The resolution of the Yang-Mills Plateau problem in super-critical dimensions

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Abstract

We study the minimization problem for the Yang-Mills energy under fixed boundary connection in supercritical dimension $n \geq 5$. We define the natural function space $\mathcal{A}_G$ in which to formulate this problem in analogy to the space of integral currents used for the classical Plateau problem. The space $\mathcal{A}_G$ can be also interpreted as a space of weak connections on a "real measure theoretic version" of reflexive sheaves from complex geometry.
We prove the weak closure result which ensures the existence of energy-minimizing weak connections in $\mathcal{A}_G$.
We then prove that any weak connection from $\mathcal{A}_G$ can be obtained as a L^2-limit of classical connections over bundles with defects. This approximation result is then extended to a Morrey analogue.
We prove the optimal regularity result for Yang-Mills local minimizers. On the way to prove this result we establish a Coulomb gauge extraction theorem for weak curvatures with small Yang-Mills density. This generalizes to the general framework of weak L^2 curvatures previous works of Meyer-Rivière and Tao-Tian in which respectively a strong approximability property and an admissibility property were assumed in addition.

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1 Introduction

1.1 A nonintegrable Plateau problem

1.1.1 The classical Plateau problem

Consider a smooth simple closed curve γ in $\mathbb{R}^3$. The classical Plateau problem can be formulated as follows:

$$\text{“Find a surface } \Sigma \subset \mathbb{R}^3 \text{ with boundary } \gamma \text{ of smallest area”.} \tag{1.1}$$

Part of the problem is giving a suitable meaning to the terms “surface”, “boundary” and “area”, in such a way as to extend the classical notions from a smooth setting to one where a minimizer is assured to exist. The *parametric approach* to problem (1.1) consists in considering immersed images of the unit disk:

$$\text{Let } u : D^2 \to \mathbb{R}^3 \text{ be a smooth immersion}$$

$$\text{such that } u|_{\partial D^2} \text{ is a parameterization of } \gamma \ .$$

One then looks for minimizers of the following area functional, defined in terms of coordinates x, y on D^2:

$$A(u) := \int_{D^2} |\partial_x u \times \partial_y u| dxdy \ .$$

An immediate difficulty which arises is the fact that the functional $A(u)$ has a *large symmetry group*: for all ϕ belonging to the group of orientation-preserving diffeomorphisms of D^2, i.e. for any immersion u as above there holds

$$A(u) = A(u \circ \phi) \quad \text{for all } \phi \in \mathcal{G}_{plat} = \mathrm{Diff}^+(D^2) \ . \tag{1.2}$$

This infinite-dimensional symmetry group $\mathcal{G}_{plat}$ is responsible for possible loss of compactness of area-minimizing sequences of maps. It is then required to break this infinite dimensional symmetry in order to hope for minimizing sequences to have some compactness. A now classical strategy introduced by J. Douglas and T. Radò consists in minimizing a more coercive functional, the Dirichlet energy E, for which

$$A(u) \le E(u) := \frac{1}{2} \int_{D^2} |Du|^2$$

instead of the area A with equality if and only if the parametrization of the immersed disc u is conformal. Such change has the effect of providing “good” minimizing sequences for $A(u)$ (so-called Coulomb immersions).

1.1.2 A nonintegrable analogue of the Plateau problem

Consider a smooth compact Riemannian n-manifold M with boundary and let G be a compact connected simply connected nonabelian Lie group with Lie algebra $\mathfrak{g}$. We assume that a principal G-bundle $P \to \partial M$ is fixed over the boundary of M. On P we consider a G-invariant connection ω, which corresponds to an equivariant horizontal n-plane distribution Q (see [32] for notations and definitions).

Analogously to the Plateau problem, we may then ask which is the "most integrable" extension of P, Q to a horizontal distribution on a principal G-bundle over M. By Frobenius' theorem, the condition for integrability in this case is that for any two horizontal G-invariant vector fields X, Y, their lie bracket $[X, Y]$ be again horizontal. The L^2-error to integrability of an extension of Q over M can be measured by taking vertical projections $\mathcal{V}$ of $[X_i, X_j]$ for X_i, X_j varying in an orthonormal basis of Q:

$$\int_M \sum_{i,j} |\mathcal{V}([X_i, X_j])|^2 . \tag{1.3}$$

Note that $F(X, Y) = \mathcal{V}([X, Y])$ is known to be a tensor, and F is nothing but the curvature of the connection.

From now on we will work on the associated vector bundle $E \to M$ corresponding to the adjoint representation of G and we identify the connection form with a covariant derivative ∇ on E. In a trivialization we have the local expression

$$\nabla \overset{loc}{=} d + A ,$$

where A is a $\mathfrak{g}$-valued 1-form on a given chart of M. The structure equation relating curvature to connection takes the form

$$F \overset{loc}{=} dA + A \wedge A \tag{1.4}$$

in a trivialization. Here $\wedge$ represents a tensorization of the usual exterior product of forms with the Lie bracket on $\mathfrak{g}$. In this setting the L^2-error in integrability (1.3) is identified with the *Yang-Mills energy*, which we consider as being a functional of the connection ∇:

$$\mathcal{YM}(\nabla) := \int_M |F_\nabla|^2 . \tag{1.5}$$

We observe that, similarly to the area functional in the Plateau problem, $\mathcal{YM}$ has again a *large invariance group* given by changing coordinates in the fibers via G. The corresponding group

$$\mathcal{G} := \{g : M \to G\} \tag{1.6}$$

acts on the curvature form $F = \sum F_{ij} dx_i \wedge dx_j$ via

$$F_{ij} \mapsto g^{-1} F_{ij} g, \quad |F| \mapsto |g^{-1} F g| = |F| ,$$

where we used the fact that the canonical norm on the Lie algebra $\mathfrak{g}$ is given by the Killing form (see again [32]).

1.2 Natural spaces of connections and the critical dimension $n = 4$

The natural function spaces in which to consider the minimization of $\mathcal{YM}$ are identified by considering the local form of the structure equation (1.4). The curvature form F is naturally required to be L^2 in order for the energy to be finite. In the abelian situation $G = U(1)$ there holds $A \wedge A = 0$ and $\int |F_\nabla|^2 = \int |dA|^2$ hence $W^{1,2}$ is a natural space to consider for the connection forms A. In a non-abelian framework the situation is more delicate due to the nonlinearity $A \wedge A$. Assuming $A \in W^{1,2}$ the linear term dA of (1.4) belongs to L^2, but the L^2 control of the quadratic nonlinearity $A \wedge A$ requires a priori $A \in L^4$.

In dimensions $n \leq 4$ the norm inequality underlying the Sobolev embedding $W^{1,2} \to L^4$ implies that we have both dA and $A \wedge A$ in L^2. This embedding is not valid anymore in dimensions $n \geq 5$, which are called *supercritical dimensions.*

Going back to the critical dimension $n = 4$ or to subcritical dimensions, K. K. Uhlenbeck [51] has proved the local existence of *good gauges*, similar to conformal parametrizations in the *classical plateau problem*, in which the L^2-norm of F controls the $W^{1,2}$ norm of A by optimizing the more coercive functional

$$\int \left(|F_\nabla|^2 + |d^* A|^2\right) \geq \int |F_\nabla|^2 \quad .$$

The class in which to formulate this Yang-Mills minimization problem is in this case the space of connections over classical bundles $E \to M$ which in each chart for some trivialization have connection forms A belonging to $W^{1,2}_{loc}$:

$$\mathcal{A}^{1,2}(E) := \{\nabla \text{ connection on } E \to M \text{ s.t. in some } W^{2,2}\text{-gauge } A \in W^{1,2}_{loc}\} \ . \tag{1.7}$$

The following result permits to solve the Yang-Mills-Plateau problem in this case:

Theorem 1.1 ([51],[45],[43],[37]). *Let M be a compact Riemannian 4-manifold and $E \to M$ a classical vector G-bundle. Consider a sequence of connections $\nabla_k \in \mathcal{A}^{1,2}(E)$ such that their curvature forms F_k are equibounded in L^2 and such that we have the weak convergence*

$$F_k \rightharpoonup F \quad \text{in } L^2 \ .$$

Then F is the curvature form of a connection $\nabla \in A^{1,2}(\tilde{E})$ where $\tilde{E} \to M$ is a classical vector G-bundle (possibly different than E).

The proof of theorem 1.1 combines the local extraction of Coulomb gauges satisfying

$$d^*A = 0$$

together with a covering argument and a point removability result. We introduce the following space, where M^4 is a compact riemannian manifold, A is a $\mathfrak{g}$-valued 1-form and F is a $\mathfrak{g}$-valued 2-form:

$$\mathcal{A}_G(M^4) := \left\{ \begin{array}{c} A \in L^2,\ F_A \stackrel{\mathcal{D}'}{=} dA + A \wedge A \in L^2 \in L^2\ , \\ \text{and loc. } \exists\ g \in W^{1,2}(M^4, G) \text{ s.t. } A^g \in W^{1,2}_{loc} \end{array} \right\} ,$$

where $A^g := g^{-1}dg + g^{-1}Ag$ is the expression of A after the gauge change g. Note that

$$\bigcup_{E \to M^4} \mathcal{A}^{1,2}(E) = \mathcal{A}_G(M^4)\ ,$$

where the union is over all smooth G-bundles $E \to M^4$.

One obtains as a direct consequence of Theorem 1.1 the following result:

Theorem 1.2. *Let M be a compact Riemannian 4-manifold with boundary and let ϕ be the connection form of a smooth connection on a classical G-bundle $E_\partial \to \partial M$. Consider the space $\mathcal{A}_{G,\phi}(M)$ consisting of all connections $\nabla \in \mathcal{A}_G(M)$ for bundles E whose restrictions over ∂M are equal to E_∂ and such that the restriction of ∇ to E_∂ is locally gauge-equivalent to $d + \phi$. Then the following holds:*

$$\inf \left\{ \int_M |F|^2 :\ F \stackrel{\mathcal{D}'}{=} dA + A \wedge A,\ A \in \mathcal{A}_{G,\phi}(M) \right\} \tag{1.8}$$

is achieved and the minimizer is the connection form corresponding to a smooth connection over a classical G-bundle $\tilde{E} \to M$.

1.3 Supercritical dimension $n = 5$

As noted above, dimensions $n \geq 5$ are more challenging because the nonlinearity of the structure equation (1.4) is not controlled by the linear part anymore in the "natural" Sobolev scpace $W^{1,2}$. The following question was at the origin of the present work:

Question 1. *Which is the correct replacement for the spaces $\mathcal{A}^{1,2}(E)$ in dimension $n \geq 5$?*

For the clarity of the presentation we restrict in this work to the case of dimension 5 and to an euclidean setting. The extension of all our results to higher dimensions $n > 5$ as well as to general Riemannian manifolds will be done in a forthcoming work [40]. One of the main achievements of the present work is to provide the following ad hoc replacement of $\mathcal{A}^{1,2}$ in supercritical dimension:

Definition 1.3 (Weak connections in dimension 5)**.** *For two L^2 connection forms A, A' over $\mathbb{B}^5$ we write $A \sim A'$ if there exists a gauge change $g \in W^{1,2}(\mathbb{B}^5, G)$ such that $A' = g^{-1}dg + g^{-1}Ag$. The class of all such L^2 connection forms A' is denoted $[A]$. We denote the class of L^2 weak connections on singular bundles over M as follows:*

$$\mathcal{A}_G(\mathbb{B}^5) := \left\{ \begin{array}{c} [A] : \ A \in L^2, \ F_A \overset{\mathcal{D}'}{=} dA + A \wedge A \in L^2 \\ \forall p \in M \ \ a.e. \ r > 0, \ \exists A(r) \in \mathcal{A}_G(\partial B_r(p)) \\ i^*_{\partial B_r(p)} A \sim A(r) \end{array} \right\} .$$

The fact that $\mathcal{A}_G$ is the correct function space for the variational study of $\mathcal{YM}$ in 5-dimensions is a consequence of the following result:

Theorem 1.4 (sequential weak closure of $\mathcal{A}_G$). *Let $[A_k] \in \mathcal{A}_G(\mathbb{B}^5)$ be a sequence of connections such that the corresponding curvature forms F_k are equibounded in $L^2(\mathbb{B}^5)$ and converge weakly to a 2-form F. Then F corresponds to $[A] \in \mathcal{A}_G(\mathbb{B}^5)$.*

Definition 1.3 and Theorem 1.4 are inspired by the slicing approach to the closure theorem for rectifiable currents, initially introduced by B. White [53], R. L. Jerrard [29] and used by L. Ambrosio and B. Kirchheim [3] for their striking proof of the closure theorem for rectifiable currents in metric spaces. The idea behind this approach is that a current is rectifiable when its slices via level sets of Lipschitz functions give a metric bounded variation (MBV, for short) function with respect to the flat metric between the sliced currents.
The closure theorem for rectifiable currents corresponds then to a compactness result for MBV functions, valid when the oscillations of slices are controlled via the overlying total mass functional for sequences of weakly convergent currents. This mass-finiteness condition was weakened by R. M. Hardt and T. Rivière [23], who introduced the notion of *rectifiable scans*.

In [38] the authors used the ideas coming from the theory of scans for defining the class of weak L^p curvatures over $U(1)$-bundles and proving the weak closure theorem relevant for minimizing the p-Yang-Mills energy $\int_M |F|^p$ in supercritical dimension 3 for $1 < p < 3/2$ (see also [31]). This class of weak curvatures is identified via Poincaré duality with the class of L^p vector fields on 3-dimensional manifolds having integer fluxes through "almost all spheres".

The new difficulty with respect to such result is mentioned in Section 1.6 and amounts to the justification of the existence of gauges $g : \mathbb{B}^5 \to G$ which are $W^{1,2}$-controlled and solve an ODE of the form $\partial_t g = -Ag$ where A is a connection form corresponding to $[A] \in \mathcal{A}_G(\mathbb{B}^5)$. Such existence result is based on the strong approximation result of Theorem 1.8.

1.4 The Yang-Mills-Plateau problem in dimension $n = 5$: a definition of weak traces

Since an element $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ is only assumed to be in L^2 it seems a priori problematic to define its trace on $\partial\mathbb{B}^5$ in order to pose the Yang-Mills Plateau problem in $\mathcal{A}_G(\mathbb{B}^5)$ and take advantage of the Sequential Weak Closure Theorem 1.4. To obtain a suitable notion of trace, the following idea introduced in [35] is used. Consider the slice equivalence class distance

$$\operatorname{dist}([A],[A']) := \min\{\|A - g^{-1}dg - g^{-1}A'g\|_{L^2(\mathbb{S}^4)} : \; g \in W^{1,2}(\mathbb{S}^4, G)\} \; .$$

Consider the boundary connection ϕ as a special slice and impose an oscillation bound for nearby slices. More precisely, we have the following definition:

Definition 1.5 (boundary trace for $\mathbb{B}^5$)**.** *For a given connection form $\phi \in \mathcal{A}^{1,2}(\mathbb{S}^4)$ we define the space of weak connection classes $[A]$ over $\mathbb{B}^5$ having trace in the class $[\phi]$ as follows:*

$$\mathcal{A}_G^{\phi}(\mathbb{B}^5) := \mathcal{A}_G(\mathbb{B}^5) \cap \left\{ \begin{array}{c} [A] \text{ s.t. for } r \uparrow 1, \quad r \notin N \\ \text{there holds } dist([A(r,0)],[\phi]) \to 0 \; . \end{array} \right\} \; , \tag{1.9}$$

where N is a Lebesgue-null set and $A(r,0)$ is the a.e.-defined L^2 form τ_r^*A on $\mathbb{S}^4$ obtained by pulling back A via the homothety $\tau_r : \mathbb{S}^4 \to \partial B_r(0)$.

The following result whose proof is similar to the one for the abelian case [35] guarantees that $\mathcal{A}_G^{\phi}(\mathbb{B}^5)$ is the right space on which to define the analogue of (1.8):

Theorem 1.6 (properties of the trace)**.** *The classes $\mathcal{A}_G^{\phi}(\mathbb{B}^5)$ satisfy the following properties:*

1. ***(closure)*** *for any 1-form $\phi \in \mathcal{A}_G(\mathbb{S}^4)$, the class $\mathcal{A}_{G,\varphi}(\mathbb{B}^5)$ is closed under sequential weak L^2-convergence of the corresponding curvature forms F.*

2. ***(nontriviality)*** *if ϕ, ψ are 1-forms in $\mathcal{A}_G(\mathbb{S}^4)$ such that $[\phi] \neq [\psi]$ as gauge-equivalence classes, then $\mathcal{A}_{G,}^{\phi}(\mathbb{B}^5) \cap \mathcal{A}_G^{\psi}(\mathbb{B}^5) = \emptyset$.*

3. ***(compatibility)*** *for any smooth connection 1-form ϕ, ∇ is a connection of a classical bundle over the finitely punctured ball $E \to \mathbb{B}^5 \setminus \{p_1, \ldots, p_k\}$ satisfying $i_{\mathbb{S}^4}^* A \in [\phi]$ if and only if the corresponding connection form A belongs to $\mathcal{A}_G^{\phi}(\mathbb{B}^5)$.*

Combining now Theorem 1.4 and Theorem 1.6 we obtain the following, which is one of the main results of the present work:

Theorem 1.7 (Yang-Mills-Plateau solution in dimension 5**).** *For all $\phi \in \mathcal{A}_G(\mathbb{S}^4)$ there exists a minimizer $[A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5)$ to the following Yang-Mills Plateau problem:*

$$\inf\left\{ \int_{\mathbb{B}^5} |F|^2 : \; F \overset{\mathcal{D}'}{=} dA + A \wedge A, \; [A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5) \right\} \; . \tag{1.10}$$

The analogous result for the case of $G = U(1)$ was proved in [35] using the result [38].

1.5 Naturality of the space $\mathcal{A}_G^{\phi}$

Our aim now is to establish a regularity result for solutions to the Yang-Mills Plateau problem as given by Theorem 1.7, corresponding to the regularity result of Theorem 1.2 in dimension $n = 4$.

The proof of the partial regularity of solutions to (1.10) goes through a more torough description of our space $\mathcal{A}_G(\mathbb{B}^5)$ as being the L^2-closure of the space of connections which are smooth away from a set of isolated points. More precisely, we introduce the class

$$\mathcal{R}^{\infty,\phi}(\mathbb{B}^5) := \left\{ \begin{array}{c} F \text{ corresponding to some } [A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5) \text{ s.t.} \\ \exists k, \exists a_1, \ldots, a_k \in \mathbb{B}^5, \quad F = F_\nabla \text{ for a smooth connection} \nabla \\ \text{on some smooth } G\text{-bundle } E \to \mathbb{B}^5 \setminus \{a_1, \ldots, a_k\} \end{array} \right\} . \tag{1.11}$$

The strong approximation will occur with respect to the following geometric distance:

$$\operatorname{dist}_F(F, F') := \min\{\|F - g^{-1} F g\|_{L^2(\mathbb{B}^5)} : \ g : \mathbb{B}^5 \to G \text{ measurable}\} . \tag{1.12}$$

We then have the following:

Theorem 1.8 (Naturality of $\mathcal{A}_G^{\phi}$). *Let $[A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5)$ and let $F \in L^2$ be the connection form of an L^2 representative A of $[A]$. Then there exist curvature forms F_k corresponding to connection forms A_k, $[A_k] \in \mathcal{R}^{\infty,\phi}(\mathbb{B}^5)$ such that*

$$A_k \to A \quad \text{in } L^2, \quad F_k \to F \quad \text{in } L^2 .$$

In particular there holds

$$dist_F(F_k, F) \to 0, \qquad \text{as } k \to \infty .$$

The strategy of proof of Theorem 1.8 is based on the strong approximation procedure that F. Bethuel introduced for his approximation results [7] for Sobolev maps into manifolds. However recall the fact that as discussed above, unlike the case of Sobolev maps (where $\|du\|_{L^p}$ controls $\|u\|_{L^{p^*}}$), here $\|F\|_{L^2}$ does not control the connection form. Hence the strategy for filling the "good cubes" differs completely from the one available in the case of Sobolev maps and requires a completely new argument.
Pushing the comparison with the case of Sobolev maps into manifolds further, the corresponding weak closure result for Sobolev maps in $W^{1,p}(\mathbb{B}^m, N^n)$ for instance is a direct consequence of Rellich-Kondrachov's theorem, whereas in our case the analogous result, Theorem 1.4 for weak connections, required a substantial amount of work.

1.6 Some consequences on weak solutions to ODE

The application of the strong density theorem 1.8 in the proof of the weak closure result theorem 1.4 goes through the result of the next proposition, which is of independent interest: the ODE (1.13) can be solved in $W^{1,2}(\mathbb{B}^5, G)$ provided the field A is given by a connection form A with $[A] \in \mathcal{A}_G(\mathbb{B}^5)$.

Corollary 1.9 (controlled solutions to the radial gauge fixing ODE)**.** *Assume that to* $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ *there corresponds a connection form* A *and a curvature form* F_A*, both of which are in* $L^2(\mathbb{B}^5)$*, as in the definition of* $\mathcal{A}_G(\mathbb{B}^5)$*. If* ρ *is the radial coordinate* $\rho(x) = 1 - |x|$ *on* $\mathbb{B}^5$ *then for fixed* $t \in [0, 1[$ *there exists a solution* $g \in W^{1,2}(\mathbb{B}^5, G)$ *to the following ODE:*

$$\begin{cases} \partial_\rho g = -A_\rho g & \text{on } \mathbb{B}_1 \setminus \mathbb{B}_{1-t} \ , \\ g(\omega, 0) = id & \text{for } \omega \in \mathbb{S}^4 \ . \end{cases} \tag{1.13}$$

In particular the form $A^g := g^{-1}dg + g^{-1}Ag$ *is still* L^2 *and has zero component in the direction* $\partial/\partial\rho$ *and the formula*

$$F_{A^g} = g^{-1} F_A g \tag{1.14}$$

holds in the sense of distributions, once we define, $F_{A^g} \overset{\mathcal{D}'}{:=} dA^g + A^g \wedge A^g$*.*

This result should be compared to the theory of [13] and [2], [12] where Lipschitz solutions $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ to the nonlinear ODE $\partial_t g(x, t) = X(t, g(x, t))$ are found, under the requirement that $X \in L^\infty, \mathrm{div} X \in L^\infty$. In that case the existence result is based on the theory of renormalized solutions for the related PDE. In our setting the ODE (1.13) is linear and the requirement for g to be a renormalized solution appears in the form (1.14) and follows from the fact that g is ensured to be $W^{1,2}$. On the other hand we don't need the incompressibility condition (see e.g. the definition of a regular Lagrangian flow in [12]).

What allows this new result is the fact that while in the cited works the existence is ensured by approximating the driving field X by smooth ones through a mollification, in our case the regularization is done via Theorem 1.8, which is better adapted to the geometry of the flows. Therefore finding a nonlinear generalization of the above result could help improving the theory of weak flows.

1.7 Coulomb gauge extraction result for weak curvatures with small densities

We first improve the result of Theorem 1.8 to an approximation result for Morrey curvatures, reading as follows:

Theorem 1.10 (Morrey counterpart of Theorem 2.10)**.** *There exist constants* C, ϵ_1 *with the following properties. Let* F *be the curvature form corresponding to an* L^2 *connection* *form A with $[A] \in F_{\mathbb{Z}}(\mathbb{B}^5)$. Assume that*

$$\sup_{x,r} \frac{1}{r} \int_{B_r(x)} |F|^2 < \epsilon_1 \ . \tag{1.15}$$

Then we can find curvature forms $\hat{F}_k$ corresponding to smooth connection forms $\hat{A}_k$ such that

$$\|\hat{F}_k - F\|_{L^2(\mathbb{B}^5)} \to 0 \ , \tag{1.16}$$

$$\|\hat{A}_k - A\|_{L^2(\mathbb{B}^5)} \to 0 \ , \tag{1.17}$$

and

$$\sup_{x,r} \frac{1}{r} \int_{B_r(x)} |\hat{F}_k|^2 < C\epsilon_1 \ . \tag{1.18}$$

We recall that the Morrey norms of a function f are defined as follows:

$$\|f\|_{M^{k,p}_\alpha(\mathbb{B}^n)} := \left(\sup_{x\in\mathbb{B}^n, r>0} \frac{1}{r^{n-\alpha p}} \int_{B_r(x)} |f|^p \right)^{\frac{1}{p}} \ .$$

Thus the above theorem asserts that for curvature forms which are $M^{0,2}_2$-small on $\mathbb{B}^5$, Theorem 1.8 can be refined to ensure uniform $M^{0,2}_2$ bounds for the curvatures of the approximating smooth connections, as well as the strong L^2-convergence of the connection forms.
Continuing the previous approximation result with the Coulomb gauge extraction method of [34] for admissible connections or the one of [47] for smooth connections in Morrey spaces, we have the following generalization of these results to our space $\mathcal{A}_G$ which is clearly much larger than the space of admissible connections:

Theorem 1.11 (Coulomb gauge extraction in Morrey norm)**.** *There exist constants ϵ, C depending only on the dimension such that the following holds. Let F be a weak curvature corresponding to an L^2 connection form A with $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ and assume that*

$$\sup_{x,r} \frac{1}{r} \int_{B_r(x)} |F|^2 := \|F\|^2_{M^{0,2}_2(\mathbb{B}^5)} \le \epsilon \ .$$

Then there exists a gauge change $g \in W^{1,2}(\mathbb{B}^5, G)$ such that the transformed connection form $A^g = g^{-1}dg + g^{-1}Ag$ satisfies

$$d^*A^g = 0 \ in\ \mathbb{B}^5 \ , \tag{1.19}$$

$$\left\langle A^g, \frac{\partial}{\partial r} \right\rangle = 0 \ on\ \partial\mathbb{B}^5 \ , \tag{1.20}$$

$$\left(\sup_{x,r} \frac{1}{r} \int_{B_r(x)} |A^g|^4 \right)^{\frac{1}{4}} + \left(\sup_{x,r} \frac{1}{r} \int_{B_r(x)} |DA^g|^2 \right)^{\frac{1}{2}} \le C\|F\|_{M^{0,2}_2(\mathbb{B}^5)} \ . \tag{1.21}$$

1.8 ϵ-regularity result for stationary weak curvatures in $\mathcal{A}_G(\mathbb{B}^5)$

The main result of [34] together with Theorem 1.11 gives the ϵ-regularity:

Theorem 1.12 (ϵ-regularity)**.** *There exists a constant $\epsilon > 0$ such that the following holds. Let F be a weak curvature corresponding to an L^2 connection form A with $[A] \in \mathcal{A}_G(\mathbb{B}^5)$, such that for all smooth perturbations $\eta \in C_0^\infty(\mathbb{B}^5, \wedge^1\mathbb{B}^5 \otimes \mathfrak{g})$ there holds*

$$\frac{d}{dt} \int_{\mathbb{B}^5} |F_{A+t\eta}|^2 \bigg|_{t=0} = 0 \tag{1.22}$$

and such that for all vector fields $X \in C_0^\infty(\mathbb{B}^5, \mathbb{R}^5)$ the function $\phi_t := id + tX$ satisfies

$$\frac{d}{dt} \int_{\mathbb{B}^5} |\phi_t^* F_A|^2 \bigg|_{t=0} = 0 \ . \tag{1.23}$$

Assume that

$$\frac{1}{r} \int_{B_r(x_0)} |F|^2 \leq \epsilon \ .$$

Then F is the curvature form of a smooth connection over $B_{r/2}(x_0)$.

Because of the above theorem we can also extend the regularity result of [34]:

Corollary 1.13 (partial regularity for stationary weak curvatures)**.** *Let F be a weak curvature corresponding to an L^2 connection form A with $[A] \in \mathcal{A}_G(\mathbb{B}^5)$, satisfying* (1.22) *and* (1.23)*.*
Then there exists a closed set $K \subset \mathbb{B}^5$ such that $\mathcal{H}^1(K) = 0$ and locally around every point in $\mathbb{B}^5 \setminus K$ there exist a gauge change such that A^g is a smooth form.

1.9 Optimal regularity result for Yang-Mills Plateau minimizers

Since we work in the natural class $\mathcal{A}_G^\phi(\mathbb{B}^5)$ in which a Yang-Mills minimizer exists according to Theorem 1.7, we may then apply Federer dimension reduction techniques and obtain:

Theorem 1.14 (optimal partial regularity for Yang-Mills-Plateau minimizers)**.** *Let ϕ be a smooth $\mathfrak{g}$-valued connection 1-form over $\partial\mathbb{B}^5$. Then the minimizer of*

$$\inf \left\{ \|F_A\|_{L^2(\mathbb{B}^5)} : \ [A] \in \mathcal{A}_{G,\phi}(\mathbb{B}^5) \right\}$$

belongs to $\mathcal{R}_\phi^\infty(\mathbb{B}^5)$, i.e. the corresponding class $[A] \in \mathcal{A}_{G,\phi}(\mathbb{B}^5)$ has a representative which is locally smooth outside a finite set.

An analogue of this result was proven by a completely different, combinatorial technique in [36] for the case of $U(1)$-curvatures.

The result of Theorem 1.14 is optimal in the following sense. Recall that in [22] it was proven that there exist smooth boundary data for harmonic maps $u : \mathbb{B}^3 \to \mathbb{S}^4$ such that the energy-minimizing harmonic map would need to have a bounded from below number of singularities. By a similar procedure it is possible to find smooth connection forms ϕ on bundles over $\partial\mathbb{B}^5$ for which the minimizers of (1.10) are forced to have singularities. Therefore in general (even in the case when the connection corresponding to ϕ does not have nontrivial topology) we cannot expect the minimizers of (1.10) to be smooth, and the optimal regularity space for them is thus $\mathcal{R}^\infty_\phi(\mathbb{B}^5)$.

1.10 Further remarks and conjectures

Note that the requirement (1.22) for all $\eta \in C^\infty_0(\mathbb{B}^5, \wedge^1\mathbb{B}^5 \otimes \mathfrak{g})$ is equivalent to the fact that the equation

$$d(*F) + [*F, A] = 0 \tag{1.24}$$

holds in the sense of distributions. We say that $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ is a *weak Yang-Mills connection* if (1.24) holds in the sense of distributions.

The related works [34], [49], [47] proved regularity results analogous to our Corollary 1.13 under stronger assumptions, e.g. requiring the limit connection to be approximable in some sense. Our main contribution in this direction is indeed the approximability Theorem 1.10, which allows to extend such results to the space of weak connections on singular bundles $\mathcal{A}_G(\mathbb{B}^5)$.

As a consequence of our strong convergence result as in Theorem 1.10 we obtain the following

Proposition 1.15 (Bianchi identity for weak curvatures)**.** *Assume that A, F are the L^2 curvature and connection forms corresponding to a weak connection class $[A] \in \mathcal{A}_G(\mathbb{R}^5)$. Then the equation*

$$d_A F := dF + [F, A] = 0 \tag{1.25}$$

holds in the sense of distributions.

Take now $G = U(n)$. Observe that in this case we have[1]

$$d\,(\mathrm{tr}(F)) = 0 \quad \text{ in } \mathcal{D}'(\mathbb{B}^5)\ ,$$

[1] This was not the case for the space of weak $U(1)$-curvatures $\mathcal{F}_\mathbb{Z}(\mathbb{B}^3)$ introduced in [38].

but if $[A] \in \mathcal{R}^\infty(\mathbb{B}^5)$ then it is not true anymore, as in the smooth case, that the form $d\,(\mathrm{tr}(F \wedge F))$ representing the second Chern classis equal to zero. We have indeed

$$d\,(\mathrm{tr}(F \wedge F)) = 8\pi^2 \sum_{i=1}^{k} d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbb{B}^5)\ ,$$

where

$$d_i = \int_{\partial B_r(a_i)} \mathrm{tr}(F \wedge F) \in \mathbb{Z}$$

represent the degrees of topological singularities situated at the points $a_1 \ldots, a_k$. For a general element $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ one can then ask "how many" such topological singularities exist.

Following the procedure of [31], [30] (in which our approximation theorem is stated as a conjecture) one obtains using the new result of Theorem 1.8 the following:

Theorem 1.16 (see [30],[31]). *If F is a curvature form of a connection A with $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ then there exists a rectifiable integral 1-current I such that*

$$\partial I = \frac{1}{8\pi^2} d(tr(F \wedge F)), \quad \mathbb{M}(I) \leq C\|F\|_{L^2(\mathbb{B}^5)}.$$

where C is a universal constant.

Following the seminal works of Brezis, Coron and Lieb [10] and of Giaquinta, Modica and Souček [19], we can define the relaxed energy for connection classes $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ in terms of their curvature form F as a supremum is taken over 1-Lipschitz functions ξ over $\mathbb{B}^5$:

$$\mathcal{YM}_{rel}(F) := \int_{\mathbb{B}^5} |F|^2 + \sup_{|d\xi|_\infty \leq 1} \left[\int_{\mathbb{B}^5} d\xi \wedge \mathrm{tr}(F \wedge F) - \int_{\mathbb{S}^4} \xi\ \mathrm{tr}(F \wedge F) \right]\ . \tag{1.26}$$

In [26] it was proven that the minimization of $\mathcal{YM}_{rel}$ over $\mathcal{R}^{\infty,\phi}(\mathbb{B}^5)$ presents a gap phenomenon analogous to the celebrated one in the theory of harmonic maps [9], [8]. We expect the relaxed energy to be lower-semicontinuous in $\mathcal{A}_G(\mathbb{B}^5)$, in particular it is natural to ask :

$$\forall\, \phi \in \mathcal{A}_G(\mathbb{S}^4) \ \text{ is } \inf_{\mathcal{A}_G^\phi(\mathbb{B}^5)} \mathcal{YM}_{rel}(F_A) \quad \text{achieved ?}$$

Using the relaxed energy

$$\mathcal{YM}_{rel}(F, G) = \int_{\mathbb{B}^5} |F|^2 + \sup_{|d\xi|_\infty \leq 1} \int_{\mathbb{B}^5} d\xi \wedge [\mathrm{tr}(F \wedge F) - \mathrm{tr}(G \wedge G)]\ ,$$

and following the main lines of [42] one should be able to construct weak Yang-Mills curvatures F corresponding to $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ of arbitrarily small Yang-Mills energy and such that the topological singular set is dense:

$$\mathrm{spt}\,(d\,(\mathrm{tr}(F \wedge F))) = \overline{\mathbb{B}^5}\ .$$

In other words, one should be able to construct *everywhere discontinuous Yang-Mills connections.*

We may define $\mathcal{A}_G(\mathbb{B}^n)$ in a stratifying way : by requiring that $A \in L^2$, $F \in L^2$ and for all centers x and almost all radii $r > 0$ the restriction $i^*_{\partial B_r(x)} A$ belongs, up to measurable gauge and rescaling, to $\mathcal{A}_G(\mathbb{S}^{n-1})$. This definition extends to compact Riemannian n-manifolds by requiring A to be locally equivalent to a form in $\mathcal{A}_G(\mathbb{B}^n)$.

We prove in a future work [40] that the techniques and proofs of our main results in the present paper extend to general compact riemannian manifolds and to higher dimension. It is then natural to adress the regularity conjecture made by Tian [49] for Ω-self dual curvatures to our $\mathcal{A}_G$-type spaces:

Conjecture 1 (**Tian's regularity conjecture**). *Assume Ω is a closed differential $(n-4)$-form on a compact n-dimensional Riemannian manifold M. Curvature forms corresponding to classes $[A] \in \mathcal{A}_G(M)$ satisfying $\Omega \wedge F = *F$ have a singular set of Hausdorff dimension $\leq n-6$.*

Since Ω-instantons belonging to $\mathcal{A}_G$ are stationary, up to now we can only prove using Corollary 1.13 that $\mathcal{H}^{n-4}(\text{sing}(F)) = 0$. The resolution of this conjecture would be of particular geometric interest on Calabi Yau 4-folds where Ω is a parallel form invariant by the special holonomy (see [16] and [49]).

1.11 Plan of the paper

The paper is organized as follows.

In Section 2 we prove the approximation results of Theorem 1.8 and of Theorem 1.10. In Section 3 we prove an extension of the point removability result in dimension 4 which is analogous to the result of [51] but relaxes the hypotheses that the connections are Yang-Mills, utilizing instead the theory from [43] based on lorentz space techniques and on the Coulomb gauge equation. This allows to obtain compactness result for general sequences of connections, which was not present in the literature before, and is needed in the proof of weak closure of section 4.
In Section 4 we prove Proposition 1.9 and the weak closure theorem 1.4.
In Section 5 we prove the regularity results of Theorem 1.12, Corollary 1.13 and Theorem 1.14. At the beginning of the section we include a short proof of Proposition 1.15.
In Section 6 we prove the properties of the trace stated in Theorem 1.6.
The Appendix A is dedicated to a modification of the Coulomb gauge extraction of K. Uhlenbeck [51] which is needed in Section 2 for the proof of the approximation under Morrey norm smallness of Theorem 1.10.

2 Approximation of nonabelian curvatures in 5 dimensions

In this section we prove the fact that weak curvatures F corresponding to classes $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ can be strongly approximated up to gauge by smooth curvatures on bundles with finitely many defects. We consider the class

$$\mathcal{R}^\infty(\mathbb{B}^5) := \left\{ \begin{array}{c} F \text{ curvature form s.t. } \exists k, \exists a_1, \ldots, a_k \in \mathbb{B}^5, \\ F = F_\nabla \text{ for a smooth connection} \nabla \\ \text{on some smooth } G\text{-bundle } E \to \mathbb{B}^5 \setminus \{a_1, \ldots, a_k\} \end{array} \right\} . \tag{2.1}$$

2.1 Approximation on balls with small boundary energy

In this section we prove the extension result which will help to define our approximating connections. We consider the scale $r = 1$.

Proposition 2.1. *Let $F \in L^2(\mathbb{B}^5_2, \wedge^2\mathbb{R}^5 \otimes \mathfrak{g})$ and $A \in L^2(\mathbb{B}^5_2, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$ be such that in the sense of distributions*

$$F = dA + A \wedge A \quad \text{on } \mathbb{B}^5_2 \ .$$

Fix also a constant $\bar{F} \in \wedge^2\mathbb{R}^5 \otimes \mathfrak{g}$ and a constant $\bar{A} \in \wedge^1\mathbb{R}^5 \otimes \mathfrak{g}$. There exists a constant $\epsilon_0 > 0$ independent of the other choices such that if

$$\int_{\mathbb{S}^4} |F|^2 < \epsilon_0, \quad \int_{\mathbb{S}^4} |A|^2 < \epsilon_0, \quad |\bar{A}|^2 < \epsilon_0$$

then there exists $\hat{A} \in L^2(\mathbb{B}^5_2, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$ and $\hat{g} : \mathbb{B}^5 \to G$ such that:

- *$i^*_{\mathbb{S}^4}\hat{A} = i^*_{\mathbb{S}^4}A$ and $\hat{A} = A$ outside $\mathbb{B}^5$,*
- *$\hat{g}(\hat{A})$ is smooth in the interior of $\mathbb{B}^5$,*
- *there holds*

$$\|d\hat{A} + \hat{A} \wedge \hat{A} - \bar{F}\|^2_{L^2(\mathbb{B}^5)} \lesssim \epsilon_0(\|\bar{F}\|^2_{L^2(\mathbb{B}^5)} + \|F\|^2_{L^2(\mathbb{S}^4)}) + \|F - \bar{F}\|^2_{L^2(\mathbb{S}^4)} \ . \tag{2.2}$$

and

$$\|\hat{A} - \bar{A}\|_{L^2(\mathbb{B}^5)} \leq C\|A - \bar{A}\|_{L^2(\mathbb{S}^4)} \ . \tag{2.3}$$

Moreover we have that

- *If $F \in \mathcal{A}_G$ then $F_{\hat{A}} \in \mathcal{A}_G$,*

- *If $U_i \subset \mathbb{B}^5_2$ is open and $i^*_{\mathbb{S}^4} A$ is continuous on $U_i \cap \mathbb{S}^4$ then $\hat{A}, \hat{g}$ are continuous on $U_i \cap \overline{\mathbb{B}^5}$.*

Proof. **Step 1.** *Coulomb gauge on the boundary.* Let g be the change of gauge g given by Theorem A.1 such that

$$\begin{cases} d^*_{\mathbb{S}^4}\pi(A^g) = d^*_{\mathbb{S}^4}(g^{-1}dg + \pi(g^{-1}Ag)) = 0 \ , \\ \|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leq C(\|F\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}) \ . \end{cases} \tag{2.4}$$

From the equation defining A^g, namely

$$A^g = g^{-1}dg + g^{-1}Ag \ ,$$

we obtain (in our notation we identify 1-forms and vector fields using the metric)

$$\begin{aligned} \Delta_{\mathbb{S}^4} g &= d^*_{\mathbb{S}^4}(g\,A^g - A\,g) \\ &= dg \cdot A^g + (g - id)\,d^*_{\mathbb{S}^4}A^g + d^*_{\mathbb{S}^4}A^g \\ &\quad - d^*_{\mathbb{S}^4}[(A - \bar{A})\,g] - d^*_{\mathbb{S}^4}[\bar{A}\,(g - id)] - d^*_{\mathbb{S}^4}\bar{A} \\ &= dg \cdot A^g + (g - id)\,d^*_{\mathbb{S}^4}A^g - d^*_{\mathbb{S}^4}[(A - \bar{A})\,g] - d^*_{\mathbb{S}^4}[\bar{A}\,(g - id)] + \\ &\quad + d^*_{\mathbb{S}^4}\left(\sum_{k=1}^{5} i^*_{\mathbb{S}^4}dx_k \fint_{\mathbb{S}^4} \langle i^*_{\mathbb{S}^4}(\bar{A} - A^g), i^*_{\mathbb{S}^4}dx_k\rangle\right) \\ &= dg \cdot A^g + (g - id)\,d^*_{\mathbb{S}^4}A^g - d^*_{\mathbb{S}^4}[(A - \bar{A})\,g] - d^*_{\mathbb{S}^4}[\bar{A}\,(g - id)] + \\ &\quad + 5\sum_{k=1}^{5} x_k \fint_{\mathbb{S}^4} \langle i^*_{\mathbb{S}^4}(\bar{A} - g^{-1}Ag), i^*_{\mathbb{S}^4}dx_k\rangle \ , \end{aligned}$$

where in the last row we used the fact that $\int_{\mathbb{S}^4}\langle i^*_{\mathbb{S}^4}(g^{-1}dg), i^*_{\mathbb{S}^4}dx_k\rangle = 0$. Note that if $\bar{g}$ is the average of g on $\mathbb{S}^4$ taken in $\mathbb{R}^5$, then using the mean value formula there exists $x \in \mathbb{S}^4$such that $|g(x) - \bar{g}| \leq C\|g - \bar{g}\|_{L^2}$ and up to changing g to gg_0 where g_0 is a constant rotation, we may also assume $g(x) = id$. Now by elliptic estimates and using the embedding $W^{-1,2} \to L^{4/3}$ and the Hölder estimate $\|ab\|_{L^{4/3}} \leq \|a\|_{L^2}\|b\|_{L^4}$ we deduce:

$$\begin{aligned} \|dg\|^2_{L^2(\mathbb{S}^4)} &\lesssim \|dg\|^2_{L^2}\|A^g\|^2_{L^4} + \|g - id\|^2_{L^4}\|A^g\|^2_{L^4} \\ &\quad + \|A - \bar{A}\|^2_{L^2} + \|g - id\|^2_{L^4}\|\bar{A}\|^2_{L^2} + \|\bar{A} - A\|^2_{L^2}\|g - id\|^2_{L^2} \ . \end{aligned}$$

Utilizing the Sobolev inequality $\|g - id\|_{L^4} \lesssim \|dg\|_{L^2}$ and the facts that

$$\begin{aligned} \|A^g\|^2_{L^4} &\lesssim \|F\|^2_{L^2} + \|A\|^2_{L^2} \lesssim \epsilon_0 \ , \\ \|\bar{A}\|^2_{L^p} &\lesssim \epsilon_0 \ , \end{aligned}$$

we absorb the terms not containing $A-\bar{A}$ from the right hand side to the left hand side. For $\epsilon_0>0$ small enough we thus obtain

$$\|dg\|_{L^2(\mathbb{S}^4)}\le C\|A-\bar{A}\|_{L^2}. \tag{2.5}$$

We have using (2.5) and the fact that $\overline{F}$ is constant

$$\int_{\mathbb{S}^4}|g^{-1}i^*_{\mathbb{S}^4}\overline{F}g-i^*_{\mathbb{S}^4}\overline{F}|^2\le 4\,|\overline{F}|^2\int_{\mathbb{S}^4}|g-id|^2\lesssim\epsilon_0\,\|\overline{F}\|^2_{L^2(\mathbb{B}^5_2)}\ .$$

Since $F_{A^g}=g^{-1}\,F\,g$, using the previous identity we obtain

$$\int_{\mathbb{S}^4}|F_{A^g}-i^*_{\mathbb{S}^4}\overline{F}|^2\lesssim\epsilon_0\,\|\overline{F}\|^2_{L^2(\mathbb{B}^5)}+\int_{\mathbb{S}^4}|F-i^*_{\mathbb{S}^4}\overline{F}|^2\ . \tag{2.6}$$

Using now the last line of (2.4) we obtain

$$\int_{\mathbb{S}^4}|F_{A^g}-dA^g|^2\ \ \le\ \ \int_{\mathbb{S}^4}|A^g|^4\lesssim\|F\|^4_{L^2(\mathbb{S}^4)}+\|A\|^4_{L^2(\mathbb{S}^4)}\ .$$

Combining this with (2.6) we obtain

$$\begin{aligned}\textstyle\int_{\mathbb{S}^4}|dA^g-i^*_{\mathbb{S}^4}\overline{F}|^2\ \ &\lesssim\ \ \epsilon_0\,\|\overline{F}\|^2_{L^2(\mathbb{B}^5)}+\textstyle\int_{\mathbb{S}^4}|F-i^*_{\mathbb{S}^4}\overline{F}|^2+\\ &\quad+\|F\|^4_{L^2(\mathbb{S}^4)}+\|A\|^4_{L^2(\mathbb{S}^4)}\ .\end{aligned} \tag{2.7}$$

Step 2. *Extension to the interior.* For any 1-form η in $W^{1,2}(\mathbb{S}^4)$ we denote by $\tilde{\eta}$ the unique solution of the following minimization problem

$$\inf\left\{\int_{\mathbb{B}^5}|dC|^2+|d^{*_{\mathbb{R}^5}}C|^2\ dx^5\quad C\in W^{1,2}(\wedge^1\mathbb{B}^5)\quad i^*_{\mathbb{S}^4}C=\eta\right\}\ . \tag{2.8}$$

A classical argument shows that it is uniquely given by

$$\left\{\begin{array}{l}d^{*_{\mathbb{R}^5}}\tilde{\eta}=0\qquad\text{in }\mathbb{B}^5\ ,\\[2mm] d^{*_{\mathbb{R}^5}}(d\tilde{\eta})=0\qquad\text{in }\mathbb{B}^5\ ,\\[2mm] i^*_{\mathbb{S}^4}\tilde{\eta}=\eta\qquad\text{on }\partial\mathbb{B}^5\ ,\end{array}\right. \tag{2.9}$$

and one has

$$\|\tilde{\eta}\|_{L^5(\mathbb{B}^5)}\le C\,\|\nabla\tilde{\eta}\|_{W^{3/2,2}(\mathbb{B}^5)}\le C\,\|\eta\|_{W^{1,2}(\mathbb{S}^4)}\ . \tag{2.10}$$

Let

$$B:=\sum_{i<j}\overline{F_{ij}}\ \frac{x_i\,dx_j-x_j\,dx_i}{2}\ . \tag{2.11}$$

Observe that

$$\begin{cases} d^{*_{\mathbb{R}^5}} B = 0 & \text{in } \mathbb{B}^5 \ , \\ d^{*_{\mathbb{R}^5}}(dB) = 0 & \text{in } \mathbb{B}^5 \ . \end{cases}$$

Thus B is the solution to (2.8) for its restriction to the boundary : $i^*_{\mathbb{S}^4} B$

$$\widetilde{i^*_{\mathbb{S}^4} B} = B \ .$$

Observe that $< B, dr > \equiv 0$ and $d^{*_{\mathbb{R}^5}} B = 0$ therefore

$$d^{*_{\mathbb{S}^4}} \left(i^*_{\mathbb{S}^4} B \right) \equiv 0 \qquad \text{on } \mathbb{S}^4 \ . \tag{2.12}$$

We apply the same extension technique $\eta \mapsto \tilde{\eta}$ to $\eta = \pi(A^g)$ obtaining a 1-form $\widetilde{\pi(A^g)}$ satisfying the analogues of (2.9). We also define the constant 1-form

$$\overline{A^g} := \sum_{k=1}^{5} dx_k \fint_{\mathbb{S}^4} \langle A^g, i^*_{\mathbb{S}^4} dx_k \rangle$$

and we note

$$\tilde{A}^g = \widetilde{\pi(A^g)} + \ \overline{A^g} \ .$$

Step 3. *Estimates on the extended curvatures.* Note that $d\pi(A^g) = dA^g$ since $\overline{A^g}$ is constant. Using (2.5), (2.12) and (2.7) we have that by Hodge inequality

$$\begin{aligned} \|\pi(A^g) - i^*_{\mathbb{S}^4} B\|^2_{W^{1,2}(\mathbb{S}^4)} &\le C \int_{\mathbb{S}^4} |d(\pi(A^g) - i^*_{\mathbb{S}^4} B)|^2 \\ &= \int_{\mathbb{S}^4} |dA^g - i^*_{\mathbb{S}^4} \overline{F}|^2 \le C \ \epsilon_0 \ \|\overline{F}\|^2_{L^2(\mathbb{B}^5)} + \\ &+ C \int_{\mathbb{S}^4} |F - i^*_{\mathbb{S}^4} \overline{F}|^2 + C \ \|F\|^4_{L^2(\mathbb{S}^4)} + C \ \|A\|^4_{L^2(\mathbb{S}^4)} \ . \end{aligned} \tag{2.13}$$

Combining now (2.10) and (2.13) we obtain

$$\begin{aligned} \|d\tilde{A}^g - \overline{F}\|^2_{L^2(\mathbb{B}^5)} &= \|d\widetilde{\pi(A^g)} - \overline{F}\|^2_{L^2(\mathbb{B}^5)} \\ &\le C \int_{\mathbb{S}^4} |d(A^g - i^*_{\mathbb{S}^4} B)|^2 \le C \ \epsilon_0 \ \|\overline{F}\|^2_{L^2(\mathbb{B}^5)} + \\ &+ C \int_{\mathbb{S}^4} |F - i^*_{\mathbb{S}^4} \overline{F}|^2 + C \ \|F\|^4_{L^2(\mathbb{S}^4)} \ + C \ \|A\|^4_{L^2(\mathbb{S}^4)} \ . \end{aligned} \tag{2.14}$$

Using (2.10) again, we obtain

$$\|\tilde{A}^g \wedge \tilde{A}^g\|^2_{L^2(\mathbb{B}^5)} \lesssim \|\tilde{A}^g\|^4_{L^4(\mathbb{B}^5)} \le \ \|A^g\|^4_{W^{1,2}(\mathbb{S}^4)} \le C \, \|F\|^4_{L^2(\mathbb{S}^4)} + C \ \|A\|^4_{L^2(\mathbb{S}^4)} \ . \tag{2.15}$$

Combining (2.14) and (2.15) we obtain

$$\begin{aligned} \|d\tilde{A}^g + \tilde{A}^g \wedge \tilde{A}^g - \overline{F}\|^2_{L^2(\mathbb{B}^5)} &\le C \ \epsilon_0 \ \|\overline{F}\|^2_{L^2(\mathbb{B}^5)} + \\ &+ C \int_{\mathbb{S}^4} |F - i^*_{\mathbb{S}^4} \overline{F}|^2 + C \ \|F\|^4_{L^2(\mathbb{S}^4)} + C \ \|A\|^4_{L^2(\mathbb{S}^4)} \ . \end{aligned} \tag{2.16}$$

Step 4. *Correcting the restriction on the boundary.* Extend now g radially in $\mathbb{B}^5$ and denote by $\hat{g}$ this extension. We have using (2.5)

$$
\begin{aligned}
\int_{\mathbb{B}^5} |\hat{g}^{-1}\overline{F}\hat{g} - \overline{F}|^2 &\leq 4\ |\overline{F}|^2 \int_{\mathbb{B}^5} |\hat{g} - id|^2\ dx^5 \\
&\leq C\,\|\overline{F}\|^2_{L^2(\mathbb{B}^5)} \int_{\mathbb{S}^4} |g - id|^2 \leq\ C\ \epsilon_0\ \|\overline{F}\|^2_{L^2(\mathbb{B}^5)}\ .
\end{aligned}
\tag{2.17}
$$

Combining (2.16) and (2.17) gives

$$
\begin{aligned}
\|d\tilde{A}^g + \tilde{A}^g \wedge \tilde{A}^g - \hat{g}^{-1}\overline{F}\hat{g}\|^2_{L^2(\mathbb{B}^5)} &\leq C\ \epsilon_0\ \|\overline{F}\|^2_{L^2(\mathbb{B}^5)} + \\
&+ C \int_{\mathbb{S}^4} |F - i^*_{\mathbb{S}^4}\overline{F}|^2 + C\ \|F\|^4_{L^2(\mathbb{S}^4)} + C\ \|A\|^4_{L^2(\mathbb{S}^4)}\ .
\end{aligned}
$$

Denote $\hat{A} := (\tilde{A}^g)_{\hat{g}^{-1}} := \hat{g}\tilde{A}^g\hat{g}^{-1} + \hat{g}d\,\hat{g}^{-1}$. Observe that with this notation one has

$$
F_{\hat{A}} = \hat{g}\,F_{A^g}\,\hat{g}^{-1}\ .
$$

This one form $\hat{A}$ extends A in $\mathbb{B}^5$, there is a gauge in which it is <u>smooth</u> and we have the desired estimate (2.2). Note also that $i^*_{\mathbb{S}^4}[\hat{A} = i^*_{\mathbb{S}^4}(\tilde{A}^g)_{\hat{g}^{-1}}] = (i^*_{\mathbb{S}^4}\tilde{A}^g)_{\hat{g}^{-1}} = i^*_{\mathbb{S}^4}A$. Then define $\hat{A} = A, \hat{g} = g$ outside $\mathbb{B}^5$. Since $i^*_{\mathbb{S}^4}(\hat{A} - A) = 0$ we obtain via integration by parts that the distributional expression of $F_{\hat{A}}$ is L^2.

Step 5. *Verifying the compatibility conditions.* We notice that if $i^*_{\mathbb{S}^4}A$ is C^0 on $U_i \cap \mathbb{S}^4$ then so is any of its Coulomb gauges g by Proposition 2.2 below and thus the radial and harmonic extensions $\hat{A}, \hat{g}$ are continuous up to the boundary, verifying our second compatibility statement.

For the first statement, suppose given $S = \partial B(x, \rho)$ such that $i^*_S A \in \mathcal{A}_G(S)$, $i^*_S F \in L^2$. Define $S^+ := \overline{S \cap \mathbb{B}^5}$. Consider a local $W^{1,2}$ gauge g_i on a chart U_i of S intersecting ∂S^+ such that $g_+(i^*_{S^+}\hat{A})$ is $W^{1,2}$ on U_i. Then $g_i\hat{g}^{-1}(\tilde{A}^g)$ is $W^{1,2}$ on $U_i \cap S^+$ and has the same trace as $g_i(A)$ on ∂S^+. Thus $g_i(\hat{A})$ is also $W^{1,2}$ on the whole of U_i as desired.

Step 6. *Verification of* (2.3). We now use the formula for $\hat{A}$ from the previous step, as well as the estimates (2.15) and (2.5) to prove the following sequence of estimates:

$$
\begin{aligned}
\|\hat{A} - \bar{A}\|^2_{L^2(\mathbb{B}^5)} &\lesssim \int_{\mathbb{B}^5} |d\hat{g}|^2 + \|\hat{g} - id\|^2_{L^4(\mathbb{L}^4)} \|\bar{A} - \tilde{A}^g\|^2_{L^4(\mathbb{B}^5)} \\
&\lesssim (1 + \epsilon_0)\left(\|dg\|^2_{L^2(\mathbb{S}^4)} + \|g - id\|^2_{L^4(\mathbb{S}^4)}\right) \\
&\lesssim \|A - \bar{A}\|_{L^2(\mathbb{S}^4)}\ .
\end{aligned}
$$

This concludes the proof. □

The following result was used in Step 5 above:

Proposition 2.2 ([30] Prop. 3.4)**.** , *Suppose that B is a smooth connection on a 4-dimensional manifold M and that $A_C = g^{-1}dg + g^{-1}Bg$ is a $W^{1,2}$ Coulomb gauge then also g (and thus B_C) is smooth.*

The proof of the above proposition goes as follows: by Lorentz space theory (see [43]) we obtain that if $A_C, B \in W^{1,2}, d^*A_C = 0$ then $g \in W^{2,2} \cap C^0$ (this is analogue to the 2-dimensional Wente lemma). This regularity for g allows to apply classical elliptic theory to the elliptic system issued from $d^*(g^{-1}dg) = d^*(g^{-1}A_C g)$ and to conclude by bootstrap.

2.1.1 Approximation under a smallness condition on F only

In this section we state a modification of Proposition 2.1 which can be applied when only a bound on F and not one on A is available. This modification will prove useful for Theorem 1.10.

Proposition 2.3 (modified version of Prop. 2.1)**.** *Let $F \in L^2(\mathbb{B}^5_2, \wedge^2\mathbb{R}^5 \otimes \mathfrak{g})$ and $A \in L^2(\mathbb{B}^5_2, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$ be such that in the sense of distributions*

$$F = dA + A \wedge A \quad \text{on } \mathbb{B}^5_2 \ .$$

Fix also a constant $\bar{F} \in \wedge^2\mathbb{R}^5 \otimes \mathfrak{g}$. There exists a constant $\epsilon_0 > 0$ independent of the other choices such that if

$$\int_{\mathbb{S}^4} |F|^2 < \epsilon_0$$

then there exists $\hat{A} \in L^2(\mathbb{B}^5_2, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$ and $\hat{g} : \mathbb{B}^5 \to G$ such that:

- *$i^*_{\mathbb{S}^4}\hat{A} = i^*_{\mathbb{S}^4}A$ and $\hat{A} = A$ outside $\mathbb{B}^5$,*
- *$\hat{g}(\hat{A})$ is smooth in the interior of $\mathbb{B}^5$,*
- *there holds*
$$\|d\hat{A} + \hat{A} \wedge \hat{A}\|^2_{L^2(\mathbb{B}^5)} \lesssim \|F\|^2_{L^2(\mathbb{S}^4)} \tag{2.18}$$
and
$$\|\hat{A}\|_{L^2(\mathbb{B}^5)} \leq \|F\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)} \ . \tag{2.19}$$

Proof. We follow the proof of Proposition 2.1, with slightly less refined estimates.
Step 1. *Classical Coulomb gauge on the boundary.* Let g be the Coulomb gauge as constructed by Uhlenbeck [51], i.e. such that

$$\begin{cases} d^*_{\mathbb{S}^4}A^g = d^*_{\mathbb{S}^4}(g^{-1}dg + g^{-1}Ag) = 0 \ , \\ \|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leq C\|F\|_{L^2(\mathbb{S}^4)} \ . \end{cases}$$

We deduce using the definition of A^g that

$$\|dg\|^2_{L^2(\mathbb{S}^4)} \leq C\left(\|A^g\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)}\right) \lesssim \|F\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)} \ .$$

Steps 2-3. *Estimates for the extensions.* We define B as in Proposition 2.1 and $\tilde{A}^g$ will be the similar extension of A^g. By elliptic and Hodge estimates using the fact that $d^*_{\mathbb{S}^4}\tilde{A}^g = 0$ we obtain

$$\|d\tilde{A}^g\|_{L^2(\mathbb{B}^5)} \lesssim \|F\|^2_{L^2(\mathbb{S}^4)}$$

and

$$\|\tilde{A}^g \wedge \tilde{A}^g\|_{L^2(\mathbb{B}^5)} \leq \|\tilde{A}^g\|^4_{L^4(\mathbb{B}^5)} \lesssim \|A^g\|^4_{L^4(\mathbb{S}^4)} \lesssim \epsilon_0 \|F\|^2_{L^2(\mathbb{S}^4)} .$$

These estimate give

$$\|F_{\tilde{A}^g}\|^2_{L^2(\mathbb{B}^5)} \lesssim \|F\|^2_{L^2(\mathbb{S}^4)} .$$

Step 4. *Correcting the extension on the boundary.* We consider the harmonic extension $\tilde{g}$ to g. Note that $W^{1,2}(\mathbb{B}^5, G)$ is the strong $W^{1,2}$-closure of $C^\infty(\mathbb{B}^5, G)$ since $\pi_2(G) = 0$, therefore the extension exists and is smooth. We also have the estimates

$$\|\tilde{g} - id\|^2_{L^2(\mathbb{B}^5)} \lesssim \|d\tilde{g}\|^2_{L^2(\mathbb{S}^4)} \lesssim \|dg\|^2_{L^2(\mathbb{S}^4)} \lesssim \|F\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)} ,$$

thus if we define $\hat{A} = \tilde{g}\tilde{A}^g\tilde{g}^{-1} + \tilde{g}d\tilde{g}^{-1}$ it follows that

$$\begin{aligned} \|\hat{A}\|^2_{L^2(\mathbb{B}^5)} &\lesssim \|\tilde{A}^g\|^2_{L^2(\mathbb{B}^5)} + \|d\hat{g}\|^2_{L^2(\mathbb{B}^5} \\ &\lesssim \|F\|^2_{L^2(\mathbb{S}^4)} + \|A\|^2_{L^2(\mathbb{S}^4)} . \end{aligned}$$

□

2.2 Smoothing in 4-dimensions

Before applying the above extension result we will always use the following classical result for $p = 2, n = 4, X = \mathbb{S}^4$:

Lemma 2.4. *Let $p \geq n/2$ and let A be a $W^{1,p}$ connection over an n-dimensional manifold X. Let K be a (possibly empty) compact set on which A is C^0. Then there exists a sequence A_η of C^0 connections over X such that $A_\eta|_K = A|_K$ and*

$$\lim_{\eta\to 0} \|A_\eta - A\|_{W^{1,p}(X)} = 0 \quad \textit{and} \quad \lim_{\eta\to 0} \|F_{A_\eta} - F_A\|_{W^{1,p}(X)} = 0 .$$

Proof. If we had just functions $f, f_\eta : X \to \wedge^1\mathbb{R}^n \otimes \mathfrak{g}$ in our statement, then the result would be classical (even without the restriction on p) and it would suffice to mollify f in order to obtain approximants $f_\eta = f * \rho_\eta$ where ρ_η is a scale-η smooth mollifier.
The problem which we face is just the fact that A is not globally defined: we have instead local expressions A_i in the chart U_i, and we must mollify A_i to $A_{i,\eta}$ for which $A_{i,\eta} = g_{ij}^{-1}dg_{ij} + g_{ij}^{-1}A_{j,\eta}g_{ij} := g_{ij}(A_{j,\eta})$ are still true. We use a partition of unity $(\theta_i)_i$ adapted to the charts U_i and define $\rho_\eta(x) = \eta_x^{-n}\rho(x/\eta_x)$, where $\eta_x := \min\{\eta, \mathrm{dist}(x, K)/2\}$. Then we define

$$(A_\eta)_i = \theta_i A_i * \rho_\eta + \sum_{i'\neq i} \theta_{i'} g_{ii'}(A_{i'} * \rho_\eta) .$$

By the cocycle condition $g_{ii'}g_{i'j} = g_{ij}$ we obtain the desired $(A_\eta)_i = g_{ij}((A_\eta)_j)$. The derivatives of θ_i enter the estimate of $\|A_\eta - A\|_{W^{1,p}(X)}$ introducing a possibly huge L^∞ factor, however this factor is independent on η. We therefore have $\lim_{\eta\to 0}\|A_{i,\eta} - A_i\|_{W^{1,p}} = 0$. The restriction on the exponent p is needed in to prove the convergence of curvatures. This is based on the following inequality:

$$\begin{aligned} \|F_A - F_B\|_{L^p} &\lesssim \|dA - dB\|_{L^p} + \|(A-B)\wedge A\|_{L^p} + \|(A-B)\wedge B\|_{L^p} \\ &\lesssim \|DA - DB\|_{L^p} + \|A-B\|_{L^{2p}}(\|A\|_{L^{2p}} + \|B\|_{L^{2p}})\ . \end{aligned}$$

We are able to conclude using the $W^{1,p}$-convergence of the A_η because we have the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$ valid precisely when $p \geq n/2$. We leave the details of the proof to the reader. $\square$

2.3 Good grids and good balls

In order to detect the regions where to apply the approximation step of the previous section we construct controlled families of balls which depend on F and on its L^2 connection A and are used for the approximation.

2.3.1 Good grids

We thus define our basic object:

Definition 2.5. *Assume that $\Lambda \subset \mathbb{R}^5$ is a discrete set and $1 < \alpha < 2$ is a constant such that the balls $B_1(p), p \in \Lambda$ cover $\mathbb{R}^5$ and for each $p \in \Lambda$ the only ball of the form $B_\alpha(q), q \in \Lambda$ covering p is the one with $q = p$. Fix a scale $r > 0$. A collection of balls $B_i = B_{r_i}(x_i)$ with $r_i \in [r, \alpha r]$ and $\{x_i\} = r\Lambda \cap \mathbb{B}^5$ will be called a* grid of balls of scale r.

$\Lambda, \alpha \in]1,2[$ as above can be found, e.g. we may take Λ to be a body-centered cubic lattice:

$$\Lambda = \beta^{-1}\left[2\mathbb{Z}^5 \cup ((1,\dots,1) + 2\mathbb{Z}^5)\right], \quad \alpha \in]1, 2/\beta[, \quad \beta \in]\sqrt{5}/2, 2[\ .$$

α, Λ will be fixed from now on; their only role is to ensure that for any choice of r_i in the allowed the balls of the grid cover $\mathbb{B}^5$. We can choose the r_i above such that a good control on the boundary of our grids is available:

Proposition 2.6. *Let $F \in L^2(\mathbb{B}^5, \wedge^2\mathbb{R}^5 \otimes \mathfrak{g})$ and $A \in L^2(\mathbb{B}^5, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$. For each fixed scale $r > 0$ pick the finitely many radii $r_i \in [r, \alpha r]$ uniformly and independently at random.*

There exist a constant C depending only on the dimension and a modulus of continuity $o(r)$ depending only on F such that at fixed r the following hold with positive probability:

$$r\sum_i \int_{\partial B_i} |F|^2 \leq C\int_{\mathbb{B}^5} |F|^2\ , \tag{2.20}$$

$$r\sum_i \int_{\partial B_i} |A|^2 \le C \int_{\mathbb{B}^5} |A|^2 \tag{2.21}$$

and, with the notation $\overline{F}_i := \fint_{B_{\alpha r}(x_i)} F$,

$$r\sum_i \int_{\partial B_i} |F - \overline{F}_i|^2 \le o(r)\ , \tag{2.22}$$

$$r\sum_i \int_{\partial B_i} |A - \overline{A}_i|^2 \le o(r)\ . \tag{2.23}$$

Proof. Since the annuli $B_{\alpha r}(x_i)\backslash B_r(x_i)$ can be divided into N families having no overlaps we obtain

$$\int_r^{\alpha r}\left(\sum_i \int_{\partial B_\rho(x_i)} |F|^2\right) d\rho \lesssim \|F\|^2_{L^2(\mathbb{B}^5)}\ ,$$

therefore for randomly picked $r_i \in [r, \alpha r]$

$$r\sum_i \int_{\partial B_{r_i}(x_i)} |F|^2 \lesssim \|F\|^2_{L^2(\mathbb{B}^5)}$$

with probability $\ge 1-X$, where C depends on X, which in turn will be fixed later. This will give (2.22), (2.23). The same reasoning can be applied also to A and we obtain that uniformly chosen $\rho \in [r, 2r]$ satisfies a (2.21) with probability $\ge 1 - X$.

Fix now smooth approximants G^k to F as a function in $L^2(\mathbb{B}^5, \Lambda^2\mathbb{R}^2 \otimes \mathfrak{g})$: assume that

$$\int_{\mathbb{B}^5} |G^k - F|^2 \le \frac{1}{k}\ .$$

Take $o_\infty(r) = \min_k o_k(r)$ for $o_k(r) := \frac{1}{k} + r^2\|G^k\|_{C^1}$. For r such that $o_\infty(r) = o_k(r)$ we apply the above argument to $G^k - F$ and obtain

$$r\sum_i \int_{\partial B_{r_i}(x_i)} |G^k - F|^2 \lesssim \int_{\mathbb{B}^5} |G^k - F|^2$$

with probability $\ge 1 - X$. Let $\bar{G}_i^k := \fint_{B_{\alpha r}(x_i)} G^k$. By a straightforward computation and by Jensen's inequality we have, independently of r,

$$\begin{aligned} r\sum_i \int_{\partial B_{r_i}(x_i)} |\bar{G}_i^k - \bar{F}|^2 &\lesssim \sum_i \int_{B_{\alpha r}(x_i)} |\bar{G}_i^k - \bar{F}_i|^2 \\ &\lesssim \sum_i \int_{B_{\alpha r}(x_i)} |G^k - F|^2 \\ &\lesssim \frac{1}{k}\ . \end{aligned}$$

We then estimate by triangle inequality between $F, \bar{F}, \bar{G}_k, G_k$

$$r \sum_i \int_{\partial B_{r_i}(x_i)} |F - \bar{F}_i|^2 \lesssim \frac{1}{k} + r \sum_i \int_{\partial B_{r_i}(x_i)} |G^k - \bar{G}_i^k|^2 \lesssim o_\infty(r) \ .$$

This shows (2.22) once we take $o(r) = C\ o_\infty(r)$. We proceed similarly to obtain also (2.23) with probability higher than X. For each r each one of the events (2.20), (2.21), (2.22), (2.23) fails with probability $\leq X$ thus their intersection fails with probability $\leq 4X$. We thus choose $X > 1/4$ and conclude the proof. □

The conditions obtained via Proposition 2.6 are contemporarily valid for a positive probability on uniformly chosen radii, thus the new condition of having a $W^{1,2}$ representative of the connection class on each $\partial B_\rho(x_i)$ keeps them valid too.

2.3.2 Good grids for Morrey curvatures

We denote $\|\cdot\|_M$ the following Morrey norm:

$$\|f\|_M^2 := \sup_{x,r} \frac{1}{r} \int_{B_r(x)} |f(y)|^2 dy \ .$$

We next extend the statement of Proposition 2.6 to a situation where we have a Morrey control on F:

Proposition 2.7 (extension of Prop. 2.6)**.** *Consider a grid as in Definition 2.5. Let $F \in L^2(\mathbb{B}^5, \wedge^2\mathbb{R}^5 \otimes \mathfrak{g})$ and $A \in L^2(\mathbb{B}^5, \wedge^1\mathbb{R}^5 \otimes \mathfrak{g})$. For each fixed scale $r > 0$ pick the finitely many radii $r_i \in [r, \alpha r]$ uniformly and independently at random.*

There exist a constant C depending only on the dimension and a modulus of continuity $o(r)$ depending only on F such that at fixed r we have (2.22), (2.23) *and the following, with positive probability:*

$$\int_{\partial B_i} |F|^2 \leq C \frac{1}{r_i} \int_{B_i} |F|^2 \quad \textit{for all } i \tag{2.24}$$

and

$$\int_{\partial B_i} |A|^2 \leq C \frac{1}{r_i} \int_{B_i} |A|^2 \quad \textit{for all } i \ . \tag{2.25}$$

Remark 2.8. *In particular if $\|F\|_M^2 < \infty$ then we directly obtain from* (2.24) *that $\|F\|^2_{L^2(\partial B_i)} \leq C \|F\|_M^2$.*

Proof. We note that in the end of the proof of Proposition 2.6 we had obtained that the estimates (2.22) and (2.23) hold contemporarily with probability at least $1-2X$. In other words the estimates hold once we choose $r_k/r \in I_k \subset [1,\alpha]$ and $\prod_k |I_k| > 1-2X$. In particular all of the I_k satisfy

$$1 \geq |I_k| \geq 1-2X \ . \tag{2.26}$$

We then obtain by Chebychev's inequality that

$$|Y_{C,k}| := \left|\left\{\rho : \int_{\partial B_\rho(x_k)} |F|^2 > \frac{C}{\alpha r}\int_{B_{\alpha r}(x_k)} |F|^2\right\}\right| \leq \frac{\alpha r}{C} \tag{2.27}$$

by recalling that α is bounded from above depending only on the dimension and using (2.26) we see that there exists a choice

$$C \sim \frac{1}{1-2X}$$

which will ensure that for each k there holds $|Y_{C,k}| \leq |I_k| r/2$. Since the number of balls is finite, with positive probability for each k we have (2.22), (2.23) and

$$\int_{\partial B_\rho(x_k)} |F|^2 \leq \frac{C}{\alpha r}\int_{B_{\alpha r}(x_k)} |F|^2 \ ,$$

which implies (2.24). We may similarly ensure (2.25) as well, up to increasing C by a controlled factor. $\square$

2.3.3 Good and bad balls

We intend to apply Proposition 2.1 to B_i belonging to grids as in Proposition 2.6, for F, A as in the definition of $\mathcal{A}_G(\mathbb{B}^5)$ and for $\bar{F} = \bar{F}_i$ on B_i with the notations of Proposition 2.6. In this situation (rescaled versions of) the estimates of Proposition 2.1 are valid for all but few "good" balls. We start by fixing the definition of "good" and "bad":

Lemma-Definition 2.9. *Fix a constant $\delta > 0$ and a scale $r > 0$. Let $A, F, B_i, o(r)$ be as in Proposition 2.6. We say that B_i is a δ-*good ball with respect to $A, F, o(r)$ *if the following bounds hold:*

$$\int_{\partial B_i} |F|^2 \leq \delta \ , \tag{2.28}$$

$$\frac{1}{r^2}\int_{\partial B_i} |A|^2 \leq \delta \ , \tag{2.29}$$

$$\frac{1}{r^2}\int_{\partial B_i} |F - \overline{F}_i|^2 \leq o(r) \ , \tag{2.30}$$

$$\frac{1}{r^2}\int_{\partial B_i} |A-\overline{A}_i|^2 \le o(r)\ . \tag{2.31}$$

In this case we will denote $\mathcal{G}_r$ the set of good balls and $\mathcal{B}_r$ the set of the remaining (so-called "bad") balls of scale r.

The cardinality of $\mathcal{B}_r$ can then be estimated as follows:

$$\#\mathcal{B}_r \lesssim \frac{\|F\|_{L^2(\mathbb{B}^5)}}{\delta r} + \frac{\|A\|_{L^2(\mathbb{B}^5)}}{\delta r^3} + \frac{1}{r}\ .$$

In particular the total volume of the bad balls vanishes as $r \to 0$.

Proof. The second statement follows from the first because the volume of each bad ball is $\sim r^5$. To prove the estimate on $\#\mathcal{B}_r$ we separately estimate the sets $\mathcal{B}_i$ of cubes for which (gi) fails.
Using Proposition 2.6 we then obtain

$$\begin{aligned}
\delta\#\mathcal{B}_1 &\lesssim \sum_{B_i\in\mathcal{B}_1}\int_{\partial B_i}|F|^2 \lesssim \frac{1}{r}\int_{\mathbb{B}^5}|F|^2\ ,\\
\delta r^2\#\mathcal{B}_2 &\lesssim \sum_{B_i\in\mathcal{B}_2}\int_{\partial B_i}|A|^2 \le \frac{1}{r}\int_{\mathbb{B}^5}|A|^2\ ,\\
o(r)\#\mathcal{B}_3 &\lesssim \sum_{B_i\in\mathcal{B}_3}\int_{\partial B_i}|F-\overline{F}_i|^2 \le \frac{o(r)}{r}\ ,\\
o(r)\#\mathcal{B}_4 &\lesssim \sum_{B_i\in\mathcal{B}_4}\int_{\partial B_i}|A-\overline{A}_i|^2 \le \frac{o(r)}{r}\ .
\end{aligned}$$

Since $\mathcal{B} = \cup_{i=1}^4 \mathcal{B}_i$ we obtain the desired result. $\square$

Going back to the r scale by pull backing all forms to the good ball C_r^i using the dilation map $x \to r^{-1}x$, denoting $\hat{A}_r = r^{-1}\sum_{j=1}^5 \hat{A}_j(r^{-1}x)\ dx_j$,

$$\begin{aligned}
&\textstyle\int_{C_r^i} |d\hat{A}_r + \hat{A}_r\wedge\hat{A}_r - \overline{F}|^2\ dx^5 \le C\ \delta\ \int_{C_r^i}|\overline{F}|^2\ dx^5 + \\
&\qquad\textstyle + C\, r\ \int_{\partial C_r^i}|F - i^*_{\partial C_r^i}\overline{F}|^2\ dvol_{\partial C_r^i} + C\ r\,\delta\ \int_{\partial C_r^i}|F|^2\ dvol_{\partial C_r^i}\ .
\end{aligned}$$

Summing up over the good balls - index i - using (2.20) and (2.22) we finally obtain the desired estimate

$$\sum_{i\in\mathcal{G}}\int_{C_r^i}|d\hat{A}_r + \hat{A}_r\wedge\hat{A}_r - \overline{F}|^2\ dx^5 \le C\ \delta + o_r(1)\ .$$

2.3.4 Good balls in the Morrey case

We now provide a version of the previous results useful for the approximation with bounds on Morrey norms. The relevant new feature is that there exists a constant ϵ_1 depending only on the underlying manifold (in our case $\mathbb{B}^5$) such that when the Morrey norm of F satisfies

$$\|F\|^2_{M(\mathbb{B}^5)} \leq \epsilon_1 \ , \tag{2.32}$$

from Remark 2.8 we automatically have the condition

$$\int_{\mathbb{S}^4} |F|^2 < \epsilon_0 \ .$$

In this case we will nevertheless fix $\delta > 0$ much smaller than ϵ_0, depending on r. The gain of the Morrey bound will be that under condition (2.32) are able to apply Proposition 2.3 in order to perform a controlled smooth extension on δ-bad balls.

2.4 Proof of Theorem 1.8

We are going to prove the following result:

Theorem 2.10. *Let F be the distributional curvature corresponding to an L^2 connection form A with $[A] \in \mathcal{A}^{\phi}_G(\mathbb{B}^5)$. Then there exist $F_n \in \mathcal{R}^{\infty,\phi}(\mathbb{B}^5)$ such that*

$$\|F - F_n\|_{L^2(\mathbb{B}^5)} \to 0, \qquad as\ n \to 0 \ .$$

Moreover we can also insure at the same time

$$\|A - A_n\|_{L^2(\mathbb{B}^5)} \to 0, \qquad as\ n \to 0 \ .$$

Proof. The proof consists in giving an "approximation algorithm" for F, which is divided into several steps. After each step the approximant connection obtained at that point will be denoted by $\hat{A}$, therefore this notation represents different connection forms at different steps of the approximation.

Step 1

Start with F, A as in the definition of $\mathcal{A}_G(\mathbb{B}^5)$ and fix $r > 0$. Apply Proposition 2.6 and choose well behaved radii r_i such that (2.20), (2.21) and (2.22) hold. We may also assume that $i^*_{\partial B_i} A \in \mathcal{A}_G(\partial B_i)$ for each i, as remarked immediately after Proposition 2.6.

Step 2

Apply Definition-Lemma 2.9 and define the families $\mathcal{G}_r, \mathcal{B}_r$ with respect to the data from Step 1 and for a small constant $\delta > 0$ to be fixed later.
The family $\mathcal{G}_r$ can be partitioned into subfamilies of disjoint balls $\mathcal{G}^1, \ldots, \mathcal{G}^N$, where N depends only on the discrete set Λ and on the constant α fixed in Definition 2.5.

Step 3

Fix $B_i = B(x_i, r_i) \in \mathcal{G}^1$. Let $(i^*_{\partial B_i} A)_{g_{B_i}} \in \mathcal{A}_G(\partial B_i)$, as in the definition of $\mathcal{A}_G(\partial B_i)$. Define then $A_{B_i} := \tau^*_{B_i} A, F_{B_i} := \tau^*_{B_i} F$, where $\tau : \mathbb{B}^5 \to B_i$ is the homothety $\tau(x) = x_i + r_i x$. From the estimates (2.28), (2.29) we obtain

$$\int_{\mathbb{S}^4} |F_{B_i}|^2 < \delta, \quad \int_{\mathbb{S}^4} |A_{B_i}|^2 < \delta \ .$$

We require δ to be smaller than the constant ϵ_0 of Proposition 2.1. Combining with (2.31) and requiring r to be sufficiently small, we also obtain

$$|\bar{A}_i|^2 < \epsilon_0 \ .$$

We may thus apply Proposition 2.1 to $A = A_{B_i}, F = F_{B_i}, \bar{F} = \bar{F}_i, \bar{A} = \bar{A}_i$. We then pull back the approximants to B_i via $\tau^{-1}_{B_i}$ and we denote the resulting approximant connection by $\hat{A}$. The error estimate (2.2) of Proposition 2.1 becomes:

$$\|d\hat{A} + \hat{A} \wedge \hat{A} - \bar{F}_i\|^2_{L^2(B_i)} \lesssim \delta \|\bar{F}_i\|^2_{L^2(B_i)} + \delta r \|F\|^2_{L^2(\partial B_i)} + r\|F - i^*_{\partial B_i} \bar{F}_i\|^2_{L^2(\partial B_i)} \ .$$

Step 4: iteration

Iterate Step 3 for all $B_i \in \mathcal{G}^1$. Since such balls are disjoint, the local replacements of A, F by $\hat{A}, F_{\hat{A}}$ are done independently. The total error that we obtain at the end is, using the estimates of Proposition 2.6,

$$\begin{aligned}
\|F_{\hat{A}} - F\|^2_{L^2(\mathbb{B}^5)} &\lesssim \sum_{B_i \in \mathcal{G}^1} \|F - \bar{F}_i\|^2_{L^2(B_i)} + \delta \sum_{B_i \in \mathcal{G}^1} \|\bar{F}_i\|^2_{L^2(B_i)} + \\
&+ \delta r \sum_{B_i \in \mathcal{G}^1} \|F\|^2_{L^2(\partial B_i)} + r \sum_{B_i \in \mathcal{G}^1} \|F - i^*_{\partial B_i} \bar{F}_i\|^2_{L^2(\partial B_i)} \\
&\lesssim \delta \|F\|_{L^2(\mathbb{B}^5)} + o(r) + \sum_{B_i \in \mathcal{G}^1} \|F - \bar{F}_i\|^2_{L^2(B_i)} \ .
\end{aligned}$$

Note that in particular the total L^2-error of averages satisfies

$$e_1 := \sum_i |B_i| \left| ⨍_{B(x_i, 2r)} F_{\hat{A}} - ⨍_{B(x_i, 2r)} F \right|^2 \leq N \|F_{\hat{A}} - F\|^2_{L^2(\mathbb{B}^5)} \ .$$

Step 5: iteration

We iterate Step 4. More precisely, we start with $\hat{A}_0 = A$ and at step $k \geq 1$ we use the balls from family $\mathcal{G}^k$ to approximate the curvature $F_{\hat{A}^{k-1}}$ obtained from step $k-1$. At step k we use the constants

$$\bar{F}^k_i := ⨍_{B(x_i, 2r)} F_{\hat{A}^{k-1}} \ .$$

Denote the new error introduced on the averages by e_k, analogously as e_1 above. Note that each B_i intersects a finite number of other balls (this number depends only on Λ, α from Definition 2.5). Therefore the total error after the final step $k = N$ is

$$
\begin{aligned}
\|F_{\hat{A}^N} - F\|^2_{L^2(\mathbb{B}^5)} &\lesssim \sum_{k=1}^{N} \|F_{\hat{A}^k} - F_{\hat{A}^{k-1}}\|^2_{L^2(\mathbb{B}^5)} \\
&\lesssim N\delta\|F\|_{L^2(\mathbb{B}^5)} + No(r) + \sum_{k=1}^{N} e_k \\
&\lesssim C(N)\left(\delta\|F\|_{L^2(\mathbb{B}^5)} + o(r) + \sum_i \|F - \bar{F}_i\|^2_{L^2(B_i)}\right) ,
\end{aligned}
$$

where the last sum is taken over all the balls B_i of our grid and $C(N)$ depends just on Λ, α from Definition 2.5. Since for any L^2 function f there holds

$$
\lim_{|h|\to 0} \int |f(x+h) - f(x)|^2 dx = 0
$$

we deduce that

$$
\sum_i \|F - \bar{F}_i\|^2_{L^2(B_i)} = o'(r) \to 0 \quad \text{as } r \to 0
$$

as well. Thus we have the following final estimate on our approximation:

$$
\|F_{\hat{A}^N} - F\|^2_{L^2(\mathbb{B}^5)} \lesssim \delta\|F\|_{L^2(\mathbb{B}^5)} + o(r) + o'(r) .
$$

Note that as a result of Proposition 2.1 we also have that $\hat{A}^N$ is continuous on the interior of $\cup\{B_i : \ B_i \in \mathcal{G}_r\}$.

Step 6

We extend $\hat{A}$ on a bad ball $B_j \in \mathcal{B}_r$ as follows. First apply Lemma 2.4 to $\hat{A}$ and to the compact $K := \partial B_j \cap \cup\mathcal{G}_r$ to obtain A_η on ∂B_j such that $A_\eta = A$ on K and A_η is C^0. Then we use the radial projection $\pi_j : B_j \setminus \{x_j\} \to \partial B_j$ and define $\hat{A}_j := \pi_j^* A_\eta$. We have the following estimate, using Step 5:

$$
\begin{aligned}
\|F_{\hat{A}_j}\|^2_{L^2(B_j)} &\lesssim r\left(\|F_{\hat{A}_j} - F_{\hat{A}}\|^2_{L^2(\partial B_j)} + \|F_{\hat{A}}\|^2_{L^2(\partial B_j)}\right) \\
&\lesssim r(o_\eta + \|F_{\hat{A}} - \bar{F}_j\|^2_{L^2(\partial B_j)}) + \|F\|^2_{L^2(B_j)} .
\end{aligned}
$$

Step 7: iteration

We iterate Step 6 for all bad balls. Since we modify at most N times the connection on each ball, the final bound for the connection $\hat{A}$ obtained after this process is still

$$\sum_{B_j \in \mathcal{B}_r} \|F_{\hat{A}}\|^2_{L^2(B_j)} \lesssim r o_\eta + o(r) + \|\bar{F}\|^2_{L^2(\cup \mathcal{B}_r)} .$$

The total error which we obtain is as follows:

$$\begin{aligned} \|F_{\hat{A}} - F\|^2_{L^2(\mathbb{B}^5)} &\lesssim \sum_{B_i \in \mathcal{G}_r} \|F_{\hat{A}} - F\|^2_{L^2(B_i)} + \sum_{B_j \in \mathcal{B}_r} \|F_{\hat{A}} - F\|^2_{L^2(B_j)} \\ &\lesssim \delta \|F\|_{L^2(\mathbb{B}^5)} + o(r) + o'(r) + r o_\eta + o(r) + \|F\|^2_{L^2(\cup \mathcal{B}_r)} . \end{aligned}$$

For r, δ, η small enough the first terms become as small as desired. The last term converges to zero by dominated convergence: indeed $|\cup \mathcal{B}_r| \to 0$ as $r \to 0$ by Lemma 2.9 and the function $\chi_{\cup B_r} F$ is dominated by $F \in L^2$.

Step 8

From the previous step we have $\hat{A}$ such that $\|F_{\hat{A}} - F\|_{L^2(\mathbb{B}^5)} \leq \frac{1}{2k}$ and $\hat{A}$ is C^0 outside the centers of bad balls by construction (see Step 3 and Step 6, and recall that by Definition 2.5 the ball $B_j \subset B_{\alpha r}(x_j)$ does not cover x_i for $j \neq i$). We now mollify $\hat{A}$ outside this finite set of centers, and we obtain the wanted curvature $F_{A_k} \in \mathcal{R}^\infty$.

By a similar reasoning we also insure $\|A_n - A\|_{L^2(\mathbb{B}^5)} \to 0$ utilizing (2.3) instead of (2.2) as above.

Utilizing the fact that the construction of Proposition 2.1 and the radial extension on the bad balls do not affect the boundary condition on our balls we obtain the approximation also in $\mathcal{R}^{\infty,\phi}(\mathbb{B}^5)$ for weak connections in $\mathcal{A}^\phi_G(\mathbb{B}^5)$. $\square$

2.5 Proof of Morrey approximation Theorem 1.10

We now provide the modifications needed to prove the Theorem 1.10 along the same steps as Theorem 2.10.

2.5.1 Strategy of L^2 approximation

It is enough to prove that for each fixed $\epsilon > 0$ we may find a smooth approximating curvature $\hat{F}$ which is closer than ϵ to F in L^2-norm and satisfies (1.18). To do this, we use the division into good and bad cubes like in the previous section and the construction for $\hat{F}$ proceeds as in the proof of Theorem 2.10 with the following modifications:

- In Step 1 we use Proposition 2.7 instead of Proposition 2.6.
- In Step 2 we further partition also the family of δ-bad balls $\mathcal{B}_r$ into disjointed subfamilies $\mathcal{B}_1, \ldots, \mathcal{B}_N$.
- In Step 3 we keep also track of the error estimate (2.3) of Proposition 2.1, which reads:
$$\|\hat{A} - \bar{A}_i\|_{L^2(B_i)} \leq Cr\|A - \bar{A}_i\|_{L^2(\partial B_i)} .$$
- The above estimate propagates through Step 4 where we obtain
$$\|\hat{A} - A\|^2_{L^2(\mathbb{B}^5)} \lesssim \sum_{B_i \in \mathcal{G}^1} \|A - \bar{A}_i\|^2_{L^2(\partial B_i)} .$$
- In Step 5 this and (2.23) gives
$$\|\hat{A}^N - A\|^2_{L^2(\mathbb{B}^5)} \lesssim \sum_i \|A - \bar{A}_i\|^2_{L^2(\mathbb{B}^5)} = o'(r) .$$
- In Step 6 we still apply Lemma 2.4 but we replace the radial extension by the application of Proposition 2.3 to the groups of bad balls $\mathcal{B}_k$ constructed in Step 2. This is allowed by the hypothesis $\|F\|^2_M < \epsilon_0$ and by the discussion of Section 2.3.4. After this procedure on each bad ball B_j we obtain the estimate
$$\|F_{\hat{A}}\|^2_{L^2(B_j)} \lesssim r(o_\eta + \|F\|^2_{L^2(\partial B_j)}) .$$
We similarly have the estimate for $\hat{A}$:
$$\|\hat{A}\|^2_{L^2(B_j)} \lesssim r(o_\eta + \|A\|^2_{L^2(\partial B_j)}) .$$
- In Step 7 we then collect the contributions from all bad balls like in Steps 4-5. We use the properties stated in Proposition 2.7 to obtain
$$\begin{aligned} \sum_{B_j \in \mathcal{B}_r} \|F_{\hat{A}}\|^2_{L^2(B_j)} &\lesssim ro_\eta + o(r) + \|F\|_{L^2(\cup \mathcal{B}_r)} + \|A\|_{L^2(\cup \mathcal{B}_r)} , \\ \sum_{B_j \in \mathcal{B}_r} \|\hat{A}\|^2_{L^2(B_j)} &\lesssim ro_\eta + o(r) + \|F\|_{L^2(\cup \mathcal{B}_r)} + \|A\|_{L^2(\cup \mathcal{B}_r)} , \end{aligned}$$
and by the same dominated convergence reasoning as in Step 7 of Theorem 2 we obtain (1.16) and (1.17).
- Step 8 proceeds exactly as in Theorem 2.

We now prove the bounds (1.18) for $\hat{F}$ constructed as above. We need to estimate
$$\frac{1}{\rho} \int_{B_\rho(x)} |\hat{F}|^2$$
uniformly in ρ, x. We consider separately the cases $\rho \gtrsim r$ and $\rho \ll r$.

2.5.2 The case $\rho \gtrsim r$

In this situation we simply estimate

$$\int_{B_\rho(x)} |\hat{F}|^2 \leq \sum_i \int_{B_\rho(x)\cap B_i} |\hat{F}|^2 \leq \sum_{i: B_{\alpha r}(x_i)\cap B_\rho(x)\neq\emptyset} \int_{B_i} |\hat{F}|^2$$

In this case we use the fact that the cover $\{B_i\}$ had the bounded intersection property, the fact that α is bounded and the fact that as a consequence of Prop. 2.1 or Prop. 2.3 (depending on the balls involved), $\|\hat{F}\|_{L^2(B_i)} \lesssim \|F\|_{L^2(B_i)}$ thus

$$\int_{B_\rho(x)} |\hat{F}|^2 \lesssim \int_{B_{c\rho}(x)} |\hat{F}|^2 \lesssim \int_{B_{c\rho}(x)} |F|^2 \ .$$

By definition of Morrey norm, we continue with

$$\frac{1}{\rho}\int_{B_\rho(x)} |\hat{F}|^2 \lesssim \frac{1}{\rho}\int_{B_{c\rho}(x)} |F|^2 \lesssim c\|F\|_M^2 \ ,$$

which finishes the proof.

2.5.3 The case $\rho \ll r$

In this case we will use elliptic regularity for the proof. We note the following scale-invariant inequalities valid for the harmonic extensions:

$$\|d\tilde{A}^g\|^2_{L^{5/2}(B_{r_i})} \leq C\int_{\partial B_{r_i}} |dA^g|^2 \ , \quad \|\tilde{A}^g\|^4_{L^5(B_{r_i})} \leq C\int_{\partial B_{r_i}} |A^g|^4 \ .$$

If $B_\rho(x) \subset B_i$ then for an application of Step 3 or 6 on B_i we can thus write:

$$\begin{aligned}
\|\hat{F}\|^2_{L^2(B_\rho(x)} &= \int_{B_\rho(x)} |d\tilde{A}^g + \tilde{A}^g \wedge \tilde{A}^g|^2 \\
&\lesssim \int_{B_\rho(x)} |d\tilde{A}^g|^2 + \int_{B_\rho(x)} |\tilde{A}^g|^4 \\
&\lesssim |B_\rho|^{\frac{1}{5}} \left(\int_{B_\rho(x)} |d\tilde{A}^g|^{5/2}\right)^{\frac{4}{5}} + |B_\rho|^{\frac{1}{5}} \left(\int_{B_\rho(x)} |\tilde{A}^g|^5\right)^{\frac{4}{5}} \\
&\lesssim \rho\left[\left(\int_{B_i} |d\tilde{A}^g|^{5/2}\right)^{\frac{4}{5}} + \left(\int_{B_i} |\tilde{A}^g|^5\right)^{\frac{4}{5}}\right] \\
&\lesssim \rho\left(\int_{\partial B_{r_i}} |dA^g|^2 + \int_{\partial B_{r_i}} |A^g|^4\right) \\
&\lesssim \rho(1+\epsilon_0)\|F\|^2_{L^2(\partial B_i)} \ ,
\end{aligned}$$

where in the first equality we used the gauge-invariance of $\hat{F}$, making the gauge change $\hat{g}$ irrelevant, and in the last estimate we use the results of Propositions 2.1, (2.3).

The desired estimate then follows similarly to the case $\rho \gtrsim r$. In the general case $B_\rho(x) \cap B_i \neq \emptyset$ we have to just replace $B_\rho(x)$ by $B_\rho(x) \cap B_i$ and the same estimates work. We note that the number of steps of type 3 or 6 in which we modify $\hat{F}$ over $B_\rho(x)$ is bounded above by a constant $C(N)$ which ultimately depends only on the dimension. $\square$

3 Coulomb gauges and point removability in 4 dimensions

In this section we prove an improved point removability result based on [43].

3.1 Uhlenbeck Coulomb gauge

In [50] Uhlenbeck proved the following point removability result:

Theorem 3.1 ([50], Thm. 4.6)**.** *Let ∇ be a Yang-Mills connection in a bundle P over $B^4 \setminus \{0\}$. If the L^2 norm of the curvature F of ∇ is finite, then there exists a gauge in which the bundle P extends to a smooth bundle $\tilde{P}$ over B^4 and the connection ∇ extends to a smooth Yang Mills connection $\tilde{\nabla}$ in B^4.*

We recall that for a connection which in local coordinates is written $\nabla = d + A$, being *Yang-Mills* means that the curvature $F = F_A$ satisfies in the weak sense

$$d_A^* F_A = 0 \ . \tag{3.1}$$

The regularity theory of Uhlenbeck allows to prove that $W^{1,2}$ Yang-Mills connections $d + A$ on trivial bundles are smooth up to a gauge change in the balls $B_\rho(x)$ such that $\int_{B_\rho(x)} |F|^2 < \epsilon_0$ for a constant ϵ_0 independent of A, F. This uses the regularity theory for the nonlinear (in A) equation (3.1), which when F does not have much energy and A is in Coulomb gauge can be seen as an elliptic system.

Therefore the main step in the proof of Theorem 3.1 is the proof that we can find a global gauge extending over a neighborhood of the origin, in which the connection is $W^{1,2}$ so that the elliptic regularity can be applied. In Uhlenbeck [50] the elliptic regularity of equation (3.1) is used on $B \setminus \{0\}$ in order to provide the needed estimates on concentric annuli. We will describe here how to proceed without this regularity.

Using a result from [43] we obtain that the analogue of Theorem 3.1 holds without the assumption that (3.1) holds. It appears that this result is not present in the literature, although it is hinted at in [5]. We will prove the following

Theorem 3.2 (Point removability [50] with no Yang-Mills assumption). *Let ∇ be a $W^{1,2}$ connection in a bundle P over $B^4 \setminus \{0\}$. If the L^2 norm of the curvature F of ∇ is finite, then there exists a gauge in which the bundle P extends to a smooth bundle $\tilde{P}$ over B^4 and the connection ∇ extends to a $W^{1,2}$ connection $\tilde{\nabla}$ in B^4.*

Theorem 3.2 allows to prove weak compactness for sequences of $W^{1,2}$ connections with curvatures bounded in L^2, again removing the assumption that the limit is Yang-Mills present in [45], [15]. The strategy in the paper [45] was to consider minimizing sequences $A_n \in \mathcal{A}^{1,2}(E)$ for the Yang-Mills functional and prove that their connections converge locally weakly in $W^{1,2}$ while the curvatures converge locally weakly in L^2, outside a finite set of "bad points" where the curvature energy density concentrates. This allowed to obtain that the limit (which corresponds to a Yang-Mills minimizer) is Yang-Mills outside those points. The point removability theorem 3.1 which worked under the Yang-Mills assumptions then provided a way for extending the limit bundle and connection over each bad point. Note that here is the only instance where the assumption of having an energy minimizing sequence was used in [45]. We can thus use our improved Theorem 3.2 to immediately obtain:

Theorem 3.3 (Bubbling [45] for general sequences). *Assume that $A_n \in \mathcal{A}^{1,2}(E)$ on a smooth bundle E over a smooth compact Riemannian 4-manifold M. If $\|F_{A_n}\|_{L^2} \leq C$ for all n then up to extracting a subsequence we have that A_n converge locally weakly in $W^{1,2}$ to a connection $A_\infty \in \mathcal{A}^{1,2}(\tilde{E})$ over a possibly different bundle.*

3.2 Coulomb gauges and Lorentz-improved regularity

We recall that the connection form A and the curvature form F are related in local coordinates by the distributional equation $F = dA + A \wedge A$. Recall that by Hodge theory the differential DA is controlled via dA and d^*A. It is then heuristically clear that if we desire a control on DA via the curvature we must therefore have some restrictions on d^*A. The estimates coming from the nonlinear elliptic system corresponding to d, d^* replaces the control via equation (3.1) as used in [51]. We recall the celebrated result of K. K. Uhlenbeck which is our starting point.

Theorem 3.4 ([51], Thm. 1.3). *There exists a constant ϵ_0 as follows. Assume that $d + A$ is the local expression of a connection of a trivial bundle $E \to \Omega$ over a compact Riemannian 4-manifold Ω such that $A \in W^{1,2}_{loc}$ and the curvature $F := F_A$ satisfies*

$$\int_\Omega |F|^2 \leq \epsilon_0 \ . \tag{3.2}$$

Then there exists a gauge $g \in W^{2,2}_{loc}(\Omega)$ such that the transformed connection form

$$A^g = g^{-1}dg + g^{-1}Ag$$

satisfies

$$d^* A^g = 0 \quad on \quad \Omega$$

and is controlled by the curvature:

$$\int_\Omega |DA^g|^2 + \int_\Omega |A^g|^4 \leq C \int_\Omega |F|^2 \ . \tag{3.3}$$

This result allows us to find controlled gauges in concentric dyadic annuli around the origin. To patch together the gauges of two overlapping annuli we use the following result, for which we use the techniques of [43] Thm. IV.1.

Proposition 3.5. *, Suppose that A and $B = g^{-1}dg + g^{-1}Ag$ are connection forms corresponding to two gauge-related connections belonging to $\mathcal{A}^{1,2}(E)$ where $E \to \Omega$ is a trivial bundle over a domain $\Omega \subset \mathbb{R}^4$ such that*

$$d^* A = d^* B = 0 \ .$$

If $A, B \in W^{1,2}$ then the gauge change g is $W^{2,2} \cap C^0$. Moreover for some $\bar{g} \in G$ we have the bound

$$\|g - \bar{g}\|_{L^\infty \cap W^{2,2}} \lesssim \|A\|^2_{W^{1,2}} + \|B\|^2_{W^{1,2}} \ . \tag{3.4}$$

Proof. From

$$dg = gB - Ag \ ,$$

since multiplication is continuous from $W^{1,2} \times (W^{1,2} \cap L^\infty)$ to $W^{1,2} \hookrightarrow L^{(4,2)}$ it follows that $dg \in W^{1,2} \hookrightarrow L^{(4,2)}$ and

$$\|dg\|_{L^{(4,2)}} \lesssim \|A\|_{W^{1,2}} + \|B\|_{W^{1,2}} \ .$$

From the above equation and using $d^*A = d^*B = 0$ and identifying 1-forms with vector fields we obtain

$$\Delta g = d^* dg = dg \cdot A - B \cdot dg \ ,$$

where both terms are products of elements of $L^{(4,2)}$ therefore belong to $L^{(2,1)}$. We have

$$\|\Delta g\|_{L^{(2,1)}} \lesssim \|dg\|_{L^{(4,2)}}(\|A\|_{L^{(4,2)}} + \|B\|_{L^{(4,2)}}) \lesssim \|A\|^2_{L^{(4,2)}} + \|B\|^2_{L^{(4,2)}} \ .$$

By the continuous embeddings $W^{2,(2,1)} \hookrightarrow W^{1,(4,1)} \hookrightarrow L^\infty$ valid in 4 dimensions, we obtain

$$\|g - \tilde{g}\|_{L^\infty \cap W^{2,2}} \lesssim \|A\|^2_{L^{(4,2)}} + \|B\|^2_{L^{(4,2)}} := (*) \ ,$$

where $\tilde{g}$ is the average of g done in the space $\mathbb{R}^N, N = k \times k$ in which the manifold G is embedded as group of matrices. Since $g \in G$ a.e., we also have

$$\mathrm{dist}_{\mathbb{R}^N}(\tilde{g}, G) \lesssim (*) \ ,$$

therefore there exists $\bar{g} \in G$ such that

$$\|g - \bar{g}\|_{L^\infty} \lesssim (*) \lesssim \|A\|^2_{W^{1,2}} + \|B\|^2_{W^{1,2}} \ ,$$

as desired. Note that $W^{1,2}$ connections in 4-dimensions can be approximated by smooth connections in $W^{1,2}$-norm (see Lemma 2.4). By applying the above result on balls $B_\rho(x)$ with $\rho \to 0$ for a.e. x, we obtain that $g \in C^0$ too. □

Notation: from now on we denote by S_k the spherical shell $B_{2^{-2k}} \setminus B_{2^{-2k-3}}$.

Lemma 3.6. *There exists a constant $\delta > 0$ such that if $\int_{S_k} |F|^2 \leq \delta$ then the bundle E is trivial over S_k and there exists a gauge g over S_k in which the connection corresponding to F is represented by a $W^{1,2}$ form A_k which satisfies*

$$d^* A_k = 0, \quad \|DA_k\|_{L^2(S_k)} + \|A_k\|_{L^4(S_k)} \leq \|F\|_{L^2(S_k)} \ . \tag{3.5}$$

Proof. Without loss of generality let $k = 0$, because the norms of F, A and DA appearing in (3.5) have the same scaling. We cover S_0 by two charts U_+, U_- which are tubular neighborhoods of opposite half-shells. In $U_\pm$ the connection has the local expression $A_\pm$. Since the bundle is trivial over $U_\pm$ we can apply Theorem 3.4 and up to a change of gauge $A_\pm$ satisfies (3.5).

On $U_+ \cap U_-$ there exists g such that $A_+ = g^{-1}dg + g^{-1}A_- g$. By Proposition 3.5 we have that $g \in C^0$ and for some $\bar{g} \in G$ there holds

$$\|g - \bar{g}\|_{L^\infty} \lesssim \delta^2 \ . \tag{3.6}$$

in particular it is not possible for g to realize a nontrivial homotopy class $[U_+ \cap U_-, G]$, provided $\delta^2 \leq C_G$ for some C_G depending on the topology of G. Therefore it is possible to extend g in a Lipschitz way over U_- and we find a global trivialization over the whole of S_0. Applying Theorem 3.4 again we find A_0 as in (3.5). □

3.3 Proof of Theorem 3.2

Proof. The bundle is non-smooth just at the origin, therefore we may work replacing $B_1(0)$ by a ball $B_\rho(0)$ with $\rho > 0$ on which $\int_{B_\rho} |F|^2 < \delta$. In other words we don't loose any generality if we assume $\int_{B_1(0)} |F|^2 < \delta$. We fix δ later, but it will be smaller than the constant δ of Lemma 3.6 and than the constant ϵ_0 of theorem 3.4.

We apply Lemma 3.6 and we start with the connections A_k defined on S_k and satisfying (3.5). On each $S_{k+1} \cap S_k$ there is a gauge change g_k such that

$$A_{k+1} = g_k^{-1} dg_k + g_k^{-1} A_k g_k \ . \tag{3.7}$$

By Proposition 3.5 there exist $\bar{g}_k \in G$ such that

$$\|g_k - \bar{g}_k\|_{L^\infty \cap W^{2,2}} \lesssim \|A_k\|^2_{W^{1,2}} + \|A_{k+1}\|^2_{W^{1,2}} \ . \tag{3.8}$$

Now we propagate the gauge along the increasing S_k's. In order to cancel the contributions of the approximating constant gauges $\bar{g}_k$, we define for example $\bar{A}_1 = \bar{g}_0 A_1 \bar{g}_0^{-1} = \bar{g}_0^{-1}(A_1) = \bar{g}_0^{-1} \circ g_0(A_0)$. This means that $\bar{A}_1$ differs from A_0 on $S_1 \cap S_0$ just by a small gauge. Similarly define

$$\bar{A}_k := \bar{h}_k(A_k), \quad \bar{h}_k := \prod_{i=0}^{k-1} \bar{g}_i^{-1} \ .$$

We use the $\bar{A}_k$'s as a reference to define a global gauge. Define $\tilde{g}_k$ on $S_{k+1} \cap S_k$ to be such that $\bar{A}_{k+1} = \tilde{g}_k(\bar{A}_k)$, i.e.

$$\tilde{g}_k := \bar{h}_k^{-1} \bar{g}_k^{-1} g_k \bar{h}_k \ . \tag{3.9}$$

The $\tilde{g}_k$'s are better than the g_k's because they don't contain the gauge jumps $\bar{g}_k$. From (3.8) and (3.5), by multiplying by constants, i.e. by isometries of G, we have

$$\begin{aligned} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k \cap S_{k+1})} &= \|g_k - \bar{g}_k\|_{L^\infty \cap W^{2,2}(S_k \cap S_{k+1})} \\ &\lesssim \int_{S_k} |F|^2 + \int_{S_{k+1}} |F|^2 \ . \end{aligned} \tag{3.10}$$

Next extend $\tilde{g}_k$ radially on $S_k^- := B_{2^{-2k-3}} \setminus B_{2^{-2k-4}}$ and on $S_k^+ := B_{2^{-2k+1}} \setminus B_{2^{-2k}}$. Call this extension $\tilde{\tilde{g}}_k$. Note that

$$\sum_{k \geq 1} \int_{S_k} |F|^2 \leq \delta \ . \tag{3.11}$$

Because of (3.11), (3.11) and because the radial extension is tame enough there holds:

$$\|\tilde{\tilde{g}}_k - id\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+)} \leq \delta \ .$$

Let δ be small enough so that $\tilde{\tilde{g}}_k = \exp_{id}(\varphi_k)$, $\|\varphi_k\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+ \cup S_k)} \sim \|\tilde{\tilde{g}}_k - id\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+ \cup S_k)}$.
This is possible because $\exp_{id}^{-1}$ is well-behaved near the identity.
We create a family of cutoff functions similar to the one used in Littlewood-Paley decompositions. Consider a function $\eta(r)$ which is smooth, decreasing, equal to 0 for $r > 2$ and to 1 for $r < 1$. We can assume $|\eta'| \leq 2$. Then define $\psi_k(x) := \eta(2^{2k}|x|) - \eta(2^{2k+4}|x|)$ and consider $\tilde{\varphi}_k := \psi_k \varphi_k$. We have

$$\begin{aligned} \|\tilde{\varphi}_k\|_{L^\infty} &\leq \|\varphi_k\|_{L^\infty(\mathbb{S}^k)} \ , \\ \|D^2 \tilde{\varphi}_k\|_{L^2} &\lesssim \|D^2 \varphi_k\|_{L^2(\mathbb{S}^k)} + \|d\psi_k\|_{L^4} \|d\varphi_k\|_{L^4(\mathbb{S}^k)} + \|D^2 \psi_k\|_{L^2} \|\varphi_k\|_{L^\infty(\mathbb{S}^k)} \\ &\lesssim \|\varphi_k\|_{L^\infty \cap W^{2,2}(\mathbb{S}^k)} \ . \end{aligned}$$

By extending $\tilde{g}_k$ via $\exp(\tilde{\varphi}_k)$ we obtain a continuous extension of $\tilde{g}_k$ on $S_k \cup S_k^- \cup S_k^+$ which still satisfies the same estimates as $\tilde{\tilde{g}}_k$. Use the notation $\hat{g}_k$. We then define on $B^4 \setminus \{0\}$

$$\lambda := \prod_{i=0}^{\infty} \hat{g}_k \ .$$

Since $\hat{g}_k$ is nonidentity on at most 5 dyadic rings, this product has locally finitely many factors different than the identity therefore it is well-defined. We also have that since $W^{2,2} \cap L^\infty$ is an algebra

$$\begin{aligned}
\|\lambda - id\|_{L^\infty \cap W^{2,2}(B_{2^{-2\bar{k}}} \setminus \{0\})} &\lesssim \sum_{k \geq \bar{k}} \|\hat{g}_k - id\|_{L^\infty \cap W^{2,2}(B^4 \setminus \{0\})} \\
&\lesssim \sum_{k \geq \bar{k}} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k \cup S_k^- \cup S_k^+)} \\
&\lesssim \sum_{k \geq \bar{k}} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k)} \\
&\lesssim \sum_{k \geq \bar{k}} \int_{S_k} |F|^2 \ .
\end{aligned}$$

In particular we see that $\lambda \to id$ at zero, therefore the bundle extends, as desired. We must now prove that in this gauge the connection form $\tilde{A}$ is $W^{1,2}$. Recall that if the gauges would be chosen all equal to $\tilde{g}_k$ then the connection would become $\bar{A}_k$ on S_k, and this is just a constant conjugation of the original A_k as in (3.5). Since the cutoff parts $\hat{g}_k$ on $S_k^- \cup S_k^+$ are controlled in $W^{2,2} \cap L^\infty$ still by the right hand side of (3.9) we obtain using (3.11) and the fact that the $\hat{g}_k$ have similar estimates as the $\tilde{g}_k$ that

$$\begin{aligned}
\|\tilde{A}\|^2_{W^{1,2}} &\lesssim \sum_{k \geq 0} \left(\|A_k\|^2_{W^{1,2}(S_k)} + \|\hat{g}_k(A_{k-1})\|^2_{W^{1,2}(S_k^-)} + \|\hat{g}_k(A_{k+1})\|^2_{W^{1,2}(S_k^+)} \right) \\
&\lesssim \sum_{k \geq 0} \left(\|A_k\|^2_{W^{1,2}(S_k)} + \|\hat{g}_k\|^2_{W^{2,2}(S_k^-)} + \|\hat{g}_k\|^2_{W^{2,2}(S_k^+)} \right) \\
&\lesssim \sum_{k \geq 0} \|A_k\|^2_{W^{1,2}(S_k)} + \sum_{k \geq 0} \|A_k\|^4_{W^{1,2}(S_k)} \\
&\lesssim \delta + \delta^2 \ .
\end{aligned}$$

In the last passage we used (3.11) and the inequality between ℓ^2 and ℓ^4. This concludes the proof of Theorem 3.2. $\square$

4 Weak closure for non-abelian curvatures in 5 dimensions

4.1 Ingredients for the proof of Theorem 1.4

We describe here what enters the proof of Theorem 1.4, while making a parallel to the works [3] and [23] on metric currents and scans, which present analogous definitions of weak objects as sets of slices "connected" via a compatibility condition based on an overlying integrable quantity (in our case this control comes from the curvature 2-form F). Our closure result comes from the interplay of three ingredients:

- A *geometric distance* on sliced 1-forms: for A, A' which are L^2 connection forms over $\mathbb{S}^4$ we use the gauge-orbit distance

  $$\mathrm{dist}([A],[A']) := \min\{\|A - g^{-1}dg - g^{-1}A'g\|_{L^2(\mathbb{S}^4)} : \ g \in W^{1,2}(\mathbb{S}^4, G)\} .$$

  This corresponds to the use of the flat distance for the closure theorem of integral currents by Ambrosio-Kirchheim [3].

- The fact that *the above distance interacts well with our energy at the level of slices*, which follows from Theorem 1.1. More precisely we have that sublevels of $A \mapsto \|F_A\|_{L^2(\mathbb{S}^4)}$ are dist-compact. In [23] a similar interaction occurs between the flat distance and the fractional mass of rectifiable currents.

- The *oscillation control on slices* of a fixed weak curvature, obtained via the overlying 2-form F. More precisely, if we identify $\mathbb{S}^4$ by homothety with each one of the spheres $S := \partial B_t(x), S' := \partial B_{t'}(x')$ then the pullbacks $A(t,x), A(t',x')$ of $i_S^*A, i_{S'}^*A$ satisfy

  $$\mathrm{dist}([A(t,x)],[A(t',x')]) \leq C\|F\|_{L^2(\mathbb{B}^5)}(|x-x'| + |t-t'|)^{1/2} .$$

  In [3] the corresponding fact is the interpretation of rectifiability as a bound of the metric variation of the slices.

We can find L^2-controlled connection forms A_n corresponding to F_n and obtain a weak limit A which will be an L^2 connection form corresponding to F. The main difficulty is to find gauges g in which the slices $i^*_{\partial B_r(x)}A$ become $W^{1,2}_{loc}$.

The above overall strategy is the one which worked in the abelian case $G = U(1)$ as well and was employed in [38].

We start by identifying the traces on lower dimensional sets $\partial B_\rho(x_0)$ with elements of a metric space $(\mathcal{Y}, \mathrm{dist})$ where $\mathcal{Y} = \mathcal{A}_G(\mathbb{S}^4)/\sim$ and $\sim$ is the guage-equivalence relation, such that we have a local control of the Hölder norm of the slice functions in terms of the

L^2-norms of the F_n. We will use Proposition 4.1 for this.

Mixing a compactness result for slice functions with respect to the distance on $\mathcal{Y}$ with the weak convergence of the A_n we will manage to obtain the convergence of a.e. slice to an element which is gauge-equivalent to an element in $\mathcal{A}^g(\mathbb{S}^4)$ as desired.

4.2 The metric space $\mathcal{Y}$

To prove the weak closure result for $\mathcal{A}_G$ we use a slicing technique. In the definition of $\mathcal{A}_G$ we required that any weak connection have a gauge on each slice in which it is represented by a $W^{1,2}$ form. Therefore we consider the following space of possible slice classes:

$$\mathcal{Y} := \mathcal{A}_G(\mathbb{S}^4)/\sim, \tag{4.1}$$

where the equivalence relation $\sim$ on global L^2 connections is

$$A \sim B \text{ if } \exists g \in W^{1,2}(\mathbb{S}^4, G) \text{ s.t. } g^{-1}dg + g^{-1}Ag = B\ .$$

We define the following gauge-invariant function:

$$\text{“dist”}(A, A') := \left(\inf\left\{\int_{\mathbb{S}^4} |A - g^{-1}dg - g^{-1}A'g|^2 :\ g \in W^{1,2}(\mathbb{S}^4, G)\right\}\right)^{\frac{1}{2}}\ .$$

For two connection forms A, A' if $g_A, g_{A'}$ are $W^{1,2}$ gauges such that

$$B = g_A^{-1}dg_A + g_A^{-1}Ag_A, \quad B' = B = g_{A'}^{-1}dg_{A'} + g_{A'}^{-1}A'g_{A'}$$

then, since $A \mapsto g^{-1}dg + g^{-1}Ag$ is a continuous group action of $\mathcal{G} \cap W^{1,2}$ on $\mathcal{A}_G(\mathbb{S}^4)$, we have

$$\text{“dist”}(A, A') = \text{“dist”}(B, B')\ .$$

“dist” then descends to a well-defined distance $\text{dist}([A], [A'])$ on equivalence classes of connection forms. Let

$$[A] = \text{ image of } A \text{ under the projection } \mathcal{A}_G(\mathbb{S}^4) \to \mathcal{A}_G(\mathbb{S}^4)/\sim\ .$$

The natural metric to impose on $\mathcal{Y}$ is the L^2-distance between (global) gauge orbits (cfr [15]):

$$\text{dist}([A], [B]) = \inf\left\{\|A' - B'\|_{L^2(\mathbb{S}^4)} :\ A' \in [A],\ B' \in [B]\right\}\ . \tag{4.2}$$

On the metric space $(\mathcal{Y}, \text{dist})$ we will study the functional

$$\mathcal{N} : \mathcal{Y} \to \mathbb{R}^+, \quad \mathcal{N}([A]) = \int_{\mathbb{S}^4} |F_A|^2\ . \tag{4.3}$$

Note that because the curvature satisfies $F_{g^{-1}dg+g^{-1}Ag} = g^{-1}F_Ag$ and since the norm on 2-forms is G-invariant, we have that $\mathcal{N}([A])$ does not depend on the representative A employed to compute F_A.

4.3 The slice a.e. convergence

We employ the following abstract theorem. See [23] Thm. 9.1 for the original inspiration. We use the notation overlapping with the previous section. The goal will be to justify this overlap in notation subsequently, by proving that the spaces and functions of Section 4.2 satisfy the hypotheses of the theorem.

Proposition 4.1. *Consider a metric space $(\mathcal{Y}, dist)$ on which a function $\mathcal{N} : \mathcal{Y} \to \mathbb{R}^+$ is defined. Suppose that the following hypothesis is met:*

$$\forall C > 0 \text{ the sublevels } \{\mathcal{N} \leq C\} \text{ are seq. compact in } \mathcal{Y} . \tag{H}$$

Suppose $f_n : [0,1] \to \mathcal{Y}$ are measurable maps such that

$$dist(f_n(t), f_n(t')) \leq C|t - t'|^{1/2} \tag{4.4}$$

and that

$$\sup_n \int_0^1 \mathcal{N}(f_n(t))dt < C .$$

Then f_n have a subsequence which converges pointwise almost everywhere. The limiting function f also satisfies

$$dist(f(t), f(t')) \leq C|t - t'|^{1/2}, \quad \int_0^1 \mathcal{N}(f(t))dt < C .$$

Proof. We divide the interval $[0,1]$ in q^2 subintervals I_i^q of equal length q^{-2}. For each n, i, by Chebychev inequality we obtain

$$\left|\left\{t \in I_i^q : \mathcal{N}(f_n(t)) < \frac{C}{q^2}\right\}\right| > 0 ,$$

therefore up to extracting a subsequence, by pigeonhole principle we may assume

$$\left|\left\{t \in I_i^q : \forall n\, \mathcal{N}(f_n(t)) < \frac{C}{q^2}\right\}\right| > 0 .$$

Consider then

$$t_i^q \in \bigcap_{n\in\mathbb{N}} \{\mathcal{N} \circ f_n > C/q^2\} \cap I_i^q .$$

Since sublevels of $\mathcal{N}$ are compact, up to extracting a subsequence we obtain

$$\forall i, n, \quad \mathrm{dist}(f_n(t_i^q), f_{n+1}(t_i^q)) \leq 2^{-n} .$$

Up to extracting a diagonal subsequence

$$\forall i, n, q, \quad \mathrm{dist}(f_n(t_i^q), f_{n+1}(t_i^q)) \leq 2^{-n} .$$

In particular, using the uniform hölderianity of f_n and the triangle inequality, we have that for all i and for $t \in I_i^q$ there holds

$$\mathrm{dist}(f_n(t), f_{n+k}(t)) \leq 2^{1-n} + q^{-1} \ .$$

Since $\{t_i^q\}_{i,q}$ form a dense subset of $[0,1]$ we deduce that for all $t \in [0,1]$ the sequence $\{f_n(t)\}_n$ is Cauchy thus it has a limit in the completion of $\mathcal{Y}$. By Fatou theorem we obtain

$$\int_0^1 \liminf_n \mathcal{N}(f_n(t))dt \leq C \ ,$$

therefore for a.e. $t \in [0,1]$ the sequence $\mathcal{N}(f_n(t))$ in bounded. Since the sublevels of $\mathcal{N}$ are compact in $\mathcal{Y}$, for such t the limit of $\{f_n(t)\}_n$ belongs to $\mathcal{Y}$. We define thus $f(t) := \lim_n f_n(t)$ and the desired properties follow by Fatou's lemma and by the pointwise dist-convergence. □

4.4 Verifying the hypothesis of Proposition 4.1

We verify that we can apply Proposition 4.1 to our situation, where the goal is to prove weak closure for the class $\mathcal{A}_G$.

4.4.1 The compactness result (H)

We start by verifying the first statement of the hypothesis (H) for $\mathcal{Y}, \mathcal{N}$ as in Section 4.2:

Proposition 4.2. *Let $\mathcal{Y}$ be the space of slices as in* (4.1) *and $\mathcal{N} : \mathcal{Y} \to \mathbb{R}^+$ be the norm of the curvature as in* (4.3)*. Then $\mathcal{N}$ has sublevels which are compact with respect to the distance dist defined in* (4.2).

Proof. We assume that we are given a sequence of curvatures F_n corresponding to connection form classes $[A_n]$, such that

$$\|F_n\|_{L^2(\mathbb{S}^4)} \leq C \ .$$

The claim of the proposition is that the $[A_n]$ have a convergent subsequence with respect to the distance dist.
Up to a global gauge change we may assume that the A_n are controlled globally in L^2 (see Lemma 4.3):

$$\|A_n\|_{L^2(\mathbb{S}^4)} \lesssim \|F_n\|_{L^2(\mathbb{S}^4)} \ .$$

Up to extracting a subsequence we have that

$$A_n \rightharpoonup A_\infty \quad , F_n \rightharpoonup F_\infty \quad \text{in } L^2(\mathbb{S}^4) \ .$$

Step 1. *Concentration points of the curvature energy and a good atlas.* By usual covering arguments we have that up to extracting a subsequence there exist a finite number of concentration points of the curvature's L^2-energy $a_1,\ldots,a_N$ in $\mathbb{S}^4$. In other words there holds

$$\forall\epsilon>0, \rho_\epsilon := \quad \liminf_{n\to\infty} \inf\left\{\rho>0, x_0\in\mathbb{S}^4\setminus\cup B_\epsilon(a_i)\ \int_{B_\rho^{\mathbb{S}^4}(x_0)}|F_n|^2\geq\delta\right\}>0\ .$$

The number N of such points is $N\leq C/\delta$ where C is the above L^2-bound on the curvatures.

Up to diminishing ϵ and $\rho:=\rho_\epsilon$ we may suppose $\epsilon+\rho_\epsilon<\rho_{inj}(\mathbb{S}^4)$ and that the balls $B_\epsilon(a_i)$ are disjoint. We can find a cover by the balls $B_\epsilon(a_i)$ and by finitely many balls $B_\rho(x_i)$ such that the maximum number of overlaps of those balls is a universal constant. The $B_\rho(x_i)$'s will be called *good balls* and they will be simply denoted B_i below.

Step 2. *Uhlenbeck Coulomb gauges converge weakly on the good balls.* Using Uhlenbeck's gauge extraction of Theorem 3.4 on each B_i one finds a gauge g_n^i such that $A_n^i:=(g_n^i)^{-1}dg_n^i+(g_n^i)^{-1}A_ng_n^i\in W^{1,2}$ and such that

$$d^*A_n^i=0,\quad \|A_n^i\|_{W^{1,2}}\lesssim\|F_n\|_{L^2}\text{ on }B_i\ .$$

Therefore up to a diagonal subsequence we also may assume that

$$A_n^i\to A^i\text{ weakly in }W^{1,2}\text{ and strongly in }L^2\ . \tag{4.5}$$

By interpolation since the g_n^i are bounded in L^∞ we see that

$$g_n^i\to g^i\text{ weakly in }W^{1,2}\text{ and strongly in }L^q,\forall q<\infty\ .$$

This strong convergence in L^q together with the weak convergence of A_n and of the dg_n^i in L^2 implies that

$$A_n=g_n^id(g_n^i)^{-1}+g_n^iA_n^i(g_n^i)^{-1}\rightharpoonup g^id(g^i)^{-1}+g^iA^i(g^i)^{-1}=A\text{ in }\mathcal{D}'$$

and by uniqueness of weak limits the A^i obtained above are the local expressions of the limit A in the limit gauges g^i.

Step 3. *Point removability and strong global gauge convergence on good part.* By Proposition 3.5 the gauge changes $g_n^{ij}:=g_n^j(g_n^i)^{-1}$ needed to pass from A_n^i to A_n^j are controlled in $W^{2,2}\cap C_0$. Therefore up to taking a diagonal subsequence we have for all i,j

$$g_n^{ij}\to g^{ij}\text{ weakly in }W^{2,2}\text{, strongly in }W^{1,2}\text{ and locally uniformly in }C^0\ .$$

In particular we can apply the gauge extension procedure of the proof of Theorem 3.2 both to g_n^{ij} and to g^{ij} on balls covering any open contractible subset U^{good} in the complement of the bad balls $B_\epsilon(a_1), \ldots, B_\epsilon(a_N)$, obtaining gauge transformations g_n^{good}, g^{good}. We recall that in this process we multiply gauges by the constants $\overline{g_n^{ij}}$ then truncate the error terms $(\overline{g_n^{ij}})^{-1} g_n^{ij}$ away from $B_i \cap B_j$. We note that up to extracting subsequences we may assume (by compactness of G and finiteness of the balls intersecting U^{good}) that the constants involved also converge:

$$\overline{g_n^{ij}} \to \overline{g^{ij}} \ .$$

This implies together with (4.5) that on U^{good}

$$g_n^{good}(A_n) \to g^{good}(A) \text{ in } L^2(U^{good}) \ .$$

Step 4. *The bad part's contribution.* The last part of the proof consists of noticing that by diminishing ϵ and by letting U^{good} increase to a set of full measure, we may find gauges $g_n^k = (g^{good})^{-1} g_n^{good}$ such that

$$(g_n^k)^{-1} dg_n^k + (g_n^k)^{-1} A_n g_n^k \to A \text{ in } L^2 \text{ outside a set of measure } \frac{1}{k} \ .$$

By extracting a diagonal subsequence we obtain g_n such that

$$g_n^{-1} dg_n + g_n^{-1} A_n g_n \to A \text{ in } L^2(\mathbb{S}^4) \ .$$

Therefore

$$\operatorname{dist}([A_n], [A]) \to 0 \ ,$$

as desired. □

4.4.2 The second hypothesis of Proposition 4.1

We now assume given a sequence of weak curvatures F_n corresponding to $[A_n] \in \mathcal{A}_G$ on $\mathbb{B}^5$ which are bounded in L^2 and converge weakly in L^2 to a 2-form F. For a fixed center $x_0 \in \mathbb{B}^5$ and for a radii $t \in [r, 2r]$ with $r > 0$, the slices of the connections A_n via spheres $\partial B_t(x_0)$ are defined and taking values in $\mathcal{Y}$ for a.e. t by the assumption that $[A_n] \in \mathcal{A}_G$. We then define (classes of) functions

$$f_n : [r, 2r] \to \mathcal{Y}, \quad f_n(t) := \left[i^*_{\partial B_t(x_0)} A_n \right] \ .$$

Notation: We denote $A(s)$ the slice along $\partial B_s(x_0)$ i.e. the pullback of $i^*_{\partial B_s(x_0)} A$ to $\mathbb{S}^4$ via the homothety $\mathbb{S}^4 \to \partial B_s(x_0)$ when it exists.

We verify that the f_n satisfy the hypothesis (4.4):

Lemma 4.3. *Assume that F is the curvature form corresponding to $[A] \in \mathcal{A}_G$ and choose a representative A which is L^2 on $B_{2r}(x_0) \setminus B_r(x_0)$. Then there exists a gauge change g*

such that $A' := g^{-1}dg + g^{-1}Ag$ has no radial component and such that for a.e. $t > t' \in [r, 2r]$

$$\int_{\mathbb{S}^4} |A'(t) - A'(t')|^2 \lesssim \frac{1}{r^2}|t - t'| \int_{B_t(x_0)\setminus B_{t'}(x_0)} |F|^2 \ , \tag{4.6}$$

for a universal implicit constant.

Proof. We will assume $x_0 = 0$ for simplicity. Note that

$$\int_{t'}^{t} \|A(t)\|^2_{L^2(\mathbb{S}^4)} dt = \int_{\mathbb{S}^4}\int_{t'}^{t} |\rho\, i^*_{\partial B_\rho} A|^2 \rho^4 d\rho d\omega \ .$$

Use Corollary 1.9 to solve the following ODE in polar coordinates:

$$\begin{cases} \partial_\rho g(\omega,\rho) = -A_\rho(\omega,\rho)g(\omega,\rho), & \text{for } \rho \in [t', t] \ , \\ g(\omega, t') = id, & \text{for all } \omega \in \mathbb{S}^4 \ . \end{cases} \tag{4.7}$$

It then follows that for $A' = g^{-1}dg + g^{-1}Ag$ there holds

$$\sum_k \frac{x_k}{\rho} A'_k := A'_\rho = 0 \ ,$$

therefore at (ω, ρ) we write

$$\sum_k x_k g^{-1} F_{ki} g = \sum_k x_k \partial_k A'_i - \sum_k x_k \partial_i A'_k + \sum_k x_k [A'_k, A'_i] = \partial_\rho(\rho A'_i) \ .$$

In other words

$$\rho \partial_\rho \llcorner (g^{-1}Fg)|_{\partial B_s(x_0)} = \partial_\rho(\rho\, i^*_{\partial B_\rho} A') \ .$$

Integrating in s we have for a.e. $t > t'$ and then in ω we obtain

$$\begin{aligned} \int_{\mathbb{S}^4} |t\, i^*_{\partial B_t} A' - t'\, i^*_{\partial B_{t'}} A'|^2 &= \int_{\mathbb{S}^4} \left| \int_{t'}^{t} \rho \partial_\rho \llcorner (g^{-1}Fg)\, d\rho \right|^2 \\ &\lesssim |t - t'| \int_{\mathbb{S}^4 \times [t', t]} \rho^2 |\partial_\rho \llcorner F|^2 \ . \end{aligned}$$

We used Jensen's inequality and the fact that the norm is G-invariant. Note that for $\omega \in \mathbb{S}^4$ there holds

$$A'(s)(\omega) = s\, i^*_{\partial B_s} A'(s\omega) \ ,$$

therefore from above it follows

$$\int_{\mathbb{S}^4} |A'(t) - A'(t')|^2 \lesssim \frac{|t - t'|}{(t')^2} \int_{B_t \setminus B_{t'}} |F|^2 \ .$$

Since $t' > r$ the thesis follows. □

In the end the functions $f_n(t)$ which will satisfy (4.4) in our situation will be the slice functions of the connection forms $A_n(t)$ in the gauges given by Lemma 4.3. Note that as a direct consequence of Lemma 4.3 we have also

$$\operatorname{dist}([A_n(t)],[A_n(t')]) \lesssim \frac{\|F_n\|_{L^2(B_{2r}\setminus B_r)}}{r}|t-t'|^{1/2} \le \frac{\|F_n\|_{L^2}}{r}|t-t'|^{1/2} . \tag{4.8}$$

4.4.3 Proof of Corollary 1.9

Proof. By Theorem 1.8 we have a sequence of connections $[A_k] \in \mathcal{R}^\infty(\mathbb{B}^5)$ such that for some L^2-representatives A_k and for their distributional curvature forms F_k there holds

$$A_k \to A \text{ in } L^2, \quad F_k \to F \text{ in } L^2 .$$

We then solve

$$\begin{cases} \partial_\rho g_k(\omega,\rho) = -(A_k)_\rho(\omega,\rho) g_k(\omega,\rho) & \text{for } \omega \in \mathbb{S}^4, \rho \in [0,t] , \\ g_k(\omega,0) = id & \text{for } \omega \in \mathbb{S}^4 , \end{cases}$$

where the solution g_k is now defined on all rays $\omega = const$ except for the (finitely many) ones which contain one of the singular points of A_k. We have then

$$\|g_k\|_{W^{1,2}(\mathbb{B}\setminus\mathbb{B}_t)} \lesssim \|(A_k)_\rho\|_{L^2(\mathbb{B}\setminus\mathbb{B}_t)} \le \|A_k\|_{L^2(\mathbb{B}\setminus\mathbb{B}_t)} . \tag{4.9}$$

Up to extracting a subsequence we may assume

$$g_k \rightharpoonup g \quad \text{weakly in } W^{1,2}$$

and thus $g_k \to g$ a.e. and strongly in all $L^p, p < \infty$ by interpolation between L^{2^*} and L^∞ (recall that $g_k \in L^\infty$ because G is compact). In particular since g_k^{-1} converges in L^2 and dg_k converges weakly in L^2 we have

$$g_k^{-1} dg_k \overset{\mathcal{D}'}{\rightharpoonup} g^{-1} dg$$

and by the above strong convergence results of A_k in L^2 and of g_k in all L^p, $p < \infty$ we have

$$g_k^{-1} A_k g_k \to g^{-1} A g \quad \text{strongly in } L^q,\ q < 2 .$$

Therefore we achieve the distributional convergence

$$A_k^{g_k} := g_k^{-1} dg_k + g_k^{-1} A_k g_k \overset{\mathcal{D}'}{\rightharpoonup} g^{-1} dg + g^{-1} A g =: A^g .$$

If we insert the above expression of A^g into the formula for the distributional curvature $F_{A^g} = dA^g + A^g \wedge A^g$ we obtain:

$$\begin{aligned} F_{A^g} &= d(g^{-1}dg + g^{-1}Ag) + (g^{-1}dg + g^{-1}Ag) \wedge (g^{-1}dg + g^{-1}Ag) \\ &= -g^{-1}dg \wedge g^{-1}dg - g^{-1}dg \wedge g^{-1}Ag + g^{-1}dA\, g - g^{-1}Ag \wedge g^{-1}dg \\ &\quad + g^{-1}dg \wedge g^{-1}dg + g^{-1}Ag \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Ag + g^{-1}A \wedge Ag \\ &= g^{-1}(dA + A \wedge A)g = g^{-1}F_A g . \end{aligned}$$

Note that the above formal calculations are actually rigourous due to the facts that $dg \in L^2$, $A \in L^2$ and $g, g^{-1} \in L^\infty$. $\square$

4.5 Proof of the Closure Theorem 1.4

We consider a sequence F_n corresponding to $[A_n] \in \mathcal{A}_G(\mathbb{B}^5)$ as in Theorem 1.4 and we construct representatives A_n such that

$$\int_{\mathbb{B}^5} |A_n|^2 \leq C \int_{\mathbb{B}^5} |F_n|^2 \ ,$$

like in Lemma 4.3. We thus have that up to extracting a subsequence there holds

$$A_n \rightharpoonup A \quad \text{ in } L^2(\mathbb{B}^5) \ . \tag{4.10}$$

As noted above it suffices that for all centers x_0 and a.e. radius $t > 0$ the homothety pullback to $\mathbb{S}^4$ of the slice $i^*_{\partial B_t} A$ of the limit connection form A is in $\mathcal{A}_G(\mathbb{S}^4)$ or equivalently corresponds to a class in $\mathcal{Y}$. Fix $x_0 \in \mathbb{B}^5$ and a range of radii $[r, 2r]$. It is sufficient to prove that

$$\text{a.e. } s \in [r, 2r], \quad A(s) \in \mathcal{A}_G(\mathbb{S}^4) \ . \tag{4.11}$$

We will assume for simplicity that $x_0 = 0$ and we apply Lemma 4.3 obtaining new gauges for the A_n in which (4.8) is valid. From now on we are going to work in these gauges only. For simplicity of notation we still denote the expressions of the A_n in these gauges by A_n. Note that we still obtain the control

$$\|A_n\|_{L^2(B_{2r} \setminus B_r)} \lesssim \|F_n\|_{L^2}$$

if in the proof of Lemma 4.3 for $A = A_n$ we replace the ODE (4.7) by

$$\begin{cases} \partial_\rho g(\omega, \rho) = -(A_n)_\rho(\omega, \rho) g(\omega, \rho), & \text{for } \rho \in [s, t] \ , \\ g(\omega, s) = id, & \text{for all } \omega \in \mathbb{S}^4 \ . \end{cases}$$

for s such that $A_n(s)$ satisfies

$$\|A_n(s)\|_{L^2} \lesssim \frac{1}{r} \|F_n\|_{L^2} \ .$$

Thus we may still suppose that (4.10) holds on $B_{2r} \setminus B_r$. We next prove that in this case we have a stronger convergence:

Lemma 4.4. *Assume that for a sequence of connection forms $A_n \in L^2(B_{2r} \setminus B_r, \wedge^1 \mathbb{R}^5 \otimes \mathfrak{g})$ there holds*

$$\|A_n(t) - A_n(t')\|_{L^2(\mathbb{S}^4)} \leq C|t - t'|^{1/2}$$

and that
$$A_n \rightharpoonup A \quad \text{weakly in } L^2 \text{ on } B_{2r} \setminus B_r \ .$$
Then there exists a subsequence n' such that
$$\text{for a.e. } s \in [r, 2r] \text{ there holds } A_{n'}(s) \rightharpoonup A(s) \quad \text{weakly in } L^2(\mathbb{S}^4) \ . \tag{4.12}$$

Proof. The weak convergence hypothesis means that
$$\int A_n \wedge \beta \to \int A \wedge \beta \text{ for all } \beta \in L^2(B_{2r} \setminus B_r, \wedge^3 \mathbb{R}^5 \otimes \mathfrak{g}) \ .$$

Consider an arbitrary 3-form ω which is L^2 on $\mathbb{S}^4$ and a test 1-form $\varphi(t)$ on $[r, 2r]$. By taking
$$\beta := h_t^* \omega \wedge \varphi(t) \quad \text{where } h_t : \mathbb{S}^4 \to \partial B_t \text{ is a homothety}$$
we obtain
$$\int_r^{2r} \int_{\mathbb{S}^4} A_n(t) \wedge \omega \wedge \varphi(t) \to \int_r^{2r} \int_{\mathbb{S}^4} A(t) \wedge \omega(x) \wedge \varphi(t) \ .$$
If we use the notation
$$f_n^\omega(t) = \int_{\mathbb{S}^4} A_n(t) \wedge \omega \ ,$$
then from the first hypothesis it follows that
$$\begin{aligned} |f_n^\omega(t) - f_n^\omega(t')| &\leq \|A_n(t) - A_n(t')\|_{L^2} \|\omega\|_{L^2} \\ &\leq C|t - t'|^{1/2} \|\omega\|_{L^2} \ . \end{aligned}$$

By Arzelà-Ascoli theorem the f_n^ω have a subsequence which converges uniformly to a 1/2-Hölder function with the same Hölder constant:
$$\sup_{t \in [r, 2r]} |f_n^\omega(t) - f^\omega(t)| \to 0 \ .$$

By applying this reasoning to a countable L^2-dense subset D of ω's in $L^2(\mathbb{S}^4, \wedge^3 T\mathbb{S}^4 \otimes \mathfrak{g})$ and by a diagonal procedure we obtain that
$$\forall \omega \in D, \quad \sup_{t \in [r, 2r]} |f_n^\omega(t) - f^\omega(t)| \to 0 \ .$$

Since the functionals $\omega \mapsto \int A_n(t) \wedge \omega$ are strongly continuous on L^2 forms for a.e. t, we obtain that the above convergence holds on all $\omega \in L^2$, completing the proof. $\square$

We are now ready to conclude the proof of the weak closure result.

End of proof of Theorem 1.4: Consider the global weak limit connection form $A \in L^2(\mathbb{B}^5)$. As said above we prove that a.e. slice of it is in $\mathcal{A}_G(\mathbb{S}^4)$ by considering separately the groups of slices with center x_0 and radii in $[r, 2r]$. We assumed $x_0 = 0$ for simplicity and we obtained that the A_n have a weakly convergent subsequence on $B_{2r} \setminus B_r$, therefore we may apply Lemma 4.4. We obtain up to extracting a subsequence the slicewise a.e. weak convergence (4.12):

$$\text{for a.e. } s \in [r, 2r] \text{ there holds } A_n(s) \rightharpoonup A(s) \quad \text{weakly in } L^2(\mathbb{S}^4) \ .$$

Note that in this case the slicewise weak limit $A(s)$ is indeed the slice of the limit connection.

On the other hand we saw in Section 4.4 that the hypotheses of Proposition 4.1 are verified for our A_n therefore we also have up to another subsequence extraction

$$\text{for a.e. } s \in [r, 2r] \text{ there holds } [A_n(s)] \to [A^d(s)] \quad \text{in } (\mathcal{Y}, \text{dist}) \ .$$

We have now to compare the slice $A(s)$ of the weak limit with the dist-limit of slices $A^d(s)$. Since

$$\text{dist}([A_n(s)], [A^d(s)]) = \inf_{g \in W^{1,2}(\mathbb{S}^4, G)} \|g^{-1}dg + g^{-1}A_n(s)g - A^d(s)\|_{L^2} \ ,$$

we obtain a sequence $g_n(s) \in W^{1,2}(\mathbb{S}^4, G)$ such that

$$g_n(s)^{-1}dg_n(s) + g_n(s)^{-1}A_n(s)g_n(s) - A^d(s) \to 0 \quad \text{strongly in } L^2 \ . \tag{4.13}$$

It follows that

$$\|dg_n(s)\|_{L^2} \lesssim \|A^d(s)\|_{L^2} + \|A_n(s)\|_{L^2} \ .$$

From

$$\|A_n(t) - A_n(t')\|_{L^2} \leq C|t - t'|^{1/2}$$

and from the fact that for all n there exists $s \in [r, 2r]$ such that

$$\|A_n(s)\|_{L^2} \lesssim \|F_n\|_{L^2} \leq C$$

it follows that $A_n(s)$ is bounded in L^2. Thus $dg_n(s)$ is also bounded in L^2. Thus up to extracting a subsequence (dependent on t)

$$dg_n(t) \rightharpoonup dg_\infty(t) \quad \text{weakly in } L^2 \ .$$

Since $g_n(s)$ is also bounded in L^∞ we obtain by Rellich's theorem and by interpolation that up to extracting a subsequence $n(t)$

$$g_n(t) \to g_\infty(t) \quad \text{in } L^q \ \forall q < \infty \ .$$

The last two facts together with the convergence $A_n(t) \stackrel{L^2}{\rightharpoonup} A(t)$ suffice to prove that

$$g_n(t)^{-1}A_n(t)g_n(t) \ \rightarrow \ g_\infty(t)^{-1}A(t)g_\infty(t) \text{ in } \mathcal{D}'(\mathbb{S}^4) \ ,$$

$$g_n(t)^{-1}dg_n(t) \ \rightarrow \ g_\infty(t)^{-1}dg_\infty(t) \text{ in } \mathcal{D}'(\mathbb{S}^4) \ .$$

This is valid for a.e. $t \in [r, 2r]$. Therefore

$$A^d(t) = g_\infty(t)^{-1}dg_\infty(t) + g_\infty(t)^{-1}A(t)g_\infty(t), \qquad \text{for a.e. } t \in [r, 2r] \ .$$

Since $A^d(t) \in \mathcal{A}_G(\mathbb{S}^4)$, this shows that for a.e. t the slice $A(t)$ of the limit connection form A belongs to $\mathcal{A}_G(\mathbb{S}^4)$, as desired. $\square$

5 Regularity results

This section is devoted to the proofs of Theorem 1.12 and its important Corollary 1.13 and the regularity of minimizers, Theorem 1.14. The structure of the proofs is analogous to the celebrated theory of harmonic maps, cfr. [46] and the references therein. We apply our new approximation and extended regularity results in order to complete all the steps for curvatures in $\mathcal{A}_G(\mathbb{B}^5)$. The analogous results hold on general Riemannian compact 5-manifolds and the proofs can be extended by working in charts and including error terms corresponding to the fact that the metric is not euclidean.

We start by proving Proposition 1.15, accoding to which the Bianchi identity $d_AF = 0$ is verified by curvature forms F and connection forms A corresponding to $[A] \in \mathcal{A}_G(\mathbb{B}^5)$.

Proof of Proposition 1.15: We use the result of Theorem 1.8, namely the existence of a sequence of connection forms A_k which are L^2 and have curvatures F_k also in L^2, such that $[A_k] \in \mathcal{R}^\infty(\mathbb{B}^5)$ and

$$A_k \to A \ \text{ in } L^2, \quad F_k \to F \ \text{ in } L^2 \ .$$

In particular we have $dF_k \stackrel{W^{-1,2}}{\rightharpoonup} dF$ and $\int_{\mathbb{B}^5} \varphi \wedge [F_k, A_k] \to \int_{\mathbb{B}^5} \varphi \wedge [F, A]$ for all $C^\infty_c(\mathbb{B}^5)$ test 1-forms ϕ. This implies in particular that

$$d_{A_k}F_k \rightharpoonup d_AF \quad \text{ in the sense of distributions} \ ,$$

thus we reduce to prove (1.25) for $[A] \in \mathcal{R}^\infty(\mathbb{B}^5)$. In this case we see directly from the classical results that $d_AF \equiv 0$ locally outside the defects $a_1, \ldots, a_k$ of the classical bundle from the definition of $\mathcal{R}^\infty$. Since we have that d_AF is a tempered distribution, it must then be locally near a_i of the form $\sum_{\alpha=0}^l c_\alpha \delta_{a_i}^{(\alpha)}$, where $\delta_x^{(\alpha)}$ is the α-th distributional derivative of the Dirac mass at x. On the other hand, since $F \in L^2$ and $[A, F] \in L^1$ we obtain that $d_AF \in W^{-1,2}_{loc}$ near a_i. Since we can construct forms ϕ_n which are bounded in $W^{1,2}$ but have values of the first l derivatives in a_i, larger than n we see that if $c_\alpha \neq 0$ for some α then

$$C \geq \langle d_A F, \phi_n \rangle = \sum_{\alpha=1}^{n} c_\alpha \phi_n^{(\alpha)} \to \infty \ ,$$

which is a contradiction. Thus $d_A F = 0$ and this concludes the proof. □

5.1 Partial regularity for stationary connections in $\mathcal{A}_G$

In this section we show how to bootstrap the results of [34] to the space $\mathcal{A}_G(\mathbb{B}^5)$, in order to prove the partial regularity result of Corollary 1.13.

The main step is to improve on the result of [34] by removing the smooth approximability requirement (cfr. Theorem I.3 of [34]). Once this proof is done, the strategy of [34] can proceed to the proof of Theorem 1.12 and to the regularity result of Corollary 1.13 with no changes.

Proof of Theorem 1.11: In [34] the existence of ϵ, C for which a gauge g in which (1.19), (1.20) and (1.21) hold was proved under the assumption that A be strongly approximable in $W^{1,2} \cap L^4$ by connection forms of smooth connections. In particular we may apply the result of [34] to the connection forms $\hat{A}_k$ furnished by Theorem 1.10. We obtain gauge changes g_k such that

$$A_k := \left(\hat{A}_k\right)_{g_k} \qquad \text{satisfies (1.19),(1.20), (1.21)}$$

with F replaced by F_k. Since $\hat{A}_k \stackrel{L^2}{\to} A, \|A_k\|_{L^2} \lesssim \|F_k\|_{L^2} \lesssim \|F\|_{L^2}$ we obtain

$$\|dg_k\|_{L^2} \leq C(\|\hat{A}_k\|_{L^2} + \|A_k\|_{L^2}) \leq C$$

therefore up to subsequence we can assume that g_k converge pointwise a.e., weakly in $W^{1,2}$ and (by interpolation with L^∞) in L^p for all $p < \infty$. Similarly we may assume that $A_k \to A_\infty$ in L^q for all $q < 2^*$. It follows from the defining equation $g_k^{-1} dg_k + g_k^{-1} \hat{A}_k g_k = A_k$ that

$$g_k^{-1} dg_k \to g_\infty^{-1} dg_\infty \quad \text{ strongly in } L^2 \ ,$$

thus we have that

$$A_{g_\infty} = A_\infty \ ,$$

in particular g_∞ is such that conditions (1.19), (1.20) and (1.21) hold, since they are stable under strong L^2 limits. □

5.2 The regularity of local minimizers of the Yang-Mills energy in dimension 5

In this section we prove Theorem 1.14, which is a new result since the existence of minimizers and thus the availability of energy comparison techniques was not available before the introduction of the class $\mathcal{A}_G$.

5.2.1 Luckhaus type lemma for weak curvatures

Our aim in this section is to prove the following proposition, using a Luckhaus-type lemma for interpolating weak connections with L^2-small curvatures while paying a small curvature cost.

Proposition 5.1. *Assume that F_k are curvature forms corresponding to local minimizers $[A_k] \in \mathcal{A}_G(\mathbb{B}^5)$ and that $F_k \rightharpoonup F$ weakly in L^2 and $\sup_k \|F_k\|_{L^2(\mathbb{B}^5)} \leq C$. Then $F_k \to F$ strongly sin L^2 on a smaller ball $\mathbb{B}^5_{\frac{1}{2}}$, and F is a local minimizer as well.*

The main tool for the proof above is the following lemma:

Lemma 5.2 (Luckhaus-type lemma for $\mathcal{A}_G$)**.** *Assume that F_0, F_1 are curvature forms on $\mathbb{B}^5_{t+4\epsilon}$ corresponding to connection forms $A_0, A_1 \in \mathcal{A}_G(\mathbb{B}_{t+3\epsilon})$ such that*

$$\|F_\alpha\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} < \epsilon_0, \quad \|A_t\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} < \epsilon_0 \ . \tag{5.1}$$

Then there exists a connection form $\hat{A}$ corresponding to $[\hat{A}] \in \mathcal{A}_G(\mathbb{B}_{t+4\epsilon})$ such that

$$\hat{A} = A_0 \text{ on } \mathbb{B}_{t-2\epsilon}, \quad \hat{A} = A_1 \text{ on } \mathbb{B}_{t+4\epsilon} \setminus \mathbb{B}_{t+3\epsilon} \tag{5.2}$$

and

$$\|F_{\hat{A}}\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} \leq C\|F_0\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} + \|F_1\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} \ . \tag{5.3}$$

Proof. **Step 1.** *Good grid of balls.* Like in Proposition 2.5 construct a good grid of balls of scale ϵ which form a cover of $\mathbb{B}_{t+\epsilon} \setminus \mathbb{B}_t$ and have centers on $\partial\mathbb{B}_{t+\epsilon/2}$. Note that since $\alpha \in]1, 2[$ such balls will stay in $\mathbb{B}_{t+3\epsilon} \setminus \mathbb{B}_{t-2\epsilon}$.

Step 2. *$W^{1,2}$ representatives on the boundary of a ball.* From now on we will work on a fixed ball B of the above-defined good grid. We want to perform a modification of the approximation procedure like in the proof of Theorem 1.8. This consists in first interpolating on the boundary ∂B and then extending the interpolant to B. We note that by the definition of $\mathcal{A}_G$, for $\alpha = 0, 1$ we may find guages $g_\alpha \in W^{1,2}(\partial B, G)$ such that $\tilde{A}_\alpha := g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} A_\alpha g_\alpha \in W^{1,2}$.

Step 3. *Interpolating gauges and connections on ∂B.* By Fubini's theorem and a pigeonhole principle we may find numbers $a_0 \in [0, 1/4], a_1 \in [3/4, 1]$ and a universal constant C such that

$$\|g_\alpha|_{W^{1,2}(\partial B \cap \partial \mathbb{B}_{t+a_\alpha \epsilon})} \leq C\|g_\alpha\|_{W^{1,2}(\partial B)} \quad \text{for } \alpha = 0, 1 \ . \tag{5.4}$$

We may then use (5.4) and apply Luchkaus' [33] procedure for the extension of $W^{1,2}$ maps into manifolds and find $\tilde{g} \in W^{1,2}(\partial B, G)$ such that

$$\begin{aligned} \tilde{g} &= g_0 \quad \text{on } \partial B \cap \mathbb{B}_{t+a_0\epsilon} \ , \\ \tilde{g} &= g_1 \quad \text{on } \partial B \setminus \mathbb{B}_{t+a_1\epsilon} \ , \\ \|\tilde{g}\|_{W^{1,2}(\partial B \cap (\mathbb{B}_{t+a_1\epsilon} \setminus \mathbb{B}_{t+a_0\epsilon})} &\leq C(\|g_0\|_{W^{1,2}(\partial B \cap (\mathbb{B}_{t+a_1\epsilon} \setminus \mathbb{B}_{t+a_0\epsilon})} + \|g1\|_{W^{1,2}(\partial B \cap (\mathbb{B}_{t+a_1\epsilon} \setminus \mathbb{B}_{t+a_0\epsilon})}) \ . \end{aligned}$$

We then extend the curvature forms simply by interpolating along meridians, i.e. we fix an increasing smooth function $\eta : [0, t + 4\epsilon] \to [0, 1]$ such that $\eta \equiv 0$ on $[0, t - a_0\epsilon]$ and $\eta \equiv 1$ on $[t + a_1\epsilon, t + 4\epsilon]$ and for polar coordinates $(\omega, \tau) = x$ centered at 0 and $(\omega, \tau) \in \partial B$ we define

$$\tilde{A}(\omega, \tau) = (1 - \eta(\tau))i^*_{\partial B}A_0 + \eta(\tau)i^*_{\partial B}A_1 \ .$$

As a consequence we obtain

$$\|\tilde{A}\|_{W^{1,2}(\partial B)} \leq C(\|A_0\|_{W^{1,2}(\partial B)} + \|A_1\|_{W^{1,2}(\partial B)}) \ .$$

Step 4. *Extension on good and bad balls.* We use the same notion of good and bad balls as in Lemma-Definition 2.9 with the exception that we require the inequalities to be contemporarily valid for both A_0, A_1. The estimates of the mentioned lemma remain true, up to changing the constants by a universal factor. In the case of a good ball B the extension of $\tilde{A}$ to the interior of B and the construction of $\hat{g}$ starting from $\tilde{g}$ are done as in Proposition 2.3. The estimates on $\tilde{g}, \tilde{A}$ from Step 3 together with the proof of Proposition 2.3 give, as a consequence of the rescaled versions of (2.18), (2.19), the estimates

$$\|d\hat{A} + \hat{A} \wedge \hat{A}\|^2_{L^2(B)} \lesssim \epsilon\|F_0\|^2_{L^2(\partial B)} + \epsilon\|F_1\|^2_{L^2(\partial B)}$$

and

$$\|\hat{A}\|_{L^2(B)} \lesssim \sum_{\alpha=0,1} \left(\epsilon\|F_\alpha\|^2_{L^2(\partial B)} + \epsilon\|A\|^2_{L^2(\partial B)}\right) \ .$$

If B is a bad ball we directly extend $\tilde{A}$ radially inside.

Step 5. *Summing up the estimates.* The conclusion of our proof consists of repeating Steps 1-5 and 8 of the proof of Theorem 1.8, i.e. we just jump the part where we perform the smoothing on the 4-skeleton of our good grid. The estimates from the previous step and the trivial estimates for the bad balls give then the desired result. □

Proof of Proposition 5.1: **Step 1.** We divide the interval $[1/2, 1-4\epsilon]$ in N equal subintervals of lenght 5ϵ, for $1/N \le \epsilon_0/C$. By pigeonhole principle there exists one of such intervals $I = [t-2\epsilon, t+3\epsilon] \subset [1/2, 1]$ such that up to subsequence we may assume

$$\|F_k\|_{L^2(\{x:|x|\in I\})} \le \epsilon_0, \quad \|F\|_{L^2(\{x:|x|\in I\})} \le \epsilon_0 \ .$$

Step 2. We may reduce to the setting of Lemma 5.2 with $F_0 = F_k, F_1 = F$. Let $\hat{F}_k$ be the interpolant produced in the Lemma 5.2. We have the following estimate:

$$\|\hat{F}_k\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} \lesssim N^{-1}(\|F_k\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})} + \|F\|_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})}) \ .$$

It is easy to check that the curvature $\hat{F}_k$ is still in $\mathcal{F}_{\mathbb{Z}}(\mathbb{B}^5)$.

Step 3. We use the fact that F_k is locally minimizing to write the following inequalities:

$$\begin{aligned}
\|F_k\|^2_{L^2(\mathbb{B}_{t-2\epsilon})} &\le \|F_k\|^2_{L^2(\mathbb{B}_{t+3\epsilon})}\\
&\le \|\tilde{F}_k\|^2_{L^2(\mathbb{B}_{t+3\epsilon})}\\
&= \|F\|^2_{L^2(\mathbb{B}_{t-2\epsilon})} + \|\hat{F}_k\|^2_{L^2(\mathbb{B}_{t+3\epsilon}\setminus\mathbb{B}_{t-2\epsilon})}\\
&= \|F\|^2_{L^2(\mathbb{B}_{t-2\epsilon})} + o_\epsilon(1) \ .
\end{aligned}$$

In particular we see that no energy is lost in the limit on $\mathbb{B}_{t-2\epsilon}$:

$$\|F_k\|_{L^2(\mathbb{B}_{t-2\epsilon})} \to \|F\|_{L^2(\mathbb{B}_{t-2\epsilon})} \ ,$$

which proves the result. □

5.2.2 Dimension reduction for the singular set

This section is devoted to the proof of Theorem 1.14. We use the following definition:

Definition 5.3. *We denote by reg(F) the set of points x such that over some neighborhood $U \ni x$ there exists a smooth classical G-bundle $P \to U$ such that F is the curvature form of a smooth connection over P. The complement of reg$F$ is denoted sing(F).*

Proof of Theorem 1.14: It can be proved (see [49] or [34]) from the monotonicity formula (see [41]) that for minimizing curvatures F, $\mathcal{H}^1(\text{sing}(F)) = 0$. If $S := \text{sing}F$ and F is a minimizing curvature we consider now $s \ge 0$ for which $\mathcal{H}^s(S \cap \Omega') > 0$. Then $\mathcal{H}^s$-a.e. x_0 there holds

$$\liminf_{\lambda\downarrow 0} \lambda^{-s}\mathcal{H}^s(S \cap B_{\lambda/2}(x_0)) > 0 \ . \tag{5.5}$$

From the monotonicity formula we have (see [49]) that for any subsequence $\lambda_i \to 0$ such that the blown-up curvature forms $F_{\lambda_i} := \tau^*_{\lambda_i,x_0}F$, the weak limit curvature form F_0 is radially homogeneous. Here $\tau_{\lambda,x}$ is the homothety of factor λ and center x. By Proposition 5.1 the convergence is also strong and F_0 is a minimizer.

$S_i := \mathrm{sing}F_{\lambda_i}$ which are the blow-ups of S, satisfy $\mathcal{H}^s(S_i \cap B_{1/2}) = \lambda_i^{-s}\mathcal{H}^s(S \cap B_{\lambda_i/2})$ thus from (5.5) we obtain

$$\mathcal{H}^s(S_0 \cap B_{1/2}) > 0 \ . \tag{5.6}$$

As in [49] from the stationarity we deduce that F_0 is radial and radially homogeneous. In particular S_0 is also radially invariant, i.e. $\lambda S_0 \subset S_0$ for $\lambda > 0$. Assume $S_0 \neq \{0\}$. In particular S_0 must then contain a line and in this case $\mathcal{H}^1(S_0) > 0$. However since F_0 is still a minimizer this contradicts Corollary 1.13.

The fact that $S_0 = \{0\}$ for blown-up curvatures implies also that for a minimizer F the singular points do not accumulate. Indeed if $x_i \to x_0$ were accumulating singular points, then by carefully choosing the blowup sequence we would be able to obtain F_0 such that $S_0 \supset \{0, u/4\}$ where u is a unit vector. □

6 Consequences of closure and approximability

We will prove here Theorem 1.6 which completes the proof of Theorem 1.7. The proofs are along the lines of the reasoning [35] done in the case of abelian curvatures.

The distance dist on gauge-equivalence classes of connections is used to compare the boundary datum with the slices of forms $F \in \mathcal{A}_G$. We abuse notation and denote by $f(x+\rho)$ the form (with variable $x \in \mathbb{S}^4$) corresponding to the restriction to $\partial B_{1-\rho}$ of the form F. This notation is inspired by the analogy to slicing via parallel hyperplanes, instead of spheres. We then define the class $\mathcal{A}_{G,\varphi}(\mathbb{B}^5)$ via the continuity requirement

$$\mathrm{dist}(f(x+\rho'), \varphi(x)) \to 0, \text{ as } \rho' \to 0^+ \ . \tag{6.1}$$

It is clear that the definition (6.1) satisfies the *nontriviality* and *compatibility* conditions, since $\mathrm{dist}(\cdot,\cdot)$ is a distance and since for $\mathcal{R}^\infty$ having smooth boundary datum implies that in a neighborhood of $\partial\mathbb{B}^5$ the slices are smooth up to gauge and converge in the smooth topology to φ. The validity of the *well-posedness* is a bit less trivial, therefore we prove it separately.

Theorem 6.1. *If $F_n \in \mathcal{A}_{G,\varphi}(\mathbb{B}^5)$ are converging weakly in L^2 to a form $F \in \mathcal{A}_G(\mathbb{B}^5)$ then also F belongs to $\mathcal{A}_{G,\varphi}(\mathbb{B}^5)$.*

Proof. By weak semicontinuity of the L^2 norm we have that F_n are bounded in this norm, $||F_n||_{L^2(B_1 \setminus B_{1-h})} \leq C$.

Therefore by Lemma 4.3 the f_n are dist-equi-Hölder, so a subsequence (which we do not relabel) of the f_n converges to a slice function f_∞ with values in Y a.e.. For all

$\rho' \in [0,\rho]$ the forms $f_n(\cdot + \rho')$ are a Cauchy sequence in n, for the distance dist. This is enough to imply that f_∞ is equal to the slice of F. Even if F is just defined up to zero measure sets, it still has a dist-continuous representative. By uniform convergence it is clear that f still satisfies (6.1). $\square$

The same proof also gives an apparently stronger result:

Theorem 6.2. *If $F_n \in \mathcal{A}_{G,\varphi_n}(\mathbb{B}^5)$ are converging weakly in L^2 to a form $F \in \mathcal{A}_G(\mathbb{B}^5)$ then the forms φ_n converge with respect to the distance dist to a form φ and also F belongs to $\mathcal{A}_{G,\varphi}(\mathbb{B}^5)$.*

Remark 6.3. *The definition of the distance can be extended as in [35] and allows to extend the definition of the boundary value to arbitrary domains.*

A Controlled gauges on the 4-sphere

Recall that $\pi : L^2(\mathbb{S}^4, \mathfrak{g}) \to \left(\mathrm{Span}\left\{i^*_{\mathbb{S}^4} dx_k,\ k = 1, \dots, 5\right\}\right)^\perp$ denotes the L^2 projection operator.

In this section we follow the overall structure of the argument from [51] to prove the following result:

Theorem A.1. *There exist constants ϵ_0, C with the following properties. If $A \in W^{1,2}(\mathbb{S}^4, \mathfrak{g})$ is a (global) connection form over $\mathbb{S}^4$ such that the corresponding curvature form F satisfies*

$$\|F\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \leq \epsilon_0$$

then there exists a gauge transformation $g \in W^{2,2}(\mathbb{S}^4, G)$ such that

$$d^*_{\mathbb{S}^4}(g^{-1}dg) = d^*_{\mathbb{S}^4}(\pi(g^{-1}dg))$$

and denoting $A^g = g^{-1}dg + g^{-1}Ag$ the new expression of the connection form after the gauge transformation g there holds

$$d^*_{\mathbb{S}^4}\left(\pi\left(A^g\right)\right) = 0 \quad \text{and} \quad \|A^g\|_{W^{1,2}(\mathbb{S}^4)} \leq C(\|F\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)})\ .$$

The proof consists in studying the case where the integrability exponent 2 is replaced by $p > 2$ first, and then obtaining the $p = 2$ cases as a limit. Note that for $p > 2$ the space $W^{2,p}(\mathbb{S}^4, G)$ embeds continuously in $C^0(\mathbb{S}^4, G)$, thus gauges g of small $W^{2,p}$-norm will be expressible as $g = \exp(v)$ for some $v \in W^{2,p}(\mathbb{S}^4, \mathfrak{g})$, due to the local invertibility of the exponential map $\exp : G \to \mathfrak{g}$.

We then consider the space

$$E_p := \left\{ v \in W^{2,p}(\mathbb{S}^4, \mathfrak{g}) : \int_{\mathbb{S}^4} v x_k = 0,\ k = 1, \ldots, 5 \right\}$$

where x_k are the ambient coordinate functions relative to the canonical immersion $\mathbb{S}^4 \to \mathbb{R}^5$. In case $p > 2$ the Banach space E_p is, by the above considerations, the local model of the Banach manifold

$$M_p := \left\{ g \in W^{2,p}(\mathbb{S}^4, G) : \int \langle g^{-1} dg, i^*_{\mathbb{S}^4} dx_k \rangle = 0,\ k = 1, \ldots, 5 \right\} .$$

We then consider the sets

$$\mathcal{U}_p^\epsilon := \left\{ A \in W^{1,p}(\mathbb{S}^4, \wedge^1 T\mathbb{S}^4 \otimes \mathfrak{g}) : \|F_A\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)} \le \epsilon_0 \right\}$$

and their subsets

$$\mathcal{V}_p^{\epsilon, C_p} := \left\{ \begin{array}{c} A \in \mathcal{U}_p^\epsilon : \exists g \in M_2 \text{ s.t. } d^*_{\mathbb{S}^4}(\pi(A^g)) = 0, \\ \|\pi(A^g)\|_{W^{1,q}} \le C_q(\|F\|_{L^q} + \|A\|_{L^q}) \text{ for } q = 2, p \\ \text{and } \|F\|_{L^2} + \|A\|_{L^2} < \epsilon \end{array} \right\} .$$

A.1 Proof of Theorem A.1

Like in [51] we prove theorem A.1 by showing that if $\epsilon_0 > 0$ is small enough then for $p \ge 2$ we may find C_p such that

$$\mathcal{V}_p^{\epsilon_0, C_p} = \mathcal{U}_p^{\epsilon_0} . \tag{A.1}$$

We are interested in (A.1) just for $p = 2$ but we use the cases $p > 2$ in the proof: we successively prove the following statements.

1. $\mathcal{U}_p^\epsilon$ is path-connected.

2. For $p \ge 2$ the set $\mathcal{V}_p^{\epsilon, C_p}$ is closed in $W^{1,p}(\mathbb{S}^4, \wedge^1 T\mathbb{S}^4 \otimes \mathfrak{g})$.

3. For $p > 2$ there exists C_p, ϵ_0 such that the set $\mathcal{V}_p^{\epsilon_0, C_p}$ is open relative to $\mathcal{U}_p^{\epsilon_0}$. In particular (A.1) is true for $p > 2$.

4. There exists K such that if $g \in M_p$, $\|A^g\|_{L^4} \le K$ and

   $$d^*_{\mathbb{S}^4}(\pi(A^g)) = 0, \quad \|F\|_{L^2} + \|A\|_{L^2} < \epsilon_0$$

   then

   $$\|A^g\|_{W^{1,2}} \le C_2(\|F\|_{L^2} + \|A\|_{L^2}) .$$

5. The case $p = 2$ of (A.1) follows from the case $p > 2$.

Proof of step 1

Fix $p \geq 2, \epsilon, A \in \mathcal{U}_p^\epsilon$. We observe that $0 \in \mathcal{U}_p^\epsilon$. Moreover the connection forms $A_t(x) := tA(tx)$ for $t \in [0,1]$ all belong to $\mathcal{U}_p^\epsilon$ as well, like in [51].

Proof of step 2

Let $A_k \in \mathcal{V}_p^{\epsilon,C_p}$ be a sequence of connection forms converging in $W^{1,p}$ to A. Consider the gauges g_k as in the definition of $\mathcal{V}_p^{\epsilon,C_p}$. We may assume that the $A_k^{g_k}$ have a weak $W^{1,p}$-limit $\tilde{A}$. The bounds and equation in the definition of $\mathcal{V}_p^{\epsilon,C_p}$ are preserved under weak limit thus we finish if we prove that $\tilde{A}$ is gauge-equivalent to A via a gauge $g \in M_p$. We note that from $dg_k = g_k A_k^{g_k} - A_k g_k$ and the fact that $G \subset \mathbb{R}^N$ is bounded it follows that $\|dg_k\|_{L^{p^*}} \lesssim \|A_k^{g_k}\|_{W^{1,p}} + \|A_k\|_{W^{1,p}}$, thus it has a weakly convergent subsequence, $g_k \overset{W^{1,p^*}}{\rightharpoonup} g$. Thus we may pass to the limit the gauge change equation and obtain indeed $\tilde{A} = A^g$ and also $g \in M_p$.

Proof of step 3

Fix $p > 2$ and let $A \in \mathcal{V}_p^{\epsilon,C_p}$. Consider the following data:

$$\begin{array}{rcl} g & \in & M_p\ , \\ \eta & \in & W^{1,p}(\mathbb{S}^4, \wedge^1 T\mathbb{S}^4 \otimes \mathfrak{g})\ . \end{array}$$

Consider the following function of such g, η, with values in $L^p \cap \{x_k,\ k = 1, \ldots, 5\}^{\perp_{L^2}}$:

$$N_A(g,\eta) := d^*_{\mathbb{S}^4}\left(\pi\left(g^{-1}dg + g^{-1}(A+\eta)g\right)\right) = d^*_{\mathbb{S}^4}\left(g^{-1}dg + \pi\left(g^{-1}(A+\eta)g\right)\right)\ .$$

Note that $N_A(id, 0) = 0$ and N_A is C^1. We want to apply the implicit function theorem in order to solve in g the equation $N_A(g,\eta) = 0$ for η in a $W^{1,p}$-neighborhood of $id \in M_p$. The implicit function theorem will imply also that the dependence of g on η will be continuous. Note that up to order 1 in t there holds $\exp(tv)^{\pm 1} \sim 1 \pm tv$. Using this and the fact that E_p is the tangent space to M_p at id we find the linearization of N_A at $(id, 0)$ in the first variable:

$$\begin{array}{rcl} H_A(v) & := & \partial_g N_A(id,0)[v] \\ & = & \left.\dfrac{\partial}{\partial t}\right|_{t=0}\left[d^*_{\mathbb{S}^4}\left(\pi\left((\exp(tv))^{-1}d\exp(tv) + \exp(tv)^{-1}(A+\eta)\exp(tv)\right)\right)\right] \\ & = & d^*_{\mathbb{S}^4}\left(dv + \pi([A,v])\right) \\ & = & d^*_{\mathbb{S}^4}dv + [\pi(A), dv]\ . \end{array}$$

In the last passage we utilized the fact that π acts only on the coefficients of A and thus $\pi[A,v] = [\pi A, v]$ and the fact that $d^*_{\mathbb{S}^4}[\pi(A), v] = [d^*_{\mathbb{S}^4}(\pi(A)), v] + [\pi(A), dv]$ where the first term vanishes by hypothesis. We see that $H_A : E_p \to L^p \cap \{x_k,\ k = 1, \ldots, 5\}^{\perp_{L^2}}$ is thus given by

$$H_A(v) = \Delta_{\mathbb{S}^4} v + [\pi(A), dv] \ .$$

By elliptic theory and Sobolev and Hölder inequalities in dimension 4 we have

$$\begin{aligned} \|H_A(v)\|_{L^p} &\geq \|\Delta_{\mathbb{S}^4} v\|_{L^p} - \|[\pi(A), dv]\|_{L^p} \\ &\geq c_p \|v\|_{W^{2,p}} - c'_p \|\pi(A)\|_{L^4} \|v\|_{W^{2,p}} \ . \end{aligned}$$

For $c'_p/c_p \|\pi(A)\|_{L^4} < \frac{1}{2}$ we find that H_A is invertible and the thesis follows.

Proof of step 4

We start by observing that since $d^*_{\mathbb{S}^4}(\pi(A^g)) = 0, \langle g^{-1}dg, i^*_{\mathbb{S}^4} dx_k \rangle_{L^2} = 0$ there holds

$$\begin{aligned} d^*_{\mathbb{S}^4} A^g &= \sum_{k=1}^{5} 5x_k \fint_{\mathbb{S}^4} \langle A^g, i^*_{\mathbb{S}^4} dx_k \rangle \\ &= \sum_{k=1}^{5} 5x_k \fint_{\mathbb{S}^4} \langle g^{-1}Ag, i^*_{\mathbb{S}^4} dx_k \rangle \ , \end{aligned}$$

thus by invariance of the norm and Jensen's inequality

$$\begin{aligned} \|d^*_{\mathbb{S}^4} A^g\|_{L^2} &= \left(\int_{\mathbb{S}^4} \left| \sum_{k=1}^{5} 5x_k \fint_{\mathbb{S}^4} \langle g^{-1}Ag, i^*_{\mathbb{S}^4} dx_k \rangle \right|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\mathbb{S}^4} |A|^2 \right)^{\frac{1}{2}} = C\|A\|_{L^2} \ . \end{aligned}$$

By Hodge inequality

$$\begin{aligned} \|\nabla A^g\|_{L^2} &\lesssim \|dA^g\|_{L^2} + \|d^*_{\mathbb{S}^4} A^g\|_{L^2} \\ &\lesssim \|F\|_{L^2} + \|A^g\|^2_{L^4} + \|A\|_{L^2} \ . \end{aligned}$$

If $\|A^g\|_{L^4} \leq K$ small enough then the second term above is estimated by $K\|\nabla A^g\|_{L^2}$ which can then be absorbed to the left side of the inequality, giving the desired estimate.

Proof of step 5

We approximate $A \in \mathcal{U}_2^{\epsilon_0}$ by smooth A_k in $W^{1,2}$ norm. In particular there holds $A_k \in W^{1,p}$ for all $p > 2$. We may obtain that $A_k \in \mathcal{U}_p^{\epsilon_0} = \mathcal{V}_p^{\epsilon_0, C_p}, p > 2$ and in particular we find $g_k \in M_p$ such that

$$\|A_k^{g_k}\|_{L^4} \lesssim \|A_k\|_{W^{1,2}} \lesssim \|F_k\|_{L^2} + \|A_k\|_{L^2} \lesssim \epsilon_0 \ ,$$

where the constants depend only on the exponents p and 2. By possibly diminishing ϵ_0 we thus achieve $\|A_k^{g_k}\|_{L^4} \leq K$ for all k. By the closure result of Step 2 for $p = 2$ we thus obtain that the same estimate holds for A and for some gauge $g \in M_2$ and by Step 4 we conclude that $A \in \mathcal{V}_p^{\epsilon_0, K}$, as desired. □

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