MONOTONICITY FORMULAS FOR OBSTACLE PROBLEMS WITH LIPSCHITZ COEFFICIENTS

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Abstract. We prove quasi-monotonicity formulas for classical obstacle-type problems with energies being the sum of a quadratic form with Lipschitz coefficients, and a Hölder continuous linear term. With the help of those formulas we are able to carry out the full analysis of the regularity of free-boundary points following the approaches in [7, 22, 28].

1. Introduction

In this note we extend the regularity theory for the obstacle problem to the case of quadratic energies with Lipschitz coefficients. The obstacle problem is a well-known topic in partial differential equations and, in its classical formulation, consists in finding the equilibrium solution for a scalar order parameter u constrained to lay above a given obstacle, $u \geq \psi$ – see, e.g., [14, 25] for several applications in physics. The analytical interests in this kind of problems are mostly related to the study of the properties of the *free boundary*, the boundary of the set where the equilibrium configuration touches the obstacle. This subject has been developed over the last 40 years by the works of many authors; it is not realistic to give here a complete account: we rather refer to the textbooks [9, 14, 18, 24, 25] for a fairly vast bibliography and its historical developments.

Very recently many authors have drawn the attention on the issue of weakening the hypotheses on the operators governing the obstacle-type problems, in order to enlarge the applicability of the results and deepen the analytical techniques introduced in the study of such problems (cp. [11, 12, 13, 21, 26, 27]). The prototype result in obstacle-type problems is a stratification of the free boundary $\partial\{u = \psi\}$ in terms of the properties of the blowup limits.

In this note we complete this program for the case of an obstacle problem with a quadratic energy having Lipschitz coefficients and suitable obstacle functions ψ (e.g., such that $\mathrm{div}(\mathbb{A}\nabla\psi) \in C^{0,\alpha}$ in the distributional sense), which can be reduced to the 0 obstacle case. We collect in the statement below the main results of our analysis, in particular the contents of Theorems 4.12 and 4.14.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and open, $\mathbb{A} \in \mathrm{Lip}(\Omega, \mathbb{R}^{n\times n})$ be symmetric and uniformly elliptic, i.e. $\lambda^{-1}|y|^2 \leq \langle \mathbb{A}(x)y, y\rangle \leq \lambda|y|^2$ for all $x \in \Omega$ and all $y \in \mathbb{R}^n$, and $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1]$ with $f \geq c_0 > 0$. Let u be the solution of the obstacle problem*

$$\min \mathscr{E}[v] := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)\, v(x)\big)\, dx,$$

where the minimum is taken in

$$\mathcal{K} := \big\{v \in H^1(\Omega) : v \geq 0 \;\; \mathcal{L}^n\text{-a.e. on } \Omega,\ \mathrm{Tr}(v) = g \;\text{ on } \partial\Omega\big\},$$

for $g \in H^{1/2}(\partial\Omega)$ a nonnegative function. Then, u is $C^{1,\gamma}_{loc}$ regular in Ω for every $\gamma \in (0, 1)$, and the free boundary decomposes as $\partial\{u = 0\} \cap \Omega = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$ denote the regular and the singular part of the free boundary, respectively, $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ and *(i)* $\mathrm{Reg}(u)$ *is relatively open in* $\partial\{u = 0\}$ *and, for every point* $x_0 \in \mathrm{Reg}(u)$*, there exist* $r = r(x_0) > 0$ *and* $\beta = \beta(x_0) \in (0, 1)$ *such that* $\mathrm{Reg}(u) \cap B_r(x_0)$ *is a* $C^{1,\beta}$ *submanifold of dimension* $n - 1$*;*

(ii) $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$*, with* S_k *contained in the union of at most countably many submanifold of dimension* k *and class* C^1*.*

The theorem above for the Dirichlet energy is the outcome of a long term program and of the efforts of many authors. It has been proved first by Caffarelli [7] under more restrictive hypothesis on f, namely $f \in C^{1,\alpha}$. Caffarelli has also streamlined his arguments to cover the case of $f \in C^{0,\alpha}$, as it has been pointed out to us by Blank (cp. [3], where further finer results are also proved). The proof in [7] is based on a monotonicity formula introduced by Alt, Caffarelli and Friedman [1] and on the regularity of harmonic functions in Lipschitz domains [2, 8, 17]. Since then, different approaches have been introduced, most remarkably the variational one by Weiss [28] and Monneau [22], who extended the techniques to deal with Hölder continuous linear terms f and simplified the arguments for the analysis of the free boundary. These improvements allowed to extend the results by Caffarelli to some other obstacle-type settings, such as the no-sign obstacle problem [10] and the two-phases membrane problem [29] – see [24] for more detailed comments, and [20] for a revisitation of such arguments in a geometric measure theory flavour.

The lack of regularity and homogeneity of the coefficients in our framework does not allow us to exploit any simple freezing argument in a way to reduce the problem to the ones above for regular operators. Indeed, in the proof of Theorem 1.1 we take advantages of the full strength of those contributions, including the remarkable epiperimetric inequality established by Weiss [28]. We prove quasi-monotonicity formulas analogous to those introduced by Weiss and Monneau for the Laplace equation. To this aim, we exploit some intrinsic computations based on a generalization of Rellich and Nečas' identity due to Payne and Weinberger (which we first learned in a paper by Kukavica [19]).

Our results leads to the stratification of the free boundary for more general obstacle problems with quasi-linear operator with $C^{1,1}$ regular solutions:

$$\min \int_\Omega \big(F(|\nabla u|^2) + G(x, u)\big)\, dx,$$

with F, G satisfying suitable assumptions, as, e.g., the ones considered in [22], which covers the case of the area functional. In particular, we point out the recent contribution by Matevosyan and Petrosyan [21], where they perform the analogous improvement of the ACF monotonicity formula for more general operators. As a byproduct of their analysis, $C^{1,1}$ regularity of solutions of a broad class of obstacle problems follows and, combining these results with our analysis, the complete stratification of the free boundary may be inferred for classical obstacle problems corresponding to a subclass of the quasi-linear operators considered by these authors, with applications to certain mean-field models for type II superconductors (cp., e.g., [13, 21]).

To conclude this introduction we describe briefly the contents of the paper: Section 2 is devoted to settle the notations, fix the main assumptions and derive the first basic results on the problem. Weiss' and Monneau's quasi-monotonicity formulas are then established in Section 3 (cp. with Theorems 3.7 and 3.8, respectively). The latter are instrumental tools to study in Section 4 the blow-up limits in free boundary points (cp. with Propositions 4.2, 4.5, 4.10 and 4.11). In turn, such an analysis leads to the regularity results stated in Theorem 1.1 (cp. with Theorems4.12 and 4.14).

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded and open set. Let $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ be a matrix-valued field and $f : \Omega \to \mathbb{R}$ be a function satisfying

(H1) $\mathbb{A} \in \mathrm{Lip}(\Omega, \mathbb{R}^{n\times n})$;
(H2) $\mathbb{A}(x) = (a_{ij}(x))_{i,j=1}^n$ is symmetric and coercive, i.e. $a_{ij} = a_{ji}$ and, for some $\lambda \geq 1$,

$$\lambda^{-1}|y|^2 \leq \langle \mathbb{A}(x)y, y\rangle \leq \lambda |y|^2 \quad \text{for all } x \in \Omega \text{ and for all } y \in \mathbb{R}^n;$$

(H3) $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ and $f \geq c_0 > 0$.

Remark 2.1. For some of the results of the paper, a weaker condition on f would suffice (e.g., a continuous function with a modulus of continuity satisfying a certain Dini-type integrability condition – cp. [23]). However, we do not pursue this issue here.

For all open subsets A of Ω and functions $v \in H^1(\Omega)$, we consider the energy

$$\mathscr{E}[v, A] := \int_A \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)\, v(x)\big)\, dx, \tag{2.1}$$

and the related minimum problem $\inf_{\mathcal{K}} \mathscr{E}[\cdot, \Omega]$, where $\mathcal{K}$ is the weakly closed convex subset of $H^1(\Omega)$ given by

$$\mathcal{K} := \big\{v \in H^1(\Omega) : \ v \geq 0 \ \ \mathcal{L}^n\text{-a.e. on } \Omega,\ \mathrm{Tr}\, v = g \ \text{ on } \partial\Omega\big\},$$

with $g \in H^{1/2}(\partial\Omega)$ a nonnegative function.

Existence and uniqueness for the above minimum problem follow straightforwardly from (H1)-(H3). In fact, the energy $\mathscr{E}$ is coercive and strictly convex in $\mathcal{K}$, which implies the lower semicontinuity for the weak topology in $H^1(\Omega)$ and the uniqueness of the minimizer, denoted in the sequel by u. Moreover, letting for any $v \in H^1(\Omega)$,

$$\mathscr{G}[v, \Omega] := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)\, v^+(x)\big)\, dx, \tag{2.2}$$

we easily infer the existence of a unique minimizer for $\mathscr{G}$ on $g + H^1_0(\Omega)$, with boundary datum $g \in H^{1/2}(\partial\Omega)$, and satisfying

$$\min_{\mathcal{K}} \mathscr{E}[\cdot, \Omega] = \min_{g+H^1_0(\Omega)} \mathscr{G}[\cdot, \Omega].$$

As in the classical case, the minimizer u satisfies a PDE both in a distributional sense and almost everywhere in Ω, as pointed out in the next proposition.

Proposition 2.2. *Let u be the minimizer of $\mathscr{E}$ in $\mathcal{K}$. Then,*

$$\mathrm{div}\,(\mathbb{A}(x)\nabla u(x)) = f(x)\, \chi_{\{u>0\}}(x) \quad \textit{a.e. in } \Omega \textit{ and in } \mathcal{D}'(\Omega). \tag{2.3}$$

Proof. Let $\varphi \in H^1_0 \cap C^0(\Omega)$ and $\varepsilon > 0$, and consider $u + \varepsilon\varphi$ as a competitor for $\mathscr{G}$. Then,

$$\begin{aligned} 0 \leq & \varepsilon^{-1}\Big(\mathscr{G}[u+\varepsilon\varphi, \Omega] - \mathscr{G}[u, \Omega]\Big) \\ = & \int_\Omega \big(\varepsilon\langle \mathbb{A}\nabla\varphi, \nabla\varphi\rangle + 2\langle \mathbb{A}\nabla u, \nabla\varphi\rangle\big)\, dx + 2\,\varepsilon^{-1}\int_\Omega f\,\big((u+\varepsilon\varphi)^+ - u\big)\, dx, \end{aligned} \tag{2.4}$$

where in the last identity we have used the positivity of u. Expanding the computation, we get

$$\int_\Omega f\,\big((u+\varepsilon\varphi)^+ - u\big)\, dx = \varepsilon \int_{\{u+\varepsilon\varphi\geq 0\}} f\,\varphi\, dx - \int_{\{u+\varepsilon\varphi<0\}} f\, u\, dx. \tag{2.5}$$

We note that

$$0 \leq \int_{\{u+\varepsilon\varphi<0\}} f\, u\, dx \leq -\varepsilon \int_{\{u+\varepsilon\varphi<0\}} f\,\varphi\, dx = o(\varepsilon).$$

Moreover, setting $A_\varphi := \{u=0\}\cap\{\varphi\geq 0\}$, it is easy to show that $\chi_{\{u+\varepsilon\varphi\geq 0\}}\to\chi_{A_\varphi\cup\{u>0\}}$ in L^1. Then, passing into the limit in (2.4), by (2.5) and the dominated convergence theorem, we deduce that

$$\int_\Omega \langle \mathbb{A}\nabla u, \nabla\varphi\rangle\,dx + \int_\Omega \varphi\, f\,\chi_{\{u>0\}\cup A_\varphi}\,dx \geq 0. \tag{2.6}$$

Set now

$$T(\varphi) := \int_\Omega \langle \mathbb{A}\nabla u, \nabla\varphi\rangle\,dx + \int_\Omega \varphi\, f\,\chi_{\{u>0\}}\,dx.$$

By applying (2.6) with $\pm\varphi$, we deduce

$$-\int_{A_\varphi}\varphi\, f\,dx \leq T(\varphi) \leq -\int_{\{u=0\}\cap\{\varphi\leq 0\}}\varphi\, f\,dx. \tag{2.7}$$

In particular, by a density argument, we deduce that

$$|T(\varphi)| \leq C\,\|\varphi\|_{L^\infty(\Omega)} \quad \text{for every } \varphi\in C^0_0(\Omega).$$

This, in turn, implies that the distribution T is a (nonpositive) Borel measure which, in view of (2.7), is dominated by an absolutely continuous measure with respect to the Lebesgue measure, so that $T=\zeta\,dx$ for some density $\zeta\in L^1_{\mathrm{loc}}(\Omega)$. Moreover, again by (2.7), we deduce that $\zeta=0$ $\mathcal{L}^n$ a.e. on $\{u>0\}$; and, since $\nabla u=0$ $\mathcal{L}^n$ a.e. on the set $\{u=0\}$, by the very definition of T we also get $\zeta=0$ $\mathcal{L}^n$ a.e. in Ω. Clearly, this shows (2.3). □

The regularity theory for uniformly elliptic equations with Lipschitz coefficients (cp. [15, Chapter III, Theorem 3.5]) and Sobolev embeddings yield that $u\in W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $p\in[1,\infty)$, and hence $u\in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ for every $\gamma\in(0,1)$. Note that, contrary to the usual obstacle-type problems, in general u fails to be $C^{1,1}_{\mathrm{loc}}$, because of the lack of regularity of the coefficients $\mathbb{A}$ (see [16, Exercise 4.9] for a related counterexample). Despite this, the sign condition on u guarantees $C^{1,1}$ regularity on the set $\{u=0\}$ (cp. with Proposition 3.2 below).

Finally, we fix the notation for the *coincidence set*, the *non-coincidence set* and the *free boundary*:

$$\Lambda_u := \{u=0\},\quad N_u := \{u>0\}\quad\text{and}\quad \Gamma_u := \partial\Lambda_u\cap\Omega.$$

3. Weiss' and Monneau's quasi-monotonicity formulas

In this section we show that the monotonicity formulas established by Weiss [28] and Monneau [22] in the standard case of the Laplace operator, i.e. $\mathbb{A}\equiv \mathrm{I}_n$, hold in an approximate way in the present setting.

3.1. **Notation and preliminary results.**

The first step towards the monotonicity formulas is to fix appropriate systems of coordinates with respect to which the formulas will be written. Let $x_0\in\Gamma_u$ be any point of the free boundary, then the affine change of variables

$$x\mapsto x_0 + f^{-1/2}(x_0)\mathbb{A}^{1/2}(x_0)x =: x_0+\mathbb{L}(x_0)\,x$$

leads to

$$\mathscr{E}[u,\Omega] = f^{1-\frac{n}{2}}(x_0)\det(\mathbb{A}^{1/2}(x_0))\,\mathscr{E}_{\mathbb{L}(x_0)}[u_{\mathbb{L}(x_0)},\Omega_{\mathbb{L}(x_0)}], \tag{3.1}$$

where, for all open subset A of $\Omega_{\mathbb{L}(x_0)} := \mathbb{L}(x_0)\,(\Omega-x_0)$, we set

$$\mathscr{E}_{\mathbb{L}(x_0)}[v,A] := \int_A\left(\langle \mathbb{C}_{x_0}\nabla v,\nabla v\rangle + 2\frac{f_{\mathbb{L}(x_0)}}{f(x_0)}\,v\right)dx, \tag{3.2}$$

with

$$u_{\mathbb{L}(x_0)}(x) := u\big(x_0+\mathbb{L}(x_0)x\big), \tag{3.3}$$

$$f_{\mathbb{L}(x_0)}(x) := f\big(x_0 + \mathbb{L}(x_0)x\big), \tag{3.4}$$

$$\mathbb{C}_{x_0}(x) := \mathbb{A}^{-1/2}(x_0)\mathbb{A}(x_0 + \mathbb{L}(x_0)x)\mathbb{A}^{-1/2}(x_0). \tag{3.5}$$

Note that $f_{\mathbb{L}(x_0)}(\underline{0}) = f(x_0)$ and $\mathbb{C}_{x_0}(\underline{0}) = \mathrm{I}_n$. Moreover, the free boundary is transformed under this map into

$$\Gamma_{u_{\mathbb{L}(x_0)}} = \mathbb{L}(x_0)(\Gamma_u - x_0),$$

and the energy $\mathscr{E}$ in (2.1) is minimized by u if and only if $\mathscr{E}_{\mathbb{L}(x_0)}$ in (3.2) is minimized by the function $u_{\mathbb{L}(x_0)}$ in (3.3).

Therefore, for a fixed base point $x_0 \in \Gamma_u$, we change the coordinates system in such a way that (with a slight abuse of notation we do not rename the various quantities) we reduce to

$$x_0 = \underline{0} \in \Gamma_u, \quad \mathbb{A}(\underline{0}) = \mathrm{I}_n \quad \text{and} \quad f(\underline{0}) = 1. \tag{3.6}$$

This convention shall be adopted throughout this section to simplify the ensuing calculations. Note that with this convention at hand $\underline{0} \in \Omega$. In this new system of coordinates we define

$$\nu(x) := \frac{x}{|x|} \quad \text{for } x \neq \underline{0}\,,$$

and

$$\mu(x) := \langle \mathbb{A}(x)\,\nu(x), \nu(x)\rangle \quad \text{for } x \neq \underline{0}, \quad \mu(\underline{0}) := 1. \tag{3.7}$$

Note that $\mu \in C^0(\Omega)$ thanks to (H1) and (3.6). Actually, μ is Lipschitz continuous.

Lemma 3.1. *If $\mathbb{A}$ satisfies (H1)-(H2) and* (3.6)*, then $\mu \in C^{0,1}(\Omega)$, and*

$$|\mu(x) - \mu(y)| \leq C\,\|\mathbb{A}\|_{W^{1,\infty}}|x-y| \quad \textit{for all } x, y \in \Omega, \tag{3.8}$$

where $C > 0$ is a dimensional constant $C > 0$, and

$$\lambda^{-1} \leq \mu(x) \leq \lambda \quad \textit{for all } x \in \mathbb{R}^n. \tag{3.9}$$

Proof. Note that in case $y = \underline{0}$ we have

$$\mu(x) - \mu(\underline{0}) = \langle (\mathbb{A}(x) - \mathrm{I}_n)\frac{x}{|x|}, \frac{x}{|x|}\rangle,$$

so that estimate (3.8) follows directly from (H1).

Then let $x, y \neq \underline{0}$ and set $z = |y|\frac{x}{|x|}$. Then, $|z| = |y|$ and by triangle inequality

$$\begin{aligned}|\mu(x) - \mu(y)| &\leq |\mu(x) - \mu(z)| + |\mu(z) - \mu(y)| \leq |\mathbb{A}(x) - \mathbb{A}(z)| + |\mu(z) - \mu(y)| \\ &\leq \|\mathbb{A}\|_{W^{1,\infty}}||x| - |y|| + |\mu(z) - \mu(y)|.\end{aligned}$$

We need only to estimate the last term. Set for simplicity $|z| = |y| = r$, and use again the triangle inequality

$$|\mu(z) - \mu(y)| \leq |\langle (\mathbb{A}(z) - \mathbb{A}(y))\frac{z}{r}, \frac{z}{r}\rangle| + |\langle \mathbb{A}(y)\frac{z}{r}, \frac{z}{r}\rangle - \langle \mathbb{A}(y)\frac{y}{r}, \frac{y}{r}\rangle|.$$

The first term is easily estimated thanks to (H1),

$$|\langle (\mathbb{A}(z) - \mathbb{A}(y))\frac{z}{r}, \frac{z}{r}\rangle| \leq \|\mathbb{A}\|_{W^{1,\infty}}|z-y|. \tag{3.10}$$

For the second term we use equality $\mathbb{A}(\underline{0}) = \mathrm{I}_n$ (see (3.6)) and $|z| = |y| = r$ to rewrite it as follows:

$$\langle \mathbb{A}(y)\frac{z}{r}, \frac{z}{r}\rangle - \langle \mathbb{A}(y)\frac{y}{r}, \frac{y}{r}\rangle = \langle \mathbb{A}(y)\frac{z+y}{r}, \frac{z-y}{r}\rangle = \langle (\mathbb{A}(y) - \mathbb{A}(\underline{0}))\frac{z+y}{r}, \frac{z-y}{r}\rangle,$$

which in turn implies

$$|\langle \mathbb{A}(y)\frac{z}{r}, \frac{z}{r}\rangle - \langle \mathbb{A}(y)\frac{y}{r}, \frac{y}{r}\rangle| \leq 2\|\mathbb{A}\|_{W^{1,\infty}}|z-y|. \tag{3.11}$$

Since $|z-y| \le |z-x| + |x-y| \le 2|x-y|$, inequalities (3.10) and (3.11) yield (3.8). Estimate (3.9) follows easily from (H2). □

Next we introduce the following notation for the rescaled functions and the rescaled energies:

$$u_r(x) := \frac{u(rx)}{r^2}, \tag{3.12}$$

$$\begin{aligned} \mathscr{E}(r) := \mathscr{E}[u, B_r] &= \int_{B_r} \left(\langle \mathbb{A}(x)\nabla u(x), \nabla u(x)\rangle + 2f\,u\right)\,dx \\ &= r^{n+2}\int_{B_1} \left(\langle \mathbb{A}(rx)\nabla u_r(x), \nabla u_r(x)\rangle + 2f(rx)\,u_r(x)\right)\,dx, \end{aligned} \tag{3.13}$$

$$\mathscr{H}(r) := \int_{\partial B_r} \mu\, u^2 d\mathcal{H}^{n-1} = r^{n+3}\int_{\partial B_1} \mu(rx)\,u_r^2(x) d\mathcal{H}^{n-1}, \tag{3.14}$$

and

$$\Phi(r) := r^{-n-2}\mathscr{E}(r) - 2\,r^{-n-3}\mathscr{H}(r). \tag{3.15}$$

Although the minimizer u is not in general globally $C^{1,1}_{\mathrm{loc}}$, the rescaled functions u_r satisfy uniform $W^{2,p}_{\mathrm{loc}}$ estimates thanks to Harnack inequality.

Proposition 3.2. *Let u be the solution to the obstacle problem* (2.1)*, and assume* (3.6) *holds. Then, for every $p \in [1,\infty)$ and $R > 0$, there exists a constant $C = C(p,R) > 0$ such that, for every $r \in (0, \frac{1}{2R}\mathrm{dist}(\underline{0}, \partial\Omega))$,*

$$\|u_r\|_{W^{2,p}(B_R)} \le C. \tag{3.16}$$

In particular, the functions u_r are equibounded in $C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\gamma \in (0,1)$.

Proof. By Proposition 2.2 and (3.12), we have that $\mathrm{div}(\mathbb{A}(rx)\nabla u_r(x)) = f(rx)\chi_{\{u_r>0\}}(x)$ in the weak sense. Since u is non-negative, we can apply the Harnack inequality (cp. [16, Theorems 8.17 and 8.18]) to infer that, for a positive constant $C = C(n,\lambda)$,

$$\|u_r\|_{L^\infty(B_{2R})} \le C\,\|f\|_{L^\infty(B_{2R})}.$$

Let now w be the harmonic function with $w|_{\partial B_{2R}} = u_r|_{\partial B_{2R}}$, and

$$g_r(x) := f(rx)\chi_{\{u_r>0\}}(x) - r\nabla\mathbb{A}(rx)\nabla w(x) - \mathbb{A}(rx) : \nabla^2 w(x),$$

where : stands for the scalar product between $n \times n$ matrices. As $\|g_r\|_{L^\infty(B_R)} \le C$ uniformly in r, and

$$\mathrm{div}(\mathbb{A}(rx)\nabla(u_r - w)(x)) = g_r(x)$$

by elliptic regularity theory (cp. [15, Chapter III Theorem 3.5], [18, Chapter IV Theorem A.1]), we deduce that

$$\|u_r\|_{W^{2,p}(B_R)} \le \|u_r - w\|_{W^{2,p}(B_R)} + \|w\|_{W^{2,p}(B_R)} \le C\,\|g_r\|_{L^\infty(B_{2R})} + C \le C. \qquad \square$$

Remark 3.3. We recall for later reference the following indentities inferred from the definitions in (3.13), (3.14) and Propositon 3.2:

$$\begin{gathered} \mathscr{E}(r) = \int_{B_r} \left(|\nabla u|^2 + 2\,u\right) + O(r^{n+2+\alpha}), \quad \mathscr{H}(r) = \int_{\partial B_r} u^2 d\mathcal{H}^{n-1} + O(r^{n+4}), \\ \int_{\partial B_r} \left(\langle \mathbb{A}\,\nabla u, \nabla u\rangle + 2\,f\,u\right) d\mathcal{H}^{n-1} = \int_{\partial B_r} \left(|\nabla u|^2 + 2\,u\right) d\mathcal{H}^{n-1} + O(r^{n+1+\alpha}). \end{gathered} \tag{3.17}$$

Moreover, we have from (3.16) that

$$\mathscr{E}(r) = O(r^{n+2}) \quad \text{and} \quad \mathscr{H}(r) = O(r^{n+3}), \tag{3.18}$$

and, since $u(\underline{0}) = 0$ and $\nabla u(\underline{0}) = \underline{0}$,

$$\|u\|_{L^\infty(B_r)} \leq C\, r^2 \quad \text{and} \quad \|\nabla u\|_{L^\infty(B_r,\mathbb{R}^n)} \leq C\, r. \tag{3.19}$$

Note that the constant C in (3.19) depends only on the constant in (3.16) and, therefore, is uniformly bounded for points $x_0 \in \Gamma_u \cap K$, for any compact $K \subset \Omega$.

3.2. **Derivatives of $\mathscr{E}$ and $\mathscr{H}$.** We provide next some estimates for the derivatives of $\mathscr{E}$ and $\mathscr{H}$. To this aim, we have benefited of some insights developed in [19], concerning Payne-Weinberger's generalization of Rellich-Nečas' identity. The symbol : below denotes the scalar product in the space of third order tensors.

Lemma 3.4. *Let $\mathbf{F} \in W^{1,\infty}(B_r, \mathbb{R}^n)$. Then, for every $w \in W^{2,p}(\Omega)$, $p \in [\frac{2n}{n+1}, \infty)$, it holds*

$$\begin{aligned}\int_{\partial B_r} &\big(\langle \mathbb{A}\nabla w, \nabla w\rangle \langle \mathbf{F}, \nu\rangle - 2\langle \mathbb{A}\,\nu, \nabla w\rangle \langle \mathbf{F}, \nabla w\rangle\big) d\mathcal{H}^{n-1}\\ &= \int_{B_r} \big(\langle \mathbb{A}\nabla w, \nabla w\rangle \mathrm{div}\mathbf{F} - 2\langle \mathbf{F}, \nabla w\rangle \mathrm{div}(\mathbb{A}\nabla w)\big) dx\\ &\quad + \int_{B_r} \big(\nabla\mathbb{A} : \mathbf{F}\otimes\nabla w\otimes\nabla w - 2\langle \mathbb{A}\nabla w, \nabla^T\mathbf{F}\nabla w\rangle\big)\, dx.\end{aligned} \tag{3.20}$$

Proof. The proof is a direct application of the Divergence theorem and the expansion of

$$\mathrm{div}\Big(\langle \mathbb{A}\nabla w, \nabla w\rangle\, \mathbf{F} - 2\,\langle \mathbf{F}, \nabla w\rangle \mathbb{A}\nabla w\Big). \qquad \square$$

In particular, we can compute the derivative of the energy $\mathscr{E}$ on balls as follows.

Proposition 3.5. *There exists a non negative constant C_1 depending on λ, and on the Lipschitz constant of $\mathbb{A}$, such that, for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega)))$,*

$$\begin{aligned}\mathscr{E}'(r) = 2\int_{\partial B_r} &\mu^{-1}\langle \mathbb{A}\,\nu, \nabla u\rangle^2 d\mathcal{H}^{n-1} + \frac{1}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle\, \mathrm{div}\left(\mu^{-1}\mathbb{A}x\right)\, dx - \frac{2}{r}\int_{B_r} f\,\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle\, dx\\ &- \frac{2}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla^T\left(\mu^{-1}\mathbb{A}x\right)\nabla u\rangle\, dx + 2\int_{\partial B_r} f\,u\, d\mathcal{H}^{n-1} + \varepsilon(r),\end{aligned} \tag{3.21}$$

with $|\varepsilon(r)| \leq C_1\,\mathscr{E}(r)$.

Proof. Consider the vector field

$$\mathbf{F}(x) := \frac{\mathbb{A}(x)x}{r\mu(x)}.$$

$\mathbf{F}$ is admissible for Lemma 3.4 because of (H1) and Lemma 3.1. Simple computations shows that

$$\langle \mathbf{F}, \nu\rangle = 1 \quad \text{on } \partial B_r \quad \text{and} \quad \langle \mathbf{F}, \nabla u\rangle = \mu^{-1}\langle \mathbb{A}\,\nu, \nabla u\rangle \quad \text{on } \partial B_r.$$

By the coarea formula, for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$, it holds

$$\mathscr{E}'(r) = \int_{\partial B_r} \left(\langle \mathbb{A}\nabla u(x), \nabla u(x)\rangle + 2f\,u\right) d\mathcal{H}^{n-1}.$$

Lemma 3.4, with the above choice of $\mathbf{F}$ and (2.3), yields

$$\begin{aligned}\mathscr{E}'(r) = 2\int_{\partial B_r} &\mu^{-1}\langle \mathbb{A}\,\nu, \nabla u\rangle^2 d\mathcal{H}^{n-1} + \frac{1}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle\, \mathrm{div}\left(\mu^{-1}\mathbb{A}x\right)\, dx - \frac{2}{r}\int_{B_r} f\,\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle\, dx\\ &+ \frac{1}{r}\int_{B_r} \mu^{-1}(\nabla\mathbb{A} : \mathbb{A}x\otimes\nabla u\otimes\nabla u)\, dx - \frac{2}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla^T\left(\mu^{-1}\mathbb{A}x\right)\nabla u\rangle\, dx + 2\int_{\partial B_r} f\,u\, d\mathcal{H}^{n-1},\end{aligned}$$

and the thesis follows thanks to the Lipschitz continuity of $\mathbb{A}$ and (3.9). $\square$

Let us now deal with the derivative of $\mathscr{H}$.

Proposition 3.6. *There exists a non negative constant C_2 depending on λ and on the Lipschitz constant of $\mathbb{A}$, such that, for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$,*

$$\left| \mathscr{H}'(r) - \frac{n-1}{r}\mathscr{H}(r) - 2\int_{\partial B_r} u\langle \mathbb{A}\,\nu, \nabla u\rangle \, d\mathcal{H}^{n-1} \right| \leq C_2 \mathscr{H}(r). \tag{3.22}$$

Proof. First note that the divergence theorem and the very definition of μ give that

$$\mathscr{H}(r) = \frac{1}{r}\int_{B_r} \mathrm{div}\left(u^2(x)\mathbb{A}(x)x\right) dx,$$

in turn implying for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$

$$\begin{aligned}\mathscr{H}'(r) &= -\frac{1}{r}\mathscr{H}(r) + \frac{1}{r}\int_{\partial B_r} \mathrm{div}\left(u^2(x)\mathbb{A}(x)x\right) d\mathcal{H}^{n-1} \\ &= -\frac{1}{r}\mathscr{H}(r) + \frac{1}{r}\int_{\partial B_r} \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x)\, x_j + \mathrm{Tr}\mathbb{A}\right) u^2 d\mathcal{H}^{n-1} + 2\int_{\partial B_r} u\langle \mathbb{A}\,\nu, \nabla u\rangle \, d\mathcal{H}^{n-1}.\end{aligned}$$

By (H1) and Lemma 3.1 we get

$$\frac{1}{r}\left| \int_{\partial B_r} \mathrm{Tr}\mathbb{A}\, u^2 d\mathcal{H}^{n-1} - n\,\mathscr{H}(r)\right| \leq C\int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \leq C\mathscr{H}(r),$$

and

$$\frac{1}{r}\left| \int_{\partial B_r} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x)\, x_j u^2(x) d\mathcal{H}^{n-1}\right| \leq C\int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \leq C\mathscr{H}(r),$$

from which the conclusion follows. □

3.3. **Weiss' monotonicity.** We begin with a Weiss' type quasi-monotonicity formula, that establishes the 2-homogeneity of blow-ups of u in free boundary points.

Theorem 3.7. *Assume that (H1)-(H3) and* (3.6) *are satisfied. There exist nonnegative constants C_3, C_4 depending on λ and on the Lipschitz constants of $\mathbb{A}$ and u, such that the function*

$$r \to e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\, t^{\alpha-1} dt$$

is non decreasing on $\left(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1\right)$.

More precisely, the following estimate holds true for $\mathcal{L}^1$-a.e. r in such an interval

$$\frac{d}{dr}\left(e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\, t^{\alpha-1} dt\right) \geq \frac{2e^{C_3 r}}{r^{n+2}}\int_{\partial B_r} \mu\left(\langle \mu^{-1}\mathbb{A}\,\nu, \nabla u\rangle - 2\,\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}. \tag{3.23}$$

In particular, the limit $\Phi(0^+) := \lim_{r\downarrow 0}\Phi(r)$ exists finite and there exists a constant $\bar{C} > 0$ such that

$$\Phi(r) - \Phi(0^+) \geq e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\, t^{\alpha-1} dt - \Phi(0^+) - \bar{C}\, r^\alpha. \tag{3.24}$$

Proof. By definition for $\mathcal{L}^1$-a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$ we have

$$\Phi'(r) = \frac{\mathscr{E}'(r)}{r^{n+2}} - (n+2)\frac{\mathscr{E}(r)}{r^{n+3}} - 2\frac{\mathscr{H}'(r)}{r^{n+3}} + 2(n+3)\frac{\mathscr{H}(r)}{r^{n+4}}. \tag{3.25}$$

First note that (3.21) in Proposition 3.5 yields

$$\begin{aligned}\frac{\mathscr{E}'(r)}{r^{n+2}}-(n+2)\frac{\mathscr{E}(r)}{r^{n+3}}\geq{}&\frac{2}{r^{n+2}}\int_{\partial B_r}\mu^{-1}\langle\mathbb{A}\,\nu,\nabla u\rangle^2 d\mathcal{H}^{n-1}+\frac{1}{r^{n+3}}\int_{B_r}\langle\mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}\left(\mu^{-1}\mathbb{A}x\right)\,dx\\&-\frac{2}{r^{n+3}}\int_{B_r}f\langle\mu^{-1}\mathbb{A}\,x,\nabla u\rangle\,dx-\frac{C_1}{r^{n+2}}\mathscr{E}(r)-\frac{2}{r^{n+3}}\int_{B_r}\langle\mathbb{A}\nabla u,\nabla^T\left(\mu^{-1}\mathbb{A}x\right)\nabla u\rangle\,dx\\&+\frac{2}{r^{n+2}}\int_{\partial B_r}f\,u\,d\mathcal{H}^{n-1}-\frac{(n+2)}{r^{n+3}}\int_{B_r}\langle\mathbb{A}\nabla u,\nabla u\rangle\,dx-\frac{2(n+2)}{r^{n+3}}\int_{B_r}f\,u\,dx.\end{aligned}$$

Then, integrating by part gives

$$\int_{B_r}\langle\mathbb{A}\nabla u,\nabla u\rangle\,dx+\int_{B_r}f\,u\,dx=\int_{\partial B_r}u\langle\mathbb{A}\,\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}.$$

Thus, we deduce

$$\begin{aligned}\frac{\mathscr{E}'(r)}{r^{n+2}}-(n+2)\frac{\mathscr{E}(r)}{r^{n+3}}\geq{}&-C_1\frac{\mathscr{E}(r)}{r^{n+2}}+\frac{2}{r^{n+2}}\int_{\partial B_r}\mu^{-1}\langle\mathbb{A}\,\nu,\nabla u\rangle^2 d\mathcal{H}^{n-1}\\&+\frac{1}{r^{n+3}}\int_{B_r}\left(\langle\mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}\left(\mu^{-1}\mathbb{A}x\right)-2\langle\mathbb{A}\nabla u,\nabla^T\left(\mu^{-1}\mathbb{A}x\right)\nabla u\rangle-(n-2)\langle\mathbb{A}\nabla u,\nabla u\rangle\right)\,dx\\&-\frac{2}{r^{n+3}}\int_{B_r}f\langle\mu^{-1}\mathbb{A}\,x,\nabla u\rangle\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r}f\,u\,d\mathcal{H}^{n-1}\\&-\frac{4}{r^{n+3}}\int_{\partial B_r}u\langle\mathbb{A}\,\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}-\frac{2n}{r^{n+3}}\int_{B_r}f\,u\,dx.\end{aligned}\tag{3.26}$$

Next we employ (3.22) in Proposition 3.6 to infer

$$-2\frac{\mathscr{H}'(r)}{r^{n+3}}+2(n+3)\frac{\mathscr{H}(r)}{r^{n+4}}\geq-2C_2\frac{\mathscr{H}(r)}{r^{n+3}}+8\frac{\mathscr{H}(r)}{r^{n+4}}-\frac{4}{r^{n+3}}\int_{\partial B_r}u\langle\mathbb{A}\,\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}.\tag{3.27}$$

Hence, by taking into account (3.26) and (3.27), equation (3.25) becomes

$$\begin{aligned}\Phi'(r)+(C_1\vee C_2)\Phi(r)\geq{}&\frac{2}{r^{n+2}}\int_{\partial B_r}\left(\mu^{-1}\langle\mathbb{A}\,\nu,\nabla u\rangle^2+4\mu\frac{u^2}{r^2}-4\frac{u}{r}\langle\mathbb{A}\,\nu,\nabla u\rangle\right)d\mathcal{H}^{n-1}\\&+\frac{1}{r^{n+3}}\int_{B_r}\left(\langle\mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}\left(\mu^{-1}\mathbb{A}x\right)-2\langle\mathbb{A}\nabla u,\nabla^T\left(\mu^{-1}\mathbb{A}x\right)\nabla u\rangle-(n-2)\langle\mathbb{A}\nabla u,\nabla u\rangle\right)\,dx\\&-\frac{2}{r^{n+3}}\left(\int_{B_r}f\left(\langle\mu^{-1}\mathbb{A}\,x,\nabla u\rangle+n\,u\right)dx-r\int_{\partial B_r}f\,u\,d\mathcal{H}^{n-1}\right)=:R_1+R_2+R_3.\end{aligned}\tag{3.28}$$

We estimate separately the R_i's. To begin with, an easy computation shows that

$$R_1=\frac{2}{r^{n+2}}\int_{\partial B_r}\mu\left(\langle\mu^{-1}\mathbb{A}\,\nu,\nabla u\rangle-2\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}.\tag{3.29}$$

Moreover, we can rewrite the second term as

$$R_2=\frac{1}{r^{n+3}}\int_{B_r}\left(\langle\mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}\left(\mu^{-1}\mathbb{A}x-x\right)-2\langle\mathbb{A}\nabla u,\nabla^T\left(\mu^{-1}\mathbb{A}x-x\right)\nabla u\rangle\right)\,dx.$$

Then, by the Lipschitz continuity of $\mathbb{A}$ and that of μ in $\underline{0}$, we get

$$\left\|\nabla\left(\mu^{-1}\mathbb{A}-\mathrm{I}_n\right)\right\|_{L^\infty(B_r,\mathbb{R}^{n\times n})}\leq C.$$

In conclusion, we infer

$$|R_2|\leq C\,\frac{\mathscr{E}(r)}{r^{n+2}}.\tag{3.30}$$

Finally, we use the identity

$$\int_{B_r} (\langle x, \nabla u\rangle + u\,\mathrm{div}x)\,dx = r\int_{\partial B_r} u\,d\mathcal{H}^{n-1},$$

that follows from the Divergence theorem, to rewrite the last term in (3.28) as

$$\begin{aligned} R_3 = &-\frac{2f(\underline{0})}{r^{n+3}}\int_{B_r}\langle \mu^{-1}\mathbb{A}\,x - x, \nabla u\rangle\,dx \\ &-\frac{2}{r^{n+3}}\left(\int_{B_r}(f(x)-f(\underline{0}))\left(\langle\mu^{-1}\mathbb{A}\,x,\nabla u\rangle + n\,u\right)dx - r\int_{\partial B_r}(f(x)-f(\underline{0}))\,u\,d\mathcal{H}^{n-1}\right). \end{aligned}$$

Hence, by the inequalities in (3.19), the Lipschitz continuity of $\mathbb{A}$ and that of μ in $\underline{0}$, and the Hölder continuity of f yield, for $r\in\left(0,\frac{1}{2}\mathrm{dist}(\underline{0},\partial\Omega)\wedge 1\right)$,

$$|R_3| \le C r^{\alpha-1}. \tag{3.31}$$

By collecting (3.29)-(3.31) we conclude

$$\Phi'(r) + C_3\Phi(r) + C_4 r^{\alpha-1} \ge \frac{2}{r^{n+2}}\int_{\partial B_r}\mu\left(\langle\mu^{-1}\mathbb{A}\,\nu,\nabla u\rangle - 2\frac{u}{r}\right)^2 d\mathcal{H}^{n-1} \tag{3.32}$$

for nonnegative constants C_3 and C_4. From this, the Weiss' type monotonicity formula (3.23) follows at once.

Note that the growth estimates in (3.19) and equalities (3.13) and (3.14) imply that $\Phi(r)$ is bounded for $r\in\left(0,\frac{1}{2}\mathrm{dist}(\underline{0},\partial\Omega)\wedge 1\right)$, so that the existence and finiteness of $\Phi(0^+)$ follows directly from (3.23). Finally, for what concerns (3.24) it is enough to estimate as follows

$$\begin{aligned} \Phi(r)-\Phi(0^+) \ge &-\left|e^{C_3 r}\Phi(r)-\Phi(r)\right| + e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\,t^{\alpha-1}dt - \Phi(0^+) \\ &- C_4\int_0^r e^{C_3 t}\,t^{\alpha-1}dt \\ \ge\ & e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\,t^{\alpha-1}dt - \Phi(0^+) - |\Phi(r)|\,C\,r - C\,r^\alpha \\ \ge\ & e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\,t^{\alpha-1}dt - \Phi(0^+) - \bar{C}\,r^\alpha, \end{aligned}$$

where we have taken into account the fact that Φ is bounded. □

3.4. **Monneau's monotonicity.** Next we prove a Monneau's type quasi-monotonicity formula for singular free boundary points (cp. with [22]). We denote by v any positive 2-homogeneous polynomial solving

$$\triangle v = 1 \quad \text{on } \mathbb{R}^n. \tag{3.33}$$

Let

$$\Psi_v(r) := \frac{1}{r^{n+2}}\int_{B_r}\left(|\nabla v(x)|^2 + 2\,v\right)\,dx - \frac{2}{r^{n+3}}\int_{\partial B_r} v^2 d\mathcal{H}^{n-1}. \tag{3.34}$$

The expression of Ψ_v is analogous to that of Φ with coefficients frozen in $\underline{0}$ (cp. with (3.15) and recall that $\mathbb{A}(\underline{0}) = \mathrm{I}_n$ and $f(\underline{0}) = \mu(\underline{0}) = 1$, by (3.6)). Moreover, since v is 2-homogeneous and (3.33) holds, we also have

$$\Psi_v(r) \equiv \Psi_v(1) = \int_{B_1} v\,dx. \tag{3.35}$$

In the next theorem we give a monotonicity formula for solutions of the obstacle problem such that $\underline{0}$ is a point of the free boundary and

$$\Phi(0^+) = \Psi_v(1) \quad \text{for some 2-homogeneous } v \text{ solving (3.33).} \tag{3.36}$$

As it will be explained in Definition 4.6, (3.36) characterizes the singular part of the free boundary $\mathrm{Sing}(u)$, therefore we will refer to it by saying that $\underline{0} \in \mathrm{Sing}(u)$.

Theorem 3.8. *Assume (H1)-(H3) and* (3.6)*. Let u be the minimizer of $\mathscr{E}$ on $\mathcal{K}$ with $\underline{0} \in \mathrm{Sing}(u)$ (i.e.* (3.36) *holds), and let v be as above. Then, there exists a nonegative constant C_5 depending on λ and on the Lipschitz constant of $\mathbb{A}$, such that*

$$r \mapsto \int_{\partial B_1} (u_r - v)^2 d\mathcal{H}^{n-1} + C_5 r^\alpha$$

is nondecreasing on $\big(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1\big)$. More precisely, $\mathcal{L}^1$-a.e. on such an interval

$$\frac{d}{dr}\left(\int_{\partial B_1} (u_r - v)^2 d\mathcal{H}^{n-1} + C_5\, r^\alpha\right) \geq \frac{2}{r}\left(e^{C_3 r}\Phi(r) + C_4 \int_0^r e^{C_3 t}\, t^{\alpha-1} dt - \Psi_v(1)\right). \tag{3.37}$$

Proof. Set $w_r := u_r - v$. By taking into account equality $\mathbb{A}(\underline{0}) = \mathrm{I}_n$ (cp. with (3.6)), the 2-homogeneity of v and the Divergence theorem, we find

$$\begin{aligned}
\frac{d}{dr}\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} &= \frac{2}{r}\int_{\partial B_1} w_r(\langle \nabla w_r, x\rangle - 2w_r) d\mathcal{H}^{n-1} = \frac{2}{r}\int_{\partial B_1} w_r(\langle \nabla u_r, x\rangle - 2u_r) d\mathcal{H}^{n-1} \\
&= \frac{2}{r}\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r) d\mathcal{H}^{n-1} + \frac{2}{r}\int_{\partial B_1} w_r \langle (\mathbb{A}(\underline{0}) - \mathbb{A}(rx))\nabla u_r, x\rangle\, d\mathcal{H}^{n-1} \\
&\geq \frac{2}{r}\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r) d\mathcal{H}^{n-1} - 2\|\nabla u_r\|_{L^2(\partial B_1)}\|w_r\|_{L^2(\partial B_1)}.
\end{aligned} \tag{3.38}$$

In view of (3.19) the latter inequality implies

$$\frac{d}{dr}\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} \geq \frac{2}{r}\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r) d\mathcal{H}^{n-1} - C. \tag{3.39}$$

Next we use an integration by parts, the identity

$$\mathrm{div}(\mathbb{A}(rx)\nabla u_r) = f(rx)\chi_{\{u_r>0\}}(x) \quad \text{a.e. and in } \mathcal{D}'(\Omega), \tag{3.40}$$

(3.33) and the positivity of u and v to rewrite the first term on the right hand side above conveniently

$$\begin{aligned}
&\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r) d\mathcal{H}^{n-1} \\
&\quad = \int_{B_1} (\langle \mathbb{A}(rx)\nabla u_r, \nabla w_r\rangle + w_r\, \mathrm{div}(\mathbb{A}(rx)\nabla u_r))\, dx - 2\int_{\partial B_1} w_r\, u_r\, d\mathcal{H}^{n-1} \\
&\quad \overset{(3.40)}{=} \int_{B_1} (\langle \mathbb{A}(rx)\nabla u_r, \nabla u_r\rangle + f(rx)u_r)\, dx \\
&\quad - \int_{B_1} \big(\langle \mathbb{A}(rx)\nabla u_r, \nabla v\rangle + v\, f(rx)\chi_{\{u_r>0\}}\big)\, dx - 2\int_{\partial B_1} w_r\, u_r\, d\mathcal{H}^{n-1} \\
&\quad = \Phi(r) - \int_{B_1} f(rx)(u_r + v\chi_{\{u_r>0\}}) dx - \int_{B_1} \langle \mathbb{A}(rx)\nabla u_r, \nabla v\rangle)\, dx \\
&\quad + 2\int_{\partial B_1} (\mu(rx) - \mu(\underline{0})) u_r^2 d\mathcal{H}^{n-1} + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1}
\end{aligned}$$

$$\overset{(H1),\,(3.8)}{\geq} \Phi(r) - \int_{B_1} f(rx)(u_r+v)dx - \int_{B_1} \langle \nabla u_r, \nabla v\rangle\, dx - r\|\mathbb{A}\|_{W^{1,\infty}} \|\nabla u_r\|_{L^2(B_1)}\|\nabla v\|_{L^2(B_1)}$$
$$- C\|\mathbb{A}\|_{W^{1,\infty}}\, r \int_{\partial B_1} u_r^2 d\mathcal{H}^{n-1} + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1}$$
$$\overset{(3.18),\,(3.33),\,(3.35)}{=} \Phi(r) - \Psi_v(1) + \int_{B_1} (f(\underline{0}) - f(rx))(u_r+v)dx + \int_{\partial B_1} u_r(2v - \langle \nabla v, x\rangle)\, d\mathcal{H}^{n-1} - Cr$$
$$\overset{(H3'),\, v \text{ 2-hom}}{\geq} \Phi(r) - \Psi_v(1) - C\, r^{\alpha}. \quad (3.41)$$

Thus, by collecting (3.39) and (3.41) we deduce

$$\frac{d}{dr}\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} \geq \frac{2}{r}(\Phi(r) - \Psi_v(1)) - C\, r^{\alpha-1}. \quad (3.42)$$

In conclusion, by (3.24), (3.41) and (3.42), (3.38) rewrites as

$$\frac{d}{dr}\left(\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} + C_5\, r^{\alpha}\right) \geq \frac{2}{r}\left(e^{C_3 r}\Phi(r) + C_4\int_0^r e^{C_3 t}\, t^{\alpha-1}dt - \Psi_v(1)\right),$$

for some nonnegative constant C_5. Inequality (3.37) is finally established. □

Remark 3.9. Alternatively, we could establish a slightly different monotonicity formula as follows: If in (3.38) we estimate the term

$$\|\nabla u_r\|_{L^2(\partial B_1)}\|w_r\|_{L^2(\partial B_1)}$$

by using Cauchy-Schwartz inequality rather than using the boundedness in C^1 of $(u_r)_{r>0}$, we infer

$$\frac{d}{dr}\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} \geq \frac{2}{r}\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r)d\mathcal{H}^{n-1} - C. \quad (3.43)$$

Thus, by collecting (3.41) and (3.43) we deduce

$$e^{-r}\frac{d}{dr}\left(e^r\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1}\right) \geq \frac{2}{r}(\Phi(r) - \Psi_v(1)) - C(r^{\alpha-1}+1). \quad (3.44)$$

Finally, from (3.42), (3.43) and (3.44) we infer that

$$e^{-r}\frac{d}{dr}\left(e^r\int_{\partial B_1} w_r^2 d\mathcal{H}^{n-1} + \zeta(r)\right) \geq \frac{2}{r}(\Phi(r) - \Psi_v(1)),$$

where $\zeta \in C^{0,\alpha}([0,\infty))$ satisfies

$$\zeta'(r) = C_5 e^r\left(r^{\alpha-1}+1\right) \quad \text{on } \big(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1\big),$$

for some nonnegative constant C_5.

4. Regularity of the free boundary

Using the quasi-monotonicity formulas above, in this section we study the regularity of the free boundary for the obstacle problem for $\mathscr{E}$ in (2.1). As discussed in the introduction, in view also of recent results by Matevosyan and Petrosyan [21], this approach applies to various obstacle problems with less regular quasi-linear operators of the type of certain mean-field models for type II superconductors (cp, e.g., [13]).

4.1. **Blow-ups.** We shall investigate in what follows the existence and uniqueness of the blow-ups. To this aim, we need to introduce new notation for the rescaled functions in any free boundary point similarly to (3.12): for every point in the free boundary $x_0 \in \Gamma_u$, set

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^2}. \tag{4.1}$$

Remark 4.1. A simple corollary of Weiss' quasi monotonicity is the precompactness of the family $(u_{x_0,r})_r$ in the topology of $C^{1,\gamma}_{\rm loc}(\mathbb{R}^n)$. Moreover, for base points x_0 in a compact set of Ω, the $C^{1,\gamma}_{\rm loc}(\mathbb{R}^n)$ norms and, thus, the constants in the various monotonicity formulas (3.23), (3.37) are uniformly bounded. Indeed, as pointed out in the corresponding statements, they depend on the distance of the point from the boundary and the Lipschitz constant of u.

We recall the notation introduced in Section 3:

$$\mathbb{L}(x_0) = f(x_0)^{-1/2}\mathbb{A}^{1/2}(x_0),$$
$$u_{\mathbb{L}(x_0)}(y) = u(x_0 + \mathbb{L}(x_0)y),$$
$$\mathbb{C}_{x_0}(y) = \mathbb{A}^{-1/2}(x_0)\mathbb{A}(x_0 + \mathbb{L}(x_0)y)\mathbb{A}^{-1/2}(x_0),$$

and in addition we set

$$u_{\mathbb{L}(x_0),r}(y) := \frac{u(x_0 + r\,\mathbb{L}(x_0)\,y)}{r^2},$$
$$\mu_{\mathbb{L}(x_0)}(y) := \langle \mathbb{C}_{x_0}(y)\nu(y), \nu(y)\rangle \quad y \neq \underline{0}, \qquad \mu_{\mathbb{L}}(\underline{0}) := 1,$$
$$\Phi_{\mathbb{L}(x_0)}(r) := \mathscr{E}_{\mathbb{L}(x_0)}[u_{\mathbb{L}(x_0),r}, B_1] + \int_{\partial B_1} \mu_{\mathbb{L}(x_0)}(ry)\, u^2_{\mathbb{L}(x_0),r}(y)\, d\mathcal{H}^{n-1}(y). \tag{4.2}$$

In passing, we note that $\lambda^{-2} \le \mu_{\mathbb{L}}(y) \le \lambda^2$ for all $y \in \mathbb{R}^n$, and that $\mu_{\mathbb{L}} \in C^{0,1}(\Omega)$.

Proposition 4.2. *Let $x_0 \in \Gamma_u$ and $(u_{x_0,r})$ be as in (4.1). Then, for every sequence $r_j \downarrow 0$ there exists a subsequence $(r_{j_k})_{k\in\mathbb{N}} \subset (r_j)_{j\in\mathbb{N}}$ such that $(u_{x_0,r_{j_k}})_{k\in\mathbb{N}}$ converges in $C^{1,\gamma}_{\rm loc}(\mathbb{R}^n)$, for all $\gamma \in (0,1)$, to a function $v(y) = w\big(\mathbb{L}^{-1}(x_0)y\big)$, where w is 2-homogeneous.*

Proof. We drop the dependence on the base point x_0 in the subscripts for the sake of convenience. Apply to $\Phi_{\mathbb{L}}$ the quasi-monotonicity formula in Theorem 3.7 on $(r_j r, r_j R)$ for $r \in (0,R)$ and get

$$\begin{aligned} e^{C_3 r_j R}\Phi_{\mathbb{L}}(r_j R) &- e^{C_3 r_j r}\Phi_{\mathbb{L}}(r_j r) + C_4 \int_{r_j r}^{r_j R} e^{C_3 t}\, t^{\alpha-1} dt \\ &\ge \int_{r_j r}^{r_j R} \frac{2}{t^{n+2}} e^{C_3 t} \int_{\partial B_t} \mu_{\mathbb{L}} \left(\langle \mu_{\mathbb{L}}^{-1}\mathbb{C}\,\nu, \nabla u_{\mathbb{L}}\rangle - 2\frac{u_{\mathbb{L}}}{t}\right)^2 d\mathcal{H}^{n-1}\, dt \\ &= \int_r^R \frac{2}{s^{n+2}} e^{C_3 r_j s} \int_{\partial B_s} \mu_{\mathbb{L}}(r_j y) \left(\langle \frac{\mathbb{C}(r_j y)\,\nu}{\mu_{\mathbb{L}}(r_j y)}, \nabla u_{\mathbb{L},r_j}\rangle - 2u_{\mathbb{L},r_j}\right)^2 d\mathcal{H}^{n-1}\, ds. \end{aligned} \tag{4.3}$$

As noticed in Proposition 3.2 above, the functions $u_{\mathbb{L},r}$ enjoy uniform $C^{1,\gamma}_{\rm loc}(\mathbb{R}^n)$ estimates, $\gamma \in (0,1)$ arbitrary. Therefore, any sequence $(u_{\mathbb{L},r_j})_{j\in\mathbb{N}}$ has a convergent subsequence in $C^{1,\gamma}_{\rm loc}$ to some function w, for all $\gamma \in (0,1)$. Thanks to inequality (4.3) and recalling that $\mathbb{C}(\underline{0}) = \mathrm{I}_n$ and $\mu_{\mathbb{L}}(\underline{0}) = 1$, we infer by the Lebesgue dominated convergence theorem that w is necessarily 2-homogeneous. Changing the coordinates back, we conclude as desired. □

4.2. **Quadratic growth.** The following simple generalization of the usual quadratic detachment property of the minimizer u from the free boundary holds true.

Lemma 4.3. *There exists a dimensional constant $\theta > 0$ such that, for every $x_0 \in \Gamma_u$ and $r \in (0, \mathrm{dist}(x_0, \partial\Omega)/2)$, it holds*

$$\sup_{x \in \partial B_r(x_0)} u(x) \geq \theta\, r^2. \tag{4.4}$$

Proof. First consider a point $y_0 \in N_u$ and $r \in (0, \mathrm{dist}(y_0, \partial\Omega))$, and define the function

$$h(x) := u(x) - u(y_0) - \theta |x - y_0|^2,$$

where $\theta > 0$ is a constant to be fixed properly. Note that $h(y_0) = 0$ and that, for some positive constant C depending only on Ω and $\|\mathbb{A}\|_{W^{1,\infty}}$, we have

$$\mathrm{div}(\mathbb{A}\nabla h) = f - 2\theta\, \mathrm{div}(\mathbb{A}(\cdot - y_0)) \overset{(H1)\,\&\,(H3)}{\geq} c_0 - C\,\theta > 0,$$

as soon as $\theta > 0$ is suitably chosen. Therefore, by the maximum principle (cp. [16]), we deduce that $\sup_{\partial(B_r(y_0) \cap N_u)} h \geq 0$. Since $h|_{B_r(y_0) \cap \Gamma_u} < 0$, it follows that $\partial B_r(y_0) \cap N_u \neq \emptyset$ and

$$\sup_{x \in \partial B_r(y_0)} u(x) \geq \theta\, r^2.$$

Since the radius does not depend on y_0 and the supremum on $\partial B_r(y_0)$ is continuous with respect to y_0, applying this reasoning to a sequence $y_k \in N_u$ converging to x_0, we conclude (4.4). □

4.3. **Classification of blow-ups.** As a simple corollary of Proposition 4.2 and Lemma 4.3, we infer that if w is a 2-homogeneous limit of a converging sequence of rescalings $(u_{x_0, r_j})_{j \in \mathbb{N}}$, in a free boundary point $x_0 \in \Gamma_u$, then $\underline{0} \in \Gamma_w$, i.e. $w \not\equiv 0$ in any neighborhood of $\underline{0}$. We show next some other properties of such limits w. To this aim we recall first the results established in the classical case.

A *global solution* to the obstacle problem is a positive function $w \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ solving (2.3) with $\mathbb{A} \equiv \mathrm{I}_n$ and $f \equiv 1$. The following theorem is due to Caffarelli [4, 7].

Theorem 4.4. *Every global solution w is convex. Moreover, if w is non-zero and homogeneous of degree 2, then one of the following two cases occur:*

(A) $w(y) = \frac{1}{2}\left(\langle y, \nu\rangle \vee 0\right)^2$ for some $\nu \in \mathbb{S}^{n-1}$;
(B) $w(y) = \langle \mathbb{B}\, y, y\rangle$ with $\mathbb{B}$ a symmetric, positive definite matrix satisfying $\mathrm{Tr}(\mathbb{B}) = \frac{1}{2}$.

Having this result at hand, a complete classification of the blow-up limits for the obstacle problem for $\mathscr{E}$ follows as in the classical setting.

Proposition 4.5 (Classification of blow-ups). *Every blow-up v_{x_0} at a free boundary point x_0 is of the form $v_{x_0}(y) = w(\mathbb{L}^{-1}(x_0) y)$ with w a non-trivial, 2-homogeneous global solution.*

Proof. We use the notation at the beginning of Section 4.1, dropping the dependence on x_0 in the subscripts. Denote by w the limit in the $C^{1,\gamma}_{\mathrm{loc}}$ topology of $(u_{\mathbb{L}, r_j})_{j \in \mathbb{N}}$, for some $r_j \downarrow 0$; and consider the energies defined on $H^1(B_1)$ by

$$\mathscr{F}_j(v) := \int_{B_1} \Big(\langle \mathbb{C}(r_j y) \nabla v(y), \nabla v(y) \rangle + 2 \frac{f_{\mathbb{L}}(r_j y)}{f(x_0)}\, v(y) \Big) dy,$$

if $v \geq 0$ $\mathcal{L}^n$ a.e. on B_1 and $v = u_{\mathbb{L}, r_j}$ on ∂B_1, ∞ otherwise. By definition, the rescaled function $u_{\mathbb{L}, r_j}$ itself is the minimizer of $\mathscr{F}_j$. Recalling that $\mathbb{C}(\underline{0}) = \mathrm{I}_n$ and $f_{\mathbb{L}}(\underline{0}) = f(x_0)$, it follows easily that $(\mathscr{F}_j)_{j\in\mathbb{N}}$ Γ-converges with respect to the strong H^1 topology to

$$\mathscr{F}(v) := \int_{B_1} \big(|\nabla v|^2 + 2\,v\big) dx,$$

if $v \geq 0$ $\mathcal{L}^n$-a.e. on B_1 and $v = w$ on ∂B_1, ∞ otherwise on $H^1(B_1)$. Therefore, according to Proposition 4.2, we infer that w is a 2-homogeneous function minimizing $\mathscr{F}$ on B_1. That is, extending w by 2-homogeneity to $\mathbb{R}^n$, w is a non-trivial, 2-homogeneous global solution. In conclusion, as $u_{x_0,r_j}(\mathbb{L}^{-1}(x_0)y) = u_{\mathbb{L},r_j}(y)$, we infer that $u_{x_0,r_j} \to v = w(\mathbb{L}^{-1}(x_0)y)$ in $C^{1,\gamma}_{\mathrm{loc}}$. □

The above proposition allows us to formulate a simple criterion to distinguish between *regular* and *singular* free boundary points.

Definition 4.6. A point $x_0 \in \Gamma_u$ is a *regular* free boundary point, and we write $x_0 \in \mathrm{Reg}(u)$, if there exist a blow-up of u at x_0 of type (A). Otherwise, we say that x_0 is *singular*, and write $x_0 \in \mathrm{Sing}(u)$.

Simple calculations show that $\Psi_w(1) = \vartheta$ for every global solution of type (A) and $\Psi_w(1) = 2\,\vartheta$ for every global solution of type (B), where Ψ_w is the energy defined in (3.34) and ϑ is a dimensional constant. Therefore, by Weiss' quasi monotonicity it follows that a point $x_0 \in \Gamma_u$ is regular if and only if $\Phi_{\mathbb{L}(x_0)}(0) = \vartheta$, or, equivalently, if and only if *every* blow-up at x_0 is of type (A).

4.4. **Uniqueness of blow-ups.** The last remarks show that the blow-up limits at the free boundary points are of a unique type: at a given point they are always either of type (A) or of type (B). Nevertheless, this does not imply the uniqueness of the limiting profile independently of the converging sequence $r_j \downarrow 0$. We show next that this is the case.

In the classical framework, the uniqueness of the blow-ups can be derived *a posteriori* from the regularity properties of the free boundary established thanks to an argument by Caffarelli employing cones of monotonicity. Those are, in turn, obtained via a PDE argument for the gradient of the solution u. In our case, due to the lack of regularity of the matrix of the coefficients $\mathbb{A}$, we need to prove it *a priori*, following the approaches by Weiss and Monneau.

For regular points, we need to introduce the following deep result by Weiss [28, Theorem 1]. For ease of readability we recall the notation introduced in (3.34) for v any positive 2-homogeneous polynomial and extended to any function $\zeta \in W^{1,2}(B_1)$:

$$\Psi_\zeta(1) = \int_{B_1} \big(|\nabla\zeta|^2 + 2\,\zeta\big)\, dx - 2\int_{\partial B_1} \zeta^2 d\mathcal{H}^{n-1}.$$

Theorem 4.7 (Weiss' epiperimetric inequality)**.** *There exist dimensional constants $\delta, \kappa > 0$ with this property: for every 2-homogeneous function $\varphi \in H^1(B_1)$ with $\|\varphi - w\|_{H^1(B_1)} \leq \delta$ for some global solution w of type (A), there exists $\zeta \in H^1(B_1)$ such that $\zeta|_{\partial B_1} = \varphi|_{\partial B_1}$ and*

$$\Psi_\zeta(1) - \vartheta \leq (1-\kappa)\big(\Psi_\varphi(1) - \vartheta\big), \tag{4.5}$$

where $\vartheta = \Psi_w(1)$ is the energy of any global solution w of type (A).

We now proceed with the proof of the uniqueness of the blow-ups at regular points. A preliminary step in this direction is the following lemma.

Lemma 4.8. *Let u be a solution of the obstacle problem with $\underline{0} \in \Gamma_u$ and assume that* (3.6) *holds. If there exist radii $0 \leq s_0 < r_0 < 1$ such that*

$$\inf_w \|u_r|_{\partial B_1} - w\|_{H^1(\partial B_1)} \leq \delta \quad \textit{for all } \; s_0 \leq r \leq r_0, \tag{4.6}$$

where the infimum is taken among all global solutions w of type (A) and $\delta > 0$ is the constant in Theorem 4.7, then for every $s_0 \le s \le t \le r_0$ we have

$$\int_{\partial B_1} |u_t - u_s| \, d\mathcal{H}^{n-1} \le C_7 \, t^{C_6}, \tag{4.7}$$

where $C_6, C_7 > 0$ are constants depending on the Lipschitz constants of $\mathbb{A}$ and u.

Proof. By means of Remark 3.3 we can compute the derivative of $\Phi(r)$ in the following way:

$$\begin{aligned}
\Phi'(r) &= \frac{\mathscr{E}'(r)}{r^{n+2}} - (n+2)\,\frac{\mathscr{E}(r)}{r^{n+3}} - 2\,\frac{\mathscr{H}'(r)}{r^{n+3}} + 2\,(n+3)\,\frac{\mathscr{H}(r)}{r^{n+4}} \\
&\overset{(3.27),(3.18)}{\ge} \frac{1}{r^{n+2}} \int_{\partial B_r} \big(\langle \mathbb{A}\,\nabla u, \nabla u\rangle + 2\,f\,u\big) - (n+2)\,\frac{\mathscr{E}(r)}{r^{n+3}} + 8\frac{\mathscr{H}(r)}{r^{n+4}} \\
&\quad - \frac{4}{r^{n+3}} \int_{\partial B_r} u\,\langle \mathbb{A}\,\nu, \nabla u\rangle - C \\
&\overset{(3.18)}{\ge} \frac{1}{r^{n+2}} \int_{\partial B_r} \big(|\nabla u|^2 + 2\,u\big) - \frac{n+2}{r}\,\Phi(r) - \frac{2}{r^{n+4}}(n-2) \int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \\
&\quad - \frac{4}{r^{n+3}} \int_{\partial B_r} u\,\langle \nu, \nabla u\rangle - C\,r^{\alpha-1} \\
&= -\frac{n+2}{r}\,\Phi(r) + \frac{1}{r} \int_{\partial B_1} \Big(\big(\langle \nu, \nabla u_r\rangle - 2\,u_r\big)^2 + |\partial_\tau u_r|^2 + 2\,u_r - 2\,n\,u_r^2\Big) d\mathcal{H}^{n-1} - C\,r^{\alpha-1},
\end{aligned}$$

where we denoted by $\partial_\tau u_r$ the tangential derivative of u_r along ∂B_1. Let w_r be the 2-homogeneous extension of $u_r|_{\partial B_1}$, then a simple integration in polar coordinates gives

$$\begin{aligned}
\int_{\partial B_1} \big(|\partial_\tau u_r|^2 + 2\,u_r - 2\,n\,u_r^2\big) d\mathcal{H}^{n-1} &= \int_{\partial B_1} \big(|\partial_\tau w_r|^2 + 2\,w_r + 4\,w_r^2 - 2\,(n+2)\,w_r^2\big) d\mathcal{H}^{n-1} \\
&= (n+2)\,\Psi_{w_r}(1).
\end{aligned}$$

Therefore, we conclude that

$$\Phi'(r) \ge \frac{n+2}{r}\,\big(\Psi_{w_r}(1) - \Phi(r)\big) + \frac{1}{r} \int_{\partial B_1} \big(\langle \nu, \nabla u_r\rangle - 2\,u_r\big)^2 d\mathcal{H}^{n-1} - C\,r^{\alpha-1}. \tag{4.8}$$

By (4.6) we can apply the epiperimetric inequality (4.5) to w_r, and find a function $\zeta \in H^1(B_1)$ with $\zeta|_{\partial B_1} = u_r|_{\partial B_1}$ such that

$$\Psi_\zeta(1) - \vartheta \le (1-\kappa)\big(\Psi_{w_r}(1) - \vartheta\big). \tag{4.9}$$

Moreover, we can assume without loss of generality (otherwise we substitute ζ with u_r itself) that $\Psi_\zeta(1) \le \Psi_{u_r}(1)$. Note that, by freezing the coefficients as usual, hypothesis (H1)-(H3) and the minimality of u_r for the energy $\mathscr{E}$ with respect to its boundary conditions, we have that

$$\begin{aligned}
\Psi_\zeta(1) &= \int_{B_1} \big(|\nabla \zeta|^2 + 2\,\zeta\big)\,dx - 2 \int_{\partial B_1} \zeta^2 d\mathcal{H}^{n-1} \\
&\ge \int_{B_1} \big(\langle \mathbb{A}(rx)\,\nabla\zeta, \nabla\zeta\rangle^2 + 2\,f(rx)\,\zeta\big)\,dx - 2 \int_{\partial B_1} \mu(rx)\,\zeta^2 d\mathcal{H}^{n-1} \\
&\quad - C\,r^\alpha \int_{B_1} \big(|\nabla\zeta|^2 + 2\,\zeta\big)\,dx - C\,r \int_{\partial B_1} \zeta^2 \\
&\ge \Phi(r) - C\,r^\alpha \int_{B_1} \big(|\nabla u_r|^2 + 2\,u_r\big)\,dx - C\,r \int_{\partial B_1} u_r^2 \\
&\ge \Phi(r) - C\,r^\alpha.
\end{aligned} \tag{4.10}$$

Combining together (4.9) and (4.10), we finally infer that

$$\Psi_{w_r}(1) - \Phi(r) \geq \frac{1}{1-\kappa}\big(\Phi(r) - \vartheta - C\, r^{\alpha}\big) + \vartheta - \Phi(r) = \frac{\kappa}{1-\kappa}\big(\Phi(r) - \vartheta\big) - C\, r^{\alpha}. \tag{4.11}$$

Therefore, we can conclude from (4.8) and (4.11) that

$$\Phi'(r) \geq \frac{n+2}{r}\,\frac{\kappa}{1-\kappa}\big(\Phi(r) - \vartheta\big) - C\, r^{\alpha-1}. \tag{4.12}$$

Let now C_6 be any exponent in $(0, \alpha \wedge (n+2)\frac{\kappa}{1-\kappa})$, then

$$\Big(\big(\Phi(r) - \vartheta\big)\, r^{-C_6}\Big)' \geq -C\, r^{\alpha-1-C_6}, \tag{4.13}$$

and by integrating in (t, r_0) for $t \geq s_0$, we finally get from (3.16)

$$\Phi(t) - \vartheta \leq C\, \left(t^{C_6} + t^{\alpha}\right) \leq C_7\, t^{C_6}. \tag{4.14}$$

Consider now $s_0 < s < t < r_0$ and estimate as follows

$$\begin{aligned}
\int_{\partial B_1} |u_t - u_s|\, d\mathcal{H}^{n-1} &\leq \int_s^t \int_{\partial B_1} r^{-2} \left| \langle \nabla u(rx), \nu(x) \rangle - 2\frac{u(rx)}{r} \right| d\mathcal{H}^{n-1}(x) \\
&= \int_s^t r^{-1} \int_{\partial B_1} |\langle \nabla u_r, \nu \rangle - 2u_r|\, d\mathcal{H}^{n-1}\, dr \\
&\leq (n\,\omega_n)^{1/2} \int_s^t r^{-1/2} \left(r^{-1} \int_{\partial B_1} (\langle \nabla u_r, \nu \rangle - 2u_r)^2\, d\mathcal{H}^{n-1} \right)^{1/2} dr.
\end{aligned}$$

Combining (3.24), (4.8), (4.11), (4.14) and Hölder inequality, we then have

$$\begin{aligned}
\int_{\partial B_1} |u_t - u_s|\, d\mathcal{H}^{n-1} &\leq C \int_s^t r^{-1/2} \left(\Phi'(r) + C\, r^{\alpha-1}\right)^{1/2} dr \\
&\leq C\, \left(\log\frac{t}{s}\right)^{\frac{1}{2}} \left(\Phi(t) - \Phi(s) + C\,(t^{\alpha} - s^{\alpha})\right)^{1/2} \leq C\, \left(\log\frac{t}{s}\right)^{1/2} t^{\frac{C_6}{2}}.
\end{aligned}$$

A simple dyadic decomposition argument then leads to the conclusion. Indeed, if $s \in [2^{-k}, 2^{-k+1})$ and $t \in [2^{-h}, 2^{-h+1})$ with $h \leq k$, applying the estimate above iteratively on dyadic intervals, we infer for $q = 2^{\frac{C_6}{2}}$ and a dimensional constant $C > 0$,

$$\int_{\partial B_1} |u_t - u_s|\, d\mathcal{H}^{n-1} \leq C \sum_{j=h}^{k} q^{-j} \leq C\, q^{-h} \leq C\, t^{\frac{C_6}{2}}. \qquad \square$$

Remark 4.9. Formula (4.8) yields Weiss' quasi-mononicity discarding both Payne-Weinberger's formula and the PDE solved by u. Indeed, by taking into account the minimality of u_r with respect to its boundary datum, directly from (4.8) we infer that

$$\Phi'(r) \geq \frac{1}{r} \int_{\partial B_1} (\langle \nu, \nabla u_r \rangle - 2u_r)^2 d\mathcal{H}^{n-1} - C\, r^{\alpha-1},$$

in turn implying

$$\Phi(r) + C\, r^{\alpha} - \Phi(s) - C\, s^{\alpha} \geq \int_s^r \frac{1}{t} \int_{\partial B_1} (\langle \nu, \nabla u_t \rangle - 2u_t)^2 d\mathcal{H}^{n-1}\, dt.$$

We can now prove the uniqueness of the blow-ups at regular points of the free boundary.

Proposition 4.10. *Let u be a solution to the obstacle problem (2.3) and $x_0 \in \mathrm{Reg}(u)$. Then, there exist constants $r_0 = r_0(x_0)$, $\eta_0 = \eta_0(x_0) > 0$ such that every $x \in \Gamma_u \cap B_{\eta_0}(x_0)$ is a regular point and, denoting by $v_x(y) = w(\mathbb{L}^{-1}(x)y)$ any blow-up of u at x, we have*

$$\int_{\partial B_1} \left| u_{\mathbb{L}(x),r} - w \right| d\mathcal{H}^{n-1}(y) \le C\, r^{\frac{C_6}{2}} \quad \text{for all } \ r \in (0, r_0), \tag{4.15}$$

where C and $\gamma > 0$ are dimensional constants. In particular, the blow-up limit v_x is unique.

Proof. Denote by $\Phi(x, r)$ the boundary adjusted energy (3.15) with base point x, i.e. with domain of integration $B_r(x)$ rather than B_r. The upper semicontinuity of $\Gamma_u \ni x \mapsto \Phi(x, 0^+)$ follows from Weiss' quasi-monotonicity, that in turn yields that $\mathrm{Reg}(u) \subset \Gamma_u$ is relatively open, thus proving the first claim if η_0 is sufficiently small.

By Proposition 3.2, given $\bar\eta > 0$ such that $B_{\bar\eta}(x_0) \subset\subset \Omega$ and $\Gamma_u \cap B_{\bar\eta}(x_0) = \mathrm{Reg}(u) \cap B_{\bar\eta}(x_0)$, then

$$C_8 := \sup_{x \in \Gamma_u \cap B_{\bar\eta}(x_0), r < \bar\eta} \|u_{\mathbb{L}(x),r}\|_{C^{1,\gamma}(\partial B_1)} < \infty.$$

Let $\delta > 0$ be the constant in Theorem 4.7. By compactness, if g is any $C^{1,\gamma}(\partial B_1)$ function satisfying $\|g\|_{C^{1,\gamma}(\partial B_1)} \le C_8$, then we can find $\varepsilon > 0$ such that

$$\|g\|_{L^1(\partial B_1)} \le \varepsilon \quad \Longrightarrow \quad \|g\|_{H^1(\partial B_1)} \le \frac{\delta}{4}. \tag{4.16}$$

Next, we fix $\bar r_0 > 0$ such that $C_7\, \bar r_0^{C_6} \le \varepsilon$ and

$$\inf_w \|u_{\mathbb{L}(x_0),\bar r_0}|_{\partial B_1} - w\|_{H^1(\partial B_1)} \le \frac{\delta}{4}, \tag{4.17}$$

where the infimum is taken among all global solutions w of type (A). To show the existence of such a treshold $\bar r_0$, we argue by contradiction: if it does not exist, we must find a sequence r_j converging to 0 such that $\|u_{\mathbb{L}(x_0),r_j} - w\|_{H^1(\partial B_1)} \ge \delta$ for every w global solution of type (A). On the other hand, since x_0 is a regular free boundary point, up to subsequences, not relabeled for conveniene, $(u_{\mathbb{L}(x_0),r_j})_{j\in\mathbb{N}}$ converges in $C^{1,\gamma}_{\mathrm{loc}}$ to a blow-up v of u at x_0 of type (A), thus giving a contradiction.

By the continuity of $\mathbb{A}$ and f, there exists $0 < \eta_0 \le \bar\eta$ such that for all $x \in \mathrm{Reg}(u) \cap B_{\eta_0}(x_0)$,

$$\inf_w \|u_{\mathbb{L}(x),\bar r_0}|_{\partial B_1} - w\|_{H^1(\partial B_1)} \le \frac{\delta}{2}, \tag{4.18}$$

where the infimum is considered in the same class of functions as above. We claim that in turn this implies that for all $x \in \mathrm{Reg}(u) \cap B_{\eta_0}(x_0)$ and $0 < r \le \bar r_0$

$$\inf_w \|u_{\mathbb{L}(x),r}|_{\partial B_1} - w\|_{H^1(\partial B_1)} \le \delta. \tag{4.19}$$

To this aim, fix $x \in \mathrm{Reg}(u) \cap B_{\eta_0}(x_0)$ and let $s_0 < \bar r_0$ be the maximal radius such that (4.19) holds for every $s_0 \le r \le \bar r_0$. Assume that $s_0 > 0$ and note that, in particular,

$$\inf_w \|u_{\mathbb{L}(x),s_0}|_{\partial B_1} - w\|_{H^1(\partial B_1)} = \delta. \tag{4.20}$$

Then, by Lemma 4.8 (recall that, being $B_{\eta_0}(x_0) \subset\subset \Omega$, the constants are uniform at points in $\Gamma_u \cap B_{\eta_0}(x_0)$ – cp. Remark 4.1), we infer that $\|u_{\mathbb{L}(x),s} - u_{\mathbb{L}(x),t}\|_{L^1(\partial B_1)} \le C_7\, \bar r_0^{C_6}$ for every $s, t \in [s_0, \bar r_0]$. Since the functions $u_{\mathbb{L}(x),s}$ are equibounded in $C^{1,\gamma}(\partial B_1)$ by C_8, (4.16) gives that

$$\|u_{\mathbb{L}(x),s} - u_{\mathbb{L}(x),t}\|_{H^1(\partial B_1)} \le \frac{\delta}{4} \quad \text{for every} \quad s, t \in [s_0, \bar r_0].$$

In particular, by (4.18) and the triangular inequality, we get a contradiction to (4.20).

We are now ready for the conclusion. Thanks to (4.19), we deduce that (4.7) in Lemma 4.8 holds for every $s, t \in (0, \bar{r}_0)$. Therefore, by passing to the limit as $s \downarrow 0$ in (4.7) we find

$$\int_{\partial B_1} \big| u_{\mathbb{L}(x),t} - w \big| \, d\mathcal{H}^{n-1} \leq C\, t^{\frac{C_6}{2}},$$

and thus the uniqueness of the blow-up limit is established. □

To prove uniqueness of blow-ups for singular point we need to establish the counterpart of Lemma 4.8 in this setting, though we do not get a rate for the convergence of the rescalings to their blow-up limits.

Proposition 4.11. *For every point x of the singular set* $\mathrm{Sing}(u)$ *there exists a unique blow-up limit* $v_x(y) = w\big(\mathbb{L}^{-1}(x)y\big)$. *Moreover, if K is a compact subset of* $\mathrm{Sing}(u)$, *then, for every point $x \in K$,*

$$\|u_{\mathbb{L}(x),r} - w\|_{C^1(B_1)} \leq \sigma_K(r) \quad \text{for all } r \in (0, r_K), \tag{4.21}$$

for some modulus of continuity $\sigma_K : \mathbb{R}^+ \to \mathbb{R}^+$ and a radius $r_K > 0$.

Proof. With no loss of generality we show the uniqueness property in case the base point $x \in \mathrm{Sing}(u)$ is actually the origin $\underline{0}$ and (3.6) holds. We use Monneau's quasi monotonicity formula in Theorem 3.8. To this aim, we suppose that $(u_{r_j})_{j\in\mathbb{N}}$ converges in $C^{1,\gamma}_{\mathrm{loc}}$, $\gamma \in (0,1)$ arbitrary, to a 2-homogeneous quadratic polynomial v with $\mathrm{Tr}(D^2 v) = 1$. Then, from (3.37) we infer that

$$\lim_j \int_{\partial B_1} (u_{r_j} - v)^2 d\mathcal{H}^{n-1} = 0.$$

In turn, this implies that the monotone function

$$r \to \int_{\partial B_1} (u_r - v)^2 d\mathcal{H}^{n-1} + C_5(r + r^\alpha)$$

is infinitesimal as $r \downarrow 0$. In particular, for all infinitesimal sequences h_j we have that $(u_{h_j})_{j\in\mathbb{N}}$ converges to v in $C^{1,\gamma}_{\mathrm{loc}}$, the uniqueness of the limit then follows at once.

Having fixed a compact subset K of $\mathrm{Sing}(u)$, to prove the uniform convergence we argue by contradiction. Assume there exist points $x_j \in K$ and radii $r_j \downarrow 0$ for which the rescalings $u_{\mathbb{L}(x_j),r_j}$ and the blow-ups v_{x_j} of u at x_j satisfy

$$\|u_{\mathbb{L}(x_j),r_j} - v_{x_j}\|_{C^1(B_1)} \geq \varepsilon > 0, \quad \text{for some } \varepsilon.$$

Thanks to Proposition 3.2 we may assume that, up to subsequences not relabeled, $(u_{\mathbb{L}(x_j),r_j})_{j\in\mathbb{N}}$ converges in $C^{1,\gamma}_{\mathrm{loc}}$ to a function w. Moreover, by taking into account that the constants in Weiss' quasi-monotonicity are bounded since the points are varying on a compact set, we may argue as in Propositions 4.2 and 4.5 to deduce that the limit w is actually a 2-homogeneous global solution (cp. (4.3)).

Let $\Phi_{\mathbb{L}(x_j)}$ be as in (4.2). It is elementary to show that $\Phi_{\mathbb{L}(x_j)}(r) \to \Psi_w(r)$ for all $r > 0$.

Then, using Lemma 4.3 and the classification of free boundary points according to the energy, we get $\underline{0} \in \mathrm{Sing}(w)$. Indeed, if not, as $\vartheta = \Psi_w(0^+) = \Psi_w(r)$ for all r, we would infer that $\Phi_{\mathbb{L}(x_j)}(\rho) \leq \frac{3}{2}\vartheta$ for j big enough for some fixed $\rho > 0$. In turn, the latter condition is a contradiction to $\Phi_{\mathbb{L}(x_j)}(0^+) = 2\,\vartheta$ that follows from the quasi-monotonicity of $\Phi_{\mathbb{L}(x_j)}$ as $x_j \in \mathrm{Sing}(u)$.

We claim next that $w(y) = \langle \mathbb{B}\, y, y\rangle$, for some positive, symmetric $\mathbb{B}$ with $\mathrm{Tr}(\mathbb{B}) = \frac{1}{2}$, i.e. w coincides with its blow-up in $\underline{0}$. To prove this, note that, Λ_w is a convex set by Theorem 4.4, and thus it is a cone since $\underline{0} \in \Lambda_w$. Therefore, $\mathcal{L}^n(\Lambda_w) = \mathcal{L}^n(\Lambda_{v_{\underline{0}}})$, where $v_{\underline{0}}$ is the blow-up of w at $\underline{0}$. As $\underline{0} \in \mathrm{Sing}(w)$, the latter equality implies that $\mathcal{L}^n(\Lambda_w) = 0$. Hence, by equation (3.33) and Lioville's theorem, w is a 2-homogeneous polynomial.

In conclusion, by taking this into account and the fact that all norms are equivalent for polynomials, Monneau's quasi monotonicity formula provides a contradiction (note that the constants therein are bounded since the points are varying in a compact set – cp. Remark 4.1):

$$\begin{aligned}0 < \varepsilon &\leq \limsup_{j\to+\infty} \|u_{\mathbb{L}(x_j),r_j} - v_{x_j}\|_{C^1(B_1)} \leq \limsup_{j\to+\infty} \|w - v_{x_j}\|_{C^1(B_1)} \\ &\leq C \limsup_{j\to+\infty} \|w - v_{x_j}\|_{L^2(\partial B_1)} \overset{(3.37)}{\leq} C \limsup_{j\to+\infty} \|w - u_{\mathbb{L}(x_j),r_j}\|_{L^2(\partial B_1)} = 0. \qquad \square\end{aligned}$$

4.5. **Regular free boundary points.** We are now ready to establish the $C^{1,\beta}$ regularity of the free boundary in a neighborhood of any point x of $\mathrm{Reg}(u)$. Recall that blow-up limits in regular points are unique (cp. Proposition 4.10), so that denoting by $n(x) \in \mathbb{S}^{n-1}$ the blow-up direction at $x \in \mathrm{Reg}(u)$, we have

$$v_x(y) = \frac{1}{2}(\langle \mathbb{L}^{-1}(x)n(x), y\rangle \vee 0)^2.$$

As usual, we shall state and prove the result below with base point the origin. We follow here the arguments in [28].

Theorem 4.12. *Let $\underline{0} \in \mathrm{Reg}(u)$. Then, there exists $r > 0$ such that $\Gamma_u \cap B_r$ is a $C^{1,\beta}$ hypersurface for some universal exponent $\beta \in (0, 1)$.*

Proof. Let $\eta_0 = \eta_0(\underline{0})$ and $r_0 = r_0(\underline{0})$ be the radii provided by Proposition 4.10. We start off showing that for a universal constant $C > 0$ and a universal (computable) exponent $\beta \in (0, 1)$

$$|\mathbb{L}^{-1}(x)\, n(x) - \mathbb{L}^{-1}(z)\, n(z)| \leq C\, |x - z|^{\beta}, \tag{4.22}$$

for every x and $z \in \mathrm{Reg}(u) \cap B_{\eta_0/2}$. To this aim, let $s \in (0, r_0)$, then

$$\begin{aligned}\|v_x - v_z\|_{L^1(\partial B_1)} &\leq \|v_x - u_{\mathbb{L}(x),s}\|_{L^1(\partial B_1)} + \|u_{\mathbb{L}(x),s} - u_{\mathbb{L}(z),s}\|_{L^1(\partial B_1)} + \|u_{\mathbb{L}(z),s} - v_z\|_{L^1(\partial B_1)} \\ &\overset{(4.15)}{\leq} C\, s^{\frac{C_6}{2}} + \|u_{\mathbb{L}(x),s} - u_{\mathbb{L}(z),s}\|_{L^1(\partial B_1)}. \end{aligned}\tag{4.23}$$

By taking into account that the map $y \to \mathbb{L}(y)$ is Hölder continuous with exponent $\theta := \alpha \wedge 1/2$ thanks to (H1) and (H3), in view of estimate (3.19) we can bound the second term above as follows

$$\begin{aligned}&\|u_{\mathbb{L}(x),s} - u_{\mathbb{L}(z),s}\|_{L^1(\partial B_1)} \\ &\leq \int_{\partial B_1} \int_0^1 s^{-2} \Big(|\nabla u(t(z + s\,\mathbb{L}(z)y) + (1-t)(x + s\,\mathbb{L}(x)y)||z - x + s(\mathbb{L}(z) - \mathbb{L}(x))|dt\Big)\, d\mathcal{H}^{n-1}(y) \\ &\leq C\, s^{-2}(|z - x| + s + s|z - x|^{\theta}) \cdot (|z - x| + s|z - x|^{\theta}) \leq C\, |z - x|^{\theta},\end{aligned}\tag{4.24}$$

if $s = |z - x|^{1-\theta}$, and $C = C(n, \|\mathbb{L}\|_{L^\infty(B_{\eta_0/2},\mathbb{R}^{n\times n})})$. Therefore, if $\beta := \theta \wedge \frac{C_6}{2}(1 - \theta)$, (4.22) follows from (4.23), (4.24) and the simple observation that for some dimensional constant $C > 0$ it holds

$$|\mathbb{L}^{-1}(x)\, n(x) - \mathbb{L}^{-1}(z)\, n(z)| \leq C\|v_x - v_z\|_{L^1(\partial B_1)},$$

as the right hand side above is a norm on $\mathbb{R}^n$.

Next, consider the cones $C^{\pm}(x, \varepsilon)$, $x \in \mathrm{Reg}(u)$, given by

$$C^{\pm}(x, \varepsilon) := \left\{ y \in \mathbb{R}^n \,:\, \pm\langle y - x, \frac{\mathbb{A}^{-1/2}(x)n(x)}{|\mathbb{A}^{-1/2}(x)n(x)|}\rangle \geq \varepsilon|y - x| \right\}.$$

We claim that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in \mathrm{Reg}(u) \cap B_{\eta_0/2}$,

$$C^{+}(x, \varepsilon) \cap B_\delta(x) \subset N_u \;\text{ and }\; C^{-}(x, \varepsilon) \cap B_\delta(x) \subset \Lambda_u. \tag{4.25}$$

For, assume by contradiction that there exist $x_j \in \mathrm{Reg}(u) \cap B_{\eta_0/2}$ with $x_j \to x \in \mathrm{Reg}(u) \cap \bar{B}_{\eta_0/2}$, and $y_j \in C^+(x_j, \varepsilon)$ with $y_j - x_j \to 0$ such that $u(y_j) = 0$. By Proposition 3.2, (4.15) and (4.23), the rescalings $u_{\mathbb{L}(x_j),r_j}$, for $r_j = |\mathbb{L}^{-1}(x_j)(y_j - x_j)|$, converge uniformly to v_x. Up to subsequences assume that $r_j^{-1}\mathbb{L}^{-1}(x_j)(y_j - x_j) \to z \in C^+(x, \varepsilon) \cap \mathbb{S}^{n-1}$, then $v_x(z) = 0$. This contradicts the fact that $x \in \mathrm{Reg}(u)$ and $v_x > 0$ on $C^+(x, \varepsilon)$ thanks to $f(x) \geq c_0 > 0$ (cp. (H3)). Clearly, we can argue analogously for the second inclusion.

We show next that $\Lambda_u \cap B_{\rho_1}$ is the subgraph of a function g, for a suitably chosen small $\rho_1 > 0$. Without loss of generality assume that $\frac{\mathbb{A}^{-1/2}(\underline{0})n(\underline{0})}{|\mathbb{A}^{-1/2}(\underline{0})n(\underline{0})|} = e_n$ and set

$$g(x') := \max\{t \in \mathbb{R} : (x', t) \in \Lambda_u\}$$

for all points $x' \in \mathbb{R}^{n-1}$ with $|x'| \leq \delta\sqrt{1 - \varepsilon^2}$. Note that by (4.25) this maximum exists and belongs to $[-\varepsilon\delta, \varepsilon\delta]$; and moreover the inclusions in (4.25) imply that $(x', t) \in \Lambda_u$ for every $-\varepsilon\,\delta < t < g(x')$, and $(x', t) \in N_u$ for every $g(x') < t < \varepsilon\,\delta$.

Eventually, by taking into account (4.22), we conclude that g is $C^{1,\beta}$ regular. □

4.6. **Singular free boundary points.** In this section we prove that the singular set of the free boundary is contained in the countable union of C^1 submanifolds.

We recall that, if $x \in \mathrm{Sing}(u)$, then the unique blow-up v_x is given by

$$v_x(y) = \langle \mathbb{L}^{-1}(x)\, \mathbb{B}_x\, \mathbb{L}^{-1}(x)y, y\rangle,$$

with $\mathbb{B}_x$ a symmetric, positive definite matrix satisfying $\mathrm{Tr}(\mathbb{B}_x) = \frac{1}{2}$ (see Proposition 4.11). We define the singular strata according to the dimension of the kernel of $\mathbb{B}_x$.

Definition 4.13. The singular stratum S_k of dimension k, for $k = 0, \ldots, n-1$, is the subset of points $x \in \mathrm{Sing}(u)$ with $\mathrm{rank}(\mathbb{B}_x) = k$.

In particular, Theorem 4.14 below shows that S_k is $\mathcal{H}^k$ rectifiable, and moreover that $\cup_{k=l}^{n-1} S_k$ is a closed set for every $l = 0, \ldots, n-1$.

Theorem 4.14. *Let $\underline{0} \in S_k$. Then, there exists $r > 0$ such that $S_k \cap B_r$ is contained in a C^1 regular k-dimensional submanifold of $\mathbb{R}^n$.*

Proof. The proof is divided into two steps. We start off proving the continuity of the map

$$\mathrm{Sing}(u) \ni x \mapsto \mathbb{L}^{-1}(x)\, \mathbb{B}_x\, \mathbb{L}^{-1}(x).$$

In turn, by taking this and Proposition 4.11 into account, we conclude by means of Whitney's extension theorem and the implicit function theorem following [7]. We give the full proof for the sake of completeness.

To establish the continuity of $\mathrm{Sing}(u) \ni x \mapsto \mathbb{L}^{-1}(x)\, \mathbb{B}_x\, \mathbb{L}^{-1}(x)$ we argue as in Theorem 4.12 by comparing two blow-ups at different points. To this aim, note that for some dimensional constant $C > 0$

$$|\mathbb{L}^{-1}(x)\, \mathbb{B}_x\, \mathbb{L}^{-1}(x) - \mathbb{L}^{-1}(z)\, \mathbb{B}_z\, \mathbb{L}^{-1}(z)| \leq C\, \|v_x - v_z\|_{L^1(\partial B_{1/2})}, \tag{4.26}$$

as the right hand side above is a norm on symmetric matrices.

Fix a compact set $K \subset \mathrm{Sing}(u)$ and let σ_K be the modulus of continuity in Proposition 4.11. Then, for all x and $z \in K$, setting $s = |x - z|^{1-\theta} \in (0, r_K)$ for $\theta = \alpha \wedge \frac{1}{2}$, we get for some dimensional constant $C > 0$

$$\begin{aligned} \|v_x - v_z\|_{L^1(\partial B_{1/2})} \leq & \|v_x - u_{x,s}\|_{L^1(\partial B_{1/2})} + \|u_{x,s} - u_{z,s}\|_{L^1(\partial B_{1/2})} + \|u_{z,s} - v_z\|_{L^1(\partial B_{1/2})} \\ \overset{(4.21)}{\leq} & C\, \sigma_K(|x - z|^{1-\theta}) + C\, |x - z|^{\theta}, \end{aligned} \tag{4.27}$$

where the difference of the two rescaled maps is estimated as in the second line of inequality (4.23) in Theorem 4.12. Inequalities (4.26) and (4.27) establish the required continuity.

Furthermore, we claim that there exists a function $g \in C^2(\mathbb{R}^n)$ such that for all $x \in K$

$$g(y) - v_x(y-x) = o(|y-x|^2) \quad \text{as } y \to x. \tag{4.28}$$

To this aim we show that the family $v_x(\cdot - x)$, $x \in K$, of translations of the blow-ups satisfies the assumptions of Whitney's extension theorem (see [30]). More precisely, we show that the polynomials $p_x(y) := v_x(y-x)$, $x \in K$, satisfies

(i) $p_x(x) = 0$ for all $x \in S_k$,

(ii) $D^l(p_x - p_z)(x) = o(|x-z|^{2-l})$ for all x and $z \in K \cap S_k$, and $l \in \{0,1,2\}$.

Condition (i) is trivially satisfied; instead for what (ii) is concerned, we note that estimate (4.21) in Proposition 4.11 rewrites, for $r \in (0, \tilde r_K)$ with $\tilde r_K$ depending only on r_K and λ, as

$$\|u - p_z\|_{C^0(B_r(z))} \le r^2\,\sigma_K(r), \quad \text{and} \quad \|\nabla u - \nabla p_z\|_{C^0(B_r(z))} \le r\,\sigma_K(r).$$

Therefore, since $u(x) = 0$ and $\nabla u(x) = \underline{0}$ imply

$$|p_x(x) - p_z(x)| = |u(x) - p_z(x)| \quad \text{and} \quad |\nabla p_x(x) - \nabla p_z(x)| = |\nabla u(x) - \nabla p_z(x)|,$$

then (ii) is verified for $l \in \{0,1\}$. In addition, if $l = 2$, condition (ii) reduces to the continuity of the map $\mathrm{Sing}(u) \ni x \mapsto f(x)\,\mathbb{A}^{-1/2}(x)\,\mathbb{B}_x\,\mathbb{A}^{-1/2}(x)$ established above.

Equality (4.28) gives that $K \subseteq \{\nabla g = \underline{0}\}$. Suppose now that $\underline{0} \in K \cap S_k$, and arrange the coordinates of $\mathbb{R}^n$ in a way that e_i, $i \in \{1,\dots,n-k\}$, are the eigenvalues of $\nabla^2 g(\underline{0})$. Then, the $(n-k)\times(n-k)$ minor of $\nabla^2 g(\underline{0})$ composed by the first $n-k$ rows and columns, is not null. Therefore, the implicit function theorem yields that $\cap_{i=1}^{n-k}\{\partial_i g = 0\}$ is a C^1 submanifold in a neighborhood of $\underline{0}$, and the conclusion follows at once noting that $K \cap S_k \subseteq \{\nabla g = \underline{0}\} \subseteq \cap_{i=1}^{n-k}\{\partial_i g = 0\}$. ☐

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