QUASISTATIC CRACK GROWTH IN FINITE ELASTICITY WITH NON–INTERPENETRATION

GIANNI DAL MASO AND GIULIANO LAZZARONI

Abstract. We present a variational model to study the quasistatic growth of brittle cracks in hyperelastic materials, in the framework of finite elasticity, taking into account the non-interpenetration condition.

Keywords: variational models, energy minimization, free-discontinuity problems, polyconvexity, quasistatic evolution, rate-independent processes, brittle fracture, crack propagation, Griffith's criterion, finite elasticity, non-interpenetration.
2000 MSC: 35R35, 74R10, 74B20, 49J45, 49Q20, 35A35, 28B20.

Contents

Introduction

In this paper we study a quasistatic evolution problem for brittle cracks in hyperelastic bodies, in the context of finite elasticity. Following the lines of [18, 6], we develop a mathematical model, based on the variational approach to fracture mechanics that goes back to Griffith [22].

All existence results in the mathematical literature on this subject [15, 8, 17, 12] were obtained using energy densities with polynomial growth. This was not compatible with the standard assumption in finite elasticity that the strain energy tends to infinity as the determinant of the deformation gradient vanishes. Our model extends the previous results to a wide class of energy densities satisfying this property; moreover, it takes into account the non-interpenetration condition, which was not considered in the above mentioned papers.

Our definition of quasistatic evolution is based on the approximation by means of solutions to incremental minimum problems obtained by time discretization (Section 2.4). This approximation method was already used in the other mathematical papers on this subject, and is common in a large class of rate-independent problems. We prove an existence result (Theorem 2.13) and show also (Theorem 2.14) that our solutions satisfy the basic properties of the energy formulation presented in [27]:

- global stability,
- energy-dissipation balance.

To simplify the functional framework, we impose a confinement condition: the deformed configuration is constrained to be contained in a prescribed compact set (Section 1.3). This allows us to formulate the problem in the space SBV of special functions of bounded variation [3], as in [17].

There are three main difficulties in passing from the polynomial growth condition to the context of finite elasticity:

- lower semicontinuity of the bulk energy,
- jump transfer,
- energy estimate.

As for the lower semicontinuity, the problem is that all theorems for quasiconvex functions require a polynomial growth, while the convexity assumption is not compatible with finite elasticity. We overcome this difficulty by assuming polyconvexity and applying a recent result [20], which requires only suitable bounds from below ((W4) in Section 1.4).

Jump transfer is a procedure introduced in [17] to prove global stability. One step of the original construction employs a reflection argument, which is forbidden by finite elasticity. We modify the jump transfer lemma, replacing the reflection argument by a suitable stretching argument (Section 4.1): the upper bounds needed in this step require a multiplicative stress estimate ((W5) in Section 1.4), already used in [4, 25].

The discrete energy inequality was obtained in [12] through an additive manipulation of the approximate solutions; moreover, the passage to the limit in this inequality was based on a lemma about the convergence of stresses, which requires a polynomial growth. In our new context, the discrete energy inequality relies on the multiplicative splitting introduced in [19], which requires a suitable continuity condition on the Kirchhoff stress ((W6) in Section 1.4); the passage to the limit is now obtained using a modification of the above mentioned lemma (Lemma 5.1), proven in [19].

The hypotheses introduced to overcome these difficulties ((W0-6) in Section 1.4) are compatible with finite elasticity and are satisfied, for instance, in the case of Ogden materials

(Example 1.8). Since in this paper we focus on the new ideas and techniques used to avoid the polynomial growth condition, we study a problem with no applied forces and with sufficiently smooth prescribed boundary conditions. The minimal regularity hypotheses on the boundary data, on the volume forces, and on the surface forces will be considered in a forthcoming paper [26].

To deal with the non-interpenetration condition, we adopt a weak formulation for SBV functions (Definition 1.1), introduced in [21], and use a stability result (Theorem 3.4) with respect to weak* convergence in SBV proven in the same paper. In the Appendix we discuss the reasons for the choice of this formulation and its physical motivation.

In Section 1 we present the hypotheses on the geometry of the body, on the strain energy, and on the prescribed deformations. In Section 2 we give the definition of quasistatic evolution and state the main theorems; first, we present their simplest form, using an auxiliary problem (Section 2.3) based on the multiplicative splitting introduced in [19]; then, we formulate these results in the original setting. Section 3 contains the proof of the existence results, while Sections 4 and 5 are devoted to the proof of the global stability and of the energy balance; moreover, in Section 5.3 we show the convergence of the energies of the approximate solutions. Section 6 contains some results on the nontrivial problem of the measurability of solutions with respect to time. In Section 7 we sketch the extension to the case of applied volume forces with smooth potentials. Finally, the Appendix contains a comparison among different notions of non-interpenetration.

1. The mechanical assumptions

1.1. **Definitions and notation.** Throughout the paper, we will consider functions defined on subsets of $\mathbb{R}^n$ (with $n \geq 2$), endowed with the Euclidean scalar product $\cdot$ and the corresponding norm $|\cdot|$. The space of $n{\times}n$ real matrices is denoted by $\mathbb{M}^{n\times n}$; SO_n stands for the subset of orthogonal matrices with determinant 1, while GL_n^+ stands for the subset of matrices with positive determinant; I is the identity matrix. The space $\mathbb{M}^{n\times n}$ is endowed with the scalar product $A : B := \mathrm{tr}\,(AB^{\mathrm{T}})$, which coincides with the Euclidean scalar product in $\mathbb{R}^{n^2}$; we denote by $|\cdot|$ the corresponding norm. Given $A \in \mathbb{M}^{n\times n}$, we define $\mathrm{adj}_j A$ as the vector composed of the minors of A of order j; its dimension is $\tau_j := \binom{n}{j}^2$.

In what follows, $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$, while $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The expression *almost everywhere*, abbreviated as *a.e.*, always refers to $\mathcal{L}^n$, unless otherwise specified. Given two sets A and B in $\mathbb{R}^n$ we say that $A \,\tilde{\subset}\, B$ whenever $\mathcal{H}^{n-1}(A\backslash B) = 0$ and we say that $A \cong B$ whenever $\mathcal{H}^{n-1}(A \,\triangle\, B) = 0$, where $A \,\triangle\, B := (A\backslash B) \cup (B\backslash A)$ denotes the symmetric difference of A and B.

We recall some notions concerning BV functions. As usual, for a bounded open set $U \subset \mathbb{R}^n$ and $m \geq 1$, $BV(U;\mathbb{R}^m)$ is the space of *functions of bounded variation*, i.e., the set of functions $u \in L^1(U;\mathbb{R}^m)$ whose distributional gradient $\mathrm{D}u$ is a Radon measure on U with $|\mathrm{D}u|\,(U) < +\infty$, where $|\mathrm{D}u|$ denotes the total variation of $\mathrm{D}u$. For a BV function u, the symbol ∇u stands for the absolutely continuous part of $\mathrm{D}u$ with respect to $\mathcal{L}^n$. We refer to [3] for the definition of the *jump set* $S(u)$, of its *unit normal vector field* ν_u, of the *jump* $[u] := u^+ - u^-$, and of the space $SBV(U;\mathbb{R}^m)$ of *special functions of bounded variation.* Given $p > 1$, we consider the space

$$SBV^p(U;\mathbb{R}^m) := \big\{ u \in SBV(U;\mathbb{R}^m) : \nabla u \in L^p\big(U;\mathbb{M}^{m\times n}\big) \,\big\}\,,$$

endowed with the norm

$$\|u\|_{SBV^p(U;\mathbb{R}^m)} := \int_U |u| \,\mathrm{d}x + \left(\int_U |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + |\mathrm{D}u|\,(U)\,, \tag{1.1}$$

which makes it a Banach space.

1.2. **The body and its cracks.** In this section we introduce a geometry modelling an elastic body with cracks, following [12]. The *reference configuration* of the body is the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$.

We will suppose that every deformation takes place in a *container* K, a compact set with Lipschitz boundary and with $\Omega \subset K$. We will assume also that every crack in the reference configuration is contained in the *brittle part* $\overline{\Omega}_B$ of $\overline{\Omega}$, and that $\overline{\Omega}_B$ is the closure of an open subset Ω_B of Ω with Lipschitz boundary.

We fix an open set Ω_D with Lipschitz boundary and with $\Omega \subset \Omega_D \subset K$, and define the *Dirichlet part* of the boundary of Ω as $\partial_D\Omega := \Omega_D \cap \partial\Omega$. The Dirichlet condition on $\partial_D\Omega$ is imposed by prescribing the deformation of $\Omega_D\backslash\Omega$, which may be considered as an *unbreakable body* in contact with Ω. The *Neumann part* of the boundary is the closed set $\partial_N\Omega := \partial\Omega\backslash\partial_D\Omega$. The case $\Omega_D = \Omega$ corresponds to a pure Neumann problem, while $\overline{\Omega} \subset \Omega_D$ corresponds to a pure Dirichlet problem (if so, it is not restrictive to take $\Omega_D = \operatorname{int} K$).

We suppose

$$\overline{\Omega}_B \cap \partial_D\Omega = \emptyset\,, \tag{1.2}$$

so that the boundary deformation acts on the brittle part Ω_B only through $\Omega\backslash\overline{\Omega}_B$, which can be regarded as a *layer of unbreakable material.* Notice that this condition does not imply that $\Omega_B \subset\subset \Omega_D$, but only that the brittle part $\overline{\Omega}_B$ does not meet the Dirichlet boundary $\partial_D\Omega = \Omega_D \cap \partial\Omega$. As a consequence, there cannot be interfacial cracks on $\partial_D\Omega$. We cannot avoid (1.2) for a technical reason, related to the non-interpenetration condition, that will appear in the proof of Lemma 4.1 about crack transfer.

A *crack* is represented in the reference configuration by a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set $\Gamma \mathrel{\tilde{\subset}} \overline{\Omega}_B \cap \Omega_D$ with $\mathcal{H}^{n-1}(\Gamma) < +\infty$. The collection of *admissible cracks* is given by

$$\mathcal{R} := \left\{\Gamma\colon (\mathcal{H}^{n-1}, n-1)\text{-rectifiable, } \Gamma \mathrel{\tilde{\subset}} \overline{\Omega}_B \cap \Omega_D\,,\ \mathcal{H}^{n-1}(\Gamma) < +\infty\right\}. \tag{1.3}$$

According to Griffith's theory, we assume that the *energy spent to produce the crack* $\Gamma \in \mathcal{R}$ is given by

$$\mathcal{K}(\Gamma) := \int_\Gamma \kappa(x, \nu_\Gamma(x))\,\mathrm{d}\mathcal{H}^{n-1}(x)\ , \tag{1.4}$$

where ν_Γ is a unit normal vector field on Γ and $\kappa\colon (\overline{\Omega}_B \cap \Omega_D) \times \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function such that

(K1) $\nu \mapsto \kappa(x,\nu)$ is a norm on $\mathbb{R}^n$ for every $x \in \overline{\Omega}_B \cap \Omega_D$,
(K2) $\kappa_1\,|\nu| \,\le\, \kappa(x,\nu) \,\le\, \kappa_2\,|\nu|$ for every $(x,\nu) \in (\overline{\Omega}_B \cap \Omega_D) \times \mathbb{R}^n$,

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$; as a consequence, we have

$$\kappa_1\,\mathcal{H}^{n-1}(\Gamma) \,\le\, \mathcal{K}(\Gamma) \,\le\, \kappa_2\,\mathcal{H}^{n-1}(\Gamma)\ . \tag{1.5}$$

To simplify the exposition of auxiliary results, we extend κ to $\Omega_D \times \mathbb{R}^n$ by setting $\kappa(x,\nu) := \kappa_2\,|\nu|$ if $x \in \Omega_D\backslash\overline{\Omega}_B$, and we define $\mathcal{K}(\Gamma)$ by (1.4) for every countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable subset Γ of $\mathbb{R}^n$.

1.3. **Admissible deformations.** A deformation of Ω_D is represented by a function u in $SBV(\Omega_D;K)$, which is defined as the set of functions $u \in SBV(\Omega_D;\mathbb{R}^n)$ such that $u(x) \in K$ for a.e. $x \in \Omega_D$. With this definition we are requiring that every deformation of the body remains in the container K. We assume that there is $\Gamma \in \mathcal{R}$ such that $S(u) \tilde{\subset} \Gamma$, so $S(u) \tilde{\subset} \overline{\Omega}_B \cap \Omega_D$.

Furthermore, we require a condition of non-interpenetration of matter in the sense of Ciarlet-Nečas [10], a notion developed first for Sobolev mappings and recently generalized to SBV functions by Giacomini-Ponsiglione [21].

Definition 1.1. A function $u \in SBV(\Omega_D;K)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if the following hold:

(CN1) u *preserves orientation*, i.e., for a.e. $x \in \Omega_D$, $\det \nabla u(x) > 0$;
(CN2) u is *a.e.-injective*, i.e., there exists a set $N \subset \Omega_D$, with $\mathcal{L}^n(N) = 0$, such that u is injective on $\Omega_D \backslash N$.

The prescribed deformation of $\Omega_D \backslash \overline{\Omega}$ is given by a function $\psi \in W^{1,1}\big(\Omega_D \backslash \overline{\Omega};K\big)$. The Dirichlet condition on u takes the form $u = \psi$ a.e. in $\Omega_D \backslash \overline{\Omega}$, i.e., we prescribe the deformation on the whole volume $\Omega_D \backslash \overline{\Omega}$ and not only on $\partial_D \Omega$. On the latter set the equality $u = \psi$ is satisfied in the sense of traces, because by (1.2) u is of class $W^{1,1}$ in the neighbourhood $\Omega_D \backslash \overline{\Omega}_B$ of $\partial_D \Omega$.

Then we define the set of *admissible deformations*, corresponding to a crack $\Gamma \in \mathcal{R}$ and a Dirichlet datum $\psi \in W^{1,1}\big(\Omega_D \backslash \overline{\Omega};K\big)$, as

$$\begin{aligned} AD(\psi,\Gamma) := \Big\{ u \in SBV(\Omega_D;K) : u \text{ satisfies (CN1), (CN2),} \\ u|_{\Omega_D \backslash \overline{\Omega}} = \psi\,, \text{ and } S(u) \tilde{\subset} \Gamma \Big\}. \end{aligned} \tag{1.6}$$

If $AD(\psi,\Gamma) \neq \emptyset$, the equality $u|_{\Omega_D \backslash \overline{\Omega}} = \psi$ implies in particular that ψ satisfies (CN1) and (CN2) in $\Omega_D \backslash \overline{\Omega}$. Moreover, if $u \in AD(\psi,\Gamma)$ there exists $N \subset \Omega_D$ with $\mathcal{L}^n(N) = 0$ such that $u(\Omega \backslash N)$ does not intersect $\psi((\Omega_D \backslash \overline{\Omega}) \backslash N)$.

Remark 1.2. The first difference from the model of [12] is the non-interpenetration requirement for the admissible deformations; this suggests to formulate the boundary conditions in terms of the leading body $\Omega_D \backslash \Omega$. Furthermore, we introduce the confinement condition $u(x) \in K$, in order to simplify the functional framework (SBV instead of $GSBV$). Another relevant difference is given by the assumptions on the bulk energy, which will be stated in the next section.

1.4. **Bulk energy.** We present the hypotheses on the bulk energy, which will allow us to deal with the case of finite elasticity. The relevant assumptions were studied in Ball [4], Francfort-Mielke [19], and Fusco-Leone-March-Verde [20].

Given a crack $\Gamma \in \mathcal{R}$, we suppose that the uncracked part $\Omega \backslash \Gamma$ is hyperelastic and that the *bulk energy* on $\Omega \backslash \Gamma$ of any deformation $u \in SBV(\Omega_D;K)$ with $S(u) \tilde{\subset} \Gamma$ can be written as

$$\mathcal{W}(u) := \int_{\Omega \backslash \Gamma} W(x, \nabla u(x))\, \mathrm{d}x = \int_{\Omega} W(x, \nabla u(x))\, \mathrm{d}x\,, \tag{1.7}$$

where $W \colon \Omega \times \mathbb{M}^{n \times n} \to [0,+\infty]$ is independent of Γ and satisfies the following properties:

(W0) *Frame indifference:* for every $(x,A) \in \Omega \times \mathbb{M}^{n \times n}$

$$W(x,QA) = W(x,A) \text{ for every } Q \in SO_n\,;$$

(W1) *Polyconvexity:* there exists a function $\widetilde{W}\colon \Omega\times\mathbb{R}^\tau \to [0,+\infty]$ such that $x\mapsto \widetilde{W}(x,\xi)$ is $\mathcal{L}^n$-measurable on Ω for every $\xi\in\mathbb{R}^\tau$, $\xi\mapsto\widetilde{W}(x,\xi)$ is continuous and convex on $\mathbb{R}^\tau$ for every $x\in\Omega$, and

$$W(x,A)=\widetilde{W}(x,M(A)) \quad \text{for every } (x,A)\in\Omega\times\mathbb{M}^{n\times n}\,,$$

where $M(A):=(\mathrm{adj}_1A,\dots,\mathrm{adj}_nA)$ is the vector (of dimension $\tau:=\tau_1+\cdots+\tau_n$) composed of all minors of A;

(W2) *Finiteness and regularity:* for every $x\in\Omega$ we have

$$W(x,A)<+\infty \quad\Longleftrightarrow\quad A\in GL_n^+$$

and $A\mapsto W(x,A)$ is of class C^1 on GL_n^+.

Furthermore, we require that there exist some constants $\beta_W^0\geq 0$, $\beta_W^1,\dots,\beta_W^n>0$, $c_W^0\geq 0$, $c_W^1>0$, and some exponents $p_1,p_2,\dots,p_n$, such that for every $x\in\Omega$:

(W3) *Bound at identity:* we have $W(x,I)\leq c_W^0$;

(W4) *Lower growth condition:* for every $A\in\mathbb{M}^{n\times n}$

$$W(x,A)\geq\sum_{j=1}^n \beta_W^j\left|\mathrm{adj}_jA\right|^{p_j}-\beta_W^0\,,$$

with

$$p_1\geq 2\,,\quad p_j\geq p_1':=\frac{p_1}{p_1-1}\ \text{for } j=2,\dots,n-1\,,\quad p_n>1\,;$$

(W5) *Multiplicative stress estimate:* for every $A\in GL_n^+$

$$\left|A^{\mathrm{T}}\,\mathrm{D}_AW(x,A)\right|\leq c_W^1(W(x,A)+c_W^0)\,;$$

(W6) *Continuity of Kirchhoff stress:* for every $\varepsilon>0$ there exists $\delta>0$, independent of x, such that for every $A\in GL_n^+$ and $B\in GL_n^+$ with $|B-I|<\delta$

$$\left|\mathrm{D}_AW(x,BA)\,(BA)^{\mathrm{T}}-\mathrm{D}_AW(x,A)\,A^{\mathrm{T}}\right|\leq\varepsilon\,(W(x,A)+c_W^0)\,.$$

Henceforth, we will set $p:=p_1$.

Remark 1.3. Hypotheses (W0), (W1), (W2), and (W5) were studied in [4], while (W6) was used in [19]. Assumptions (W5) and (W6) involve two stress tensors:

$$K(x,A):=\mathrm{D}_A\,W(x,A)A^{\mathrm{T}}\,, \tag{1.8}$$

sometimes called *Kirchhoff stress tensor*, and

$$L(x,A):=A^{\mathrm{T}}\,\mathrm{D}_AW(x,A)\,, \tag{1.9}$$

which appears in the expression of the so called *energy–momentum tensor*

$$W(x,A)\,I-A^{\mathrm{T}}\,\mathrm{D}_AW(x,A)\,. \tag{1.10}$$

In all these formulas, $\mathrm{D}_AW(x,A)$ denotes the matrix whose entries are the partial derivatives of W with respect to the corresponding entries of A.

Remark 1.4. Hypotheses (W1) and (W4) guarantee lower semicontinuity for $\mathcal{W}$, thanks to Theorem 3.1 below, due to [20]. When $p>n$, it suffices to suppose $W(x,A)\geq\beta_W^1\,|A|^p$, instead of (W4), thanks to a result by Ambrosio [2, Corollary 4.9]. Notice that, if $\mathcal{W}(u)<+\infty$ for a function $u\in SBV(\Omega_D;K)$, then $u\in SBV^p(\Omega_D;K)$ by (W4).

In the next proposition, we state a consequence of hypothesis (W5) for L; moreover, we highlight the counterpart of (W5) in the case of K. For the proofs and a deeper discussion, we refer to [4, Section 2.4].

Proposition 1.5. *Let W satisfy* (W5)*. Then there exists $\gamma \in (0,1)$ such that, for every $(x, A) \in \Omega \times GL_n^+$ and every $B \in GL_n^+$ with $|B - I| < \gamma$,*

$$W(x, AB) + c_W^0 \leq \frac{n}{n-1} \left(W(x, A) + c_W^0 \right) . \tag{1.11}$$

Moreover, if W satisfies also (W0)*, then for every $(x, A) \in \Omega \times GL_n^+$*

$$\left| \mathrm{D}_A W(x, A)\, A^{\mathrm{T}} \right| \leq \left| A^{\mathrm{T}}\, \mathrm{D}_A W(x, A) \right| ,$$

so that

$$\left| \mathrm{D}_A W(x, A)\, A^{\mathrm{T}} \right| \leq c_W^1 \left(W(x, A) + c_W^0 \right) . \tag{1.12}$$

Remark 1.6. There are examples of functions satisfying (1.12) but not (W5); instead, these properties are equivalent when the material is isotropic, i.e.,

$$W(x, A\, Q) = W(x, A) \text{ for every } Q \in SO_n . \tag{1.13}$$

If either (W5) or (1.12) holds, there exists $c_W^2 > 0$ such that for every $(x, A) \in \Omega \times GL_n^+$

$$W(x, A) \leq c_W^2 \left(|A|^s + \left| A^{-1} \right|^s \right) , \tag{1.14}$$

where $s := n\, c_W^1$. All these properties can be found in [4].

Remark 1.7. In [19, Proposition 5.2] it is proven that K satisfies (W6) whenever (1.12) holds, all entries $K_{ij}(x, A)$ of $K(x, A)$ are differentiable in A, and there exists $c_W^3 > 0$ such that

$$|\mathrm{D}_A\, K_{ij}(x, A) \colon (CA)| \leq c_W^3 (W(x, A) + c_W^0)\, |C| \tag{1.15}$$

for every $C \in \mathbb{M}^{n \times n}$ and $(x, A) \in \Omega \times GL_n^+$.

Example 1.8 (Ogden Materials)**.** An important class of hyperelastic isotropic materials in dimension $n = 3$ was studied by Ogden in 1972 [29, 30] to describe the behaviour of natural rubbers. These materials provide a classical example in finite elasticity [9, Section 4.10]; the strain-energy associated with $A \in GL_3^+$ is given by

$$W(A) = \sum_{i=1}^{M} a_i\, |A|^{\gamma_i} + \sum_{j=1}^{N} b_j\, |\mathrm{cof} A|^{\delta_j} + h(\det A) ,$$

where several material parameters appear: $M, N \geq 1$, $a_i, b_j > 0$, $\gamma_i, \delta_j \geq 1$. Moreover, $h \colon (0, \infty) \to \mathbb{R}$ is a convex function satisfying $h(t) \to +\infty$ as $t \to 0^+$. Here, $\mathrm{cof} A := (\det A)\, A^{-\mathrm{T}}$ stands for the cofactor matrix of A.

In general, the strain-energy considered in this example is polyconvex and satisfies inequality (W4) [9, 20]. Moreover, in [4] it is proven that W satisfies (W5) and (1.12), whenever the following growth condition holds for every $t > 0$, with $C > 0$:

$$|t\, h'(t)| \leq C(h(t) + 1) .$$

Now we show a simple example of Ogden material satisfying all the properties we are requiring for W. A similar example is presented in [19], in the case $p > n$, where Ambrosio's result is sufficient to prove lower semicontinuity, so that one can take $\beta_W^1 = \cdots = \beta_W^{n-1} = 0$ in (W4). In our example $\beta_W^j > 0$ for every j, which allows us to consider the case $2 \leq p \leq n$, where Ambrosio's semicontinuity result cannot be applied. Another example can be found in [24].

Let $n = 3$ again and take, for $A \in \mathbb{M}^{3 \times 3}$,

$$W(A) := \begin{cases} \beta_W^1\, |A|^{p_1} + \beta_W^2\, |\mathrm{cof} A|^{p_2} + \beta_W^3\, |\det A|^{p_3} + \gamma\, |\det A|^{-q} & \text{if } \det A > 0 , \\ +\infty & \text{otherwise,} \end{cases}$$

where $p_1 = p \geq 2$, $p_2 \geq p'$, $p_3 > 1$, and $\beta_W^j > 0$, as in (W4), and $q > 0$, $\gamma > 0$.

Let us verify that properties (W0–6) hold: polyconvexity (W1) and lower growth estimate (W4) are clear by construction; moreover, one can see that W satisfies frame indifference (W0), local non-interpenetration (W2), and isotropy (1.13). To check the other properties, we must compute the derivative of W for $A \in GL_3^+$; for this, we need the expression of the differential $\mathrm{d}_A \operatorname{cof} A$, considered as a linear map from $\mathbb{M}^{3\times 3}$ into $\mathbb{M}^{3\times 3}$:

$$\mathrm{d}_A \operatorname{cof} A\,[B] = \left[\operatorname{tr}\left(A^{-1}B\right)\, I - A^{-\mathrm{T}}B^{\mathrm{T}}\right] \operatorname{cof} A\,,$$

whence we conclude that $\mathrm{d}_A \operatorname{cof} A$ is symmetric, i.e.,

$$\mathrm{d}_A \operatorname{cof} A\,[B] : C = \mathrm{d}_A \operatorname{cof} A\,[C] : B\,. \tag{1.16}$$

Then we see that

$$\left(\mathrm{d}_A \operatorname{cof} A\,[\operatorname{cof} A]\right) A^{\mathrm{T}} = |\operatorname{cof} A|^2\, I - \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}}\,, \tag{1.17}$$

$$\mathrm{d}_A \operatorname{cof} A\,[CA] = \left(\operatorname{tr}(C)\, I - C^{\mathrm{T}}\right) \operatorname{cof} A\,. \tag{1.18}$$

Using (1.8), (1.16), and (1.17), we get

$$\begin{aligned} K(A) = {} & \beta_W^1 p_1\, |A|^{p_1-2}\, AA^{\mathrm{T}} + \beta_W^2 p_2 \left[|\operatorname{cof} A|^{p_2}\, I - |\operatorname{cof} A|^{p_2-2} \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}}\right] + \\ & + \left(\beta_W^3 p_3\, |\det A|^{p_3} - \gamma q\, |\det A|^{-q}\right) I\,. \end{aligned}$$

We compute its differential $\mathrm{d}_A K(A)$, considered as a linear map from $\mathbb{M}^{3\times 3}$ into $\mathbb{M}^{3\times 3}$. Using (1.18) we obtain

$$\begin{aligned} \mathrm{d}_A K(A)\,[CA] = {} & \beta_W^1\, p_1 \left[(p_1-2)\, |A|^{p_1-4} \left(AA^{\mathrm{T}} : C\right) AA^{\mathrm{T}} + |A|^{p_1-2} \left(CAA^{\mathrm{T}} + AA^{\mathrm{T}}C\right)\right] + \\ & + \beta_W^2\, p_2{}^2 \left[|\operatorname{cof} A|^{p_2} \operatorname{tr}(C) - |\operatorname{cof} A|^{p_2-2} \left(\operatorname{cof} A^{\mathrm{T}}\, \operatorname{cof} A\right) : C\right] I + \\ & - \beta_W^2\, p_2 (p_2-2)\, |\operatorname{cof} A|^{p_2-2} \operatorname{tr}(C)\, \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}} + \\ & + \beta_W^2\, p_2 (p_2-2)\, |\operatorname{cof} A|^{p_2-4} \left[\left(\operatorname{cof} A^{\mathrm{T}}\, \operatorname{cof} A\right) : C\right]\, \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}} + \\ & - \beta_W^2\, p_2\, |\operatorname{cof} A|^{p_2-2} \left[\operatorname{tr}(C)\, I - C^{\mathrm{T}}\right] \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}} + \\ & - \beta_W^2\, p_2\, |\operatorname{cof} A|^{p_2-2} \operatorname{cof} A\, \operatorname{cof} A^{\mathrm{T}} \left[\operatorname{tr}(C)\, I - C\right] + \\ & + \left[\beta_W^3\, p_3\, |\det A|^{p_3} - \gamma q\, |\det A|^{-q}\right] \operatorname{tr}(C)\, I\,. \end{aligned}$$

The formulas for $K(A)$ and $\mathrm{d}_A K(A)$ immediately show that (1.12) and (1.15) hold; then, by Remarks 1.6 and 1.7, (W5) and (W6) hold.

With the same procedure one can treat *Mooney-Rivlin materials* [9], where

$$W(A) := \begin{cases} a\, |A|^2 + b\, |\operatorname{cof} A|^2 + c\, |\det A|^2 - d \log \det A & \text{if } \det A > 0\,, \\ +\infty & \text{otherwise,} \end{cases}$$

where a, b, c, d are positive constants. Also in this case, because of the exponent $p = 2$, Ambrosio's result does not apply.

1.5. **Prescribed deformations.** We prescribe a time-dependent deformation of $\Omega_D \backslash \Omega$, requiring that $u(x) = \psi(t,x)$ for a.e. $x \in \Omega_D \backslash \Omega$, at every time $t \in [0,1]$. For technical reasons, we have to assume that $x \mapsto \psi(t,x)$ is defined for every $x \in K$, takes values in K, and has an inverse function on K, denoted by $y \mapsto \phi(t,y)$. This determines two functions

$$\psi,\ \phi\colon [0,1] \times K \to K\,.$$

With a small abuse of notation, the functions $x \mapsto \psi(t,x)$ and $y \mapsto \phi(t,y)$ are denoted by $\psi(t)\colon K \to K$ and $\phi(t)\colon K \to K$, respectively. At each time t they satisfy

$$\text{(BC1)} \qquad \psi(t) \circ \phi(t) = I = \phi(t) \circ \psi(t)\,,$$

where I denotes the identical function in K.

We require that for every $i, j = 1, \dots n$

$$\text{(BC2)} \qquad \mathrm{D}_t\psi\,,\ \mathrm{D}_{x_i}\psi\,,\ \mathrm{D}_{x_i}\mathrm{D}_{x_j}\psi\,,\ \mathrm{D}_t\mathrm{D}_{x_i}\psi \text{ exist, continuous on } [0,1]\times K$$

and

$$\text{(BC3)} \qquad \mathrm{D}_t\phi\,,\ \mathrm{D}_{y_i}\phi\,,\ \mathrm{D}_{y_i}\mathrm{D}_{y_j}\phi\,,\ \mathrm{D}_t\mathrm{D}_{y_i}\phi \text{ exist, continuous on } [0,1]\times K\,.$$

This implies that the mixed derivative $\mathrm{D}_{x_i}\mathrm{D}_t\psi$ exists and coincides with $\mathrm{D}_t\mathrm{D}_{x_i}\psi$; the same is true for ϕ. We use the following notation: $\nabla\psi$ and $\nabla\phi$ are the Jacobian matrices with respect to x or y; moreover, $\dot\psi := \mathrm{D}_t\psi$, $\nabla\dot\psi := \nabla\mathrm{D}_t\psi = \mathrm{D}_t\nabla\psi$, and the same for ϕ.

We need a uniform bound on the energy of the prescribed deformation: we suppose that there exists a constant M such that

$$\text{(BC4)} \qquad W(x, \nabla\psi(t,x)) \le M$$

for every $(t,x) \in [0,1]\times\Omega$ (for example, this holds when $\psi(t) = I$). This assumption, together with (W2), gives

$$\det\nabla\psi(t,x) > 0 \text{ for a.e. } x \in K\,.$$

Since by (BC1) and (BC2) $\det\nabla\psi(t,x) \neq 0$ for every $t \in [0,1]$ and $x \in K$, by continuity one has

$$\det\nabla\psi(t,x) > 0 \text{ for every } x \in K\,, \tag{1.19}$$

which in turn implies

$$\det\nabla\phi(t,y) > 0 \text{ for every } y \in K\,. \tag{1.20}$$

Notice that (1.19) and the invertibility of $\psi(t)$ imply that $\psi(t)$ satisfies the Ciarlet-Nečas condition; as $S(\psi(t)) = \emptyset$, this implies that $\psi(t) \in AD(\psi(t),\Gamma)$ for every $\Gamma \in \mathcal{R}$.

2. Evolution of stable equilibria

The aim of this paper is to study the evolution of stable equilibria for the physical system introduced in the previous section: an elastic body with cracks, subjected to a general strain energy, compatible with the non-interpenetration hypotheses (W2).

In the present section, we define the notion of *incrementally-approximable quasistatic evolution of global minimizers* for the total energy $\mathcal{E}$. Our main results are the existence of such a quasistatic evolution with prescribed initial conditions (Theorem 2.18) and the analysis of its properties (Theorem 2.19).

2.1. **Minimum energy configurations.** We begin by discussing the notion of stable equilibrium, first considering only the bulk energy $\mathcal{W}$. For a fixed time $t \in [0,1]$ and a given crack $\Gamma \in \mathcal{R}$, a deformation u corresponding to an equilibrium is a critical point of the functional $\mathcal{W}$ on the set $AD(\psi(t),\Gamma)$ defined in (1.6). Among such critical points, we select the minimum points of the problem

$$\min_{u \in AD(\psi(t),\Gamma)} \mathcal{W}(u)\,, \tag{2.1}$$

which are called the *minimum energy deformations at time t with crack* Γ. Their existence is guaranteed by the following theorem, which will be proven in Section 3.3.

Theorem 2.1 (Minimization of the elastic energy). *Let $\mathcal{W}$ satisfy* (W0–6)*. Consider the prescribed deformations defined in* (BC1–4)*. Then for every $t \in [0,1]$ and every $\Gamma \in \mathcal{R}$ the minimum problem* (2.1) *has a solution.*

Next, we define the total energy

$$\mathcal{E}(u,\Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma)\,. \tag{2.2}$$

In Griffith's theory, an *equilibrium configuration* at a fixed time $t \in [0,1]$ is an admissible configuration $(u(t),\Gamma(t))$ which is a "critical point" of the functional $\mathcal{E}(u,\Gamma)$ on the set of configurations (u,Γ) with $\Gamma \in \mathcal{R}$, $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and $u \in AD(\psi(t),\Gamma)$. Unfortunately, the definition of "critical point" in this context has never been made mathematically precise.

Following [18], among these equilibrium configurations we will consider only *minimum energy configurations*, which are defined as those admissible configurations $(u(t),\Gamma(t))$, with $\Gamma(t) \in \mathcal{R}$ and $u(t) \in AD(\psi(t),\Gamma(t))$, such that the unilateral minimality condition holds:

$$\mathcal{E}(u(t),\Gamma(t)) \le \mathcal{E}(u,\Gamma) \tag{2.3}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and every $u \in AD(\psi(t),\Gamma)$.

The next theorem ensures that for every $t \in [0,1]$ and for every initial datum $\Gamma_0 \in \mathcal{R}$ there exists at least a minimum energy configuration $(u(t),\Gamma(t))$ such that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma(t)$; the proof is in Section 3.3.

Theorem 2.2 (Minimization of the total energy). *Let $\mathcal{E}$ be the energy defined in* (2.2)*, where $\mathcal{W}$ satisfies* (W0–6) *and $\mathcal{K}$ satisfies* (K1–2)*. Consider the prescribed deformations defined in* (BC1–4)*. Then, for every $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem*

$$\min\left\{\mathcal{E}(u,\Gamma)\colon \Gamma \in \mathcal{R},\ \Gamma_0 \mathrel{\tilde{\subset}} \Gamma,\ u \in AD(\psi(t),\Gamma)\right\} \tag{2.4}$$

has a solution.

2.2. **The discrete-time problems.** To define a quasistatic evolution, we employ a standard method for rate-independent processes [27], developed in [18, 15, 12, 17] for problems in fracture mechanics: first, we consider a time-discretization of the problem and find some *incremental approximate solutions*; the desired *incrementally-approximable quasistatic evolution* will then be the limit of the discrete solutions.

Let us fix a sequence of subdivisions $\{t_k^i\}_{0\le i\le k}$ of the interval $[0,1]$, with

$$0 = t_k^0 < t_k^1 < \dots < t_k^{k-1} < t_k^k = 1 \tag{2.5}$$

and

$$\lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0\,. \tag{2.6}$$

We will call such a sequence a *time discretization.*

As a datum of the problem, we are given an initial condition (u_0,Γ_0), satisfying $\Gamma_0 \in \mathcal{R}$, $u_0 \in AD(\psi(0),\Gamma_0)$, and the unilateral minimality condition

$$\mathcal{E}(u_0,\Gamma_0) \le \mathcal{E}(u,\Gamma) \tag{2.7}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma$ and every $u \in AD(\psi(0),\Gamma)$.

For every time subdivision, we define a corresponding incremental approximate solution, whose existence is guaranteed by Theorem 2.2.

Definition 2.3. Fix $k \in \mathbb{N}$. An *incremental approximate solution* for $\mathcal{E}$ corresponding to the time subdivision $\{t_k^i\}_{0\le i\le k}$ with initial datum (u_0,Γ_0) is a function $t \mapsto (u_k(t),\Gamma_k(t))$, such that

(a) $(u_k(0), \Gamma_k(0)) = (u_0, \Gamma_0)$;
(b) $u_k(t) = u_k(t_k^i)$ and $\Gamma_k(t) = \Gamma_k(t_k^i)$ for $t \in \big[t_k^i, t_k^{i+1}\big)$ and $i = 0, \dots, k-1$;
(c) for $i = 1, \dots, k$, $(u_k(t_k^i), \Gamma_k(t_k^i))$ is a solution of

$$\min\big\{\mathcal{E}(u,\Gamma)\colon \Gamma \in \mathcal{R},\ \Gamma(t_k^{i-1}) \,\tilde{\subset}\, \Gamma,\ u \in AD(\psi(t_k^i), \Gamma)\big\}\,. \tag{2.8}$$

Notice that, if $t \mapsto (u_k(t), \Gamma_k(t))$ is an incremental approximate solution, by the minimality and by (BC4) we have $\mathcal{E}(u_k(t), \Gamma_k(t)) < +\infty$ for every t, hence $u_k \in SBV^p(\Omega_D; K)$ by (W4), with $p = p_1$. To study the limit of these objects, we recall a notion of convergence for sequences in $SBV^p(\Omega_D; \mathbb{R}^m)$, usually called weak* convergence, in spite of the fact that it does not involve any predual space.

Definition 2.4. A sequence u_k *converges to* u *weakly** *in* $SBV^p(\Omega_D; \mathbb{R}^m)$ if

- $u_k, u \in SBV^p(\Omega_D; \mathbb{R}^m)$;
- $u_k \to u$ in measure;
- $\|u_k\|_{L^\infty(\Omega_D;\mathbb{R}^m)}$ is bounded uniformly with respect to k;
- $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega_D; \mathbb{M}^{m\times n})$;
- $\mathcal{H}^{n-1}(S(u_k))$ is bounded uniformly with respect to k.

As for the cracks, we need a notion of convergence for sets, called σ^p-convergence, introduced in [12].

Definition 2.5. A sequence Γ_k σ^p*-converges to* Γ if $\Gamma_k, \Gamma \subset \Omega_D$, $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded uniformly with respect to k, and the following conditions are satisfied:

- if u_j converges weakly* to u in $SBV^p(\Omega_D)$ and $S(u_j) \,\tilde{\subset}\, \Gamma_{k_j}$ for some sequence $k_j \to \infty$, then $S(u) \,\tilde{\subset}\, \Gamma$;
- there exist a function $u \in SBV^p(\Omega_D)$ and a sequence u_k converging to u weakly* in $SBV^p(\Omega_D)$ such that $S(u) \cong \Gamma$ and $S(u_k) \,\tilde{\subset}\, \Gamma_k$ for every k.

2.3. **Formulation with time-independent prescribed deformations.** Now we pass to an alternative formulation of the problem, where the Dirichlet conditions are time-independent, whilst the time-dependence is transferred to the energy terms; this approach is based on [19]. We look for a solution $u \in AD(\psi(t), \Gamma)$ to (2.4) of the form $u = \psi(t) \circ z$, with $z \in SBV(\Omega_D; K)$; this request implies $z \in AD(I, \Gamma)$. The chain rule in BV [3, Theorem 3.96] gives $\nabla u(x) = \nabla\psi(t, z(x))\, \nabla z(x)$ for a.e. $x \in \Omega_D$, so that we define the auxiliary volume energy

$$\mathcal{V}(t)(z) := \int_\Omega V(t, x, z(x), \nabla z(x))\, \mathrm{d}x\,, \tag{2.9}$$

where

$$V(t, x, y, A) := W(x, \nabla\psi(t, y)\, A)\,. \tag{2.10}$$

Hence,

$$\mathcal{W}(u) = \mathcal{V}(t)(\phi(t) \circ u)\,, \quad \mathcal{V}(t)(z) = \mathcal{W}(\psi(t) \circ z)\,. \tag{2.11}$$

This leads to introduce a class of functions $V\colon [0,1]\times\Omega\times K\times\mathbb{M}^{n\times n} \to [0,+\infty]$ satisfying the following properties:

(V1) *Polyconvexity:* there exists a function $\widetilde{V}\colon [0,1] \times \Omega \times K \times \mathbb{R}^\tau \to [0,+\infty]$ such that $x \mapsto \widetilde{V}(t, x, y, \xi)$ is $\mathcal{L}^n$-measurable on Ω for every $(t, y, \xi) \in [0,1] \times K \times \mathbb{R}^\tau$, $(t, y, \xi) \mapsto \widetilde{V}(t, x, y, \xi)$ is continuous on $[0,1] \times K \times \mathbb{R}^\tau$ for every $x \in \Omega$, $\xi \mapsto \widetilde{V}(t, x, y, \xi)$ is convex on $\mathbb{R}^\tau$ for every $(t, x, y) \in [0,1] \times \Omega \times K$, and

$$V(t, x, y, A) = \widetilde{V}(t, x, y, M(A)) \quad \text{for every } (t, x, y, A) \in [0,1] \times \Omega \times K \times \mathbb{M}^{n\times n}\,,$$

where $M(A)$ is defined as in (W1);

(V2) *Finiteness and regularity:* for every $(t,x,y)\in[0,1]\times\Omega\times K$ we have

$$V(t,x,y,A)<+\infty \quad\Longleftrightarrow\quad A\in GL_n^+\,,$$

and $(t,y,A)\mapsto V(t,x,y,A)$ is of class C^1 on $[0,1]\times K\times GL_n^+$ for every $x\in\Omega$;

furthermore, there exist some constants $\beta_V^0\geq 0$, $\beta_V^1,\dots,\beta_V^n>0$, $c_V^0\geq 0$, $c_V^1>0$, and some exponents $p_1,p_2,\dots,p_n$, such that for every $(t,x,y)\in[0,1]\times\Omega\times K$:

(V3) *Bound at identity:* we have $V(t,x,x,I)\leq c_V^0$;

(V4-5) *Dependence on the matricial term:* $A\mapsto V(t,x,y,A)$ satisfies (W4–5);

(V6) *Estimate on the time derivative:* for every $A\in GL_n^+$

$$|\mathrm{D}_t V(t,x,y,A)|\leq c_V^1(V(t,x,y,A)+c_V^0)\,;$$

(V7) *Continuity of the time derivative:* for every $\varepsilon>0$ there exists $\delta>0$, independent of (t,x,y), such that for every $s\in[0,1]$ with $|t-s|<\delta$ and every $A\in GL_n^+$

$$|\mathrm{D}_t V(t,x,y,A)-\mathrm{D}_t V(s,x,y,A)|\leq \varepsilon\,(V(t,x,y,A)+c_V^0)\,;$$

(V8) *Estimate on spatial derivatives:* for every $A\in GL_n^+$

$$|\mathrm{D}_y V(t,x,y,A)|\leq c_V^1(V(t,x,y,A)+c_V^0)\,.$$

Proposition 2.6. *If* (W0–6) *and* (BC1–4) *hold, then the function* V *defined in* (2.10) *satisfies properties* (V1–8).

Proof. Properties (V1–2) are obvious.

Checking property (V4) reduces to estimate $\left|\mathrm{adj}_j\nabla\psi(t,y)\,A\right|$ from below in terms of $\left|\mathrm{adj}_j A\right|$, for given $t\in[0,1]$, $y\in K$, and $A\in GL_n^+$. Let $B\in GL_n^+$; then

$$\left|\mathrm{adj}_j(BA)\right|\leq\left|\mathrm{adj}_j B\right|\left|\mathrm{adj}_j A\right|\leq C_j\sup_{l,m}|b_{lm}|^j\,\left|\mathrm{adj}_j A\right|\leq C_j\,|B|^j\left|\mathrm{adj}_j A\right|\,,$$

where the first inequality is given by [11, Proposition 5.66], b_{lm} are the elements of B, and $C_j>0$ depends only on n and j. This is equivalent to

$$\left|\mathrm{adj}_j(B^{-1}A)\right|\geq\frac{1}{C_j}\,|B|^{-j}\left|\mathrm{adj}_j A\right|\,.$$

For $B^{-1}=\nabla\psi(t,y)$, employing the hypotheses of boundedness (BC2), (BC3), and the invertibility condition (1.19), we conclude, modifying the constants properly.

We take

$$c_V^0\geq c_W^0\vee M\,,\quad c_V^1\geq\max_{[0,1]\times K}\Big\{c_W^1,\ 1+|\dot\psi|,\ |\nabla\psi|\,,\ c_W^1\,|\nabla\phi|\,|\nabla\dot\psi|,\ c_W^1\,|\nabla\phi|\,|\nabla^2\psi|\Big\}\,,$$

where M is the constant of (BC4). Then (V3) comes from (BC4), while (V5), (V6), and (V8) follow from (W5), (BC2), and (BC3), using the following consequence of (1.12): for every $(x,A)\in\Omega\times GL_n^+$,

$$\left|\mathrm{D}_A W(x,\nabla\psi(t,y)A)\,A^{\mathrm{T}}\right|\leq c_W^1\left(W(x,\nabla\psi(t,y)A)+c_W^0\right)|\nabla\phi(t,\psi(t,y))|\,.$$

Similarly, (V7) follows from (W6), thanks again to (1.12) and to the properties of ψ (see also [19, Lemma 5.5]). ☐

Remark 2.7. Frame indifference is not preserved under (2.10).

The previous proposition allows us to leave the setting introduced in Section 1 and consider the more general class of functions satisfying (V1–8). Here we underline some consequences of these properties.

Remark 2.8. Property (V5) implies (1.11) for V. Furthermore, (V6) gives, via the Gronwall Lemma,

$$V(t_2,x,y,A)+c_V^0 \le (V(t_1,x,y,A)+c_V^0)\, e^{c_V^1|t_2-t_1|} \tag{2.12}$$

for every $t_1,t_2\in[0,1]$ and $(x,y,A)\in\Omega\times K\times GL_n^+$, which ensures the uniform continuity of $t\mapsto V(t,x,y,A)$ on the sublevels of V. Analogously, (V8) implies

$$V(t,x,y_2,A)+c_V^0 \le (V(t,x,y_1,A)+c_V^0)\, e^{c_V^1|y_2-y_1|} \tag{2.13}$$

for every $y_1,y_2\in K$ and $(t,x,A)\in[0,1]\times\Omega\times GL_n^+$.

Estimate (2.12) has the following consequence: if $\mathcal{V}(t_0)(z)<+\infty$ for a fixed time $t_0\in[0,1]$ and a function $z\in SBV(\Omega_D;K)$, then $\mathcal{V}(t)(z)<+\infty$ for every $t\in[0,1]$; then, by (V6), $t\mapsto\mathcal{V}(t)(z)$ is well defined and C^1 on [0,1], and its derivative $\dot{\mathcal{V}}(t)(z)$ is given by

$$\dot{\mathcal{V}}(t)(z)=\int_\Omega \mathrm{D}_t V(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,. \tag{2.14}$$

We regard $\dot{\mathcal{V}}(t)$ as a functional defined on

$$\mathcal{U}_{\mathcal{V}} := \big\{z\in SBV(\Omega_D;K) : \mathcal{V}(0)(z)<+\infty\big\}\,. \tag{2.15}$$

Finally, we define

$$\mathcal{F}(t)(z,\Gamma):=\mathcal{V}(t)(z)+\mathcal{K}(\Gamma)\,. \tag{2.16}$$

Using the new formulation, (2.4) is equivalent to the auxiliary problem

$$\min\big\{\mathcal{F}(t)(u,\Gamma)\colon \Gamma\in\mathcal{R},\ \Gamma_0\,\tilde{\subset}\,\Gamma,\ u\in AD(I,\Gamma)\big\}\,. \tag{2.17}$$

Also in this case, we provide two minimization results, proven in Section 3.3.

Theorem 2.9 (Minimization of the elastic energy)**.** *Let $\mathcal{V}(t)$ satisfy* (V1–8)*. Then for every $t\in[0,1]$ and every $\Gamma\in\mathcal{R}$ the minimum problem*

$$\min_{u\in AD(I,\Gamma)} \mathcal{V}(t)(u) \tag{2.18}$$

has a solution.

Theorem 2.10 (Minimization of the total energy)**.** *Let $\mathcal{F}(t)$ be the energy defined in* (2.16)*, where $\mathcal{V}(t)$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Then, for every $t\in[0,1]$ and $\Gamma_0\in\mathcal{R}$, the minimum problem* (2.17) *has a solution.*

2.4. **Quasistatic evolution.** Let us fix an initial condition (u_0,Γ_0). We suppose that it is a minimum energy configuration at time 0, i.e., $\Gamma_0\in\mathcal{R}$, $u_0\in AD(I,\Gamma_0)$, and

$$\mathcal{F}(0)(u_0,\Gamma_0)\le\mathcal{F}(0)(u,\Gamma) \tag{2.19}$$

for every $\Gamma\in\mathcal{R}$ with $\Gamma_0\,\tilde{\subset}\,\Gamma$ and every $u\in AD(I,\Gamma)$.

We define the notion of incremental approximate solution for $\mathcal{F}(t)$, corresponding to a time subdivision $\big\{t_k^i\big\}_{0\le i\le k}$ (see (2.5) and (2.6)). The existence of such solutions is guaranteed by Theorem 2.10.

Definition 2.11. Fix $k\in\mathbb{N}$. An *incremental approximate solution* for $\mathcal{F}(t)$ corresponding to the time subdivision $\big\{t_k^i\big\}_{0\le i\le k}$ with initial datum (u_0,Γ_0) is a function $t\mapsto(u_k(t),\Gamma_k(t))$, such that

(a) $(u_k(0),\Gamma_k(0))=(u_0,\Gamma_0)$;
(b) $u_k(t)=u_k(t_k^i)$ and $\Gamma_k(t)=\Gamma_k(t_k^i)$ for $t\in\big[t_k^i,t_k^{i+1}\big)$ and $i=0,\dots,k-1$;
(c) for $i=1,\dots,k$, $(u_k(t_k^i),\Gamma_k(t_k^i))$ is a solution of

$$\min\big\{\mathcal{F}(t_k^i)(u,\Gamma)\colon \Gamma\in\mathcal{R},\ \Gamma_k^{i-1}\,\tilde{\subset}\,\Gamma,\ u\in AD(I,\Gamma)\big\}\,. \tag{2.20}$$

An incrementally-approximable quasistatic evolution for (2.17) is the limit of a sequence of incremental approximate solutions, as in the next definition.

Definition 2.12. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally-approximable quasistatic evolution* of minimum energy configurations for problem (2.17) with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\{t_k^i\}_{0 \le i \le k}$, and a corresponding sequence of incremental approximate solutions $t \mapsto (u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0,1]$:

(a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;

(b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$ and $\lim_{k\to\infty} \theta_{k_j}(t) = \limsup_{k\to\infty} \theta_k(t)$, where

$$\theta_k(t) := \dot{\mathcal{V}}(t)(u_k(t)) \,. \tag{2.21}$$

We state the existence result for incrementally-approximable quasistatic evolutions, which will be proven in Section 3.4.

Theorem 2.13 (Existence of quasistatic evolutions)**.** *Let $\mathcal{F}(t)$ be the energy defined in* (2.16)*, where $\mathcal{V}(t)$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let (u_0, Γ_0) be a minimum energy configuration at time 0 as in* (2.19)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0).*

Notice that in the definition of quasistatic evolution we make no measurability assumptions on the function $t \mapsto u(t)$. We will prove later, in Section 6, that there exists a quasistatic evolution such that the function $t \mapsto u(t)$ is strongly measurable, regarded as a function from $[0,1]$ into $SBV^p(\Omega_D; \mathbb{R}^n)$.

The next theorem guarantees that the definition of incrementally-approximable quasistatic evolution fits in with the general scheme of the energy formulation of *rate-independent processes*, developed by Mielke (see [27] and the references therein); for the proof, see Section 5.2.

Theorem 2.14 (Properties of quasistatic evolutions)**.** *For every incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ for $\mathcal{F}(t)$, the following hold:*

(1) Global stability: *for every $t \in [0,1]$ the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(I, \Gamma(t))$, and*

$$\mathcal{F}(t)(u(t), \Gamma(t)) \le \mathcal{F}(t)(v, \Gamma) \tag{2.22}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and every $v \in AD(I, \Gamma)$;

(2) Energy balance: *the function $F(t) := \mathcal{F}(t)(u(t), \Gamma(t))$ is absolutely continuous on $[0,1]$ and its time derivative satisfies*

$$\dot{F}(t) = \dot{\mathcal{V}}(t)(u(t), \Gamma(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,1] \,. \tag{2.23}$$

Remark 2.15. Notice that in these hypotheses $\mathcal{V}(u(t))$ is finite for every t, because I is a competitor in (2.22) and has finite energy by (V3).

In Section 5.3 we provide a further result about the convergence of the energy terms of the incremental approximate solutions: the elastic and the crack energy of an incrementally-approximable quasistatic evolution are the limit of the corresponding energies of the associated sequence of incremental approximate solutions; this holds for the whole sequence and not only for a subsequence.

In order to come back to the original energy $\mathcal{E}$, we compute the partial time derivative $\dot{\mathcal{V}}(t)$ when $\mathcal{V}(t)$ is given by (2.10). The functionals will be defined on

$$\mathcal{U}_{\mathcal{W}} := \big\{ v \in SBV(\Omega_D; K) : \mathcal{W}(v) < +\infty \big\} . \tag{2.24}$$

Fix $t \in [0,1]$; if $u \in \mathcal{U}_{\mathcal{W}}$, then $z := \phi(t) \circ u \in \mathcal{U}_{\mathcal{V}}$, so by (2.10), (2.14), and Remark 2.8 $s \mapsto \mathcal{V}(s)(z)$ is well defined and C^1 on $[0,1]$, with derivative

$$\dot{\mathcal{V}}(s)(z) = \int_\Omega \mathrm{D}_A W(x, \nabla(\psi(s) \circ z)) : \nabla\big(\dot{\psi}(s) \circ z\big) \, \mathrm{d}x \, .$$

For $s = t$, recalling that $u = \psi(t) \circ z$, we conclude that

$$\dot{\mathcal{V}}(t)(\phi(t) \circ u) = \mathcal{P}(t)(u) \, , \tag{2.25}$$

where $\mathcal{P}(t)$ represents the power of the system and is given by

$$\mathcal{P}(t)(v) := \int_\Omega \mathrm{D}_A W(x, \nabla v) : \nabla\big(\dot{\psi}(t) \circ \phi(t) \circ v\big) \, \mathrm{d}x \, . \tag{2.26}$$

Remark 2.16. The integral appearing in the definition of $\mathcal{P}(t)(v)$ is well defined for every v in $\mathcal{U}_{\mathcal{W}}$: indeed, it can be rewritten as

$$\int_\Omega \mathrm{D}_A W(x, \nabla v)(\nabla v)^{\mathrm{T}} : \nabla\big(\dot{\psi}(t) \circ \phi(t)\big)(v) \, \mathrm{d}x \, ,$$

so that the existence of the integral can be deduced from (1.12), (BC2), (BC3), and (2.24).

Furthermore, if W, Ω, K, $u(t)$, and $\Gamma(t)$ are regular enough, we have

$$\mathcal{P}(t)(u(t)) = \int_{\partial_D \Omega} \mathrm{D}_A W(x, \nabla u(t)) \, \nu_\Omega(x) \cdot \dot{\psi}(t) \, \mathrm{d}x \, , \tag{2.27}$$

so that $\mathcal{P}(t)(u(t))$ can be interpreted as the power of the surface forces acting on $\partial_D \Omega$ at time t.

To prove (2.27), one considers the Euler conditions of (2.4), taking into account the reaction forces generated by the confinement constraint K. Formula (2.27) is then obtained multiplying the Euler equations by $\dot{\psi}(t) \circ \phi(t) \circ u(t)$ and integrating by parts, as in [12, Section 3.8]. Indeed, the additional terms due to the reaction forces give no contribution, since they are orthogonal to $\partial\Omega_0$, while $\dot{\psi}(t) \circ \phi(t)$ is tangential at each point of $\partial\Omega_0$.

This discussion leads to the following definition of incrementally-approximable quasistatic evolution for $\mathcal{E}$ with initial condition (u_0, Γ_0), satisfying (2.7).

Definition 2.17. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally-approximable quasistatic evolution* of minimum energy configurations for problem (2.4) with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t)$, a time discretization $\big\{t_k^i\big\}_{0 \le i \le k}$, and a corresponding sequence of incremental approximate solutions $t \mapsto (u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0,1]$:

- (a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;
- (b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j} \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$ and $\lim\limits_{k\to\infty} \eta_{k_j}(t) = \limsup\limits_{k\to\infty} \eta_k(t)$, where

$$\eta_k(t) := \mathcal{P}(t)(u_k(t)) \, . \tag{2.28}$$

Theorems 2.13 and 2.14 have the following counterparts when dealing with $\mathcal{E}$; the proofs follow from (2.11) and (2.25).

Theorem 2.18 (Existence of quasistatic evolutions). *Let $\mathcal{E}$ be the energy defined in* (2.2)*, where $\mathcal{W}$ satisfies* (W0–6) *and $\mathcal{K}$ satisfies* (K1–2). *Consider the prescribed deformations defined in* (BC1–4). *Let (u_0, Γ_0) be a minimum energy configuration at time 0, i.e., assume $\Gamma_0 \in \mathcal{R}$, $u_0 \in AD(\psi(0), \Gamma_0)$, and* (2.7). *Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0).*

Theorem 2.19 (Properties of quasistatic evolutions). *For every incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ for $\mathcal{E}$, the following hold:*

(1) Global stability: *for every $t \in [0,1]$ the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(\psi(t), \Gamma(t))$, and*

$$\mathcal{E}(u(t), \Gamma(t)) \leq \mathcal{E}(v, \Gamma) \tag{2.29}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and every $v \in AD(\psi(t), \Gamma)$;

(2) Energy balance: *the function $E(t) := \mathcal{E}(u(t), \Gamma(t))$ is absolutely continuous on $[0,1]$ and its time derivative satisfies, for $\mathcal{L}^1$-a.e. $t \in [0,1]$,*

$$\dot{E}(t) = \mathcal{P}(t)(u(t)), \tag{2.30}$$

where $\mathcal{P}(t)$ is defined by (2.26).

3. Existence results

This section is devoted to proving Theorem 2.13. Beforehand, we must show the existence of *minimum energy configurations*, in order to make rigorous Definition 2.12. For this, we will use some recent semicontinuity theorems for SBV functions, together with the properties of σ^p-convergence.

3.1. **Semicontinuity and compactness.** We provide a lower semicontinuity property for the volume energy $\mathcal{V}(t)$ with respect to the weak* convergence in $SBV^p(\Omega_D; K)$ (Definition 2.4). This is guaranteed by the polyconvexity and the growth inequality (V4), thanks to a result by Fusco-Leone-March-Verde [20]. We adapt the proof to treat the case of functionals which may assume the value $+\infty$.

Theorem 3.1 (Semicontinuity). *Let $\mathcal{V}(t)$ be defined as in* (2.9)*, where V satisfies* (V1–4) *and* (V6). *Let $t_k \to t_\infty$ and let $u_k \rightharpoonup u_\infty$ weakly* in $SBV^p(\Omega_D; K)$. Then*

$$\mathcal{V}(t_\infty)(u_\infty) \leq \liminf_{k \to \infty} \mathcal{V}(t_k)(u_k). \tag{3.1}$$

Proof. First we prove the theorem for $t_k = t_\infty$. We claim that there exists a nondecreasing sequence of everywhere finite functions V_j satisfying (V1) and converging pointwise to V. Let $\mathcal{V}_j(t)$ be the corresponding integral functionals. By [20, Theorem 3.5] we have

$$\mathcal{V}_j(t_\infty)(u_\infty) \leq \liminf_{k \to \infty} \mathcal{V}_j(t_\infty)(u_k) \leq \liminf_{k \to \infty} \mathcal{V}(t_\infty)(u_k).$$

Passing to the limit with respect to j, we get (3.1) when $t_k = t_\infty$. The general case is obtained using (2.12).

It remains only to prove the claim. This will be done by constructing the sequence $\widetilde{V}_j$ associated to V_j by (V1). To this end we consider the convex conjugate $\widetilde{V}^*$ of $\widetilde{V}$ with respect to ξ, defined by

$$\widetilde{V}^*(t, x, y, \xi^*) := \sup_{\xi \in \mathbb{R}^\tau} \left[\xi^* \cdot \xi - \widetilde{V}(t, x, y, \xi) \right].$$

By (V3), we have $\widetilde{V}^*(t,x,y,\xi^*) > -\infty$ for every (t,x,y,ξ^*). Using (V3) and (V4), it is easy to see that for every $M>0$ there exists $R>0$ such that, if $|\xi^*|\le M$, then

$$\widetilde{V}^*(t,x,y,\xi^*) = \sup_{|\xi|\le R}\Big[\xi^*\cdot\xi - \widetilde{V}(t,x,y,\xi)\Big] \tag{3.2}$$

for every (t,x,y). By continuity, the supremum is attained, so that $\widetilde{V}^*(t,x,y,\xi^*) < +\infty$.

For every x, the function $(t,y,\xi^*)\mapsto \widetilde{V}^*(t,x,y,\xi^*)$ is lower semicontinuous, since the functions $(t,y,\xi^*)\mapsto \xi^*\cdot\xi - \widetilde{V}(t,x,y,\xi)$ are continuous for every ξ. To prove the continuity of $(t,y,\xi^*)\mapsto \widetilde{V}^*(t,x,y,\xi^*)$, it is enough to show that

$$\widetilde{V}^*(t_\infty,x,y_\infty,\xi^*_\infty) \ge \limsup_{k\to\infty} \widetilde{V}^*(t_k,x,y_k,\xi^*_k) \tag{3.3}$$

for every $(t_k,y_k,\xi^*_k)\to(t_\infty,y_\infty,\xi^*_\infty)$. Let $M>0$ be a constant such that $|\xi_k|\le M$ for every k, let $R>0$ be a constant such that (3.2) is satisfied, and let ξ_k, with $|\xi_k|\le R$, be a point where the supremum in (3.2) is attained for $(t,x,y,\xi^*)=(t_k,x,y_k,\xi^*_k)$. Passing to a subsequence, we may assume that $\xi_k\to\xi_\infty$, so that

$$\begin{aligned}\widetilde{V}^*(t_\infty,x,y_\infty,\xi^*_\infty) &\ge \xi^*_\infty\cdot\xi_\infty - \widetilde{V}(t_\infty,x,y_\infty,\xi_\infty) = \\ = \lim_{k\to\infty}\Big[\xi^*_k\cdot\xi_k - \widetilde{V}(t_k,x,y_k,\xi_k)\Big] &= \lim_{k\to\infty}\widetilde{V}^*(t_k,x,y_k,\xi^*_k),\end{aligned}$$

which proves (3.3) and concludes the proof of the continuity of $(t,y,\xi^*)\mapsto\widetilde{V}^*(t,x,y,\xi^*)$.

We now define

$$\widetilde{V}_j(t,x,y,\xi) := \max_{|\xi^*|\le j}\Big[\xi^*\cdot\xi - \widetilde{V}^*(t,x,y,\xi^*)\Big]\,.$$

Arguing as before, it can be proven that $(t,y,\xi)\mapsto\widetilde{V}_j(t,x,y,\xi)$ is continuous. Moreover, $\xi\mapsto\widetilde{V}_j(t,x,y,\xi)$ is convex, being a supremum of affine functions. Finally, it is well known from Convex Analysis that

$$\widetilde{V}(t,x,y,\xi) = \sup_{\xi^*\in\mathbb{R}^\tau}\Big[\xi^*\cdot\xi - \widetilde{V}^*(t,x,y,\xi^*)\Big]\,.$$

This implies that $\widetilde{V}_j \nearrow \widetilde{V}$ and concludes the proof of the claim. ☐

We will need also the following fact, which is proven in [20, Theorem 3.4] as an intermediate step to show Theorem 3.1; we recall that τ_j is the dimension of the vector $\mathrm{adj}_j A$ for $A\in\mathbb{M}^{n\times n}$.

Theorem 3.2. *Let u_k be a sequence in $SBV(\Omega_D;K)$, converging in measure to a function $u_\infty\in SBV(\Omega_D;K)$. Suppose that, for $j=1,\dots,n$, $\|\mathrm{adj}_j\nabla u_k\|_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})}$ and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k, where the exponents p_j satisfy* (W4). *Then, for $j=1,\dots,n$, $\mathrm{adj}_j\nabla u_k \rightharpoonup \mathrm{adj}_j\nabla u_\infty$ weakly in $L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})$.*

Exploiting (V4), we get the following coercivity estimate for $\mathcal{V}(t)$:

$$\mathcal{V}(t)(u) \ge \sum_{j=1}^n \beta_V^j\,\|\mathrm{adj}_j\nabla u\|^{p_j}_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})} - \beta_V^0\,\mathcal{L}^n(\Omega_D)\,. \tag{3.4}$$

This allows us to employ, with $p=p_1$, the following compactness theorem, proven in [1, Proposition 4.3] (see also [3, Theorem 4.8]).

Theorem 3.3 (Compactness)**.** *Let u_k be a sequence in $SBV^p(\Omega_D;K)$ such that $\|\nabla u_k\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$ and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly* in $SBV^p(\Omega_D;K)$.*

Finally, we recall from [21, Theorem 4.4] a stability property of the Ciarlet-Nečas non-interpenetration condition (Definition 1.1) under weak* convergence in $SBV^p(\Omega_D;K)$.

Theorem 3.4 (Stability of the Ciarlet-Nečas condition). *Let u_k converge to u weakly* in $SBV^p(\Omega_D;K)$. Suppose that every u_k satisfies* (CN1) *and* (CN2)*, u satisfies* (CN1)*, and $\det\nabla u_k \rightharpoonup \det\nabla u$ weakly in $L^1(\Omega_D)$. Then u satisfies* (CN2).

3.2. **The σ^p-convergence of sets.** We state the basic properties of the σ^p-convergence (see Definition 2.5); as before, we will use the exponent $p=p_1$ given in (W4). The lower semicontinuity theorem and the compactness property were proven in [12, Theorems 4.3 and 4.7].

Theorem 3.5 (Semicontinuity). *Let κ satisfy* (K1–2)*, let Γ_0, Γ_k, and Γ be countably $(\mathcal{H}^{n-1},n-1)$-rectifiable subsets of Ω_D with $\mathcal{H}^{n-1}(\Gamma_0)<+\infty$, and let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E)<+\infty$. If Γ_k σ^p-converges to Γ, then*

$$\int_{(\Gamma\cup\Gamma_0)\setminus E}\kappa(x,\nu)\,\mathrm{d}\mathcal{H}^{n-1}(x)\le\liminf_{k\to\infty}\int_{(\Gamma_k\cup\Gamma_0)\setminus E}\kappa(x,\nu_k)\,\mathrm{d}\mathcal{H}^{n-1}(x)\;, \tag{3.5}$$

where ν and ν_k are unit normal vector fields on $\Gamma\cup\Gamma_0$ and $\Gamma_k\cup\Gamma_0$, respectively.

Theorem 3.6 (Compactness). *Every sequence $\Gamma_k\subset\Omega_D$ with $\mathcal{H}^{n-1}(\Gamma_k)$ uniformly bounded has a σ^p-convergent subsequence.*

Remark 3.7. Let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E)<+\infty$ and let u_k be a sequence converging to u weakly* in $SBV^p(\Omega_D;K)$. Applying Theorems 3.5 and 3.6 with $\Gamma_k=S(u_k)$ and $\Gamma_0=\emptyset$, we can prove that, if $S(u_k)\,\tilde{\subset}\,E$ for every k, then $S(u)\,\tilde{\subset}\,E$.

In the following remark, we state some properties of σ^p-convergence, referring to [12, Section 4] for the proofs.

Remark 3.8. If Γ_k σ^p-converges to Γ, then

- Γ is countably $(\mathcal{H}^{n-1},n-1)$-rectifiable;
- $\mathcal{H}^{n-1}(\Gamma)<+\infty$;
- if in addition $\Gamma_k\,\tilde{\subset}\,\Gamma'_k$ and Γ'_k σ^p-converges to Γ', then $\Gamma\,\tilde{\subset}\,\Gamma'$;
- if C is relatively closed in Ω_D and $\Gamma_k\,\tilde{\subset}\,C$ for every k, then $\Gamma\,\tilde{\subset}\,C$; in particular, if $\Gamma_k\in\mathcal{R}$, then $\Gamma\in\mathcal{R}$.

On the contrary, it can be shown that in general the inclusion $C\,\tilde{\subset}\,\Gamma_k$ for every k does not imply $C\,\tilde{\subset}\,\Gamma$, even if C is a compact subset of a $(n-1)$-dimensional manifold. This is because, when C is irregular, there is no $u\in SBV(\Omega_D)$ with $S(u)\cong C$ (see [12]).

The following theorem, proven in [13, 12, Theorem 4.8], is the analogue of Helly's theorem for this set convergence.

Theorem 3.9 (Helly property). *Let $t\mapsto\Gamma_k(t)$ be a sequence of increasing set functions defined on an interval $I\subset\mathbb{R}$ with values in $\mathcal{R}$, i.e., $\Gamma_k(s)\,\tilde{\subset}\,\Gamma_k(t)\in\mathcal{R}$ for every $s,t\in I$ with $s<t$. Assume that the measures $\mathcal{H}^{n-1}(\Gamma_k(t))$ are bounded uniformly with respect to k and t. Then there exist a subsequence Γ_{k_j} and an increasing set function $t\mapsto\Gamma(t)$ on I such that $\Gamma_{k_j}(t)$ σ^p-converges to $\Gamma(t)$ for every $t\in I$.*

3.3. **Existence of minima.** Now we can prove Theorems 2.9 and 2.10, adapting the arguments of [12, Theorem 3.9 and 3.10]. Theorems 2.1 and 2.2 are an immediate consequence of these results.

Proof of Theorem 2.9. Let us fix $t \in [0,1]$ and $\Gamma \in \mathcal{R}$. Let u_k be a minimizing sequence of problem (2.18). The infimum in (2.18) is finite, because of (V3); then, a uniform bound holds for $\mathcal{V}(t)(u_k)$ for k large enough, too.

Combining this bound with (3.4), we conclude that there exists $C > 0$ such that

$$\sum_{j=1}^{n} \beta_V^j \left\| \mathrm{adj}_j \nabla u_k \right\|_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})}^{p_j} \leq C \tag{3.6}$$

for k large; in particular, $u_k \in SBV^p(\Omega_D; K)$. Then, by the Compactness Theorem 3.3 there exists a subsequence, still denoted by u_k, which converges weakly* in $SBV^p(\Omega_D; K)$ to a function u. By Remark 3.7, we have $S(u) \tilde{\subset} \Gamma$; moreover, $u = I$ a.e. on $\Omega_D \backslash \Omega$.

By (3.1) we obtain

$$\mathcal{V}(t)(u) \leq \liminf_{k\to\infty} \mathcal{V}(t)(u_k) < +\infty\,. \tag{3.7}$$

Finally, we notice that u satisfies the orientation preserving condition (CN1): in $\Omega_D \backslash \Omega$ because $u = I$ a.e. on this set, in Ω because of (3.7) and (V2). Moreover, (3.6) and Theorem 3.2 imply that $\det \nabla u_k \rightharpoonup \det \nabla u$ weakly in $L^1(\Omega_D)$, hence Theorem 3.4 shows that u satisfies (CN2); then $u \in AD(I, \Gamma)$. The minimality follows from (3.7) and from the fact that u_k is a minimizing sequence. □

Proof of Theorem 2.10. Let us fix $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, and let (u_k, Γ_k) be a minimizing sequence of problem (2.17). Again, the infimum in (2.17) is finite by (V3). Moreover, by (1.5) and (3.4), there exists a constant $C \geq 0$ such that

$$\sum_{j=1}^{n} \beta_V^j \left\| \mathrm{adj}_j \nabla u_k \right\|_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})}^{p_j} + \mathcal{H}^{n-1}(\Gamma_k) \leq C$$

for every k, which implies that $u_k \in SBV^p(\Omega_D; K)$ and $\mathcal{H}^{n-1}(\Gamma_k)$ is uniformly bounded. By the Compactness Theorem 3.3 there exists a subsequence, still denoted by u_k, which converges weakly* in $SBV^p(\Omega_D; K)$ to a function u which satisfies $u = I$ a.e. on $\Omega_D \backslash \Omega$.

On the other hand, by the Compactness Theorem 3.6 and Remark 3.8, there exists a subsequence, still denoted by Γ_k, σ^p-converging to a set $\Gamma^* \in \mathcal{R}$. By Definition 2.5 we have $S(u) \tilde{\subset} \Gamma^*$. Finally, we take $\Gamma = \Gamma^* \cup \Gamma_0$, in order to get $\Gamma_0 \tilde{\subset} \Gamma$.

By Theorem 3.5 we have

$$\mathcal{K}(\Gamma) = \mathcal{K}(\Gamma^* \cup \Gamma_0) \leq \liminf_{k\to\infty} \mathcal{K}(\Gamma_k \cup \Gamma_0) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k)\,.$$

Arguing as in the proof of Theorem 2.9 we conclude that

$$\mathcal{F}(t)(u, \Gamma) \leq \liminf_{k\to\infty} \mathcal{F}(t)(u_k, \Gamma_k) < +\infty$$

and that u satisfies (CN1) and (CN2). Then we have $u \in AD(I, \Gamma)$, so that the last inequality implies that (u, Γ) is a minimum point of (2.17). □

3.4. **Existence of quasistatic evolutions.** The proof of Theorem 2.13 follows a scheme developed in [15, 12, 17, 19]: problem (2.17) is approximated via time discretization, then the existence result is obtained by passing to the limit as the time steps tend to zero.

First, we show that an incremental approximate solution satisfies an a-priori bound. Then, we will prove Theorem 2.13 as a consequence of Compactness Theorem 3.3 and Helly Theorem 3.9.

Henceforth, given a time discretization $\left\{t_k^i\right\}_{0\leq i\leq k}$ of $[0,1]$, we will use the following notation:

$$\tau_k(t) := t_k^i\,, \quad \mathcal{V}_k(t) := \mathcal{V}(t_k^i)\,, \quad \text{and} \quad \mathcal{F}_k(t) := \mathcal{F}(t_k^i) \quad \text{for } t \in \left[t_k^i, t_k^{i+1}\right)\,. \tag{3.8}$$

Proposition 3.10 (Discrete energy inequality). *Let $t \mapsto (u_k(t), \Gamma_k(t))$ be a sequence of incremental approximate solutions to* (2.17)*, corresponding to a time discretization $\{t_k^i\}_{0\le i\le k}$ of $[0,1]$. Let $\theta_k(t)$ be as in* (2.21)*, $\tau_k(t)$ and $\mathcal{F}_k(t)$ as in* (3.8)*. Then $\mathcal{H}^{n-1}(\Gamma_k(t))$, $\|\nabla u_k(t)\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$, and $\theta_k(t)$ are bounded uniformly in k and t; in particular, $u_k(t) \in SBV^p(\Omega_D; K)$. Moreover, for every $t \in [0,1]$*

$$\mathcal{F}_k(t)(u_k(t), \Gamma_k(t)) \le \mathcal{F}(0)(u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\, \mathrm{d}s\,. \tag{3.9}$$

Proof. We recall the definition of $(u_k(t), \Gamma_k(t))$: for $i = 1, \dots, k$ the pair $(u_k^i, \Gamma_k^i) := (u_k(t_k^i), \Gamma_k(t_k^i))$ is a solution of (2.20); the definition is completed by setting $u_k(t) = u_k^i$ and $\Gamma_k(t) = \Gamma_k^i$ for $t \in [t_k^i, t_k^{i+1})$.

Taking $(u, \Gamma) = (I, \Gamma_k^{i-1})$ in (2.20), we get $\mathcal{V}(t_k^i)(u_k^i) \le \mathcal{V}(t_k^i)(I)$, thanks to the monotonicity of $\mathcal{K}$. Hence by (V3)

$$\mathcal{V}(t_k^i)(u_k^i) < C\,, \tag{3.10}$$

for some constant C independent of k, i, and t, so that $\|\nabla u_k^i\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$ is bounded uniformly in k and i by (3.4); in particular, $u_k \in SBV^p(\Omega_D; K)$.

Now we can compare (u_k^i, Γ_k^i) with $(u_k^{i-1}, \Gamma_k^{i-1})$: as $u_k^{i-1} \in AD(I, \Gamma_k^{i-1})$, by (2.20)

$$\mathcal{F}(t_k^i)(u_k^i, \Gamma_k^i) \le \mathcal{F}(t_k^i)(u_k^{i-1}, \Gamma_k^{i-1})\,. \tag{3.11}$$

Then, we rewrite the right-hand side in terms of $\mathcal{F}(t_k^{i-1})(u_k^{i-1}, \Gamma_k^{i-1})$. By (V6), (2.14), and (3.10) we get, modifying the value of C,

$$\left|\dot{\mathcal{V}}(t)(u_k^i)\right| \le C \tag{3.12}$$

so that $\theta_k(t)$ is bounded uniformly in k and t. Therefore, we have

$$\mathcal{V}(t_k^i)(u_k^{i-1}) - \mathcal{V}(t_k^{i-1})(u_k^{i-1}) = \int_{t_k^{i-1}}^{t_k^i} \dot{\mathcal{V}}(t)(u_k^{i-1})\, \mathrm{d}t\,. \tag{3.13}$$

Summing up (3.11) and (3.13) and using (2.16), we obtain for every $t \in [0,1]$ the discrete energy inequality (3.9).

By (3.9) and (3.12), $\mathcal{F}_k(t)(u_k(t), \Gamma_k(t))$ is bounded uniformly with respect to k and t. Hence the nonnegativity of V and (1.5) give a bound on $\mathcal{H}^{n-1}(\Gamma_k(t))$, uniform in k and t. □

Proof of Theorem 2.13. Take any time discretization $\{t_k^i\}_{0\le i\le k}$ of $[0,1]$ and consider the corresponding incremental approximate solutions. By Proposition 3.10, thanks to the uniform bound on $\mathcal{H}^{n-1}(\Gamma_k(t))$, we can use the Helly Theorem 3.9 to find a subsequence, still denoted Γ_k, and an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, such that $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ for every $t \in [0,1]$; we define $\Gamma(t) := \Gamma^*(t) \cup \Gamma_0$. This determines the sequence $(u_k(t), \Gamma_k)$ and the set function $\Gamma(t)$ of Definition 2.12.

Consider the quantity $\theta_k(t)$ defined in (2.21). For every $t \in [0,1]$, we can extract a subsequence k_j, depending on t, such that

$$\limsup_{k\to\infty} \theta_k(t) = \lim_{j\to\infty} \theta_{k_j}(t)\,.$$

By Proposition 3.10 and the Compactness Theorem 3.3, there exists a further subsequence, still denoted by u_{k_j}, and a function $u(t)$ such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$. By Definition 2.5 we have $S(u(t)) \mathrel{\tilde{\subset}} \Gamma^*(t)$. This determines the subsequence $u_{k_j}(t)$ and the function $u(t)$ of Definition 2.12; the proof is concluded. □

4. Stability of the limit process

In this section we obtain a stability result for the minimizers of problem (2.17), stated in Theorem 4.3 and proven after the Crack Transfer Lemma 4.1. This allows us to prove property (1) in Theorem 2.14.

4.1. **Crack transfer.** An important tool in the proof of the stability result is the *Crack Transfer Lemma* due to Francfort-Larsen [17, Theorem 2.1]. In the original version of the lemma, the jump set of a displacement u is modified ("transferred" into a fixed set) by replacing u with its reflection in some regions. In our framework, reflections are forbidden by non-interpenetration (see (1.6) and (V2)), so we adapt the proof using a suitable stretching as a substitute for the reflection.

Lemma 4.1 (Crack Transfer)**.** *Assume that $t_k \to t_\infty$ and $\Gamma_k \in \mathcal{R}$ σ^p-converges to $\Gamma^* \in \mathcal{R}$. Let $\Gamma \in \mathcal{R}$ with $\Gamma^* \tilde{\subset} \Gamma$. Assume that $\mathcal{V}(t)$ satisfies* (V1–6) *and* (V8)*. Let $v \in AD(I,\Gamma)$ be such that $\mathcal{V}(t_\infty)(v) < +\infty$. Then there exist a sequence $\Gamma'_k \in \mathcal{R}$ with $\Gamma_k \tilde{\subset} \Gamma'_k$, a sequence $v_k \in AD(I,\Gamma'_k)$, and a sequence of closed sets $C_k \subset \Omega$ such that the following properties hold:*

(a) $\mathcal{L}^n(C_k) \to 0$*;*
(b) $v_k = v$ *a.e. in* $\Omega_D \backslash C_k$*;*
(c) $\int_{C_k} V(t_k, x, v_k(x), \nabla v_k(x))\, dx \to 0$*;*
(d) $\mathcal{H}^{n-1}(\Gamma^* \backslash C_k) \to 0$*;*
(e) $(\Gamma'_k \backslash \Gamma_k)\backslash C_k \tilde{\subset} \Gamma \backslash C_k$*;*
(f) $\mathcal{H}^{n-1}((\Gamma'_k \backslash \Gamma_k) \cap C_k) \to 0$*.*

Proof. We modify the proof of [17, Theorem 2.1], with Ω and Ω' replaced by Ω_B and Ω_D (the fact that $\overline{\Omega}_B$ is not necessarily contained in Ω_D is irrelevant). According to Definition 2.5 there exist $u, u_k \in SBV^p(\Omega_D)$ such that $S(u) \cong \Gamma^*$, $S(u_k) \tilde{\subset} \Gamma_k$ for every k, and $u_k \rightharpoonup u$ weakly* in $SBV^p(\Omega_D)$; by Definition 2.4, u and u_k satisfy the hypotheses of [17], except possibly for the weak convergence of $|\nabla u_k|$ in $L^1(\Omega_D)$, replaced here by the equiintegrability, which is sufficient to obtain the results.

Let E_t be the set of the Lebesgue-density-one points for $\{x \colon u(x) > t\}$ and E^k_t the set of the Lebesgue-density-one points for $\{u_k > t\}$. It is possible to find a countable dense set $D \subset \mathbb{R}$ such that for every $t \in D$ the set E_t has finite perimeter and $\mathcal{L}^n(\{u = t\}) = 0$. Then

$$S(u) \cong G := \bigcup_{\substack{t_1, t_2 \in D \\ t_1 < t_2}} (\partial^* E_{t_1} \cap \partial^* E_{t_2}), \tag{4.1}$$

where ∂^* denotes the reduced boundary. For each $x \in G$, we can choose $t_1(x) < t_2(x)$ in D so that $x \in \partial^* E_{t_1(x)} \cap \partial^* E_{t_2(x)}$ and $t_2(x) - t_1(x) \geq \frac{1}{2}\,|[u](x)|$, where $[u]$ denotes the jump of u. It is possible to show that $\partial^* E_{t_1(x)}$ and $\partial^* E_{t_2(x)}$ have a common outward unit normal $\nu(x)$ at x. We refer to [17] for the details.

For every $x \in G$ and $r > 0$, we fix a closed cube $Q_r(x)$ centred at x, with side length $2r$, and with a face perpendicular to $\nu(x)$. We consider also the half-cubes

$$Q^+_r(x) := \{y \in Q_r(x) \colon (y - x)\cdot \nu(x) > 0\},$$
$$Q^-_r(x) := \{y \in Q_r(x) \colon (y - x)\cdot \nu(x) < 0\}$$

and the $(n-1)$-dimensional cubes

$$H_r(x) := \{y \in Q_r(x) \colon (y - x)\cdot \nu(x) = 0\},$$
$$H_r(x, s) := \{y \in Q_r(x) \colon (y - x)\cdot \nu(x) = s\}.$$

We fix a constant λ, with

$$1 < \lambda < \frac{1}{1-\gamma}\,, \tag{4.2}$$

where γ is given by Proposition 1.5.

Let N be the set of points where $\partial\Omega_B$ is not differentiable; we set

$$G_j := \Big\{ x \in G\backslash N \colon \lim_{r\to 0} \frac{\mathcal{H}^{n-1}\big([S(u)\backslash \partial^* E_{t_1(x)}] \cap Q_r(x)\big)}{(2r)^{n-1}} = 0\,, \\ |[u](x)| > \frac{1}{j}\,, \quad \mathrm{dist}(x, \partial\Omega_D) > \frac{1}{j} \Big\}\,,$$

so that $G_j \subset\subset \Omega_D$. As in [17], it can be proven that $G \cong \bigcup G_j$. Given $\varepsilon \in (0, \frac{\lambda-1}{\lambda+1})$, we fix $j = j(\varepsilon)$ such that

$$\mathcal{H}^{n-1}(G\backslash G_j) < \varepsilon\,. \tag{4.3}$$

Arguing as in [17], we consider a fine cover of $\mathcal{H}^{n-1}$-almost all of G_j, composed of a suitable collection of cubes $Q_r(x)$. Employing the Morse-Besicovitch Theorem [5, 16, 28], we can find $C > 0$, $m = m(\varepsilon) \in \mathbb{N}$, $k(\varepsilon) \in \mathbb{N}$, and, for $i = 1, \dots, m$, $x_i \in \overline{\Omega}_B$, $r_i > 0$, and $t_i \in [t_1(x_i), t_2(x_i)]$, and, for every $k \geq k(\varepsilon)$, we can find $\delta_i^+, \delta_i^- > 0$, such that, setting $Q_i := Q_{r_i}(x_i)$, $Q_i^+ := Q_{r_i}^+(x_i)$, $Q_i^- := Q_{r_i}^-(x_i)$, $H_i := H_{r_i}(x_i)$, $H_i^+ := H_{r_i}(x_i, \delta_i^+)$, $H_i^- := H_{r_i}(x_i, -\delta_i^-)$, and R_i the open rectangle between H_i^+ and H_i^-, the following hold:

(1) $\mathcal{L}^n(\bigcup_{i=1}^m Q_i) < \varepsilon$;
(2) if $x_i \in \Omega_B$, then $Q_i \subset \Omega_B$; if $x_i \in \partial\Omega_B$, then $Q_i \subset \Omega$;
(3) if $x_i \in \partial\Omega_B$, then $\partial\Omega_B \cap Q_i$ is a Lipschitz graph contained in R_i;
(4) if $x_i \in \partial\Omega_B$, then $\mathcal{H}^{n-1}(\partial\Omega_B \cap Q_i) - (2r_i)^{n-1} < \varepsilon r_i^{n-1}$;
(5) $\mathcal{H}^{n-1}(S(u) \cap \partial Q_i) = 0$;
(6) $r_i^{n-1} \leq C\,\mathcal{H}^{n-1}(S(u) \cap Q_i)$;
(7) $\mathcal{H}^{n-1}((S(v)\backslash S(u)) \cap Q_i) < \varepsilon r_i^{n-1}$;
(8) $\sum_{i=1}^m \mathcal{H}^{n-1}\big((\partial^* E_{t_i}^k \cap Q_i)\backslash S(u_k)\big) < \varepsilon$ for $k \geq k(\varepsilon)$;
(9) $\mathcal{L}^n((E_{t_i}^k \cap Q_i) \,\triangle\, Q_i^-) < \varepsilon(2r_i)^n$ for $k \geq k(\varepsilon)$;
(10) $\mathcal{L}^n((E_{t_i} \cap Q_i) \,\triangle\, Q_i^-) < \varepsilon(2r_i)^n$;
(11) $\mathcal{H}^{n-1}\big(H_i^\pm \cap E_{t_i}^k\big) < 8\varepsilon(2r_i)^{n-1}$ for $k \geq k(\varepsilon)$;
(12) $\mathcal{H}^{n-1}\big(H_i^\pm \cap E_{t_i}\big) < 8\varepsilon(2r_i)^{n-1}$;
(13) $\delta_i^\pm \in [\frac{\varepsilon}{2} r_i, \varepsilon r_i]$;
(14) $\mathcal{H}^{n-1}(G_j\backslash(\bigcup_{i=1}^m R_i)) < C\varepsilon$.

In (3) by Lipschitz graph we mean that there exists a Lipschitz function $g_i \colon H_i \to \mathbb{R}$ such that $\partial\Omega_B \cap Q_i = \{x + g_i(x)\,\nu(x_i) \colon x \in H_i\}$.

Finally, we set

$$d_i^+ := \frac{\lambda\delta_i^+ + \delta_i^-}{\lambda - 1}\,, \quad d_i^- := \frac{\lambda\delta_i^- + \delta_i^+}{\lambda - 1}$$

and note that $\delta_i^\pm < d_i^\pm < r_i$, where the second inequality follows from (13) and the choice of ε. We define the $(n-1)$-dimensional cubes $J_i^+ := H_{r_i}(x_i, d_i^+)$, $J_i^- := H_{r_i}(x_i, -d_i^-)$, and the following n-dimensional open rectangles: S_i between J_i^+ and J_i^-, S_i^+ between J_i^+ and H_i^+, S_i^- between H_i^- and J_i^-, R_i^+ between J_i^+ and H_i^-, and R_i^- between H_i^+ and J_i^-, so that $R_i = R_i^+ \cap R_i^-$ (see Figure 1). We fix in Q_i an orthogonal system of coordinates (x', x_n) such that $H_i \subset \{x_n = 0\}$. The stretching $(x', x_n) \mapsto (x', \lambda(x_n - d_i^+) + d_i^+)$ maps S_i^+ into R_i^+; the stretching $(x', x_n) \mapsto (x', \lambda(x_n + d_i^-) - d_i^-)$ maps S_i^- into R_i^-.

Now we transfer the jump set $S(v)$ from $G_j \cap \bigcup_i Q_i$ to $\bigcup_i(\partial^* E_{t_i}^k \cap Q_i)$. For every i we consider the stretched version $v_i^\oplus$ of v, defined in R_i^+ by $v_i^\oplus(x', x_n) := v(x', \frac{1}{\lambda}(x_n - d_i^+) +$

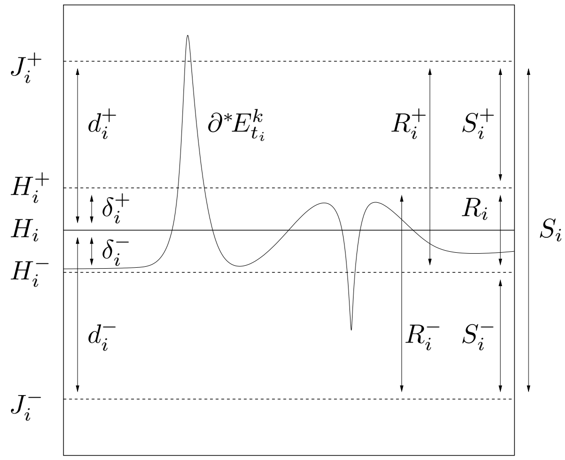

FIGURE 1. The cube Q_i.

d_i^+); analogously we consider the stretched version $v_i^\ominus$ of v, defined in R_i^- by $v_i^\ominus(x', x_n) := v(x', \frac{1}{\lambda}(x_n + d_i^-) - d_i^-)$. If $x_i \notin \partial\Omega_B$ we consider the functions v_k^ε defined in Q_i by

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \backslash E^k_{t_i})\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \cap E^k_{t_i})\,. \end{cases}$$

If $x_i \in \partial\Omega_B$, by (3) there are two cases: either $Q_i^+ \backslash R_i \subset \Omega_B$ or $Q_i^- \backslash R_i \subset \Omega_B$. In the former, we define v_k^ε on Q_i by

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \cap (\Omega_B \backslash E^k_{t_i}))\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \backslash (\Omega_B \backslash E^k_{t_i}))\,; \end{cases}$$

in the latter, we set

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \backslash (E^k_{t_i} \cap \Omega_B))\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \cap E^k_{t_i} \cap \Omega_B)\,. \end{cases}$$

We complete the definition of v_k^ε in Ω_D by $v_k^\varepsilon := v$ in $\Omega_D \backslash \bigcup_i Q_i$.

Now we fix an arbitrary decreasing sequence $\varepsilon_h \to 0$, with $\varepsilon_h < \frac{\lambda - 1}{\lambda + 1}$, and apply the previous construction with $\varepsilon = \varepsilon_h$. For $k \ge k(\varepsilon_1)$ we define v_k, j_k, and m_k by setting $v_k := v_k^{\varepsilon_h}$, $j_k := j(\varepsilon_h)$, and $m_k := m(\varepsilon_h)$ for $k \in [k(\varepsilon_h), k(\varepsilon_{h+1}))$. Moreover we define $\Gamma'_k := S(v_k) \cup \Gamma_k$ and $C_k := \bigcup_{i=1}^{m_k} \overline{S}_i$.

Let us prove that Γ'_k, v_k, and C_k satisfy the properties (a)–(f) required in the statement. By construction $\Gamma'_k \in \mathcal{R}$ and $\Gamma_k \mathrel{\tilde{\subset}} \Gamma'_k$; moreover, as stretching preserves the non-interpenetration condition by (1.2), it is easy to see that $v_k \in AD(I, \Gamma'_k)$.

Condition (a) is a consequence of (1) and (b) is guaranteed by the definition of v_k. To prove (c), notice that in C_k we have $\nabla v_k(x) = \nabla v(x)\,\Lambda$, where Λ is the diagonal $n \times n$ matrix with entries $1, \ldots, 1$, and $\frac{1}{\lambda}$. As $|\Lambda - I| \le \gamma$ by (4.2), in C_k we have

$$V(t_k, x, v_k(x), \nabla v_k(x)) + c_V^0 \le \frac{n}{n-1}\left[V(t_k, x, v_k(x), \nabla v(x)) + c_V^0\right]$$

thanks to (1.11). Moreover, by Remark 2.8 there exists a constant $C>0$ (depending on the diameter of K) such that

$$V(t_k,x,v_k(x),\nabla v(x))+c_V^0\le C\,\left[V(t_\infty,x,v(x),\nabla v(x))+c_V^0\right]\,.$$

As $\mathcal{V}(t_\infty)(v)<+\infty$ and $\mathcal{L}^n(C_k)\to 0$, this shows (c).

Part (d) is a consequence of (4.1), (4.3), and (14), with $j=j_k$, while (e) follows from (b) and the definition of Γ'_k. To prove (f), it is enough to show that

$$\mathcal{H}^{n-1}\left((S(v_k)\backslash S(u_k))\cap\bigcup_{i=1}^{m_k}\overline{S}_i\right)\to 0\,.$$

Arguing like in [17], we consider a partition $\overline{S}_i=P_i^1\cup P_i^2\cup P_i^3\cup P_i^4\cup P_i^5$, where

$$\begin{aligned}
P_i^1 &:= \overline{S}_i\cap\partial^*E^k_{t_i}\,,\\
P_i^2 &:= (S_i\cup J_i^+\cup J_i^-)\backslash(H_i^+\cup H_i^-\cup\partial\Omega_B\cup\partial^*E^k_{t_i})\,,\\
P_i^3 &:= (H_i^+\cup H_i^-)\backslash\partial^*E^k_{t_i}\,,\\
P_i^4 &:= \partial S_i\backslash(J_i^+\cup J_i^-\cup\partial^*E^k_{t_i})\,,\\
P_i^5 &:= (\partial\Omega_B\cap S_i)\backslash\partial^*E^k_{t_i}\,.
\end{aligned}$$

By (8) we have

$$\sum_i\mathcal{H}^{n-1}\big(P_i^1\backslash S(u_k)\big)\to 0\,.$$

By the construction of v_k, we have

$$\mathcal{H}^{n-1}\big(P_i^2\cap S(v_k)\big)\le\lambda\,\mathcal{H}^{n-1}\big(S(v)\cap(\overline{S}_i\backslash R_i)\big)\;;$$

by (4.3) and (14)

$$\mathcal{H}^{n-1}\Big((S(v)\cap S(u))\backslash\bigcup R_i\Big)\to 0\,,$$

while by (6) and (7)

$$\mathcal{H}^{n-1}\Big((S(v)\backslash S(u))\cap\bigcup\overline{S}_i\Big)\to 0\,,$$

so that

$$\sum_i\mathcal{H}^{n-1}\big(P_i^2\cap S(v_k)\big)\to 0\,.$$

As for P_i^3, the parts of $S(v_k)$ lying in $H_i^+\backslash E^k_{t_i}$ and in $H_i^-\cap E^k_{t_i}$ can be controlled like those in P_i^2. Thanks to (11), the remaining parts $H_i^+\cap E^k_{t_i}$ and $H_i^-\backslash E^k_{t_i}$ have $\mathcal{H}^{n-1}$-measure less than $C\varepsilon r_i^{n-1}$, hence by (6)

$$\sum_i\mathcal{H}^{n-1}\big(P_i^3\cap S(v_k)\big)\to 0\,.$$

By (13) $d_i^\pm\le\frac{\lambda+1}{\lambda-1}\,\varepsilon r_i$, so that using again (6) we see that

$$\sum_i\mathcal{H}^{n-1}\big(P_i^4\big)\to 0\,.$$

Finally, we need a bound on P_i^5 when $x_i\in\partial\Omega_B$. Assume that $Q_i^-\backslash R_i\subset\Omega_B$ (the other possibility, $Q_i^+\backslash R_i\subset\Omega_B$, is treated in the same way); then for the parts of $S(v_k)$ lying in $(S_i\cap\partial\Omega_B)\backslash E^k_{t_i}$ we can argue like in the case of P_i^2. To estimate the jumps in $F:=S_i\cap\partial\Omega_B\cap E^k_{t_i}$, we consider its partition $F=F_i^1\cup F_i^2$, with $F_i^1:=\pi(\partial^*E^k_{t_i}\cap(Q_i\backslash\overline{\Omega}_B))$ and $F_i^2:=F\backslash F_i^1$, where π is the projection of $Q_i\backslash\overline{\Omega}_B$ onto $\partial\Omega_B$, parallel to ν_i. If L denotes the Lipschitz constant of Ω_B (uniform with respect to k and i), we have

$$\mathcal{H}^{n-1}\big(F_i^1\big)\le\sqrt{1+L^2}\;\mathcal{H}^{n-1}\big(\partial^*E^k_{t_i}\cap(Q_i\backslash\overline{\Omega}_B)\big)\;,$$

so that, using (8) and recalling that $S(u_k) \mathrel{\tilde{\subset}} \overline{\Omega}_B$,

$$\sum_{x_i \in \partial\Omega_B} \mathcal{H}^{n-1}\big(F_i^1\big) \to 0\,.$$

As for F_i^2, let $\widetilde{F}_i^2 := \pi^{-1}(F_i^2)$: by (3) and (13)

$$\mathcal{H}^{n-1}\big(F_i^2\big) \le \frac{\sqrt{1+L^2}}{r_i(1-\varepsilon_h)}\,\mathcal{L}^n(\widetilde{F}_i^2)\,.$$

As $Q_i^- \backslash R_i \subset \Omega_B$, by (3) and (13) we have $\mathcal{L}^n((Q_i \cap \Omega_B) \triangle Q_i^-) < (2r_i)^{n-1}\varepsilon$; by (9) and (6),

$$\sum_{x_i \in \partial\Omega_B} \frac{1}{r_i}\,\mathcal{L}^n(E_{t_i}^k \cap (Q_i \backslash \overline{\Omega}_B)) \to 0\,.$$

Now one can see that $\widetilde{F}_i^2 \subset E_{t_i}^k \cap (Q_i \backslash \overline{\Omega}_B)$, except at most for a set of null Lebesgue-measure (for instance, apply Ambrosio's method of one-dimensional sections [3, Section 3.11]), hence

$$\sum_{x_i \in \partial\Omega_B} \mathcal{H}^{n-1}\big(F_i^2\big) \to 0\,.$$

We have shown

$$\sum_{x_i \in \partial\Omega_B} \mathcal{H}^{n-1}\big(S_i \cap \partial\Omega_B \cap E_{t_i}^k\big) \to 0\,,$$

so that

$$\sum_i \mathcal{H}^{n-1}\big(P_i^5 \cap S(v_k)\big) \to 0\,.$$

Collecting the last results, we get (f) and complete the proof. □

The Crack Transfer Lemma implies the following consequences.

Corollary 4.2. *Let t_∞, t_k, Γ^*, Γ_k, Γ, Γ'_k, $\mathcal{V}(t)$, v, v_k be as in Lemma 4.1. Moreover, let $\Gamma_0 \in \mathcal{R}$ such that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k$ for every k; let $\Gamma_\infty := \Gamma^* \cup \Gamma_0 \in \mathcal{R}$. Then*

(1) *$v_k \to v$ in measure;*
(2) *$\nabla v_k \to \nabla v$ strongly in $L^p(\Omega_D; \mathbb{M}^{n\times n})$;*
(3) *$\mathcal{V}(t_k)(v_k) \to \mathcal{V}(t_\infty)(v)$;*
(4) *$\mathcal{H}^{n-1}((\Gamma'_k \backslash \Gamma_k)\backslash(\Gamma\backslash\Gamma_\infty)) \to 0$;*
(5) *$\limsup_{k\to\infty} \mathcal{K}(\Gamma'_k \backslash \Gamma_k) \le \mathcal{K}(\Gamma\backslash\Gamma_\infty)$.*

Proof. Properties (1), (2), and (3) are given by the consequences (a)–(c) of the Lemma, with the aid of (V4). To get (4), use (f) for the part of $\Gamma'_k \backslash \Gamma_k$ contained in C_k; use (d) for the part contained in Γ_∞, recalling that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k$; use (e) for the remaining part. Employing (1.5) we see that $\mathcal{K}((\Gamma'_k \backslash \Gamma_k)\backslash(\Gamma\backslash\Gamma_\infty)) \to 0$, which implies (5). □

4.2. **Stability of minimizers.** Thanks to the Crack Transfer Lemma, we are now able to prove the stability of the minimizers of problem (2.4) with respect to the σ^p-convergence, adapting the arguments of [12, Theorem 5.5].

Theorem 4.3 (Stability of minimizers)**.** *Let $\mathcal{F}(t)$ be the energy defined in* (2.16)*, where $\mathcal{V}(t)$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let $t_k \to t_\infty \in [0,1]$. Let $\Gamma_k \in \mathcal{R}$ be a sequence such that Γ_k σ^p-converges to a set $\Gamma^* \in \mathcal{R}$; let $\Gamma_0 \in \mathcal{R}$ such that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k$ for every k. Let $u_k \in AD(I, \Gamma_k)$ be a sequence such that*

$$\mathcal{F}(t_k)(u_k, \Gamma_k) \le \mathcal{F}(t_k)(v, \Gamma) \tag{4.4}$$

for every $\Gamma \in \mathcal{R}$ *with* $\Gamma_k \,\widetilde{\subset}\, \Gamma$ *and every* $v \in AD(I,\Gamma)$. *Assume that* u_k *converges to a function* u_∞ *weakly* in* $SBV^p(\Omega_D;K)$. *Then* $u_\infty \in AD(I,\Gamma_\infty)$, *where* $\Gamma_\infty := \Gamma^* \cup \Gamma_0 \in \mathcal{R}$; *moreover*

$$\mathcal{F}(t_\infty)(u_\infty,\Gamma_\infty) \le \mathcal{F}(t_\infty)(v,\Gamma) \tag{4.5}$$

for every $\Gamma \in \mathcal{R}$, *with* $\Gamma_\infty \,\widetilde{\subset}\, \Gamma$, *and every* $v \in AD(I,\Gamma)$; *in addition,*

$$\mathcal{V}(t_k)(u_k) \to \mathcal{V}(t_\infty)(u_\infty)\,. \tag{4.6}$$

Proof. The fact that $u_k \in SBV^p(\Omega_D;K)$ comes from (4.4) with $\Gamma = \Gamma_k$ and $v = I$, with the aid of (V3) and (V4), recalling that $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded by definition of σ^p-convergence. By Definition 2.5, we have $S(u) \,\widetilde{\subset}\, \Gamma_\infty$; moreover, by the weak* convergence in $SBV^p(\Omega_D;K)$ we get $u = I$ a.e. on $\Omega_D \backslash \Omega$.

To show the minimality property (4.5), let us fix $\Gamma \in \mathcal{R}$ with $\Gamma_\infty \,\widetilde{\subset}\, \Gamma$ and $v \in AD(I,\Gamma)$. By the Crack Transfer Lemma 4.1, we find a sequence $\Gamma'_k \in \mathcal{R}$ with $\Gamma_k \,\widetilde{\subset}\, \Gamma'_k$, a sequence $v_k \in AD(I,\Gamma'_k)$, and a sequence of closed sets $C_k \subset \Omega$ such that (a)–(f) hold. By the minimality condition (4.4) we have

$$\mathcal{V}(t_k)(u_k) + \mathcal{K}(\Gamma_k) \le \mathcal{V}(t_k)(v_k) + \mathcal{K}(\Gamma'_k)\,,$$

which implies

$$\mathcal{V}(t_k)(u_k) \le \mathcal{V}(t_k)(v_k) + \mathcal{K}(\Gamma'_k \backslash \Gamma_k)\,.$$

Let $k \to \infty$: thanks to the weak* convergence in $SBV^p(\Omega_D;K)$ we get (3.1). In the right-hand side, we can pass to the $\limsup$ by Corollary 4.2, obtaining

$$\limsup_{k\to\infty} \mathcal{V}(t_k)(v_k) + \mathcal{K}(\Gamma'_k \backslash \Gamma_k) \le \mathcal{V}(t_\infty)(v) + \mathcal{K}(\Gamma \backslash \Gamma_\infty)\,.$$

Hence we get (4.5), which in turn implies (CN1) for u_∞ (by (V2) and (V3)); arguing as in the proof of Theorem 2.9, we conclude that $u_\infty \in AD(I,\Gamma_\infty)$.

Repeating the construction with $v = u_\infty$ and $\Gamma = \Gamma_\infty$, we get (4.6). □

Remark 4.4. Let $t \mapsto (u(t),\Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}(t)$. Definition 2.12 provides a sequence $t \mapsto (u_k(t),\Gamma_k(t))$ of incremental approximate solutions and, fixed t, a subsequence $(u_{k_j}(t),\Gamma_{k_j}(t))$ satisfying the hypotheses of Theorem 4.3 with $t_{k_j} = \tau_{k_j}(t)$ (see (3.8) and recall that $\Gamma_0 \,\widetilde{\subset}\, \Gamma_k(t)$). Hence the stability result guarantees that

$$\mathcal{V}(\tau_{k_j}(t))(u_{k_j}(t)) \to \mathcal{V}(t)(u(t)) \tag{4.7}$$

and that $(u(t),\Gamma(t))$ satisfies (2.22).

5. Energy balance

In this section we show property (2) of Theorem 2.14. The first step is passing to the limit in (3.9) to get the so called energy inequality, then the opposite inequality is obtained via a standard method based on stability. This procedure was developed in [15, 12, 17, 19].

5.1. **The energy inequality.** Let $t \mapsto (u(t),\Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}(t)$ and let $t \mapsto (u_k(t),\Gamma_k(t))$ be an associated sequence of incremental approximate solutions as in Definition 2.12. Recall that $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$, $\Gamma_0 \,\widetilde{\subset}\, \Gamma_k(t)$, and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$. Let $\theta_k(t)$ be as in (2.21), $\tau_k(t)$ and $\mathcal{F}_k(t)$ as in (3.8).

We have already seen in Proposition 3.10 that, for every sequence of incremental approximate solutions, $\mathcal{H}^{n-1}(\Gamma_k(t))$, $\|\nabla u_k(t)\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$, and $\theta_k(t)$ are bounded uniformly in k and t. By Theorem 3.5 we have for every $t \in [0,1]$

$$\mathcal{K}(\Gamma(t)) = \mathcal{K}(\Gamma^*(t) \cup \Gamma_0) \le \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t) \cup \Gamma_0) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t))\,; \tag{5.1}$$

moreover, Fatou's lemma implies that the function

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t) \tag{5.2}$$

belongs to $L^1([0,1])$ and

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s \le \int_0^t \theta_\infty(s)\,\mathrm{d}s\,. \tag{5.3}$$

Fixed $s\in[0,1]$, by Definition 2.12 there is a subsequence $(u_{k_j}(s),\Gamma_{k_j}(s))$ such that

$$u_{k_j}(s) \rightharpoonup u(s) \text{ weakly}^* \text{ in } SBV^p(\Omega_D;K) \tag{5.4}$$

and

$$\theta_\infty(s) = \lim_{k\to\infty} \theta_{k_j}(s)\,. \tag{5.5}$$

By Remark 4.4 and (2.12) we have

$$\mathcal{V}(s)(u_{k_j}(s)) \to \mathcal{V}(s)(u(s))\,, \tag{5.6}$$

so that the function $s\mapsto \mathcal{V}(s)(u(s))$ is measurable.

Now we would like to pass to the limit as $k_j\to\infty$ in (3.9): this is possible thanks to the following result. In our setting, hypothesis (5.7) is a consequence of (V7).

Lemma 5.1. *Let $\mathcal{V}\colon [0,1]\times SBV^p(\Omega_D;K)\to[0,+\infty]$ be a functional, differentiable in the first variable and lower semicontinuous with respect to the weak* convergence in $SBV^p(\Omega_D;K)$. Assume that for every $M>0$ there is a modulus of continuity $\omega_M\colon[0,1]\to[0,+\infty)$ (i.e., a nondecreasing function of t, vanishing for $t\to 0$), such that*

$$\left|\dot{\mathcal{V}}(t)(u) - \dot{\mathcal{V}}(s)(u)\right| \le \omega_M(|t-s|) \tag{5.7}$$

for every $s,t\in[0,1]$ and every $u\in SBV^p(\Omega_D;K)$ such that $\mathcal{V}(0)(u)\le M$. Fix $s\in[0,1]$ and let u_j be a sequence converging to u_∞ weakly* in $SBV^p(\Omega_D;K)$. Assume that $\mathcal{V}(s)(u_j)\to\mathcal{V}(s)(u_\infty)<+\infty$. Then $\dot{\mathcal{V}}(s)(u_j)\to\dot{\mathcal{V}}(s)(u_\infty)$.

Proof. See [19, Proposition 3.3]. □

Applying this lemma, from (5.4) and (5.6) we deduce that

$$\dot{\mathcal{V}}(s)(u_{k_j}(s)) \to \dot{\mathcal{V}}(s)(u(s))\,.$$

Hence, by (2.21) and (5.5), for every $s\in[0,1]$ we get

$$\theta_\infty(s) = \dot{\mathcal{V}}(s)(u(s))\,, \tag{5.8}$$

which is thus measurable.

By (2.16), (4.7), and (5.1) we have

$$\mathcal{F}(t)(u(t),\Gamma(t)) \le \liminf_{j\to\infty} \mathcal{F}_{k_j}(t)(u_{k_j}(t),\Gamma_{k_j}(t)) \le \limsup_{k\to\infty} \mathcal{F}_k(t)(u_k(t),\Gamma_k(t))\,. \tag{5.9}$$

From (3.9), (5.2), (5.3), and (5.8) we obtain

$$\limsup_{k\to\infty} \mathcal{F}_k(t)(u_k(t),\Gamma_k(t)) \le \mathcal{F}(0)(u_0,\Gamma_0) + \int_0^t \dot{\mathcal{V}}(s)(u(s))\,\mathrm{d}s\,. \tag{5.10}$$

This leads to the energy inequality

$$\mathcal{F}(t)(u(t),\Gamma(t)) \le \mathcal{F}(0)(u_0,\Gamma_0) + \int_0^t \dot{\mathcal{V}}(s)(u(s))\,\mathrm{d}s\,. \tag{5.11}$$

5.2. **The energy equality.** The last point in the proof of Theorem 2.14 is the opposite of (5.11); we argue again by discretization and employ the stability property.

Proof of Theorem 2.14. Let $t \mapsto (u(t), \Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}(t)$. Global stability property (1) has been proven in Remark 4.4.

Since a Lebesgue integral can be approximated by a suitable Riemann sum (see [23] and [12, Lemma 4.12]), there exists a sequence of subdivisions $\{s_k^i\}_{0\le i\le i_k}$, satisfying

$$0 = s_k^0 < s_k^1 < \dots < s_k^{i_k-1} < s_k^{i_k} = t$$

and

$$\lim_{k\to\infty} \max_{1\le i\le i_k} (s_k^i - s_k^{i-1}) = 0\,,$$

such that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \left| (s_k^i - s_k^{i-1}) \dot{\mathcal{V}}(s_k^i)(u(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{V}}(s)(u(s))\,\mathrm{d}s \right| = 0\,. \tag{5.12}$$

Comparing $(u(t), \Gamma(t))$ with $(I, \Gamma(t))$, by (2.22) and (V3) we find a uniform bound

$$\mathcal{V}(t)(u(t)) < M\,. \tag{5.13}$$

For $i = 1, \dots, i_k$, we can compare $(u(s_k^{i-1}), \Gamma(s_k^{i-1}))$ with $(u(s_k^i), \Gamma(s_k^i))$: as $u(s_k^i) \in AD(I, \Gamma(s_k^i))$ and $\Gamma(s_k^{i-1}) \subset \Gamma(s_k^i)$, the stability result (2.22) guarantees that

$$\mathcal{F}(s_k^{i-1})(u(s_k^{i-1}), \Gamma(s_k^{i-1})) \le \mathcal{F}(s_k^{i-1})(u(s_k^i), \Gamma(s_k^i))\,.$$

Arguing as in Proposition 3.10, by (5.13) and (V6) we see that

$$\mathcal{F}(s_k^{i-1})(u(s_k^i), \Gamma(s_k^i)) = \mathcal{F}(s_k^i)(u(s_k^i), \Gamma(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{V}}(s)(u(s_k^i))\,\mathrm{d}s\,.$$

Summing up,

$$\mathcal{F}(t)(u(t), \Gamma(t)) \ge \mathcal{F}(0)(u_0, \Gamma_0) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{V}}(s)(u(s_k^i))\,\mathrm{d}s\,.$$

Finally,

$$\mathcal{F}(t)(u(t), \Gamma(t)) \ge \mathcal{F}(0)(u_0, \Gamma_0) + \sum_{i=1}^{i_k} (s_k^i - s_k^{i-1}) \dot{\mathcal{V}}(s_k^i)(u(s_k^i)) - \omega_k(t)\,,$$

where

$$\omega_k(t) := \sum_{i=1}^{i_k} \left| (s_k^i - s_k^{i-1}) \dot{\mathcal{V}}(s_k^i)(u(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{V}}(s)(u(s_k^i))\,\mathrm{d}s \right|\,.$$

By (V7) and (5.13) we have $\omega_k(t) \to 0$; hence, by (5.12) we find, recalling (5.11),

$$\mathcal{F}(t)(u(t), \Gamma(t)) = \mathcal{F}(0)(u_0, \Gamma_0) + \int_0^t \dot{\mathcal{V}}(s)(u(s))\,\mathrm{d}s\,, \tag{5.14}$$

which leads to the energy balance property (2). □

Remark 5.2. Let $(u(t), \Gamma(t))$ and $(u_{k_j}(t), \Gamma_{k_j}(t))$ be as in Definition 2.12; let $\mathcal{V}_k(t)$ and $\mathcal{F}_k(t)$ be as in (3.8). By (5.9), (5.10), and (5.14) we obtain

$$\mathcal{F}(t)(u(t), \Gamma(t)) = \lim_{j\to\infty} \mathcal{F}_{k_j}(t)(u_{k_j}(t), \Gamma_{k_j}(t))\,.$$

As by Remark 4.4

$$\mathcal{V}(t)(u(t)) = \lim_{j\to\infty} \mathcal{V}_{k_j}(t)(u_{k_j}(t))\,,$$

we get

$$\mathcal{K}(\Gamma(t)) = \lim_{j\to\infty} \mathcal{K}(\Gamma_{k_j}(t)).$$

5.3. **Convergence of the discrete-time problems.** In the last remark we have seen that the elastic energy and the crack energy of an incrementally-approximable quasistatic evolution are the limits of the corresponding energies for the associated subsequence of incremental approximate solutions. Now we show that the convergence holds for the whole sequence of incremental approximate solutions, adapting [12, Theorem 8.1].

Theorem 5.3 (Convergence of energies)**.** *Let* $\mathcal{F}(t)$ *be the energy defined in* (2.16)*, where* $\mathcal{V}(t)$ *satisfies* (V1–8) *and* $\mathcal{K}$ *satisfies* (K1–2)*. Let* $(u(t),\Gamma(t))$*,* (u_0,Γ_0)*,* $\Gamma^*(t)$*, and* $(u_k(t),\Gamma_k(t))$ *be as in Definition 2.12; let* $\mathcal{V}_k(t)$ *and* $\mathcal{F}_k(t)$ *be as in* (3.8)*. Then for every* $t\in[0,T]$

$$\mathcal{V}(t)(u(t)) = \lim_{k\to\infty} \mathcal{V}_k(t)(u_k(t)), \tag{5.15}$$

$$\mathcal{K}(\Gamma(t)) = \lim_{k\to\infty} \mathcal{K}(\Gamma_k(t)). \tag{5.16}$$

Moreover, the functions $\theta_k(t)$ *defined in* (2.21) *satisfy*

$$\theta_k \to \theta_\infty \quad \textit{in } L^1([0,T]), \tag{5.17}$$

where $\theta_\infty(t)$ *is given by* (5.8)*.*

Proof. Let us fix $t\in[0,T]$ and let $u_{k_l}(t)$ be a subsequence of $u_k(t)$ such that

$$\lim_{l\to\infty} \mathcal{V}_{k_l}(t)(u_{k_l}(t)) = \liminf_{k\to\infty} \mathcal{V}_k(t)(u_k(t)).$$

By Proposition 3.10 and the Compactness Theorem 3.3, there exists a further subsequence, still denoted by u_{k_l}, and a function $u^*(t)$ such that $u_{k_l} \rightharpoonup u^*(t)$ weakly in $SBV^p(\Omega_D;K)$. Since $\Gamma_{k_l}(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t)=\Gamma^*(t)\cup\Gamma_0$, using (2.20) we can apply Theorem 4.3 to $\Gamma_{k_l}(t)$, $u_{k_l}(t)$, and to the sequence $\tau_{k_l}(t)$ defined in (3.8). Therefore $u^*(t)\in AD(I,\Gamma(t))$,

$$\mathcal{V}(t)(u^*(t)) = \lim_{l\to\infty} \mathcal{V}_{k_l}(u_{k_l}(t)),$$

and

$$\mathcal{F}(t)(u^*(t),\Gamma(t)) \le \mathcal{F}(t)(v,\Gamma)$$

for every $\Gamma\in\mathcal{R}$, with $\Gamma(t)\,\tilde{\subset}\,\Gamma$, and for every $v\in AD(I,\Gamma)$. Since $(u(t),\Gamma(t))$ satisfies the same minimality property by (2.22), we have

$$\mathcal{V}(t)(u(t)) = \mathcal{V}(t)(u^*(t)).$$

Collecting these facts we get

$$\mathcal{V}(t)(u(t)) = \liminf_{k\to\infty} \mathcal{V}_k(t)(u_k(t)), \tag{5.18}$$

so that by (5.1)

$$\mathcal{F}(t)(u(t),\Gamma(t)) \le \liminf_{k\to\infty} \mathcal{F}_k(t)(u_k(t),\Gamma_k(t))$$

and from (5.10) and (5.14) we obtain

$$\mathcal{F}(t)(u(t),\Gamma(t)) = \lim_{k\to\infty} \mathcal{F}_k(t)(u_k(t),\Gamma_k(t)). \tag{5.19}$$

Hence, (5.15) and (5.16) follow from (5.1), (5.18), and (5.19).

Moreover, by (3.9), (5.3), and (5.14) we get

$$\int_0^t \theta_\infty(s)\,\mathrm{d}s = \lim_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s$$

for every $t \in [0,T]$; in particular,

$$\int_0^1 \theta_\infty(t)\,\mathrm{d}t = \lim_{k\to\infty}\int_0^1 \theta_k(t)\,\mathrm{d}t\,.$$

By (5.2) $\theta_k \vee \theta_\infty$ converges to θ_∞ pointwise on $[0,T]$, so that $\theta_k \vee \theta_\infty$ converges to θ_∞ in $L^1([0,T])$ thanks to the uniform bound on $\theta_k(t)$ (see Proposition 3.10). Since $\theta_k + \theta_\infty = (\theta_k \vee \theta_\infty) + (\theta_k \wedge \theta_\infty)$, we conclude

$$\int_0^1 \theta_\infty(t)\,\mathrm{d}t = \lim_{k\to\infty}\int_0^1 (\theta_k \wedge \theta_\infty)(t)\,\mathrm{d}t\,.$$

As $\theta_k \wedge \theta_\infty \le \theta_\infty$, this implies that $\theta_k \wedge \theta_\infty$ converges to θ_∞ in $L^1([0,T])$, which, together with the convergence of $\theta_k \vee \theta_\infty$, gives (5.17). □

6. Measurable evolutions

So far we have not taken care of the measurability properties of $t \mapsto u(t)$. The following result ensures that, during the limit process described in Section 3.4, it is possible to select an incrementally-approximable quasistatic evolution $(u(t),\Gamma(t))$ so that the function $t \mapsto u(t)$ is measurable from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$, endowed with the norm (1.1).

Theorem 6.1 (Measurability of quasistatic evolutions)**.** *Let $\mathcal{F}(t)$ be the energy defined in* (2.16)*, where $\mathcal{V}(t)$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let (u_0,Γ_0) be a minimum energy configuration at time* 0 *as in* (2.19)*, let $t \mapsto (u_k(t),\Gamma_k(t))$ be a sequence of incremental approximate solutions with initial datum (u_0,Γ_0), such that $\Gamma_k(t)$ σ^p-converges to a set $\Gamma^*(t) \in \mathcal{R}$, and let $\Gamma(t) := \Gamma^*(t) \cup \Gamma_0$. Then there exists a measurable function $t \mapsto u(t)$ from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$ such that $u(t)$ satisfies condition (b) of Definition 2.12.*

In view of the previous fact, repeating the proof of Theorem 2.13 we obtain an existence result for measurable evolutions.

Corollary 6.2 (Existence of measurable quasistatic evolutions)**.** *Let $\mathcal{F}(t)$ be as before. Let (u_0,Γ_0) be a minimum energy configuration at time* 0 *as in* (2.19)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t),\Gamma(t))$ with initial datum (u_0,Γ_0), such that $t \mapsto u(t)$ is measurable as a function from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$.*

The first step in the proof of Theorem 6.1 is the measurability in L^p: the following lemma is an adaptation of [14, Theorem 3.5].

Lemma 6.3. *In the hypotheses of Theorem 6.1, there exists a function $t \mapsto u(t)$, satisfying condition (b) of Definition 2.12, such that the function $t \mapsto (\nabla u(t), u(t))$ is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$.*

Proof. Let $(u_k(t),\Gamma_k(t))$ be a sequence of incremental approximate solutions associated to $(u(t),\Gamma(t))$ as in Definition 2.12. Let $\theta_k(t)$ be as in (2.21) and $\theta_\infty(t)$ as in (5.2). For every $t \in [0,1]$, let us consider the sets

$$\begin{aligned}\mathcal{A}(t) := \big\{(\nabla u, u)\colon u \in SBV^p(\Omega_D;K) &\text{ and there is a subsequence } k_j \text{ such that} \\ u_{k_j}(t) \rightharpoonup u \text{ weakly}^* \text{ in } SBV^p(\Omega_D;K) &\text{ and } \theta_{k_j}(t) \to \theta_\infty(t)\big\}\,.\end{aligned}$$

By Definition 2.12, for any selection $t \mapsto (\nabla u(t), u(t))$ the function $t \mapsto (u(t),\Gamma(t))$ is an incrementally-approximable quasistatic evolution.

By the Dominated Convergence Theorem and the Compactness Theorem 3.3, $(\nabla u, u) \in \mathcal{A}(t)$ if and only if there is a subsequence k_j such that $\nabla u_{k_j}(t)$ converges to ∇u weakly in $L^p(\Omega_D;\mathbb{M}^{n\times n})$, $u_{k_j}(t)$ converges to u weakly in $L^p(\Omega_D;\mathbb{R}^n)$, and $\theta_{k_j}(t) \to \theta_\infty(t)$. Moreover, as the gradients $\nabla u_k(t)$ are bounded in $L^p(\Omega_D;\mathbb{M}^{n\times n})$ uniformly in k and t and the functions $u_k(t)$ take value in K, there exists a bounded closed convex set $B \subset L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$ such that $(\nabla u_k(t), u_k(t)) \in B$ for every k and t. This leads to regard B as a compact metrizable space, endowed with the weak topology of $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$.

Thanks to [14, Lemma 3.6], the multifunction $t \mapsto \mathcal{A}(t)$ is measurable from $[0,1]$ to B. By the Aumann-von Neumann Selection Theorem [7, Theorem III.6], we can select $t \mapsto (\nabla u(t), u(t))$ in such a way that it is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$, endowed with the weak topology. The passage to the strong topology is an application of the Pettis Theorem [31, Chapter 5, Section 4]. □

Proof of Theorem 6.1. Consider the function $t \mapsto u(t)$ found in the previous lemma; we want to show that it is measurable from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$. Let $M_b(\Omega_D;\mathbb{M}^{n\times n})$ be the Banach space of all bounded $\mathbb{M}^{n\times n}$-valued Radon measures on Ω_D, endowed with the norm $\|\mu\|_{M_b(\Omega_D;\mathbb{M}^{n\times n})} := |\mu|(\Omega_D)$. Since $SBV^p(\Omega_D;\mathbb{R}^n)$ is isometric to a closed subspace of $L^1(\Omega_D;\mathbb{R}^n) \times L^p(\Omega_D;\mathbb{M}^{n\times n}) \times M_b(\Omega_D;\mathbb{M}^{n\times n})$ by (1.1), the measurability from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$ is equivalent to requiring that

- $t \mapsto u(t)$ is measurable from $[0,1]$ to $L^1(\Omega_D;\mathbb{R}^n)$,
- $t \mapsto \nabla u(t)$ is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n})$,
- $t \mapsto \mathrm{D}u(t)$ is measurable from $[0,1]$ to $M_b(\Omega_D;\mathbb{M}^{n\times n})$.

As $t \mapsto (\nabla u(t), u(t))$ is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$ and $S(u(t)) \mathrel{\tilde{\subset}} \Gamma(t)$, we must only prove the measurability of $t \mapsto [u(t)] \otimes \nu_{u(t)}\, \mathcal{H}^{n-1} \llcorner \Gamma(t)$, the jump part of $\mathrm{D}u(t)$, as a function from $[0,1]$ to $M_b(\Omega_D;\mathbb{M}^{n\times n})$. Notice that, by the monotonicity of $\Gamma(t)$, the unit normal vector $\nu_{u(t)}$ can be regarded as a time-independent term, equal to a prescribed unit normal ν to $\Gamma := \Gamma(1)$. Hence, $[u(t)] \otimes \nu_{u(t)}\, \mathcal{H}^{n-1} \llcorner \Gamma(t) = [u(t)] \otimes \nu\, \mathcal{H}^{n-1} \llcorner \Gamma$.

We are left to show the measurability of $t \mapsto [u(t)]\, \mathcal{H}^{n-1} \llcorner \Gamma$ from $[0,1]$ to $M_b(\Omega_D;\mathbb{R}^n)$. To this aim it is sufficient to prove that the function $t \mapsto [u(t)]$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$. For every $r > 0$, we consider the bounded linear operator $\Phi_r : L^1(\Omega_D;\mathbb{R}^n) \to L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ defined by

$$\Phi_r(u)(x) := \frac{2}{\mathcal{L}^n(B_r(x))}\Big(\int_{B_r^+(x)} u(y)\,\mathrm{d}y - \int_{B_r^-(x)} u(y)\,\mathrm{d}y\Big),$$

where $B_r^\pm(x)$ denotes the half-ball with centre x and radius r, oriented as $\pm\nu(x)$. Since $t \mapsto u(t)$ is measurable from $[0,1]$ to $L^1(\Omega_D;\mathbb{R}^n)$, the function $t \mapsto \Phi_r(u(t))$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ for every $r > 0$. As $u(t) \in BV(\Omega_D;\mathbb{R}^n) \cap L^\infty(\Omega_D;\mathbb{R}^n)$ for every t, we have $\Phi_r(u(t)) \to [u(t)]$ strongly in $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ as $r \to 0$. We conclude that $t \mapsto [u(t)]$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$. □

Remark 6.4. We have proven the measurability in the sense of SBV^p as a consequence of the measurability in the sense of L^p. Viceversa, one can see that, for every measurable map $t \mapsto u(t)$ from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$, the function $t \mapsto (\nabla u(t), u(t))$ is also measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$, so that the conclusion of Lemma 6.3 follows from Theorem 6.1.

7. Extension to volume forces

For the sake of simplicity, we have treated the case without applied forces, where the time dependence is given only by the boundary data. Actually, with elementary modifications to the proofs presented here, it is possible to consider smooth volume forces, depending on time.

We assume that the applied forces are conservative, i.e., there exists a function $G\colon [0,1]\times\Omega\times K\to\mathbb{R}$ such that the force density per unit volume in the reference configuration corresponding to a deformation $u\in SBV(\Omega_D;K)$ is given by $\mathrm{D}_yG(t,x,u(x))$, where $\mathrm{D}_yG(t,x,y)$ denotes the partial gradient of G with respect to y. So, the work done by the body forces is given up to an additive constant by

$$\mathcal{G}(t)(u) := \int_\Omega G(t,x,u(x))\,\mathrm{d}x\,. \tag{7.1}$$

We suppose that G satisfies the following properties:

(G1) $x\mapsto G(t,x,y)$ is $\mathcal{L}^n$-measurable on Ω for every $(t,y)\in[0,1]\times K$;
(G2) $(t,y)\mapsto G(t,x,y)$ is C^1 on $[0,1]\times K$ for every $x\in\Omega$;
(G3) there exists a constant $a_G>0$ such that

$$|G(t,x,y)|+|\mathrm{D}_tG(t,x,y)|+|\mathrm{D}_yG(t,x,y)|\le a_G$$

for every $(t,x,y)\in[0,1]\times\Omega\times K$.

Under these assumptions, for any $u\in SBV(\Omega_D;K)$ the function $t\mapsto\mathcal{G}(t)(u)$ is C^1 on $[0,1]$ and its derivative $\dot{\mathcal{G}}(t)(u)$ is given by

$$\dot{\mathcal{G}}(t)(u) = \int_\Omega \mathrm{D}_tG\left(t,x,u(x)\right)\,\mathrm{d}x\,. \tag{7.2}$$

Notice that the presence of the confinement hypothesis $u(x)\in K$ allows us to avoid the growth conditions with respect to y, required in [12].

We add the force term in (2.2) and redefine the total energy of the system, which now depends also on t:

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{W}(u)-\mathcal{G}(t)(u)+\mathcal{K}(\Gamma)\,. \tag{7.3}$$

Following the technique of multiplicative splitting (see Section 2.3), we look for a solution $u\in AD(\psi(t),\Gamma)$ to (2.4) in the form $u=\psi(t)\circ z$, with $z\in SBV(\Omega_D;K)$. To treat the case of the volume forces, we substitute (2.10) with

$$V(t,x,y,A) := W(x,\nabla\psi(t,y)\,A)-G(t,x,\psi(t,y))+a_G\,. \tag{7.4}$$

The term a_G, which has no influence on the solution, has been added in order to get $V\ge 0$. As always, given $u\in SBV(\Omega_D;K)$, $\mathcal{V}(t)(u)$ represents the integral of $V(t,x,u(x),\nabla u(x))$. We have

$$\mathcal{W}(u)-\mathcal{G}(t)(u) \;=\; \mathcal{V}(t)(\phi(t)\circ u)-b_G\,, \tag{7.5}$$

$$\mathcal{V}(t)(z)-b_G \;=\; \mathcal{W}(\psi(t)\circ z)-\mathcal{G}(t)(\psi(t)\circ z)\,, \tag{7.6}$$

where $b_G := a_G\,\mathcal{L}^n(\Omega)$. The last expression suggests that the minimal hypotheses on $\mathcal{G}$ depend on the assumptions on the prescribed deformation $\psi(t)$: they will be studied in [26].

It is possible to prove that the new functional $\mathcal{V}(t)$ satisfies the same properties (V1–8) stated in Section 2.3. Hence, the results concerning the existence and the main properties of quasistatic evolutions still hold. When coming back to the original formulation with time-dependent prescribed deformations, one should take into account the force term in the definition of the power of the system, which becomes

$$\begin{aligned} \mathcal{P}(t)(u) := & \int_\Omega \mathrm{D}_A W(x,\nabla u) : \nabla\big(\dot\psi(t)\circ\phi(t)\circ u\big)\,\mathrm{d}x \,+ \\ & - \int_\Omega \mathrm{D}_y G(t,x,u)\cdot\big(\dot\psi(t)\circ\phi(t)\circ v\big)\,\mathrm{d}x\,. \end{aligned} \tag{7.7}$$

The rule for the change of variables in the derivative of $\mathcal{V}(t)$ is now

$$\dot{\mathcal{V}}(t)(\phi(t)\circ u) = \mathcal{P}(t)(u) - \dot{\mathcal{G}}(t)(u)\,, \tag{7.8}$$

so that Definition 2.17 is modified by setting

$$\eta_k(t) := \mathcal{P}(t)(u_k(t)) - \dot{\mathcal{G}}(t)(u_k(t))\,. \tag{7.9}$$

Finally, Theorems 2.18 and 2.19 also hold for the system with applied forces, with the energy balance law

$$\dot E(t) = \mathcal{P}(t)(u(t)) - \dot{\mathcal{G}}(t)(u(t))\,, \tag{7.10}$$

where $E(t) := \mathcal{E}(t)(u(t),\Gamma(t))$. We leave the details to the reader.

Appendix A. Some remarks about non-interpenetration

Besides the Ciarlet-Nečas condition for cracked bodies, adopted in the present paper (see Definition 1.1), two other notions of non-interpenetration can be considered for a function $u\in SBV(\Omega;\mathbb{R}^n)$:

(a) *Linearized self-contact condition:* for $\mathcal{H}^{n-1}$-a.e. $x\in S(u)$

$$[u(x)]\cdot\nu_u(x)\ge 0\,;$$

(b) *Progressive non-interpenetration:* there exists a "continuous" function $\lambda\mapsto u(\lambda)$, defined for $\lambda\in[0,1]$ and with values in $SBV(\Omega;\mathbb{R}^n)$, such that $u(0)$ is the identity map, $u(1)=u$, and $u(\lambda)$ satisfies the Ciarlet-Nečas condition of Definition 1.1 for every $\lambda\in[0,1]$.

Condition (b) clearly depends on the choice of the notion of continuity: ideally, it should be selected so that $\lambda\mapsto u(\lambda)$ is continuous if and only if the associated motion can be realized by a physical process.

In [21, Section 6], Definition 1.1 and condition (a) have been compared, showing that neither property implies the other one. Moreover, if $u\in SBV^q(\Omega;\mathbb{R}^n)$ for some $q>n$, it is proven in [21, Proposition 6.2] that (a) holds whenever the functions

$$u(\lambda,x) := x+\lambda\, v(x) \tag{A.1}$$

satisfy Definition 1.1 for every $\lambda\in[0,1]$, where $v(x):=u(x)-x$. Since this property usually holds when the displacement $v(x)$ is "small", this result suggests that the linearized self-contact condition is natural for linearized elasticity. It also proves that (b) implies (a) in the special case where $u(\lambda)$ is given by (A.1).

The following examples show that, in the general case, the progressive non-interpenetration does not imply the linearized self-contact condition, even if $u(\lambda,x)$ is smooth out of the jump set. In both examples $n=2$ and Ω is the open ball with centre 0 and radius 2.

Example A.1. For every $\lambda\in[0,1]$ and $x\in\Omega$, let

$$u(\lambda,x) := \begin{cases} x & \text{if } |x|\le 1\,,\\ R_\lambda\, x & \text{if } |x|>1\,, \end{cases}$$

where R_λ is the rotation of angle λ. Then for every λ the Ciarlet-Nečas condition is satisfied, the jump set $S(u(\lambda,\cdot))$ coincides with $\Gamma := \{|x|=1\}$, and

$$[u(\lambda,x)]\cdot\nu(x) = (R_\lambda\, x - x)\cdot x = \cos\lambda - 1 < 0 \quad \text{for every } x\in\Gamma\,.$$

In this case the lips of the crack in the deformed configuration remain in contact for every λ. However, we can obtain a similar example with an opening crack, defining

$$u(\lambda,x) := \begin{cases} x & \text{if } |x|\le 1\,,\\ a_\lambda\, R_\lambda\, x & \text{if } |x|>1\,,\end{cases}$$

where $\lambda\mapsto a_\lambda$ is continuous and $1<a_\lambda<1/\cos\lambda$ for $0<\lambda<1$.

In the previous example, the crack set in the reference configuration does not depend on λ, and $u(\lambda,x)=x$ on one of the regions determined by the crack set. The violation of (a) is obtained by exploiting the strict convexity of this region. Instead, the next example achieves the same result with a rectilinear crack.

Example A.2. Let $\zeta\in C^\infty(\mathbb{R})$ be a nondecreasing function such that $\zeta(s)=0$ for $s\le 0$ and $\zeta(s)>0$ if $s>0$, and let $\Gamma := \{(x_1,0) : 0<x_1<1\}$. For every $\lambda\in[0,1]$ and every $x=(x_1,x_2)\in\Omega\setminus\Gamma$ we define

$$u(\lambda,x) := \begin{cases} (x_1, x_2+\lambda\, x_1^2) & \text{if } x_2>0\,,\\ (x_1,\lambda\, x_1^2) & \text{if } x_1\le 0 \text{ and } x_2=0\,,\\ (x_1[1+\lambda\,\zeta(x_1)], x_2+\lambda\, x_1^2[1+\lambda\,\zeta(x_1)]) & \text{if } x_2<0\,.\end{cases}$$

First of all, we observe that $u(\lambda,\cdot)$ is injective in each of the three regions used for the definition (thanks to the monotonicity of ζ). To prove the injectivity on the whole domain, it is enough to show that these regions are mapped into pairwise disjoint sets. The image of $\{x_2>0\}$ lies strictly above the parabola $\Pi := \{(x_1,\lambda x_1^2)\colon x_1\in\mathbb{R}\}$; the region $\{x_1\le 0\,,\, x_2=0\}$ is mapped into Π, while the image of the third region $\{x_2<0\}$ lies strictly below the curve $\{(x_1[1+\lambda\,\zeta(x_1)],\lambda x_1^2[1+\lambda\,\zeta(x_1)])\colon x_1\in\mathbb{R}\}$. The branch of this curve corresponding to $x_1\le 0$ is contained in Π, while the branch corresponding to $x_1>0$ lies strictly below Π for $\lambda>0$, since $1<1+\lambda\,\zeta(x_1)$. This shows that $u(\lambda,\cdot)$ is injective and that the crack lips in the deformed configuration overlap only at the crack tip $(0,0)$, except for $\lambda=0$. Moreover, u belongs to $C^\infty([0,1]\times(\Omega\backslash\Gamma))$ and all its partial derivatives have a finite limit on both sides of Γ. For every λ the jump set $S(u(\lambda,\cdot))$ coincides with Γ, and

$$[u(\lambda,x)]\cdot\nu(x) = -\lambda^2\, x_1^2\,\zeta(x_1) < 0 \quad \text{for every } x\in\Gamma\,.$$

In both cases condition (a) is violated not only by $u(1,\cdot)$, but also by $u(\lambda,\cdot)$ for every $\lambda>0$. Hence, (a) may not hold even if the deformation satisfies (b) and is very close to the identity in a C^∞ sense. Notice that, if λ is interpreted as time, the function $u(\lambda,x)$ represents a physically admissible motion of the cracked body $\Omega\backslash\Gamma$, starting from the undeformed configuration $u(0,x)=x$. Therefore, requiring (a) appears to be unnatural, unless one linearizes with respect to λ at $\lambda=0$.

In our opinion, the correct notion of non-interpenetration in nonlinear fracture mechanics is condition (b), since it takes into account the fact that the deformation is always the result of a "continuous" evolution through non-interpenetrating intermediate states, starting from an initial condition, that may be taken as reference configuration. Unfortunately, up to now, there are no mathematical results concerning the stability of this property: this is the reason why we adopted instead the Ciarlet-Nečas condition.

However, if we consider an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$, according to Definition 2.17, such that the initial datum u_0 satisfies (b) and $t \mapsto u(t)$ is continuous on some interval $[0, \tau]$, in the same sense chosen for (b), it follows immediately from the definition that $u(t)$ satisfies also the progressive non-interpenetration condition for every $t \in [0, \tau]$.

Acknowledgments. This work is part of the Project “Variational problems with multiple scales” 2006, supported by the Italian Ministry of University and Research.

References

[1] L. Ambrosio: *A compactness theorem for a new class of functions of bounded variation.* Boll. Un. Mat. Ital. B **3** (1989), 857–881.

[2] L. Ambrosio: *On the lower semicontinuity of quasiconvex integrals in* $SBV(\Omega, \mathbb{R}^k)$. Nonlinear Anal. **23** (1994), 405–425.

[3] L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[4] J. M. Ball: *Some open problems in elasticity.* In: P. Newton, P. Holmes, and A. Weinstein, editors, *Geometry, mechanics, and dynamics*, 3–59. Springer, New York, 2002.

[5] W. W. Bledsoe, A. P. Morse: *Some aspects of covering theory.* Proc. Amer. Math. Soc. **3** (1952), 804–812.

[6] B. Bourdin, G. A. Francfort, J.-J. Marigo: *The variational approach to fracture.* J. Elasticity **91** (2008), 5–148.

[7] C. Castaing, M. Valadier: *Convex analysis and measurable multifunctions.* Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-New York, 1977.

[8] A. Chambolle: *A density result in two-dimensional linearized elasticity, and applications.* Arch. Ration. Mech. Anal. **167** (2003), 211–233.

[9] P. G. Ciarlet: *Mathematical elasticity – Vol. I: Three-dimensional elasticity.* Studies in Mathematics and its Applications **20**, North-Holland Publishing Co., Amsterdam, 1988.

[10] P. G. Ciarlet, J. Nečas: *Injectivity and self-contact in nonlinear elasticity.* Arch. Ration. Mech. Anal. **97** (1987), 171–188.

[11] B. Dacorogna: *Direct methods in the calculus of variations.* Applied Mathematical Sciences **78**, Springer, New York, 2008.

[12] G. Dal Maso, G. A. Francfort, R. Toader: *Quasistatic crack growth in nonlinear elasticity.* Arch. Ration. Mech. Anal. **176** (2005), 165–225.

[13] G. Dal Maso, G. A. Francfort, R. Toader: *Quasistatic crack growth in finite elasticity.* Preprint SISSA, Trieste, 2004 (`http://www.sissa.it/fa/`).

[14] G. Dal Maso, A. Giacomini, M. Ponsiglione: *A variational model for quasi-static growth in nonlinear elasticity: some qualitative properties of the solutions.* Boll. Unione Mat. Ital. (9) **2** (2009), 371–390.

[15] G. Dal Maso, R. Toader: *A model for the quasi-static growth of brittle fractures: existence and approximation results.* Arch. Ration. Mech. Anal. **162** (2002), 101–135.

[16] I. Fonseca, G. Leoni: *Modern methods in the calculus of variations:* L^p *spaces.* Springer, New York, 2007.

[17] G. A. Francfort, C. J. Larsen: *Existence and convergence for quasi-static evolution in brittle fracture.* Comm. Pure Appl. Math. **56** (2003), 1465–1500.

[18] G. A. Francfort, J.-J. Marigo: *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids **46** (1998), 1319–1342.

[19] G. A. Francfort, A. Mielke: *Existence results for a class of rate-independent material models with nonconvex elastic energies.* J. Reine Angew. Math. **595** (2006), 55–91.

[20] N. Fusco, C. Leone, R. March, A. Verde: *A lower semi-continuity result for polyconvex unctionals in* SBV. Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 321–336.

[21] A. Giacomini, M. Ponsiglione: *Non interpenetration of matter for* SBV*-deformations of hyperelastic brittle materials.* Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 1019–1041.

[22] A. A. Griffith: *The phenomena of rupture and flow in solids.* Philos. Trans. Roy. Soc. London Ser. A **221** (1921), 163–198.

[23] H. Hahn: *Über Annäherung an Lebesgue'sche Integrale durch Riemann'sche Summen*. Sitzungsber. Math. Phys. Kl. K. Akad. Wiss. Wien **123** (1914), 713–743.

[24] D. Knees, A. Mielke: *Energy release rate for cracks in finite-strain elasticity*. Math. Methods Appl. Sci. **31** (2008), 501–528.

[25] D. Knees, C. Zanini, A. Mielke: *Crack growth in polyconvex materials*. To appear on Phys. D.

[26] G. Lazzaroni: *Quasistatic crack growth in finite elasticity with Lipschitz data*. In preparation.

[27] A. Mielke: *Evolution of rate-independent systems*. In: C. M. Dafermos and E. Feireisl, editors, *Evolutionary equations – Vol. II*, 461–559. Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2005.

[28] A. P. Morse: *Perfect blankets*. Trans. Amer. Math. Soc. **61** (1947) 418–442.

[29] R. W. Ogden: *Large deformation isotropic elasticity – on the correlation of theory and experiment for incompressible rubberlike solids*. Proc. Roy. Soc. London A **326** (1972), 565–584.

[30] R. W. Ogden: *Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubberlike solids*. Proc. Roy. Soc. London A **328** (1972), 567–583.

[31] K. Yosida, *Functional analysis*. Grundlehren der Mathematischen Wissenschaften **123**, Springer-Verlag, Berlin-New York, 1980.

(G. Dal Maso, G. Lazzaroni) SISSA, via Beirut 2–4, 34151 Trieste, Italy

E-mail address, G. Dal Maso: dalmaso@sissa.it

E-mail address, G. Lazzaroni: giuliano.lazzaroni@sissa.it