# SYNCHRONIC AND ASYNCHRONIC DESCRIPTIONS OF IRRIGATION PROBLEMS

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Abstract. In this paper we complete the work started with [29], in which the present paper was announced, in order to set a unified theory of the irrigation problem. In particular, we show the equivalence of the various formulations introduced so far as well as a new one introduced here. Moreover we formalize several geometric and analytical concepts which play an important role in the theory and may deserve an intrinsic interest in themselves.

## Contents

## 1. Introduction

In recent years *Optimal Transportation Theory* has become an active area of interest and it has reached deep results involving different branches of mathematical analysis (see the seminal works [26], [31], [36], the more recent [1], [2], [18], [19], [23] and the comprehensive books [39], [40]). In this context, starting from [37], [27], a variant of the Monge-Kantorovitch transport problem leading to variational models describing ramified structures has been investigated. The starting point in this theory relies on the concavity properties of the model function cost $|x|^\alpha$, involved in the functionals, with $0 < \alpha < 1$. The first idea in this direction is due to Gilbert ([24], [25]) and in [37] a continuous version of this problem is studied in the framework of geometric measure theory, i.e. the minimization problem for a functional defined on flat chains while in [27] the proposed functional is defined on families of curves parametrized on a set $\Omega$ equipped with a probability measure or, equivalently, as remarked in [3] and [5] where other formulations have been proposed, for measures defined on a space of curves.
Many papers concerned with one of the previous formalization of the problem ([3], [4], [5],[6]) contain results which cannot be directly applied to the other contexts, although they can almost always be adapted with a minor effort. In this situation, one obviously feels the necessity of establishing a general enough theory to cast the various results in a common and unified setting.
The aim of this paper relies in establishing such a general theory for irrigation problems completing the first step achieved in [29], in which we promised to answer completely the question of the equivalence of the approaches in [27], [3] and in [24], [37], in the present work.
To this aim it is necessary to establish some orderings and equivalences between particle motions (see Definition 2.2 below). Some of these relations are deduced from analogous concepts regarding orbits and this leads us to deal with the general problem of transferring binary relations from a points set to the corresponding space of probability measures (this is achieved in Subsection 3.1). We will consider other relations which are connected to the notion of density. Subsequently, we will introduce the *irrigation problem* in its various formulations and we will establish some geometrical properties in the same spirit of [15] (in Subsection 4.5 we introduce the *flow order* which is crucial to study the no-cycle property). Finally we will show the equivalence of the various irrigation models.
Some of the concepts introduced here have been considered by other authors using the

same name and different but substantially equivalent definitions. We have preferred to make the paper self-contained rather than using previously established results, indeed in some cases these would not directly apply to our context. Some general properties, more or less available in literature, such as the inversion of the integration order in spaces with a non $\sigma$-finite measure, which we have employed in the paper and which seem useful for understanding some concepts, have been grouped for the reader's convenience in an articulated appendix.

## 2. Preliminary results and main notation

2.1. **Measure theoretic tools.** Let $X$ be a Polish space, we denote by $\mathcal{P}(X)$ the space of the probability measures defined on $X$ (see [14]). Moreover, we denote by $\Pi(\mu,\nu) \subset \mathcal{P}(X \times Y)$ the set of the probability measures on $X \times Y$ with marginals $\mu$ and $\nu$, that is, for $\pi \in \Pi(\mu,\nu)$, we have $p^0_\# \pi = \mu$, $p^1_\# \pi = \nu$, where $p^0 : (x,y) \mapsto x$ and $p^1 : (x,y) \mapsto y$ are the projections. A measure $\pi \in \Pi(\mu,\nu)$ is called a *transport plan* and $\Pi(\mu,\nu)$ is the set of the transport plans between $\mu$ and $\nu$.
Let us recall that, by applying Skorokhod convergence Theorem (see [14, Theorem 11.7.5], [5, Theorem A.8]) to a constant sequence of probability measures, one can deduce that any probability measure can be *parametrized* on any probability space endowed with a non-atomic measure, i.e. $\mu = f_\# \mu_\Omega$, where $\mu \in \mathcal{P}(X)$, $f : \Omega \to X$ and $(\Omega, \mu_\Omega)$ is a non-atomic probability space. We shall refer to the mapping $f$ as to a *lagrangian parametrization* of $\mu$ on the *reference space* $\Omega$.

Let $X$ be a Polish space and let $C_b(X)$ be the space of bounded and continuous real-valued functions defined on $X$. $\mathcal{B}(\mathbb{R}^N)$ denotes the set of the Borel subsets of $\mathbb{R}^N$.

**Definition 2.1.** *A sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(X)$ is narrowly convergent to $\nu \in \mathcal{P}(X)$, in symbols $\nu_n \rightharpoonup \nu$, if*
$$\lim_{n\to\infty} \int_X f \, d\nu_n = \int_X f \, d\nu \quad \forall f \in C_b(X).$$

Let $\Gamma$ be the space of the orbits, i.e. absolutely continuous maps $\gamma : I \to \mathbb{R}^N$ defined on a generic interval $I \subset \mathbb{R}$, equipped with the topology of the locally uniform convergence and let $\Gamma_c$ be the subspace of $\Gamma$ constituted by the orbits defined on a compact interval $I$. Sometimes we will use the symbol $\Gamma_I$ instead of $\Gamma$ to the aim of distinguishing the interval $I$.

Let us recall the notion of microscopic motion introduced in [29].

**Definition 2.2.** *Let $\Sigma = \mathcal{P}(\Gamma)$ be the space of probability measures on $\Gamma$, we shall call microscopic motion or, equivalently, particle motion any $\sigma \in \Sigma$.*

For every $t \in I$, let $p_t : \Gamma_I \to X$ be given by $p_t(\gamma) = \gamma(t)$ and let $\sigma \in \mathcal{P}(\Gamma_I)$ be a given microscopic motion. Then for every $t \in I$, by setting
$$\tilde{\sigma}(t) = (p_t)_\# \sigma, \tag{2.1}$$

we get the macroscopic motion $\tilde{\sigma} : I \to \mathcal{P}(X)$ induced by the particle motion $\sigma$. When no confusion is possible we shall write, with a clear abuse of notation, $\sigma$ instead of $\tilde{\sigma}$ also to denote the macroscopic motion induced by $\sigma$, as for instance in the following definition.

**Definition 2.3.** *Let $\mu, \nu \in \mathcal{P}(X)$ be two given measures. We shall say that $\sigma \in \Sigma$ is an admissible motion between $\mu$ and $\nu$ on the interval $I = [a, b]$ if $\sigma(a) = \mu, \sigma(b) = \nu$. Let $\Sigma_I(\mu, \nu)$ denote the set of the admissible $\sigma$ and $\Sigma(\mu, \nu)$ be the union of $\Sigma_I(\mu, \nu)$ for all the closed bounded intervals $I \subset \mathbb{R}$.*

We can obtain a lagrangian parametrization of $\sigma \in \Sigma$ by using a measurable map $\hat{\chi} : \Omega \to \Gamma$. Actually, assigning $\hat{\chi}$ is equivalent to give a measurable map $\chi : \Omega \times I \to \chi(p, t) \in X$ such that for a.e. material point $p \in \Omega$, $\chi_p : t \mapsto \chi(p, t)$ is absolutely continuous. Indeed, $\chi$ is induced by $\hat{\chi}$ by setting $\chi(p, t) = [\hat{\chi}(p)](t)$ and, conversely, $\hat{\chi}$ is induced by $\chi$ as the map from $\Omega$ to $\Gamma$ defined by $\hat{\chi} : p \to \chi_p$. Then we shall say that $\chi : \Omega \times I \to \mathbb{R}^N$ is a lagrangian parametrization of $\sigma$ if

$$\sigma = \hat{\chi}_{\#}\mu_\Omega. \tag{2.2}$$

If $\chi : \Omega \times I \to \mathbb{R}^N$ is a lagrangian parameterization of $\sigma$, $\Omega' \subset \Omega$ and $I' \subset I$, the restriction of $\chi$ to $\Omega' \times I'$, denoted by $\chi_{|_{\Omega' \times I'}}$, induces the particle motion $\sigma'$ which will be called a sub-motion of $\sigma$.

Let $(\Omega, \mu)$ be any measure space, we shall often use the symbol $\int_{(\Omega,\mu)} f(x)dx$ with the same meaning of the more usual $\int_\Omega f(x)d\mu(x)$.

2.2. **Truncations and restrictions.** If $\sigma \in \Sigma_I$ is given, let $\tau : \Gamma \to I$ be a $\sigma$-measurable function, which in the sequel will be called a *truncation mapping*. Given $\gamma \in \Gamma_I$, we consider the two *complementary truncations* $\gamma_1$ (left truncation), $\gamma_2$ (right truncation), defined as follows.

$$\gamma_1(t) = \begin{cases} \gamma(t) & \text{if } t \leq \tau(\gamma) \\ \gamma(\tau) & \text{if } t > \tau(\gamma), \end{cases} \tag{2.3}$$

$$\gamma_2(t) = \begin{cases} \gamma(\tau) & \text{if } t < \tau(\gamma) \\ \gamma(t) & \text{if } t \geq \tau(\gamma). \end{cases} \tag{2.4}$$

Moreover, for any $\gamma \in \Gamma$ such that $\tau(\gamma) \in I$, we set $b_\tau(\gamma) = \gamma(\tau(\gamma))$, which will be called *breaking point*. For any given truncation mapping $\tau : \Gamma \to \overline{\mathbb{R}}$, we set

$$p^i_\tau(\gamma) = \gamma_i, \quad i = 1, 2. \tag{2.5}$$

It is readily seen that the mappings $p^i_\tau : \Gamma \to \Gamma$ are Borel measurable. Then, given a particle motion $\sigma$, the measures

$$\sigma_i = (p^i_\tau)_{\#}\sigma, \quad i = 1, 2, \tag{2.6}$$

will be called *complementary truncations of* $\sigma$. The previous operation can be iterated, leading to *multiple complementary truncations*. Indeed, if for instance two truncation mappings $\tau_1 \leq \tau_2$ are given, we can split a particle motion $\sigma$ into the three complementary truncations $\sigma_1$, $\sigma_2$ and $\sigma_3$ consecutively operating with the two truncation mappings.

If $\sigma \in \Sigma$ is a microscopic motion, we can consider the restriction $\sigma_s$ of $\sigma$ to the constant orbits $\{\gamma \in \Gamma \mid \gamma(t) = const.\ \forall t \in I\}$ and set $\sigma_m = \sigma - \sigma_s$ as the restriction of $\sigma$ to the non constant orbits. With this notation we can split $\sigma$ as

$$\sigma = \sigma_s + \sigma_m. \tag{2.7}$$

## 3. General tools

3.1. **Extension of binary relations to probability measures.** In this section we shall be concerned with the extension of binary relations defined between a pair of Polish spaces $X, Y$ to the spaces of probability measures $\mathcal{P}(X), \mathcal{P}(Y)$. Notice that any space $X$ can be considered as a subspace of $\mathcal{P}(X)$ through the identification of a generic point $x$ with the Dirac measure $\delta_x$. Therefore any binary relation on a pair of measure spaces $\mathcal{P}(X)$, $\mathcal{P}(Y)$ trivially induces a relation between $X$ and $Y$ by a restriction. So, when a relation between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ is given, we shall use it, with the same symbol, also as a relation between $X$ and $Y$. On the other hand, the *pointwise extension of a relation* from points to measures spaces will be introduced in this section, as specified in the following definition.

**Definition 3.1.** *(Pointwise extension of relations) Let $\mathcal{R} \subset X \times Y$, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ be given. We say that $\mu \tilde{\mathcal{R}} \nu$ if there exists $\pi \in \Pi(\mu, \nu)$ concentrated on $\mathcal{R}$.*

According to the previous definition, we shall say that any binary relation $\mathcal{R} \subset X \times Y$ induces a relation $\tilde{\mathcal{R}}$ on the probability measures on such spaces.

**Remark 3.2.** It is easily seen that $\tilde{\mathcal{R}}$ is a real extension of $\mathcal{R}$ in the sense that restricted to pair of points (identified with Dirac masses) agrees with $\mathcal{R}$. On the contrary, if we pointwise extend to a pair of measure spaces the restriction to the points of a given relation $\mathcal{R}$, defined between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, we obtain a relation $\widetilde{\mathcal{R}}$ which can be different from $\mathcal{R}$. We shall widely see in the following (see, for instance, Theorem 3.62 below) meaningful examples in which one of the two inclusions $\mathcal{R} \subset \widetilde{\mathcal{R}}$ or $\widetilde{\mathcal{R}} \subset \mathcal{R}$ holds true.

**Remark 3.3.** It is readily seen that the following properties hold true.

$$\mu_1 \tilde{\mathcal{R}} \nu_1 \text{ and } \mu_2 \tilde{\mathcal{R}} \nu_2 \Longrightarrow (t_1\mu_1 + t_2\mu_2) \tilde{\mathcal{R}} (t_1\nu_1 + t_2\nu_2) \quad \forall t_1, t_2 \in \mathbb{R}; \tag{3.8}$$

$$\mathcal{R}_1 \subset \mathcal{R}_2 \Longrightarrow \widetilde{\mathcal{R}}_1 \subset \widetilde{\mathcal{R}}_2. \tag{3.9}$$

**Proposition 3.4.** *Let $\mathcal{R} \subset X \times Y$, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ be given and let $(\Omega, \mu_\Omega)$ be a given reference space. Then $\mu \mathcal{R} \nu$ if and only if there exist two lagrangian parameterizations $f : \Omega \to X$ and $g : \Omega \to Y$ of $\mu$ and $\nu$ respectively such that for $\mu_\Omega$-a.e. $p \in \Omega$ $(f(p), g(p)) \in \mathcal{R}$, i.e. $f(p)\mathcal{R}g(p)$.*

*Proof.* Assume there exist $f, g$ as above, then we take the map $k : \Omega \to X \times Y$ such that $k(p) = (f(p), g(p))$ for a.e. $p \in \Omega$. So let $\pi = k_\# \mu_\Omega$. To prove the converse implication, let $\pi \in \Pi(\mu, \nu)$ be concentrated on $\mathcal{R}$ and let $k : \Omega \to X \times Y$ be a lagrangian parametrization of $\pi$ on $\Omega$, $k = (k^1, k^2)$. Then we take $f = k^1$ and $g = k^2$. Since $\pi$ is concentrated on $\mathcal{R}$ then for $\mu_\Omega$-a.e. $p \in \Omega$ $f(p)\mathcal{R}g(p)$. □

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Polish spaces, for every $n$ let $\mathcal{R}_n \subset X_n \times X_{n+1}$ be given and let $X = \prod_{i\in\mathbb{N}} X_i$. Let $p^n$, $p^{n,m}$ be the projection operators defined on $X$ by

$$p^n : (x_1, \ldots, x_n, \ldots) \mapsto x_n \in X_n, \quad p^{n,m} : (x_1, \ldots, x_n, \ldots) \mapsto (x_n, x_m) \in X_n \times X_m.$$

A sequence of transport plans $\tilde{\pi} = (\pi_1, \pi_2, \ldots, \pi_n, \ldots) \in \mathcal{P}(X_1 \times X_2) \times \mathcal{P}(X_2 \times X_3) \times \ldots \mathcal{P}(X_n \times X_{n+1}) \times \ldots$ will be said a *compatible chain* if, for every $i \in \mathbb{N} \setminus 0$, $p^1_\#(\pi_i) = p^0_\#(\pi_{i+1})$. Let us recall that a multiple plan (see [2],[29]) is a measure $\pi \in \mathcal{P}(X)$.

By [2, Lemma 5.3.4], given any compatible chain of transport plans $(\pi_1, \pi_2, \ldots, \pi_k, \ldots)$, there exists a multiple plan $\pi \in \mathcal{P}(X)$ such that for every $n \in \mathbb{N}$ $p^{n,n+1}_\# \pi = \pi_n \in \Pi(\mu_n, \mu_{n+1})$, where the measures $\mu_n$ are the marginals of $\pi$, i.e. $\mu_n = p^n_\# \pi$.

By keeping this notation we state the next assertion which extends Proposition 3.4.

**Proposition 3.5.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures such that for every $n \in N$ $\mu_n \mathcal{R}_n \mu_{n+1}$ for a given $(\mathcal{R}_n)_{n\in\mathbb{N}}$. Then for any given reference space $(\Omega, \mu_\Omega)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of lagrangian parameterizations of the measures $(\mu_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ and for $\mu_\Omega$-a.e. $p \in \Omega$ we have $f_n(p)\mathcal{R}_n f_{n+1}(p)$.*

*Proof.* Since, for every $n$, $\mu_n \mathcal{R}_n \mu_{n+1}$, by Definition 3.1 there exists a transport plan $\pi_n \in \Pi(\mu_n, \mu_{n+1})$ concentrated on $\mathcal{R}_n$. By virtue of the above mentioned [2, Lemma 5.3.4] we take a multiple plan $\pi \in \mathcal{P}(X)$ such that, for every $n \in \mathbb{N} \setminus 0$, $p^{n,n+1}_\# \pi = \pi_n$, then we take a lagrangian parametrization of $\pi$ on $\Omega$ denoted by $k$ and for every natural $n$ we set $f_n = p^n \circ k$. It is easy to see that $(f_n)_{n\in\mathbb{N}}$ is as required. □

**Remark 3.6.** The construction in the previous proof also says that $\forall n \in \mathbb{N}$ $(f_n, f_{n+1})$ is a parameterization of the transport plan $\pi_n$. If anyone of these transport plans is assigned, then such an information is a meaningful completion of the thesis.

**Lemma 3.7.** *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of multiple plans in $\mathcal{P}(X)$ such that the sequences of marginals $(p^i_\# \pi_n)_{n\in\mathbb{N}}$ are compact. Then $(\pi_n)_{n\in\mathbb{N}}$ has a narrowly converging subsequence.*

*Proof.* For every $i$, let $\mu^i_n = p^i_\# \pi_n$. Since $(\mu^i_n)_{n\in\mathbb{N}}$ is compact, by applying Prokhorov Theorem ([2, Theorem 6.1.1]) we know that, for every $\varepsilon > 0$, there exists a compact set $C_i \subset X_i$ such that $\mu_n^i(X_i \setminus C_i) < \frac{\varepsilon}{2^i}$ for every $n \in \mathbb{N}$. Now, for every $i$ we set $A_i = \prod_j Y_j$, with $Y_j = X_j$ if $j \neq i$ and $Y_j = X_i \setminus C_i$ if $j = i$. Then, for every $n$ and for every $i$, we have $\pi_n(A_i) = \mu_n^i(X_i \setminus C_i) < \frac{\varepsilon}{2^i}$. We set $C = \prod_i C_i$. $C$ is compact and $X \setminus C = \bigcup_i A_i$. Therefore we get that, for every $n \in \mathbb{N}$, $\pi_n(X \setminus C) < \varepsilon$ and thus, by applying again Prokhorov Theorem, we get the thesis. □

**Proposition 3.8.** *For every $n \in \mathbb{N}$ let $\mu_n^1 \tilde{\mathcal{R}}_n^1 \mu_n^2 \tilde{\mathcal{R}}_n^2 \mu_n^3 \dots \tilde{\mathcal{R}}_n^{k-1} \mu_n^k \dots$ be a sequence of measures in the spaces $\mathcal{P}(X_1), \dots, \mathcal{P}(X_k), \dots$ and related by $\mathcal{R}_n^i \subset X_i \times X_{i+1}$. We suppose that $\mu_n^i \rightharpoonup \mu^i$ for every $i \in \mathbb{N}$. Then, by passing to a subsequence, there exist lagrangian parameterizations $(f_n^i)_{n \in N}$ of the measures $(\mu_n^i)_{n \in \mathbb{N}}$ and $f^i$ of the measure $\mu^i$ such that, for every $i$, $f_n^i \to f^i$ a.e. and for every $n$ and for every $i$: $f_n^i(p) \mathcal{R}_n^i f_n^{i+1}(p)$ for a.e. $p$.*

*Proof.* For every $n \in \mathbb{N}$, by virtue of [2, Lemma 5.3.4] we take a multiple plan $\pi_n \in \mathcal{P}(X)$ such that, for every $i$, $p_{\#}^{i,i+1} \pi_n \in \Pi(\mu_n^i, \mu_n^{i+1})$ is concentrated on $\mathcal{R}_n^i$. By Lemma 3.7 we have that, by passing to a subsequence, $\pi_n \rightharpoonup \pi$. By Skorohod Theorem there exist $(\varphi_n)_{n \in \mathbb{N}}$ and $\varphi$, which are lagrangian parameterizations of $(\pi_n)_{n \in \mathbb{N}}$ and $\pi$ respectively, such that $\varphi_n \to \varphi$ a.e.. By taking, $\forall n \in \mathbb{N}$ $f_n^i = p^i \circ \varphi_n$ and $f^i = p^i \circ \varphi$, we get the thesis. □

Now we study some properties of the extension to probability measures of orderings. Firstly we observe that the antisymmetry property alone does not pass to the extension to probability measures, while it is preserved for closed relations when the transitivity property also holds, as we shall see later.

**Example 3.9.** Let $A = \{p_1, p_2, p_3\}$. We consider the following (cyclic order) relation

$$\mathcal{R} = \{(p_1, p_2), (p_2, p_3), (p_3, p_1), (p_1, p_1), (p_2, p_2), (p_3, p_3)\},$$

which is antisymmetric. Let

$$\mu_1 = \frac{1}{2}\delta_{p_1} + \frac{1}{2}\delta_{p_2}, \quad \mu_2 = \frac{1}{2}\delta_{p_1} + \frac{1}{2}\delta_{p_3}.$$

We have that $\mu_1 \tilde{\mathcal{R}} \mu_2$ by taking $\pi \in \Pi(\mu_1, \mu_2)$ defined by $\pi = \frac{1}{2}(\delta_{p_1} \otimes \delta_{p_1}) + \frac{1}{2}(\delta_{p_2} \otimes \delta_{p_3})$. Moreover, we have that $\mu_2 \tilde{\mathcal{R}} \mu_1$ by taking $\pi = \frac{1}{2}(\delta_{p_3} \otimes \delta_{p_1}) + \frac{1}{2}(\delta_{p_1} \otimes \delta_{p_2}) \in \Pi(\mu_2, \mu_1)$. Thus $\{(\mu_1, \mu_2), (\mu_2, \mu_1)\} \subset \tilde{\mathcal{R}}$ and so $\tilde{\mathcal{R}}$ is not antisymmetric.

**Proposition 3.10.** *Let $\mathcal{R} \subset X \times Y$ be closed. Then $\tilde{\mathcal{R}}$ is closed with respect to the narrow convergence.*

*Proof.* Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ and $(\nu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(Y)$ be two sequences such that $\forall n \in \mathbb{N}$ $\mu_n \tilde{\mathcal{R}} \nu_n$. Let us suppose that $\mu_n \rightharpoonup \mu$ and $\nu_n \rightharpoonup \nu$ narrowly. By Lemma 3.7 we know that there exists $\pi_n \in \Pi(\mu_n, \nu_n)$, concentrated on $\mathcal{R}$, which has a narrowly converging subsequence to some $\pi \in \Pi(\mu, \nu)$. Moreover, $\pi_n$ being concentrated on the closed set $\mathcal{R}$, we can conclude that also $\pi$ is concentrated on $\mathcal{R}$ and so $\mu \tilde{\mathcal{R}} \nu$. □

**Proposition 3.11.** *Let $\mathcal{R} \subset X \times Y$ be transitive. Then $\tilde{\mathcal{R}}$ is transitive.*

*Proof.* Let $\mu_1 \tilde{\mathcal{R}} \mu_2$ and $\mu_2 \tilde{\mathcal{R}} \mu_3$. Then there exist $\pi_1 \in \Pi(\mu_1, \mu_2)$ and $\pi_2 \in \Pi(\mu_2, \mu_3)$ both concentrated on $\mathcal{R}$. Since $\mathcal{R}$ is transitive, the composition $\pi_3 = \pi_1 \otimes \pi_2$ is concentrated on $\mathcal{R}$ (see [29, Section 1.2]) and so we get $\mu_1 \tilde{\mathcal{R}} \mu_3$. □

**Definition 3.12.** *Let $X$ be a Polish space and let $\leq$ be a given ordering. We shall say that $\leq$ is compatible with the topology of $X$ if it is closed and satisfies the condition*

$$\text{If for every } n \in \mathbb{N}\ x_n \leq z_n \leq y_n \text{ and } lim_n x_n = lim_n y_n = z \text{ then } z_n \to z. \tag{3.10}$$

**Proposition 3.13.** *Let $X$ be a Polish space and let $\leq$ be a closed ordering. If $(x_n)_{n \in \mathbb{N}}$ is any monotone increasing sequence with respect to $\leq$, that is, for every $n$, $x_n \leq x_{n+1}$, and $x_n \to x$, then $x = \sup_n x_n$.*

*Proof.* For every fixed $n \in N$ and for any large $m \in \mathbb{N}$ we have $x_n \leq x_m$, then by passing to the limit with respect to $m$ we get, by the closeness of the ordering, $x_n \leq x$ for every $n \in \mathbb{N}$. So $x$ is an upper bound. Moreover, if for every $n \in \mathbb{N}$ $x_n \leq y$, by passing to the limit, we get $x \leq y$. Thus $x$ is the least upper bound of the sequence and the thesis follows. □

**Theorem 3.14.** *Let $\pi \in \mathcal{P}(X \times X)$ be concentrated on a closed transitive relation $\mathcal{R}$ and such that $p^0_\# \pi = p^1_\# \pi$. Then $\pi$ is concentrated on $\mathcal{R} \cap \mathcal{R}^{-1}$.*

*Proof.* Let us suppose that $\pi$ is not concentrated on $\mathcal{R}^{-1}$. In such a case, since $X$ is a Polish space (so it has a countable basis) there exists $\bar{z} \in \mathcal{R} \setminus \mathcal{R}^{-1}$ such that every neighborhood $W$ of $\bar{z}$ has a positive measure $\pi(W) > 0$. By the closeness of $\mathcal{R}^{-1}$ we can choose $W = W_1 \times W_2$ such that $W_1 \cap W_2 = \emptyset$ and $W \cap \mathcal{R}^{-1} = \emptyset$ and, by applying Ulam Theorem ([14, Theorem 7.1.4]), we can find two compact sets $U, V \subset X$ such that $U \times V \subset W$ and $\pi(U \times V) > 0$. Let $\tilde{V} = V \cup \mathcal{R}(V)$ be the stable part with respect to $\mathcal{R}$ generated by $V$. Since $U \cap V \subset W_1 \cap W_2 = \emptyset$ and $(U \times V) \cap \mathcal{R}^{-1} = \emptyset$, we get $U \cap \tilde{V} = \emptyset$. Since $V$ is compact and $\mathcal{R}$ is closed then $\tilde{V}$ is closed and therefore measurable. We deduce

$$(p^1_\# \pi)(\tilde{V}) = \pi(X \times \tilde{V}) \geq \pi(U \times \tilde{V}) + \pi(\tilde{V} \times \tilde{V}) > \pi(\tilde{V} \times \tilde{V}), \tag{3.11}$$

since $\pi(U \times \tilde{V}) \geq \pi(U \times V) > 0$. On the other hand,

$$\pi(\tilde{V} \times \tilde{V}) = \pi(\tilde{V} \times X) - \pi(\tilde{V} \times (X \setminus \tilde{V})).$$

Now $\pi(\tilde{V} \times (X \setminus \tilde{V}) = 0$, since $\pi$ is concentrated on the transitive relation $\mathcal{R}$ which has empty intersection with $\left(\tilde{V} \times (X \setminus \tilde{V})\right)$ because $\mathcal{R}(\tilde{V}) \subset \tilde{V}$. Then we get

$$\pi(\tilde{V} \times \tilde{V}) = (p^0_\# \pi)(\tilde{V}),$$

in contradiction to (3.11) since we have assumed $p^0_\# \pi = p^1_\# \pi$. □

Let $f, g : \Omega \to X$, we shall say that $f\mathcal{R}g$ if $f(p)\mathcal{R}g(p)$ for $\mu_\Omega$-a.e. $p \in \Omega$. In the next statement, let $\mathcal{R}$ be as in the previous theorem.

**Corollary 3.15.** *Let $f, g : \Omega \to X$ be two lagrangian parameterizations of $\mu \in \mathcal{P}(X)$ such that $f\mathcal{R}g$ for a given closed transitive relation $\mathcal{R}$. Then $f(p)\mathcal{R}^{-1}g(p)$ for $\mu_\Omega$-a.e. $p \in \Omega$.*

*Proof.* Just apply Theorem 3.14 to the transport plan $(f, g)_\# \mu_\Omega$. $\square$

**Corollary 3.16.** *Let $\pi \in \mathcal{P}(X \times X)$ be concentrated on a closed transitive relation $\mathcal{R}$ and such that $p^1_\# \pi \tilde{\mathcal{R}} p^0_\# \pi$. Then $\pi$ is concentrated on $\mathcal{R} \cap \mathcal{R}^{-1}$.*

*Proof.* Let $\mu = p^0_\# \pi$, $\nu = p^1_\# \pi$. By Proposition 3.5 and Remark 3.6 let $f, h, g : \Omega \to X$ be three lagrangian parameterizations such that $f_\# \mu_\Omega = h_\# \mu_\Omega = \mu$, $g_\# \mu_\Omega = \nu$, satisfying $f\mathcal{R}g$, $g\mathcal{R}h$ and such that $(f, g)$ is a parameterization of $\pi$. Then, by transitivity, $f\mathcal{R}h$ holds so, by Corollary 3.15 $(f, h) \in \mathcal{R}^{-1}$ and, by transitivity, $(g, f) \in \mathcal{R}$ i.e. $(f, g) \in \mathcal{R} \cap \mathcal{R}^{-1}$. Since $(f, g)$ parametrizes $\pi$ we get the thesis. $\square$

**Corollary 3.17.** *Let $\mathcal{R} \subset X \times X$ be a closed transitive relation. Then $\widetilde{\mathcal{R} \cap \mathcal{R}^{-1}} = \tilde{\mathcal{R}} \cap \tilde{\mathcal{R}}^{-1}$.*

*Proof.* Let $(\mu, \nu) \in \tilde{\mathcal{R}} \cap \tilde{\mathcal{R}}^{-1}$ and let $\pi$ be a transport plan concentrated on $\mathcal{R}$ such that $p^0_\# \pi = \mu$, $p^1_\# \pi = \nu$. Then by Corollary 3.16 $\pi$ is concentrated on $\mathcal{R} \cap \mathcal{R}^{-1}$ and so $(\mu, \nu) \in \widetilde{\mathcal{R} \cap \mathcal{R}^{-1}}$. This proves one of the inclusions, the other one trivially follows by (3.9). $\square$

**Proposition 3.18.** *Let $\leq$ be a closed ordering in $X \times X$, then its extension to probability measures is a closed ordering with respect to the narrow convergence topology.*

*Proof.* By Proposition 3.10 and Proposition 3.11 we just have to prove that the extended $\leq$ remains antisymmetric. This last property trivially follows from Corollary 3.17. $\square$

**Proposition 3.19.** *Let $\leq$ be a compatible ordering with the topology of $X$. Then its extension to the probability measures is a compatible ordering with the narrow convergence topology of $\mathcal{P}(X)$.*

*Proof.* We are going to prove (3.10) for the extension of the ordering to the probability measures. Let us suppose that $\lambda_n \leq \nu_n \leq \mu_n$ for every $n \in \mathbb{N}$ and that $\lambda_n, \mu_n \rightharpoonup \mu$. Using Lemma 3.8, for every $n \in \mathbb{N}$ we find $f_n, g_n, h_n$ which are respectively lagrangian parameterizations of $\lambda_n, \mu_n, \nu_n$, satisfying $f_n \leq g_n \leq h_n$ and such that $f_n \to f$, $h_n \to h$, where $f$ and $h$ are both lagrangian parameterizations of $\mu$. By passing to the limit as $n \to \infty$ we get $f \leq h$ and by applying Corollary 3.15 we get $f(p) = h(p)$ for $\mu_\Omega$-a.e. $p$. Then for $\mu_\Omega$-a.e. $p$ the sequences $f_n(p)$ and $h_n(p)$ both converge to $f(p)$ and, since for every $n$ $f_n(p) \leq g_n(p) \leq h_n(p)$, by (3.10) we get $g_n(p) \to f(p)$ for

$\mu_\Omega$-a.e. $p$. By applying Skorohod Theorem ([14, Theorem 11.7.2]) we finally obtain $\nu_n = g_{n\#}\mu_\Omega \rightharpoonup f_\#\mu_\Omega = \mu$. $\square$

3.2. **Density functions and tracks.** We shall denote by $\#$ the counting measure and by $\mathbf{1}_A$ the characteristic function of the set $A$.

**Definition 3.20.** *For every $\gamma \in \Gamma_I$ and for $\mathcal{H}^1$ a.e. $x \in \mathbb{R}^N$ we define the multiplicity function*

$$m_\gamma(x) = \#\{t \in I \mid \gamma(t) = x\}$$

*and the affiliation function $a_\gamma(x) = \mathbf{1}_{\gamma(I)}(x) = \inf(m_\gamma(x), 1)$.*

Let us introduce the following set

$$\Gamma_0 = \{\gamma \in \Gamma \mid m_\gamma < +\infty,\ \mathcal{H}^1- \text{ a.e.}\}. \tag{3.12}$$

We shall often work with curves $\gamma \in \Gamma_0$. Many times this fact will not represent a real restriction. Indeed, since absolutely continuous curves defined on a compact interval have a finite total length, by the Area Formula [21, Theorem 3.2.6] we have

$$\Gamma_c \subset \Gamma_0. \tag{3.13}$$

So in particular every $\gamma \in \Gamma$ is locally in $\Gamma_0$.

**Theorem 3.21.** *The mapping $(\gamma, x) \mapsto m_\gamma(x)$ is a Borel measurable function on $\Gamma \times \mathbb{R}^N$.*

*Proof.* Let $\delta > 0$ be fixed and let $T \subset I$ be a given compact subset. For any given $\gamma \in \Gamma_I$ and $x \in \mathbb{R}^N$ we define

$$S_{\delta,T}(\gamma, x) = \{X \subset \gamma^{-1}(x) \cap T \mid \forall y, z \in X,\ y \neq z : |y - z| \geq \delta\},$$

$$m_{\delta,T}(\gamma, x) = \sup_{X \in S_{\delta,T}(\gamma,x)} \#X.$$

Then we have $m_\gamma(x) = \sup_{\delta,T} m_{\delta,T}(\gamma, x)$. We claim that for every $\delta, T$, the mapping $(\gamma, x) \mapsto m_{\delta,T}(\gamma, x)$ is upper semicontinuous in the product topology. Indeed, let $(\bar\gamma, \bar x) \in \Gamma \times \mathbb{R}^N$ and $k \leq \limsup_{(\gamma,x)\to(\bar\gamma,\bar x)} m_{\delta,T}(\gamma, x)$. We choose a sequence $(\gamma_n, x_n)$ such that $(\gamma_n, x_n) \to (\bar\gamma, \bar x)$ and $m_{\delta,T}(\gamma_n, x_n) \geq k$. Notice that for every $n \in \mathbb{N}$ there exists $X_n = \{t_n^1, \dots, t_n^k\} \subset T$ such that $|t_n^i - t_n^j| \geq \delta$ for every $i \neq j$ and $x_n = \gamma_n(t_n^i)$ for every $i = 1, \dots, k$. Since $T$ is a compact set, by passing ($k$ times) to subsequences we get, for every $i$, $t_n^i \to t^i \in T$. By definition of locally uniform convergence we have $\gamma_n(t_n^i) \to \bar\gamma(t^i)$, that is $x_n \to \bar\gamma(t^i)$ and, since $x_n \to \bar x$, we get $\bar\gamma(t_i) = \bar x$. If $i \neq j$ by passing to the limit we also have $|t^i - t^j| \geq \delta$. Then we obtain $m_{\delta,T}(\bar\gamma, \bar x) \geq k$ and so $m_{\delta,T}$ is upper semicontinuous and therefore measurable. Finally, the multiplicity function turns out to be the supremum of a countable family of measurable functions $m_{\delta,T}$ and so it is measurable. $\square$

Notice that, since $a_\gamma = \inf\{m_\gamma, 1\}$, $a_\gamma$ is also measurable. Let $\sigma \in \Sigma$ be given, we introduce the following functions

$$m_\sigma(x) = \int_{(\Gamma,\sigma)} m_\gamma(x)\, d\gamma, \tag{3.14}$$

$$a_\sigma(x) = \int_{(\Gamma,\sigma)} a_\gamma(x)\, d\gamma. \tag{3.15}$$

Obviously by the above definition we have $a_\sigma(x) = \sigma(\{\gamma \in \Gamma \mid x \in \gamma(I)\})$.

**Remark 3.22.** Let us observe that, by the linearity of the integral with respect to the measure, if $\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2$, by (3.14) (3.15) we have $m_\sigma = \lambda_1 m_{\sigma_1} + \lambda_2 m_{\sigma_2}$ and $a_\sigma = \lambda_1 a_{\sigma_1} + \lambda_2 a_{\sigma_2}$.

**Definition 3.23.** *(Track) Let $\sigma \in \Sigma$ be given. We shall say that $T \subset \mathbb{R}^N$ is a track of $\sigma$ if $\mathcal{H}^1(\gamma(I) \setminus T) = 0$ for $\sigma$-a.e. $\gamma \in \Gamma$.*

We observe that (unless $\sigma$ is concentrated on constant orbits) the set of the tracks is a filter. In particular, if $T_1$ and $T_2$ are two tracks of $\sigma$, then $T_1 \cap T_2$ is a track of $\sigma$ too.

**Definition 3.24.** *(Non-spread particle motions) We shall say that $\sigma \in \Sigma$ is a non-spread particle motion if there exists a $\mathcal{H}^1$ $\sigma$-finite track of $\sigma$.*

We set

$$\Sigma_0 = \{\sigma \in \Sigma \mid \sigma \text{ non-spread}, m_\sigma < +\infty \ \mathcal{H}^1{-}a.e.\}. \tag{3.16}$$

Note that the condition $m_\sigma < +\infty$ $\mathcal{H}^1$−a.e. in the above definition holds on the whole $\mathbb{R}^N$. The following proposition shows, in particular, that we just need to check it on any track.

**Proposition 3.25.** *Let $\sigma \in \Sigma$ be given and let $T$ be a Borel track for $\sigma$. Then $m_\sigma(x) = 0$ for $\mathcal{H}^1$ a.e. $x \in \mathbb{R}^N \setminus T$.*

*Proof.* By Definition 3.23 and by Integration Order Inequality (Theorem B.5), we compute

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus T} m_\sigma(x) d\mathcal{H}^1 &= \int_{\mathbb{R}^N\setminus T} \left(\int_{(\Gamma,\sigma)} m_\gamma(x) d\gamma\right) d\mathcal{H}^1 \\
&\leq \int_{(\Gamma,\sigma)} \left(\int_{\mathbb{R}^N\setminus T} m_\gamma(x) d\mathcal{H}^1\right) d\gamma = 0.
\end{aligned}$$

□

**Proposition 3.26.** *If $\sigma \in \Sigma_0$ then $\sigma$ is concentrated on $\Gamma_0$.*

*Proof.* Since $\sigma$ is a non-spread particle motion it admits a track $T = \bigcup_n T_n$ with $\mathcal{H}^1(T_n) < +\infty$, $T_n \subset T_{n+1}$ $\forall n \in \mathbb{N}$. Observe that for $\mathcal{H}^1$-a.e. $\bar{x} \in T$ there is $m \in \mathbb{N}$ and $k \in \mathbb{R}$ so that $\bar{x} \in T_m$ and $m_\sigma(\bar{x}) \le k$. Let $n = \max\{m, k\}$, $\bar{x} \in S_n = \{x \in T_n \mid m_\sigma(x) \le n\}$. Hence $T = \bigcup_n S_n$ up to a $\mathcal{H}^1$ negligible set. Since $\forall n \in \mathbb{N}$ we have by Fubini Theorem

$$\int_{(\Gamma,\sigma)} \left( \int_{S_n} m_\gamma(x) d\mathcal{H}^1 \right) d\gamma = \int_{S_n} m_\sigma(x) d\mathcal{H}^1 \le n\mathcal{H}^1(T_n) < +\infty,$$

then for $\sigma$-a.e. $\gamma \in \Gamma$, $m_\gamma(x) < +\infty$ for $\mathcal{H}^1$-a.e. $x \in S_n$. Therefore $\sigma$ is concentrated on $\Gamma_0$. $\square$

**Proposition 3.27.** *Let $\sigma \in \Sigma$ be non-spread. Then the set*

$$T_0 = \{x \in \mathbb{R}^N \mid m_\sigma(x) > 0\}$$

*is a Borel track for $\sigma$ and it is a minimal track of for set inclusion modulo an $\mathcal{H}^1$-negligible set.*

*Proof.* Let $T$ be a $\sigma$-finite track for $\sigma$. The $\sigma$-finiteness of $T$ allows the use of Fubini Theorem to get

$$\int_{(\Gamma,\sigma)} \left( \int_{T \setminus T_0} m_\gamma(x) d\mathcal{H}^1 \right) d\gamma = \int_{T \setminus T_0} m_\sigma(x) d\mathcal{H}^1 = 0.$$

Thus, for $\sigma$ a.e. $\gamma \in \Gamma$, we have $\int_{T \setminus T_0} m_\gamma(x) d\mathcal{H}^1 = 0$. The minimality of $T_0$ follows from Proposition 3.25. Indeed, if $T'$ is any other track, $T \cap T'$ is a $\sigma-$finite track and therefore it is a Borel track modulo a negligible set. So, by Proposition 3.25, $T_0 \subset T \cap T' \subset T'$, modulo a negligible set. $\square$

**Theorem 3.28.** *(Dominated density convergence) Let $(\sigma_n)_{n \in \mathbb{N}} \subset \Sigma$ be a sequence of non-spread particle motions. Let us suppose that there exists a $\mathcal{H}^1$-measurable mapping $a : \mathbb{R}^N \to \mathbb{R}_+$ with $\int_{\mathbb{R}^N} a(x)\, d\mathcal{H}^1 < +\infty$ so that, for every $n \in \mathbb{N}$, $a_{\sigma_n} \le a$. If $\sigma_n \rightharpoonup \sigma$, then $\sigma$ is a non-spread particle motion.*

*Proof.* For any fixed $\varepsilon > 0$, since $a \in L^1$, there exists a compact subset $A_\varepsilon \subset \mathbb{R}^N$ with $\mathcal{H}^1(A_\varepsilon) < +\infty$ (see [33, Theorem 27]), such that

$$\int_{\mathbb{R}^N \setminus A_\varepsilon} a(x) d\mathcal{H}^1 \le \varepsilon. \tag{3.17}$$

Then, for every $n \in \mathbb{N}$, by Proposition D.12 we have

$$\int_{(\Gamma,\sigma_n)} \mathcal{H}^1(\gamma(I) \setminus A_\varepsilon)\, d\gamma = \int_{\mathbb{R}^N \setminus A_\varepsilon} a_{\sigma_n}(x) d\mathcal{H}^1 \le \int_{\mathbb{R}^N \setminus A_\varepsilon} a(x) d\mathcal{H}^1 \le \varepsilon. \tag{3.18}$$

Since $\sigma_n \rightharpoonup \sigma$, by Proposition D.8, (3.18) and by Theorem C.3, we get

$$\int_{(\Gamma,\sigma)} \mathcal{H}^1(\gamma(I) \setminus A_\varepsilon)\, d\gamma \leq \liminf_n \int_{(\Gamma,\sigma_n)} \mathcal{H}^1(\gamma(I) \setminus A_\varepsilon)\, d\gamma \leq \varepsilon.$$

Now we take an infinitesimal sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ ad set $A = \bigcup_n A_{\varepsilon_n}$. Notice that $A$ is a $\mathcal{H}^1$ $\sigma$-finite set. Then, for every $n \in \mathbb{N}$ we have

$$\int_{(\Gamma,\sigma)} \mathcal{H}^1(\gamma(I) \setminus A)\, d\gamma \leq \int_{(\Gamma,\sigma)} \mathcal{H}^1(\gamma(I) \setminus A_{\varepsilon_n})\, d\gamma \leq \varepsilon_n.$$

From $\varepsilon_n \to 0$ we can conclude that $A$ is a track and so $\sigma$ is non-spread. □

## 3.3. **Scaling and density relations.**

### 3.3.1. *Scaling orderings.*

**Definition 3.29.** *Let $I_1, I_2 \subset \mathbb{R}$ be two given intervals of positive measure. Let $\varphi : I_1 \to I_2$ be a given function. We define the mapping $\varphi^* : \Gamma_{I_2} \to \Gamma_{I_1}$ as $\varphi^*(\gamma_2) = \gamma_2 \circ \varphi$ for every $\gamma_2 \in \Gamma_{I_2}$.*

**Definition 3.30.** *Let $I_1, I_2 \subset \mathbb{R}$ be two given intervals. We shall denote by $\Phi(I_1, I_2)$ the set of monotone functions $\varphi : I_1 \to I_2$ such that* $\inf \varphi = \inf I_2$, $\sup \varphi = \sup I_2$.

**Remark 3.31.** Assume, as we will do in this section, that $I_1, I_2 \subset \mathbb{R}$ are bounded intervals. This is not a restriction since it can be always achieved by a nonlinear homeomorphic scaling. In order to define $\Phi(I_1, I_2)$ it is not restrictive to assume that if one of the two intervals contains one of its extreme points then the other one enjoys the same property (the same extreme point if $\varphi$ is increasing, the other one if $\varphi$ is decreasing). Indeed, if $I_1$, for instance, has a minimum and $\varphi$ is, for instance, monotone increasing, by definition of $\Phi(I_1, I_2)$ we have $\inf I_2 = \inf \varphi = \varphi(\min I_1) \in I_2$. Conversely, if $I_2$ has a minimum we have $\min I_2 = \inf I_2 = \inf \varphi = \lim_{t\to\inf I_1} \varphi(t)$. So $\varphi$ can be considered to be defined on $I_1 \cup \{\inf I_1\}$ In particular, if one of the two intervals is compact so is the other one. We will assume that this is always the case from now on in this section, adding this requirement to the definition of $\Phi(I_1, I_2)$.

**Definition 3.32.** *(Scaling ordering on curves) Let $\gamma_1 : I_1 \to \mathbb{R}^N$ and $\gamma_2 : I_2 \to \mathbb{R}^N$. We say $\gamma_1 \leq_S \gamma_2$ if there exists a monotone increasing map $\varphi \in \Phi(I_1, I_2)$, such that $\gamma_1 = \gamma_2 \circ \varphi = \varphi^*(\gamma_2)$.*

**Remark 3.33.** If the two intervals are compact (see the previous remark) and $\gamma_1 \leq_S \gamma_2$, by the definition of $\Phi(I_1, I_2)$ $\gamma_1$ and $\gamma_2$ have the same boundary values.

Let us point out that, in spite of its name, the relation $\leq_S$ is not an ordering since it only satisfies the reflexivity and transitivity properties. Hence we are led to define the following equivalence classes which allow to deal with an ordering.

**Definition 3.34.** *(Scaling equivalence on curves) We shall say that two curves $\gamma_1, \gamma_2$ are scaling equivalent, in symbols $\gamma_1 \cong_S \gamma_2$, if $\gamma_1 \leq_S \gamma_2$ and $\gamma_2 \leq_S \gamma_1$.*
*We shall denote by $\mathcal{T}$ the quotient topological space of $\Gamma$ with respect to $\cong_S$ and we shall call oriented trajectories the elements of $\mathcal{T}$. For any $\gamma \in \Gamma$, the equivalence class of $\gamma$ will be denoted by $[\gamma]_S$.*

Notice that the relation $\leq_S$ induces an ordering on $\mathcal{T}$. We shall say that any mapping $c : \Gamma \to \mathbb{R}$ is *strictly monotone increasing* with respect to $\leq_S$ if for every $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leq_S \gamma_2$ we have $c(\gamma_1) \leq c(\gamma_2)$ and, if in addition $c(\gamma_1) = c(\gamma_2)$, then $\gamma_2 \leq_S \gamma_1$. If $c$ is monotone increasing with respect to $\leq_S$ then it is constant on the equivalence classes above introduced and therefore it can be identified with an increasing map defined on the quotient space $\mathcal{T}$.

The analogous version of definitions 3.32, 3.34 can be formulated in terms of particle motions as follows.

**Definition 3.35.** *(Uniform scaling relation on particle motions) Let $\sigma_1 \in \mathcal{P}(\Gamma_{I_1})$ and $\sigma_2 \in \mathcal{P}(\Gamma_{I_2})$. We say that $\sigma_1 \leq_S \sigma_2$ if there exists $\varphi \in \Phi(I_1, I_2)$ such that $\sigma_1 = \varphi^*_{\#}\sigma_2$.*

**Definition 3.36.** *(Uniform scaling equivalence of particle motions) We shall say that two particle motions $\sigma_1$ and $\sigma_2$ are uniform scaling equivalent, in symbols $\sigma_1 \cong_S \sigma_2$, if $\sigma_1 \leq_S \sigma_2$ and $\sigma_2 \leq_S \sigma_1$.*

**Remark 3.37.** We shall see in the following (Theorem 3.65) that, for $\sigma_1, \sigma_2 \in \Sigma_0$, $\sigma_1 \cong_S \sigma_2$ means that we can find a monotone change of variable $\varphi$ such that $\sigma_1 = \varphi^*_{\#}\sigma_2$ and $\sigma_2 = (\varphi^{-1})^*_{\#}\sigma_1$ for any $\varphi^{-1}$ in the inverse class (see the next subsection) of $\varphi$.

As just observed in the beginning of Subsection 3.1, the scaling ordering on curves can be regarded as the restriction to the curves of the uniform scaling on the particle motions, while according to Definition 3.1, another relation on the space of particle motions is provided by the above introduced pointwise extension.

**Definition 3.38.** *(Extension of scaling ordering to particle motions) The pointwise extension of $\leq_S$ to the particle motions will be denoted by $\leq_{\widetilde{S}}$ and will be called fiberwise scaling relation.*

**Definition 3.39.** *(Fiberwise scaling equivalence on particle motions) The pointwise extension of the scaling equivalence to the particle motions will be called fiberwise scaling equivalence and will be denoted by $\cong_{\widetilde{S}}$.*

**Remark 3.40.** We can easily deduce from Remark 3.33 that, if $\sigma_1$ and $\sigma_2$ are comparable by $\leq_S$ and $\sigma_1 \in \Sigma(\mu, \nu)$ then also $\sigma_2 \in \Sigma(\mu, \nu)$.

We shall see in the sequel (Proposition 3.42) that when $I_1$ and $I_2$ are compact then $\leq_S$ is closed and therefore by Proposition 3.17 two particle motions $\sigma_1$ and $\sigma_2$ are scaling equivalent if and only if $\sigma_1 \leq_{\widetilde{S}} \sigma_2$ and $\sigma_2 \leq_{\widetilde{S}} \sigma_1$. We shall also see that this last conclusion holds true even when $I_1$ and $I_2$ are not compact if $\sigma_1, \sigma_2 \in \Sigma_0$.

3.3.2. *Monotone change of variables.* Let us point out a few facts about the monotone functions $\varphi : I_1 \to I_2$

**Lemma 3.41.** *Let $I_1, I_2 \subset \mathbb{R}$ be two given intervals with $I_2$ compact and $\forall n \in \mathbb{N}$ let $\varphi_n : I_1 \to I_2$ be a monotone function. Then $(\varphi_n)_{n\in\mathbb{N}}$ admits a pointwise converging subsequence.*

*Proof.* Let us assume, by passing to a subsequence, that the functions are monotone increasing (or decreasing with a similar argument). By a diagonal selection we can extract from $(\varphi_n)_{n\in\mathbb{N}}$ a converging subsequence to a function $\varphi$ defined on $(\mathbb{Q}\cap I_1)\cup E$, where $E$ is the set of the (eventual) extrema of $I_1$ which belong to $I_1$. This function $\varphi$ admits an extension to $I_1$ defined as

$$\bar\varphi(t) = \sup\{\varphi(q) \mid q \in (\mathbb{Q}\cap I_1)\cup E, q \le t\}$$

which is monotone increasing and so it is continuous except at most a countable set. Let $t \in I_1$ a continuity point for $\bar\varphi$. Fix $\varepsilon > 0$, there exist $q_1, q_2 \in (\mathbb{Q}\cap I_1)\cup E$, $q_1 \le t$, $q_2 \ge t$ such that $\varphi$ is defined on $q_1, q_2$ and $\varphi(q_1) > \varphi(q_2) - \varepsilon$. Now, we know that $\varphi_n(q_1) \to \varphi(q_1)$ and $\varphi_n(q_2) \to \varphi(q_2)$ and so we get by monotonicity

$$\liminf_n \varphi_n(t) \ge \lim_n \varphi_n(q_1) = \varphi(q_1) > \varphi(q_2) - \varepsilon \ge \bar\varphi(t) - \varepsilon,$$
$$\limsup_n \varphi_n(t) \le \lim_n \varphi_n(q_2) = \varphi(q_2) < \varphi(q_1) + \varepsilon \le \bar\varphi(t) + \varepsilon.$$

Then $\lim_n \varphi_n(t) = \bar\varphi(t)$ out of a countable set $D$ (the set of discontinuity points of $\bar\varphi$). Finally, a further diagonal selection argument allows to extract a subsequence also converging on $D$. □

**Proposition 3.42.** *Assume $I_1, I_2 \subset \mathbb{R}$ are compact intervals. Then the relation $\le_S$ on the curves is closed with respect to the locally uniform convergence.*

*Proof.* For every $n \in \mathbb{N}$ let ${\gamma^1}_n = {\gamma^2}_n \circ \varphi_n$ and suppose that ${\gamma^1}_n \to \gamma_1$, ${\gamma^2}_n \to \gamma_2$ (locally) uniformly. By applying Lemma 3.41 to the sequence $(\varphi_n)_{n\in N}$, since $I_2$ is compact, we have that there exists $\varphi \in \Phi(I_1, I_2)$ such that, for every $t \in I_1$, $\varphi_n(t) \to \varphi(t)$. Then ${\gamma^2}_n(\varphi_n(t)) \to \gamma_2(\varphi(t))$ which means ${\gamma^1}_n(t) \to \gamma_2(\varphi(t))$ pointwise. Since ${\gamma^1}_n(t) \to \gamma_1(t)$ locally uniformly we get $\gamma_1 = \gamma_2 \circ \varphi$, so $\gamma_1 \le_S \gamma_2$. □

**Remark 3.43.** Observe that $\le_S$ is not closed if $I_1$ and $I_2$ are not compact. Indeed, let us consider $\gamma : [0, 1[\to \mathbb{R}^N$ and $\forall n \in \mathbb{N}$ set $\gamma_n : [0, 1[\to \mathbb{R}^N$,

$$\gamma_n(t) = \begin{cases} \gamma(0) & \text{if } t \in [0, 1 - \frac{1}{n}] \\ \gamma(nt - n + 1) & \text{if } t \in [1 - \frac{1}{n}, 1[. \end{cases}$$

It is readily seen that for every $n \in \mathbb{N}$: $\gamma \le_S \gamma_n$. By passing to the limit we have that $\gamma_n$ converges to the curve $\bar\gamma$ which takes the constant value $\gamma(0)$ and thus $\gamma \not\le_S \bar\gamma$.

**Proposition 3.44.** *Assume $I_1, I_2 \subset \mathbb{R}$ are compact intervals. Then the relation $\le_{\widetilde S}$ is closed in the narrow convergence with respect to the locally uniform convergence on $\Gamma$.*

*Proof.* The thesis follows by Proposition 3.10. □

As it is well known, any monotone (for instance, increasing) function a.e. defined on an interval $I_1$ of $\mathbb{R}$ represents an equivalence class of monotone functions which agree almost everywhere except for a countable set. In particular, the functions $\varphi_-$, $\varphi_+$, respectively defined as the lower semicontinuous and upper semicontinuous (or, equivalently, left and right continuous) envelopes of $\varphi$, belong to the equivalence class of $\varphi$. Thus for every $t \in I_1$: $\varphi_-(t) \leq \varphi(t) \leq \varphi_+(t)$. Moreover, for any such $\varphi$ we can define the inverse equivalence class $\varphi^{-1}$, to which belong in particular the left and right inverses $\varphi_-^{-1}$ and $\varphi_+^{-1}$ given by $\varphi_-^{-1}(s) = \sup\{t \in I_1 \mid \varphi(t) < s\}$, $\varphi_+^{-1}(s) = \inf\{t \in I_1 \mid \varphi(t) > s\}$ (where $\sup\emptyset = \inf I_1$ and $\inf\emptyset = \sup I_1$. They belong to $I_1$ whenever $\varphi \in \Phi(I_1, I_2)$ and they need to be assumed, thanks to Remark 3.31).

**Lemma 3.45.** *Let $\varphi$ be as above. Then, for every $t \in I_1$ we have $\varphi_-^{-1} \circ \varphi(t) \leq t \leq \varphi_+^{-1} \circ \varphi(t)$ and for every $s \in I_2$ we have $\varphi_- \circ \varphi^{-1}(s) \leq s \leq \varphi_+ \circ \varphi^{-1}(s)$.*

*Proof.* The left inequality follows by observing that by definition $\varphi_-^{-1}(\varphi(t)) \leq t$ if and only if for every $s > t$: $\varphi(s) \geq \varphi(t)$ and this is true since $\varphi$ is an increasing function. The other inequalities are proved in a similar way. □

**Lemma 3.46.** *Let $\gamma_1 \in \Gamma_{I_1}$, $\gamma_2 \in \Gamma_{I_2}$ such that $\gamma_1 \leq_S \gamma_2$ with a monotone change of variable $\varphi \in \Phi(I_1, I_2)$, i.e. $\gamma_1 = \gamma_2 \circ \varphi$. If, for every $t \in I_1$, $\gamma_2$ is constant on the interval $[\varphi_-(t), \varphi_+(t)]$, then for any $\varphi^{-1}$ in the inverse class of $\varphi$, $\gamma_2 = \gamma_1 \circ \varphi^{-1} = \gamma_2 \circ \varphi \circ \varphi^{-1}$ and so, in particular, $\gamma_2 \leq_S \gamma_1$.*

*Proof.* For every $s \in I_2$, by Lemma 3.45, $s, \varphi(\varphi^{-1}(s)) \in [\varphi_-(\varphi^{-1}(s)), \varphi_+(\varphi^{-1}(s))]$ on which $\gamma_2$ is required to be constant. So $\gamma_2(s) = \gamma_2(\varphi(\varphi^{-1}(s))) = \gamma_1(\varphi^{-1}(s))$. □

**Remark 3.47.** Let us observe that, on the contrary, the final conclusion stating that also $\gamma_2 \leq_S \gamma_1$ does not imply that $\gamma_2$ is constant on the interval $[\varphi_-(t), \varphi_+(t)]$ neither that $\gamma_1 = \gamma_2 \circ \varphi^{-1}$. Indeed, let us consider, for instance, the case $\gamma_1(t) = \gamma_2(t) = \sin t$ and

$$\varphi(t) = \begin{cases} t & \text{if } t \leq 0 \\ t + 2\pi & \text{if } t > 0. \end{cases}$$

In such a case, obviously, we have $\gamma_1 = \gamma_2 \circ \varphi$ and also $\gamma_2 = \gamma_1 \circ \varphi \neq \gamma_1 \circ \varphi^{-1}$.

3.3.3. *Monotonicity of the length.* We shall make use of the notation introduced in Appendix D. In particular, for any $\gamma \in \Gamma_I$ we need to use the local orbit-length $l_\gamma$ defined in (D.54) and the total length $L(\gamma)$ defined in (D.55).

**Lemma 3.48.** *Let $\gamma_1 \in \Gamma_{I_1}$, $\gamma_2 \in \Gamma_{I_2}$ such that $\gamma_1 \leq_S \gamma_2$. Then $m_{\gamma_1}(x) \leq m_{\gamma_2}(x)$, for every $x \in \mathbb{R}^N$ up to a countable set. Moreover, if $\gamma_1 = \gamma_2 \circ \varphi$ for a monotone change of variable $\varphi \in \Phi(I_1, I_2)$, $\gamma_2 \in \Gamma_0$ and $m_{\gamma_1} = m_{\gamma_2}$ $\mathcal{H}^1$-a.e., then $\gamma_2$ is constant on $[\varphi_-(t), \varphi_+(t)] \subset I_2$ $\forall t \in I_1$ and $\gamma_2 = \gamma_1 \circ \varphi^{-1}$ for every $\varphi^{-1}$ in the inverse class of $\varphi$, hence $\gamma_1 \cong_S \gamma_2$.*

*Proof.* Suppose that $m_{\gamma_1}(x) > m_{\gamma_2}(x)$. Then $\varphi$ is not injective on $\gamma_1^{-1}(x)$. Indeed, if $t \in \gamma_1^{-1}(x)$, then $\varphi(t) \in \gamma_2^{-1}(x)$. Therefore either $m_{\gamma_1}(x) \leq m_{\gamma_2}(x)$ or there exist $t_1 < t_2$ with $t_1, t_2 \in \gamma_1^{-1}(x)$ such that $\varphi(t_1) = \varphi(t_2)$. Since $\varphi$ is monotone increasing, this last case implies that $\varphi$ takes a constant value on the interval $[t_1, t_2]$, then $\varphi(t_1) \in A$ where $A$ is at most a countable set. Hence $x \in \gamma_2(A)$ which is at most a countable set.
Assume that for some $t \in I_1$ $\psi_1(t) = \gamma_2|_{[\varphi_-(t),\varphi_+(t)]} \neq const$ and let us denote by $\psi_2$ the function $\gamma_2$ redefined on the interval $[\varphi_-(t), \varphi_+(t)]$ as the constant value assumed in the two boundary points $(\gamma_2(\varphi_-(t)) = \gamma_2(\varphi_+(t))$ by the continuity of $\gamma_1 = \gamma_2 \circ \varphi)$. Then $m_{\gamma_2} = m_{\psi_1} + m_{\psi_2}$ $\mathcal{H}^1$-a.e. and we have $\int_{\mathbb{R}^N} m_{\psi_1}(x) d\mathcal{H}^1 > 0$, $\gamma_1 = \psi_2 \circ \varphi$. Therefore we get

$$m_{\gamma_1}(x) \leq m_{\psi_2}(x) = m_{\gamma_2}(x) - m_{\psi_1}(x) < m_{\gamma_2}(x)$$

on a set of positive $\mathcal{H}^1$ measure. The last assertion follows by Lemma 3.46. □

**Corollary 3.49.** *Assume $\gamma_1 \leq_S \gamma_2$. Then $l_{\gamma_1} \leq l_{\gamma_2}$. In particular we get the strict monotonicity of the length functional.*

**Corollary 3.50.** *Let $\gamma_1 \in \Gamma_{I_1}$, $\gamma_2 \in \Gamma_{I_2}$ such that $\gamma_1 \leq_S \gamma_2$ with a monotone change of variable $\varphi \in \Phi(I_1, I_2)$, i.e. $\gamma_1 = \gamma_2 \circ \varphi$. Then $L(\gamma_1) \leq L(\gamma_2)$. If $L(\gamma_1) = L(\gamma_2) < +\infty$, or, equivalently, $m_{\gamma_1} = m_{\gamma_2}$ $\mathcal{H}^1-$a.e., then $\gamma_2$ is constant on $[\varphi_-(t), \varphi_+(t)] \subset I_2$ $\forall t \in I_1$ and $\gamma_2 = \gamma_1 \circ \varphi^{-1}$.*

By Lemma 3.48 it follows that if $\gamma_2 \in \Gamma_0$ and $\gamma_1 \leq_S \gamma_2$ then $\gamma_1 \in \Gamma_0$.
We can therefore introduce the set

$$\mathcal{T}_0 = \{[\gamma]_S \mid \gamma \in \Gamma_0\} \tag{3.19}$$

and note that (see (3.13))

$$\mathcal{T}_c = \{[\gamma]_S \mid \gamma \in \Gamma_c\} \subset \mathcal{T}_0. \tag{3.20}$$

**Definition 3.51.** *We shall say that $\gamma_1$ (equivalently $[\gamma_1]_S \in \mathcal{T}$) is minimal for $\leq_S$ if for every $\gamma_2 \leq_S \gamma_1$ we have $\gamma_1 \leq_S \gamma_2$.*

**Lemma 3.52.** *Every nonconstant $[\gamma]_S \in \mathcal{T}_c$ has a representative, still denoted by $\gamma$, enjoying the property that there is no interval $I'$ with a positive measure such that $\gamma(t) = const.$ on $I'$.*

*Proof.* We suppose that $\gamma \in \Gamma_{I_1}$, with $I_1 = [a, b]$, takes constant values on some maximal disjoint open sub-intervals $A_i \subset I_1$ $i = 1, \ldots, n_j, \ldots$. Let $\beta = \mathcal{H}^1(I_1 \setminus \bigcup_i A_i) > 0$ and $\varphi$ be the mapping from $I_1$ to $[0, \beta]$ defined as

$$\varphi : t \mapsto \mathcal{H}^1(I_t \setminus \bigcup_i A_i), \;\; I_t = \{x \in I \mid x \leq t\}.$$

Since $\varphi$ is continuous, the inverses $\varphi_-^{-1}$ and $\varphi_+^{-1}$ are injective mappings. For every $s \in I_2 = [0, \beta] = \varphi(I_1)$, the open interval $]\varphi_-^{-1}(s), \varphi_+^{-1}(s)[$ is either empty or it is one of the intervals $A_i$ on which $\varphi$ is constant. Since $\gamma$ is constant too on $A_i$, it results that $\gamma\circ\varphi^{-1}$ is absolutely continuous. Obviously, $\gamma\circ\varphi^{-1}\leq_S\gamma$, moreover $\gamma\circ\varphi^{-1}\circ\varphi=\gamma$ by Lemma 3.46 and so $\gamma\leq_S\gamma\circ\varphi^{-1}$. Since $\varphi(\bigcup_i A_i)$ is a countable set, for every interval of positive length $I'\subset I_2$, $\gamma$ cannot be constant on $\varphi^{-1}(I')$. □

**Proposition 3.53.** *$[\gamma]_S\in\mathcal{T}_c$ is minimal if and only if it has an injective representative or it is constant.*

*Proof.* Let $\gamma: I\to\mathbb{R}^N$ be a representative as in Proposition 3.52. If $\gamma$ is not injective then there exist $t_1<t_2$ in $I$ such that $\gamma(t_1)=\gamma(t_2)$. We set $I^*=I\cap I-(t_2-t_1)$ and take the function $\varphi: I^*\to I$ defined as follows

$$\varphi(t)=\begin{cases} t & \text{if } t\leq t_1\\ t+t_2-t_1 & \text{if } t>t_1.\end{cases}$$

Let us consider $\gamma^*=\gamma\circ\varphi$. Notice that that $\gamma^*$ is continuous, since $\gamma(t_1)=\gamma(t_2)$. Obviously we have $\gamma^*\leq_S\gamma$. Since $\gamma$ is not constant on $[t_1,t_2]$ we have $m_{\gamma^*}<m_\gamma$ on a set of positive $\mathcal{H}^1$ measure. Then, by applying Lemma 3.48 we have that the relation $\gamma\leq_S\gamma^*$ cannot hold and thus $\gamma$ is not minimal. To prove the reverse implication, let us assume that $\gamma$ is an injective map and $\psi\leq\gamma$, then there is an increasing map $\varphi$ such that $\psi=\gamma\circ\varphi$. Let us notice that, since $\psi$ is continuous and $\gamma$ is injective, then $\varphi$ must be continuous. So, by Lemma 3.46 $\gamma=\psi\circ\varphi^{-1}\leq_S\psi$. □

3.3.4. *Density orderings.*

**Definition 3.54.** *(Density ordering on curves) Let $\gamma_1: I_1\to\mathbb{R}^N$ and $\gamma_2: I_2\to\mathbb{R}^N$. We say $\gamma_1\leq_D\gamma_2$ if $m_{\gamma_1}(x)\leq m_{\gamma_2}(x)$, for $\mathcal{H}^1$ a.e. $x\in\mathbb{R}^N$.*

**Definition 3.55.** *(Density equivalence of curves) We shall say that two curves $\gamma_1$ and $\gamma_2$ are density-equivalent, in symbols $\gamma_1\cong_D\gamma_2$, if $m_{\gamma_1}(x)=m_{\gamma_2}(x)$, for $\mathcal{H}^1$ a.e. $x\in\mathbb{R}^N$.*

The analogous version of Definitions 3.54 and 3.55 can be formulated in terms of particle motions and, according to Subsection 3.1, we shall use the same symbols to denote such relations.

**Definition 3.56.** *(Density Ordering on particle motions) Let $\sigma_1,\sigma_2\in\Sigma$ be non-spread particle motions. We say that $\sigma_1\leq_D\sigma_2$ if $m_{\sigma_1}\leq m_{\sigma_2}$ for $\mathcal{H}^1$ a.e. $x$.*

**Remark 3.57.** Definition 3.56 is not the only possible way to reformulate Definition 3.54 in terms of particle motions. Indeed the relation defined by requiring both $m_{\sigma_1}\leq m_{\sigma_2}$ and $a_{\sigma_1}\leq a_{\sigma_2}$ still agrees with Definition 3.54 in the case of a pair of curves, since it is readily seen that $m_{\gamma_1}\leq m_{\gamma_2}$ implies $a_{\gamma_1}\leq a_{\gamma_2}$ but in general $m_{\sigma_1}\leq m_{\sigma_2}$ does not imply $a_{\sigma_1}\leq a_{\sigma_2}$.

**Definition 3.58.** *(Density equivalence of particle motions) We shall say that two particle motions $\sigma_1$ and $\sigma_2$ are density-equivalent, in symbols $\sigma_1 \cong_D \sigma_2$, if $m_{\sigma_1} = m_{\sigma_2}$ for $\mathcal{H}^1$ a.e. $x$.*

According to Definition 3.1, the pointwise extension of the density ordering to the space of particle motions is given in the following definition.

**Definition 3.59.** *(Extension of density ordering to measures) The pointwise extension of $\leq_D$ to the particle motions will be denoted by $\leq_{\widetilde{D}}$ and will be called fiberwise density relation.*

**Lemma 3.60.** *Let $\sigma_i \in \Sigma$ for $i = 1, 2$. If $\sigma_1 \leq_{\widetilde{D}} \sigma_2$ then $\sigma_1 \leq_D \sigma_2$.*

*Proof.* By virtue of Proposition 3.4 we take two lagrangian parameterizations $\hat{\chi}_i$, $i = 1, 2$, such that $\sigma_i = \hat{\chi}_{i\#}\mu_\Omega$ and such that for a.e. $p \in \Omega$: $\hat{\chi}_1(p) \leq_D \hat{\chi}_2(p)$. Now, let $m_i(p, x) = m_\gamma(x)$ for $\gamma = \hat{\chi}_i(p)$. By Theorem 3.21 the mappings $m_i$ are Borel measurable on $\Omega \times \mathbb{R}^N$. By Theorem B.5 we get from the previous proposition

$$\begin{aligned}
\int_{\mathbb{R}^N} (m_{\sigma_1}(x) - m_{\sigma_2}(x))^+ \, d\mathcal{H}^1 &= \int_{\mathbb{R}^N} \left( \int_{(\Omega,\mu_\Omega)} m_1(p, x) dp - \int_{(\Omega,\mu_\Omega)} m_2(p, x) dp \right)^+ d\mathcal{H}^1 \\
&\leq \int_{\mathbb{R}^N} \left( \int_{(\Omega,\mu_\Omega)} (m_1(p, x) - m_2(p, x))^+ \, dp \right) d\mathcal{H}^1 \\
&\leq \int_{(\Omega,\mu_\Omega)} \left( \int_{\mathbb{R}^N} (m_1(p, x) - m_2(p, x))^+ \, d\mathcal{H}^1 \right) dp = 0,
\end{aligned}$$

which finally gives $m_{\sigma_1}(x) \leq m_{\sigma_2}(x)$ for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^N$. □

**Lemma 3.61.** *Let $\sigma_i \in \Sigma$ for $i = 1, 2$. If $\sigma_1 \leq_{\widetilde{S}} \sigma_2$ then $\sigma_1 \leq_{\widetilde{D}} \sigma_2$.*

*Proof.* Assume $\gamma_1 \leq_S \gamma_2$, hence by Lemma 3.48 $m_{\gamma_1} \leq m_{\gamma_2}$ up to a countable set and then $\gamma_1 \leq_D \gamma_2$. Therefore by (3.9) we get the thesis. □

**Theorem 3.62.** *For any $\sigma_1, \sigma_2 \in \Sigma$ the following implications hold true.*

$$\sigma_1 \leq_S \sigma_2 \Rightarrow \sigma_1 \leq_{\widetilde{S}} \sigma_2 \Rightarrow \sigma_1 \leq_{\widetilde{D}} \sigma_2 \Rightarrow \sigma_1 \leq_D \sigma_2. \tag{3.21}$$

*Proof.* The first implication easily follows by observing that the uniform scaling relation implies, in particular, the existence of a transport plan concentrated on the scaling ordering on curves. The second and third implications respectively follow by Lemma 3.61 and Lemma 3.60 . □

**Lemma 3.63.** *Let $\sigma_1 \in \Sigma$, $\sigma_2 \in \Sigma_0$, $\sigma_1 \leq_{\widetilde{D}} \sigma_2$ and $\sigma_2 \leq_D \sigma_1$. Then $\sigma_1 \cong_{\widetilde{D}} \sigma_2$.*

*Proof.* By using Theorem 3.62 we get $\sigma_1 \cong_D \sigma_2$, i.e. $m_{\sigma_1} = m_{\sigma_2}$ $\mathcal{H}^1$-a.e.. Taking into account that $\sigma_2 \in \Sigma_0$ is a non-spread particle motion, we can apply Fubini Theorem and

Theorem B.5 so, with the construction and the notation in the proof of Lemma 3.60,

$$\begin{aligned}\int_{(\Omega,\mu_\Omega)}\left(\int_{\mathbb{R}^N}(m_2(p,x)-m_1(p,x))\,d\mathcal{H}^1\right)dp &\le \int_{\mathbb{R}^N}\left(\int_{(\Omega,\mu_\Omega)}(m_2(p,x)-m_1(p,x))\,dp\right)d\mathcal{H}^1\\ &= \int_{\mathbb{R}^N}(m_{\sigma_2}(x)-m_{\sigma_1}(x))\,d\mathcal{H}^1=0.\end{aligned}$$

Hence since $m_1 \le m_2$, for a.e. $p\in\Omega$

$$\int_{\mathbb{R}^N}(m_2(p,x)-m_1(p,x))\,d\mathcal{H}^1=0.$$

So, for a.e. $p\in\Omega$: $m_2(p,x)-m_1(p,x)=0$ for $\mathcal{H}^1$ a.e. $x\in\mathbb{R}^N$, i.e. $\sigma_1 \cong_{\widetilde{D}} \sigma_2$. □

**Definition 3.64.** *(Fiberwise density equivalence of particle motions) The pointwise extension of* $\cong_D$ *to the particle motions will be called fiberwise density equivalence and will be denoted by* $\cong_{\widetilde{D}}$.

Let us remark that, though we cannot directly apply Proposition 3.17, by Lemma 3.63 we deduce that $\forall\sigma_1,\sigma_2\in\Sigma_0$ $\sigma_1\cong_{\widetilde{D}}\sigma_2$ if and only if $m_{\sigma_1}\le_{\widetilde{D}} m_{\sigma_2}$ and $m_{\sigma_2}\le_{\widetilde{D}} m_{\sigma_1}$.

**Theorem 3.65.** *Let* $\sigma_1\in\Sigma$, $\sigma_2\in\Sigma_0$, $\sigma_1\le\sigma_2$, *where* $\le$ *denotes any one of the relations* $\le_S,\le_{\widetilde{S}},\le_{\widetilde{D}}$ *and assume* $\sigma_2\le_D\sigma_1$. *Then* $\sigma_2\le\sigma_1$.

*Proof.* The case $\sigma_1\le_{\widetilde{D}}\sigma_2$ is proved in Lemma 3.63, so let us assume $\sigma_1\le_{\widetilde{S}}\sigma_2$. By applying Theorem 3.62 we have $\sigma_1\le_{\widetilde{D}}\sigma_2$, so the conclusions of Lemma 3.63 hold true and so, with the notation in the proof of Lemma 3.60, for a.e. $p\in\Omega$: $\hat\chi_1(p)\le_S\hat\chi_2(p)$ and $m_2(p,x)-m_1(p,x)=0$ for $\mathcal{H}^1$ a.e. $x\in\mathbb{R}^N$. By Proposition 3.26 $\sigma_2$ is concentrated on $\Gamma_0$, then by Lemma 3.48 we get $\hat\chi_1(p)\cong_S\hat\chi_2(p)$ for a.e. $p\in\Omega$, hence $\sigma_1\cong_{\widetilde{S}}\sigma_2$.
Assume $\sigma_1\le_S\sigma_2$. By definition $\sigma_1=\varphi^*_\#\sigma_2$, for a continuous and strictly increasing map $\varphi: I\to\overline{J}$. By Lemma 3.63 for $\sigma_1$-a.e. $\gamma\in\Gamma$ $m_\gamma=m_{\varphi^*(\gamma)}$ and $\gamma\in\Gamma_0$. So Lemma 3.48 implies $\gamma=(\varphi^{-1})^*(\varphi^*(\gamma))$. Therefore $\sigma_2=\varphi^{-1}_\#\sigma_1$. □

**Corollary 3.66.** *Let* $\sigma\in\Sigma_0$ *such that* $\sigma$ *is minimal for the relation* $\le_D$ *on a given class of competitors* $\Sigma'\subset\Sigma$. *Then* $\sigma$ *is minimal for* $\le_{\widetilde{D}}$ *Furthermore, if* $\sigma$ *is minimal for* $\le_{\widetilde{D}}$ *then it is minimal for* $\le_{\widetilde{S}}$ *and if* $\sigma$ *is minimal for* $\le_{\widetilde{S}}$ *then it is minimal for* $\le_S$.

*Proof.* We only prove the first one of the above implications, the other ones being similar. Let $\sigma'\in\Sigma$ such that $\sigma'\le_{\widetilde{D}}\sigma$. Then by applying Theorem 3.62 we get $\sigma'\le_D\sigma$ and by the minimality of $\sigma$ with respect to $\le_D$ we deduce $\sigma\le_D\sigma'$. Finally, by Lemma 3.63 the two conditions $\sigma'\le_{\widetilde{D}}\sigma$, $\sigma\le_D\sigma'$ simultaneously fulfilled imply $\sigma\le_{\widetilde{D}}\sigma'$. □

### 3.4. **Splitting orbits.**

**Lemma 3.67.** *Let $\sigma \in \Sigma$ be given and let $\sigma_1$ and $\sigma_2$ be two complementary truncations of $\sigma$. Then $m_\sigma(x) = m_{\sigma_1}(x) + m_{\sigma_2}(x)$ except at most a countable set.*

*Proof.* Let $\tau : \Gamma \to I$ be the truncation mapping inducing $\sigma_1$ and $\sigma_2$ (see subsection 2.2). For a.e. $\gamma \in \Gamma$ and for every $x$ so that $x \neq b_\tau(\gamma)$, we have $m_\gamma(x) = m_{\gamma_1}(x) + m_{\gamma_2}(x)$. Then we can state that

$$\begin{aligned}\int_{(\Gamma,\sigma)} m_\gamma(x)\, d\gamma &= \int_{(\Gamma,\sigma)} m_{\gamma_1}(x)\, d\gamma + \int_{(\Gamma,\sigma)} m_{\gamma_2}(x)\, d\gamma \\ &= \int_{(\Gamma,\sigma_1)} m_\gamma(x)\, d\gamma + \int_{(\Gamma,\sigma_2)} m_\gamma(x)\, d\gamma\end{aligned}$$

except the case $x = b_\tau(\gamma)$ for $\gamma$ belonging to some $\Gamma'$ with $\sigma(\Gamma') > 0$. Since the sets $\Gamma'$ are disjoint, so they form at most a countable set, we get the thesis. □

**Remark 3.68.** Let us observe that the analogous property of the one stated in the previous lemma holds when dealing with the composition of particle motions, i.e. $m_\sigma = m_{\sigma_1} + m_{\sigma_2}$ when $\sigma$ is the composition of $\sigma_1$ and $\sigma_2$ (see [29, Section 1.2]).

**Lemma 3.69.** *Let $x \in \mathbb{R}^N$ be given. The map $\tau : \Gamma_I \to \overline{\mathbb{R}}$ defined as*

$$\tau : \gamma \mapsto \inf\{t \in I \mid \gamma(t) = x\},$$

*with $\tau(\gamma) = \sup I$ if $x \notin \gamma(I)$, is Borel measurable.*

*Proof.* Let us firstly assume $I = [a, b]$. Observe that $\tau$ is measurable if and only if for every $\bar{t} \in ]a, b[$ the set $A = \{\gamma \mid \tau(\gamma) \leq \bar{t}\}$ is Borel measurable. Notice that $\tau(\gamma) \leq \bar{t}$ means $x \in \gamma([a, \bar{t}])$. Let us show that the set $A = \{\gamma \mid x \in \gamma([a, \bar{t}])\}$ is measurable. Indeed $\gamma \in A$ if and only if for every integer $h > 0$ there exists $s \in [a, \bar{t}] \cap \mathbb{Q}$ such that $\gamma(s) \in B_{\frac{1}{h}}(x)$. Therefore

$$A = \bigcap_h \bigcup_{s \in [a,\bar{t}] \cap \mathbb{Q}} \{\gamma \in \Gamma_I | \gamma(s) \in B_{\frac{1}{h}}(x)\},$$

which is Borel measurable.
If $I$ is not compact, then $I = \bigcup_n I_n$ with $I_n$ compact. Let $\tau_n$ be analogously defined, with $I$ replaced by $I_n$. Since $\tau = \inf \tau_n$ and $\tau = \sup \tau_n$ ($= \sup I$) on two complementary Borel subsets of $\mathbb{R}$, $\tau$ is Borel measurable. □

**Remark 3.70.** Let $\sigma \in \Sigma(\mu, \nu)$ and $\bar{x} \in \mathbb{R}^N$ be given. If for $\sigma$-a.e. $\gamma \in \Gamma$ we have $\bar{x} \in \gamma(I)$, then we set

$$\tau_1(\gamma) = \inf\{t \in I \mid \gamma(t) = \overline{x}\},$$
$$\tau_2(\gamma) = \sup\{t \in I \mid \gamma(t) = \overline{x}\}.$$

Then by applying Lemma 3.69 to $\tau_1$ and (with trivial changes) to $\tau_2$, we can split $\sigma$ in the three following complementary truncations (see Section 2.2) $\sigma_1 \in \Sigma(\mu, \delta_{\bar{x}})$, $\sigma_2 \in \Sigma(\delta_{\bar{x}}, \delta_{\bar{x}})$, $\sigma_3 \in \Sigma(\delta_{\bar{x}}, \nu)$ and, by Lemma 3.67, we have $m_\sigma = m_{\sigma_1} + m_{\sigma_2} + m_{\sigma_3}$.

**Proposition 3.71.** *Let $\sigma \in \Sigma$, $\overline{x} \in \mathbb{R}^N$ be given. Assume that $\bar{x} \in \gamma(I)$ for $\sigma$-a.e. $\gamma \in \Gamma$. Then there exist $\sigma' \in \Sigma$ and $t_{\overline{x}} \in \mathbb{R}$ such that $\sigma \cong_{\widetilde{S}} \sigma'$ and $\gamma(t_{\overline{x}}) = \overline{x}$ for $\sigma'$-a.e. $\gamma \in \Gamma$.*

*Proof.* Assume, without any restriction, $I = [0, 1]$. Thanks to Lemma 3.69 we define a map $\tau : \Gamma \to I$ by setting for any $\gamma \in \Gamma$ such that $\bar{x} \in \gamma(I)$

$$\tau_1(\gamma) = \inf\{t \in I \mid \gamma(t) = \overline{x}\}.$$

We claim that we can find a fiberwise reparametrization $\sigma' \cong_{\widetilde{S}} \sigma$ in such a way that every $\gamma$ is reparametrized on the interval $I' = [0, 2]$ so that $\gamma(t) = \overline{x}$ for every $t \in [\tau^1(\gamma), \tau^1(\gamma) + 1]$. Then the thesis follows by taking $t_{\overline{x}} = 1$.
The claim can be achieved, for instance, by employing some above introduced construction. We just need to take the two complementary truncations induced by $\tau^1$ (see Subsection 2.2) : $\sigma_i = (p^i_{\tau_1})_\# \sigma$, $i = 1, 2$, where the mappings $p^i_{\tau_1}$ are defined in (2.5). Then we shift $\sigma_2$ on $[1, 2]$ and take as $\sigma'$ the composition of $\sigma_1$ and $\sigma_2$, as defined in [29, Sect. 1.2], by using the transport plan $\pi = (p^1_{\tau_1} \times p^2_{\tau_1})_\# \sigma$. □

**Remark 3.72.** Given $\overline{x}$ as in Proposition 3.71, the same construction as in the above proof can be repeated by using the map $\tau_2 : \Gamma \to I$, $\tau_2(\gamma) = \sup\{t \in I \mid \gamma(t) = \overline{x}\}$. In this way we get the existence of two points $t_1$, $t_2$ such that the set $I_1 = \{t \in I \mid \gamma(t) = \overline{x}, t \leq t_1\}$ is an interval containing $t_1$ as right boundary point and analogously $I_2 = \{t \in I \mid \gamma(t) = \overline{x}, t \geq t_2\}$ is an interval containing $t_2$ as left boundary point, for $\sigma'-$a.e. $\gamma \in \Gamma$. We shall use this construction in the sequel.

**Remark 3.73.** The construction in the proof of Proposition 3.71 can be easily iterated. If $\sigma$-a.e. $\gamma$ is such that $\gamma(t^\gamma_1) = x$ and $\gamma(t^\gamma_2) = y$ for $t^\gamma_1 < t^\gamma_2$, by using the same technique one can find $t_x < t_y$ (not depending on $\gamma$) such that for every $\gamma$ $\gamma(t_x) = x$ and $\gamma(t_y) = y$, for $\sigma'-$a.e. $\gamma \in \Gamma$.

**Lemma 3.74.** *Let $\sigma \in \Sigma$ and $x, y, z \in \mathbb{R}^N$ be given and let*

$$\Gamma' = \{\gamma \in \Gamma \mid \exists\, t_1 < t_2 : \gamma(t_1) = x, \gamma(t_2) = y\},$$
$$\Gamma'' = \{\gamma \in \Gamma \mid \exists\, t_2 < t_3 : \gamma(t_2) = y, \gamma(t_3) = z\},$$
$$\Gamma''' = \{\gamma \in \Gamma \mid \exists\, t_1 < t_2 < t_3 : \gamma(t_1) = x, \gamma(t_2) = y, \gamma(t_3) = z\}.$$

*Then if $\sigma(\Gamma') > 0$, $\sigma(\Gamma'') > 0$ there exists $\bar{\sigma} \cong_D \sigma$ such that $\bar{\sigma}(\Gamma''') > 0$.*

*Proof.* Let $\sigma'$, $\sigma''$ be two Borel positive measure, respectively concentrated on $\Gamma'$ and $\Gamma''$, satisfying the inequality $\sigma' + \sigma'' \leq \sigma$ and such that $\sigma'(\Gamma) = \sigma''(\Gamma) > 0$. (We can take $\sigma' = c_1 \sigma \llcorner \Gamma'$, $\sigma'' = c_2 \sigma \llcorner \Gamma''$ with $c_1, c_2$ such that $c_1 + c_2 \leq 1$ and $c_1 \sigma(\Gamma') = c_2 \sigma(\Gamma'')$).
We have $\sigma = \sigma' + \sigma'' + (\sigma - (\sigma' + \sigma''))$, we set $p^0_\# \sigma' = \mu'$, $p^1_\# \sigma' = \nu'$, $p^0_\# \sigma'' = \mu''$,

$p^1_{\#}\sigma'' = \nu''$. By arguing as in the final argument of the proof of Proposition 3.71, we can split $\sigma'$ as $\sigma' = \sigma'_1 \otimes \sigma'_2$ with $\sigma'_1 \in \Sigma(\mu', \delta_y)$ and $\sigma'_2 \in \Sigma(\delta_y, \nu')$. Analogously we have $\sigma'' = \sigma''_1 \otimes \sigma''_2$ with $\sigma''_1 \in \Sigma(\mu'', \delta_y)$ and $\sigma''_2 \in \Sigma(\delta_y, \nu'')$.
We set

$$\bar{\sigma} = \sigma'_1 \otimes \sigma''_2 + \sigma''_1 \otimes \sigma'_2 + \sigma - (\sigma' + \sigma'').$$

It is easy to see that $\sigma'_1 \otimes \sigma''_2 \in \Sigma(\mu', \nu'')$, $\sigma''_1 \otimes \sigma'_2 \in \Sigma(\mu'', \nu')$ and $\sigma - (\sigma' + \sigma'') \in \Sigma(\mu - (\mu' + \mu''), \nu - (\nu' + \nu''))$. Thanks to the additivity of the multiplicity function (see Remark 3.70), the equivalence between $\sigma$ and $\bar{\sigma}$ follows. □

3.5. **Length Parameterizations.** In the following, for a sake of simplicity and also in view of successive applications, we assume that $I \subset \mathbb{R}$ contains the minimum, though with slight modifications one can consider a general interval by fixing an initial point. On the other side, in order to allow the scaling equivalence between $\Gamma_I$ and $\Gamma_{\mathbb{R}_+}$ we shall assume $I$ is half-open at the right end. Otherwise we shall reparametrize the curves on $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ rather than $\mathbb{R}_+$ (see Remark 3.31).

**Definition 3.75.** *Let us define the mapping $\mathscr{L} : \Gamma_I \to \Gamma_{\mathbb{R}_+}$ which associates to any curve $\gamma \in \Gamma_I$, with length $L(\gamma)$, the curve $\mathscr{L}(\gamma) \in [\gamma]_S$ which has $|\frac{d}{ds}\mathscr{L}(\gamma)(s)| = 1$ a.e. in $[0, L(\gamma)]$ (so it takes a constant value on $[L(\gamma), +\infty[$).*

Notice that for every $\gamma_1, \gamma_2 \in [\gamma]_s$ we have $\mathscr{L}(\gamma_1) = \mathscr{L}(\gamma_2)$, so $\mathscr{L}$ can be considered as defined on the oriented trajectories. The main result of this section is the following theorem whose proof will follow as an immediate consequence of the subsequent lemmas.

**Theorem 3.76.** *The mapping $\mathscr{L} : \Gamma_I \to \Gamma_{\mathbb{R}_+}$ is Borel measurable.*

**Lemma 3.77.** *Let $I, J \subset \mathbb{R}$ be two given intervals and let $(\varphi_n)_{n\in\mathbb{N}}$ be any given sequence of continuous (increasing) functions from $J$ to $I$ and let $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma_I$ be a given sequence of orbits. If $\varphi_n \to \varphi$ and $\gamma_n \to \gamma$ locally uniformly, then $\varphi_n^*(\gamma_n) \to \varphi^*(\gamma)$.*

*Proof.* Let $(t_n)_{n\in\mathbb{N}} \subset J$ be such that $t_n \to t \in J$. Then $\varphi_n(t_n) \to \varphi(t)$ and so $\gamma_n \circ \varphi_n(t_n) \to \gamma \circ \varphi(t)$, i.e. $\varphi_n^*(\gamma_n) \to \varphi^*(\gamma)$. □

Let $J, I \subset \mathbb{R}$ be two given intervals. We consider a finite partition of $J$ in subintervals $J_i$, namely $J = \bigcup_i J_i$ and we set $l_i = |J_i|$ for every $i$. Moreover, for every $i$ let $c_i \in \mathbb{R}$ be given so that $|I| = \sum_i c_i l_i$. We consider the piecewise linear function $\varphi$ from $J$ to $I$ such that for every $i$ the slope is equal to $c_i$ on the interval $J_i$.

**Remark 3.78.** We assume without any restriction (see Remark 3.31) that $I$ is always bounded, while, to the aim of taking $J = \mathbb{R}_+$, we let $J$ be unbounded on one side and, if $J$ is not bounded below, then in the partition we must take the first length $l_1 = +\infty$ and the corresponding coefficient $c_1 = 0$. Analogously we must proceed with the last index when $J$ is not bounded above.

For every $k \in \mathbb{N}$ we take a partition of $J$ in disjoint intervals having length $l_i^k$, so $|J| = \sum_i l_i^k$, then we take $c_i^k$ so that $\sum_i c_i^k l_i^k = |I|$ and $\varphi_k : J \to I$ as above.

**Lemma 3.79.** *Let $c_i^k \to c_i$ and $l_i^k \to l_i$ as $k \to \infty$. Then $c_i$ and $l_i$ determine $\varphi : J \to I$ such that $\varphi_k^*(\gamma) \to \varphi^*(\gamma)$ for every $\gamma \in \Gamma_I$.*

*Proof.* Since $c_i^k \to c_i$ and $l_i^k \to l_i$ for every $i$, then $\varphi_k \to \varphi$ locally uniformly and so, by Lemma 3.77 $\varphi_k^*(\gamma) \to \varphi^*(\gamma)$, for every $\gamma \in \Gamma_I$. □

Let $I \subset \mathbb{R}$ and $\gamma \in \Gamma_I$ be given. For any fixed $n \in \mathbb{N}$, let us consider a partition of $I$ into $n$ subintervals $I_i$ with the same measure. Let $J = \mathbb{R}_+$, $l_i = L(\gamma_{|I_i})$ for $i = 1, \dots, n$ and $l_{n+1} = +\infty$, as in Remark 3.78. Let us denote by $\bar{\varphi}_n$ the piecewise linear function from $J$ to $I$ under the choice $c_i = (nl_i)^{-1}|I|$, for $i = 1, \dots, n$ and $c_{n+1} = 0$. For every $\gamma \in \Gamma_I$ we set $\mathscr{L}_n(\gamma) = \bar{\varphi}_n^*(\gamma)$. Let us prove the following result.

**Lemma 3.80.** *For every $\gamma \in \Gamma_I$: $\mathscr{L}_n(\gamma) \to \mathscr{L}(\gamma)$ as $n \to +\infty$ .*

*Proof.* Let $\gamma \in \Gamma_I$ be fixed. Let $\varepsilon > 0$, by the absolute continuity of $\gamma$ there exists $\delta > 0$ such that the restriction $\gamma_{|I'}$ of $\gamma$ to the intervals $I'$ such that $|I'| < \delta$ satisfies $L(\gamma_{|I'}) < \varepsilon$. Let $n > \delta^{-1}|I|$, then for every $i$ $l_i \leq \varepsilon$ and so for every $s, t \in I$ we have

$$|\bar{\varphi}_n^*(\gamma)(s) - \bar{\varphi}_n^*(\gamma)(t)| \leq |s - t| + 2\varepsilon. \tag{3.22}$$

By the last estimate we deduce that the sequence $(\bar{\varphi}_n^*(\gamma))_{n\in\mathbb{N}}$ is equicontinuous. By applying Lemma 3.41 we get, by passing to a subsequence, that $\bar{\varphi}_n$ pointwise converges to some increasing function $\varphi$. So $\varphi_n^*(\gamma)$ pointwise converges to $\varphi^*(\gamma)$ and by the equicontinuity the convergence is locally uniform. By (3.22) $\varphi^*(\gamma)$ is a 1-Lipschitz function. By taking the converging subsequence from dyadic partitions of the interval $I$, by the definition of length functional we get $L(\varphi^*(\gamma)) \geq L(\gamma)$. So, since $\varphi^*(\gamma)$ is constant for $t \geq L(\gamma)$, $|(\varphi^*(\gamma))'| = 1$ on $[0, L(\gamma)]$ and therefore $\varphi^*(\gamma) = \mathscr{L}(\gamma)$. □

Let $n \in \mathbb{N}$ be given as before, for any fixed $k \in \mathbb{N}$ let us partition $\mathbb{R}_+$ in subintervals $A_j$ with $|A_j| \leq \frac{1}{k}$ $\forall j$ and let $l_i^k$ be the middle point of the interval $A_j$ which contains $l_i$. Then we take $c_i^k = (nl_i^k)^{-1}|I|$. Let us define $\varphi_n^k$ analogously to $\varphi_n$ with the coefficients $c_i$ replaced by $c_i^k$, moreover let us define $\mathscr{L}_n^k(\gamma) = (\varphi_n^k)^*(\gamma)$.

**Lemma 3.81.** *For every $k, n \in \mathbb{N}$, $\mathscr{L}_n^k$ is a measurable map.*

*Proof.* Since the length functional is l.s.c., for any given $i \leq n$ and for any $j$, the set of the curves $\gamma$ such that $L(\gamma_{|I_i}) \in A_j$ is Borel measurable. So $\Gamma_I$ can be partitioned as the union of a sequence of disjoint Borel measurable sets on which the coefficients $c_i^k$ are fixed. Then the thesis follows from the continuity of the maps $\varphi^*$, a particular case of Lemma 3.77. □

**Lemma 3.82.** *$\mathscr{L}_n^k \to \mathscr{L}_n$ pointwise as $k \to +\infty$.*

*Proof.* Since $l_i^k \to l_i$ and $c_i^k \to c_i$ as $k \to \infty$, by applying Lemma 3.79 we have $\varphi_k^*(\gamma) \to \varphi^*(\gamma)$, namely $\mathscr{L}_n^k \to \mathscr{L}_n$. □

*Proof of Theorem 3.76.* The thesis follows by subsequently applying Lemma 3.81, Lemma 3.82 and Lemma 3.80. □

For every $\sigma \in \Sigma$, since $\mathscr{L} : \Gamma_I \to \Gamma_{\mathbb{R}_+}$ is measurable, we can define

$$\sigma_{\mathscr{L}} := \mathscr{L}_{\#}\sigma \in \mathcal{P}(\Gamma_{\mathbb{R}_+}) \tag{3.23}$$

and we shall call $\sigma_{\mathscr{L}}$ the *length parametrized* particle motion corresponding to $\sigma$.

**Remark 3.83.** Let us note that, given any $\sigma \in \Sigma$, if $L(\gamma) < +\infty$ for $\sigma-$a.e. $\gamma$ then the length parametrized $\sigma_{\mathscr{L}}$ can be assumed by taking $\mathscr{L} : \Gamma_{\overline{\mathbb{R}}_+} \to \Gamma_{\mathbb{R}_+}$.

**Theorem 3.84.** *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a given sequence of length parametrized particle motions. If $(\sigma_n(0))_{n\in\mathbb{N}}$ is tight then $(\sigma_n)_{n\in\mathbb{N}}$ has a narrowly converging subsequence. Moreover, if $L(\gamma) < +\infty$ for $\sigma_n$-a.e. $\gamma$ then tightness holds true with respect to the (locally) uniform convergence on $\overline{\mathbb{R}}_+$.*

*Proof.* Since $(\sigma_n(0))_{n\in\mathbb{N}}$ is tight, for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathbb{R}^N$ such that, for every $n$, $[\sigma_n(0)](\mathbb{R}^N \setminus K_\varepsilon) < \varepsilon$. The set

$$H_\varepsilon = \{\gamma \in \Gamma_{\mathbb{R}_+} \mid \gamma(0) \in K_\varepsilon,\ \gamma \text{ is Lipschitz cont. with const} \le 1\}$$

is a compact subset of $\Gamma_{\mathbb{R}_+}$, as it is readily seen by locally applying Ascoli-Arzelá Theorem. By hypotheses $\forall n$: $\sigma_n(\Gamma_{\mathbb{R}_+} \setminus H_\varepsilon) = \sigma_n(\{\gamma \in \Gamma_{\mathbb{R}_+} \mid \gamma(0) \notin K_\varepsilon\}) < \varepsilon$. By the arbitrariness of $\varepsilon$, $(\sigma_n)_{n\in\mathbb{N}}$ turns out to be a tight sequence, hence by Prokhorov Theorem it has a narrowly converging subsequence. □

3.6. **No-cycle properties.** In the following we propose a notion of *no-cycle property*, just exploited by other authors (see [5], [6], [37]) in the context of curves or graphs and recognized as a key property of branched minimizing structures, which works at the general level of ordered sets. Indeed, in the present approach the no-cycle property is linked to the flow ordering introduce below in Section 4.

Given any $n \in \mathbb{N}$, we use the notation $I_n = \{1, 2, \dots, n\}$. We shall give weak and strong notions of the no-cycle property.

**Definition 3.85.** *(Weak no-cycle property) We shall say that an ordered set $(S, \le)$ satisfies the weak no-cycle property if for every $x, y \in S$ the order interval $[x, y]$ is totally ordered by $\le$.*

The main tool in the following construction is represented by a finite sequence of points enjoying a suitable property and called oscillating chain.

**Definition 3.86.** *(Oscillating chain). We shall say that a finite sequence $C = (x_i)_{i\in I_n}$ of terms of $(S, \le)$ is an oscillating chain if any two consecutive terms are comparable.*

**Definition 3.87.** *Let $C = (x_i)_{i\in I_n}$ be a given oscillating chain and let $I \subset I_n$ be a subset made of $k$ consecutive indexes. Then the restriction of $C$ to $I$ (translated to $I_k$) is called a sub-chain of $C$.*

**Definition 3.88.** *Let $C = (x_1, x_2, \dots, x_n)$ be an oscillating chain, the $n$ terms $x_i$ will be called vertices. The terms $x_1$ and $x_n$ will be called extreme points of the chain. For any $i \le n-1$ let $S_i = [x_i, x_{i+1}]$ if $x_i \le x_{i+1}$, or $S_i = [x_{i+1}, x_i]$ if $x_{i+1} \le x_i$. We shall call support of $C$ the set:* $\mathrm{spt} C = \bigcup_{i=1}^{n-1} S_i$.

The closeness of two indexes in $I_n$ and, consequently, of two terms of the chain requires a suitable definition.

**Definition 3.89.** *The cyclic distance in $I_n$ is defined by $d_n(i,j) = (i-j) \bmod n$. We shall say that two indexes $i, j \in I_n$ are close if $d_n(i,j) \le 1$.*

The optimality of oscillating chains can be formulated as follows.

**Definition 3.90.** *Let $C$, $D$ be two oscillating chains with the same extreme points. We say that $C$ is more convenient than $D$ if the number of vertices of $C$ is strictly smaller than the number of vertices of $D$ and $\mathrm{spt}\, C \subset \mathrm{spt}\, D$.*

**Definition 3.91.** *We say that an oscillating chain $C$ is optimal if there are no more convenient chains than $C$.*

**Definition 3.92.** *We shall say that $C = (x_i)_{i \in I_n}$ is a simple oscillating chain if it is an optimal oscillating chain with even number of terms, moreover, for every $i$, $x_i \le x_{i+1}$ if $i$ is odd and $x_i \ge x_{i+1}$ if $i$ is even and any two terms are comparable if and only if they are close.*

**Remark 3.93.** Note that in particular, if $(x_i)_{i \in I_n}$ is a simple chain $x_1$ and $x_n$ must be comparable. Moreover, if $x_n < x_1$ then $x_n < x_2$ and this is a contradiction. So $x_1 \le x_n$ must hold true.

**Remark 3.94.** Note that every sub-chain of an oscillating chain is an oscillating chain and every sub-chain of an optimal chain is an optimal chain.

**Remark 3.95.** Note that the notions of oscillating chain and optimal oscillating chain are invariant by reversing the relation $\le$ in $\ge$. On the contrary, the notion of simple chain is not invariant since we have assumed the convention $x_i \le x_{i+1}$ if $i$ is odd. Nevertheless, if the chain is not simple because such a convention does not hold, then it can be made simple by reversing either the order $\le$ or the sequential order of the terms.

**Lemma 3.96.** *If an ordering enjoys the weak no-cycle property, then every optimal oscillating chain satisfies the following property:*

$$\forall i: \quad i \le n-2 \Longrightarrow x_i \quad \textit{and } x_{i+2} \textit{ are not comparable.} \tag{3.24}$$

*Proof.* Assume by contradiction that $x_i$ and $x_{i+2}$ are comparable. We can always assume (by, eventually changing $\le$ with $\ge$ and reversing the order of the chain) that $x_i \le x_{i+1}$ and that $x_i \le x_{i+2}$. Then $[x_i, x_{i+2}] \subset [x_i, x_{i+1}] \cup [x_{i+1}, x_{i+2}]$, since $x_{i+1}$ and

$x_{i+2}$ are comparable and $[x_i, x_{i+2}]$ is totally ordered by virtue of the weak no-cycle property. So, by eliminating $x_{i+1}$ we get a more convenient chain, in contradiction to the optimality assumption. □

**Remark 3.97.** The term *oscillating chain* is fully meaningful when we deal with optimal chains with respect to an ordering satisfying the weak no-cycle property. Indeed, (3.24) shows in particular that the ordering between $x_i$ and $x_{i+1}$ is opposite to the one existing between $x_{i+1}$ and $x_{i+2}$. In particular, when the weak no-cycle property holds true, no optimal chain with more than two elements can be monotone.

**Remark 3.98.** On the contrary, if an ordering does not satisfy the weak no-cycle property, then, given an order interval $[x, y]$ which is not totally ordered and $z \in [x, y]$ which is not comparable with any other element of the interval, the chain $(x, z, y)$ is an optimal monotone chain. Indeed the only possible competitor would be $(x, y)$ but this chain is not more convenient since $\mathrm{spt}(x, y) = [x, y] \not\subset [x, z] \cup [z, y] = \mathrm{spt}(x, z, y)$. So (3.24) does not hold.

**Remark 3.99.** Combining the two above remarks we can conclude that the weak no-cycle property can equivalently be formulated by asking that every optimal chain satisfies (3.24), that every optimal chain with three elements has not comparable extreme points, or that there exist no totally ordered optimal chains with more than two elements.

The previous remark suggests the following definition as a stronger case of Definition 3.85.

**Definition 3.100.** *(Strong no-cycle property) We shall say that an ordered set $(S, \leq)$ satisfies the strong no-cycle property if any optimal chain having more than two terms has not comparable extreme points .*

**Remark 3.101.** We see from Remark 3.99 that if an ordered set $(S, \leq)$ enjoys the strong no-cycle property, then it also satisfies the weak no-cycle property but, in general, the reverse implication does not hold as shown in the the following example.

**Example 3.102.** Let $S = \{x_1, x_2, x_3, x_4\}$, with $x_1 = \{0\}$, $x_2 = \{0, 1, 2\}$, $x_3 = \{1\}$, $x_4 = \{0, 1, 3\}$ and $\leq$ given by the set inclusion $\subset$. Then $S$ satisfies the weak no-cycle property but it does not satisfy the strong no-cycle property since $x_1 \leq x_4$.

**Definition 3.103.** *Let $C = (x_i)_{i \in I_n}$ be a simple oscillating chain in $(S, \leq)$. Extending the notation introduced in Definition 3.88, we set $S_n = [x_1, x_n]$.*

**Proposition 3.104.** *If $(S, \leq)$ satisfies the weak no-cycle property but not the strong no-cycle property, then it contains a simple oscillating chain having at least four elements.*

*Proof.* Assume $\leq$ enjoys the weak no-cycle property and let $C$ be an optimal chain with more than two elements and comparable extrema. If $n$ is the number of vertices of $C$, then by Remark 3.99 it follows that $n \geq 4$. By restricting the chain to a minimal sub-chain among the ones having more than two elements and comparable extreme points, we can obtain that any two elements which are not close are not comparable. We observe that by Remark 3.97 for two consecutive indexes either the even index is $\leq$ than the odd one or vice versa. Without any restriction (see Remark 3.95), we suppose that every even term is $\geq$ than the previous odd one. Finally, we have $x_n \geq x_1$, otherwise we would have $x_n \leq x_1 \leq x_2$, in contradiction to the minimality of $C$. For the same reason we have that $n$ is even, otherwise $x_1 \leq x_n \leq x_{n-1}$. □

Note that the next statement is empty when $n < 4$, so we shall assume $n \geq 4$ in the proof.

**Proposition 3.105.** *Let $C = (x_i)_{i \in I_n}$ be any simple oscillating chain and let $i, j$ be two non-close indexes. Then $S_i \cap S_j = \emptyset$.*

*Proof.* We begin by assuming $i, j \neq n$. Let $i < j$ and suppose by contradiction that $z \in S_i \cap S_j$, then the sequence $(x_1, x_2, \dots, x_i, z, x_{j+1}, \dots x_n)$ has $n + i - j + 1 < n$ terms and so it is more convenient than $C$, in contradiction with the assumption $C$ simple oscillating chain, thus the claim follows. Now let us suppose $j = n$ (and so $i \neq 1, n, n-1$) and distinguish the two following cases. If $n = 4$ then $i = 2$ necessarily. If $z \in S_2 \cap S_4$ we deduce $x_3 \leq z$ from $z \in S_2$ and $z \leq x_4$ from $z \in S_4$. Therefore we get $x_3 \leq z \leq x_4$ which means $z \in S_3$. By arguing in the same way one can see that $z \in S_1$ and so one recovers the previous case with $i = 1$ and $j = 3$. If $n > 4$ then $i \neq 2$ or $i \neq n-2$ (we can assume $i \neq 2$ eventually changing $\leq$ with $\geq$ and the order of the sequence, see Remark 3.95). Let $k$ be the even number in $\{i, i+1\}$, since $i \geq 3$ we get $k > 2$ and since $i < n-1$ we get $k < n$. So if $z \in S_i \cap S_n$ we have $x_1 \leq z \leq x_k$, in contradiction with the assumption $C$ simple oscillating chain, thus the thesis follows. □

## 4. Irrigation models

4.1. **The irrigation problem.** By *irrigation problem* we mean a special kind of mass transportation problem characterized by a transport cost which depends on a concave function of the transported mass. This leads to variational problems whose minimizers are given by branching curves due to the opposite energetic requirements of keeping the mass together (the concave cost) and spreading the mass as required by the target measure. Starting from the papers [37] and [27], various approaches to the problem have been proposed in the literature (see [5]). In this section we shall show the equivalence of the minimization problems obtained by taking four different functionals related to the irrigation problem.

4.2. **Synchronous and asynchronous functionals.**

**Definition 4.1.** *The functional $\mathcal{F} : \Sigma \to \mathbb{R}$ is said to be speed-invariant if $\mathcal{F}(\sigma) = \mathcal{F}(\nu)$ for every $\sigma, \nu$ such that $\sigma \cong_S \nu$ (uniformly scaling equivalent).*

We shall consider minimization problems involving speed-invariant functionals with prescribed boundary conditions on a compact interval. Of course the speed-invariance makes the minimization problems invariant with respect to the choice of the interval. So we shall speak of min $\mathcal{F}$ on $\Sigma(\mu, \nu)$ and we shall actually work on $\Sigma_I(\mu, \nu)$ for any convenient choice of the compact interval $I$ ($I = [0, 1]$ in most of the cases).

A particular class of speed-invariant functionals is here defined.

**Definition 4.2.** *A (speed invariant) functional $\mathcal{F} : \Sigma \to \mathbb{R}$ is said to by asynchronous if $\mathcal{F}(\sigma) = \mathcal{F}(\nu)$ for every $\sigma, \nu$ such that $\sigma \cong_{\widetilde{S}} \nu$ (fiberwise scaling equivalent).*

Let us observe that an asynchronous functional is fully characterized through its behavior on the set of oriented trajectories, then its analysis can be carried out by considering length parameterizations. Indeed, for every $\sigma \in \Sigma$, we know (see (3.23)) that there exists the length parametrized particle motion $\sigma_{\mathscr{L}}$ fiberwise equivalent to $\sigma$. Therefore an asynchronous functional can be studied, without loss of generality, on the set of length parameterized measures.

**Definition 4.3.** *Any speed-invariant functional $\mathcal{F} : \Sigma \to \mathbb{R}$ which is not asynchronous will be called synchronous.*

**Definition 4.4.** *A functional $\mathcal{F} : \Sigma \to \mathbb{R}$ is said to be scaling monotone if $\mathcal{F}(\sigma) \leq \mathcal{F}(\nu)$ for every $\sigma \leq_{\widetilde{S}} \nu$.*

The next statement is trivially proved.

**Proposition 4.5.** *If the functional $\mathcal{F} : \Sigma \to \mathbb{R}$ is scaling monotone then it is asynchronous.*

4.3. **Irrigation functionals.** Let us consider a microscopic motion $\sigma \in \Sigma$, to the aim of introducing the functional costs modeling the irrigation problem, we consider equivalence classes of fibers as representing the mass carried by the irrigation structure parameterized with the variable $t \in \mathbb{R}_+$. More precisely, the equivalence class of fibers to be considered joint to $\gamma$ at the time $t$ is given by

$$[\gamma]_t^0 = \{\hat{\gamma} \in \Gamma \mid \hat{\gamma}(s) = \gamma(s) \; \forall s \leq t\} ,$$

(as in [27]) or by

$$[\gamma]_t^1 = \{\hat{\gamma} \in \Gamma; \mid \hat{\gamma}(t) = \gamma(t)\} .$$

For $\alpha \in ]0, 1[$ and $k = 0, 1$, we introduce the following cost densities.

$$s_\sigma^k(\gamma, t) = \sigma([\gamma]_t^k))^{\alpha - 1}. \tag{4.25}$$

Moreover, we define

$$s_\sigma^2(x) = (a_\sigma(x))^{\alpha - 1} \tag{4.26}$$

and

$$s^3_\sigma(x) = s^3(x) = \begin{cases} (m_\sigma(x))^{\alpha-1} & \text{if } m_\sigma(x) < +\infty \\ 1 & \text{otherwise.} \end{cases} \tag{4.27}$$

Note that (4.27) implies in any case

$$s^3_\sigma(x) = m^\alpha_\sigma(x). \tag{4.28}$$

Then, for $k = 0, 1, 2, 3$ and $\alpha \in ]0, 1[$ we introduce the functionals $J^k_\alpha$ defined as follows:

$$J^k_\alpha(\sigma) = \int_{(\Gamma,\sigma)\times I} s^k_\sigma(\gamma(t))\, |\gamma'(t)|\ d\gamma\ dt. \tag{4.29}$$

By taking a lagrangian parametrization $\chi : \Omega \times I \to \mathbb{R}^N$ on the reference space $(\Omega, \mu_\Omega)$, we can express the irrigation functionals (4.29) in terms of $\chi$ as

$$J^k_\alpha(\chi) = \int_{\Omega\times I} s^k_\alpha(p,t) \left| \frac{\partial \chi}{\partial t}(p,t) \right|\ dp\ dt, \tag{4.30}$$

where

$$s^k_\alpha(p,t) = |\mu_\Omega([p]^k_t)|^{\alpha-1} \quad k = 0, 1, \tag{4.31}$$

$$[p]^0_t = \{q \in \Omega \mid \chi(q,s) = \chi(p,s)\ \forall s \le t\}\,, \tag{4.32}$$

$$[p]^1_t = \{q \in \Omega \mid \chi(q,t) = \chi(p,t)\}\,, \tag{4.33}$$

$$s^2_\alpha(p,t) = s^2_\sigma(\chi(p,t)) = a_\sigma(\chi(p,t))^{\alpha-1}, \tag{4.34}$$

$$s^3_\alpha(p,t)) = s^3_\alpha(\chi(p,t)). \tag{4.35}$$

Let us observe that $J^0_\alpha$ is the cost functional of [27], $J^1_\alpha$ concerns the approach based on transport distances of [29] and $J^2_\alpha$ is the cost functional of [3].

The functional $J^3_\alpha$ is introduced here and it furnishes another description of the irrigation problem. Let us note that all the above functionals do not depend on the choice of the lagrangian parametrization $\chi$ and are speed-invariant but only $J^0_\alpha$ and $J^1_\alpha$ are synchronous functionals

Let us remark that, as one easily sees from the definition, for every $\alpha \in ]0, 1[$, the following inequalities hold true

$$J^0_\alpha \ge J^1_\alpha \ge J^2_\alpha \ge J^3_\alpha. \tag{4.36}$$

Now we are going to study some properties of the the above functionals.

**Lemma 4.6.** *If $\sigma \in \Sigma$ is a non-spread particle motion then the following representation formula holds true*

$$J^3_\alpha(\sigma) = \int_{\mathbb{R}^N} m_\sigma(x)^\alpha\ d\mathcal{H}^1. \tag{4.37}$$

*Proof.* By virtue of Area Formula (see [21, Theorem 3.2.6]), we have

$$\int_I s^3(\gamma(t))|\gamma'(t)|\,dt = \int_{\mathbb{R}^N} s^3(x)m_\gamma(x)d\mathcal{H}^1$$

and so we get

$$J^3_\alpha(\sigma) = \int_{(\Gamma,\sigma)} \left( \int_{\mathbb{R}^N} s^3(x)m_\gamma(x)d\mathcal{H}^1 \right) d\gamma. \tag{4.38}$$

By Definition 3.24, $\sigma$ admits a $\mathcal{H}^1$ $\sigma$-finite track $T$, therefore we can apply Fubini Theorem and then, using (4.28),

$$J^3_\alpha(\sigma) = \int_T \left( s^3(x) \int_{(\Gamma,\sigma)} m_\gamma(x)d\gamma \right) d\mathcal{H}^1 = \int_{\mathbb{R}^N} s^3(x)m_\sigma(x)d\mathcal{H}^1 = \int_{\mathbb{R}^N} m_\sigma(x)^\alpha d\mathcal{H}^1.$$

□

Let us notice that if $\sigma \in \Sigma$ is spread, using Theorem B.5 instead of Fubini Theorem in the above argument, we get

$$\int_{\mathbb{R}^N} m_\sigma(x)^\alpha d\mathcal{H}^1 \leq J^3_\alpha(\sigma). \tag{4.39}$$

Through the same argument used in the proof of the previous lemma, one can prove the following representation formula.

**Lemma 4.7.** *If $\sigma \in \Sigma$ is a non-spread particle motion then the following representation formula holds true*

$$J^2_\alpha(\sigma) = \int_{\mathbb{R}^N} a_\sigma(x)^{\alpha-1} m_\sigma(x)\,d\mathcal{H}^1. \tag{4.40}$$

We are going to see in the sequel that the right-hand side of (4.39) is actually equal to $+\infty$ when $\sigma$ is spread.

**Lemma 4.8.** *Every $\sigma \in \Sigma$ such that $J^3_\alpha(\sigma) < +\infty$ is a non-spread particle motion.*

*Proof.* For every Borel set $T \subset \mathbb{R}^N$ let us define

$$\mathcal{R}(T) = \int_T m^\alpha_\sigma d\mathcal{H}^1.$$

Notice that, by (4.39)

$$\mathcal{R}(T) \leq J^3_\alpha(\sigma) < +\infty.$$

We claim that $\mathcal{R}$ admits a maximizer $\hat{T}$ among the Borel sets having $\mathcal{H}^1$ $\sigma$-finite measure, indeed if $(T_n)_{n\in\mathbb{N}}$ is any maximizing sequence in that class, we take $\hat{T} = \bigcup_n T_n$ and this set maximizes the functional. We are going to prove that $\hat{T}$ is a track. Indeed, if this were not true, using (4.38) we could find $\hat{\gamma} \in \Gamma_I$ such that

$$\int_{\hat{\gamma}(I)} s^3_\alpha(x)d\mathcal{H}^1 < +\infty, \quad \mathcal{H}^1(\hat{\gamma}(I) \setminus \hat{T}) > 0.$$

In such a case there exists a set of positive measure $A \subset I$ such that, for every $t \in A$, $\hat{\gamma}(t) \notin \hat{T}$, $\hat{\gamma}'(t) \neq 0$ and $s^3_\alpha(\hat{\gamma}(t)) < +\infty$. Let $C = \{(\gamma, t) \in \Gamma \times A \mid \gamma(t) = \hat{\gamma}(t)\}$, $C$ is Borel measurable. Moreover, since for $t \in A$ the $t$-section of $C$ has a positive measure because $s^3_\alpha(\hat{\gamma}(t)) < +\infty$, then $C$ has a measure different from zero. By slicing $C$ with $\gamma$-sections we can state that there exists $\hat{\Gamma} \subset \Gamma$ with $\sigma(\hat{\Gamma}) > 0$ and such that, for every $\gamma \in \hat{\Gamma}$, the set $A_\gamma = \{t \in A \mid \gamma(t) = \hat{\gamma}(t)\}$ has a positive measure. For $\gamma \in \hat{\Gamma}$ we have

$$\mathcal{H}^1(\gamma(I) \cap (\hat{\gamma}(I) \setminus \hat{T})) \geq \mathcal{H}^1(\hat{\gamma}(A_\gamma)) > 0.$$

Since, being $s^3_\alpha(\hat{\gamma}(t)) > 0$, by (4.27) $m^\alpha_\sigma(\hat{\gamma}(t)) > 0$, for every $\gamma \in \hat{\Gamma}$, $t \in A_\gamma$, we deduce

$$\int_{\gamma(I) \cap (\hat{\gamma}(I) \setminus \hat{T})} m_\sigma(x) d\mathcal{H}^1 > 0$$

and so by Fubini Theorem

$$\mathcal{R}(\hat{\gamma}(I) \setminus \hat{T}) = \int_{(\hat{\Gamma}, \sigma)} \left( \int_{\gamma(I) \cap (\hat{\gamma}(I) \setminus \hat{T})} m^\alpha_\sigma(x) d\mathcal{H}^1 \right) d\gamma > 0.$$

Then

$$\mathcal{R}(\hat{\gamma}(I) \cup \hat{T}) > \mathcal{R}(\hat{T}),$$

in contradiction to the maximality of $\hat{T}$. □

For $i = 0, 1, 2, 3$ we introduce the set

$$\Sigma^i_0(\mu, \nu) = \{\sigma \in \Sigma(\mu, \nu) \mid J^i_\alpha(\sigma) < +\infty)\}. \tag{4.41}$$

We have by Lemma 4.8 and by (4.39)

$$\Sigma^0_0 \subset \Sigma^1_0 \subset \Sigma^2_0 \subset \Sigma^3_0 \subset \Sigma_0.$$

### 4.4. **Loop free particle motions.**

**Definition 4.9.** *Let $\gamma \in \Gamma$ be given. We shall say that $x \in \mathbb{R}^N$ is a loop-point for $\gamma$ if there exist $t_1, t_2 \in I$ such that $\gamma(t_1) = \gamma(t_2) = x$ and there exists $t \in [t_1, t_2]$ such that $\gamma(t) \neq x$.*

**Proposition 4.10.** *Let $\gamma \in \Gamma$ and $x \in \mathbb{R}^N$ be given. If $a_\gamma(x) < m_\gamma(x) < +\infty$, then $x$ is a loop-point for $\gamma$.*

*Proof.* Since $m_\gamma(x) > a_\gamma(x)$ there exist $t_1 < t_2$ so that $\gamma(t_1) = \gamma(t_2) = x$ and since $m_\gamma(x) < +\infty$ we have $\gamma(t) \neq x$ for some $t \in [t_1, t_2]$. □

**Definition 4.11.** *Let $\sigma \in \Sigma$ be given. We shall say that $x \in \mathbb{R}^N$ is a loop-point for $\sigma$ if there exists $\Gamma' \subset \Gamma$ with $\sigma(\Gamma') > 0$ and such that for every $\gamma \in \Gamma'$ $x$ is a loop-point for $\gamma$. The set of the loop-points of $\sigma$ will be denoted by $\mathcal{L}_\sigma$.*

The analogous statement to Proposition 4.10 also holds for particle motions.

**Proposition 4.12.** *Let $\sigma \in \Sigma$ and $x \in \mathbb{R}^N$ be given. If $a_\sigma(x) < m_\sigma(x) < +\infty$, then $x$ is a loop-point for $\sigma$.*

*Proof.* Let $\Gamma' = \{\gamma \in \Gamma \mid a_\gamma(x) < m_\gamma(x) < +\infty\}$. By Proposition 4.10 $x$ is a loop point of every $\gamma \in \Gamma'$. □

**Definition 4.13.** *We shall say that $\sigma \in \Sigma$ is loop-free if $\mathcal{L}_\sigma = \emptyset$.*

**Corollary 4.14.** *Let $\sigma \in \Sigma_0^3$ be a loop-free particle motion. Then $m_\sigma = a_\sigma$ $\mathcal{H}^1$-a.e..*

*Proof.* By (4.39) we have $m_\sigma < +\infty$ $\mathcal{H}^1$ a.e., then the thesis follows from Proposition 4.12. □

**Theorem 4.15.** *If $\sigma \in \Sigma_0$ is minimal for $\leq_{\widetilde{S}}$, then it is loop-free.*

*Proof.* We argue by contradiction, hence assume $\overline{x}$ is a loop point for $\sigma$, so by Definition 4.11 there exists $\Gamma' \subset \Gamma$ with $\sigma(\Gamma') > 0$ and such that for every $\gamma \in \Gamma'$ $\overline{x}$ is a loop-point for $\gamma$. Let us decompose $\sigma$ as $\sigma = \sigma_1 + \sigma_2$, where $\sigma_1 = \sigma \llcorner \Gamma'$ and $\sigma_2 = \sigma - \sigma_1$. By using the construction in Remark 3.73 we have $t_1, t_2$ so that $\gamma(t_1) = \gamma(t_2) = \bar{x}$ for $\sigma_1'$-a.e. $\gamma$, where $\sigma_1' \cong_{\widetilde{S}} \sigma_1$.
Assume $\sigma \in \Sigma_I$ with $I = [0,1]$ and define $\varphi : I \to I$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{1-(t_2-t_1)} & \text{if } t \leq t_1 \\ \\ \frac{t+t_2-t_1}{1-(t_2-t_1)} & \text{if } t > t_1. \end{cases} \tag{4.42}$$

Let $\sigma_1'' = \varphi^*_\# \sigma_1'$, so $\sigma_1'' \leq_S \sigma_1' \cong_{\widetilde{S}} \sigma_1$. Since $\sigma_2 \leq_S \sigma_2$, by taking $\sigma' = \sigma_1'' + \sigma_2$ and applying (3.9) we get $\sigma' \leq_{\widetilde{S}} \sigma$.
The inverse relation between $\sigma$ and $\sigma'$ does not hold true because one can easily see that $m_{\sigma'} < m_\sigma$ on a set of positive measure and so by Theorem 3.62 $\sigma \not\leq_{\widetilde{S}} \sigma'$. Indeed, by the additivity of the density, stated in Lemma 3.67, and its finiteness due to the condition $\sigma \in \Sigma_0$, we just have to show that $m_{\sigma_1''} < m_{\sigma_1}$. Let $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ be the complementary restrictions of $\sigma_1$ on $[\min I, t_1]$, $[t_1, t_2]$ and $[t_2, \max I]$, respectively. By Lemma 3.67 we get $m_{\sigma_1'} = m_{\sigma_1} - m_{\tilde{\sigma}_2}$ and $m_{\sigma_1'}$ is not zero a.e. because a.e. $\gamma \in \Gamma'$ has a positive length between $\tau_1(\gamma)$ and $\tau_2(\gamma)$ by definition of loop-point, so $J^3_\alpha(\tilde{\sigma}_2) > 0$. Moreover $\tilde{\sigma}_2$ is non-spread because it is a restriction of $\sigma$ which is known to enjoy such a property, so $\int_{\mathbb{R}^N} m_{\tilde{\sigma}_2}(x)\, d\mathcal{H}^1 > 0$ follows from (4.37). □

**Theorem 4.16.** *Let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$ be a decreasing sequence with respect to the relation $\leq_D$ and let $\sigma_n \rightharpoonup \sigma$ non-spread. Then $m_\sigma(x) \leq m_{\sigma_1}(x)$ for $\mathcal{H}^1$-a.e. $x$.*

*Proof.* Firstly we observe that if $(\sigma_n)_{n\in\mathbb{N}}$ is decreasing with respect to $\leq_D$, then the sequence of weighted local set-lengths $(l_{\sigma_n})_{n\in\mathbb{N}}$, introduced in Definition D.9, is decreasing in the ordered space of measures. Fix any open subset $B \subset \mathbb{R}^N$, by Definition D.1

and Lemma D.4, since the map $\gamma \mapsto l_\gamma$ is l.s.c., we can apply Lemma C.3 thus obtaining

$$l_\sigma(B) = \int_{(\Gamma,\sigma)} l_\gamma(B)d\gamma \le \liminf_n \int_{(\Gamma,\sigma_n)} l_\gamma(B)d\gamma \le \liminf_n l_{\sigma_n}(B).$$

Since $(\sigma_n)_{n\in\mathbb{N}}$ is decreasing we have $l_\sigma \le l_{\sigma_n}$ for every $n \in \mathbb{N}$, then, by Proposition D.10 we get $m_\sigma \le m_{\sigma_n}$ for every $n$. □

**Lemma 4.17.** *$(\Sigma_0(\mu,\nu), \ge_D)$ is a countably inductive set (see Definition A.1).*

*Proof.* Let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma_0(\mu,\nu)$ be a decreasing sequence with respect to the relation $\le_D$, which is not restrictive to assume to be length parametrized on $\overline{\mathbb{R}}_+$ (see (3.23)). By Theorem 3.84 there exists a narrowly converging subsequence to a limit $\sigma$ and by Theorem 3.28 we know that $\sigma$ is a non-spread particle motion. Finally, by Theorem 4.16 we get $\sigma \le_D \sigma_n$ for every $n \in \mathbb{N}$. □

**Theorem 4.18.** *For every $\sigma \in \Sigma_0(\mu,\nu)$ there exists a loop-free $\bar\sigma \in \Sigma_0$ so that $\bar\sigma \le_D \sigma$.*

*Proof.* Let $\sigma \in \Sigma_0(\mu,\nu)$ be given. By the previous lemma we know that $(\Sigma_0(\mu,\nu), \ge_D)$ is a countably inductive so, by Theorem A.2 considering that $J^3_\alpha : \Sigma_0(\mu,\nu) \to \mathbb{R}$ is an increasing function, we have that there exists $\bar\sigma \in \Sigma_0(\mu,\nu)$, minimal for $\le_D$ so that $\bar\sigma \le_D \sigma$. By Theorem 4.15 and Corollary 3.66 we know that $\bar\sigma$ is loop-free. □

4.5. **Flow ordering.** Let $\mathcal{M}$ be the set of the multiplicity functions induced by the minimizers of $J^2_\alpha$ (or $J^3_\alpha$, by Theorem 5.1 below) in $\Sigma(\mu,\nu)$. For any fixed $a \in \mathcal{M}$, let $M_a$ be the set of the non-spread particle motions $\sigma$ such that $m_\sigma(x) = a(x)$ for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^N$. Let (coherently with Proposition 3.27)

$$T_0 = \{x \in \mathbb{R}^N \mid a(x) > 0\}. \tag{4.43}$$

By Proposition 3.25 or by Lemma 4.7 $T_0$ has $\sigma$-finite $\mathcal{H}^1$ measure and by Proposition 3.27, for every $\sigma \in M_a$, $T_0$ is the minimal track of $\sigma$ up to a $\mathcal{H}^1$ negligible set. By Theorem 4.15, every particle motion in $M_a$ is loop-free. We also notice that $J^3_\alpha$ is constant on $M_a$ and $M_a$ is a closed convex set.

**Definition 4.19.** *Let $a \in \mathcal{M}$ and $x, y \in \mathbb{R}^N$ be given. We say that $x \le_a y$ if there exists $\sigma \in M_a$ and there exists $A \subset \Gamma$ with $\sigma(A) > 0$ such that for every $\gamma \in A$ there exist $t_1 \le t_2$ such that $\gamma(t_1) = x$ and $\gamma(t_2) = y$.*

**Remark 4.20.** By Proposition 3.71, Remark 3.72 and Theorem 3.62, $t_1$ and $t_2$ can be taken the same for all the orbits $\gamma \in A$. Let us observe that two points $x$ and $y$ are in the relation $\le_a$ (in unspecified order) if and only if there exists $\sigma \in M_a$ and $\Gamma' \subset \Gamma$ with $\sigma(\Gamma') > 0$ such that $x, y \in \gamma(I)$, $\forall \gamma \in \Gamma'$.

**Lemma 4.21.** *Let $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ be two sequences such that, for every $k \in \mathbb{N}$, $x_k \le_a y_k$. Then there exists $\sigma \in \mathcal{M}$ such that, for every $k$, there exists $A_k \subset \Gamma$ with $\sigma(A_k) > 0$ and such that for every $\gamma \in A_k$ there exist $t_1 < t_2$ such that $\gamma(t_1) = x_k$ and $\gamma(t_2) = y_k$.*

*Proof.* By definition for every $k$ there exists $\sigma_k$ as in Definition 4.19. Then the thesis follows by taking the measure $\sigma = \sum_{k=0}^{\infty} \frac{1}{2^k}\sigma_k$ and observing that, by convexity, $\sigma \in M_a$. □

**Proposition 4.22.** *The relation $\leq_a$ is an ordering on $T_0$ given by* (4.43)*.*

*Proof.* To prove the transitivity property, let us observe that if $x \leq_a y$ and $y \leq_a z$ then by using Lemma 4.21 we know that there exists $\sigma \in \Sigma$ with a track $T_\sigma$ such that, setting

$$\Gamma' = \{\gamma \in \Gamma \mid \exists\, t_1 < t_2 : \gamma(t_1) = x, \gamma(t_2) = y\},$$

$$\Gamma'' = \{\gamma \in \Gamma \mid \exists\, t_1 < t_2 : \gamma(t_1) = y, \gamma(t_2) = z\},$$

we have $\sigma(\Gamma') > 0$, $\sigma(\Gamma'') > 0$. Then by Lemma 3.74 there exists $\bar\sigma \in M_a$ and $\Gamma''' \subset \Gamma$, with $\bar\sigma(\Gamma''') > 0$, such that every curve in $\Gamma'''$ connects $x$ to $z$.

To prove that $\leq_a$ is antisymmetric let us observe that if $x \leq_a y$, $y \leq_a x$ and $x \neq y$, then by arguing as above with $z = x$, it results that $x$ is a loop-point (see Definition 4.11), in contradiction to the minimality of $\sigma$ as stated in Theorem 4.15. Finally, the last statement in Remark 4.20 shows that $T_0$ is the domain of the relation $\leq$ which is reflexive on such a set. □

**Definition 4.23.** *(Geometric flow curve). Let $x, y \in \mathbb{R}^N$ such that $x \leq_a y$. We shall call geometric flow curve $G$ with extreme points $x$ to $y$ the order interval $[x, y]$ with respect to $\leq_a$.*

**Definition 4.24.** *(Flow connectedness). Let $a \in \mathcal{M}$ be given. We say that $A \subset T_0$ is flow connected if any two $x, y \in A$ belong to an oscillating chain with respect to $\leq_a$ whose support is contained in $A$.*

**Remark 4.25.** If $A$ is flow connected then every $x, y \in A$ are the end points of an optimal oscillating chain $C$ with respect to $\leq_a$, with a support contained in $A$.

**Theorem 4.26.** *Let $a \in \mathcal{M}$ and let $x, y, z \in \mathbb{R}^N$ such that $x \leq_a y \leq_a z$. Then for every $\sigma \in M_a$ and for $\sigma$-a.e. $\gamma$ such that $x, z \in \gamma(I)$, it results $y \in \gamma(I)$.*

*Proof.* Given $\sigma$ as in the thesis, we claim that we can assume, without any restriction, that the set $\Gamma'''$ of the curves $\gamma$ so that $x, y, z \in \gamma(I)$ satisfies $\sigma(\Gamma''') > 0$. To this aim we can use Lemma 4.21 and Lemma 3.74 to find a $\tilde\sigma \in M_a$ such that $\tilde\sigma(\Gamma''') > 0$, then the claim follows by replacing $\sigma$ with $\frac{1}{2}(\sigma + \tilde\sigma)$. By Remark 3.72, arguing as in Remark 4.20, $t_1$ and $t_2$ can be taken the same for all the orbits in $\Gamma'''$. Let $\sigma_3 = \sigma \llcorner \Gamma'''$, $\sigma_* = \sigma - \sigma_3$ and for every $\bar\gamma \in \Gamma'''$ let $P_{\bar\gamma} : \Gamma''' \to \Gamma'''$ be defined by

$$P_{\bar\gamma}(\gamma) = \begin{cases} \gamma(t) & \text{if } t \leq t_1,\ t \geq t_2 \\ \bar\gamma(t) & \text{if } t_1 \leq t \leq t_2. \end{cases}$$

We set $\sigma_{\bar\gamma} = (P_{\bar\gamma})_{\#}\sigma_3 + \sigma_*$. We claim that $\forall y \in \mathbb{R}^N$

$$\fint_{(\Gamma''',\sigma_3)} m_{\sigma_{\bar\gamma}}(y)d\bar\gamma = m_{\sigma_3}(y).$$

Indeed, since $P_{\bar\gamma}$ does not change the multiplicity $m_\gamma$ out of the interval $[t_1,t_2]$ we shall restrict the orbits to $[t_1,t_2]$ and we shall use the notation $\overline{m}_\gamma$, $\overline{m}_\sigma$ to denote the multiplicity function of a curve $\gamma$ or a particle motion $\sigma$ after a restriction to the interval $[t_1,t_2]$, i.e. $\overline{m}_\gamma := m_{\gamma_{|[t_1,t_2]}}$ By the additivity of the multiplicity function (Lemma 3.67) we deduce

$$\fint_{(\Gamma''',\sigma_3)} m_{\sigma_{\bar\gamma}}(y)d\bar\gamma = m_{\sigma_3}(y).$$

Using the additivity of $m$ again, we have $\forall y \in \mathbb{R}^N$

$$\fint_{(\Gamma''',\sigma_3)} \overline{m}_{\sigma_{\bar\gamma}}(y)d\bar\gamma = |\sigma_3| \fint_{(\Gamma''',\sigma_3)} \overline{m}_{\bar\gamma}(y)d\bar\gamma = \int_{(\Gamma''',\sigma_3)} \overline{m}_{\bar\gamma}(y)d\bar\gamma = \overline{m}_{\sigma_3}(y),$$

proving the claim. By the concavity of $m_\sigma^\alpha$, we get

$$J_\alpha^3(\sigma) = \int_{R^N} m_\sigma^\alpha(x)d\mathcal{H}^1 \geq \fint_{(\Gamma''',\sigma_3)} \left(\int_{R^N} \overline{m}_{\sigma_{\bar\gamma}}^\alpha(y)d\mathcal{H}^1\right) d\bar\gamma = \fint_{(\Gamma''',\sigma_3)} J_\alpha^3(\sigma_{\bar\gamma})d\bar\gamma,$$

where the equality holds if and only if, for $\mathcal{H}^1$-a.e. $x$: $\overline{m}_{\sigma_{\bar\gamma}}(x) = \overline{m}_\sigma(x)$ for $\sigma_3$-a.e. $\bar\gamma$, i.e. $\overline{m}_\gamma(x) = \overline{m}_\sigma(x)$ for $\sigma$-a.e. $\gamma \in \Gamma'''$. Since $\sigma$ is a minimizer for $J_\alpha^3$, we can conclude that this must be the case.

We set $C = \{(\gamma,\xi) \in \Gamma''' \times T_0 \mid \overline{m}_\gamma(\xi) \neq \overline{m}_\sigma(\xi)\}$. Notice that, by Theorem 3.21 $C$ is a measurable set, then we can apply Fubini Theorem and so, by the previous argument we have $(\sigma \otimes \mathcal{H}^1)(C) = 0$. Then, by integrating the $\mathcal{H}^1$ measure of the $\gamma$ sections we get that for $\sigma$-a.e. $\gamma$ $m_\gamma(\xi) = m_\sigma(\xi)$ for $\mathcal{H}^1$-a.e. $\xi$. Now let us suppose that there exists a set $A \subset \Gamma$ with $\sigma(A) > 0$ so that $y \notin \gamma([t_1,t_3])$ for every $\gamma \in A$. Then we can claim that there is $r > 0$ and $B_r(y)$, so that the subset $A_r \subset A$, consisting of the curves $\gamma$ such that $\gamma([t_1,t_3]) \cap B_r(y) = \emptyset$, satisfies $\sigma(A_r) > 0$. Indeed, if for every $n \in \mathbb{N}$ the set $A_{\frac{1}{n}}$ has null measure, then $\sigma(\bigcup_n A_{\frac{1}{n}}) = 0$ and, since $[t_1,t_3]$ is compact, $y \in \gamma([t_1,t_3])$ for almost every $\gamma \in A$. By fixing $r > 0$ with $\sigma(A_r) > 0$ as claimed, we can take $\gamma_1$ such that $x,y,z \in \gamma_1(([t_1,t_3])$ and $\gamma_2$ such that $x,z \in \gamma_2(([t_1,t_3])$, $\gamma_2(([t_1,t_3]) \cap B_r(y) = \emptyset$ and such that $m_{\gamma_1}(\xi) = m_\sigma$, $m_{\gamma_2}(\xi) = m_\sigma(\xi)$ for $\mathcal{H}^1$-a.e. $\xi$ and thus $m_{\gamma_1}(\xi) = m_{\gamma_2}(\xi)$ for $\mathcal{H}^1$-a.e. $\xi$. Then we have that $x,y,z \in \gamma_1(([t_1,t_3])$ and so, if $D = \{\xi \in \gamma(I) \cap B_r(y) \mid m_{\gamma_1} \geq 1\}$, then $\mathcal{H}^1(D) \geq 2r$, but $\gamma_2(([t_1,t_3]) \cap B_r(y) = \emptyset$ and this is in contradiction to $m_{\gamma_1}(\xi) = m_{\gamma_2}(\xi)$ for $\mathcal{H}^1$-a.e. $\xi$.

□

**Corollary 4.27.** *(Weak no-cycle property). For every $a \in \mathcal{M}$ the flow order $\leq_a$ enjoys the weak no-cycle property.*

*Proof.* Let $G$ be a given geometric flow curve with extreme points $x$ and $y$, induced by $a$ and let $\xi, \eta \in G$. By definition, we take $\sigma \in M_a$, so there is a set $A$ with $\sigma(A) > 0$ made of curves $\gamma$ connecting $x$ and $y$. Then by Theorem 4.26 $\xi, \eta \in \gamma(I)$ for $\sigma$ a.e. $\gamma \in A$ and therefore, by Remark 4.20, they are comparable by $\leq_a$. □

**Definition 4.28.** *Let $x, y \in \mathbb{R}^N$ with $x \leq_a y$ be given. We denote by $\Gamma_{xy} \subset \Gamma$ the set of curves $\gamma$ such that $x, y \in \gamma(I)$ and $\forall \gamma \in \Gamma_{xy}$ define $K_x^y(\gamma) = \gamma([\tau_1, \tau_2])$, where $\tau_1 = \inf\{t \in I \mid \gamma(t) = x\}$ and $\tau_2 = \sup\{t \in I \mid \gamma(t) = y\}$.*

**Remark 4.29.** By Proposition 3.27 for every $\sigma \in M_a$ and for $\sigma$-a.e. $\gamma \in \Gamma_{xy}$: $K_x^y(\gamma) \subset T_0$, modulo a $\mathcal{H}^1$-negligible set.

**Lemma 4.30.** *With the notation introduced in Definition 4.28, let $a \in \mathcal{M}$, $\sigma \in M_a$ be given and let $[x, y]$ be a given order interval for $\leq_a$. Then for $\sigma$-a.e. $\gamma \in \Gamma_{xy}$: $K_x^y(\gamma) \subset [x, y]$ $\mathcal{H}^1$-a.e..*

*Proof.* Let $B = \{(\gamma, z) \in \Gamma_{xy} \times [x, y] \mid z \notin K_x^y(\gamma)\}$. We observe that, by Theorem 3.21 $B$ is Borel measurable and $[x, y] \subset T_0$ has a $\sigma$-finite $\mathcal{H}^1$ measure. Then applying Fubini Theorem, by Theorem 4.26, we deduce $(\sigma \otimes \mathcal{H}^1)(B) = 0$. Hence we can state that for $\sigma$-a.e. $\gamma \in \Gamma_{xy}$ the order interval $[x, y]$ is contained in $K_x^y(\gamma)$ modulo a negligible $\mathcal{H}^1$ set. □

**Lemma 4.31.** *With the notation introduced in Definition 4.28, let $a \in \mathcal{M}$, $\sigma \in M_a$ be given, let $[x, y]$ be a given order interval. Then for $\sigma$-a.e. $\gamma \in \Gamma_{xy}$ it results $K_x^y(\gamma) \subset [x, y]$ $\mathcal{H}^1$ a.e..*

*Proof.* We consider the set $C = \{(\gamma, z) \in \Gamma_{xy} \times (T_0 \setminus [x, y]) \mid z \in K_x^y(\gamma)\}$. We observe that, by the definition of $\leq_a$, for any fixed $z$ the section $C^z$ satisfies $\sigma(C^z) = 0$. Then by applying Fubini Theorem we get, for $\sigma - a.e. \gamma \in \Gamma_{xy}$, $\mathcal{H}^1(K_x^y(\gamma)) \cap (T_0 \setminus [x, y]) = 0$. By Remark 4.29 we have that $K_x^y(\gamma) \subset [x, y]$, $\mathcal{H}^1$-a.e. □

**Corollary 4.32.** *With the notation introduced in Definition 4.28, let $a \in \mathcal{M}$ and let $[x, y]$ be a given order interval for $\leq_a$. Then for every $\sigma \in M_a$, for $\sigma$ a.e. $\gamma_1, \gamma_2 \in \Gamma_{xy}$ it results $K_x^y(\gamma_1) = K_x^y(\gamma_2)$.*

*Proof.* By applying the two previous lemmas we have $K_x^y(\gamma_1) = K_x^y(\gamma_2)$ $\mathcal{H}^1$-a.e.. Since the curves $\gamma_i$ are continuous, by employing a straightforward continuity arguments we reach the thesis. □

**Corollary 4.33.** *Let $a \in \mathcal{M}$, $\sigma \in M_a$ be given, let $[x, y]$ be a given order interval and let $\Gamma_{xy} \subset \Gamma$ the set of curves connecting $x$ and $y$. Then for $\sigma$ a.e. $\gamma \in \Gamma_{xy}$ it results $K_x^y(\gamma) = [x, y]$.*

*Proof.* The thesis follows from Corollary 4.32 by the definition of flow ordering and by Theorem 4.26. □

**Corollary 4.34.** *Let $a \in \mathcal{M}$, $\sigma \in M_a$ be given. Then every order interval is a connected compact set with finite $\mathcal{H}^1$ measure.*

**Corollary 4.35.** *If $x \leq_a y$ then the order interval $[x, y]$ is isomorphic to a closed bounded interval of $\mathbb{R}$.*

**Lemma 4.36.** *Let $C$ be a given simple oscillating chain with respect to $\leq_a$ with more than two terms. By keeping the notation in Definitions 3.88 and 3.103 we have, for every $i$,*

$$\mathcal{H}^1(S_i \setminus \bigcup_{i\neq j} S_j) > 0.$$

*Proof.* If $i$ and $j$ are non-close indexes then $S_i \cap S_j = \emptyset$ by Proposition 3.105. Note that $x_i \in S_{i-1} \cap S_i \neq \emptyset$ and $x_{i+1} \in S_i \cap S_{i+1} \neq \emptyset$ and that, by Corollary 4.34, $S_i$ is connected, $S_{i+1} \cap S_{i-1} = \emptyset$ by Proposition 3.105 and $S_{i+1}$ and $S_{i-1}$ are compact. Then by using Proposition 3.105 we have

$$\mathcal{H}^1(S_i \setminus \bigcup_{i\neq j} S_j) = \mathcal{H}^1(S_i \setminus (S_{i+1} \cup S_{i-1})) \geq \operatorname{dist}(S_{i+1}, S_{i-1}) > 0.$$

□

**Theorem 4.37.** *(Strong no-cycle property) Let $a \in \mathcal{M}$ and $T_0$ given by (4.43). Then the ordered set $(T_0, \leq_a)$ satisfies the strong no-cycle property (Definition 3.100).*

*Proof.* By Corollary 4.27 the ordered set $(T_0, \leq_a)$ enjoys the weak no-cycle property. If $(T_0, \leq_a)$ does not enjoy the strong no-cycle property then by Proposition 3.105 we can find in such a set a simple oscillating chain having at least four terms. We fix such an oscillating chain $C = (x_i)_{i\in I_n}$. By applying Lemma 4.21 we deduce the existence of $\sigma \in M_a$ such that, for every $i$, there is a set $\Gamma_i$ with $m_i = \sigma(\Gamma_i) > 0$ made of curves connecting $x_i$ to $x_{i+1}$. Let $c = \min_i m_i$. We claim that for $i \neq j$ we have $\sigma(\Gamma_i \cap \Gamma_j) = 0$, as follows from Remark 4.20 because two terms of the chain which are not close are not comparable. Let $\sigma_i = \sigma \llcorner \Gamma_i$ and $\sigma_0 = \sigma - \sum_{i=1}^n \sigma_i$. Then $m_\sigma = \sum_{i=0}^n m_{\sigma_i}$. For $i > 0$, $[a, b] \subset \mathbb{R}$, we define the functions $\tau_i^1, \tau_i^2 : \Gamma_i \to [a, b]$ as follows

$$\tau_i^1 : \gamma \mapsto \inf\{t \in [a, b] \mid \gamma(t) \in \{x_i, x_{i+1}\}\},$$
$$\tau_i^2 : \gamma \mapsto \sup\{t \in [a, b] \mid \gamma(t) \in \{x_i, x_{i+1}\}\}.$$

By Remark 3.70 we have that $\tau_i^1$ and $\tau_i^2$ induce three complementary truncations of $\sigma_i$, say $\sigma_i^1$, $\sigma_i^2$, $\sigma_i^3$, hence for every $i$: $m_{\sigma_i} = m_{\sigma_i^1} + m_{\sigma_i^2} + m_{\sigma_i^3}$, $\mathcal{H}^1$-a.e. (see Remark 3.70). By additivity, setting for $j = 1, 2, 3$ $\sigma^j = \sum_{i=1}^n \sigma_i^j$, we get that the measure $\sum_{i=1}^n \sigma_i$ is decomposed (through the previous mappings) in the complementary truncations $\sigma^1, \sigma^2, \sigma^3$. For every $i \geq 1$ let

$$\sigma_i^{\pm} = \frac{m_i \pm (-1)^i c}{m_i} \sigma_2^i.$$

Observe that for every $i$ the measures $\sigma^2_\pm = \sum_{i=1}^n \sigma_i^\pm$ and $\sigma_0^2$ have the same marginals and so $\sigma^2_\pm$ are two competitors of $\sigma_2$ . Let $\sigma^\pm$ be the sum of $\sigma_0$ with the composition of $\sigma^2_\pm$ with $\sigma_1$ and $\sigma_3$. We have that both $\sigma^+$ and $\sigma^-$ are competitors of $\sigma$ and that $\sigma = \frac{1}{2}(\sigma^+ + \sigma^-)$, so we have $m_\sigma(x) = \frac{1}{2}(m_{\sigma^+}(x) + m_{\sigma^-}(x)) = a$ for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^N$. By the concavity of the functional we have

$$J^3_\alpha(\sigma) \geq \frac{1}{2} J^3_\alpha(\sigma^+) + \frac{1}{2} J^3_\alpha(\sigma^-), \tag{4.44}$$

where the equality holds if and only if $m_{\sigma^+} = m_{\sigma^-}$ $\mathcal{H}^1$-a.e.. Let us show that this last case does not hold. Indeed, let $i$ be even. On the set $\Gamma_i \setminus \bigcup_{j\neq i} \Gamma_j$ we have $m_{\sigma^+} > m_{\sigma^-}$ and this set is not $\mathcal{H}^1$-negligible by Lemma 4.36. So the strict inequality holds true in (4.44), in contradiction to the minimality of $\sigma$. □

**Definition 4.38.** *We say that a sequence of geometric flow curves $(G_n)_{n\in\mathbb{N}}$ is linked if for every $n \geq 1$ $G_n \cap \bigcup_{i=0}^{n-1} G_i \neq \emptyset$.*

**Lemma 4.39.** *Let $(G_n)_{n\in\mathbb{N}}$ be any linked sequence of geometric flow curves. Then $\forall n \in \mathbb{N}$: $\mathcal{G} = \cup_{k=0}^n G_k$ is flow connected.*

*Proof.* We proceed by induction on $n$. If $n = 0$ the thesis obviously holds. Let us suppose that the thesis holds for a certain value of $n$ and fix $x, y \in \cup_{i=0}^{n+1} G_i$ such that $x \in \cup_{i=0}^n G_i$, $y \in G_{n+1}$. By Definition 4.38 we have that there exists $z \in G_n \cap \cup_{i=0}^n G_i$ and, by the induction hypotheses, there exists an oscillating chain $C$ connecting $x, z$ in $\cup_{i=0}^n G_i$. Since $z$ also is connectible to $y$ on $G_{n+1}$, the thesis follows. □

**Theorem 4.40.** *Let $(G_n)_{n\in\mathbb{N}}$ be any linked sequence of geometric flow curves. Then for every $n \in \mathbb{N}$ the set $G_n \cap \cup_{i=0}^{n-1} G_i$ is an order interval with respect to $\leq_a$, i.e. it is a geometric flow curve.*

*Proof.* Let $x, y \in G_n \cap \cup_{i=0}^{n-1} G_i$. By Lemma 4.39 $x, y \in \bigcup_{i=0}^{n-1} G_i$ are extreme points of an oscillating chain whose support is contained in $\bigcup_{i=1}^{n-1} G_i$ and which, by Remark 4.25 can be assumed to be optimal. Then the strong no-cycle property (Theorem 4.37) says that such a chain has only the extreme points $x$ and $y$. Indeed these points are comparable by virtue of the weak no-cycle property because they are in the same geometric flow curve $G_n$. Since the support of the chain is contained in $\bigcup_{i=0}^{n-1} G_i$, then $[x, y] \subset \bigcup_{i=0}^{n-1} G_i$. Since by definition $[x, y] \subset G_n$, it follows that $[x, y] \subset G_n \cap \bigcup_{i=0}^{n-1} G_i$. Finally, by Corollaries 4.33 and 4.34 we see that $G_n \cap \bigcup_{i=0}^{n-1} G_i$ has a minimum and a maximum so it is a geometric flow curve. □

## 5. Equivalence of the irrigation models

### 5.1. Equivalence between $J^2_\alpha$ and $J^3_\alpha$.

**Theorem 5.1.** *For every $\alpha \in ]0, 1[$ the following properties hold true.*

1) $\inf_{\sigma\in\Sigma} J^2_\alpha = \inf_{\sigma\in\Sigma} J^3_\alpha$.
2) *The functionals* $J^2_\alpha$ *and* $J^3_\alpha$ *have the same minima.*

*Proof.* 1) By (4.36) we deduce $\inf_{\sigma\in\Sigma} J^3_\alpha \le \inf_{\sigma\in\Sigma} J^2_\alpha$. On the other hand, let $(\sigma_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $J^3_\alpha$, by applying Theorem 4.18 we have the existence of a loop-free sequence $(\sigma'_n)_{n\in N}$ such that, for every $n\in\mathbb{N}$, $J^2_\alpha(\sigma'_n) = J^3_\alpha(\sigma'_n) \le J^3_\alpha(\sigma_n)$. Then

$$\inf_{\sigma\in\Sigma} J^2_\alpha \le \inf_n J^2_\alpha(\sigma'_n) = \inf_n J^3_\alpha(\sigma'_n) \le \inf_n J^3_\alpha(\sigma_n) = \inf_{\sigma\in\Sigma} J^3_\alpha.$$

2) Let $\sigma \in \mathrm{argmin} J^2_\alpha$, then by (4.36) and by assertion 1) we have that $J^3_\alpha(\sigma) \le J^2_\alpha(\sigma) = \inf_{\sigma\in\Sigma} J^2_\alpha = \inf_{\sigma\in\Sigma} J^3_\alpha$ and so $\sigma \in \mathrm{argmin} J^3_\alpha$. On the other hand, if $\sigma \in \mathrm{argmin} J^3_\alpha$ then it is minimal with respect to $\le_D$ and by Theorem 4.15 and Corollary 3.66 it is loop-free and thus $J^2_\alpha(\sigma) = J^3_\alpha(\sigma) = \inf_{\sigma\in\Sigma} J^3_\alpha = \inf_{\sigma\in\Sigma} J^2_\alpha$. □

5.2. **Existence of minimizers.** A simpler version of Theorem 3.21 leads to the following statement.

**Lemma 5.2.** *Let* $I \subset \mathbb{R}$ *be a compact interval. Then the mapping* $(\gamma, x) \mapsto a_\gamma(x)$ *is upper semicontinuous in* $\Gamma_I \times \mathbb{R}^N$.

*Proof.* Let us take a sequence $(\gamma_n, x_n)_{n\in\mathbb{N}} \subset \Gamma_I \times \mathbb{R}^N$ so that $(\gamma_n, x_n) \to (\gamma, x)$. If $a_{\gamma_n}(x_n) = 0$ definitively, we have done. Otherwise, if there are infinitely many $n\in\mathbb{N}$ so that $a_{\gamma_n}(x_n) = 1$, namely there exist $t_n$ so that $\gamma_n(t_n) = x_n$. From the sequence $(t_n)_{n\in\mathbb{N}}$ we can extract a subsequence, relabeled as $(t_n)_{n\in\mathbb{N}}$, so that $t_n \to t \in I$. By (locally) uniform convergence we have $\gamma_n(t_n) = x_n \to x = \gamma(t)$ and so $a_\gamma(x) = 1$. □

**Lemma 5.3.** *Let* $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$ *so that* $J^2_\alpha(\sigma_n) \le c$, *for every* $n \in \mathbb{N}$ *for a given positive constant* $c$ *and so that* $\sigma_n \rightharpoonup \sigma$. *Then* $s^2_\sigma \le \Gamma - \liminf_n s^2_{\sigma_n}$.

*Proof.* Let us take $\bar{x} \in \mathbb{R}^N$ and a sequence $x_n \to \bar{x}$. Then, by applying the previous lemma and item 2) of Theorem C.3, since $\sigma_n \otimes \delta_{x_n} \rightharpoonup \sigma \otimes \delta_{\bar{x}}$, we get

$$\begin{aligned} a_\sigma(\bar{x}) &= \int_{(\Gamma,\sigma)} a_\gamma(\bar{x}) d\gamma = \int_{(\Gamma'\times\mathbb{R}^N, \sigma\otimes\delta_{\bar{x}})} a_\gamma(x) d\gamma dx \\ &\ge \limsup_n \int_{(\Gamma'\times\mathbb{R}^N, \sigma_n\otimes\delta_{x_n})} a_\gamma(x) d\gamma dx = \limsup_n a_{\sigma_n}(x_n) \end{aligned}$$

and so, by (4.26), since $\alpha < 1$, we have

$$s^2_\sigma(\bar{x}) \le \liminf_n s^2_{\sigma_n}(x_n).$$

Finally, by the arbitrariness of $x_n \to \bar{x}$, we have the thesis. □

**Lemma 5.4.** *Let* $\sigma_n \rightharpoonup \sigma$. *Then, if* $\gamma_n \to \gamma$ *we have*

$$\int_I s^2_\sigma(\gamma(t))|\gamma'(t)|dt \le \liminf_n \int_I s^2_{\sigma_n}(\gamma_n(t))|\gamma'_n(t)|dt.$$

*Proof.* By Lemma D.4 we know that $l_{\gamma_n}$ lower semiconverges to $l_\gamma$ and by using the previous lemma we know that $s_\sigma^2 \leq \Gamma - \liminf s_{\sigma_n}^2$, so we can apply 3) of Theorem C.3 from which the thesis follows. □

**Lemma 5.5.** *For any $\sigma \in \Sigma$ let us define the mapping*

$$f_\sigma : \gamma \mapsto \int_I s_\sigma^2(\gamma(t))|\gamma'(t)|dt.$$

*If $\sigma_n \rightharpoonup \sigma$ then $f_\sigma \leq \Gamma - \liminf_n f_{\sigma_n}$.*

*Proof.* The thesis follows from the previous lemma, passing to the infimum for all the sequences $\gamma_n \to \gamma$. □

The following result easily follows from the previous lemmas by Theorem C.3, 3).

**Corollary 5.6.** *The functional $J_\alpha^2$ is lower semicontinuous in $\Sigma$, i.e. for every $\sigma_n \rightharpoonup \sigma$ it results $J_\alpha^2(\sigma) \leq \liminf_n J_\alpha^2(\sigma_n)$.*

**Theorem 5.7.** *For every $\alpha \in ]0,1[$ the functional $J_\alpha^2$ admits minimizers.*

*Proof.* If $J_\alpha^2 = +\infty$ then the thesis trivially follows since any particle motion $\sigma$ can be considered to be a minimizer. Then we work with particle motions $\sigma$ such that $J_\alpha^2 < +\infty$ which trivially implies (since $a_\sigma \leq 1$) that $L(\gamma) < +\infty$ for $\sigma$-a.e. $\gamma$. Since $J_\alpha^2$ is asynchronous then it can be evaluated on the set of length parametrized measures on $\overline{\mathbb{R}}_+$. If $(\sigma_n)_{n\in\mathbb{N}}$ is any minimizing sequence, by Theorem 3.84 we have compactness since $(\sigma_n(0))_{n\in N} = \mu$ is tight. By Corollary 5.6 we have that $J_\alpha^2$ is lower semicontinuous and so we get the thesis. □

5.3. **Synchronization.**

**Definition 5.8.** *Let $\sigma \in \Sigma$ be given. We say that $\sigma$ is synchronized if $J_\alpha^1(\sigma) = J_\alpha^2(\sigma)$.*

**Definition 5.9.** *Let $\sigma \in \Sigma$ be given. We say that $\sigma$ is synchronizable if there exists $\overline{\sigma} \cong_{\widetilde{S}} \sigma$ which is synchronized.*

**Theorem 5.10.** *(Discrete synchronization) Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$ be countably discrete measures and let $\sigma$ be a minimizer of $J_\alpha^2$ in $\Sigma(\mu,\nu)$. Then $\sigma$ is synchronizable.*

*Proof.* Let $\mu = \sum_i m_i \delta_{x_i}$, $\nu = \sum_j m'_j \delta_{y_j}$. As a first step we replace $\sigma$ with a (finitely or countably) discrete measure equivalent to $\sigma$ for $\cong_{\widetilde{S}}$ (see Definition 3.39). To this aim, for every $i,j$, we set $\Gamma_{ij} = \{\gamma \in \Gamma_{[a,b]} \mid \gamma(a) = x_i, \gamma(b) = y_j\}$, $m_{ij} = \sigma(\Gamma_{ij})$. Let $I$ be the set of pairs $(i,j)$ such that $m_{ij} > 0$. By virtue of Corollary 4.33, $\forall (i,j) \in I$, $\sigma$-a.e. $\gamma \in \Gamma_{ij}$ enjoys the property $\gamma([a,b]) = [x_i, y_j]$ and so $\gamma \cong_S \gamma_{ij}$ for a fixed $\gamma_{ij} \in [\gamma]_S$. Then we have $\sigma \cong_{\widetilde{S}} \sigma' = \sum_{(i,j)\in I} m_{ij}\delta_{\gamma_{ij}}$. We proceed by selecting a sequence $(\gamma_n)_{n\in I}$ according to the following rule.

Let $\gamma_0$ be one of the curves $\gamma_{ij}$ such that $m_{ij}$ is maximum. Given $\gamma_n$, let

$$I_n = \left\{ (i,j) \in I \mid \gamma_{ij} \neq \gamma_k \ \forall k \leq n \text{ and } \gamma_{ij}([a,b]) \cap \bigcup_{k=0}^{n} \gamma_k([a,b]) \neq \emptyset \right\}.$$

If $I_n \neq \emptyset$, we choose as $\gamma_{n+1}$ one of the curves $\gamma_{ij}$ with $(i,j) \in I_n$, such that the corresponding mass $m_{ij}$, which will be denoted by $m_{n+1}$, is such that $m_{n+1} = \max_{(i,j)\in I_n} m_{i,j}$. By proceeding recursively in such a way, we get a (possibly finite, if we find $\overline{n} \in \mathbb{N}$ such that $I_{\overline{n}} = \emptyset$) sequence $(\gamma_n)_{n\in I}$ and we claim that every curve $\gamma_{ij}$ which has not been selected as one of the curves $\gamma_n$ satisfies $\gamma_{ij}([a,b]) \cap \gamma_n([(a,b)]) = \emptyset$ for every $n$. Indeed, if we have selected a finite sequence and $\overline{n}$ is the last index, we should have $I_{\overline{n}} = \emptyset$. Then the assertion trivially holds, otherwise $(i,j) \in I_{\overline{n}} \neq \emptyset$. If $\gamma_n$ has been defined $\forall n \in \mathbb{N}$ then $\gamma_{ij}([a,b]) \cap \bigcup_n \gamma_n([(a,b)]) \neq \emptyset$ implies the existence of $n_1$ such that $\gamma_{ij}([a,b]) \cap \gamma_{n_1}([(a,b)]) \neq \emptyset$. By denoting with $m_n$ the mass carried by the curve $\gamma_n$ for every $n \in \mathbb{N}$, since $\sum_{n\in\mathbb{N}} m_n \leq 1$ then $m_n \to 0$ and so there exists $n_2$ such that for every $n > n_2$ one has $m_n < m_{ij}$. By setting $\overline{n} = \max\{n_1, n_2\}$, we can conclude that $\gamma_{ij}$ should have been selected instead of $\gamma_{\overline{n}+1}$ and so the claim is proved.

Now we pass to select a new sequence of curves according to the previous rule. No curve of the new sequence intersects any curve of the previous one. Then we keep up to select sequences in the same way, a simpler variant of the argument used to prove the above claim shows that after an at most countable sequence of sequences we employ all the curves $\gamma_{ij}$. Afterward we go to synchronize each one of the selected sequences. Since any two curves belonging to different sequences do not intersect, they do not need to be synchronized among themselves.

Let us show how to synchronize any selected sequence $(\gamma_n)_{n\in I}$. We recursively proceed by synchronizing all the curves $\gamma_i$ by only using injective parameterizations and we don't care if they are defined on possibly different intervals since in the end of the construction we can extend them by a constant value on a common interval. Thus, given $k$, we can assume that all the curves $\gamma_j$ with $j < k$ have been synchronized. For every $i$ let $G_i = \gamma_i([a,b])$. By Theorem 4.40, the set $G = G_k \cap \bigcup_{i\leq k} G_i$ is an order interval with respect to $\leq_a$, since all the parameterizations $\gamma_i$ are injective, each one of them admits an inverse $T_i : G_i \cap G_k \to \gamma_i^{-1}(G_i \cap G_k) \subset \mathbb{R}$. By virtue of the synchronization of the curves $\gamma_i$, all the mappings $T_i$ are coherent, so they admit a common extension $T : G \to \mathbb{R}$.

We claim that $T$ is a strictly monotone mapping from the flow order to the natural order of $\mathbb{R}$. Indeed by Corollary 4.35 $G$ is isomorphic to an interval of $\mathbb{R}$ and $\forall i$ $G_i \cap G_k$ is isomorphic to a (possibly empty) subinterval. Since for every $i$ $T_i := T_{|G_i\cap G_k}$ is strictly increasing, then the claim follows from elementary arguments. Therefore $T$ is injective and its inverse $\gamma = T^{-1} : T(G) \to G$ is well defined. Since every $\gamma_i$ is strictly monotone from the order of $\mathbb{R}$ to the flow order, the sets $T(G_i) = T_i(G_i)$ are closed intervals covering $T(G)$. On each one of such intervals $\gamma \equiv \gamma_i$, is absolutely continuous, so $\gamma$ is absolutely continuous. Then $\gamma$ is a parameterization of $G$ on $T(G)$ which is bounded and so it can be extended to a parameterization $\gamma_k$ of $G_k$ on a larger bounded interval. $\square$

**Remark 5.11.** The above theorem has been stated for a minimizer $\sigma$ of $J^2_\alpha$ because this is the most natural context. However it can be extended without any effort (the same proof works) to the more general case in which $\sigma$ is a truncation of a minimizer $\sigma'$. Indeed, in such a case, every curve $G_i$ considered in the proof is also a geometric flow curve induced by $\sigma'$.

Under the assumption that $\sigma$ is concentrated on non-constant orbits, which will be kept until the proof of the following Theorem 5.16, we state the following variant of the Pruning Theorem ([29, Theorem 4.2]).

**Theorem 5.12.** *Let $\sigma \in \Sigma(\mu, \nu)$ be a given particle motion concentrated on non-constant orbits. There exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of truncations of $\sigma$ with (countably) discrete marginals $\mu_n$ and $\nu_n \in \mathcal{P}(\mathbb{R}^N)$ such that $\sigma_n \rightharpoonup \sigma$.*

Before proving the previous theorem we establish some preliminary results. Let $I = [a, b]$. For a.e. $\gamma \in \Gamma$ let us define the stopping time maps

$$T^-(\gamma) = \sup\{t \in [0,1] \mid \gamma(s) = const. \ \ \forall s \in [a,t]\},$$
$$T^+(\gamma) = \inf\{t \in [0,1] \mid \gamma(s) = const. \ \ \forall s \in [t,b]\}.$$

Fix $\varepsilon > 0$ and let $\mathcal{A}$ be the set of the Borel measurable subsets $A$ of $\Gamma$ such that there exists a measurable mapping $\tau_- : A \to [a, b]$, $T_- \le \tau_- \le T_- + \varepsilon$, and a countable set $X$ such that for $\sigma$-a.e. $\gamma \in A$: $\gamma(\tau_-(\gamma)) \in X$.

**Lemma 5.13.** *$\mathcal{A}$ is closed with respect to the countable union.*

*Proof.* Let $(A_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$, for every $n \in \mathbb{N}$ let $(\tau_-)_n$ be as above defined. Let $A = \bigcup_n A_n$, we define the function

$$\tau(\gamma) = \tau_k(\gamma) \ \text{ if } \ k = \min\{n \in \mathbb{N} \mid \gamma \in A_n\}.$$

The map $\tau$ is measurable since for every $n$ $A_n$ and $(\tau_-)_n$ are measurable. $\square$

**Corollary 5.14.** *There exists $\overline{A} \in \mathcal{A}$ which is maximum for inclusion, modulo a $\sigma$-negligible set.*

**Lemma 5.15.** $\sigma(\overline{A}) = 1$.

*Proof.* If the assertion does not hold the set $\Gamma' = \Gamma\setminus\overline{A}$ has a positive measure $\sigma(\Gamma') > 0$. Let, for every $\delta \le \varepsilon$, $T_\delta = \inf\{T_-(\gamma) + \delta, b\}$ and let $\sigma'$ be the $T_\delta$-left truncation of $\sigma \llcorner \Gamma'$. For $\delta$ small enough, since $\sigma$ is concentrated on nonconstant orbits, $\sigma'$ cannot be concentrated on constant orbits. Moreover we have $J^2_\alpha(\sigma') \le J^2_\alpha(\sigma)$ and so $J^2_\alpha(\sigma') < +\infty$ which in turn implies that there exists $x \in \mathbb{R}^N$ such that $a_{\sigma'}(x) > 0$. Notice that $a_{\sigma'}(x)$ actually is the measure of the set $A'$ of the curves $\gamma \in \Gamma'$ such that $\gamma(t) = x$ for some

$t \in [T_-(\gamma), T_-(\gamma)+\varepsilon]$. By Proposition 3.69 we can define a Borel measurable function $\vartheta : A' \to \mathbb{R}$ such that for every $\gamma \in A'$: $\gamma(\vartheta(\gamma)) = x$. Thus the existence of such a function shows that $A' \in \mathcal{A}$ and this is in contradiction to $\sigma(A') > 0$, $A' \cap \overline{A} = \emptyset$ and to the maximality of $\overline{A}$ established in Corollary 5.14. □

*Proof of Theorem 5.12.* Fix $\varepsilon > 0$. By Lemma 5.15 fix $\tau_\pm : \Gamma \to [a,b]$ with $T_- \leq \tau_- \leq T_- + \varepsilon$ and $T_+ - \varepsilon \leq \tau_+ \leq T_+$ such that, if $\overline{\sigma}$ is the $(\tau_-, \tau_+)$-truncation of $\sigma$, the marginals are discrete measures. Now, since for $\varepsilon \to 0$ $\overline{\sigma}$ weakly converge to $\sigma$, the thesis follows. □

**Theorem 5.16.** *(Synchronization Theorem) If $\sigma \in \Sigma(\mu,\nu)$ is a minimizer for $J^2_\alpha$ then it is synchronizable.*

*Proof.* Firstly we observe that the restriction of $J^2_\alpha$ to particle motions concentrated on constant orbits has a density which is different from zero at most on a countable set and, consequently, this set of particle motions does not affect the value of $J^2_\alpha$ and $J^1_\alpha$. Indeed, by (2.7) we have $\sigma = \sigma_m + \sigma_s$ and by (4.25), (4.26), (4.29) we have $J^2_\alpha(\sigma) = J^2_\alpha(\sigma_m)$, $J^1_\alpha(\sigma) = J^1_\alpha(\sigma_m)$. So, it is not restrictive to prove the thesis by assuming $\sigma$ is concentrated on non-constant orbits. Then we can apply Theorem 5.12 and so we have a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of truncations of $\sigma$ with countably discrete marginals $\mu_n$ and $\nu_n \in \mathcal{P}(\mathbb{R}^N)$ such that $\sigma_n \rightharpoonup \sigma$. By applying Theorem 5.10 (see Remark 5.11) to such a sequence we get a new sequence $(\sigma'_n)_{n\in\mathbb{N}}$ such that for every $n$ $\sigma'_n \cong_{\widetilde{S}} \sigma_n$ (see Definition 3.39) and so

$$J^1_\alpha(\sigma'_n) = J^2_\alpha(\sigma'_n) = J^2_\alpha(\sigma_n) \leq J^2_\alpha(\sigma).$$

By applying [29, Corollary 3.3], for every natural $n$ we can select $\sigma''_n$ so that $\sigma''_n \cong_S \sigma'_n \cong_{\widetilde{S}} \sigma_n$ and $\sigma''_n \rightharpoonup \sigma'$. By applying Proposition 3.44 and Proposition 3.10 we deduce $\sigma' \cong_{\widetilde{S}} \sigma$.

By [29, Lemma 4.5] we know that $J^1_\alpha$ is lower semicontinuous and so, from $\sigma''_n \rightharpoonup \sigma'$ we have

$$J^1_\alpha(\sigma') \leq \liminf_{n\to\infty} J^1_\alpha(\sigma''_n) = \liminf_{n\to\infty} J^1_\alpha(\sigma'_n) = \liminf_{n\to\infty} J^2_\alpha(\sigma_n) \leq J^2_\alpha(\sigma).$$

Therefore we can conclude that $J^1_\alpha(\sigma') \leq J^2_\alpha(\sigma) = J^2_\alpha(\sigma')$ with $\sigma' \cong_{\widetilde{S}} \sigma$. □

5.4. **Equivalence of the irrigation functionals.** The previous results concerning the equivalence of the irrigation functionals are summarized in the following statement.

**Theorem 5.17.** *We have*

$$M = argmin_{\Sigma(\mu,\nu)} J^3_\alpha = argmin_{\Sigma(\mu,\nu)} J^2_\alpha \neq \emptyset,$$

*such a set is closed with respect to $\cong_{\widetilde{S}}$ and, more precisely, it is the closure of $argmin_{\Sigma(\mu,\nu)} J^1_\alpha$ with respect to $\cong_{\widetilde{S}}$. Finally, if $\mu = \delta_S$ then*

$$argmin_{\Sigma(\mu,\nu)} J^1_\alpha = argmin_{\Sigma(\mu,\nu)} J^0_\alpha$$

*and therefore*

$$\inf_{\Sigma(\mu,\nu)} J_\alpha^0 = \inf_{\Sigma(\mu,\nu)} J_\alpha^1 = \inf_{\Sigma(\mu,\nu)} J_\alpha^2 = \inf_{\Sigma(\mu,\nu)} J_\alpha^3.$$

## Appendix A. Countably inductive sets

**Definition A.1.** *We shall say that a partially ordered set $(E, \leq)$ is countably inductive if every increasing sequence in $E$ has an upper bound.*

Let $(E, \leq)$ be a countably inductive set and let $f : E \to \mathbb{R}$ be a given bounded function. Let us define the mapping $s : E \to \mathbb{R}$ as follows

$$\forall x \in E : \quad s(x) = \sup_{y \geq x} f(y). \tag{A.45}$$

Note that $s$ is a monotone decreasing function and that for every $x \in E$: $f(x) \leq s(x)$, where the equality holds if $x$ is a maximal element. Moreover, if $f$ is a strictly increasing function, then $s(x) \leq f(x)$ is equivalent to the maximality of $x$.

**Theorem A.2.** *If $(E, \leq)$ is a countably inductive set and there exists a strictly increasing function $f : E \to \mathbb{R}$, then for every $x \in E$ there exists $m \in E$ such that $x \leq m$ and $m$ is maximal.*

*Proof.* Of course it is not restrictive to assume $f$ bounded. Given $x \in E$, let us recursively choose an increasing sequence $(x_n)_{n \in N}$ by setting $x_0 = x$ and by taking, for any $n \geq 1$, using the definition of $s$, an element $x_n \geq x_{n-1}$ so that

$$f(x_n) > s(x_{n-1}) - \frac{1}{n}. \tag{A.46}$$

Since $(E, \leq)$ is countably inductive, the sequence $(x_n)_{n \in N}$ has an upper bound $m$ and since $s$ and $f$ are monotone we have from (A.46) $s(m) \leq f(m)$. Thus $m$ is maximal. □

## Appendix B. Integration Order Inequality

**Definition B.1.** *Let $(E, \mathcal{B})$ be a measurable space. We shall say that a positive measure $\mu$ is semifinite if for every measurable subset $A \in \mathcal{B}$ such that $\mu(A) > 0$ there exists a measurable subset $X \subset A$ such that $0 < \mu(X) < +\infty$.*

Note that the semifiniteness property is implied by the $\sigma$-finiteness property. A remarkable example of a semifinite measure which is not $\sigma$-finite is given by the Hausdorff measure $\mathcal{H}^\alpha$ on the $\sigma$-algebra of Borel subsets of the euclidean space $\mathbb{R}^N$ when $\alpha < N$ (see [30, Theorem 8.13]). The result is sharp in the sense that $\mathcal{H}^\alpha$ is not semifinite on the $\sigma$-algebra of Carathéodory measurable sets (see also [22, Corollary 439H]).

**Lemma B.2.** *If $\mu$ is a semifinite positive measure on $(E, \mathcal{B})$ and if $A \in \mathcal{B}$, $\mu(A) = +\infty$, $\forall a \in \mathbb{R}_+$ there exists $X \in \mathcal{B}$, $X \subset A$ such that $a < \mu(X) < +\infty$.*

*Proof.* If the thesis is not true, we have that $a = \sup \mu(X) < +\infty$, where the sup is taken over the set $\{X \in \mathcal{B} \mid X \subset A, \mu(X) < +\infty\}$. So $\forall n \in \mathbb{N}$ there exists $X_n \in \mathcal{B}$ such that $X_n \subset A$, $a - \frac{1}{n} \leq \mu(X_n) \leq a$ and one can easily prove that $\overline{X} = \bigcup_n X_n$ satisfies $\overline{X} \in \mathcal{B}$, $\overline{X} \subset A$ and $\mu(\overline{X}) = a$. So $\mu(A \setminus \overline{X}) = +\infty$ and therefore the semifiniteness property implies that there is $Y \in \mathcal{B}$, $Y \subset A \setminus \overline{X}$ with $0 < \mu(Y) < +\infty$. By additivity, $a < \mu(\overline{X} \cup Y) < +\infty$, in contradiction to the definition of $a$. □

Let $(E, \mathcal{B})$ be a measurable space and let $\mathcal{M}(E)$ be the space of the positive measures defined on the $\sigma$-algebra $\mathcal{B}$. Given $\mu \in \mathcal{M}(E)$, we introduce the following notation

$$\mathcal{F}_\mu = \{\nu \in \mathcal{M}(E) \mid \ \nu \leq \mu,\ \nu(E) < +\infty\}. \tag{B.47}$$

**Lemma B.3.** *If $\mu$ is a positive semi-finite measure then it is the supremum of a family of measures with finite masses, i.e. $\mu = \sup \mathcal{F}_\mu$.*

*Proof.* Let $A \in \mathcal{B}$ and $a < \mu(A)$. By Lemma B.2 there is $X \subset A$ with $a < \mu(X) < +\infty$. So we have $a < \mu(X) = \mu \llcorner X(A) \leq \sup_{\nu \in \mathcal{F}_\mu} \nu(A)$. By the arbitrariness of $a$, $\mu(A) \leq \sup_{\nu \in \mathcal{F}_\mu} \nu(A)$. □

**Proposition B.4.** *Let $(E, \mathcal{B})$ be a measurable space, $\mu \in \mathcal{M}(E)$ and let $f : E \to \mathbb{R}$ be a positive and measurable mapping. If $\mu$ is semi-finite, then*

$$\int_E f\, d\mu = \sup_{\nu \in \mathcal{F}_\mu} \int_E f\, d\nu. \tag{B.48}$$

*Proof.* Since $\mu = \sup \mathcal{F}_\mu$, then for every measurable subset $A$, by taking $f = \mathbf{1}_A$ we have

$$\int_E f\, d\mu = \mu(A) = \sup_{\nu \in \mathcal{F}_\mu} \nu(A) = \sup_{\nu \in \mathcal{F}_\mu} \int_E f\, d\nu.$$

If $s : E \to \mathbb{R}$ is any positive measurable step function, by additivity we get

$$\sup_{\nu \in \mathcal{F}_\mu} \int_E s\, d\nu = \int_E s\, d\mu.$$

Then, given any positive measurable function $f$, we get by definition

$$\int_E f\, d\mu = \sup_{s \leq f} \int s\, d\mu = \sup_{s \leq f} \sup_{\nu \in \mathcal{F}_\mu} \int_E s\, d\nu \leq \sup_{\nu \in \mathcal{F}_\mu} \int_E f\, d\nu,$$

where the sup is taken on all the positive measurable step functions $s \leq f$. □

**Theorem B.5.** *(Integration Order Inequality) Let $(E_1, \mathcal{B}_1, \mu)$, $(E_2, \mathcal{B}_2, \nu)$ be two measure spaces. Assume $\mu$ is semi-finite, $\nu$ $\sigma$-finite and let $f : E_1 \times E_2 \to \mathbb{R}_+$ be $\mathcal{B}_1 \times \mathcal{B}_2$ measurable. Then*

$$\int_{E_1} \left( \int_{E_2} f(x, y)\, d\nu \right) d\mu \leq \int_{E_2} \left( \int_{E_1} f(x, y)\, d\mu \right) d\nu. \tag{B.49}$$

*Proof.* Since $\mu$ is semi-finite, we have from Proposition B.4

$$\int_{E_1}\left(\int_{E_2} f(x,y)\,d\nu\right)d\mu = \sup_{\mu'\in\mathcal{F}_\mu}\int_{E_1}\left(\int_{E_2} f(x,y)\,d\nu\right)d\mu'.$$

By applying Fubini-Tonelli Theorem ([34, Theorem 8.8]) to the integral on the right hand side, we get

$$\int_{E_1}\left(\int_{E_2} f(x,y)\,d\nu\right)d\mu = \sup_{\mu'\in\mathcal{F}_\mu}\int_{E_2}\left(\int_{E_1} f(x,y)\,d\mu'\right)d\nu \le \int_{E_2}\left(\int_{E_1} f(x,y)\,d\mu\right)d\nu.$$

□

Let us note that in general the reverse inequality of (B.49) does not hold (see [34, Counterexample 8.9 b)] )

## Appendix C. Semiconvergence of measures

**Definition C.1.** *Let $(\mu_n)_{n\in\mathbb{N}}$ and $\mu$ be given Radon measures on a metric space $X$. We shall say that $\mu_n$ is lower semiconvergent (respectively upper semiconvergent) to $\mu$ if the following conditions respectively hold:*

*Lsc) For every open set $A\subset X$: $\mu(A)\le \liminf_{n\to\infty}\mu_n(A)$.*

*Usc) For every closed set $C\subset X$: $\mu(C)\ge \limsup_{n\to\infty}\mu_n(C)$.*

Notice that if $\mu$ and $(\mu_n)_{n\in\mathbb{N}}$ are probability measures then *Lsc* and *Usc* are both equivalent to the narrow convergence (see [14, Theorem 11.1.1]), whereas, in general, the narrow convergence is equivalent to *Lsc* and *Usc* simultaneously satisfied (see [28]).

Let us recall (see [12, Definition 4.1]) that the $\Gamma$-*lower limit* of a sequence of functions $(F_h)_{h\in\mathbb{N}}$ from a topological space $X$ into $\overline{\mathbb{R}}$ is defined as the function

$$(\Gamma-\liminf_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)}\ \liminf_{h\to\infty}\ \inf_{y\in U} F_h(y), \tag{C.50}$$

where $\mathcal{N}(x)$ denotes the set of all open neighborhoods of $x$ in $X$. The following characterization of the $\Gamma$-lower limit will be useful in the sequel.

**Proposition C.2.** *Let $(F_h)_{h\in\mathbb{N}}$ be a sequence of functions from a topological space $X$ in $\overline{\mathbb{R}}$ and let $F=\Gamma-\liminf_{h\to\infty}F_h$. Then*

$$F = \sup\{\varphi: X\to\overline{\mathbb{R}} \mid \varphi\ l.s.c.,\ \varphi\le F_h \text{ def.}\} = \sup_{k\in\mathbb{N}}\varphi_k, \tag{C.51}$$

*where $\varphi_k = sc^-(\inf_{h\ge k}F_h)$ and $sc^- f$ denotes the lower semicontinuous envelope of $f$.*

*Proof.* Let us observe that by (C.50) if $(G_h)_{h\in\mathbb{N}}$ is a sequence of functions from $X$ into $\overline{\mathbb{R}}$ such that $G_h \le F_h$ definitively, then

$$\Gamma - \liminf_{h\to\infty} G_h \le \Gamma - \liminf_{h\to\infty} F_h.$$

Moreover we know (see [12, Remark 4.1]) that if the functions $F_h(x)$ are a constant sequence, i.e. $F_h(x) = f(x)$ $\forall x \in X$ and $\forall h \in \mathbb{N}$, then $\Gamma - \liminf_{h\to\infty} F_h = sc^- f$. Therefore if $\varphi : X \to \overline{\mathbb{R}}$ is any lower semicontinuous function such that $\varphi \le F_h$ definitively, then

$$\varphi(x) \le \Gamma - \liminf_{h\to\infty} F_h. \tag{C.52}$$

We fix $x \in X$ such that $F(x) < +\infty$ and $\varepsilon > 0$. By (C.50) there exists an open set $U \in \mathcal{N}(x)$ such that

$$\liminf_{k\to\infty} \inf_{y\in U} F_k(y) \ge F(x) - \varepsilon$$

and so there is $\overline{k} \in \mathbb{N}$ such that $\forall k > \overline{k}$: $\inf_{y\in U} F_k(y) > F(x) - \varepsilon$.
Let

$$\varphi(x) = \begin{cases} F(x) - \varepsilon & \text{if } x \in U \\ -\infty & \text{if } x \notin U. \end{cases}$$

We have $\varphi \le F_k$ $\forall k > \overline{k}$ and, since $\varphi$ is lower semicontinuous, we have $\varphi \le \varphi_{\overline{k}}$. So $\forall x \in X$ and $\forall \varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $F(x) - \varepsilon \le \varphi_k(x)$ which implies by the arbitrariness of $x$ and $\varepsilon$

$$F \le \sup_{k\in\mathbb{N}} \varphi_k. \tag{C.53}$$

The thesis follows by (C.52) and (C.53). □

**Theorem C.3.** *Let $(\mu_n)_{n\in\mathbb{N}}$ and $\mu$ be given Radon measures on a metric space $X$. The following assertions are equivalent.*

1) *$\mu_n$ is lower semiconvergent to $\mu$;*

2) *For every positive lower semicontinuous $f : X \to \mathbb{R}$: $\displaystyle\int_X f\, d\mu \le \liminf_{n\to\infty} \int_X f\, d\mu_n$;*

3) *For every $(f_n)_{n\in\mathbb{N}}$ and $f$ positive functions defined on $X$ and satisfying the inequality: $f \le \Gamma\text{–}\liminf\limits_{n\to\infty} f_n$, the following inequality holds true*

$$\int_X f\, d\mu \le \liminf_{n\to\infty} \int_X f_n\, d\mu_n.$$

*Proof.* 1) ⇒ 2). Let $f : X \to \mathbb{R}$ be positive and lower semicontinuous. By definition, $\int_X f\, d\mu = \sup \int_X s\, d\mu$, among the simple functions $s \le f$. Since $f$ is lower semicontinuous we can easily restrict to consider $s = \sum_i c_i \mathbf{1}_{A_i}$, where $c_i \ge 0$ and the $A_i$ are open sets. Then by Definition C.1

$$\int_X s\, d\mu = \sum_i c_i \mu(A_i) \leq \liminf_n \sum_i c_i \mu_n(A_i) = \liminf_n \int_X s\, d\mu_n \leq \liminf_{n\to\infty} \int_X f\, d\mu_n.$$

2) $\Rightarrow$ 3). Let $\varphi : X \to \mathbb{R}$ be any lower semicontinuous function such that $\varphi \leq f_n$ definitively. Then by item 2)

$$\int_X \varphi\, d\mu \leq \liminf_{n\to\infty} \int_X \varphi\, d\mu_n \leq \liminf_{n\to\infty} \int_X f_n\, d\mu_n.$$

Therefore by (C.51) and by monotone convergence

$$\int_X f\, d\mu \leq \int_X \sup_{k\to\infty} \varphi_k\, d\mu = \lim_{k\to\infty} \int_X \varphi_k\, d\mu \leq \liminf_{n\to\infty} \int_X f_n\, d\mu_n.$$

3) $\Rightarrow$ 1). Let $A \subset X$ be an open set, then the characteristic function $\mathbf{1}_A$ is lower semicontinuous. So if we set, for every $n \in \mathbb{N}$, $f_n = \mathbf{1}_A$, then we have by [12, Remark 4.1] $\mathbf{1}_A = \Gamma - \liminf_{n\to\infty} f_n$. So by 3) we get $\mu(A) = \int_X \mathbf{1}_A d\mu \leq \liminf_{n\to\infty} \int_X \mathbf{1}_A d\mu_n = \liminf_n \mu_n(A)$. $\square$

We notice that in [8, Proposition 5.5] it was essentially proved that if $X$ is compact and $\mu_n$ weakly converges to $\mu$, than statement 3) of the previous theorem holds true. It is worth comparing this result with Fatou Lemma: the pointwise limit of the functions is replaced by $\Gamma$-lower limit but one can work with a weakly converging sequence of measures instead of a fixed measure.

## Appendix D. Length measures

In this section we will discuss the two notions of length measures which have been used in this paper. Let us recall that $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure ([21], [30]).
Given an orbit $\gamma$, we introduce the following Borel measure, used in Section 3.

**Definition D.1.** *For any $\gamma \in \Gamma(I)$ we define the local orbit-length $l_\gamma$ as follows*

$$\forall B \in \mathcal{B}(\mathbb{R}^N): \quad l_\gamma(B) = \int_{\gamma^{-1}(B)} |\gamma'(t)| dt. \tag{D.54}$$

*The total length of $\gamma$ is given by*

$$L(\gamma) = l_\gamma(\mathbb{R}^N). \tag{D.55}$$

**Proposition D.2.** *For every $\gamma \in \Gamma(I)$, $l_\gamma \ll \mathcal{H}^1$ and $\dfrac{dl_\gamma}{d\mathcal{H}^1} = m_\gamma$.*

*Proof.* The thesis follows by applying the Area Formula (see [21, Theorem 3.2.6]). Indeed, for any $B \in \mathcal{B}(R^N)$, let $A = \gamma^{-1}(B)$, the Area Formula gives

$$l_\gamma(B) = \int_A |\gamma'(t)|dt = \int_{R^N} N(\gamma|A, x)d\mathcal{H}^1 = \int_B m_\gamma(x)d\mathcal{H}^1, \tag{D.56}$$

where $N(\gamma|A, x) = m_{\gamma_{|_A}}(x)$ is the the multiplicity function. □

**Definition D.3.** *The restriction $\mathcal{H}^1 \llcorner B$ to any $B \in \mathcal{B}(\mathbb{R}^N)$, defined by*

$$\mathcal{H}^1 \llcorner B(A) = \mathcal{H}^1(B \cap A), \quad \forall A \in \mathcal{B}(\mathbb{R}^N),$$

*will be called local* set-length. *For $B = \gamma(I)$, $\gamma \in \Gamma(I)$, we introduce the notation $h_\gamma(A) = [\mathcal{H}^1 \llcorner \gamma(I)](A)$.*

The local set-length is absolutely continuous with respect to $\mathcal{H}^1$ and we have

$$\mathcal{H}^1 \llcorner B(A) = \int_A \mathbf{1}_B d\mathcal{H}^1,$$

that is $\mathbf{1}_B$ is the Radon-Nikodym derivative of $\mathcal{H}^1 \llcorner B$. Therefore

$$\frac{dh_\gamma}{d\mathcal{H}^1} = \mathbf{1}_{\gamma(I)} = a_\gamma. \tag{D.57}$$

**Proposition D.4.** *If $\gamma_n$ is locally uniformly convergent to $\gamma$ then $l_{\gamma_n}$ is lower semi-convergent to $l_\gamma$.*

*Proof.* Let $B \subset \mathbb{R}^N$ be a fixed open set and let $\gamma_n \to \gamma$ locally uniformly. Since $\gamma^{-1}(B)$ is an open set, we can set $\gamma^{-1}(B) = \bigcup_h \overset{\circ}{K}_h$, where $(K_h)_{h \in \mathbb{N}}$ is an increasing sequence of compact sets. The locally uniform convergence of $\gamma_n$ implies

$$\gamma^{-1}(B) \subset \bigcup_n \overbrace{\bigcap_{i \geq n} \gamma_i^{-1}(B)}^{\circ}.$$

Therefore, for every $h \in \mathbb{N}$ there exists $n_h \in \mathbb{N}$ such that $K_h \subset \gamma_n^{-1}(B)$ for every $n \geq n_h$. For every fixed $K_h$ we can state

$$\int_{\overset{\circ}{K}_h} |\gamma^{'}(t)|\, dt \quad \leq \quad \liminf_{n \to \infty} \int_{\overset{\circ}{K}_h} |\gamma_n^{'}(t)|\, dt \leq \liminf_{n \to \infty} \int_{\gamma_n^{-1}(B)} |\gamma_n^{'}(t)|\, dt = l_{\gamma_n(B)}.$$

By passing to the supremum with respect to $h$ we get the thesis. □

Given any subset $X \subset \mathbb{R}^N$, we define the distance function $d_X(x) = d(x, X)$. There exists a strict relation between the distance function and the Hausdorff metric $\mathbf{d}$, namely

$$\| d_X - d_Y \|_{\infty, \mathbb{R}^N} = \sup_{x \in \mathbb{R}^N} |d_X(x) - d_Y(x)| = \mathbf{d}(X, Y).$$

**Definition D.5.** *We shall say that a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^N$ locally converges in the Hausdorff distance to a set $K$, in symbols $K_n \xrightarrow{\mathbf{d}_{loc}} K$, if $(d_{K_n})_{n\in\mathbb{N}}$ locally uniformly converges to the function $d_K$, i.e. if, for every compact $C \subset \mathbb{R}^N$, $\| d_{K_n} - d_K \|_{\infty,C} \to 0$ as $n \to \infty$.*

Analogously to Proposition D.4 we prove the following statement.

**Proposition D.6.** *(Local Golab Theorem) Let $(K_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ be any sequence of connected sets such that $K_n \xrightarrow{\mathbf{d}_{loc}} K$ and let $\mu_n = \mathcal{H}^1 \llcorner K_n$ $\forall n \in \mathbb{N}$, $\mu = \mathcal{H}^1 \llcorner K$. Then $\mu_n$ is lower semiconvergent to $\mu$.*

*Proof.* We refer to the proof of [32, Theorem 10.19] (Golab Theorem) where it is proved that the connectedness implies the uniform concentration property with respect to one dimensional Hausdorff measure, which lets to see Golab Theorem as a particular case of [32, Theorem 10.14]. Notice that such a theorem is stated in the global case ($A = \mathbb{R}^N$) but, since the uniform concentration property has a local nature (it automatically extends to the traces on any open set), then also the thesis trivially holds in the local case. □

**Remark D.7.** Regarding the previous theorem, we also refer to the papers [28] and [13] for a more explicit (in this sense) statement of [32, Theorem 10.14] and to [8, Theorem 3.3].

Notice that though uniform convergence of orbits implies Hausdorff convergence of the trajectories, in general it is not true that local uniform convergence of orbits implies local Hausdorff convergence of the trajectories. Nevertheless, we get lower semicontinuity of the local set-length $h_\gamma$ by arguing as follows.

Let $B \subset \mathbb{R}^N$ be any given open set. For every compact $J \subset I$ we define the mapping $F_J : \Gamma \to \mathbb{R}$ as follows:

$$F_J : \gamma \mapsto \mathcal{H}^1(\gamma(J) \cap B).$$

**Proposition D.8.** *For every open set $B \subset \mathbb{R}^N$ the mapping*

$$F : \gamma \mapsto \mathcal{H}^1(\gamma(I) \cap B) = \sup\{F_J(\gamma) \mid J \subset I,\ \ J \text{ compact}\} \tag{D.58}$$

*is lower semicontinuous with respect to the locally uniform convergence.*

*Proof.* Since the local uniform convergence of the orbits implies the uniform convergence of the restrictions to any compact set $J$ and, subsequently, the Hausdorff convergence of the image set $\gamma(J)$, by the local semicontinuity of the length (Proposition D.6) we can state that for every $J$ the mapping $F_J$ is l.s.c. with respect to the locally uniform convergence. The thesis follows. □

The previous notions of length measures lead to corresponding notions in terms of particle motions.

**Definition D.9.** *Let $\sigma \in \Sigma$ be a non-spread particle motion. We define the weighted local orbit-length $l_\sigma$ as follows*

$$\forall B \in \mathcal{B}(\mathbb{R}^N): \quad l_\sigma(B) = \int_{(\Gamma,\sigma)} l_\gamma(B) d\gamma.$$

**Proposition D.10.** *For every non-spread $\sigma \in \Sigma$, $l_\sigma \ll \mathcal{H}^1$ and $\dfrac{dl_\sigma}{d\mathcal{H}^1} = m_\sigma$.*

*Proof.* For any fixed Borel set $B \in \mathcal{B}(\mathbb{R}^N)$, using Proposition 3.25 and Proposition D.2, we have, by using a $\mathcal{H}^1$ $\sigma$-finite track $T$ for $\sigma$,

$$\begin{aligned} l_\sigma(B) &= \int_{(\Gamma,\sigma)} l_\gamma(B) d\gamma = \int_{(\Gamma,\sigma)} \left( \int_B m_\gamma(x) d\mathcal{H}^1 \right) d\gamma \\ &= \int_{(\Gamma,\sigma)} \left( \int_{B\cap T} m_\gamma(x) d\mathcal{H}^1 \right) d\gamma \\ &= \int_{B\cap T} \left( \int_{(\Gamma,\sigma)} m_\gamma(x) d\gamma \right) d\mathcal{H}^1 = \int_B m_\sigma(x) d\mathcal{H}^1. \end{aligned}$$

□

**Definition D.11.** *Let $\sigma \in \Sigma$ be a non-spread particle motion. We define the weighted local set-length $h_\sigma$ as follows*

$$\forall B \in \mathcal{B}(\mathbb{R}^N): \quad h_\sigma(B) = \int_{(\Gamma,\sigma)} h_\gamma(B) d\gamma.$$

Analogously to Proposition D.10 we state the following property.

**Proposition D.12.** *For every non-spread $\sigma \in \Sigma$, $h_\sigma \ll \mathcal{H}^1$ and $\dfrac{dh_\sigma}{d\mathcal{H}^1} = a_\sigma$.*

*Proof.* Let $B \subset \mathbb{R}^N$ be any Borel set and $T$ be a $\mathcal{H}^1$ $\sigma-$finite track of $\sigma$, by applying Proposition 3.25, (D.57) and Fubini Theorem, we have, as in Proposition D.10,

$$\begin{aligned} h_\sigma(B) &= \int_{(\Gamma,\sigma)} \mathcal{H}^1(\gamma(I) \cap B)\, d\gamma = \int_{(\Gamma,\sigma)} \left( \int_{B\cap T} a_\gamma(x)\, d\mathcal{H}^1 \right) d\gamma \\ &= \int_{B\cap T} \left( \int_{(\Gamma,\sigma)} a_\gamma(x)\, d\gamma \right) d\mathcal{H}^1 = \int_B a_\sigma(x)\, d\mathcal{H}^1. \end{aligned}$$

□

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