that, due to a lack of semicontinuity, the functional $\mathcal{E}$ leads to ill-posed minimum problems. They explicitly computed the relaxation of $\mathcal{E}$ and showed that the total energy can be properly written by introducing, besides the usual bulk phase $A \subset \Omega$, an additional variable $A' \subset \partial\Omega$, independent of ∂A, with its complement $B' := \partial\Omega \setminus A'$: the *boundary phases*.

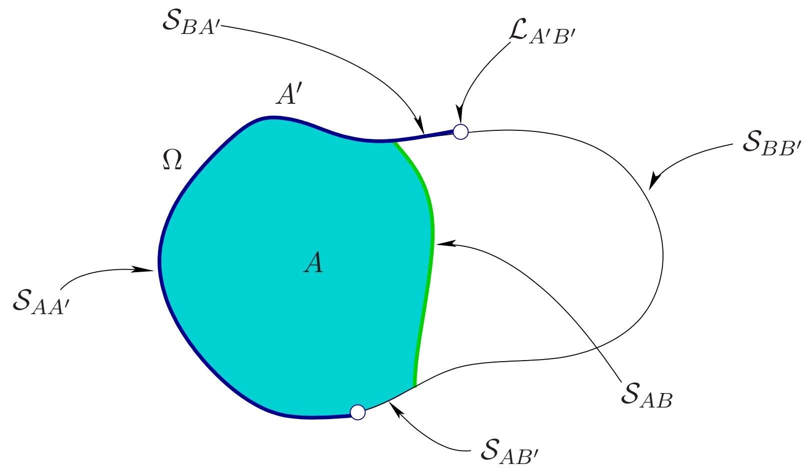

Figure 0.4: An arbitrary configuration (A, A').

In this view, the total energy $\tilde{\mathcal{E}}$ of the configurations (A, A') is given by the sum of three different terms: the classical surface tension on the interface between the bulk phases A and B, a surface density on the wall of the container (depending on which bulk phase and which boundary phase meet together) and a line density along the line $\mathcal{L}_{A'B'}$ (the *dividing line*) which separates the boundary phases A' and B' (see Fig. 0.4):

$$\begin{aligned}\tilde{\mathcal{E}}(A, A') \;:=\; & \sigma\mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW}\mathcal{H}^2(\mathcal{S}_{AA'}) + (\sigma + \sigma_{AW})\mathcal{H}^2(\mathcal{S}_{AB'}) + \sigma_{BW}\mathcal{H}^2(\mathcal{S}_{BB'}) \\ & + (\sigma + \sigma_{BW})\mathcal{H}^2(\mathcal{S}_{BA'}) + \tau\mathcal{H}^1(\mathcal{L}_{A'B'}). \qquad (0.5)\end{aligned}$$

Therefore $\mathcal{E}$ can be written in terms of $\tilde{\mathcal{E}}$ by choosing $A' = \mathcal{S}_{AW}$. It is important to stress that the boundary phase A' may differ from the interface $\mathcal{S}_{AW}$ between the bulk phase A and the wall of the container, at equilibrium. Hence, the line tension is located on the dividing line $\mathcal{L}_{A'B'}$, which in general does not agree with the contact line $\mathcal{L}_c$ (see [9], Example 5.2, p. 35). In this case, Alberti, Bouchitté and Seppecher speak of "dissociation of the contact line and the dividing line".

In order to properly establish a connection with the associated model for capillarity with line tension, they studied the asymptotic behavior of the following functional

$$\tilde{E}_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)d\mathcal{H}^2, \tag{0.6}$$

where V is a double well potential with wells α' and β' (corresponding to the boundary phases A' and B'), and λ_ε satisfies:

$$\varepsilon \log \lambda_\varepsilon \to k \in (0, +\infty) \text{ as } \varepsilon \text{ goes to } 0. \tag{0.7}$$

This logarithmic scaling[2] provides a uniform control on the oscillation of Tu_ε, the traces of minimizing sequences u_ε, and ensures that the transition of Tu_ε from α' to β' takes place in a thin layer. In fact, Alberti, Bouchitté and Seppecher proved that, under (0.7), the traces Tu_ε converge (up to a subsequence) to a function v in $BV(\partial\Omega, \{\alpha', \beta'\})$ and then the boundary phases $\{v = \alpha'\}$ and $\{v = \beta'\}$ are divided by the line Sv. Namely, the asymptotic behavior of $\tilde{E}_\varepsilon$ is described by a functional Ψ which depends on the two variables u and v:

$$\begin{aligned} \Psi(u, v) \ := \ & \sigma\mathcal{H}^2(Su) + \int_{\partial\Omega} |\tilde{H}(Tu) - \tilde{H}(v)|d\mathcal{H}^2 + \tau\mathcal{H}^1(Sv), \\ & \forall \ (u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}), \end{aligned} \tag{0.8}$$

where $\sigma := |\tilde{H}(\beta) - \tilde{H}(\alpha)|$, being $\tilde{H}$ a primitive of $2W^{1/2}$; and $v : \partial\Omega \to \mathbb{R}$ is the so-called boundary mass density.

The proof of this Γ-convergence result requires several steps in which different effects are analyzed and then different terms of the limit energy Ψ are deduced. The first term of the limit energy can be evaluate like in [48], while the second term is obtained by adapting the approach by Modica in [49]. Via “localization” and slicing techniques it is possible to reduce the analysis of the line tension effect to the asymptotic analysis of the following functional defined on a two-dimensional half-disk:

$$E^2_\varepsilon(u) := \varepsilon \int_{D_r} |Du|^2 dx + \lambda_\varepsilon \int_{E_r} V(Tu)d\mathcal{H}^1, \tag{0.9}$$

where, for every $r > 0$, we denote by

$$\begin{aligned} D_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 > 0\right\}, \\ E_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 = 0\right\}. \end{aligned} \tag{0.10}$$

[2]The choice of this scaling will be explained in Section 2.5, Remark 2.7.

Then the two-dimensional Dirichlet energy (0.9) is replaced on the half-disk D_r by the $H^{1/2}$ intrinsic norm on the "diameter" E_r. This is possible thanks to the existence of an optimal constant for the trace inequality involving the L^2 norm of the gradient of a function defined on a two-dimensional domain and the $H^{1/2}$ norm of its trace on a line. Hence, the original problem is reduced to the analysis of a new kind of perturbation problem involving a non-local term:

$$E^1_\varepsilon(v) := \varepsilon \iint_{I\times I} \left| \frac{v(t)-v(t')}{t-t'} \right|^2 dt dt' + \lambda_\varepsilon \int_I V(v)\, dt, \tag{0.11}$$

where I is an open interval of $\mathbb{R}$ and λ_ε satisfies the condition (0.7).

Alberti, Bouchitté and Seppecher analyze the asymptotic behavior of (0.11) in [8] and prove that

$$E^1_\varepsilon \xrightarrow{\Gamma} 2k(\beta'-\alpha')^2 \mathcal{H}^0(Sv),$$

and strongly use this result to obtain the boundary term in (0.8).

Note that the qualitative asymptotic behavior of (0.11) departs from all the examples that we mentioned before. In this case the logarithmic scaling has many special effects. In contrast with what happens to the classical Modica-Mortola functional (and similar), in this case all the energy of the limit comes from the non-local term; so that (0.11) does not produce "equi-partition of energy". Moreover the limit line tension energy is not characterized by an optimal profile problem, which instead is the case of the Modica-Mortola energy and all the variants that we recalled above; so that the transition between two boundary phases is always optimal as far as it occurs on a layer of order $1/\lambda_\varepsilon$.

The same phemonena has been observed in other variants of the energy (0.11), related to the study of boundary vortices (see Kurzke [46], [47]), or to the study of a phase field model for defects in crystals (see Garroni and Müller [39]). What those results have in common is the presence of a non-local singular (non L^1) regularization of $H^{1/2}$ type.

Other results concerning a functional of the type (0.6) are obtained replacing the Dirichlet energy $\varepsilon^2 \int_\Omega |Du|^2 dx$ by the singular perturbation $\varepsilon^2 \int_\Omega |D^2u|^2 dx$ (see Sousa [55]).

The analysis of a non-local singular perturbation problem involving a functional of the form (0.11) is also the first contribution of this thesis. In Chapter 3, we investigate the asymptotic behavior in terms of Γ-convergence of the following functional

$$K_\varepsilon(v) := \varepsilon^{p-2} \iint_{I\times I} \left| \frac{v(t)-v(t')}{t-t'} \right|^p dt dt' + \frac{1}{\varepsilon} \int_I V(v)\, dt \quad (p>2), \tag{0.12}$$

as $\varepsilon \to 0$.

In contrast with (0.11), the functional K_ε shares similar properties with the Modica-Mortola functional, such as the following scaling property

$$K_\varepsilon(v, I) = K_1(v^{(\varepsilon)}, I/\varepsilon), \tag{0.13}$$

where $v^{(\varepsilon)}(t) := v(\varepsilon t)$ and $I/\varepsilon := \{t : \varepsilon t \in I\}$. In view of this scaling property, it is natural to consider the optimal profile problem

$$\gamma := \inf \left\{ \iint_{\mathbb{R}\times\mathbb{R}} \left| \frac{w(t) - w(t')}{t - t'} \right|^p dt dt' + \int_{\mathbb{R}} V(w)\, dt : w \in W^{1-\frac{1}{p},p}_{\text{loc}}(\mathbb{R}), \lim_{t\to-\infty} w(t) = \alpha', \lim_{t\to+\infty} w(t) = \beta' \right\}. \tag{0.14}$$

In [40], we prove that the asymptotic behavior of K_ε is described by the following functional

$$K(v) := \gamma \mathcal{H}^0(Sv), \qquad v \in BV(I, \{\alpha', \beta'\}). \tag{0.15}$$

Theorem 0.1. *Let $K_\varepsilon : W^{1-\frac{1}{p},p}(I) \to \mathbb{R}$ and $K : BV(I, \{\alpha', \beta'\}) \to \mathbb{R}$ be defined by (0.12) and (0.15).*

Then

(i) [Compactness] *If $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ is a sequence such that $K_\varepsilon(v_\epsilon)$ is bounded, then (v_ϵ) is pre-compact in $L^1(I)$ and every cluster point belongs to $BV(I, \{\alpha', \beta'\})$.*

(ii) [Lower bound inequality] *For every $v \in BV(I, \{\alpha', \beta'\})$ and every sequence $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $v_\epsilon \to v$ in $L^1(I)$,*

$$\liminf_{\varepsilon\to 0} K_\varepsilon(v_\epsilon) \geq K(v).$$

(iii) [Upper bound inequality] *For every $v \in BV(I, \{\alpha', \beta'\})$ there exists a sequence $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $v_\epsilon \to v$ in $L^1(I)$ and*

$$\limsup_{\varepsilon\to 0} K_\varepsilon(v_\epsilon) \leq K(v).$$

In the proof of the theorem, we strongly use the "localization" and the scaling property of K_ε. Moreover, an important role will be played by the monotonicity properties of K_ε with respect to truncations and monotone rearrangements (see Section 1.4.1). Using a monotone rearrangement result we will also prove that the infimum in (0.14) is not trivial and is achieved.

The study of the functional K_ε has its own interest, showing that the analysis performed by Modica and Mortola is stable under a much larger class of perturbations, including non-local singular perturbations, as far as they are not "critical" in the sense of trace imbedding. On the other hand it has also been the first step towards the comprehension of the main problem of this thesis, that concerns the study of a functional similar to (0.6), but with a super-quadratic growth in the perturbation term.

For every $\varepsilon > 0$, we consider the functional F_ε defined by

$$F_\varepsilon(u) := \varepsilon^{p-2} \int_\Omega |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u)dx + \frac{1}{\varepsilon} \int_{\partial\Omega} V(Tu) d\mathcal{H}^2. \tag{0.16}$$

This functional still describes a capillarity problem (0.5), but its asymptotic behavior brings out different characteristics with respect to the energy (0.6).

Let us briefly analyze the asymptotic behavior of the functional F_ε. If (u_ε) is a sequence with equi-bounded energy, we observe that the term $\frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u_\varepsilon)dx$ forces u_ε to take values close to α and β, while the term $\varepsilon^{p-2} \int_\Omega |Du_\varepsilon|^p dx$ penalizes the oscillations of u_ε. We will see that when ε tends to 0, the sequence (u_ε) converges (up to a subsequence) to a function u, that belongs to $BV(\Omega)$, which takes only the values α and β. Moreover each u_ε has a transition from the value α to the value β in a thin layer close to the surface Su, which separates the bulk phases $\{u = \alpha\}$ and $\{u = \beta\}$. Similarly, the boundary term of F_ε forces the traces Tu_ε to take values close to α' and β', and the oscillations of the traces Tu_ε are again penalized by the integral $\varepsilon^{p-2} \int_\Omega |Du_\varepsilon|^p dx$. Then, as for the case of the functional studied by Alberti, Bouchitté and Seppecher, we expect that the sequence (Tu_ε) converges to a function v in $BV(\partial\Omega)$ which takes only the values α' and β', and that a concentration of energy occurs along the line Sv, which separates the boundary phases $\{v = \alpha'\}$ and $\{v = \beta'\}$.

In view of possible "dissociation of the contact line and the dividing line", we recall that Tu may differ from v. Since the total energy $F_\varepsilon(u_\varepsilon)$ is partly concentrated in a thin layer close to Su (where u_ε has a transition from α to β), partly in a thin layer close to the boundary (where u_ε has a transition from Tu to v), and partly in the vicinity of Sv (where Tu_ε has a transition from α' to β'), we expect that the limit energy is the sum of a surface energy concentrated on Su, a boundary energy on $\partial\Omega$ (with density depending on the gap between Tu and v), and a line energy concentrated along Sv.

The asymptotic behavior of the functional F_ε is described by a functional Φ which depends on the two variables u and v. If $\mathcal{W}$ is a primitive of $W^{(p-1)/p}$, we prove that for every $(u,v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$

$$\Phi(u,v) = \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 + \gamma_p \mathcal{H}^1(Sv), \tag{0.17}$$

where Su and Sv denote the jump set of u and v, respectively; c_p and σ_p are two positive constants defined by $c_p := \dfrac{p}{(p-1)^{p/(p-1)}}$; $\sigma_p := c_p|\mathcal{W}(\beta) - \mathcal{W}(\alpha)|$; γ_p is given by the optimal profile problem

$$\gamma_p := \inf \left\{ \int_{\mathbb{R}^2_+} |Du|^p dx + \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 : u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} |Du|^p dx \text{ is finite}, \right. \\ \left. \lim_{t\to-\infty} Tu(t) = \alpha', \ \lim_{t\to+\infty} Tu(t) = \beta' \right\}. \tag{0.18}$$

In Chapter 6 we prove the main convergence result, stated in the following theorem.

Theorem 0.2. *Let $F_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ and $\Phi : BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\}) \to \mathbb{R}$ defined by (0.16) and (0.17).*

Then

(i) [Compactness] *If $(u_\varepsilon) \subset W^{1,p}(\Omega)$ is a sequence such that $F_\varepsilon(u_\varepsilon)$ is bounded, then $(u_\varepsilon, Tu_\varepsilon)$ is pre-compact in $L^1(\Omega) \times L^1(\partial\Omega)$ and every cluster point belongs to $BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$.*

(ii) [Lower Bound Inequality] *For every $(u,v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$,*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v).$$

(iii) [Upper Bound Inequality] *For every $(u,v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$ and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u,v).$$

Notice that the limit functional Φ is of the same form of the functional Ψ defined by (0.8); that is, the Γ-limit of $\tilde{E}_\varepsilon$. Nonetheless, the variation in the power of the gradient in the perturbation term is not a simple generalization of the quadratic case, since the structure of these two problems is different. In fact, in the quadratic case the natural scaling of the energy is logarithmic and this implies that the profile in which the phase transition occurs on the boundary is not important for the first order of the energy. While, the super-quadratic case is characterized by an optimal profile problem which determines the line tension in the limit. This characteristic will be a double-edged sword in the proof of Theorem 0.2: some arguments will be simplified by the presence of an optimal profile problem, some other will require more care.

The proof of Theorem 0.2 requires several steps. We can deduce the terms of the limit energy Φ, localizing three effects: the bulk effect, the wall effect and the boundary effect.

In the bulk term, the limit energy can be evaluate like in [48]. We will use the super-quadratic version of the Modica-Mortola functional (see Section 2.2).

The second term of Φ can be obtained by adapting the approach by Modica in [49]. Since the results by Modica concerns a functional with quadratic growth in the singular perturbation term and with a boundary contribution of the form $\lambda \int_{\partial\Omega} g(Tu)d\mathcal{H}^2$, with λ not depending on ε and g a positive continuous function, we need to adapt part of the results in [49] to our goal (see Chapter 4).

Finally, the boundary effect requires a deeper analysis. The main strategy consists in: first reducing to the case in which the boundary is "flat"; hence studying the behavior of the original energy in the three-dimensional half ball; then reducing the problem of one dimension via a slicing argument. Thus, the main problem becomes the analysis of the asymptotic behavior of the following two-dimensional functional

$$H_\varepsilon(u) := \varepsilon^{p-2} \int_{D_1} |Du|^p dx + \frac{1}{\varepsilon} \int_{E_1} V(Tu) d\mathcal{H}^1, \tag{0.19}$$

where D_1 and E_1 are defined by (0.10). Chapter 5 is devoted to the analysis of the asymptotic behavior of (0.19) and to the proof of the existence of a minimum for the optimal profile (0.18). We remark that we can not reduce further to a one dimensional problem, like in the case $p = 2$, where one can use the optimal trace imbedding. In spite of that, as a consequence of equi-partition of the energy, the optimal profile problem plays an important role in the proof. In this respect, the case $p = 2$ represent the critical case in the context of this type of non-local perturbations.

A similar dichotomy occurs in the case of Ginzburg-Landau problems (see for instance Alberti, Baldo and Orlandi [4] versus Desenzani and Fragalà [29]).

Plan of the thesis

Chapter 1. We introduce the notion of Γ-convergence, with its main properties, and we state some preliminary results about Young measures, the slicing method and monotone rearrangements.

Chapter 2. We briefly explain the Cahn-Hilliard model for phase transitions and state some preliminary convergence results for Cahn-Hilliard functionals, with or without taking into account the interactions with the wall.

Chapter 3. We study the asymptotic behavior of the non-local perturbed energy (0.12), providing a complete proof of the Γ-convergence result stated in Theorem 0.1 and a characterization of the optimal profile problem (0.14).

Chapter 4. We introduce the main problem of this thesis, namely the analysis of the asymptotic behavior of the functional (0.16); we also exhibit the strategy of the proof of Theorem 0.2.

Chapter 5. We study the asymptotic behavior of the two-dimensional functional (0.19) and we provide a proof of the existence of a minimum for the optimal profile problem (0.18).

Chapter 6. We prove the main results in this thesis, namely the compactness and the Γ-convergence results stated in Theorem 0.2.

Chapter 1

Preliminaries

In this chapter, we briefly give a definition of Γ-convergence and we state some results about Young measures, the slicing method and the behavior of certain classes of integral functionals with respect to the monotone rearrangement of functions. Every statement in this chapter will be presented in the form that better fits to our purposes.

Notation

In this work, we consider different domains A with dimensions $N = 1, 2, 3$; more precisely, A will always be a bounded open set either of $\mathbb{R}^N$. We denote by ∂A the boundary of A relative to the ambient manifold; ∂A is always assumed to be Lipschitz regular. Unless otherwise stated, A is endowed with the corresponding N-dimensional Hausdorff measure, $\mathcal{H}^N$ (see [32], Chapter 2). We write $\int_A f dx$ instead of $\int_A f d\mathcal{H}^N$, and $|A|$ instead of $\mathcal{H}^N$.

The N-dimensional density of A at x is the limit (if it exists) of $\mathcal{H}^N(A \cap B_r(x))/\omega_N r^N$, where $B_r(x)$ is the ball centered in x with radius r and ω_N is the measure of the unit ball in $\mathbb{R}^N$.

The *essential boundary* of A is the set of all points where A has neither density 0 nor 1 and where the density does not exist. Since the essential boundary agrees with the topological boundary when the latter is Lipschitz regular, we also denote the essential boundary by ∂A.

For every $u \in L^1_{\text{loc}}(A)$, we denote by Su the complement of the set of Lebesgue points of u; i.e., the *jump set*, the set where the upper and lower approximate limits of u differ or are not finite. We denote by Du the derivative of u in the sense of distributions. As usual, for every $p \geq 1$, $W^{1,p}(A)$ is the Sobolev space of all $u \in L^p(A)$ such that $Du \in L^p(A)$; $BV(A)$ is the space of all $u \in L^1(A)$ with bounded variation; i.e., such that Du is a bounded Borel measure on A.

For every $s \in (0,1)$ and every $p \geq 1$, $W^{s,p}$ is the space of all $u \in L^p(A)$ such that the fractional semi-norm $\displaystyle\iint_{A\times A} \frac{|u(x)-u(x')|^p}{|x-x'|^{sp/N}} dxdx'$ is finite.

We denote by T the trace operator which maps $W^{1,p}(A)$ onto $W^{1-1/p,p}(\partial A)$ and $BV(A)$ onto $L^1(\partial A)$. For details and results about the theory of BV functions and Sobolev spaces we refer to [32] and [1].

Throughout this thesis, all the functions and sets are assumed to be Borel measurable. Moreover, we always use the term "sequence" also to denote families (of functions) labelled by continuous parameter ε, which tends to 0. Thus, a subsequence of (u_ε) is any sequence (u_{ε_k}) such that $\varepsilon_k \to 0$ as $k \to +\infty$, and we say that (u_ε) is pre-compact if every subsequence admits a convergent sub-subsequence. To simplify the notation, we often do not relabel subsequences.

1.1 Γ-Convergence

Γ-convergence was introduced by De Giorgi in the early 70's. Its first definition was stated in [28], where all the main properties were presented. Γ-convergence is linked to previous notions of convergence such as Mosco's convergence (see [51]) or Kuratowski's convergence of sets. Indeed, Γ-convergence of a sequence of functions can be viewed as a convergence of their epigraphs (*epiconvergence*), just like semicontinuity can be seen as a property of the epigraphs.

We give the definition and the main properties of Γ-convergence as a notion of convergence for functions on a generic metric space X. Therefore, in the following, u is an element of X and F a function from X to $\overline{\mathbb{R}} := [-\infty, +\infty]$. Here, we present a simplified version of the original definition; we refer to [2] (see [27] and [23] for a detailed treatment of the general theory of Γ-convergence and various applications).

Definition 1.1. *We say that a sequence $F_\varepsilon : X \to \overline{\mathbb{R}}$* Γ-converges *to F on X as ε to 0 if for every $u \in X$ the following conditions hold:*

(i) [Lower bound inequality] *for every sequence* (u_ε) *converging to* u *in* X

$$F(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon); \tag{1.1}$$

(ii) [Upper bound inequality] *there exists a sequence* (u_ε) *converging to* u *in* X *such that*

$$F(u) = \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon). \tag{1.2}$$

Condition (i) means that whatever sequence we choose to approximate u, the value of $F_\varepsilon(u_\varepsilon)$ is, in the limit, larger than $F(u)$. On the other hand, condition (ii) implies that this bound is sharp; that is, there always exists a sequence (u_ε) which approximates u so that $F_\varepsilon(u_\varepsilon) \to F(u)$.

When proving a Γ-convergence result, it is convenient to reduce the amount of verifications and constructions. To this aim, note that if (i) holds, then equality (1.2) can be replaced by

$$F(u) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

From Definition 1.1, we may deduce the following properties of Γ-convergence:

(p1) The Γ-limit F is always lower semicontinuous on X;

(p2) Γ-limits are stable under continuous perturbations. This means that one Γ-limit is computed we do not have to redo all computations if "lower-order terms" are added. Conversely, we can always remove such terms to simplify calculations;

(p3) Under suitable conditions Γ-convergence implies convergence of minimum values and minimizers. Note that some minimizers of the Γ-limit may not be limit of minimizers, so that Γ-convergence may be interpreted as a choice criterion.

Now, we pass to describe how this notion of variational convergence will be used.

Assume that for every $\varepsilon > 0$ we are given a function u_ε which minimizes the functional F_ε on X, and that we want to know what happens to u_ε as ε goes to 0. Sometimes, the minimizers u_ε can be written via some explicit formula from which we can deduce all information about its asymptotic behavior. In many instances, no such representation of u_ε is available and then we can exploit the fact that each u_ε solves the Euler-Lagrange equation associated with F_ε and try to understand which kind of limit equation is verified by a limit point u of (u_ε). Another possibility is to use the notion of Γ-convergence.

Suppose that we have computed the Γ-limit F of the functional F_ε, by property (p3) we can conclude that any limit point u is a minimizer of F, and in particular solves the Euler-Lagrange equation associated with F.

Notice that such a strategy makes sense only if we know *a priori* that the minimizing sequence (u_ε) is pre-compact in X. A Γ-convergence result for the functional F_ε should always be paired with a "compactness result" for the corresponding minimizing sequences (u_ε). According to this viewpoint, the Definition 1.1 have to be completed by the following equi-coercivity of F_ε:

(iii) [Compactness] *If (u_ε) is a sequence such that $F_\varepsilon(u_\varepsilon)$ is bounded, then (u_ε) is pre-compact in X.*

1.1.1 Choice of interesting rescalings

If u_ε minimizes F_ε, then it also minimizes $\lambda_\varepsilon F_\varepsilon$, for every positive λ_ε. Hence, we can recover information about the limit points of (u_ε) also by the Γ-limit of $\lambda_\varepsilon F_\varepsilon$. Notice that different choices of the scaling factor λ_ε generate different Γ-limits, which provide different information. For instance, it may happen that the functional F_ε converges to a constant functional F, and consequently we have no information about the limit points u, while the Γ-limit of functionals $\lambda_\varepsilon F_\varepsilon$ may be less trivial. Therefore, before trying to verify Definition 1.1 and a compactness result, it is important to find a suitable λ_ε so that the Γ-limit of the rescaled functional $\lambda_\varepsilon F_\varepsilon$ gives the largest amount of information. Sometimes this optimal rescaling is evident but sometimes it is not.

1.2 Young Measures

Young measures (also called "generalized functions") were introduced by Young in the 30's to solve optimal control problems that have no classical solution. The idea is to replace functions which take values in a set $A \subset \mathbb{R}^N$ by functions that take values in the space of probability measures. For instance, a mixture of two states can be represented by a convex combination of two Dirac measures (see [58]). Since then, Young measures arguments have been used in various problems.

In the following, we introduce Young measures associated to a sequence, that are appropriate to describe the limit oscillation behavior of the sequence self. In few words, under mild assumption, any sequence (u_n) of functions has a subsequence which converges to some Young measure.

We state some results, which will be useful to our purposes. We refer to Müller in [53] and Valadier in [56].

1.2.1 The fundamental theorem on Young measures and applications

By $C_0(\mathbb{R}^N)$ we denote the closure of continuous functions on $\mathbb{R}^N$ with compact support. The dual of $C_0(\mathbb{R}^N)$ can be identified with the space $\mathcal{M}(\mathbb{R}^N)$ of signed Radon measures with finite mass via the pairing

$$\langle \mu, f\rangle = \int_{\mathbb{R}} f d\mu.$$

A map $\mu : A \to \mathcal{M}(\mathbb{R}^N)$ is called "weak-$*$ measurable" if the functions $x \to \langle \mu(x), f\rangle$ are measurable for all $f \in C_0(\mathbb{R}^N)$.

Theorem 1.2. ([53], Theorem 3.1, p. 31). *Let $A \subset \mathbb{R}^N$ be a measurable set of finite measure and let $u_n : A \to \mathbb{R}^N$ be a sequence of measurable functions. Then there exists a subsequence (u_{n_k}) and a weak-$*$ measurable map $\nu : A \to \mathcal{M}(\mathbb{R}^N)$ such that the following statements hold:*

(i) $\nu_x \equiv \nu(x) \geq 0,\ \|\nu_x\|_{\mathcal{M}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} d\nu_x \leq 1,$ *for a.e.* $x \in A$.

(ii) For all $f \in C_0(\mathbb{R}^N)$

$$f(u_{n_k}) \overset{*}{\rightharpoonup} \bar{f} \ \textit{in}\ L^\infty(A),$$

where

$$\bar{f} = \langle \nu_x, f\rangle = \int_{\mathbb{R}^N} f d\nu_x.$$

(iii) Let $K \subset \mathbb{R}^N$ *be compact. Then*

$$\mathrm{supp}(\nu_x) \subset K \ \textit{for a.e.} x \in A \ \textit{if}\ \mathrm{dist}(u_{n_k}, K) \to 0 \ \textit{in measure.}$$

(iv) Furthermore, one has

$$\textit{(i')} \|\nu_x\|_{\mathcal{M}(\mathbb{R})} = 1 \ \textit{for a.e.}\ x \in A$$

if and only if

$$\lim_{M\to+\infty} \sup_k |\{|u_{n_k}| \geq M\}| = 0.$$

(v) *If (i') holds, if $B \subset A$ is measurable, if $f \in C(\mathbb{R}^N)$ and if $f(u_{n_k})$ is relatively weakly compact in $L^1(B)$, then*

$$f(u_{n_k}) \rightharpoonup \bar{f} \text{ in } L^1(B), \ \bar{f}(x) = \langle \nu_x, f \rangle.$$

(vi) *If (i') holds, then in (iii) one can replace "if" by "if and only if".*

Definition 1.3. *The map $\nu : E \to \mathcal{M}(\mathbb{R}^N)$ is called the Young measure* associated to *(or* generated by*) the sequence (u_{n_k}).*

Note that every weak-$*$ measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^N)$ that satisfies (i) of Theorem 1.2 is generated by some sequence (u_n).

Example.
Let us show a typical application of Theorem 1.2.
If (u_n) is bounded in $L^p(A)$ and the continuous function f has a certain growth to $+\infty$, like $|f(t)| \leq C(1 + |t|^q)$, with $q < p$.
Then, by (v), it follows

$$f(u_{n_k}) \rightharpoonup \bar{f} \text{ in } L^{p/q}(A).$$

In particular, for $p > 1$, choosing $f \equiv \text{id}$, we have

$$u_{n_k} \rightharpoonup u \text{ with } u(x) = \langle \nu_x, \text{id} \rangle, \quad \forall x \in A.$$

The measure ν_x describes the probability of finding a certain value in the sequence (u_{n_k}) in a small neighborhood $B_r(x)$ in the limits $k \to \infty$ and $r \to 0$.

Corollary 1.4. ([53], Corollary 3.2, p. 34). *Let (u_n) be a sequence of measurable functions from A to $\mathbb{R}^N$ that generates the Young measures $\nu : A \to \mathcal{M}(\mathbb{R}^N)$. Then*

$$u_n \to u \text{ in measure if and only if } \nu_x = \delta_{u(x)} \text{ a.e. in } A.$$

Another property of the Young measures associated to a sequence is stated in the following theorem, which we will use to recover important compactness results. We denote by $\mathcal{Y}(A)$ the family of all weakly-$*$ measurable maps $\nu : A \to \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$.

Theorem 1.5. ([56], Theorem 16, p. 166). *Let u_n be a bounded sequence in $L^1(A)$. There exists a subsequence u_{n_k} and a map $\nu \in \mathcal{Y}(A)$ such that, for every Carathéodory function $f : \mathbb{R} \to [0, +\infty)$ we have*

$$\liminf_{k\to+\infty} \int_A f(x, u_{n_k}(x))dx \geq \int_A \bar{f}dx, \tag{1.3}$$

where $\bar{f}(x) := \int_{\mathbb{R}} f(x,t)d\nu_x(t)$.

1.3 The slicing method

In this section, we describe a well-known method often used to recover compactness and lower bound inequalities through the study of problems of lower dimension via a slicing argument. The slicing method has been introduced by Ambrosio to treat free-discontinuity problems (see [10]) and has been used to prove various results also within the theory of phase transitions (see [5] and [9]).

Let us try to understand how the slicing method will be used by looking at the Modica-Mortola functional

$$E_\varepsilon(u) := \varepsilon \int_A |Du|^2 dx + \frac{1}{\varepsilon}\int_A W(u)dx,$$

defined on $H^1(A)$, with A bounded open set of $\mathbb{R}^N$. We may examine the behavior of E_ε on one-dimensional sections as follows:

for each e unit vector in $\mathbb{R}^N$ we consider the hyperplane

$$\Pi_e := \left\{z \in \mathbb{R}^N : \langle z, e\rangle = 0\right\}$$

passing through 0 and orthogonal to e. We denote by A_e the projection of A onto Π_e;

for every $y \in \Pi_e$ we consider the one-dimensional set (see Fig.1.1)

$$A_e^y := \{t \in \mathbb{R} : y + te \in A\};$$

for every function u defined in A we consider the trace of u on A_e^y, i.e., the one-dimensional function

$$u_e^y(t) := u(y + te).$$

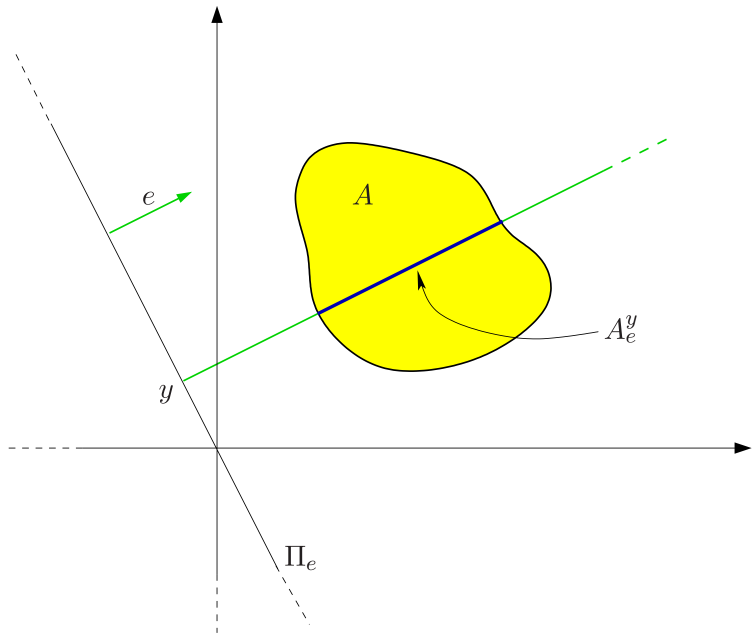

Figure 1.1: A section of the domain A.

Now, we are ready to obtain a lower bound for the Γ-liminf of E_ε by looking at the limit of the functionals induced by E_ε on the one-dimensional sections. Thanks to Fubini's Theorem, we rewrite E_ε as

$$E_\varepsilon(u) = \int_{\Pi_e}\int_{A_e^y}\left(\varepsilon|Du(y+te)|^2 + \frac{1}{\varepsilon}W(u(y+te))\right)dtdy. \tag{1.4}$$

The main idea of the slicing method is using the Γ-limit of the one-dimensional functionals

$$v \mapsto \int_{A_e^y}\left(\varepsilon|v'(t)|^2 + \frac{1}{\varepsilon}W(v(t))\right)dt$$

and the inequality from (1.4)

$$E_\varepsilon(u) \geq \int_{\Pi_e}\int_{A_e^y}\left(\varepsilon|(u_e^y)'(t)|^2 + \frac{1}{\varepsilon}W(u_e^y(t))\right)dtdy,$$

to obtain a lowerbound for the Γ-liminf of E_ε, by Fatou's Lemma and optimizing the choice of e.

In the following subsections, we recall some general results regarding the slicing of Sobolev functions.

1.3.1 Some slicing results

For the sake of simplicity, we work only with one-dimensional slicing, but the following results are true for slicing with arbitrary dimension. Let A, e, A_e and A_e^y be given as before and take a Borel function u in A. By Fubini's Theorem u belongs to L^p, $1 \leq p < \infty$, if and only if u_e^y belongs to $L^p(A_e^y)$ for a.e. $y \in A_e$ and the function $y \to \|u_e^y\|_{L^p}$ belongs to $L^p(A_e)$. Similarly, given a sequence $(u_k) \subset L^p(A)$ which converges to u in $L^p(A)$, we have that (up to a subsequence) $(u_k)_e^y$ converges to u_e^y in $L^p(A_e^y)$ for a.e. $y \in A_e^y$. Moreover, if $(u_k)_e^y$ converges to u_e^y for a.e. $y \in A_e$ and the functions $|u_k|^p$ are equi-integrable, then u_k converges to u in $L^p(A)$.

Proposition 1.6. ([32], Theorem 2, p. 164). *Let $u \in L^p(A)$ be given. If e is an arbitrary unit vector and u belongs to $W^{1,p}(A)$, then $u_e^y \in W^{1,p}(A_e^y)$ for a.e. $y \in A_e$, and the derivative $(u_e^y)'(t)$ agrees with the partial derivative $\partial_e u(y + te)$ for a.e. $y \in A_e$ and $t \in A_e^y$. Conversely, u belongs to $W^{1,p}(A)$ if there exist N linearly independent unit vectors e such that $u_e^y \in W^{1,p}(A_e^y)$ for a.e. $y \in A_e$ and the function $y \to \|(u_e^y)'\|_{L^p}$ belongs to $L^p(A_e)$.*

Proposition 1.7. ([32], Section 5.10, p. 216). *Let a Borel set $E \subset A$ be given. If E has finite perimeter in A, then E_e^y has finite perimeter in A_e^y and $\partial(E_e^y \cap A_e^y) = (\partial E \cap A)_e^y$ for a.e. $y \in A_e$, and*

$$\int_{A_e} \mathcal{H}^0(\partial E_e^y \cap A_e^y)dy = \int_{\partial E \cap A} \langle v_E, e \rangle. \tag{1.5}$$

Conversely, E has finite perimeter in A if there exist N linearly independent unit vectors e such that the integral of $\mathcal{H}^0(\partial E_e^y \cap A_e^y)$ over all $y \in A_e$ is finite.

We establish a connection between the compactness of a family of functions in $L^1(\mathbb{R}^N)$ and the compactness of the traces of these functions. We need to recall the definition of δ-dense family of functions.

Definition 1.8. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two families of functions on A. For every $\delta > 0$, we say that the family $\mathcal{F}'$ is* δ-dense *in $\mathcal{F}$ if $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}'$ with respect to the $L^1(A)$ topology.*

According to previous notation, for every family $\mathcal{F}$ of functions on A, we denote by $\mathcal{F}_e^y := (u_e^y)_{u \in \mathcal{F}}$ the family of functions on A_e^y.

Theorem 1.9. ([9], Theorem 6.6, pag. 42). *Let $\mathcal{F}$ be a family of functions $v : A \to [-m, m]$ and assume that there exist N linearly independent unit vectors e which satisfy the following property:*

$$\begin{aligned}&\textit{For every } \delta > 0 \textit{ there exists a family } \mathcal{F}_\delta \ \delta\textit{-dense in } \mathcal{F} \textit{ such that}\\ &(\mathcal{F}_\delta)^y_e \textit{ is pre-compact in } L^1(A) \textit{ for } \mathcal{H}^{N-1}\textit{-a.e. } y \in A_e.\end{aligned} \tag{1.6}$$

Then $\mathcal{F}$ is pre-compact in $L^1(A)$.

We will use the L^1-pre-compactness criterion by slicing of Theorem 1.9 to prove the pre-compactness of the traces of the minimizing sequence of the functional F_ε, defined by (0.16).

1.3.2 Isometry defect

When we work with slicings, we may also want to evaluate "the error we make" when we perturb a three-dimensional domain to get a two-dimensional one. To this aim, we define the "isometry defect", introduced by Alberti, Bouchitté and Seppecher in [9].

As usual, we denote by $O(3)$ the set of linear isometries on $\mathbb{R}^3$.

Definition 1.10. *Let $A_1, A_2 \subset \mathbb{R}^3$ and let $\Psi : \overline{A_1} \to \overline{A_2}$ bi-Lipschitz homeomorphism. Then the* "isometry defect $\delta(\Psi)$ of Ψ" *is the smallest constant δ such that*

$$\operatorname{dist}(D\Psi(x), O(3)) \leq \delta, \quad \textit{for a.e. } x \in A_1. \tag{1.7}$$

Here $D\Psi(x)$ is regarded as a linear mapping of $\mathbb{R}^3$ into $\mathbb{R}^3$. The distance between linear mappings is induced by the norm $\|\cdot\|$, which, for every L, is defined as the supremum of $|Lv|$ over all v such that $|v| \leq 1$. Hence, for every $L_1, L_2 : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\operatorname{dist}(L_1, L_2) := \sup_{x:|x|\leq 1} |L_1(x) - L_2(x)|.$$

Given $L_1, L_2 : \overline{A_1} \to \overline{A_2}$, with L_1 isometry, such that there exists $\delta < 1$ such that

$$\|L_1 - L_2\| \leq \delta.$$

Then, L_2 is invertible and $\|L_1^{-1} - L_2^{-1}\| \leq \delta/(1-\delta)$.

By (1.7), it follows

$$\operatorname{dist}(D\Psi^{-1}(y), O(3)) \leq \delta/(1-\delta), \quad \text{for a.e. } y \in A_2,$$

and then
$$\delta(\Psi^{-1}(y)) \leq \delta(\Psi(y))/(1-\delta(\Psi(y))), \quad \text{for a.e. } y \in A_2.$$

Inequality (1.7) also implies that

$$\|D\Psi(x)\| \leq 1+\delta(\Psi) \text{ for a.e. } x \in A_1,$$

and then Ψ is $(1+\delta(\Psi))$-Lipschitz continuous on every convex subset of A_1. Similarly, Ψ^{-1} is $(1-\delta(\Psi))^{-1}$-Lipschitz continuous on every convex subset of A_2.

For every $A \subset \mathbb{R}^3$ and every $A' \subset \partial A$, let $F_\varepsilon : W^{1,p}(A) \to \mathbb{R}$ be the functional defined by (0.16); that is:

$$F_\varepsilon(u, A, A') := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) dx + \frac{1}{\varepsilon} \int_{A'} V(Tu) d\mathcal{H}^2.$$

The following proposition holds.

Proposition 1.11. *Let $A_1, A_2 \subset \mathbb{R}^3, \Psi : \overline{A_1} \to \overline{A_2}$ a bi-Lipschitz homeomorphism, $A'_1 \subset \partial A_1, A'_2 \subset \partial A_2$, be given such that $\Psi(A'_1) = A'_2$ and $\delta(\Psi) < 1$. Then for every $u \in W^{1,p}(A_2)$*

$$F_\varepsilon(u, A_2, A'_2) \geq (1-\delta(\Psi))^{p+3} F_\varepsilon(u \circ \Psi, A_1, A'_1).$$

Proof. The proof is a simple modification of the one by Alberti Bouchitté and Seppecher in [9] (Proposition 4.9, p. 25), where they treat the case $p = 2$.

By (1.10), we get $\|D\Psi(x)\| \leq 1+\delta(\Psi)$ for x a.e. in A_1, that implies

$$|D(u \circ \Psi)(x)| \leq (1+\delta(\Psi))|((Du) \circ \Psi)(x)|, \quad \text{a.e. in } A_1. \tag{1.8}$$

Let g_1 and g_2 denote the inverse of Ψ and the restriction to ∂A_2 of the inverse of Ψ, respectively:

$$g_1(x) := (\Psi)^{-1}(x), \quad g_2(x) := (\Psi|\partial A_2)^{-1}(x).$$

g_1 and g_2 are locally Lipschitz and such that

$$|Jg_1| \leq (1-\delta(\Psi))^3 \text{ a.e. on } A_2 \text{ and } |Jg_2| \leq (1-\delta(\Psi))^3 \text{ a.e. on } \partial A_2. \tag{1.9}$$

Since $\delta(\Psi) < 1$ then

$$(1+\delta(\Psi)) \leq (1-\delta(\Psi))^{-1}. \tag{1.10}$$

Thus, using the estimates on the Jacobian determinants (1.9) and the inequality (1.10), we obtain the desired conclusion by changing-variable formula. □

Proposition 1.12. ([9], Proposition 4.10, p. 25). *For every $x \in \partial\Omega$ and every positive r smaller than a certain critical value $r_x > 0$, there exists a bi-Lipschitz map $\Psi_r : \overline{D_r} \to \overline{\Omega \cap B_r(x)}$ such that*

(a) *Ψ_r takes D_r onto $\Omega \cap B_r(x)$ and E_r onto $\partial\Omega \cap B_r(x)$;*

(b) *Ψ_r is of class C^1 on D_r and $\|D\Psi_r - I\| \le \delta_r$ everywhere in D_r, where $\delta_r \to 0$ as $r \to 0$.*

Note that, in particular, the isometry defect of Ψ_r vanishes as $r \to 0$.

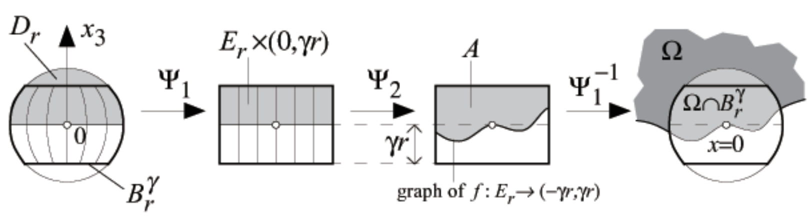

Figure 1.2: Construction of $\Psi := \Psi_1^{-1} \circ \Psi_2 \circ \Psi_1$ ([9], Fig. 5, p. 26).

1.4 Rearrangement results

In this section we state some rearrangement results that we will use in the sequel. Rearrangement problems have been widely studied in the literature (see for instance [45], [15], [14], [3] and [17]). Our main concern is the behavior with respect to the rearrangement of certain classes of integral functionals.

Before starting definitions and results, we stress that we use the terms increasing and decreasing in the weak sense, that is, to mean non-decreasing and non-increasing respectively.

1.4.1 Monotone rearrangement in one-dimension

We refer to [5] and [38].

We denote by I the open bounded interval (a, b) in $\mathbb{R}$.

Definition 1.13. *For every measurable function* $v : I \to \mathbb{R}$ *we define the* monotone increasing rearrangement of v *as the function* $v^* : I \to \mathbb{R}$ *by*

$$v^*(t+a) := \sup\{\lambda : |\{s \in (a,b) : v(s) \le \lambda\}| \le t\}, \quad \forall t \in (0, b-a).$$

The following theorems hold.

Theorem 1.14. [[5], Theorem 5.6, p. 557]. *Let W be a non-negative continuous function on $[\alpha', \beta']$ such that $W(\alpha') = W(\beta') = 0$. For every $v : I \to \mathbb{R}$, the following equality holds:*

$$\int_I W(v)dx = \int_I W(v^*)dx, \tag{1.11}$$

where v^* is the increasing rearrangement of v.

Theorem 1.15. [[38], Theorem I.1, p. 67]. *Let $J, P : \mathbb{R} \to \mathbb{R}$ be two continuous even functions such that*

(i) *$J(e^s)$ is convex in $\mathbb{R}$ and $J(|s|) \to +\infty$ as $|s| \to +\infty$;*

(ii) *$P(|s|) \to 0$ as $|s| \to +\infty$.*

Then, for every measurable function $v : I \to \mathbb{R}$ such that $\displaystyle\iint_{I\times I} J\left(\frac{v(t') - v(t)}{P(t'-t)}\right) dt'dt$ is finite, the following inequality holds:

$$\iint_{I\times I} J\left(\frac{v^*(t') - v^*(t)}{P(t'-t)}\right) dt'dt \le \iint_{I\times I} J\left(\frac{v(t') - v(t)}{P(t'-t)}\right) dt'dt,$$

where v^* is the monotone increasing rearrangement of v in I.

As an immediate consequence of Theorem 1.15, we have that replacing a function v by its increasing rearrangement decreases the p-power of the fractional semi-norm of $W^{1-1/p,p}(I)$:

$$\iint_{I\times I} \frac{|v^*(t') - v^*(t)|^p}{|t'-t|^p} dt'dt \le \iint_{I\times I} \frac{|v(t') - v(t)|^p}{|t'-t|^p} dt'dt. \tag{1.12}$$

Hence, by (1.11) and (1.12) with $J(\cdot) = |\cdot|^p$ and $P(\cdot) = |\cdot|$, we obtain that the same monotonicity property with respect to monotone increasing rearrangements holds for the following non-local functional:

$$K_\varepsilon(v) := \varepsilon^{p-2}\iint_{I\times I}\left|\frac{v(t) - v(t')}{t-t'}\right|^p dtdt' + \frac{1}{\varepsilon}\int_I V(v)\, dt.$$

1.4.2 Monotone rearrangement in one direction

We state a rearrangement result for the energy of Sobolev functions defined on bounded cylinders; we refer to Kawohl [45] and Berestycky and Lachand-Robert [17].

Let ω be a smooth bounded domain in $\mathbb{R}^{N-1}$ and let I be a bounded open interval of $\mathbb{R}$, we denote by $Q := I \times \omega$ the bounded cylinder in $\mathbb{R}^N$. For every measurable function $u : Q \to \mathbb{R}$, we denote by $u^\star$ the *monotone rearrangement in direction* x_1 of u; i.e., the function $u^\star : Q \to \mathbb{R}$ which is increasing in I with respect to x_1 (for almost all $x' \in \omega$), and such that for every $\lambda \in \mathbb{R}$ and for every $x' \in \omega$:

$$| \left\{ x_1 \in I : u(x_1, x') \geq \lambda \right\} | = | \left\{ x_1 \in I : u^\star(x_1, x') \geq \lambda \right\} |.$$

Theorem 1.16. [[45], Corollary 2.14 , p. 51, and [17], Theorem 3, p. 10]. *If u belongs to* $W^{1,p}(Q)$*, then its monotone increasing rearrangement in direction* x_1 $u^\star$ *belongs to* $W^{1,p}(Q)$ *and*

$$\int_Q |Du^\star|^p dx \leq \int_Q |Du|^p dx.$$

This result will be a key point in the proof of the existence of a minimum for the optimal profile problem (0.14).

Chapter 2

Phase transitions and known results

A phase transition is a change of a thermodynamic system from one phase to another. The main characteristic of a phase transition is a sudden transformation in one or more physical properties, like heat capacity, with a change in a thermodynamic variable such as the temperature. We can find many phase transition events, like the transitions between the solid, liquid and gaseous phases, due to effect of temperature and pressure, the transitions between the ferromagnetic and paramagnetic phases of magnetic materials at the Curie point, the emergence of superconductivity in certain metals when cooled below a critical temperature, and so.

In the following, we pay attention to the classical model for two-phase fluids and its mathematical approach initiated by Gibbs and revisited by Cahn and Hilliard.

2.1 The classical model for phase transitions

In the classical theory of phase transitions, a two-phase systems is modeled as follows: the container is represented by a bounded regular domain Ω in $\mathbb{R}^3$; every configuration of the fluid is described by a mass density u on Ω which takes only two values, α and β, corresponding to the phases $A := \{u = \alpha\}$ and $B := \{u = \beta\} = \Omega \setminus A$. The singular set of u (the set of discontinuity points of u) is the interface between the two phases, that we denote by Su.

The space of admissible configurations is given by all $u : \Omega \to \{\alpha, \beta\}$ under some volume constraint. The energy is located on the interface $\mathcal{S}_{AB} := Su$ which separates the

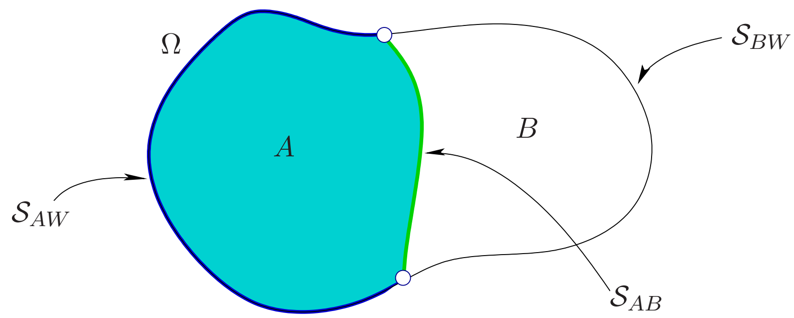

Figure 2.1: The classical model for phase transitions.

two phases and on the contact surfaces $\mathcal{S}_{AW} := \partial A \cap \partial\Omega$ and $\mathcal{S}_{BW} := \partial B \cap \partial\Omega$ between the wall of the container and the phases A and B (see Fig. 2.1).

The equilibrium configurations are assumed to minimize the capillary energy

$$\mathcal{E}_0(A) := \sigma\mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW}\mathcal{H}^2(\mathcal{S}_{AW}) + \sigma_{BW}\mathcal{H}^2(\mathcal{S}_{BW}), \tag{2.1}$$

where the positive constants σ, σ_{AW} and σ_{BW} are referred to as **surface tension**. The problem of minimizing (2.1) is called **liquid-drop problem** and the existence of a solution is assured by the **wetting condition**:

$$\sigma \geq |\sigma_{AW} - \sigma_{BW}|.$$

At the equilibrium, the interface $\mathcal{S}_{AB}$ has constant mean curvature and meets the wall of the container with a constant **contact angle** ϑ, which satisfies the *Young's law*

$$\vartheta = \arccos\frac{\sigma_{AW} - \sigma_{BW}}{\sigma}.$$

See, for instance Finn in [33].

2.2 The Cahn-Hilliard model for phase transitions

In the late 50's, Cahn and Hilliard [24] proposed an alternative way to study two-phase fluids. They followed the continuum mechanics approach by Gibbs and assumed that the transition is not given by a separating interface, but is a continuous phenomenon occurring in a thin layer in which is identified with the interface.

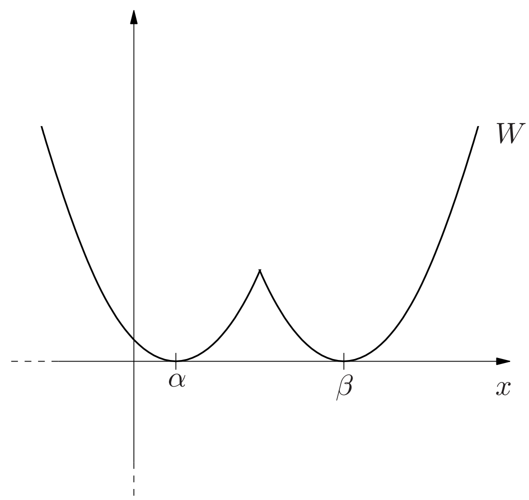

Figure 2.2: A double-well type potential W.

Hence, the mass density u varies continuously from the value α to the value β. Neglecting the interactions between the fluid and the wall of the container, the energy associated to u is the sum of a free energy $\int_\Omega W(u)dx$, where W is a "double-well potential" (a continuous positive function which vanished only at α and β; see Fig. 2.2), and a perturbation $\varepsilon^2 \int_\Omega |Du|^2 dx$ which penalizes the non-homogeneity of the fluid:

$$\varepsilon^2 \int_\Omega |Du|^2 dx + \int_\Omega W(u)dx, \tag{2.2}$$

where ε is a small parameter, giving the length of the thickness of the interface. This model is known as "diffuse interface model" for phase transitions. The length ε is much smaller than the size of the container. It is natural to study the equilibrium of the fluid in an asymptotic way, as ε goes to 0; i.e., by considering the limit as ε tends to 0 of a minimizer u_ε of a suitable rescaling of the energy (2.2).

2.3 The Modica-Mortola Theorem

A connection between the classical sharp interface model and the diffuse interface model, without taking into account the interactions with the wall, was established by

Modica in [48](1987). He proved that a suitable rescaling of the energy (2.2) Γ-converges to the surface energy functional $u \mapsto \sigma\mathcal{H}^2(Su)$; He also proved that at equilibrium the two phases arrange themselves in order to minimize the area of the separation interface. This result was conjectured by De Giorgi. A great contribute to the work of Modica was already given by Modica himself and Mortola in [50] (1977).

For every $\varepsilon > 0$, let us consider the functional E_ε defined in $H^1(\Omega)$, given by the following rescaling of (2.2)

$$E_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx. \tag{2.3}$$

To explain the scale of this penalization, we show a heuristic scaling argument in dimension one.

Consider an interval $(t, t+\delta)$ and suppose that u is close to α and β at the endpoints of this interval, respectively. We can show that the contribution on this interval of the first integral of (2.2) is of order ε/δ (since the gradient is of order ε), while the contribution of the second integral is of order δ.
Hence

$$\varepsilon^2 \int_t^{t+\delta} |u'|^2 ds + \int_t^{t+\delta} W(u)ds \cong \frac{\varepsilon^2}{\delta} + \delta,$$

and the minimization in δ of this quantity gives $\delta = \varepsilon$ and a contribution of order ε. This implies that if the energy is bounded then the number of such intervals is bounded and hence u resembles a piecewise-constant function. This argument suggests a scaling of the problem and to consider the minimum problem

$$\min\left\{\varepsilon\int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx : \int_\Omega u\, dx = C\right\},$$

whose minimizers are clearly the same as the energy (2.2).

Theorem 2.1. *The functional $E_\varepsilon : H^1(\Omega) \to \mathbb{R}$ defined by (2.3) Γ-converges with respect to the $L^1(\Omega)$ convergence to the functional*

$$E(u) := \begin{cases} \sigma\mathcal{H}^2(Su), & \text{if } u \in BV(\Omega, \{\alpha, \beta\}), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.4}$$

where $\sigma := 2\int_\alpha^\beta W^{1/2} dt.$

Since the Modica-Mortola Theorem, several results were given which extend theorem 2.1 in different directions. In particular, we are interested in the possibility of replacing the perturbation term by a p-energy of Dirichlet. In this case, a similar heuristic argument like in the quadratic case, gives the following rescaled energy

$$\delta^{p-1}\int_\Omega |Du|^p dx + \frac{1}{\delta}\int_\Omega W(u)dx.$$

Anyway, choosing ε such that $\delta^{p-1} = \varepsilon^{p-2}$ better fits to our purposes. So that, for every open set $A \subset \mathbb{R}^3$ and every real function $u \in W^{1,p}(A)$, we consider the following functional

$$G_\varepsilon(u, A) := \varepsilon^{p-2}\int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_A W(u)dx. \tag{2.5}$$

Let $\mathcal{W}$ be a primitive of $W^{p/(p-1)}$, we denote by $\sigma_p := \dfrac{p}{(p-1)^{p/(p-1)}}|\mathcal{W}(\beta) - \mathcal{W}(\alpha)|$.

Theorem 2.2. *For every domain $A \subset \mathbb{R}^3$ the following statements hold.*

(i) *If $(u_\varepsilon) \subset W^{1,p}(A)$ is a sequence with uniformly bounded energies $G_\varepsilon(u_\varepsilon, A)$. Then (u_ε) is pre-compact in $L^1(A)$ and every cluster point belongs to $BV(A, \{\alpha, \beta\})$.*

(ii) *For every $u \in BV(A, \{\alpha, \beta\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$,*

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, A) \geq \sigma_p \mathcal{H}^2(Su),$$

(iii) *For every $u \in BV(A, \{\alpha, \beta\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and*

$$\limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, A) \leq \sigma_p \mathcal{H}^2(Su).$$

Moreover, when Su is a closed Lipschitz surface in A, the functions u_ε may be required to be $(C_W/\varepsilon^{\frac{p-2}{p-1}})$-Lipschitz continuous, and to converge to u uniformly on every set with positive distance from Su (here C_W is the supremum of $W^{1/p}$ in $[\alpha, \beta]$).

A proof can be obtained thanks to simple modifications to the proof of the Modica-Mortola Theorem by Modica in [48] (see also [22], Theorem 3.10, p. 42). We preferred to enunciate the results as in the Theorem 2.2, because they will be useful in this form to our goal.

Remark 2.3. Either in the quadratic version or in the super-quadratic one of the Modica-Mortola functional, the optimal constant in the limit energy comes from an optimal profile problem. For instance, in the quadratic case, we have

$$\sigma := \left\{ \int_{\mathbb{R}} |u'|^2 dt + \int_{\mathbb{R}} W(u) dt : u \in H^1(\mathbb{R}), \lim_{t\to-\infty} u(t) = \alpha, \lim_{t\to+\infty} u(t) = \beta \right\}. \tag{2.6}$$

The minimum problem (2.6) represents the minimal cost we pay each time we have a transition from α to β (or conversely) in the real line. Moreover, we explicitly know the optimal profile; it is given by the solution of the following differential equation

$$\begin{cases} u' = \sqrt{W(u)}, \\ u(0) = \dfrac{\alpha+\beta}{2}; \end{cases}$$

that is, the Euler-Lagrange equation of the problem.This fact is strictly linked to the structure of the Modica-Mortola functional, which is characterized by the "equi-partition of the energy" between its two terms.

2.4 The interactions between the fluids and the wall of the container

We stress that Theorem 2.1 and Theorem 2.2 do not take into account the interaction of the fluid with the wall of the container. In this sense, an extension of the classical model for two-phase fluids is obtained by adding to the energy (2.1) a **line tension** energy[1] with density τ concentrated along the line $\mathcal{L}_c \equiv \partial \mathcal{S}_{AW}$, where $\mathcal{S}_{AB}$ meets the wall of the container; i.e., along the "contact line" (see Fig. 2.3). In this model, the capillary energy becomes

$$\mathcal{E}(A) := \sigma \mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW} \mathcal{H}^2(\mathcal{S}_{AW}) + \sigma_{BW} \mathcal{H}^2(\mathcal{S}_{BW}) + \tau \mathcal{H}^1(\mathcal{L}_c). \tag{2.7}$$

A partial rigorous connection with this classical model is provided by Modica at the end of 80's. In [49], Modica added to (2.3) a boundary contribution of the form $\lambda \int_{\partial\Omega} g(Tu) d\mathcal{H}^2$, where Tu denotes the trace of u on $\partial\Omega$, λ does not depend on ε, and g is a non-negative continuous function. He proved that a sequence of minimizers u_ε for

[1] In Gibbs' original formulation, the concept of line tension was introduced to describe the excess free energy arising a three-phase line, that is, along a curve where three distinct phases coexist (see [41]).

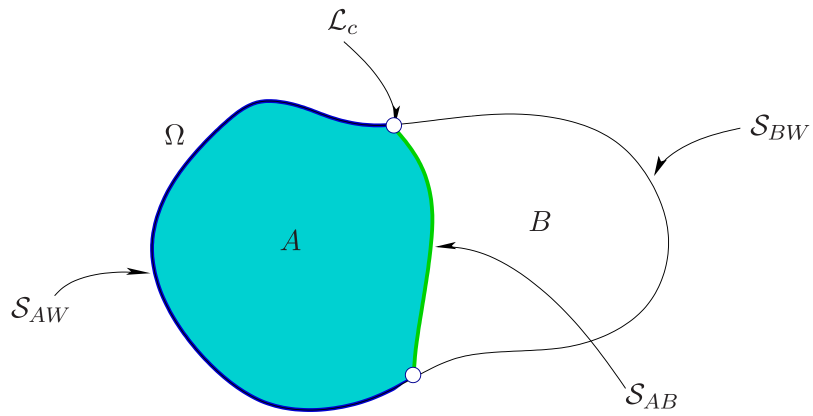

Figure 2.3: The line tension effect.

this functional is pre-compact in $L^1(\Omega)$, each limit point u takes only the values α and β, and the corresponding phase $A := \{u = \alpha\}$ is a solution of the liquid-drop problem associated to the energy (2.1).

Modica also proved that the asymptotic behavior of the energy is related with the following geometric minimization problem

$$\min \left\{ P_\Omega(E) + \gamma \mathcal{H}^{N-1}(\partial E \cap \partial\Omega) : E \subseteq \Omega, |E| = m_1 \right\}, \tag{2.8}$$

with γ and m_1 fixed real constants.

For every $\varepsilon > 0$, we set

$$E_\varepsilon^g(u) := \varepsilon \int_\Omega |Du|^2 + \frac{1}{\varepsilon} \int_\Omega W(u)dx + \int_{\partial\Omega} g(Tu) d\mathcal{H}^{N-1}, \ \forall u \in H^1(\Omega),$$

with g a non-negative continuous function.

For every $t > 0$,

$$H(t) := \int_0^t \sqrt{W} ds,$$

$$\hat{g}(t) := \inf \left\{ g(s) + 2|H(s) - H(t)| : s \geq 0 \right\}.$$

The main results in [49] are stated in the following theorem.

Theorem 2.4. [[49], Theorem 2.1, p. 497]. *Fix $m \in [\alpha|\Omega|, \beta|\Omega|]$ and let $(u_\varepsilon) \subset C^1$ be a minimizing sequence of E^g_ε among the class of functions with fixed volume m. If $u_\varepsilon \to u_0$ in $L^1(\Omega)$, then*

(i) $W(u_0(x)) = 0, \quad a.e.\ x \in \Omega;$

(ii) *The set $E_0 := \{x \in \Omega : u_0(x) = \alpha\}$ is a solution of the minimum problem (2.8), with* $\gamma := \dfrac{\hat{g}(\alpha) - \hat{g}(\beta)}{2\int_\alpha^\beta \sqrt{W}dt}$ *and* $m_1 := \dfrac{\beta|\Omega| - m}{\beta - \alpha}$;

(iii) $\lim\limits_{\varepsilon \to 0} E^g_\varepsilon(u_\varepsilon) = \left(2\displaystyle\int_\alpha^\beta \sqrt{W}dt\right) P_\Omega(E_0) + \hat{g}(\alpha)\mathcal{H}^{N-1}(\partial E_0 \cap \partial\Omega) + \hat{g}(\beta)\mathcal{H}^{N-1}(\partial\Omega \setminus \partial E_0).$

2.5 Phase transitions with line tension effect

In the 90's, Alberti, Bouchitté and Seppecher in [9] proved that, due to a lack of semicontinuity, the functional $\mathcal{E}$ leads to ill-posed minimum problems. They explicitly compute the relaxation of $\mathcal{E}$ and show that the total energy can be properly written by introducing, besides the usual bulk phase $A \subset \Omega$, an additional variable $A' \subset \partial\Omega$, independent of ∂A, with its complement $B' := \partial\Omega \setminus A'$: the *boundary phases*.

According to this viewpoint, the total energy $\tilde{\mathcal{E}}$ of the configurations (A, A') is given by the sum of three different terms: the classical surface tension on the interface between the bulk phases A and B, a surface density on the wall of the container (depending on which bulk phase and which boundary phase meet together) and a line density along the line $\mathcal{L}_{A'B'}$ (the *dividing line*) which separates the boundary phases A' and B' (see Fig. 2.4)

$$\begin{aligned}\tilde{\mathcal{E}}(A, A') \ := \ & \sigma\mathcal{H}^2(\mathcal{S}_{AB}) + \sigma_{AW}\mathcal{H}^2(\mathcal{S}_{AA'}) + (\sigma + \sigma_{AW})\mathcal{H}^2(\mathcal{S}_{AB'}) + \sigma_{BW}\mathcal{H}^2(\mathcal{S}_{BB'}) \\ & + (\sigma + \sigma_{BW})\mathcal{H}^2(\mathcal{S}_{BA'}) + \tau\mathcal{H}^1(\mathcal{L}_{A'B'}).\end{aligned} \tag{2.9}$$

Therefore $\mathcal{E}$ can be written in terms of $\tilde{\mathcal{E}}$ by choosing $A' = \mathcal{S}_{AW}$. We stress that the boundary phase A' may differ from the interface $\mathcal{S}_{AW}$ between the bulk phase A and the wall of the container, at equilibrium. Hence, the line tension is located on the dividing line $\mathcal{L}_{A'B'}$, which in general does not agree with the contact line $\mathcal{L}_c$ (see [9], Example 5.2, p. 35). In this case, Alberti, Bouchitté and Seppecher speak of "dissociation of the contact line and the dividing line".

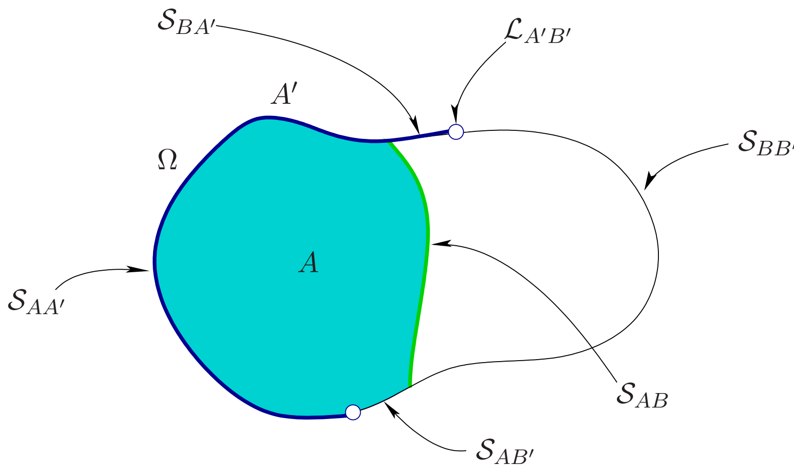

Figure 2.4: An arbitrary configuration (A, A').

In order to properly establish a connection with the associated model for capillarity with line tension, they study the asymptotic behavior of the following functional

$$\tilde{E}_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon} \int_\Omega W(u) dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) d\mathcal{H}^2, \quad \forall\, u \in H^1(\Omega), \tag{2.10}$$

where V is a double well potential with wells α' and β' (corresponding to the boundary phases A' and B'), and λ_ε satisfies

$$\varepsilon \log \lambda_\varepsilon \to k \in (0, +\infty) \text{ as } \varepsilon \text{ goes to } 0. \tag{2.11}$$

The logarithmic scaling (2.10) will be explained in the following pages (see Remark 2.7). In the meantime, notice that it provides a uniform control on the oscillation of Tu_ε, the traces of minimizing sequences u_ε, and ensures that the transition of Tu_ε from α' to β' takes place in a thin layer. In fact, Alberti, Bouchitté and Seppecher proved that, under (2.11), the traces Tu_ε converge (up to a subsequence) to a function v in $BV(\partial\Omega, \{\alpha', \beta'\})$ and then the boundary phases $\{v = \alpha'\}$ and $\{v = \beta'\}$ are divided by the line Sv. Namely, the asymptotic behavior of $\tilde{E}_\varepsilon$ is described by a functional Ψ which depends on the two variables u and v:

$$\begin{aligned} \Psi(u, v) \;:=\; & \sigma \mathcal{H}^2(Su) + \int_{\partial\Omega} |\tilde{H}(Tu) - \tilde{H}(v)| d\mathcal{H}^2 + \tau \mathcal{H}^1(Sv), \\ & \forall\, (u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}), \end{aligned} \tag{2.12}$$

where $\tau = \frac{k(\beta'-\alpha')^2}{\pi}$ and $\sigma := |\tilde{H}(\beta) - \tilde{H}(\alpha)|$, being $\tilde{H}$ a primitive of $2W^{1/2}$. Note that Ψ reduces to (2.9) taking into account the definition of τ, σ and $\tilde{H}$, if we set $\sigma_{AA'} := |\tilde{H}(\alpha) - \tilde{H}(\alpha')|$, $\sigma_{AB'} := |\tilde{H}(\alpha) - \tilde{H}(\beta')|$, $\sigma_{BA'} := |\tilde{H}(\beta) - \tilde{H}(\alpha')|$ and $\sigma_{BB'} := |\tilde{H}(\beta) - \tilde{H}(\beta')|$.

Theorem 2.5. [[9], Theorem 2.6, p. 10] *Let $\tilde{E}_\varepsilon : H^1(\Omega) \to \mathbb{R}$ and $\Psi : BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}) \to \mathbb{R}$ be defined by (2.10) and (2.12) respectively. Then*

(i) *If $(u_\varepsilon) \subset H^1(\Omega)$ is a sequence with uniformly bounded energies $\tilde{E}_\varepsilon(u_\varepsilon)$, then the sequence $(u_\varepsilon, Tu_\varepsilon)$ is pre-compact in $L^1(\Omega) \times L^1(\partial\Omega)$ and every cluster point belongs to $BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$.*

(ii) *For every $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset H^1(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$,*

$$\liminf_{\varepsilon \to 0} \tilde{E}_\varepsilon(u_\varepsilon) \geq \Psi(u, v),$$

(iii) *For every $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$ there exists a sequence $(u_\varepsilon) \subset H^1(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$ and*

$$\limsup_{\varepsilon \to 0} \tilde{E}_\varepsilon(u_\varepsilon) \leq \Psi(u, v).$$

The proof of Theorem 2.5 requires several steps in which different effects are analyzed and then different terms of the limit energy Ψ can be deduced. In the bulk term, the limit energy can be evaluate like in [48], while the second term in Ψ is obtained by adapting the approach by Modica in [49]. The main step in the proof concerns the analysis of the line tension effect. Via “localization” and slicing techniques, Alberti, Bouchitté and Seppecher reduces the analysis of the line tension to the asymptotic analysis of the following functional defined on a two-dimensional half-disk

$$E^2_\varepsilon(u) := \varepsilon^2 \int_{D_r} |Du|^2 dx + \lambda_\varepsilon \int_{E_r} V(Tu) d\mathcal{H}^1, \tag{2.13}$$

where, for every $r > 0$, we denote by

$$\begin{aligned} D_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 > 0\right\}, \\ E_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 = 0\right\} \cong (-r, r). \end{aligned} \tag{2.14}$$

Then the two-dimensional Dirichlet energy (2.13) is replaced on the half-disk D_r by the $H^{1/2}$ intrinsic norm on the "diameter" E_r. This is possible thanks to the following lemma, concerning with the optimal constant for the trace inequality involving the L^2 norm of the gradient of a function defined on a two-dimensional domain and the $H^{1/2}$ norm of its trace on a line.

Lemma 2.6. [[9], Corollary 6.4, p. 41]. *Let u be a function in $H^1(D_r)$, then the trace of u on the segment $E_r \times \{0\}$ belongs to $H^{1/2}(E_r)$ and*

$$\iint_{E_r\times E_r} \left|\frac{Tu(t')-Tu(t)}{t'-t}\right|^2 dt'dt \leq 2\pi \int_{D_r} |Du|^2 dx. \tag{2.15}$$

Hence, the original problem is reduced to the study of a new kind of perturbation problem involving a non-local term. Let I be an open interval in $\mathbb{R}$, for every $v \in H^{1/2}$, we set

$$E^1_\varepsilon(v) := \varepsilon \iint_{I\times I} \left|\frac{v(t)-v(t')}{t-t'}\right|^2 dtdt' + \lambda_\varepsilon \int_I V(v)\, dt, \tag{2.16}$$

where λ_ε satisfies the condition (2.11).

Remark 2.7. The singular perturbation problem involving the energies (2.16) brings to the fore the right scaling for (2.10); i.e., the choice of λ such that $\log \lambda_\varepsilon \approx 1/\varepsilon$.

Let us consider the following energy:

$$\bar{E}^1_\varepsilon(v) := \varepsilon^2 \iint_{I\times I} \left|\frac{v(t')-v(t)}{t'-t}\right|^2 dt'dt + \int_I V(v)dt,$$

on a one-dimensional set $I = (a, b)$.

Adapting the argument in Section 2.3, we look at a transition from α' to β' taking place on an interval $(t, t+\delta)$. We then have

$$\begin{aligned} \bar{E}^1_\varepsilon(v) &\geq 2\varepsilon^2 \iint_{(a,t)\times(t+\delta,b)} \left|\frac{1}{s'-s}\right|^2 ds'ds + C\delta \\ &\geq -2\varepsilon^2(\log\delta + C) + C\delta. \end{aligned}$$

By optimizing the last expression we get $\delta = 2\varepsilon^2/C$ and hence

$$\bar{E}^1_\varepsilon(v) \geq 4\varepsilon^2 |\log\varepsilon| + O(\varepsilon^2).$$

We are then led to the scaled energy

$$\frac{1}{|\log\varepsilon|}\iint_{I\times I}\left|\frac{v(t')-v(t)}{t'-t}\right|^2 dt'dt+\frac{1}{\varepsilon^2|\log\varepsilon|}\int_I V(v)dt,$$

and hence, by renaming ε the scaling factor $1/|\log\varepsilon|$, we obtain the energy (2.16).

Notice that this natural scaling signs the main differences with the classical Modica-Mortola problem. In fact, the asymptotic behavior of the Modica-Mortola functional (2.3) is characterized by the equi-partition of the energy between the two terms in the functional and by a suitable scaling property which provides an optimal profile problem describing the shape of the optimal transition. While, the logarithmic scaling for the functionals (2.16) produces no equi-partition of the energy at all; the limit comes only from the non-local part of the energy and any profile is optimal as far as transition occurs in a layer of order ε.

Similar effects can be found in other recent results for phase transition problems with non local singular perturbation (see Garroni and Müller[39] and Kurzke[46], [47]).

Alberti, Bouchitté and Seppecher analyze the asymptotic behavior of (0.11) in [8], proving that

$$E^1_\varepsilon \xrightarrow{\Gamma} 2k(\beta'-\alpha')\mathcal{H}^0(Sv),$$

and strongly use this result to obtain the boundary term in (0.8).

Theorem 2.8. [[8], Theorem 1, p. 334, and [9], Theorem 4.4, p. 20.] *Let $E^1_\varepsilon : H^{1/2}(I)\to\mathbb{R}$ be defined by (2.16) and, for every $v\in BV(I,\{\alpha',\beta'\})$, set $E^1(v):=2k(\beta'-\alpha')^2\mathcal{H}^0(Sv)$. Then*

(i) *If $(v_\varepsilon)\subset H^{1/2}(I)$ is a sequence such that $E^1_\varepsilon(v_\varepsilon)$ is bounded, then (v_ε) is pre-compact in $L^1(I)$ and every cluster point belongs to $BV(I,\{\alpha',\beta'\})$.*

(ii) *For every $v\in BV(I,\{\alpha',\beta'\})$ and every sequence $(v_\varepsilon)\subset H^{1/2}(I)$ such that $v_\varepsilon\to v$ in $L^1(I)$,*

$$\liminf_{\varepsilon\to 0} E^1_\varepsilon(v_\varepsilon)\geq E^1(v).$$

(iii) *For every $v\in BV(I,\{\alpha',\beta'\})$ there exists a sequence $(v_\varepsilon)\subset H^{1/2}(I)$ such that $v_\varepsilon\to v$ in $L^1(I)$ and*

$$\limsup_{\varepsilon\to 0} E^1_\varepsilon(v_\varepsilon)\leq E^1(v).$$

Chapter 3

A singular perturbation result with a fractional norm

We study a problem involving a non-local singular perturbation for a Cahn-Hilliard functional of the type of the energy (2.16), seen in the previous chapter:

$$E_\varepsilon^1(v) := \varepsilon \iint_{I\times I} \left| \frac{v(x) - v(x')}{x - x'} \right|^2 dx dx' + \lambda_\varepsilon \int_I V(v)\, dx. \qquad (3.1)$$

Let I be an open bounded interval of $\mathbb{R}$ and V a non-negative continuous function vanishing only at $\alpha', \beta' \in \mathbb{R}$ $(0 < \alpha' < \beta')$, with growth at least linear at infinity. We investigate the asymptotic behavior in terms of Γ-convergence of the following functional

$$K_\varepsilon(v) := \varepsilon^{p-2} \iint_{I\times I} \left| \frac{v(x) - v(x')}{x - x'} \right|^p dx dx' + \frac{1}{\varepsilon} \int_I V(v)\, dx, \ \forall v \in W^{1-\frac{1}{p},p}(I), \quad (p > 2), \qquad (3.2)$$

as $\varepsilon \to 0$.

We recall that the natural logarithmic scaling in (2.16) signs the main differences with the classical Modica-Mortola problem and, also, with our functional (3.2). In fact, the logarithmic scaling produces no equi-partition of the energy; all the limit comes only from the non-local part of the energy and any profile is optimal as far as transition occurs in a layer of order ε.

If we make the same computation seen in Chapter 2, we note that both the terms in functional (3.2) are of the first order; i.e., both the terms are important in the limit. We can bring out this characteristic of (3.2) in view of the following scaling property (3.5) and hence the limit is characterized by the following optimal profile problem

$$\gamma := \inf \left\{ \iint_{\mathbb{R}\times\mathbb{R}} \left| \frac{w(x)-w(x')}{x-x'} \right|^p dx dx' + \int_{\mathbb{R}} V(w)\, dx : w \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R}), \right. \\ \left. \lim_{x\to-\infty} w(x) = \alpha', \ \lim_{x\to+\infty} w(x) = \beta' \right\}. \tag{3.3}$$

3.1 The Γ-convergence result

The asymptotic behavior in term of Γ-convergence of K_ε is described by the functional

$$K(v) := \gamma \mathcal{H}^0(Sv), \qquad v \in BV(I, \{\alpha', \beta'\}), \tag{3.4}$$

where γ is given by the optimal profile problem (3.3).

The Γ-convergence result is precisely stated in the following theorem.

Theorem 3.1. [[39], Theorem , p. 113]. *Let $K_\varepsilon : W^{1-\frac{1}{p},p}(I) \to \mathbb{R}$ and $K : BV(I, \{\alpha', \beta'\}) \to \mathbb{R}$ be defined by (3.2) and (3.4).*

Then

(i) [Compactness] *If $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ is a sequence such that $K_\varepsilon(v_\epsilon)$ is bounded, then (v_ϵ) is pre-compact in $L^1(I)$ and every cluster point belongs to $BV(I, \{\alpha', \beta'\})$.*

(ii) [Lower bound inequality] *For every $v \in BV(I, \{\alpha', \beta'\})$ and every sequence $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $v_\epsilon \to v$ in $L^1(I)$,*

$$\liminf_{\varepsilon\to 0} K_\varepsilon(v_\epsilon) \geq K(v).$$

(iii) [Upper bound inequality] *For every $v \in BV(I, \{\alpha', \beta'\})$ there exists a sequence $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $v_\epsilon \to v$ in $L^1(I)$ and*

$$\limsup_{\varepsilon\to 0} K_\varepsilon(v_\epsilon) \leq K(v).$$

3.2 The optimal profile problem

In this section we will study the main features of our functional, namely the scaling property and the optimal profile problem.

It is useful to introduce the localization of the functional K_ε. For every open set $J \subseteq I$ and every function $v \in W^{1-\frac{1}{p},p}(J)$ we will denote

$$K_\varepsilon(v,J) := \varepsilon^{p-2} \iint_{J\times J} \left| \frac{v(x)-v(x')}{x-x'} \right|^p dx dx' + \frac{1}{\varepsilon} \int_J V(v)\, dx.$$

Clearly, $K_\varepsilon(v) = K_\varepsilon(v,I)$, for every $v \in W^{1-\frac{1}{p},p}(I)$.

Given $J \subseteq I$ and $v \in W^{1-\frac{1}{p},p}(J)$ we set $v^{(\varepsilon)}(x) := v(\varepsilon x)$ and $J/\varepsilon := \{x : \varepsilon x \in J\}$. By scaling it is immediately seen that

$$K_\varepsilon(v,J) = K_1(v^{(\varepsilon)}, J/\varepsilon). \tag{3.5}$$

In view of this scaling property, it is now natural to consider the optimal profile problem (3.3). The constant γ represents the minimal cost in the term of the non-scaled energy K_1 for a transition from α' to β' on the whole real line. By (3.5) γ will also give the cost of one jump from α' to β'.

Using the monotone rearrangement result of Section 1.4, we will prove that the infimum in (3.3) is not trivial and is achieved.
We recall that the increasing rearrangement v^* of a function $v \in W^{1-\frac{1}{p},p}(J)$, with $J = (a,b)$, is defined by

$$v^*(a+x) := \sup\left\{\lambda : |\{t \in (a,b) : v(t) < \lambda\}| \le x\right\}, \quad \forall x \in (0, b-a), \tag{3.6}$$

and satisfies

$$K_\varepsilon(v^*, J) \le K_\varepsilon(v,J) \tag{3.7}$$

(see Theorem 1.14, p. 30 and Theorem 1.15, p. 30).

In order to prove the upper bound it is convenient to introduce an auxiliary optimal profile problem. For every $T > 0$, we consider

$$\gamma^T := \inf\left\{ K_1(w,\mathbb{R}) : w \in W^{1-\frac{1}{p},p}_{\text{loc}}(\mathbb{R}), w(x) = \alpha' \ \forall x \le -T, w(x) = \beta' \ \forall x \ge T \right\}. \tag{3.8}$$

By the compactness of the embedding of $W^{1-\frac{1}{p},p}((-2T,2T))$ in $L^p((-2T,2T))$, it is easy to prove that the minimum in (3.8) is achieved. By truncation and rearrangement it also follows that the minimum can be achieved by a function $\varphi^T \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R})$ which is non-decreasing and satisfies $\alpha' \leq \varphi^T \leq \beta'$.

Proposition 3.2. *The sequence γ^T is non-increasing in T and* $\lim_{T\to+\infty} \gamma^T = \gamma$.

Proof. By the definition of γ^T, it immediately follows that γ^T is monotone and is greater than or equal to γ. Hence, the limit exists and satisfies

$$\lim_{T\to+\infty} \gamma^T \geq \gamma.$$

It remains to prove the reverse inequality. For every $\mu > 0$, let us fix $\psi \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R})$ such that

$$\lim_{x\to-\infty} \psi(x) = \alpha', \quad \lim_{x\to+\infty} \psi(x) = \beta' \quad \text{and} \quad K_1(\psi, \mathbb{R}) \leq \gamma + \mu.$$

Moreover, by truncation we may always assume that $\alpha' \leq \psi \leq \beta'$.

The idea is to modify ψ in order to construct a function φ which is a good competitor for γ^T. To this aim we consider

$$\Psi(x) := \int_{\mathbb{R}} \left| \frac{\psi(x) - \psi(x')}{x - x'} \right|^p dx'.$$

Since $\Psi \in L^1(\mathbb{R})$ we can choose a sequence $\{T_n\}_{n\in\mathbb{N}}$, with $T_n \to +\infty$, such that

$$\Psi(-T_n) \to 0 \quad \text{and} \quad \Psi(T_n) \to 0 \quad \text{as } n \to +\infty.$$

For every $\delta > 0$, due to the asymptotic behavior of ψ, we can find $n_\delta \in \mathbb{N}$ such that

$$\psi(-T_n) \leq \alpha' + \delta \quad \text{and} \quad \psi(T_n) \geq \beta' - \delta, \quad \forall n \geq n_\delta. \tag{3.9}$$

For every $M > 0$, we define a function φ which coincides with ψ in $[-T_n, T_n]$, satisfies $\varphi(x) = \alpha'$ if $x < -T_n - M$ and $\varphi(x) = \beta'$ if $x > T_n + M$ and it is affine in $(-T_n - M, -T_n)$ and $(T_n, T_n + M)$ (see Fig. 3.1).

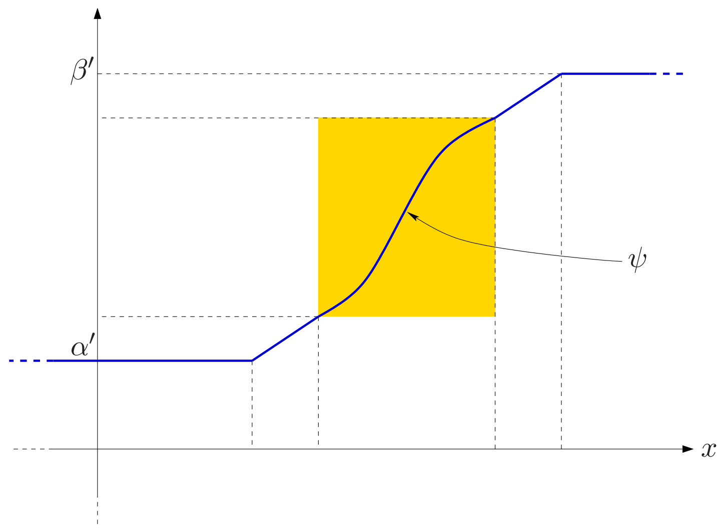

Figure 3.1: The competitor φ.

Namely,

$$\varphi(x) := \begin{cases} \alpha' & \text{if } x \in (-\infty, -T_n - M], \\ \dfrac{\psi(-T_n) - \alpha'}{M}(x + T_n) + \psi(-T_n) & \text{if } x \in (-T_n - M, -T_n), \\ \psi(x) & \text{if } x \in [-T_n, T_n], \\ \dfrac{\beta' - \psi(T_n)}{M}(x - T_n) + \psi(T_n) & \text{if } x \in (T_n, T_n + M), \\ \beta' & \text{if } x \in [T_n + M, +\infty). \end{cases}$$

Clearly, φ is a good competitor for γ^{T_n+M}. Let us compute its energy, denoting $J_n :=$

$(-T_n, T_n)$,

$$\begin{aligned}
\gamma^{T_n+M} &\leq K_1(\varphi, \mathbb{R}) \\
&= K_1(\psi, J_n) + K_1(\varphi, \mathbb{R} \setminus J_n) + 2\iint_{(\mathbb{R}\setminus J_n)\times J_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' \\
&\leq \gamma + \mu + \iint_{(\mathbb{R}\setminus J_n)\times(\mathbb{R}\setminus J_n)} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' + \int_{\mathbb{R}\setminus J_n} V(\varphi)\, dx \\
&\quad +2\iint_{(\mathbb{R}\setminus J_n)\times J_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' \\
&= \gamma + \mu + I_1 + I_2 + I_3. \qquad (3.10)
\end{aligned}$$

The first two integrals in the right hand side of (3.10) can be easily estimated as follows

$$\begin{aligned}
I_1 := \iint_{(\mathbb{R}\setminus J_n)\times(\mathbb{R}\setminus J_n)} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' &\leq 2(\beta'-\alpha')^p \int_{-\infty}^{-T_n}\int_{T_n}^{+\infty} \frac{dxdx'}{|x-x'|^p} \\
&= \frac{(\beta'-\alpha')^p}{(p-1)(p-2)(2T_n)^{p-2}}
\end{aligned}$$

and

$$I_2 := \int_{\mathbb{R}/J_n} V(\varphi)\, dx \leq 2M\omega_\delta,$$

where

$$\omega_\delta := \max_{s\in[\alpha',\alpha'+\delta]\cup[\beta'-\delta,\beta']} V(s). \qquad (3.11)$$

Instead, an upper bound for the last integral requires more attention in computation. Let us show it in detail.

$$\begin{aligned}
I_3 := &\ 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' + 2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' \\
&+2\int_{T_n+M}^{+\infty}\int_{-T_n}^{T_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' + 2\int_{T_n}^{T_n+M}\int_{-T_n}^{T_n} \left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx'.
\end{aligned}$$

We have

$$
\begin{aligned}
2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' &= 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\left|\frac{\psi(x')-\alpha'}{x-x'}\right|^p dxdx' \\
&\leq 2(\beta'-\alpha')^p\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\frac{dxdx'}{|x-x'|^p} \\
&= \frac{2(\beta'-\alpha')^p}{(p-1)(p-2)M^{p-2}}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(x')}{x-x'}\right|^p dxdx' \\
&= 2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\frac{\left|\psi(x')-\psi(-T_n)-\frac{\psi(-T_n)-\alpha'}{M}(x+T_n)\right|^p}{|x-x'|^p}dxdx' \\
&\leq 2^p\int_{-T_n-M}^{-T_n}\Psi(-T_n)dx+2^p\frac{|\psi(-T_n)-\alpha'|^p}{M^p}\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\left|\frac{x+T_n}{x-x'}\right|^p dxdx' \\
&\leq 2^pM\Psi(-T_n)+\frac{2^{p-1}\delta^p}{(p-1)M^{p-2}}, \quad \forall n\geq n_\delta\,,
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\frac{|x+T_n|^p}{|x-x'|^p}dxdx' &= \frac{1}{p-1}\int_{-T_n-M}^{-T_n}\left(|x+T_n|-\frac{|x+T_n|^p}{|T_n-x|^{p-2}}\right)dx \\
&\leq \frac{M^2}{2(p-1)}.
\end{aligned}
$$

Similarly, we can estimate the third and the fourth integrals of I_3 and we get

$$
I_3\leq 2^pM(\Psi(-T_n)+\Psi(T_n))+\frac{2^p\delta^p}{(p-1)M^{p-2}}+\frac{4(\beta'-\alpha')^p}{(p-1)(p-2)M^{p-2}}.
$$

Finally, by (3.10), we obtain

$$
\gamma^{T_n+M}\leq\gamma+\mu+r_n+r_\delta+\frac{4(\beta'-\alpha')^p}{(p-1)(p-2)M^{p-2}}, \quad \forall n\geq n_\delta, \tag{3.12}
$$

where

$$r_n := \frac{2(\beta' - \alpha')^p}{(p-1)(p-2)(2T_n)^{p-2}} + 2^p M \left(\Psi(-T_n) + \Psi(T_n)\right)$$

and

$$r_\delta := \frac{2^p}{(p-1)M^{p-2}}\delta^p + 2M\omega_\delta.$$

Taking the limit as $n \to +\infty$ and then $\delta \to 0$ and $M \to +\infty$, we get

$$\lim_{T\to+\infty} \gamma^T = \lim_{n\to+\infty} \gamma^{T_n+M} \leq \gamma + \mu,$$

which concludes the proof by the arbitrariness of μ. □

Let us conclude this section with the proof of the existence of an optimal profile.

Proposition 3.3. *The minimum for γ defined by (3.3) is achieved by a non-decreasing function φ satisfying $\alpha' \leq \varphi \leq \beta'$.*

Proof. Let $T > 0$ and let φ^T be a non-decreasing minimizer for γ^T. Since the functions φ^T are monotone and bounded, by Helly's Theorem, there exist a subsequence φ^{T_k} of φ^T and a non-decreasing function φ, bounded by α' and β', such that φ^{T_k} converges pointwise in $\mathbb{R}$ to φ. By Fatou's Lemma and Proposition 3.2 we also have

$$\iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{\varphi(x') - \varphi(x')}{x - x'}\right|^p dx dx' + \int_{\mathbb{R}} V(\varphi)\, dx \leq \lim_{k\to\infty} \gamma^{T_k} = \gamma\,.$$

This implies that φ is a minimizer for γ. □

3.3 Compactness

The proof of the compactness follows the lines of the proof of Alberti, Bouchitté and Seppecher in [8], requiring a Young measure argument. We use the following lemma which gives a (non-optimal) lower bound for K_ε.

Lemma 3.4. *Let $(v_\epsilon) \subset W^{1-\frac{1}{p},p}(I)$ and let $J \subset I$ be an open interval. For every δ such that $0 < \delta < (\beta' - \alpha')/2$, let us define*

$$A_\varepsilon := \{x \in I : v_\epsilon(x) \leq \alpha' + \delta\} \quad \textit{and} \quad B_\varepsilon := \{x \in I : v_\epsilon(x) \geq \beta' - \delta\}.$$

Let us set

$$a_\varepsilon := \frac{|A_\varepsilon \cap J|}{|J|} \quad \text{and} \quad b_\varepsilon := \frac{|B_\varepsilon \cap J|}{|J|}. \tag{3.13}$$

Then

$$K_\varepsilon(v_\epsilon, J) \geq \left(\frac{2(\beta' - \alpha' - 2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}} \right) \right) \varepsilon^{p-2} + c_\delta, \tag{3.14}$$

where c_δ does not depend on ε.

Proof. Let $x_0, x_0' \in \mathbb{R}$ be such that $J = (x_0, x_0')$; we obtain

$$\begin{aligned} K_\varepsilon(v_\varepsilon, J) &\geq K_\varepsilon(v_\varepsilon^*, J) \\ &\geq 2\varepsilon^{p-2}(\beta' - \alpha' - 2\delta)^p \int_{x_0}^{x_0 + a_\varepsilon |J|} \int_{x_0' - b_\varepsilon |J|}^{x_0'} \frac{dx dx'}{|x - x'|^p} + \frac{1}{\varepsilon} m_\delta |J| (1 - a_\varepsilon - b_\varepsilon) \\ &= \frac{2\varepsilon^{p-2}(\beta' - \alpha' - 2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}} + \frac{1}{(1-a_\varepsilon - b_\varepsilon)^{p-2}} \right) \\ &\quad + \frac{1}{\varepsilon} m_\delta |J| (1 - a_\varepsilon - b_\varepsilon), \end{aligned}$$

where v_ε^* denote the non-decreasing rearrangement of v_ε in (x_0, x_0') defined by (3.6) and $m_\delta := \min\{V(s) : \alpha' + \delta \leq s \leq \beta' - \delta\}$.

Minimizing with respect to $|J|(1 - a_\varepsilon - b_\varepsilon)$, we get

$$\begin{aligned} K_\varepsilon(v_\varepsilon, J) \ &\geq \ \varepsilon^{p-2} \left(\frac{2(\beta' - \alpha' - 2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}} \right) \right) \\ &\quad + 2^{\frac{1}{p-1}} \frac{(p-1)^{\frac{p-2}{p-1}}}{p-2} (\beta' - \alpha' - 2\delta)^{\frac{p}{p-1}} m_\delta^{\frac{p-2}{p-1}}, \end{aligned}$$

for every $0 < \delta < (\beta' - \alpha')/2$, and hence (3.14) is proved. □

We are now in a position to prove the compactness result (i.e., Theorem 3.1(i)).

Let $(v_\epsilon) \subset W^{1-1/p,p}(I)$ be a sequence with equi-bounded energy; i.e., a sequence satisfying $\sup_{\varepsilon > 0} K_\varepsilon(v_\epsilon) \leq C$. In particular

$$\int_I V(v_\epsilon)\, dx \leq C\varepsilon \tag{3.15}$$

and this implies that

$$V(v_\epsilon) \to 0 \text{ in } L^1(I). \tag{3.16}$$

Thanks to the growth assumptions on V, (v_ϵ) is equi-integrable. Hence, by Dunford-Pettis' Theorem, (v_ϵ) is weakly relatively compact in $L^1(I)$; i.e., there exists $v \in L^1(I)$ such that (up to subsequences) $(v_\epsilon) \rightharpoonup v$ in $L^1(I)$

We have to prove that this convergence is strong in $L^1(I)$ and that $v \in BV(I, \{\alpha, \beta\})$. Let ν_x be the Young measure associated to (v_ϵ). Since $V \geq 0$, we can use Theorem 1.5 (see Section 1.2, p. 24). We have

$$\int_I \int_{\mathbb{R}} V(t)\, d\nu_x(t) \leq \liminf_{\varepsilon \to 0} \int_I V(v_\varepsilon)\, dx$$

Hence, by (3.16), it follows that

$$\int_{\mathbb{R}} V(t)\, d\nu_x(t) = 0, \quad \text{a.e. } x \in I,$$

which implies the existence of a function ϑ on $[0,1]$ such that

$$\nu_x(dt) = \vartheta(x)\delta_{\alpha'}(dt) + (1 - \vartheta(x))\delta_{\beta'}(dt)\,, \quad x \in I$$

and

$$v(x) = \vartheta(x)\alpha' + (1 - \vartheta(x))\beta'\,, \quad x \in I.$$

It remains to prove that ϑ belongs to $BV(I, \{0,1\})$. Let us consider the set $S\vartheta$ of the points where the approximate limits of ϑ is neither 0 nor 1. For every $N \leq \mathcal{H}^0(S\vartheta)$ we can find N disjoint intervals $\{J_n\}_{n=1,\ldots,N}$ such that $J_n \cap S\vartheta \neq \emptyset$ and such that the quantities a_ε^n and b_ε^n, defined by (3.13) replacing J by J_n, satisfy

$$a_\varepsilon^n \to a^n \in (0,1) \quad \text{and} \quad b_\varepsilon^n \to b^n \in (0,1) \quad \text{as } \varepsilon \text{ goes to zero.}$$

We can now apply Lemma 3.4 in the interval J_n and, taking the limit as $\varepsilon \to 0$ in the inequality (3.14), we obtain

$$\liminf_{\varepsilon \to 0} K_\varepsilon(u_\varepsilon, J_n) \geq c_\delta.$$

Finally, we use the sub-additivity of $K_\varepsilon(u, \cdot)$ and we get

$$\liminf_{\varepsilon \to 0} K_\varepsilon(v_\epsilon, I) \geq \sum_{n=1}^{N} \liminf K_\varepsilon(v_\epsilon, J_n) \geq N c_\delta. \tag{3.17}$$

Since (v_ϵ) has equi-bounded energy, this implies that $S\vartheta$ is a finite set and, since a function $w : I \to \{0,1\}$ is in $BV(I, \{0,1\})$ if and only if $\mathcal{H}^0(Sw) < \infty$, it follows $\theta \in BV(I, \{0,1\})$. The proof of the compactness for K_ε is complete.

3.4 Lower bound inequality

In this section, we prove the Γ-liminf inequality. An optimal lower bound for $K_\varepsilon(v_\epsilon)$ is a consequence of the following proposition.

Proposition 3.5. *Let J be an open interval of $\mathbb{R}$. Let (v_ϵ) be a sequence of non-decreasing functions in $W^{1-\frac{1}{p},p}(J)$ and assume that there exist $\bar{a}, \bar{b} \in J$, $\bar{a} < \bar{b}$, such that for every $\delta > 0$ there exists ε_δ such that*

$$v_\epsilon(\bar{a}) \leq \alpha' + \delta \quad \textit{and} \quad v_\epsilon(\bar{b}) \geq \beta' - \delta \quad \forall \varepsilon \leq \varepsilon_\delta.$$

Then

$$\liminf_{\varepsilon \to 0} K_\varepsilon(v_\varepsilon, J) \geq \gamma.$$

Proof. Let $J = (a, b)$. It is clearly enough to consider the case

$$\liminf_{\varepsilon \to 0} K_\varepsilon(v_\epsilon, (a, b)) < +\infty.$$

By a truncation argument, without loss of generality, we may also assume that

$$\alpha' \leq v_\epsilon(x) \leq \beta', \quad \forall x \in (a, b).$$

Let us define

$$U_\varepsilon(x) := \varepsilon^{p-2} \int_a^b \left| \frac{v_\epsilon(x) - v_\epsilon(x')}{x - x'} \right|^p dx'.$$

By the fact that

$$\liminf_{\varepsilon \to 0} \int_a^b U_\varepsilon(x)\, dx$$

is finite, we get that there exist $\tilde{x} \in (a, \bar{a})$ and $\tilde{x}' \in (\bar{b}, b)$ such that

$$\liminf_{\varepsilon \to 0} U_\varepsilon(\tilde{x}) \leq C \quad \text{and} \quad \liminf_{\varepsilon \to 0} U_\varepsilon(\tilde{x}') \leq C \quad \text{for some } C > 0. \tag{3.18}$$

Fix $M > 0$. We now extend v_ϵ on the whole $\mathbb{R}$ as follows

$$\tilde{v}_\epsilon(x) := \begin{cases} \alpha' & \text{if } x \in (-\infty, \tilde{x} - M\varepsilon), \\ \dfrac{v_\epsilon(\tilde{x}) - \alpha'}{M\varepsilon}(x - \tilde{x}) + v_\epsilon(\tilde{x}) & \text{if } x \in [\tilde{x} - M\varepsilon, \tilde{x}], \\ v_\epsilon(x) & \text{if } x \in (\tilde{x}, \tilde{x}'), \\ \dfrac{\beta' - v_\epsilon(\tilde{x}')}{M\varepsilon}(x - \tilde{x}') + v_\epsilon(\tilde{x}') & \text{if } x \in [\tilde{x}', \tilde{x}' + M\varepsilon], \\ \beta' & \text{if } x \in (\tilde{x}' + M\varepsilon, +\infty). \end{cases}$$

Denote $\tilde{J} := (\tilde{x}, \tilde{x}') \subseteq (a, b)$. We have

$$\begin{aligned} K_\varepsilon(v_\epsilon, \tilde{J}) &\geq \gamma - K_\varepsilon(\tilde{v}_\epsilon, \mathbb{R} \setminus \tilde{J}) - 2\varepsilon^{p-2} \iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}} \left| \frac{\tilde{v}_\epsilon(x) - \tilde{v}_\epsilon(x')}{x - x'} \right|^p dx dx' \\ &= \gamma - \varepsilon^{p-2} \iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})} \left| \frac{\tilde{v}_\epsilon(x) - \tilde{v}_\epsilon(x')}{x - x'} \right|^p dx dx' - \frac{1}{\varepsilon} \int_{\mathbb{R}\setminus\tilde{J}} V(\tilde{v}_\epsilon)\, dx \\ &\quad - 2\varepsilon^{p-2} \iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}} \left| \frac{\tilde{v}_\epsilon(x) - \tilde{v}_\epsilon(x')}{x - x'} \right|^p dx dx' \\ &= \gamma - I_1 - I_2 - I_3. \end{aligned} \tag{3.19}$$

Using the definition of $\tilde{v}_\epsilon$, we easily get

$$\begin{aligned} I_1 &:= \varepsilon^{p-2} \iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})} \left| \frac{\tilde{v}_\epsilon(x) - \tilde{v}_\epsilon(x')}{x - x'} \right|^p dx dx' \\ &\leq \varepsilon^{p-2} (\beta' - \alpha')^p \iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})} \frac{dx dx'}{|x - x'|^p} \\ &= \frac{(\beta' - \alpha')^p}{(p-1)(p-2)|\tilde{J}|^{p-2}} \varepsilon^{p-2}. \end{aligned}$$

Moreover, since v_ϵ is non-decreasing,

$$v_\epsilon(x) \leq \alpha' + \delta \quad \forall\, x \leq \bar{a} \quad \text{and} \quad v_\epsilon(x) \geq \beta' - \delta \quad \forall\, x \geq \bar{b}$$

and, in particular,

$$I_2 := \frac{1}{\varepsilon}\int_{\mathbb{R}\setminus\tilde{J}} V(\tilde{v}_\epsilon)\,dx \le 2M\omega_\delta,$$

where ω_δ is defined in (3.11).

Finally, using the fact that $v_\epsilon(\tilde{x}) \le \alpha' + \delta$ and $v_\epsilon(\tilde{x}') \ge \beta' - \delta$, we can estimate the third integral

$$\begin{aligned}I_3 &:= 2\varepsilon^{p-2}\iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx' = 2\varepsilon^{p-2}\int_{-\infty}^{\tilde{x}-M\varepsilon}\int_{\tilde{x}}^{\tilde{x}'}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx' \\ &+2\varepsilon^{p-2}\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\int_{\tilde{x}}^{\tilde{x}'}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx' + 2\varepsilon^{p-2}\int_{\tilde{x}'+M\varepsilon}^{+\infty}\int_{\tilde{x}}^{\tilde{x}'}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx' \\ &+2\varepsilon^{p-2}\int_{\tilde{x}'}^{\tilde{x}'+M\varepsilon}\int_{\tilde{x}}^{\tilde{x}'}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx'. \end{aligned} \tag{3.20}$$

We have

$$\begin{aligned}2\varepsilon^{p-2}\int_{-\infty}^{\tilde{x}-M\varepsilon}\int_{\tilde{x}}^{\tilde{y}}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-y}\right|^p dxdy \;&\le\; 2\varepsilon^{p-2}(\beta'-\alpha')^p\int_{-\infty}^{\tilde{x}-M\varepsilon}\int_{\tilde{x}}^{\tilde{y}}\frac{dxdy}{|x-y|^p} \\ &\le\; \frac{2(\beta'-\alpha')^p}{(p-1)(p-2)M^{p-2}}.\end{aligned}$$

Moreover,

$$\begin{aligned}&2\varepsilon^{p-2}\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\int_{\tilde{x}}^{\tilde{x}'}\left|\frac{\tilde{v}_\epsilon(x)-\tilde{v}_\epsilon(x')}{x-x'}\right|^p dxdx' \\ &\qquad= 2\varepsilon^{p-2}\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\int_{\tilde{x}}^{\tilde{x}'}\frac{|v_\epsilon(x')-v_\epsilon(\tilde{x})-\frac{v_\epsilon(\tilde{x})-\alpha'}{M\varepsilon}(x-\tilde{x})|^p}{|x-x'|^p}dxdx' \\ &\qquad\le 2^p\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}U_\varepsilon(\tilde{x})dx + 2^p\frac{|v_\epsilon(\tilde{x})-\alpha'|^p}{M^p\varepsilon^2}\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\int_{\tilde{x}}^{\tilde{x}'}\frac{|\tilde{x}-x|^p}{|x-x'|^p}dxdx' \\ &\qquad\le 2^pM\varepsilon U_\varepsilon(\tilde{x}) + \frac{2^{p-1}\delta^p}{(p-1)M^{p-2}}\,, \quad \forall\,\varepsilon\le\varepsilon_\delta\,,\end{aligned}$$

where we used the fact that

$$\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\int_{\tilde{x}}^{\tilde{x}'}\frac{|\tilde{x}-x|^p}{|x-x'|^p}dxdx' = \frac{1}{(p-1)}\int_{\tilde{x}-M\varepsilon}^{\tilde{x}}\left(|x-\tilde{x}| - \frac{|x-\tilde{x}|^p}{|\tilde{x}'-x|^{p-1}}\right)dx \le \frac{(M\varepsilon)^2}{2(p-1)}.$$

Similarly, we can estimate the third and the fourth integrals of I_3 and we get

$$I_3 \leq 2^p M(U_\varepsilon(\tilde{x}) + U_\varepsilon(\tilde{y}))\varepsilon + \frac{2^p \delta^p}{(p-1)M^{p-2}} + \frac{4(\beta' - \alpha')^p}{(p-1)(p-2)M^{p-2}}\,, \quad \forall\, \varepsilon \leq \varepsilon_\delta.$$

Hence, by (3.19), we obtain

$$\begin{aligned} K_\varepsilon(v_\epsilon, \tilde{J}) \ &\geq \ \gamma - \left(\frac{(\beta' - \alpha')^p}{(p-1)(p-2)|\tilde{J}|^{p-2}} \varepsilon^{p-2} + 2^p M(U_\varepsilon(\tilde{x}) + U_\varepsilon(\tilde{y}))\varepsilon - r_\delta \right) \\ &\quad - \frac{4(\beta' - \alpha')^p}{(p-1)(p-2)M^{p-2}}\,, \quad \forall\, \varepsilon \leq \varepsilon_\delta\,, \end{aligned}$$

with

$$r_\delta := \frac{2^p \delta^p}{(p-1)M^{p-2}} \delta^p + 2M\omega_\delta$$

vanishing as $\delta \to 0$.

Thus, by (3.18) and taking the liminf as $\varepsilon \to 0$ and then as $\delta \to 0$, we get

$$\liminf_{\varepsilon \to 0} K_\varepsilon(v_\epsilon, \tilde{J}) \geq \gamma - \frac{4(\beta' - \alpha')^p}{(p-1)(p-2)M^{p-2}},$$

which concludes the proof by the arbitrariness of M. □

Remark 3.6. *Clearly an analogue proposition holds in the case of v_ϵ non-increasing satisfying the hypotheses with $\bar{a} > \bar{b}$.*

In order to conclude, let us first observe that, thanks to the compactness result for K_ε, we may assume that the sequence (v_ϵ) converges in $L^1(I)$ to some $u \in BV(I, \{\alpha', \beta'\})$. Hence, the jump set Sv is finite and we can find $N := \mathcal{H}^0(Sv)$ disjoint subintervals $\{I_i\}_{i=1,...,N}$ such that $Sv \cap I_i \neq \emptyset$, for every $i = 1, ..., N$.

Now, let us consider the monotone rearrangement $v^*_{\varepsilon,i}$ of v_ϵ in I_i. The rearrangement $v^*_{\varepsilon,i}$ is non-decreasing if u is non-decreasing in I_i and non-increasing otherwise. With this choice clearly $v^*_{\varepsilon,i}$ converges to v in $L^1(I_i)$ and thus it satisfies the assumptions of Proposition 3.5 (see also Remark 3.6) with J replaced by I_i. Then, for every $i = 1, ..., N$, we may conclude that

$$\liminf_{\varepsilon \to 0} K_\varepsilon(v_\epsilon, I_i) \geq \liminf_{\varepsilon \to 0} K_\varepsilon(v^*_{\varepsilon,i}, I_i) \geq \gamma.$$

Finally, using the sub-additivity of $K_\varepsilon(v_\epsilon,\cdot)$, we get

$$\liminf_{\varepsilon\to 0} K_\varepsilon(v_\epsilon, I) \geq \liminf_{\varepsilon\to 0} \sum_{i=1}^{N} K_\varepsilon(v_\epsilon, I_i) \geq N\gamma = \gamma\mathcal{H}^0(Sv)$$

and hence the lower bound stated by Theorem 3.1, (ii), is proved. □

3.5 Upper bound inequality

In this section, we conclude the proof of the Theorem 3.1, proving the limsup inequality. Let us first construct an optimal sequence for v of the form

$$v(x) = \begin{cases} \alpha', & \text{if } x \leq x_0, \\ \beta', & \text{if } x > x_0. \end{cases}$$

Let $T > 0$ be fixed and let $\varphi^T \in W^{1-\frac{1}{p},p}_{\text{loc}}(\mathbb{R})$ be the minimizer for γ^T defined by (3.8); i.e.,

$$\varphi^T(x) = \alpha' \quad \forall\, x \leq -T, \quad \varphi^T(x) = \beta' \quad \forall\, x \geq T \quad \text{and} \quad K_1(\varphi,\mathbb{R}) = \gamma^T.$$

Let us define, for every $\varepsilon > 0, v_\epsilon(x) := \varphi^T\left(\frac{x-x_0}{\varepsilon}\right)$, for every $x \in I$. We have

$$v_\epsilon \to v \text{ in } L^1(I)$$

and

$$\begin{aligned} K_\varepsilon(v_\epsilon) \;&=\; \varepsilon^{p-2}\iint_{I\times I}\left|\frac{\varphi^T(\frac{x-x_0}{\varepsilon}) - \varphi^T(\frac{x'-x_0}{\varepsilon})}{x-x'}\right|^p dxdx' + \frac{1}{\varepsilon}\int_I V\left(\varphi^T\left(\frac{x-x_0}{\varepsilon}\right)\right)dx \\ &=\; K_1(\varphi^T, (I-x_0)/\varepsilon) \leq K_1(\varphi^T,\mathbb{R}) = \gamma^T. \end{aligned} \tag{3.21}$$

By Proposition 3.2 we get

$$\lim_{T\to+\infty}\limsup_{\varepsilon\to 0} K_\varepsilon(v_\epsilon) \leq \gamma.$$

Then by a diagonalization argument we can construct a sequence $\tilde{v}_\epsilon$ converging to v in $L^1(I)$, which satisfies

$$\limsup_{\varepsilon\to 0} K_\varepsilon(\tilde{v}_\epsilon) \leq \gamma.$$

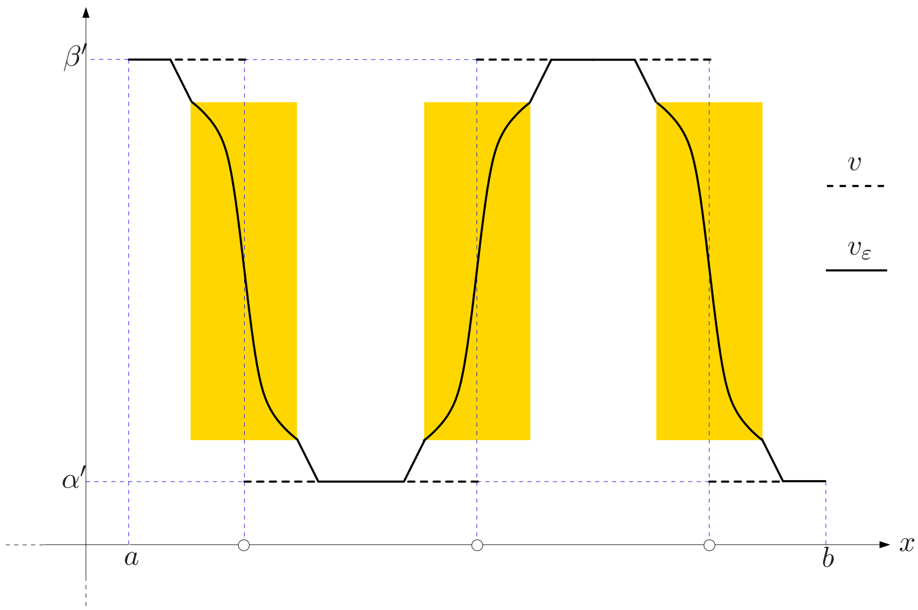

Figure 3.2: Construction of v_ε, with $I = (a, b)$ and $\mathcal{H}^0(Sv) = 3$.

The optimal sequence for an arbitrary $v \in BV(I, \{\alpha', \beta'\})$ can be easily obtained gluing the sequences constructed above for each single jump of v and taking into account that, thanks to the scaling ε^{p-2}, the long range interactions between two different recovery sequences decay as $\varepsilon \to 0$. See Fig. 3.2.

Chapter 4

A class of phase transition problems with line tension effect

In this chapter, we introduce the main problem of this thesis, that concerns the study of a functional similar to (2.10), but with a super-quadratic growth in the perturbation term.

Let Ω be a bounded open subset of $\mathbb{R}^3$ with smooth boundary; let W and V be non-negative continuous functions on $\mathbb{R}$ with growth at least linear at infinity and vanishing respectively only in the "double well" $\{\alpha, \beta\}$, with $\alpha < \beta$, and $\{\alpha', \beta'\}$, with $\alpha' < \beta'$. We also suppose that W and V are convex near their respective wells. Let $p > 2$ be a real number, for every $\varepsilon > 0$, we consider the functional F_ε defined in $W^{1,p}(\Omega)$, given by

$$F_\varepsilon(u) := \varepsilon^{p-2} \int_\Omega |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u)dx + \frac{1}{\varepsilon} \int_{\partial\Omega} V(Tu) d\mathcal{H}^2, \tag{4.1}$$

where as usual Tu denotes the trace of u on $\partial\Omega$.

Note that the choice of the scaling in (4.1) comes from the super-quadratic version of the Modica-Mortola functional; i.e.,

$$\varepsilon^{p-1} \int_\Omega |Du|^p dx + \frac{1}{\varepsilon} \int_\Omega W(u)dx,$$

but we also take into account the scaling of the functional K_ε studied in the previous chapter

$$\varepsilon^{p-2} \iint_{I\times I} \left| \frac{v(t) - v(t')}{t - t'} \right|^p dt dt' + \frac{1}{\varepsilon} \int_I V(v)\, dt.$$

We analyze the asymptotic behavior of the functional F_ε in terms of Γ-convergence. Let (u_ε) be an equi-bounded sequence for F_ε; i.e., there exists a constant C such that

$F(u_\varepsilon) \le C$. We observe that the term $\dfrac{1}{\varepsilon^{\frac{p-2}{p-1}}}\displaystyle\int_\Omega W(u_\varepsilon)dx$ forces u_ε to take values close to α and β, while the term $\varepsilon^{p-2}\displaystyle\int_\Omega |Du_\varepsilon|^p dx$ penalizes the oscillations of u_ε. We will see that when ε tends to 0, the sequence (u_ε) converges (up to a subsequence) to a function u, that belongs to $BV(\Omega)$, which takes only the values α and β. Moreover each u_ε has a transition from the value α to the value β in a thin layer close to the surface Su, which separates the bulk phases $\{u=\alpha\}$ and $\{u=\beta\}$. Similarly, the boundary term of F_ε forces the traces Tu_ε to take values close to α' and β', and the oscillations of the traces Tu_ε are again penalized by the integral $\varepsilon^{p-2}\displaystyle\int_\Omega |Du_\varepsilon|^p dx$. Then, we expect that the sequence (Tu_ε) converges to a function v in $BV(\partial\Omega)$ which takes only the values α' and β', and that a concentration of energy occurs along the line Sv, which separates the boundary phases $\{v=\alpha'\}$ and $\{v=\beta'\}$.

In view of possible "dissociation of the contact line and the dividing line", we recall that Tu may differ from v.[1] Since the total energy $F_\varepsilon(u_\varepsilon)$ is partly concentrated in a thin layer close to Su (where u_ε has a transition from α to β), partly in a thin layer close to the boundary (where u_ε has a transition from Tu to v), and partly in the vicinity of Sv (where Tu_ε has a transition from α' to β'), we expect that the limit energy is the sum of a surface energy concentrated on Su, a boundary energy on $\partial\Omega$ (with density depending on the gap between Tu and v), and a line energy concentrated along Sv.

The asymptotic behavior of the functional F_ε is described by a functional Φ which depends on the two variables u and v. Let $\mathcal{W}$ be a primitive of $W^{(p-1)/p}$. For every $(u,v) \in BV(\Omega,\{\alpha,\beta\}) \times BV(\partial\Omega,\{\alpha',\beta'\})$, we will prove that

$$\Phi(u,v) := \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 + \gamma_p \mathcal{H}^1(Sv), \tag{4.2}$$

where as usual the jump sets Su and Sv are the complement of the set of Lebesgue points of u and v, respectly; c_p and σ_p are the constants defined in Chapter 2; that are, $c_p := \dfrac{p}{(p-1)^{p/(p-1)}}$; $\sigma_p := c_p|\mathcal{W}(\beta) - \mathcal{W}(\alpha)|$; γ_p is given by the optimal profile problem

$$\gamma_p := \inf\left\{ \int_{\mathbb{R}^2_+} |Du|^p dx + \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 : u \in L^1_{\text{loc}}(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} |Du|^p dx \text{ is finite}, \right.$$
$$\left. \lim_{t\to-\infty} Tu(t) = \alpha', \ \lim_{t\to+\infty} Tu(t) = \beta' \right\}. \tag{4.3}$$

[1] See Chapter 2, p. 39.

The main convergence result is precisely stated in the following theorem.

Theorem 4.1. *Let* $F_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ *and* $\Phi : BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}) \to \mathbb{R}$ *defined by (6.1) and (6.2).*

Then

(i) [Compactness] *If* $(u_\varepsilon) \subset W^{1,p}(\Omega)$ *is a sequence such that* $F_\varepsilon(u_\varepsilon)$ *is bounded, then* $(u_\varepsilon, Tu_\varepsilon)$ *is pre-compact in* $L^1(\Omega) \times L^1(\partial\Omega)$ *and every cluster point belongs to* $BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$.

(ii) [Lower Bound Inequality] *For every* $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$ *and every sequence* $(u_\varepsilon) \subset W^{1,p}(\Omega)$ *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$ *and* $Tu_\varepsilon \to v$ *in* $L^1(\partial\Omega)$,

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

(iii) [Upper Bound Inequality] *For every* $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$ *there exists a sequence* $(u_\varepsilon) \subset W^{1,p}(\Omega)$ *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$, $Tu_\varepsilon \to v$ *in* $L^1(\partial\Omega)$ *and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u, v).$$

We can easily rewrite this theorem in term of Γ-convergence. To this aim, we extend each F_ε to $+\infty$ on $L^1(\Omega) \backslash W^{1,p}(\Omega)$ and, from Theorem 4.1, we briefly deduce the following remark.

Remark 4.2. F_ε *Γ-converges on* $L^1(\Omega)$ *to* F, *given by*

$$F(u) := \begin{cases} \inf\{\Phi(u, v) : v \in BV(\partial\Omega, \{\alpha', \beta'\})\} & \text{if } u \in BV(\Omega, \{\alpha, \beta\}), \\ +\infty & \text{elsewhere in } L^1(\Omega). \end{cases}$$

Note that the limit functional Φ is of the same type of the functional Ψ (defined by (2.12), Γ-limit of $\tilde{E}_\varepsilon$. Nevertheless, the variation of the power of the gradient in the perturbation term is not a simple generalization with respect to the quadratic case. The structure of these two problems is different. In the quadratic case, the natural scaling of the energy is logarithmic: this implies that the profile in which the phase transition occurs on the boundary is not important for the first order of energy. Instead, the super-quadratic case is characterized by an optimal profile problem which determines the line tension in the limit. In view of this, some arguments in the proof of Theorem 4.1 will be simplified, some other will require more care.

4.1 Strategy of the proof and some preliminary results

The proof of Theorem 4.1 requires several steps in which we have to analyze different effects. Then, we can deduce the terms of the limit energy Ψ, localizing three effects: the bulk effect, the wall effect and the boundary effect.

4.1.1 The bulk effect

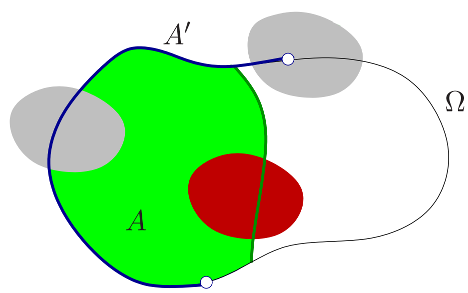

Figure 4.1: The bulk.

In the bulk term, the limit energy can be evaluate like in [48]. Of course, we will use the super-quadratic version of the Modica-Mortola functional, like seen in Chapter 2, p. 36.

4.1.2 The wall effect

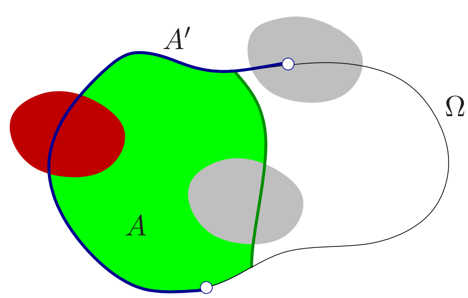

Figure 4.2: The wall.

The second term of Φ can be obtained by adapting the approach by Modica in [49].

We recall that the results by Modica concern a Cahn-Hilliard functional with quadratic growth in the singular perturbation term and with a boundary contribution of the form $\lambda \int_{\partial\Omega} g(Tu)d\mathcal{H}^2$, with λ not depending on ε and g a positive continuous function. Hence, we need to adapt part of Proposition 1.2 ([49], p. 492) and Proposition 1.4 ([49], p. 494) to our goal.

For every open set $A \subset \mathbb{R}^3$ and every real function $u \in W^{1,p}(A)$, we consider the functional

$$G_\varepsilon(u, A) := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u)dx. \tag{4.4}$$

Proposition 4.3. *For every domain $A \subset \mathbb{R}^3$ with boundary piecewise of class C^1 and for every $A' \subset \partial A$ with Lipschitz boundary, the following statements hold.*

(i) *For every $(u, v) \in BV(A, \{\alpha, \beta\}) \times BV(A', \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and $Tu_\varepsilon \to v$ in $L^1(A')$,*

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, A) \geq c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

(ii) *Let a function v, constant on A', and a function u, constant on A, such that $u \equiv \alpha$ or $u \equiv \beta$, be given. Then there exists a sequence (u_ε) such that $Tu_\varepsilon = v$ on A', u_ε converges uniformly to u on every set with positive distance from A' and*

$$\limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, A) \leq c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

Moreover, the function u_ε may be required to be $\frac{C'_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous (where C'_W is the maximum of $W^{(1)/p}$ over any interval which contains the values of u and v).

Proof. We may assume that $G_\varepsilon(u_\varepsilon) \leq C$. For every $\varepsilon > 0$, let us denote by

$$w_\varepsilon(x) := (\mathcal{W} \circ u_\varepsilon)(x), \quad \forall x \in A. \tag{4.5}$$

Step 1: $\int_A |Dw_\varepsilon| dx \leq$ *constant.*

By Young's Inequality, we have:

$$\begin{aligned}\int_A |Dw_\varepsilon| dx &= \int_A |\mathcal{W}'(u_\varepsilon)||Du_\varepsilon| dx \\ &= \int_A W^{\frac{p-1}{p}}(u_\varepsilon)|Du_\varepsilon| dx \\ &\leq \frac{G_\varepsilon(u_\varepsilon)}{c_p} \leq \frac{C}{c_p}.\end{aligned} \tag{4.6}$$

Step 2: $w_\varepsilon \to \mathcal{W} \circ u \in BV(A)$ *in* $L^1(A)$*.*

For every $\varepsilon > 0$ let us define the function

$$\bar{w}_\varepsilon(x) := w_\varepsilon(x) - \theta_\varepsilon, \quad \forall x \in A,$$

where

$$\theta_\varepsilon := \frac{1}{|A|} \int_A w_\varepsilon dx.$$

Then (see [10], Theorem 3.44, p. 148) there exists a constant c_A, depending on A, such that

$$\int_A |\bar{w}_\varepsilon| dx \leq c_A \int_A |D\bar{w}_\varepsilon| dx. \tag{4.7}$$

Since

$$\int_A |D\bar{w}_\varepsilon| dx = \int_A |Dw_\varepsilon| dx,$$

by (4.6), we obtain

$$\int_A |D\bar{w}_\varepsilon| dx < \text{const}. \tag{4.8}$$

By (4.7) and (4.8) follows that the sequence $(\bar{w}_\varepsilon)$ is bounded in $W^{1,1}(A)$. Hence, by Rellich's Theorem, there exists a function $\bar{w} \in BV(A)$ such that (up to a subsequence)

$$\bar{w}_\varepsilon \to \bar{w} \ \text{ in } L^1(A).$$

Since w_ε is bounded, it is not restrictive to assume that both $(\bar{w}_\varepsilon)$ and (w_ε) converge almost everywhere in A, so we have that θ_ε converges to θ in $\mathbb{R}$, and, finally, that (w_ε) converges to $\bar{w}+\theta$ in $L^1(A)$. Thus, since u_ε converges to u in $L^1(A)$, by (4.5), we conclude that (w_ε) converges to $\mathcal{W} \circ u$ in $L^1(A)$.

Moreover, by lower semicontinuity, we have

$$\int_A |D(\mathcal{W} \circ u)(x)| \leq \liminf_{\varepsilon \to 0} \int_A |Dw_\varepsilon(x)| \leq \text{ const.}$$

Step 3: $G_0(z) := \int_A |Dz(x)| + \int_{\partial A} |Tz - \mathcal{W}(v)| d\mathcal{H}^2$ *is l.s.c. on* $BV(A)$ *with respect to the topology* $L^1(A)$.

Let us fix $z \in BV(A)$ and let (z_ε) be a sequence in $BV(A)$ converging to z in $L^1(A)$. We will prove that

$$\limsup_{\varepsilon \to 0} (G_0(z) - G_0(z_\varepsilon)) \leq 0, \tag{4.9}$$

where

$$G_0(z) := \int_A |Dz(x)| + \int_{\partial A} |Tz - \mathcal{W}(v)| d\mathcal{H}^2, \ \forall z \in BV(A). \tag{4.10}$$

We can estimate the boundary part of the functional (4.10) with the L^1-norm of the difference of the traces fo z and z_ε:

$$\int_{\partial A} |Tz - \mathcal{W}(v)| d\mathcal{H}^2 - \int_{\partial A} |Tz_\varepsilon - \mathcal{W}(v)| d\mathcal{H}^2 \leq \int_{\partial A} |Tz - Tz_\varepsilon| d\mathcal{H}^2.$$

Thus, we have

$$G_0(z) - G_0(z_\varepsilon) \leq \int_A |Dz| dx - \int_A |Dz_\varepsilon(x)| + \int_{\partial A} |Tz - Tz_\varepsilon| d\mathcal{H}^2. \tag{4.11}$$

For every $\delta > 0$, take a cut-off function $\xi_\delta \in C_0^\infty(A)$ such that $0 \leq \xi_\delta(x) \leq 1$, $\xi_\delta(x) = 1$ if $\text{dist}(x, \partial A) \geq \delta$, $|D\xi_\delta| \leq 2/\delta$.

Let us define

$$z_\varepsilon^\delta(x) := (1 - \xi_\delta(x))(z - z_\varepsilon), \quad \forall x \in A,$$

Applying to z_ε^δ the trace inequality for BV functions by Anzellotti and Giaquinta (see [11], Teorema 5, p. 13); we obtain

$$\int_{\partial A} |Tz - Tz_\varepsilon| d\mathcal{H}^2 \leq \int_{A_\delta} |D(z - z_\varepsilon)(x)| + \left(\frac{2}{\delta} + c'\right) \int_{A_\delta} |z - z_\varepsilon| dx, \tag{4.12}$$

where $A_\delta := \{x \in A : \text{dist}(x, \partial A) \leq \delta\}$ and c is a constant that does not depend on δ. Moreover,

$$\int_{A_\delta} |D(z - z_\varepsilon)(x)| \leq \int_{A_\delta} |Dz(x)| + \int_{A_\delta} |Dz_\varepsilon(x)|, \tag{4.13}$$

where we used that $(z - z_\varepsilon) \in BV(A)$. Then we have

$$\int_{\partial(A_\delta^c)} |D(z - z_\varepsilon)(x)| \quad \text{(for every } \varepsilon > 0\text{)},$$

Hence, from (4.11), (4.12) and (4.13), we obtain

$$\begin{aligned} G_0(z) - G_0(z_\varepsilon) &\leq \int_A |Dz(x)| + \int_{A_\delta} |Dz(x)| - \int_{A_\delta^c} |Dz_\varepsilon(x)| + \left(\frac{2}{\delta} + c'\right) \int_{A_\delta} |z - z_\varepsilon| dx \\ &= 2\int_{A_\delta} |Dz(x)| + \int_{A_\delta^c} |Dz(x)| - \int_{A_\delta^c} |Dz_\varepsilon(x)| + \left(\frac{2}{\delta} + c'\right) \int_{A_\delta} |z - z_\varepsilon| dx \end{aligned}$$

Finally, using the lower semicontinuity in $L^1(A)$ of the functional

$$z \mapsto \int_{A_\delta^c} |Dz(x)|$$

and the fact that z_ε converges to z in $L^1(A)$, we conclude that

$$\limsup_{\varepsilon \to 0} \left(G_0(z) - G_0(z_\varepsilon)\right) \leq 2 \int_{A_\delta} |Dz(x)| \tag{4.14}$$

for almost every $\delta > 0$. By taking $\delta \to 0$ in (4.14), we obtain the inequality (4.9).

Step 4: Proof of Statement (i).

Applying the lower semicontinuity of the functional G_0 to the sequence (w_ε) defined by (4.5), we obtain the following inequality

$$\begin{aligned} c_p \int_A |D(\mathcal{W} \circ u)(x)| \; &+ \; c_p \int_{\partial A} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 \\ &\leq \liminf_{\varepsilon \to 0} c_p \left(\int_A |D(\mathcal{W} \circ u_\varepsilon)(x)| + \int_{\partial A} |\mathcal{W}(Tu_\varepsilon) - \mathcal{W}(v)| d\mathcal{H}^2 \right) \\ &\leq \liminf_{\varepsilon \to 0} \left(G_\varepsilon(u_\varepsilon) + c_p \int_{\partial A} |\mathcal{W}(Tu_\varepsilon) - \mathcal{W}(v)| d\mathcal{H}^2 \right). \end{aligned} \tag{4.15}$$

Since $Tu_\varepsilon \to v$ in $L^1(A')$, we deduce that

$$\lim_{\varepsilon \to 0} c_p \int_{\partial A} |\mathcal{W}(Tu_\varepsilon) - \mathcal{W}(v)| d\mathcal{H}^2 = 0. \tag{4.16}$$

Hence, the lower bound inequality of statement (i) follows from (4.15) and (4.16).

Step 5: Proof of Statement (ii). Let us consider the case $u = \beta$ and $v = \gamma$, with $\alpha < \gamma < \beta$;

the other cases can be treated in a similar way.

Let $\varphi : [0, +\infty) \to [\gamma, \beta]$ be a solution of the ordinary differential equation

$$\begin{cases} \varphi(t)' = \frac{p^{1/(p-1)}}{(p-1)^p} W^{1/p}(\varphi(t)), & \text{on } \mathbb{R}, \\ \varphi(0) = \gamma. \end{cases}$$

Then, φ is increasing, converges to β at $+\infty$, and satisfies

$$\begin{aligned} \int_{\mathbb{R}} |\varphi'|^p dt + \int_{\mathbb{R}} W(\varphi) dt &= c_p \int_{\mathbb{R}} W^{(p-1)/p}(\varphi)|\varphi'|^p dt \\ &= c_p |\mathcal{W}(\beta) - \mathcal{W}(\gamma)| \end{aligned} \tag{4.17}$$

Now, let us denote by $d(x)$ the distance of x from A'. For every $\varepsilon > 0$ and every $x \in A$, let us set

$$u_\varepsilon(x) := \varphi\left(\frac{d(x)}{\varepsilon^{\frac{p-2}{p-1}}}\right).$$

Since φ and d are C'_W and 1-Lipschitz continuous, respectively, u_ε is $\dfrac{C'_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous. Moreover, u_ε converges to u uniformly on every set with positive distance from A' and satisfies

$$G_\varepsilon(u_\varepsilon, A) = \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \left(\int_A |\varphi'(d/\varepsilon^{\frac{p-2}{p-1}})|^p dx + \int_A W(\varphi(d/\varepsilon^{\frac{p-2}{p-1}}))dx \right). \tag{4.18}$$

Hence, using the Co-Area Formula, it follows

$$G_\varepsilon(u_\varepsilon, A) = \int_{\mathbb{R}} \left(|\varphi'|^p + W(\varphi) \right) \mathcal{H}^2(\Sigma_{\varepsilon^{(p-2)/(p-1)}}) dt, \tag{4.19}$$

with $\Sigma_s := \{x \in A : d(x, A') = s\}$. Finally, using (4.17) and the Dominated Convergence Theorem in (4.19), we obtain that $G_\varepsilon(u_\varepsilon, A)$ converges to $c_p \displaystyle\int_{A'} |\mathcal{W}(\beta) - \mathcal{W}(\gamma)| d\mathcal{H}^2$. □

4.1.3 The boundary effect

This is a delicate step, that requires a deeper analysis. The main strategy is the following

- we reduce to the case in which the boundary is flat;
- hence we study the behavior of the energy in the three-dimensional half ball;

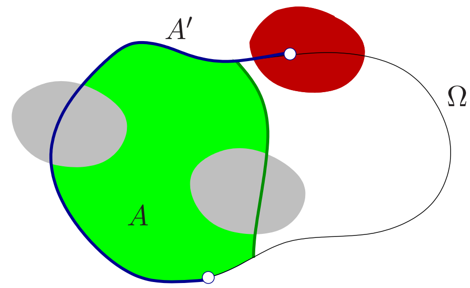

Figure 4.3: The boundary.

- then we reduce the problem of one dimension via a slicing argument.

Thus, the main problem becomes the analysis of the asymptotic behavior of the following two-dimensional functional

$$H_\varepsilon(u) := \varepsilon^{p-2} \int_{D_1} |Du|^p dx + \frac{1}{\varepsilon} \int_{E_1} V(Tu) d\mathcal{H}^1, \ \forall u \in W^{1,p}(D_1),$$

where D_1 and E_1 are defined by (2.14).

The asymptotic analysis of H_ε will be the subject of the next chapter.

4.2 Some remark about the structure of F_ε

The methods used in the proof strongly requires the "localization" of the functional F_ε; i.e., looking at F_ε as a function of sets. By fixing u we will be able to characterize the various effects of the problem, in the spirit of the classical "blow-up" method, developed by Fonseca and Müller in [36]. In this sense, for every open set $A \subset \mathbb{R}^3$, every set $A' \subseteq \partial A$ and every function $u \in W^{1,p}(A)$, we will denote

$$F_\varepsilon(u, A, A') := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) dx + \frac{1}{\varepsilon} \int_{A'} V(Tu) d\mathcal{H}^2.$$

Clearly, $F_\varepsilon(u) = F_\varepsilon(u, \Omega, \partial\Omega)$ for every $u \in W^{1,p}(\Omega)$.

Let us observe that, thanks to the growth hypothesis on the potentials W and V, we may assume that there exists a constant m such that:

$$\begin{gathered} -m \leq \alpha, \alpha', \beta, \beta' \leq m, \\ W(t) \geq W(m) \text{ and } V(t) \geq V(m) \text{ for } t \geq m, \\ W(t) \geq W(-m) \text{ and } V(t) \geq V(-m) \text{ for } t \leq -m. \end{gathered} \tag{4.20}$$

In particular, assumption (4.20) will allow us to use the truncation argument given by the following Lemma.

Lemma 4.4. *Let a domain $A \subset \mathbb{R}^3$, a set $A' \subseteq \partial A$, and a sequence $(u_\varepsilon) \subset W^{1,p}(A)$ with uniformly bounded energies $F_\varepsilon(u_\varepsilon, A, A')$ be given. If we set $\bar{u}_\varepsilon(x) := (u_\varepsilon(x) \wedge m) \vee -m)$, then*

(i) $F_\varepsilon(\bar{u}_\varepsilon, A, A') \le F_\varepsilon(u_\varepsilon, A, A')$,

(ii) $\|\bar{u}_\varepsilon - u_\varepsilon\|_{L^1(A)}$ *and* $\|T\bar{u}_\varepsilon - Tu_\varepsilon\|_{L^1(A')}$ *vanish as* $\varepsilon \to 0$.

Proof. The inequality $F_\varepsilon(\bar{u}_\varepsilon, A, A') \le F_\varepsilon(u_\varepsilon, A, A')$ follows immediately from (4.20). Statement (ii) follows from the fact that both W and V have growth at least linear at infinity and the integrals $\int W(u_\varepsilon)dx$ and $\int V(Tu_\varepsilon)dx$ vanish as ε goes to 0.

Since W is strictly positive and continuous out of α and β, for every $\delta > 0$, there exist $a > 0$ and $M > 0$ such that

$$W(t) \ge a \quad \forall t \in [-M, \alpha - \delta] \cup [\beta + \delta, M].$$

Moreover, since W has growth at least linear at infinity, we can find $b > 0$ such that

$$W(t) \ge b|t| \text{ when } |t| > M.$$

For every $\delta > 0$, we define

$$A_m := \{x \in A : m + \delta \le |u_\varepsilon(x)| \le M\} \text{ and } A_M := \{x \in A : |u_\varepsilon(x)| \ge M\}.$$

We have

$$\begin{aligned} \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^1(A)} &= \int_{A \setminus (A_m \cap A_M)} |u_\varepsilon - \bar{u}_\varepsilon| dx + \int_{A_m} |u_\varepsilon - \bar{u}_\varepsilon| dx + \int_{A_M} |u_\varepsilon - \bar{u}_\varepsilon| dx \\ &\le \delta|A| + M|A_m| + \int_{A_M} |u_\varepsilon| dx \\ &\le \delta|A| + \frac{M}{a} \int_{A_m} W(u_\varepsilon) dx + \frac{1}{b} \int_{A_M} W(u_\varepsilon) dx \\ &\le \delta|A| + \left(\frac{M}{a} + \frac{1}{b}\right) \int_A W(u_\varepsilon) dx. \end{aligned} \tag{4.21}$$

Since the sequence (u_ε) has uniformly bounded energy, there exists a constant C such that

$$\frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u_\varepsilon) dx \leq C. \tag{4.22}$$

Then, by (4.21) and (4.22), if follows

$$\|u_\varepsilon - \bar{u}_\varepsilon\|_{L^1(A)} \leq \delta |A| + C \left(\frac{M}{a} + \frac{1}{b} \right) \varepsilon^{\frac{p-2}{p-1}}. \tag{4.23}$$

Passing to the limit as ε goes to 0 in (4.23), we obtain

$$\limsup_{\varepsilon \to 0} \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^1(A)} \leq \delta |A|.$$

The proof is complete by the arbitrariness of δ. $\square$

Chapter 5

Recovering the "contribution of the wall": the flat case

We will obtain "the contribution of the wall" to the limit energy Φ, defined by (4.2), namely $\gamma_p \mathcal{H}^1(Sv)$, by estimating the asymptotic behavior of the functional

$$F_\varepsilon(u, B\cap\Omega, B\cap\partial\Omega) = \varepsilon^{p-2}\int_{B\cap\Omega}|Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{B\cap\Omega}W(u)dx + \int_{B\cap\partial\Omega}V(Tu)d\mathcal{H}^2,$$

when B is a small ball centered on $\partial\Omega$ and $B\cap\partial\Omega$ is a flat disk. We will follow the idea of Alberti, Bouchitté and Seppecher in [9], using a suitable slicing argument. Later on we will show that the flatness assumption on $B\cap\partial\Omega$ can be dropped when B is sufficiently small. Hence, we need to prove a compactness result and a lower bound inequality for the following two-dimensional functional

$$H_\varepsilon(u) := \varepsilon^{p-2}\int_{D_1}|Du|^p dx + \frac{1}{\varepsilon}\int_{E_1}V(Tu)d\mathcal{H}^1,\ \forall u\in W^{1,p}(D_1;[-m,m]), \tag{5.1}$$

where E_r and D_r are defined by (2.14). We recall that we will always study H_ε like a reduction of F_ε. Hence there will be some hypotheses inherited by this reduction. In particular, the hypothesis $u\in[-m,m]$ in (5.1) is justified by Lemma 4.4.

Let us introduce the "localization" of the functional H_ε. For every open set $A\subset\mathbb{R}^2$, every set $A'\subset\partial A$ and every function $u\in W^{1,p}(A)$, we will denote

$$H_\varepsilon(u, A, A') := \varepsilon^{p-2}\int_A |Du|^p dx + \frac{1}{\varepsilon}\int_{A'}V(Tu)d\mathcal{H}^1. \tag{5.2}$$

If we set $u^{(\varepsilon)}(x) := u(\varepsilon x)$ and $A/\varepsilon := \{x : \varepsilon x\in A\}$, by scaling it is immediately seen that

$$H_\varepsilon(u, A, A') = H_1(u^{(\varepsilon)}, A/\varepsilon, A'/\varepsilon). \tag{5.3}$$

In view of this scaling property, we consider the optimal profile problem, introduced in the previous chapter; that is,

$$\gamma_p = \inf\left\{\int_{\mathbb{R}^2_+} |Du|^p dx + \int_{\mathbb{R}} V(Tu)d\mathcal{H}^1 : u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} |Du|^p dx \text{ is finite,} \right.$$
$$\left. \lim_{t\to-\infty} Tu(t) = \alpha',\ \lim_{t\to+\infty} Tu(t) = \beta' \right\} \tag{5.4}$$

and determines the line tension on the limit energy Φ.

5.1 Compactness of the traces

We prove the pre-compactness of the traces of the sequences equi-bounded for H_ε, using the trace imbedding of $W^{1-1/p,p}(\partial D_1)$ in $W^{1,p}(D_1)$ and Theorem 3.1(i); that is, the compactness result for the one-dimensional functional K_ε (see Section 3.3, p. 51).

Proposition 5.1. *If $(u_\varepsilon) \subset W^{1,p}(D_1; [-m, m])$ is a sequence such that $H_\varepsilon(u_\varepsilon)$ is bounded, then (Tu_ε) is pre-compact in $L^1(E_1)$ and every cluster point belongs to $BV(E_1, \{\alpha', \beta'\})$.*

Proof. By hypothesis, there exists a constant C such that $H_\varepsilon(u_\varepsilon) \le C$.

By the trace imbedding of $W^{1-1/p,p}(\partial D_1)$ in $W^{1,p}(D_1)$, there exists a constant C_p such that for every $u \in W^{1,p}(D_1)$

$$\|Tu\|_{W^{1-1/p,p}(\partial D_1)} \le C_p \|u\|_{W^{1,p}(D_1)}.$$

It follows that there exists a constant (still denoted by C_p) such that

$$\begin{aligned} \iint_{E_1\times E_1} \frac{|Tu_\varepsilon(t) - Tu_\varepsilon(t')|^p}{|t-t'|^p} dt'dt &\le C_p \int_{D_1} |u_\varepsilon|^p dx + C_p \int_{D_1} |Du_\varepsilon|^p dx \\ &\le C_p \frac{\pi}{2} m^p + C_p \int_{D_1} |Du_\varepsilon|^p dx. \end{aligned} \tag{5.5}$$

It follows that

$$K_\varepsilon(Tu_\varepsilon, E_1) \le \frac{C_p \pi m^p}{2} \varepsilon^{p-2} + (1 \wedge C_p) H_\varepsilon(u_\varepsilon) \le C, \tag{5.6}$$

where we used the equi-boundedness of u_ε. Hence, by (5.6), we have that the sequence (Tu_ε) is equi-bounded for K_ε and then we can use Theorem 3.1(i) to obtain the desired conclusion. □

5.2 Lower bound inequality

Now, we will prove an optimal lower bound for H_ε.

Proposition 5.2. *For every (u, v) in $BV(D_1, \{\alpha, \beta\}) \times BV(E_1, \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(D_1; [-m, m])$ such that $u_\varepsilon \to u$ in $L^1(D_1)$ and $Tu_\varepsilon \to v$ in $L^1(E_1)$*

$$\liminf_{\varepsilon \to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p \mathcal{H}^0(Sv). \tag{5.7}$$

Proof. We will prove the lower bound inequality (5.7) for v such that

$$v(t) = \begin{cases} \alpha', & \text{if } t \in (-1, 0], \\ \beta', & \text{if } t \in (0, 1). \end{cases}$$

Let us consider the natural extension of v to the whole real line $\mathbb{R}$ (still denoted by v); i.e.,

$$v(t) = \begin{cases} \alpha', & \text{if } t \leq 0, \\ \beta', & \text{if } t > 0. \end{cases}$$

Step 0: Strategy of the proof. We are looking for an extension of u_ε to the whole $\mathbb{R}^2_+$, namely w_ε, such that w_ε is a competitor for (5.4) and $H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) \simeq H_\varepsilon(u_\varepsilon, D_1, E_1)$ as $\varepsilon \to 0$ in a precise sense. More exactly, we will able to find $s < 1$ and we will construct a competitor w_ε such that, for any given $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that

$$\begin{aligned} H_\varepsilon(u_\varepsilon) &\geq H_\varepsilon(u_\varepsilon, D_s, E_s) \\ &= H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) - H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+ \setminus D_s, \mathbb{R} \setminus E_s) \\ &\geq \gamma_p - \delta, \quad \forall \varepsilon \leq \varepsilon_\delta. \end{aligned}$$

Step 1: Construction of the competitor. For every $s > 0$, we define the harmonic extension of v from $\mathbb{R} \setminus E_s$ to $\mathbb{R}^2_+ \setminus D_s$, namely $\bar{u}$, defined in polar coordinates by

$$\bar{u}(\rho, \theta) := \frac{\theta}{\pi} \alpha' + \left(1 - \frac{\theta}{\pi}\right) \beta', \quad \forall \theta \in [0, \pi), \ \forall \rho \geq s.$$

We will construct the competitor w_ε simply gluing the function $\bar{u}$ and the function u_ε. Hence, for every $\varepsilon > 0$, we consider the cut-off function φ in $C^\infty(\mathbb{R}^2_+)$, such that $\varphi \equiv 1$ in

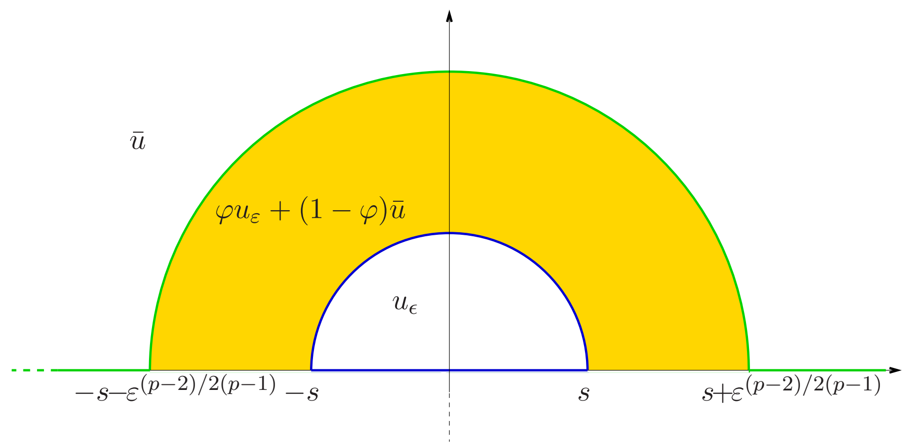

Figure 5.1: The competitor w_ε.

D_s, $\varphi \equiv 0$ in $\mathbb{R}^2_+ \setminus D_{s(\varepsilon)}$ and $|D\varphi| \leq 1/\varepsilon^{\frac{p-2}{2(p-1)}}$, where we denote by

$$s(\varepsilon) := s + \varepsilon^{\frac{(p-2)}{2(p-1)}}.$$

Thus the function w_ε can be defined as

$$w_\varepsilon := \begin{cases} u_\varepsilon & \text{in } D_s, \\ \varphi u_\varepsilon + (1-\varphi)\bar{u} & \text{in } D_{s(\varepsilon)} \setminus \ D_s, \\ \bar{u} & \text{in } \mathbb{R}^2_+ \setminus D_{s(\varepsilon)}. \end{cases}$$

Note that w_ε belongs to $W^{1,p}_{\text{loc}}(\mathbb{R}^2_+)$, $\lim\limits_{t\to-\infty} Tw_\varepsilon(t) = \alpha'$ and $\lim\limits_{t\to+\infty} Tw_\varepsilon(t) = \beta'$. Clearly, w_ε is a good competitor for (5.4).

Step 2: Choice of the annulus. We need to choose an annulus in the half-disk, in which we can recover a suitable quantity of energy of u_ε: there exists $L > 0$ such that for every $\varepsilon > 0$ there exists $s \in \left(\frac{1}{2}, 1 - \varepsilon^{\frac{p-2}{2(p-1)}}\right)$ such that

$$H_\varepsilon(u_\varepsilon, D_{s(\varepsilon)} \setminus D_s, E_{s(\varepsilon)} \setminus E_s) \leq L\varepsilon^{\frac{p-2}{2(p-1)}}. \tag{5.8}$$

Let us prove claim (5.8). By contradiction, for every $L > 0$ there exists ε_L such that for every $s \in \left(\frac{1}{2}, 1 - \varepsilon_L^{\frac{p-2}{2(p-1)}}\right)$

$$H_{\varepsilon_L}(u_{\varepsilon_L}, D_{s(\varepsilon_L)} \setminus D_s, E_{s(\varepsilon_L)} \setminus E_s) > L\varepsilon_L^{\frac{p-2}{2(p-1)}}. \tag{5.9}$$

In particular, we can choose a finite set (s_k), for $k = 1, 2, ..., \left[\varepsilon_L^{-\frac{p-2}{2(p-1)}}\right]$, such that

$$\bigcup_k (D_{s_k(\varepsilon_L)} \setminus D_{s_k}) = D_1 \setminus D_{1/2}, \tag{5.10}$$

with $D_{s_k(\varepsilon_L)} \setminus D_{s_k}$ disjoint sets.
By (5.9) and (5.10), we get

$$\begin{aligned} H_{\varepsilon_L}(u_{\varepsilon_L}, D_1, E_1) &\geq \sum_{k=1}^{\left[\varepsilon_L^{-\frac{p-2}{2(p-1)}}\right]} H_{\varepsilon_L}(u_{\varepsilon_L}, D_{s_k(\varepsilon_L)} \setminus D_{s_k}, E_{s_k(\varepsilon_L)} \setminus E_{s_k}) \\ &> \sum_{k=1}^{\left[\varepsilon_L^{-\frac{p-2}{2(p-1)}}\right]} L\varepsilon_L^{\frac{p-2}{2(p-1)}} = L\left[\varepsilon_L^{-\frac{p-2}{2(p-1)}}\right]\varepsilon_L^{\frac{p-2}{2(p-1)}} \geq L. \end{aligned} \tag{5.11}$$

Since (u_ε) is equi-bounded, by taking the limit of L to $+\infty$ in (5.11), we have a contradiction.

Step 3: Estimates. By the scaling property of H_ε (see (5.3)), we have

$$\begin{aligned} \gamma_p &\leq H_1(w_\varepsilon^{(\varepsilon)}, \mathbb{R}^2_+, \mathbb{R}) = H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+/\varepsilon, \mathbb{R}/\varepsilon) \\ &\leq H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) = H_\varepsilon(w_\varepsilon, D_s, E_s) + H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+ \setminus D_s, \mathbb{R} \setminus E_s) \\ &\leq H_\varepsilon(u_\varepsilon) + \varepsilon^{p-2}\int_{\mathbb{R}^2_+ \setminus D_s} |Dw_\varepsilon|^p dx + \frac{1}{\varepsilon}\int_{\mathbb{R} \setminus E_s} V(Tw_\varepsilon) d\mathcal{H}^1 \\ &= H_\varepsilon(u_\varepsilon) + I_1 + I_2. \end{aligned} \tag{5.12}$$

By definition of w_ε, the first integral in the right hand side of (5.12) can be easily estimated as follows

$$\begin{aligned} I_1 &= \varepsilon^{p-2}\int_{\mathbb{R}^2_+ \setminus D_{s(\varepsilon)}} |D\bar{u}|^p dx + \varepsilon^{p-2}\int_{D_{s(\varepsilon)} \setminus D_s} |D(\varphi u_\varepsilon + (1-\varphi)\bar{u})|^p dx \\ &\leq 3^{p-1}\varepsilon^{p-2}\int_{\mathbb{R}^2_+ \setminus D_s} |D\bar{u}|^p dx + 3^{p-1}\varepsilon^{p-2}\int_{D_{s(\varepsilon)} \setminus D_s} |Du_\varepsilon|^p dx \\ &\quad + 6^{p-1} m^p \pi \varepsilon^{\frac{p(p-2)}{2(p-1)}} + 3^{p-1}(2m)^p \pi s \varepsilon^{\frac{p-2}{2}}, \end{aligned}$$

where we used the fact that

$$\begin{aligned}\int_{D_{s(\varepsilon)}\setminus D_s} |D\varphi|^p |u_\varepsilon - \bar{u}|^p dx &\leq \frac{1}{\varepsilon^{\frac{p(p-2)}{2(p-1)}}} \int_{D_{s(\varepsilon)}\setminus D_s} |u_\varepsilon - \bar{u}|^p \\ &\leq 2^{p-1} m^p \pi \left(\frac{1}{\varepsilon^{\frac{(p-2)^2}{2(p-1)}}} + \frac{2s}{\varepsilon^{\frac{p-2}{2}}} \right)\end{aligned}$$

and that $|u_\varepsilon| < m$ and $|\bar{u}| < m$.

By definition of $\bar{u}$, we have

$$\int_{\mathbb{R}^2_+\setminus D_s} |D\bar{u}|^p dx = \frac{|\beta' - \alpha'|^p}{\pi^{p-1}(p-2)s^{p-2}}.$$

Hence,

$$\begin{aligned}I_1 &\leq 3^{p-1}\varepsilon^{p-2} \int_{D_{s(\varepsilon)}\setminus D_s} |Du_\varepsilon|^p dx + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-2}s^{p-2}} \varepsilon^{p-2} \\ &\quad + 6^{p-1} m^p \pi \varepsilon^{\frac{p(p-2)}{2(p-1)}} + 3^{p-1}(2m)^p \pi s \varepsilon^{\frac{p-2}{2}}. \qquad (5.13)\end{aligned}$$

Let us estimate the second integral in the right hand side of (5.12). Since $Tw_\varepsilon = \alpha'$ and $Tw_\varepsilon = \beta'$ on $\mathbb{R} \setminus E_{s(\varepsilon)}$, we have

$$I_2 = \frac{1}{\varepsilon} \int_{\mathbb{R}\setminus E_{s(\varepsilon)}} V(T\bar{u}) d\mathcal{H}^1 + \frac{1}{\varepsilon} \int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon) d\mathcal{H}^1 \equiv \frac{1}{\varepsilon} \int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon) d\mathcal{H}^1.$$

For every $\delta > 0$, let us define

$$E_\delta := \left\{ x \in E_{s(\varepsilon)} \setminus E_s : |Tu_\varepsilon - \beta'| > \delta \text{ and } |Tu_\varepsilon - \alpha'| > \delta \right\}.$$

Thanks to Step 2, we get: there exists $N > \dfrac{L}{\omega_\delta}$ (where we denote by $\omega_\delta := \min\limits_{\substack{|t-\alpha'|\geq\delta \\ |t-\beta'|\geq\delta}} V(t)$) such that for every $\delta > 0$ there exists ε_δ such that

$$|E_\delta| \leq N \varepsilon^{\frac{p-2}{2(p-1)}} \varepsilon, \ \forall \varepsilon \leq \varepsilon_\delta, \qquad (5.14)$$

In particular, choosing δ small, the convexity of V near its wells provides

$$\begin{aligned}I_2 &= \frac{1}{\varepsilon} \int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tw_\varepsilon) d\mathcal{H}^1 + \frac{1}{\varepsilon} \int_{E_\delta} V(Tw_\varepsilon) d\mathcal{H}^1 \\ &\leq \frac{1}{\varepsilon} \int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tu_\varepsilon) d\mathcal{H}^1 + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}},\end{aligned} \qquad (5.15)$$

where $\omega_m := \max_{|t|<m} V(t)$ and we used the inequality (5.14).

Finally, by (5.12), (5.13) and (5.15), we obtain, for every $\delta > 0$

$$\begin{aligned} H_\varepsilon(u_\varepsilon) \;&\geq\; \gamma_p - \Bigg(3^{p-1} H_\varepsilon(w_\varepsilon, D_{s(\varepsilon)} \setminus D_s, E_{s(\varepsilon)} \setminus E_s) + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1} s^{p-2}} \varepsilon^{p-2} \\ &\qquad 6^{p-1} m^p \pi \varepsilon^{\frac{p(p-2)}{2(p-1)}} + 3^{p-1}(2m)^p \pi s \varepsilon^{\frac{p-2}{2}} + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}} \Bigg), \\ &\geq\; \gamma_p - \Bigg(3^{p-1} L\varepsilon + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1} s^{p-2}} \varepsilon^{p-2} \\ &\qquad 6^{p-1} m^p \pi \varepsilon^{\frac{p(p-2)}{2(p-1)}} + 3^{p-1}(2m)^p \pi s \varepsilon^{\frac{p-2}{2}} + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}} \Bigg), \quad \forall \varepsilon \leq \varepsilon_\delta. \end{aligned} \tag{5.16}$$

Notice that for every $\varepsilon > 0$, $s \in \left(1/2, 1 - \varepsilon^{\frac{p-2}{2(p-1)}}\right)$. Hence, taking the limit as $\varepsilon \to 0$, we get $\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p$, which concludes the proof. □

5.3 Reduction to the flat case

We prove compactness and a lower bound inequality for the following energies

$$F_\varepsilon(u_\varepsilon, D, E) = \varepsilon^{p-2} \int_D |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_D W(u) dx + \frac{1}{\varepsilon} \int_E V(Tu) d\mathcal{H}^2,$$

where $D \subset \mathbb{R}^3$ is the open half-ball centered in 0 with radius $r > 0$ and $E \subset \mathbb{R}^2$ is its "diameter"; that is,

$$E := \left\{(x_1, x_2, x_3) \in \mathbb{R}^2 : |x| \leq r, x_3 = 0\right\}.$$

We will reduce to Proposition 5.1 and Proposition 5.2 via a suitable slicing argument. In the following we use the notation introduced in Section 1.3: e is an unit vector in the plane $P := \{x_3 = 0\}$; M is the orthogonal complement of e in P; π is the projection of $\mathbb{R}^3$ onto M; for every $y \in E_e := \pi(E)$, we denote by $E^y := \pi^{-1}(y) \cap E$, $D^y := \pi^{-1}(y) \cap D$ (see Fig. 5.2).

Proposition 5.3. *Let $(u_\varepsilon) \subset W^{1,p}(D; [-m, m])$ be a sequence with uniformly bounded energies $F_\varepsilon(u_\varepsilon, D, E)$. Then the traces Tu_ε are pre-compact in $L^1(E)$ and every cluster*

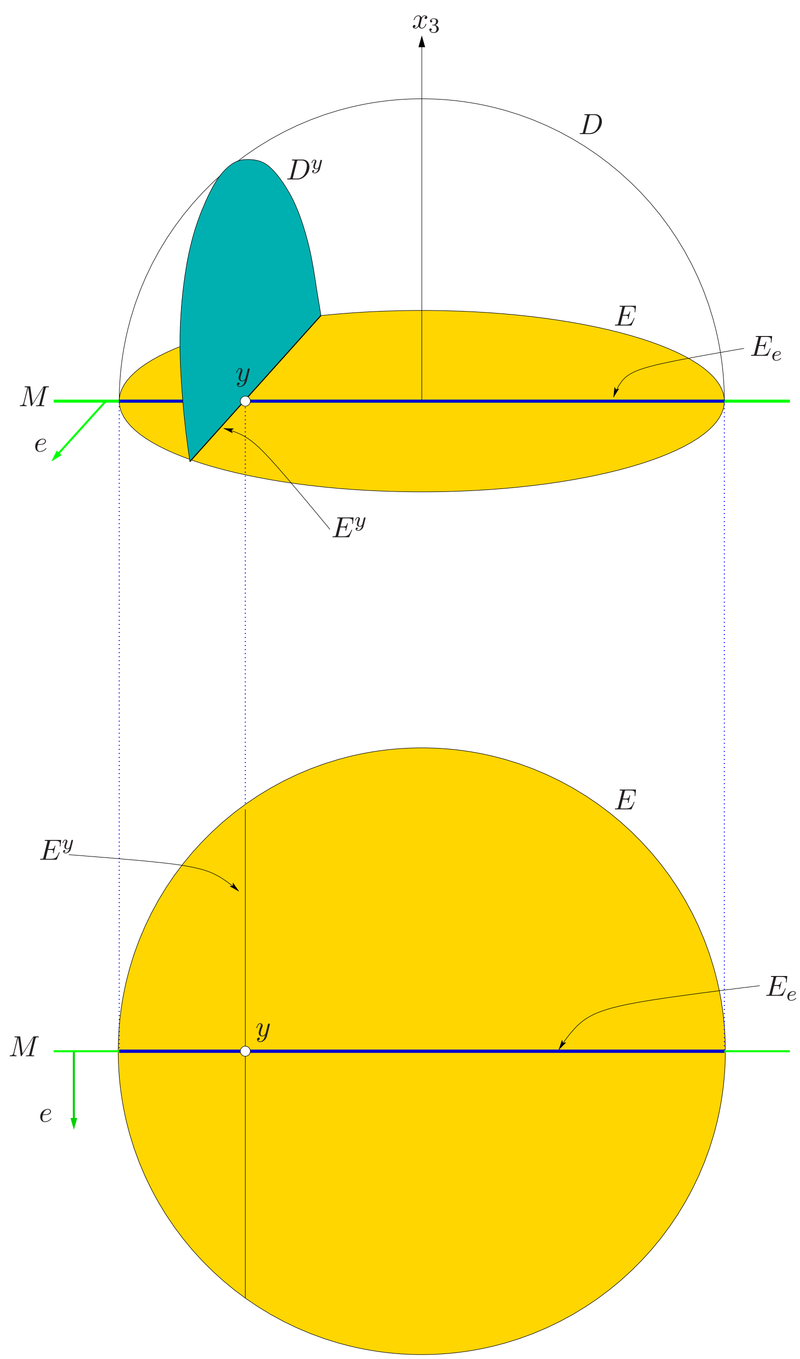

Figure 5.2: The sets D, E, E_e, E^y and D^y.

point belongs to $BV(E,\{\alpha',\beta'\})$*. Moreover, if* $Tu_\varepsilon \to v$ *in* $L^1(E)$*, then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \geq \gamma_p \left| \int_{E\cap Sv} \nu_v \right| d\mathcal{H}^1. \tag{5.17}$$

Proof. By Fubini's Theorem, for every $\varepsilon > 0$, we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, D, E) &\geq \varepsilon^{p-2} \int_D |Du_\varepsilon|^p dx + \frac{1}{\varepsilon} \int_E V(Tu_\varepsilon) d\mathcal{H}^2 \\ &\geq \int_{E_e} \left[\varepsilon^{p-2} \int_{D^y} |Du_\varepsilon^y|^p dx + \frac{1}{\varepsilon} \int_{E^y} V(Tu_\varepsilon^y) d\mathcal{H}^1 \right] dy \\ &= \int_{E_e} H_\varepsilon(u_\varepsilon^y, D^y, E^y) dy \end{aligned} \tag{5.18}$$

We first prove that (Tu_ε) is pre-compact in $L^1(E)$. In view of Theorem 1.9, it suffices to show that the family $\mathcal{F} := (Tu_\varepsilon)$ satisfies the following property: for every $\delta > 0$ there exists a family $\mathcal{F}_\delta$ δ-dense in $\mathcal{F}$ such that $(\mathcal{F}_\delta)_e^y$ is pre-compact in $L^1(E)$ for $\mathcal{H}^2$-a.e. $y \in E_e$. By assumption $F_\varepsilon(u_\varepsilon, D, E) \leq C$, by (5.18) we have

$$\int_{E_e} H_\varepsilon(u_\varepsilon^y, D^y, E^y) dy \leq C. \tag{5.19}$$

Fix $\delta > 0$ and, for every $\varepsilon > 0$, define $v_\varepsilon : E \to [-m, m]$ such that

$$v_\varepsilon^y := \begin{cases} Tu_\varepsilon^y & \text{if } y \in E_e \text{ and } H_\varepsilon(u_\varepsilon^y, D^y, E^y) \leq 2mrC/\delta, \\ \alpha' & \text{otherwise.} \end{cases} \tag{5.20}$$

By (5.19), we have $v_\varepsilon^y = Tu_\varepsilon^y$ for every $y \in E_e$ apart from a subset of measure smaller than $\delta/2mr$. Hence, $v_\varepsilon = Tu_\varepsilon$ in E up to a set of measure smaller than δ/m. So, from $|Tu_\varepsilon| \leq m$, we deduce

$$\|v_\varepsilon - Tu_\varepsilon\|_{L^1(E)} \leq \delta.$$

Therefore, the family $\mathcal{F}_\delta := (v_\varepsilon)$ is $\delta-$dense in $\mathcal{F}$.

By (5.20), $H_\varepsilon(v_\varepsilon^y, D^y, E^y) \leq 2mrC/\delta$ for every $y \in E_e$ and this implies that the sequence (v_ε^y) is pre-compact in $L^1(E^y)$. By Theorem 1.9(i), the sequence (Tu_ε) is pre-compact in $L^1(E)$; i.e., there exists a function $v \in L^1(E)$ such that

$$Tu_\varepsilon \to v \text{ in } L^1(E).$$

Let us show that v belongs to $BV(E, \{\alpha', \beta'\})$. We have that (up to a subsequence)

$$Tu_\varepsilon^y \to v^y \text{ in } L^1(E) \text{ for a.e. } y \in E_e.$$

Then, by Proposition 5.1, we have that $v^y \in BV(E^y, \{\alpha', \beta'\})$. Hence, by slicing property for BV functions, it follows that $v \in BV(E, \{\alpha', \beta'\})$ (see [32], Section 5.10, for details).

It remains to prove that inequality (5.17) holds. Taking the limit as $\varepsilon \to 0$ in (5.18), by Fatou's Lemma, we deduce that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \geq \int_{E_e} \liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon^y, D^y, E^y) dy.$$

Then, using the Proposition 5.2, we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \geq \int_{E_e} \gamma_p \mathcal{H}^0(Sv^y) dy. \tag{5.21}$$

The right-hand side of (5.21) is finite and Sv^y agrees with $Sv \cap E^y$ for a.e. $y \in E_e$. By (1.5)[1] we may rewrite (5.21) as

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \geq \gamma_p \int_{E\cap Sv} \langle \nu_v, e\rangle d\mathcal{H}^1. \tag{5.22}$$

Finally, (5.17) follows from (5.22) by choosing a suitable unit vector e. □

5.4 Existence of an optimal profile problem

We conclude this chapter with the proof of the existence of a minimum for the optimal profile problem (5.4). We will use a rearrangement result in one direction to show that the minimum for γ_p is achieved by a function with non-decreasing trace.

Proposition 5.4. *The minimum for γ_p defined by (5.4) is achieved by a function u such that Tu is a non-decreasing function in $\mathbb{R}$.*

Proof. Note that, since the energy H_1 is decreasing under truncation by α' and β', it is not restrictive to minimize the problem (5.4) with the additional condition $\alpha' \leq u \leq \beta'$.

[1] See Proposition 1.7, p. 26.

We denote by X the class of all $w : \mathbb{R} \to [\alpha', \beta']$ such that $w \in L^1_{loc}(\mathbb{R}^2_+)$, $\int_{\mathbb{R}^2_+} |Dw|^p dx$ is finite, $\lim_{t\to-\infty} Tw(t) = \alpha'$, $\lim_{t\to+\infty} Tw(t) = \beta'$; we denote by X^* the class of all $w \in X$ such that Tw is non-decreasing, $Tw(t) \geq \frac{\alpha'+\beta'}{2}$ for $t > 0$ and $Tw(t) \leq \frac{\alpha'+\beta'}{2}$ for $t < 0$.

Step 1: The infimum of H_1 on X is equal to the infimum of H_1 on X^*.

Since $X^* \subset X$ we have

$$\inf_{w\in X^*} H_1(w, \mathbb{R}^2_+, \mathbb{R}) \geq \inf_{w\in X} H_1(w, \mathbb{R}^2_+, \mathbb{R}). \tag{5.23}$$

Fix u in X, we claim that for every $\delta > 0$ there exists a function u_δ in X^* such that

$$H_1(u_\delta, \mathbb{R}^2_+, \mathbb{R}) \leq H_1(u, \mathbb{R}^2_+, \mathbb{R}) + o(\delta). \tag{5.24}$$

Once we have (5.24), for every $\delta > 0$, we get

$$\inf_{w\in X^*} H_1(w, \mathbb{R}^2_+, \mathbb{R}) \leq H_1(u, \mathbb{R}^2_+, \mathbb{R}) + o(\delta), \ \forall u \in X.$$

Taking the limit for $\delta \to 0$ and then the infimum on $u \in X$, we obtain

$$\inf_{w\in X^*} H_1(w, \mathbb{R}^2_+, \mathbb{R}) \leq \inf_{u\in X} H_1(u, \mathbb{R}^2_+, \mathbb{R})$$

and this together with (5.23) conclude the proof of the step. It remain to prove (5.24).

For every $S > 0$, we denote by

$$Q_S := [-S, S] \times [0, S]$$

and by $u^\star$ the monotone increasing rearrangement in x_1 of u in Q_S (see Section 1.4.2, p. 31).

For every $R > 0$, we define the harmonic extension of the function

$$\alpha' \chi_{(-\infty,0)}(t) + \beta'(1 - \chi_{[0,+\infty)(t)})$$

from $\mathbb{R} \setminus E_R$ to $\mathbb{R}^2_+ \setminus D_R$; i.e., the function $\bar{u}$ that expressed in polar coordinates is given by

$$\bar{u}(\rho, \theta) := \frac{\theta}{\pi}\alpha' + \left(1 - \frac{\theta}{\pi}\right)\beta', \quad \forall \theta \in [0, \pi], \ \forall \rho \geq R.$$

We will construct a function $\tilde{u} \in X^*$ gluing the function $\bar{u}$ and the function $u^\star$. Hence, for every $S \geq R > 0$, we consider the cut-off function φ such that

$$\varphi = 0 \text{ in } D_R, \quad \varphi = 1 \text{ in } \mathbb{R}^2_+ \setminus D_S \text{ and } |D\varphi| \leq 1/(S - R) \text{ in } D_S \setminus D_R.$$

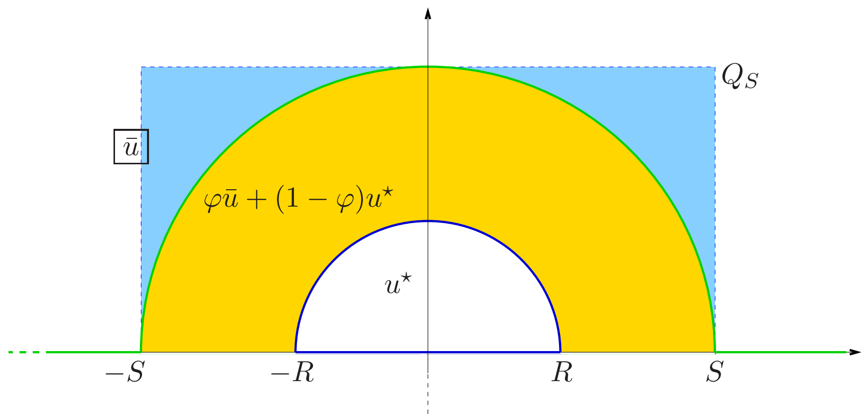

Figure 5.3: The competitor $\tilde{u}$.

The function $\tilde{u}$ can be defined as

$$\tilde{u} := \begin{cases} u^\star & \text{in } D_R, \\ \varphi\bar{u} + (1-\varphi)u^\star & \text{in } D_S \setminus\ D_R, \\ \bar{u} & \text{in } \mathbb{R}^2_+ \setminus D_S. \end{cases}$$

Note that $\tilde{u}$ belongs to $W^{1,p}_{\text{loc}}(\mathbb{R}^2_+)$, $\lim_{t\to-\infty} T\tilde{u}(t) = \alpha'$, $\lim_{t\to+\infty} T\tilde{u}(t) = \beta'$ and $T\tilde{u}$ is non-decreasing in $\mathbb{R}$. Let us compute its energy.

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx = \int_{D_R} |Du^\star|^p dx + \int_{D_S\setminus D_R} |D(\varphi\bar{u} + (1-\varphi)u^\star)|^p dx + \int_{\mathbb{R}^2_+\setminus D_S} |D\bar{u}|^p dx \tag{5.25}$$

We estimate the integral in the set $D_S \setminus D_R$, using the fact that for every $\delta \in (0,1)$ there exists $a(\delta) \to 0$ as $\delta \to 0$, such that

$$(A + B + C)^p \leq (1+\delta)A^p + a(\delta)B^p + a(\delta)C^p,$$

for every non negative A, B, C.

Hence, for every $\delta \in (0,1)$, we have

$$\int_{D_S\setminus D_R} |D(\varphi\bar{u} + (1-\varphi)u^\star)|^p dx \leq (1+\delta)\int_{D_S\setminus D_R} |Du^\star|^p dx + a(\delta)\int_{D_S\setminus D_R} |D\bar{u}|^p dx + a(\delta)\frac{|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}, \tag{5.26}$$

where we used that $|D\varphi| \leq 1/(S-R)$ and $\bar{u}, u^\star \in [\alpha', \beta']$. By (5.25) and (5.26), we obtain

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx \leq (1+\delta)\int_{D_S} |Du^\star|^p dx + a(\delta)\int_{\mathbb{R}^2_+\setminus D_R} |D\bar{u}|^p dx + \frac{a(\delta)|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}. \tag{5.27}$$

We can estimates the first term in the right hand side of (5.27) using the fact that monotone increasing rearrangement in one direction decreases the L^p-norm of the gradient (see Theorem 1.16, p. 31).

$$\int_{D_S} |Du^\star|^p dx \leq \int_{Q_S} |Du^\star|^p dx \leq \int_{Q_S} |Du|^p dx. \tag{5.28}$$

The second term in the right hand side of (5.27) can be explicitly computed

$$\int_{\mathbb{R}^2_+\setminus D_R} |D\bar{u}|^p dx = \frac{\pi^{p-1}|\beta' - \alpha'|^p}{(p-2)R^{p-2}}. \tag{5.29}$$

Finally, putting together (5.27), (5.28) and (5.29), we obtain that, for every $S \geq R > 0$ and every $\delta \in (0,1)$, the following estimate holds

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx \leq (1+\delta)\int_{\mathbb{R}^2_+} |Du|^p dx + \frac{a(\delta)\pi^{p-1}|\beta' - \alpha'|^p}{(p-2)R^{p-2}} + \frac{a(\delta)|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}, \tag{5.30}$$

with $a(\delta) \to +\infty$ as $\delta \to 0$. In particular, $\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx$ is finite.

Now, we evaluate the second term of $H_1(\tilde{u}, \mathbb{R}^2_+, \mathbb{R})$.

$$\int_{\mathbb{R}} V(T\tilde{u})d\mathcal{H}^1 = \int_{E_S\setminus E_R} V\big(\varphi\bar{u} + (1-\varphi)Tu^\star\big)d\mathcal{H}^1 + \int_{E_R} V(Tu^\star)d\mathcal{H}^1, \tag{5.31}$$

where we used that, by definition, $\bar{u}(t) = \alpha'$ if $t < -S$ and $\bar{u}(t) = \beta'$ if $t > S$.

Since $Tu(t)$ tends to α' and β' as $t \to -\infty$ and $+\infty$ respectively and V is convex near the wells, there exists $R_0 > 0$ such that $Tu(t)$ lies in the convex wells of V for all $t \in \mathbb{R} \setminus E_R$, for every $R > R_0$, and the same does $Tu^\star(t)$. This implies that for any $S > R > R_0$ we have

$$\begin{aligned}\int_{E_S\setminus E_R} V\big(\varphi\bar{u} + (1-\varphi)Tu^\star\big)d\mathcal{H}^1 &\leq \int_{E_S\setminus E_R} \varphi V(\bar{u})d\mathcal{H}^1 + \int_{E_S\setminus E_R} (1-\varphi)V(Tu^\star)d\mathcal{H}^1 \\ &\leq \int_{E_S\setminus E_R} V(Tu^\star)d\mathcal{H}^1. \end{aligned} \tag{5.32}$$

By (5.31) and (5.32), we have

$$\int_{\mathbb{R}} V(T\tilde{u})d\mathcal{H}^1 \leq \int_{E_S} V(Tu^\star)d\mathcal{H}^1 = \int_{E_S} V(Tu)d\mathcal{H}^1 \leq \int_{\mathbb{R}} V(Tu)d\mathcal{H}^1. \tag{5.33}$$

Finally, by (5.30) and (5.33), we have the following estimate, for every $S \geq R > R_0 > 0$ and every $\delta \in (0,1)$

$$H_1(\tilde{u}, \mathbb{R}^2_+, \mathbb{R}) \leq (1+\delta)H_1(u) + \frac{a(\delta)\pi^{p-1}|\beta'-\alpha'|^p}{(p-2)R^{p-2}} + \frac{a(\delta)|\beta'-\alpha'|^p\pi^2(S^2-R^2)}{(S-R)^p}. \tag{5.34}$$

This proves the claim, taking R and $S - R$ large enough.

Step 2: The infimum of H_1 on X^* is achieved.
We use the Direct Method. Take a minimizing sequence $(u_n) \subset X^*$. In particular,

$H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) \leq C$, Du_n converges weakly to Du in $L^p(R^2_+)$ and u_n converges to u weakly in $W^{1,p}_{\text{loc}}(\mathbb{R}^2_+)$. Since $\int_{\mathbb{R}^2_+} |Du_n|^p dx$ is bounded, we can find a function $u \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ and $\int_{\mathbb{R}^2_+} |Du|^p dx$ is finite, such that (up to a subsequence)

$$Du_n \rightharpoonup Du \text{ in } L^p(\mathbb{R}^2_+) \text{ and } u_n \rightharpoonup u \text{ in } L^p_{\text{loc}}(\mathbb{R}^2_+).$$

By the trace imbedding of $W^{1-1/p,p}$ in $W^{1,p}$, we have

$$Tu_n \rightharpoonup Tu \text{ in } W^{1-1/p,p}_{\text{loc}}(\mathbb{R}).$$

By the compact embedding of $W^{1-1/p,p}_{\text{loc}}(\mathbb{R})$ in $C^0_{\text{loc}}(\mathbb{R})$ (see [1], Theorem 7.34, p. 231), we have that, up to a subsequence, Tu_n uniformly converges to Tu. Thus Tu is non-decreasing and satisfies

$$Tu(t) \geq \frac{\alpha' + \beta'}{2} \text{ for } t > 0 \quad \text{and} \quad Tu(t) \leq \frac{\alpha' + \beta'}{2} \text{ for } t < 0.$$

Let us show that $\lim_{t \to -\infty} Tu(t) = \alpha'$ and $\lim_{t \to +\infty} Tu(t) = \beta'$. Since Tu is non-decreasing in $[\alpha', \beta']$, there exist $a \leq \frac{\alpha' + \beta'}{2}$ and $b \geq \frac{\alpha' + \beta'}{2}$ such that

$$a := \lim_{t \to -\infty} Tu(t) \text{ and } b := \lim_{t \to +\infty} Tu(t).$$

By contradiction, we assume that either $a \neq \alpha'$ or $b \neq \beta'$. Then, since V is continuous and strictly positive in (α', β'), we obtain

$$\int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 = +\infty,$$

This is impossible, because, by Fatou's Lemma, we have

$$\int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} V(Tu_n) d\mathcal{H}^1 < \liminf_{n \to +\infty} H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) < +\infty.$$

Hence, u is in X^*. Since H_1 is clearly lower semicontinuous on sequences such that $Du_n \rightharpoonup Du$ in L^p and $Tu_n \to Tu$ pointwise, this concludes the proof. □

Chapter 6

Proof of the main result

In this chapter, we will prove the main result of this thesis, namely the compactness, the lower bound inequality and the upper bound inequality stated in Theorem 4.1.

For the sake of simplicity, we recall the definition of the functionals which we deal with:

$$F_\varepsilon(u) = \varepsilon^{p-2} \int_\Omega |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u)dx + \frac{1}{\varepsilon} \int_{\partial\Omega} V(Tu)d\mathcal{H}^2,\ \forall u \in W^{1,p}(\Omega), \quad (6.1)$$

and

$$\Phi(u,v) = \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 + \gamma_p \mathcal{H}^1(Sv),$$
$$\forall (u,v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\}). \quad (6.2)$$

Theorem 6.1. *Let $F_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ and $\Phi : BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\}) \to \mathbb{R}$ defined by (6.1) and (6.2).*

Then

(i) [Compactness] *If $(u_\varepsilon) \subset W^{1,p}(\Omega)$ is a sequence such that $F_\varepsilon(u_\varepsilon)$ is bounded, then $(u_\varepsilon, Tu_\varepsilon)$ is pre-compact in $L^1(\Omega) \times L^1(\partial\Omega)$ and every cluster point belongs to $BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$.*

(ii) [Lower Bound Inequality] *For every $(u,v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$,*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v).$$

(iii) [Upper Bound Inequality] *For every* $(u,v) \in BV(\Omega,\{\alpha,\beta\}) \times BV(\partial\Omega,\{\alpha',\beta'\})$ *there exists a sequence* $(u_\varepsilon) \subset W^{1,p}(\Omega)$ *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$, $Tu_\varepsilon \to v$ *in* $L^1(\partial\Omega)$ *and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le \Phi(u,v).$$

6.1 Compactness

Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ be given such that $F_\varepsilon(u_\varepsilon)$ is bounded. Since $F_\varepsilon(u_\varepsilon) \ge F_\varepsilon(u_\varepsilon,\Omega,\emptyset) \equiv G_\varepsilon(u_\varepsilon,\Omega)$, by the statement (i) of Theorem 2.2, the sequence (u_ε) is pre-compact in $L^1(\Omega)$ and there exists $u \in BV(\Omega,\{\alpha,\beta\})$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$.

It remains to prove that (Tu_ε) is pre-compact in $L^1(\partial\Omega)$ and that its cluster points are in $BV(\partial\Omega,\{\alpha',\beta'\})$. Thanks to Proposition 1.12 we can cover $\partial\Omega$ with finitely many balls $(B_i)_{i\in I}$ centered on $\partial\Omega$, of radius r_i such that for every $i \in I$ there exists a bi-Lipschitz map Ψ_i, with isometry defect $\delta(\Psi_i) < 1$, which satisfies $\Psi_i(D_{r_i} \cap B_i) = \Omega \cap B_i$ and $\Psi_i(E_{r_i} \cap B_i) = \partial\Omega \cap B_{r_i}$ (see Section 1.3.2, p. 29). We show that (Tu_ε) is pre-compact in $L^1(\partial\Omega \cap B_i)$ for every $i \in I$.

For every fixed i, let us set

$$u_\varepsilon^i := u_\varepsilon \circ \Psi_i.$$

Since the isometry defect of Ψ_i is smaller than 1, Proposition 1.11 implies

$$F_\varepsilon(u_\varepsilon, \Omega\cap B_i, \partial\Omega \cap B_i) \ge (1-\delta(\Psi_i))^{p+3} F_\varepsilon(u_\varepsilon^i, D_{r_i}\cap B_i, E_{r_i}\cap B_i),$$

so $F_\varepsilon(u_\varepsilon^i, D_{r_i}\cap B_i, E_{r_i}\cap B_i)$ is bounded. Hence, the compactness of the traces Tu_ε^i in $L^1(E_{r_i})$ follows from Proposition 5.3. Finally, using the invertibility of Ψ_i, we have that (Tu_ε) is pre-compact in $L^1(\partial\Omega)$ and that its cluster points are in $BV(\partial\Omega,\{\alpha',\beta'\})$. □

6.2 Lower bound inequality

The proof of the lower bound inequality of Theorem 6.1 follows the lines of the proof of Theorem 2.6(ii) by Alberti, Bouchitté and Seppecher in [9], but for the estimate of the boundary effect we will use the optimal profile problem (5.4) in connection with the results proved in the previous chapter.

Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ be given such that $u_\varepsilon \to u \in BV(\Omega, \{\alpha, \beta\})$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v \in BV(\partial\Omega, \{\alpha'\beta'\})$ in $L^1(\partial\Omega)$. We have to prove that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v), \tag{6.3}$$

where Φ is given by (6.2).
Clearly, we can assume that $\liminf\limits_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$.

For every $\varepsilon > 0$, let μ_ε be the energy distribution associated to F_ε with configuration u_ε; i.e., μ_ε is the positive measure given by

$$\mu_\varepsilon(B) := \varepsilon^{p-2} \int_{\Omega\cap B} |Du_\varepsilon|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_{\Omega\cap B} W(u_\varepsilon) dx + \frac{1}{\varepsilon} \int_{\partial\Omega\cap B} V(Tu_\varepsilon) d\mathcal{H}^2, \tag{6.4}$$

for every $B \subset \mathbb{R}^3$ Borel set.

Similarly, let us define

$$\begin{aligned} \mu^1(B) &:= \sigma_p \mathcal{H}^2(Su \cap B), \\ \mu^2(B) &:= c_p \int_{\partial\Omega\cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2, \\ \mu^3(B) &:= \gamma_p \mathcal{H}^1(Sv \cap B). \end{aligned}$$

The total variation $\|\mu_\varepsilon\|$ of the measure μ_ε is equal to $F_\varepsilon(u_\varepsilon)$, and $\|\mu^1\| + \|\mu^3\| + \|\mu^3\|$ is equal to $\Phi(u, v)$. $\|\mu_\varepsilon\|$ is bounded and we can assume that μ_ε converges in the sense of measure to some finite measure μ in $\mathbb{R}^3$. Then, by the lower semicontinuity of the total variation, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \equiv \liminf_{\varepsilon\to 0} \|\mu_\varepsilon\| \geq \|\mu\|.$$

Since the measures μ^i are mutually singular, we obtain the lower bound inequality (6.3) if we prove that

$$\mu \geq \mu^i, \quad \text{for } i = 1, 2, 3. \tag{6.5}$$

We prove that $\mu \geq \mu^i$ by showing that $\mu(B) \geq \mu^i(B)$ for all sets $B \subset \mathbb{R}^3$ such that $B \cap \Omega$ is a Lipschitz domain and $\mu(\partial B) = 0$. This class is large enough to imply the inequality (6.5) for all Borel sets B.

We have

$$\mu(B) = \lim_{\varepsilon\to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega \cap B, \emptyset) \geq \sigma_p \mathcal{H}^2(Su \cap B) \equiv \mu^1(B),$$

where the last inequality follows from statement (ii) of Theorem 2.2.

Similarly, we can prove that $\mu \geq \mu^2$. We have

$$\mu(B) = \lim_{\varepsilon\to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega\cap B, \emptyset) \geq c_p \int_{\partial\Omega\cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 \equiv \mu^2(B),$$

where we used Proposition 4.3(i) with $A := B\cap\Omega$ and $A' := B\cap\partial\Omega$.

The inequality $\mu \geq \mu^3$ requires a different argument. Notice that μ^3 is the restriction of $\mathcal{H}^1$ to the set Sv, multiplied by the factor γ_p. Thus, if we prove that

$$\liminf_{r\to 0} \frac{\mu(B_r(x))}{2r} \geq \gamma_p, \quad \mathcal{H}^1\text{-a.e. } x\in Sv, \tag{6.6}$$

we obtain the required inequality. In fact, the left-hand side of (6.6) represents the "one-dimensional density of the measure μ at x", and it agrees with the Radon-Nykodim derivative of the measure μ with respect to $\mathcal{H}^1 \llcorner Sv$ for $\mathcal{H}^1$-a.e. $x \in Sv$.

Let us fix $x\in Sv$ such that there exists $\lim\limits_{r\to 0}\dfrac{\mu(B_r(x))}{2r}$ and Sv has one-dimensional density equal to 1. We denote by ν_v the unit normal at x.

For r small enough, we choose a map Ψ_r such as in Proposition 1.12. Thus we have $\Psi_r(\overline{D_r}) = \Omega\cap B_r(x)$, $\Psi_r(E_r) = \partial\Omega\cap B_r(x)$ and $\delta(\Psi_r)\to 0$ as $r\to 0$.

Let us set

$$\bar{u}_\varepsilon := u_\varepsilon\circ\Psi_r \text{ and } \bar{v} := v\circ\Psi_r.$$

Hence, $T\bar{u}_\varepsilon \to \bar{v}$ in $L^1(E_r)$ and $\bar{v}\in BV(E_r, \{\alpha', \beta'\})$. So, thanks to Proposition 1.11, we obtain

$$\begin{aligned} \mu(B_r(x)) &= \lim_{\varepsilon\to 0}\mu_\varepsilon(B_r(x)) \\ &= \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega\cap B_r(x), \partial\Omega\cap B_r(x)) \\ &\geq \liminf_{\varepsilon\to 0}(1-\delta(\Psi_r))^{p+3} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r). \end{aligned} \tag{6.7}$$

Moreover, by Proposition 5.3, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r) \geq \gamma_p \left|\int_{S\bar{v}\cap E_r} \nu_v d\mathcal{H}^1\right| \tag{6.8}$$

Finally, we notice that $\delta(\Psi_r)$ vanishes and $\left|\int_{S\bar{v}\cap E_r}\nu_v d\mathcal{H}^1\right| = 2r + o(r)$ as r goes to 0. So (6.7) and (6.8) give the following inequality

$$\frac{\mu(B_r(x))}{2r} \geq \gamma_p\left(1 + \frac{o(r)}{2r}\right) \text{ as } r\to 0,$$

that implies $\mu \geq \mu^3$. This concludes the proof of the lower bound inequality. $\square$

6.3 Upper bound inequality

We will construct an optimal sequence u_ε according to Theorem 6.1(iii) in a suitable partition of Ω. To this aim, and in order to use the preliminary convergence results stated in the previous chapters, we need the following lemma

Lemma 6.2. *Let A be a domain in $\mathbb{R}^3, A' \subset \partial A, v : A' \to [-m, m]$ a Lipschitz function (where m is given by (4.20)) and G_ε defined by (4.4).*
Then, for every $\varepsilon > 0$, there exists an extension $u : \overline{A} \to [-m, m]$ such that

$$\mathrm{Lip}(u) \leq \varepsilon^{-\frac{p-2}{p-1}} + \mathrm{Lip}(v)$$

and

$$G_\varepsilon(u, A) \leq \left((\varepsilon^{\frac{p-2}{p-1}}\mathrm{Lip}(v) + 1)^p + C_m\right)\left(\mathcal{H}^2(\partial A) + o(1)\right)\omega, \quad \text{as } \varepsilon \to 0, \tag{6.9}$$

where $C_m := \max\limits_{t\in[-m,m]} W(t)$, $\omega := \|v - \alpha\|_\infty \wedge \|v - \beta\|_\infty$.

Proof. It is not restrictive to assume that $A' = \partial A$; in fact, we can extend v to ∂A without increasing its Lipschitz constant (defining $v(x) := \inf\limits_{x'\in A'} v(x') + \mathrm{Lip}(v)|x - y|$ for every x in ∂A). We additionally suppose that $\omega = \|v - \alpha\|_\infty$ (the case $\omega = \|v - \beta\|_\infty$ being similar).

Let us set

$$u(x) := \begin{cases} v(x) & \text{on } \partial A, \\ \alpha & \text{on } A \setminus A_{\omega\varepsilon^{(p-2)/(p-1)}}, \end{cases}$$

where A_t is the set of all x in A such that $0 < \mathrm{dist}(x, \partial A) < t$.
Then, u is $\left(\varepsilon^{-\frac{p-2}{p-1}} + \mathrm{Lip}(v)\right)$-Lipschitz continuous on $\overline{A} \setminus A_{\omega\varepsilon^{(p-2)/(p-1)}}$. Finally u can be extended to $\overline{A}$, without increasing its Lipschitz constant.
We have

$$\begin{aligned} G_\varepsilon(u, A) &= \varepsilon^{p-2}\int_{A_{\omega\varepsilon^{(p-2)/(p-1)}}} |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-1}{p-2}}}\int_{A_{\omega\varepsilon^{(p-2)/(p-1)}}} W(u)dx \\ &\leq |A_{\omega\varepsilon^{(p-2)/(p-1)}}|\left(\varepsilon^{p-2}(\mathrm{Lip}(v) + \frac{1}{\varepsilon^{(p-1)/(p-2)}})^p + \frac{1}{\varepsilon^{(p-1)/(p-2)}}C_m\right) \\ &= \left((\mathcal{H}^2(\partial A) + o(1))(\varepsilon^{\frac{p-1}{p-2}}\mathrm{Lip}(v) + 1)^p + C_m\right)\omega, \quad \text{as } \varepsilon \to 0, \end{aligned}$$

where we used that $|A_t| = (\mathcal{H}^2(\partial A) + o(1))t$ as $t \to 0$. □

Proof of the upper bound inequality. We assume that u and v (up to modifications on negligible sets) are constant in each connected component of $\Omega \setminus Su$ and $\partial\Omega \setminus Sv$ respectively.

The idea is to construct a *partition* of Ω in four subsets, and to use the preliminary convergence results of previous chapters to obtain the upper bound inequality.

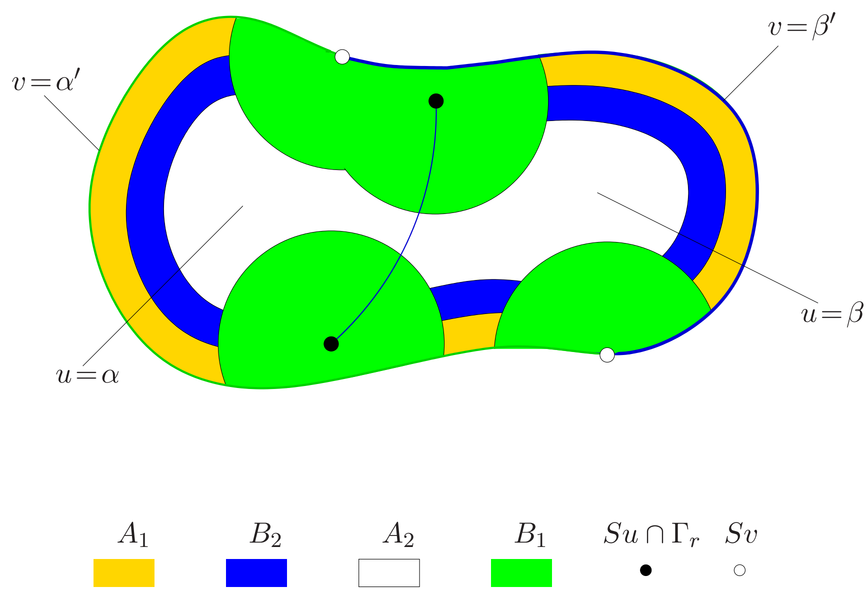

Figure 6.1: Upper bound inequality - partition of Ω.

For every $r > 0$, we set

$$\Gamma_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) = r\}.$$

Step 1: Partition of Ω. Fix $r > 0$ such that Γ_r and Γ_{2r} are Lipschitz surfaces and $Su \cap \Gamma_r$ is a Lipschitz curve.

Now, we are ready to construct the following partition of Ω:

$$\begin{aligned} B_1 &:= \{x \in \Omega : \operatorname{dist}(x, Sv \cup (Su \cap \Gamma_r)) < 3r\}, \\ A_1 &:= \left\{x \in \Omega \setminus \overline{B}_1 : \operatorname{dist}(x, \partial\Omega) < r\right\}, \\ B_2 &:= \left\{x \in \Omega \setminus \overline{B}_1 : r < \operatorname{dist}(x, \partial\Omega) < 2r\right\}, \\ A_2 &:= \left\{x \in \Omega \setminus \overline{B}_1 : \operatorname{dist}(x, \partial\Omega) > 2r\right\}. \end{aligned}$$

(See Fig. 6.1)

For every $r > 0$ and every $\varepsilon < r^{\frac{p-1}{p-2}}$ we construct a Lipschitz function $u_{\varepsilon,r}$ in each subset.

Step 2: Construction of $u_{\varepsilon,r}$ in A_2.

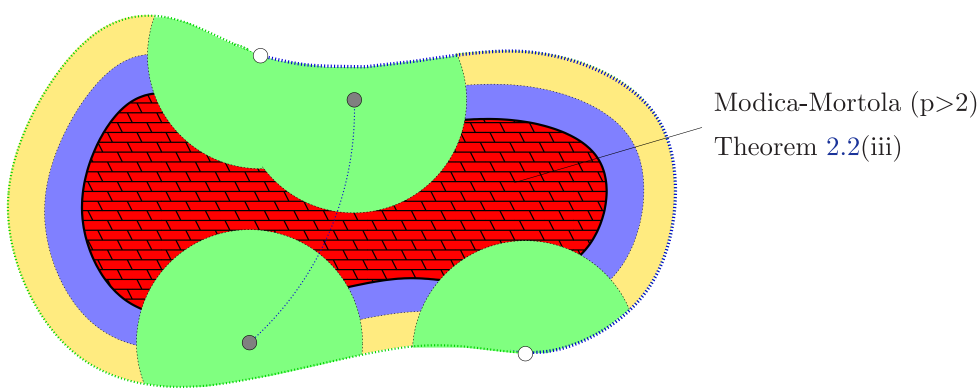

Figure 6.2: Construction of $u_{\varepsilon,r}$ in A_2.

We take $u_{\varepsilon,r}$ being the optimal sequence for the Modica-Mortola functional G_ε in the set A_2 (see Theorem 2.2(iii), p. 36) and we extend it to ∂A_2 by continuity. Hence, $u_{\varepsilon,r}$ is $\dfrac{C_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz in $\overline{A}_2$ (here C_W is the maximum of $W^{p/(p-1)}$ in $[\alpha, \beta]$), $u_{\varepsilon,r}$ converges to u pointwise on A_2 and uniformly on $\partial A_2 \cap \partial B_2$, and

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,r}, A_2, \emptyset) \equiv G_\varepsilon(u_{\varepsilon,r}, A_2) &\geq \sigma_p \mathcal{H}^2(Su \cap A_2) + o(1) \\ &= \sigma_p \mathcal{H}^2(Su) - \sigma_p \mathcal{H}^2(Su \setminus A_2) + o(1), \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{6.10}$$

Step 3: Construction of $u_{\varepsilon,r}$ in A_1.

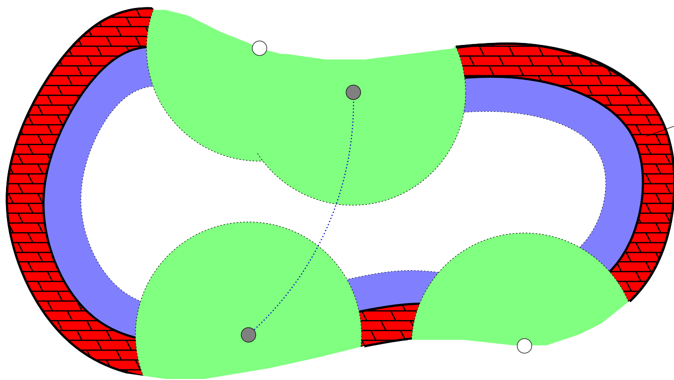

Figure 6.3: Construction of $u_{\varepsilon,r}$ in A_1.

The function u is constant (equal to α or β) on every connected component A of A_1, and the function v is constant (equal to α' or β') on $\partial A \cap \partial\Omega$. So, we can use Proposition 4.3 to get a function $u_{\varepsilon,r}$ such that $Tu_{\varepsilon,r} = v$ on $\partial A \cap \partial\Omega$ and $u_{\varepsilon,r}$ converges to u pointwise on A_1 and uniformly on every subset with positive distance from $\partial A \cap \partial\Omega$.

By Proposition 4.3(ii), $u_{\varepsilon,r}$ is $\dfrac{C'_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous on $\overline{A}_1$ and we can extend it to ∂A_1 with continuity. Since the distance between two different connected components of A_1 is larger than r and $\dfrac{1}{\varepsilon^{(p-2)/(p-1)}} > \dfrac{1}{r}$, choosing $C \geq 2m \vee C'_W$ it follows that $u_{\varepsilon,r}$ is $\dfrac{C}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous on $\overline{A}_1$ and agrees with v on $\partial A_1 \cap \partial\Omega$. Moreover, the function $u_{\varepsilon,r}$ satisfies

$$F_\varepsilon(u_{\varepsilon,r}, A_1, \partial A_1 \cap \partial\Omega) \equiv G_\varepsilon(u_{\varepsilon,r}, A_1) \leq c_p \int_{\partial A_1 \cap \partial\Omega} |\mathcal{W}(Tu(x)) - \mathcal{W}(v(x))| d\mathcal{H}^2 + o(1), \quad \text{as } \varepsilon \to 0. \tag{6.11}$$

Step 4: Construction of $u_{\varepsilon,r}$ in B_2.

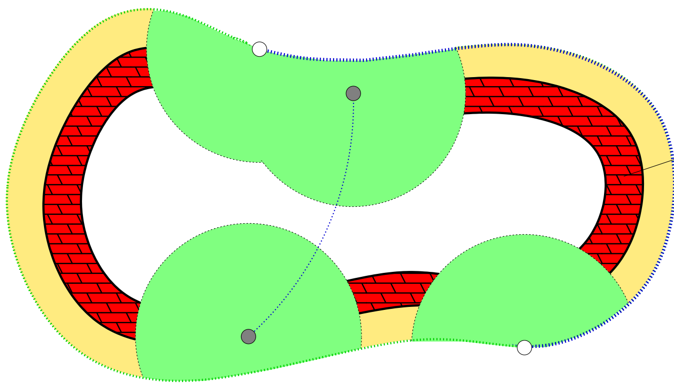

Figure 6.4: Construction of $u_{\varepsilon,r}$ in B_2.

Note that in the previous steps we have constructed an optimal sequence in $\overline{A}_1 \cup \overline{A}_2$ that is $\dfrac{C}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous; in particular it is defined and Lipschitz on $((\partial A_1 \cup \partial A_2) \cap \partial B)$, for every connected component B of B_2. By virtue of Lemma 6.2 we can extend $u_{\varepsilon,r}$ to every B, obtaining a $\dfrac{C+1}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz function that satisfies

$$F_\varepsilon(u_{\varepsilon,r}, B_2, \emptyset) \equiv G_\varepsilon(u_{\varepsilon,r}, B_2) \leq \big(((C+2)^p + C_m)(\mathcal{H}^2(\partial B_2) + o(1))\big)\omega_\varepsilon = o(1) \quad \text{as } \varepsilon \to 0, \tag{6.12}$$

where we used that $\omega_\varepsilon := \inf_{(\partial A_1 \cup \partial A_2)\cap \partial B_2} |u_{\varepsilon,r} - u| = o(1)$ as $\varepsilon \to 0$ (since $u_{\varepsilon,r}$ is constant on each connected components of B_2).

Step 5: Construction of $u_{\varepsilon,r}$ in B_1.

We will use an optimal profile for the minimum problem (4.3). By Proposition 5.4, there exists $\psi \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ such that $\int_{\mathbb{R}^2_+} |D\psi|^p dx < +\infty$, $T\psi(t) \to \alpha'$ as $t \to -\infty$, $T\psi(t) \to \beta'$ as $t \to +\infty$ and $H_1(\psi, \mathbb{R}^2_+, \mathbb{R}) = \gamma_p$. Now, we construct a function $w_\varepsilon : \mathbb{R}^2_+ \to \mathbb{R}$ via the same method used to provide a good competitor u_δ in the proof of Proposition 5.4(Step 1).

For every $\varepsilon > 0$, $\rho_\varepsilon, \sigma_\varepsilon \in \mathbb{R}$, we take a cut-off function $\xi \in C^\infty(\mathbb{R}^2_+)$ such that $\xi \equiv 1$

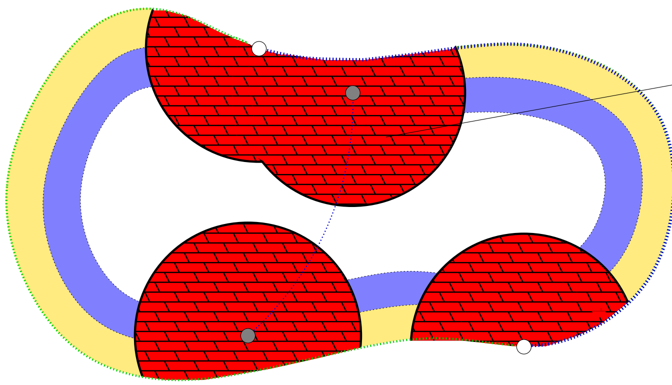

Figure 6.5: Construction of $u_{\varepsilon,r}$ in B_1.

on $(\mathbb{R}^2_+) \setminus D_{\rho_\varepsilon}$ and $\xi \equiv 0$ on D_{σ_ε} such that $|D\xi| \leq \frac{1}{|\rho_\varepsilon - \sigma_\varepsilon|}$. We denote by $\bar{u}$ the function expresses in polar coordinates $\theta \in [0, \pi]$, $\rho \in [0, +\infty)$, as follows:

$$\bar{u}(\theta, \rho) := \frac{\theta}{\pi}\alpha' + \left(1 - \frac{\theta}{\pi}\right)\beta'.$$

We define w_ε as

$$w_\varepsilon(x) := \begin{cases} \psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\sigma_\varepsilon}, \\ \xi(x)\bar{u}(x) + (1 - \xi(x))\psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\rho_\varepsilon} \setminus D_{\sigma_\varepsilon}, \\ \bar{u}(x) & \text{if } x \in (\mathbb{R} \times [0, +\infty)) \setminus D_{\rho_\varepsilon}, \end{cases}$$

Let us show that we can choose ρ_ε and σ_ε such that w_ε satisfies the following inequality

$$H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) \leq \gamma_p + o(1), \text{ as } \varepsilon \to 0. \tag{6.13}$$

We have

$$\begin{aligned} H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) &= H_\varepsilon(\psi^{(\varepsilon)}, D_{\sigma_\varepsilon}, E_{\sigma_\varepsilon}) + \varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} |Dw_\varepsilon|^p dx \\ &\quad + \frac{1}{\varepsilon}\int_{E_{\rho_\varepsilon}\setminus E_{\sigma_\varepsilon}} V(Tw_\varepsilon) d\mathcal{H}^1 \\ &=: H_\varepsilon(\psi^{(\varepsilon)}, D_{\sigma_\varepsilon}, E_{\sigma_\varepsilon}) + I_1 + I_2, \end{aligned} \tag{6.14}$$

where $\psi^{(\varepsilon)}(x) := \psi(\frac{x}{\varepsilon})$ (see Section 5.3, p. 72).

The first integral in the right hand side of (6.14) can be easily estimated as follows

$$\begin{aligned} I_1 &\leq 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} |D\psi(\frac{x}{\varepsilon})|^p dx + 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} |D\xi|^p|\psi(\frac{x}{\varepsilon}) - \bar{u}(x)|^p dx \\ &\quad + 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} |D\bar{u}|^p dx \\ &\leq 3^{p-1}\int_{D_{\rho_\varepsilon/\varepsilon}\setminus D_{\sigma_\varepsilon/\varepsilon}} |D\psi|^p dx + 3^{p-1}C^p\frac{\varepsilon^{p-2}(\rho_\varepsilon^2 - \sigma_\varepsilon^2)}{(\rho_\varepsilon - \sigma_\varepsilon)^p} + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1}}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^{p-2}. \end{aligned} \tag{6.15}$$

While using the convexity of V near its wells and the asymptotic behavior of $T\psi(\frac{x}{\varepsilon})$, for ε small, we have

$$I_2 \leq \frac{1}{\varepsilon}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} V(T\psi(\frac{x}{\varepsilon}))d\mathcal{H}^1. \tag{6.16}$$

Thus, by (6.14), (6.15), (6.16) and suitably choosing of ρ_ε and σ_ε[1], we get

$$\begin{aligned} H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) &\leq H_\varepsilon(\psi^{(\varepsilon)}, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + 3^{p-1}\int_{D_{\rho_\varepsilon/\varepsilon}\setminus D_{\sigma_\varepsilon/\varepsilon}} |D\psi|^p dx + 3^{p-1}C^p\frac{\varepsilon^{p-2}(\rho_\varepsilon^2 - \sigma_\varepsilon^2)}{(\rho_\varepsilon - \sigma_\varepsilon)^p} \\ &\quad + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1}}\left(\frac{\varepsilon}{\rho_\varepsilon}\right)^{p-2} \\ &\leq \gamma_p + o(1) \text{ as } \varepsilon \to 0. \end{aligned} \tag{6.17}$$

We now define a function $\bar{w}_\varepsilon$ on $Sv \times \mathbb{R}^2_+$, by

$$\bar{w}_\varepsilon(x, y) := w_\varepsilon(x), \quad \text{for every } x \in Sv \text{ and every } y \in \mathbb{R}^2_+. \tag{6.18}$$

[1] For instance, we can choose $\rho_\varepsilon = \varepsilon^{1/3} + \varepsilon^{1/2}$ and $\sigma_\varepsilon = \varepsilon^{1/2}$.

Using (6.17) and Fubini's Theorem, we obtain[2]

$$F_\varepsilon(\bar{w}_\varepsilon, Sv\times D_{\rho_\varepsilon}, Sv\times E_{\rho_\varepsilon}) = \mathcal{H}^1(Sv)\left(H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + \frac{1}{\varepsilon^{(p-2)/(p-1)}}\int_{D_{\rho_\varepsilon}} W(w_\varepsilon)dx\right)$$

$$\leq \mathcal{H}^1(Sv)\big(\gamma_p + o(1)\big) \text{ as } \varepsilon \to 0. \tag{6.19}$$

Since Sv is a boundary in $\partial\Omega$, we can construct a diffeomorfism between the intersection of a tubular neighborhood of Sv and Ω and the product of Sv with an half-disk.

For every x in Ω, let us define the oriented distance from Sv as

$$d'(x) := \begin{cases} \operatorname{dist}(x, Sv) & \text{if } x \in \{v = \beta'\}, \\ \\ -\operatorname{dist}(x, Sv) & \text{if } x \in \{v = \alpha'\}. \end{cases}$$

For every $r > 0$, we set

$$\mathcal{S}_r := \{x \in \Omega : 0 < \operatorname{dist}(x, Sv) < r\}\,. \tag{6.20}$$

For every $x \in \overline{\Omega}$, we define

$$\Psi(x) := (x'', d'(x'), \operatorname{dist}(x, \partial\Omega)), \tag{6.21}$$

where x' is a projection of x on $\partial\Omega$ and x'' is a projection of x' on Sv. The function Ψ is well-defined and is a diffeomorfism of class C^2 on $\overline{\Omega} \cap U$ for some neighborhood U of Sv; and satisfies the following properties: $\Psi(\Omega \cap U) = Sv \times \mathbb{R}^2_+$; $\Psi(\partial\Omega \cap U) = Sv \times \mathbb{R} \times \{0\}$; $\Psi(x) = x$, for every $x \in \partial\Omega$; $D\Psi(x)$ is an isometry.

We have that

$$\lim_{r\to 0} \delta_r = 0,$$

where δ_r is the isometry defect of the restriction of Ψ to $\mathcal{S}_r$ (see Section 1.3.2).

We construct $u_{\varepsilon,r}$ on $\overline{\mathcal{S}}_{\rho_\varepsilon/2}$ as

$$u_{\varepsilon,r} := \bar{w}_\varepsilon \circ \Psi,$$

where $\bar{w}_\varepsilon$, $\mathcal{S}_r$ and Ψ are defined by (6.18), (6.20) and (6.21) respectively. For ε small, the function Ψ maps $\mathcal{S}_{\rho_\varepsilon/2}$ into $Sv \times D_{\rho_\varepsilon}$ and $\partial\mathcal{S}_{\rho_\varepsilon/2} \cap \partial\Omega$ into $Sv \times E_{\rho_\varepsilon}$, so we can use

[2] Note that we can define the functional F_ε by (6.1) also on functions $u \in W^{1,p}(A)$, where $A \subset \mathbb{R}^N$, with $N \geq 3$.

Proposition 1.11 and, by (6.19), we obtain

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,r}, \mathcal{S}_{\rho_\varepsilon/2}, \partial\mathcal{S}_{\rho_\varepsilon/2} \cap \partial\Omega) &\le (1-\delta_\varepsilon)^{-(p+3)} F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{\rho_\varepsilon}, Sv \times E_{\rho_\varepsilon}) \\ &\le \mathcal{H}^1(Sv)\big(\gamma_p + o(1)\big) \text{ as } \varepsilon \to 0, \end{aligned} \tag{6.22}$$

where we also used that $\delta_\varepsilon := \delta(\Psi\lfloor\mathcal{S}_{\rho_\varepsilon})$ tends to 0 as $\varepsilon \to 0$.

Notice that for ε small enough, Ψ is 2-Lipschitz continuous. Using again Lemma 6.2, we can extend $u_{\varepsilon,r}$ by setting $u_{\varepsilon,r} := v$ on the remaining part of $\partial B_1 \cap \partial\Omega$; we have that $u_{\varepsilon,r}$ is equal to v on $\partial\Omega \setminus \partial\mathcal{S}_{\rho_\varepsilon/2}$. Thus, we can extend $u_{\varepsilon,r}$ on the whole $B_1 \setminus \mathcal{S}_{\rho_\varepsilon/2}$ to a $\dfrac{2C+1}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous function, which satisfies

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,r}, B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}, \partial(B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}) \cap \partial\Omega) &= G_\varepsilon(u_{\varepsilon,r}, B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}) \\ &\le \big((2C+2)^p + C_m\big)\big(\mathcal{H}^2(\partial B_1) + o(1)\big)2m \text{ as } \varepsilon \to 0, \end{aligned} \tag{6.23}$$

where we used $\|u_{\varepsilon,r} - \alpha\|_\infty \wedge \|u_{\varepsilon,r} - \beta\|_\infty \le 2m$.

Step 6: Upper bound inequality. We recall that for every $r > 0$ and every $\varepsilon < r^{\frac{p-2}{p-1}}$ we have constructed a function $u_{\varepsilon,r}$ defined on the whole Ω such that

$$\limsup_{\varepsilon\to 0} \|u_{\varepsilon,r} - u\|_{L^1(\Omega)} \le 2m(|B_1| + |B_2|) \text{ and } \limsup_{\varepsilon\to 0} \|Tu_{\varepsilon,r} - v\|_{L^1(\partial\Omega)} = 0.$$

Since $|B_1|$ and $|B_2|$ have order r^2 and r respectively, we get that $u_{\varepsilon,r} \to u$ in $L^1(\Omega)$, first taking $\varepsilon \to 0$ and then $r \to 0$.

Combining (6.10), (6.11), (6.12), (6.22) and (6.23), we obtain

$$\begin{aligned} \limsup_{\varepsilon\to 0} F_\varepsilon(u_{\varepsilon,r}) \le\ & \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu(x)) - \mathcal{W}(v(x))| d\mathcal{H}^2 + \gamma_p \mathcal{H}^1(Sv) \\ & -\sigma_p \mathcal{H}^2(Su \setminus A_2) + \big((2C+2)^p + C_m\big)\big(\mathcal{H}^2(\partial B_1) + o(1)\big)2m. \end{aligned} \tag{6.24}$$

Since $\mathcal{H}^2(\partial B_1)$ has order r, taking r to 0 in (6.24), we deduce the upper bound inequality (iii). Finally, applying a suitable diagonalization argument[3] to the sequence $u_{\varepsilon,r}$, we obtain the desired recovery sequence u_ε. This concludes the proof. □

[3] See for instance Attouch [12], Corollary 1.18, p. 37.

List of Symbols

Sets, numbers, measures

$a \vee b$ ($a \wedge b$) the maximum (minimum) between a and b

$B_r(x)$ the open ball of centre x and radius r

C (if not otherwise stated) a strictly positive constant independent from the parameters

$|E|$ the Lebesgue measure of the set E

$\mathcal{H}^k$ the k-dimensional Hausdorff measure

$\mu \llcorner E$ the restriction of the measure μ to E

$[t]$ the integer part of $t \in \mathbb{R}$

Function spaces

$O(3)$ the set of linear isometries on $\mathbb{R}^3$

$C^k(\Omega)$ the space of k-times differentiable real valued functions on Ω

$C_0(\mathbb{R})$ the closure of continuous functions on $\mathbb{R}$ with compact support

$\mathcal{M}(\mathbb{R})$ ($\mathcal{M}^+(\mathbb{R})$) the space of signed (positive) Radon measure with finite mass

$\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

$\mathcal{Y}(A)$ the family of all weakly-$*$ measurable maps $\nu : A \to \mathcal{P}(\mathbb{R})$

$L^p(\Omega)$ the space of real valued p-summable functions on Ω

$W^{1,p}(\Omega)$ the space of Sobolev functions with p-summable derivatives on Ω

$\|u\|_{L^p(\Omega)}$ or simply $\|u\|_p$ the L^p norm of u

$\|u\|_{W^{1,p}(\Omega)}$ the $W^{1,p}$ norm of u

$W^{k,p}(\omega)$ the fractional space of order $\{k, s\}$ of functions on ω

$\|v\|_{W^{k,p}(\omega)}$ or simply $\|v\|_{k,p}$ the $W^{k,p}$ norm of v

$X_{\rm loc}(\Omega)$ $\{u : \Omega \to \mathbb{R} : u \in X(A)$ for all open $A \subset\subset \Omega\}$, where X is a generic notation for a function space

Functions

χ_E the characteristic function of the set E $\big(\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if $x \notin E\big)$

Su the set of essential discontinuity points of u (jump set)

$\nu_u(x)$ the normal to Su at x

$u_n \to u$ u_n converges strongly to u

$u_n \rightharpoonup u$ u_n converges weakly to u

$u_n \overset{*}{\rightharpoonup} u$ u_n converges weakly-$*$ to u

List of Figures

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