ASYMPTOTIC ANALYSIS OF AMBROSIO-TORTORELLI ENERGIES IN LINEARIZED ELASTICITY

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Abstract. We provide an approximation result in the sense of Γ-convergence for energies of the form

$$\int_\Omega \mathscr{Q}_1(e(u))\,dx + a\,\mathcal{H}^{n-1}(J_u) + b\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot\nu_u)\,d\mathcal{H}^{n-1},$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $\mathscr{Q}_0$ and $\mathscr{Q}_1$ are coercive quadratic forms on $\mathbb{M}^{n\times n}_{sym}$, a, b are positive constants, and u runs in the space of fields $SBD^2(\Omega)$, i.e., it's a special field with bounded deformation such that its symmetric gradient $e(u)$ is square integrable, and its jump set J_u has finite $(n-1)$-Hausdorff measure in $\mathbb{R}^n$.

The approximation is performed by means of Ambrosio-Tortorelli type elliptic regularizations, the prototype example being

$$\int_\Omega \Big(v|e(u)|^2 + \frac{(1-v)^2}{\varepsilon} + \gamma\,\varepsilon|\nabla v|^2\Big)\,dx,$$

where $(u,v)\in H^1(\Omega,\mathbb{R}^n)\times H^1(\Omega)$, $\varepsilon\le v\le 1$ and $\gamma>0$.

1. Introduction

The variational approximation of free discontinuity energies via families of elliptic functionals has turned out to be an efficient analytical tool and numerical strategy to analyze the behaviour of those energies and of their minimizers (see the book [14] for more detailed references). The prototype result is the approximation by means of Γ-convergence in the strong L^1 topology of the Mumford and Shah energy defined as

$$\int_\Omega |\nabla u|^2\,dx + a\,\mathcal{H}^{n-1}(J_u),$$

a any positive constant and u in the space of *(generalised) special functions with bounded variation*, i.e. $u\in(G)SBV(\Omega)$ (we refer to Section 2 for all the notations and the functional spaces introduced throughout this section). The two-fields functionals introduced by Ambrosio and Tortorelli [8] for this purpose are of the type

$$E_k(u,v) := \int_\Omega \Big((v+\eta_k)|\nabla u|^2 + \frac{(1-v)^2}{\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)dx, \tag{1.1}$$

if $(u,v)\in H^1(\Omega,\mathbb{R}^n)\times H^1(\Omega,[0,1])$ and ∞ otherwise in $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$, with $\eta_k = o(\varepsilon_k)\ge 0$.

The quoted result has been later extended into several directions with different aims: for the purpose of approximating either energies arising in the theory of nematic liquid crystals [9], or general free discontinuity functionals defined over vector-valued fields [24, 25], or the Blake and Zisserman second order model in computer vision [5], or fracture models for brittle linearly elastic materials [16, 17, 29], to provide a common framework for curve evolution and image segmentation [31, 1, 2], to study the asymptotic behaviour of gradient damage models under different regimes [22, 28], and to give a regularization of variational models for plastic slip [7].

The condition $\eta_k = o(\varepsilon_k)$ is instrumental for the quoted Γ-convergence statement, this can be easily checked by a simple calculation in $1d$. In addition, choosing the infinitesimal η_k to be strictly positive makes each functional E_k in (1.1) coercive, thus ensuring the existence of a minimizer by adding suitable boundary conditions or lower order terms. The convergence of the sequence of

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minimizers of the E_k's to the counterparts of the Mumford and Shah functional is then a consequence of classical Γ-convergence theory (see [20]).

Instead, the regime $\eta_k \sim \varepsilon_k$ has been investigated only recently in the papers [22, 28] to study the asymptotics of some mechanical models proposed by Pham, Marigo, and Maurini [30] in the gradient theory of incomplete damage in the isotropic and homogeneous antiplane case. To investigate those models the functionals above are equivalently redefined as

$$E_k(u, v) = \int_\Omega \Big(v|\nabla u|^2 + \frac{(1-v)^2}{\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)dx \tag{1.2}$$

if $(u, v) \in H^1(\Omega, \mathbb{R}^n)\times V_{\varepsilon_k}$, where $V_{\varepsilon_k} := \big\{v \in H^1(\Omega) : \varepsilon_k \leq v \leq 1 \ \mathcal{L}^n\text{-a.e. in } \Omega\big\}$, ∞ otherwise in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$. The constraint on the auxiliary variable v has the interpretation that complete damage is forbidden (we refer to the paper [30] for more insight on the mechanical model, see also [22, 28]). In this new regime an additional term in the limit energy appears in a way that not only the measure of the jump set of the corresponding deformation is taken into account, but also a term depending on the opening of the crack is present. More in details, from the variational point of view of Γ-convergence, the asymptotic behaviour of the sequence (E_k) is described by the energy

$$\int_\Omega |\nabla u|^2\, dx + a\,\mathcal{H}^{n-1}(J_u) + b\int_{J_u} |[u]|d\mathcal{H}^{n-1},$$

for some positive constants a and b, and for all deformations $u \in SBV(\Omega)$.

In this paper we are concerned with studying the complete case of linearized elasticity, for which several additional difficulties arise. Let us stress that we carry out our analysis for a broad class of families of quadratic forms rather than the perturbation of the euclidean one in (1.2) (see the definition of the energy F_k in formula (3.1) and the successive assumptions (H1)-(H4)). Though, in this introduction we stick to the simple case analogous to (1.2) for the sake of clarity:

$$F_k(u, v) := \int_\Omega \Big(v|e(u)|^2 + \frac{(1-v)^2}{\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)dx \tag{1.3}$$

if $(u, v) \in H^1(\Omega, \mathbb{R}^n)\times V_{\varepsilon_k}$, where $V_{\varepsilon_k} = \big\{v \in H^1(\Omega) : \varepsilon_k \leq v \leq 1 \ \mathcal{L}^n\text{-a.e. in } \Omega\big\}$, ∞ otherwise in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$. Recall that $e(u)$ denotes the symmetric part of the gradient field of u, i.e., $e(u) = (\nabla u + \nabla^T u)/2$.

In what follows we shall prove that the asymptotic behaviour of the sequence (F_k) is described, in the sense of Γ-convergence, by the energy

$$F(u) := \int_\Omega |e(u)|^2\, dx + a\,\mathcal{H}^{n-1}(J_u) + b\int_{J_u} |[u]\odot\nu_u|d\mathcal{H}^{n-1}, \tag{1.4}$$

for suitable positive constants a and b and for all fields u in $SBD(\Omega)$, the space of *special functions with bounded deformation*, and F is ∞ otherwise in $L^1(\Omega, \mathbb{R}^n)$. The symbol $\odot$ in (1.4) denotes the symmetrized tensor product between vectors.

A first interpretation for the last integral in (1.4) can be given using the terminology of fracture mechanics. A constant force acts between the lips of the crack J_u, whose displacements are u^+ and u^-; therefore the energy per unit area spent to create the crack is proportional to $|[u]\odot\nu_u|$. This interpretation is not properly covered by the classical Barenblatt's cohesive crack model [10], due to the presence of an activation energy $\mathcal{H}^{n-1}(J_u)$ and to the fact that the cohesive force bridging the crack lips is not decreasing with respect to the crack opening and does not vanish for large values of the opening itself.

The functional in (1.4) and its regularization via Γ-convergence have been recently investigated in [7] in connection with a variational model for plastic slip in the antiplane case. The different approximations of the energy (1.4) introduced in that paper are obtained by perturbing the Ambrosio-Tortorelli's elliptic functionals in (1.1) as follows

$$\int_\Omega \Big((v+\eta_k)|\nabla u|^2 + \frac{(1-v)^2}{\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)dx + \int_\Omega (v-1)^2|\nabla u|dx,$$

with $u, v \in H^1(\Omega)$, $0 \le v \le 1$, and $\eta_k = o(\epsilon_k) \ge 0$. The unpinned surfaces J_u, after the overcoming of the energy barrier, are now seen in terms of sliding surfaces in a strain localization plastic process. Therefore $|[u]|$ here represents the surface plastic energy, that is the work per unit area that must be expended in order to produce plastic slip, supposed to occur at constant yielding shear stress. The model neglects the final failure stage eventually leading to fracture, so that infinite energy would be necessary to produce a complete separation of the body.

Going back to the discussion of the contents of our paper, we note that the natural compactness for the problem and the identification of the domain of the possible limits are two main issues. To deal with the former, one is naturally led to fix the strong L^1 topology, actually any strong L^p topology would work for $p \in [1, 1^*)$; while the latter is given by the space $SBD^2(\Omega)$, an appropriate subset of $SBD(\Omega)$. To prove such assertions we establish first the equi-coercivity in the space BD of the energies F_k in (1.3) (see (4.7)). Given this, we use a global technique introduced by Ambrosio in [3] (see also [24, 25]) to gain coercivity in the space SBD. To this aim we construct a new sequence of displacements, with SBV regularity, by cutting around suitable sublevel sets of v in order to decrease the elastic contribution of the energy at the expense of introducing a surface term that can be kept controlled (see (4.14)). Thus, the SBD compactness result leads to the identification of the domain of the Γ-limit, and it provides the necessary convergences to prove the lower bound inequality for the volume term in (1.4) simply by applying a classical lower semicontinuity result due to De Giorgi and Ioffe (see estimate (4.4)).

From a technical point of view, the preliminary BD-compactness step is instrumental for two main reasons. On one hand, it allows us to fulfill the assumptions of the compactness theorem in SBD without imposing L^∞ bounds on the relevant sequences as it typically happens in problems of this kind (see for instance [16, 17] and the related comments in [21]); on the other hand it enables us to develop our proof completely within the theory of the space SBD, without making use of its extension $GSBD$, i.e. the space of *generalised special functions with bounded deformation*. Recently, the latter space has been introduced in [21] as the natural functional framework for weak formulations of variational problems arising in fracture mechanics in the setting of linearized elasticity. Roughly speaking, it provides the natural completion of SBD when no uniform bounds in L^∞ can be assumed for the problem at hand, analogously to SBV and its counterpart $GSBV$.

The two $(n-1)$-dimensional terms in the target functional in (1.4) are the result of different contributions: the $\mathcal{H}^{n-1}$ measure of the jump set is detected as in the standard case by the Modica-Mortola type term in (1.3) and it quantifies the energy paid by the function v, being forced to make a transition from values close to 1 to values close to ε_k (see (4.5)); the term depending on the opening of the crack, instead, is associated to the size of the zone where v takes the minimal value ε_k, and, in the general case, it is related to the behaviour close to 0 of the family of quadratic forms in (3.1) (see assumption (H4)). A refinement of the arguments developed in establishing the compactness properties referred to above and the blow-up technique by Fonseca and Müller are then used to infer the needed estimate (cp. with (4.6)). All these issues are dealt with in the proof of Theorem 3.3 below.

Technical problems of different nature arise when we want to show that the lower bound that we have established is matched. Recovery sequences in Γ-convergence problems are built typically for classes of fields that are dense in energy and having more regular members. Recently, this issue has been investigated for linearly elastic brittle materials in the paper [29] in the functional framework of $GSBD$ fields. Such a result allows the proof of the full Γ-convergence statement in the regime $\eta_k = o(\varepsilon_k)$, thus completing the conclusions obtained in the papers [16, 17] under the usual L^∞ restriction. In our setting the density result established in [29] enables us to prove the sharpness of the estimate from below only for bounded fields in $SBD^2(\Omega)$ (see Theorem 3.4). Actually, we can extend it also to *all* fields in $SBV^2(\Omega, \mathbb{R}^n)$ by means of classical density theorems (see Remark 4.5 for more details). Clearly, these are strong hints that the lower bound we have derived is optimal, and that we cannot draw the conclusion in the general case for difficulties probably only of technical nature.

Eventually, let us resume briefly the structure of the paper: Section 2 is devoted to fixing the notations and recalling some of the prerequisites needed in what follows; the main result of the paper, Theorem 3.2, is stated in Section 3, where some comments on the imposed hypotheses are also discussed; finally, in Section 4 the proofs of Theorems 3.3 and 3.4 are presented, from which that of Theorem 3.2 eventually follows.

2. Notation and Preliminaries

Let $n \geq 2$ be a fixed integer. The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively. For every set A the characteristic function χ_A is defined by $\chi_A(x) := 1$ if $x \in A$ and by $\chi_A(x) := 0$ if $x \notin A$.

Throughout the paper Ω is a bounded open subset of $\mathbb{R}^n$, and c denotes a generic positive constant that can vary from line to line. We shall always indicate the parameters on which each constant c depends in the related estimate.

Let us denote by $\mathcal{M}_b(\Omega)$ the set of all bounded Radon measures in Ω and by $\mathcal{M}_b^+(\Omega)$ the set of nonnegative ones. Given $\mu_k, \mu \in \mathcal{M}_b(\Omega)$, we say that $\mu_k \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(\Omega)$ if

$$\int_\Omega \varphi \, d\mu_k \to \int_\Omega \varphi \, d\mu \quad \text{for every } \varphi \in C_0^0(\Omega),$$

where $C_0^0(\Omega)$ is the completion of continuous and compactly supported functions in Ω with respect to the supremum norm.

For the definitions, the notations and the main properties of the spaces BV and SBV we refer to the book [6]. Here, we only recall the definition of the space $SBV^2(\Omega, \mathbb{R}^n)$ used in the sequel:

$$SBV^2(\Omega, \mathbb{R}^n) := \big\{ u \in SBV(\Omega, \mathbb{R}^n) : \nabla u \in L^2(\Omega, \mathbb{M}^{n\times n}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty \big\},$$

being $\mathbb{M}^{n\times n}$ the space of all $n\times n$ matrices.

Instead, we recall briefly some notions related to the spaces $BD(\Omega)$ and to its subspace $SBD(\Omega)$. For complete results we refer to [33], [32], [11], [4], [12], and [23].

The symmetrized distributional derivative Eu of a function $u \in BD(\Omega)$ is by definition a finite Radon measure on Ω. Its density with respect to the Lebesgue measure on Ω is represented by the approximate symmetric gradient $e(u)$, the approximate jump set J_u is a $(\mathcal{H}^{n-1}, n-1)$ rectifiable set on which a measure theoretic normal and approximate one-sided limits $u^\pm$ can be defined $\mathcal{H}^{n-1}$-a.e.. Furthermore, we denote by $[u] := u^+ - u^-$ the related jump function.

For $u_k, u \in BD(\Omega)$, we say that $u_k \rightharpoonup u$ weakly* in $BD(\Omega)$ if $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(\Omega, \mathbb{M}^{n\times n}_{sym})$, where $\mathbb{M}^{n\times n}_{sym}$ is the space of all $n\times n$ symmetric matrices.

We define $SBD^2(\Omega)$ by

$$SBD^2(\Omega) := \big\{ u \in SBD(\Omega) : \, e(u) \in L^2(\Omega, \mathbb{M}^{n\times n}_{sym}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty \big\}. \tag{2.1}$$

Fixed $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$, let π_ξ be the orthogonal projection onto the hyperplane $\Pi^\xi := \big\{ y \in \mathbb{R}^n : y \cdot \xi = 0 \big\}$, and for every subset $A \subset \mathbb{R}^n$ set

$$A^\xi_y := \big\{ t \in \mathbb{R} : y + t\xi \in A \big\} \qquad \text{for } y \in \Pi^\xi.$$

Let $v : \Omega \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}^n$, then define the slices $v^\xi_y, u^\xi_y : \Omega^\xi_y \to \mathbb{R}$ by

$$v^\xi_y(t) := v(y + t\xi) \qquad \text{and} \qquad u^\xi_y(t) := u(y + t\xi) \cdot \xi. \tag{2.2}$$

We recall next the slicing theorem in SBD (see [4]).

Theorem 2.1. *Let $u \in L^1(\Omega, \mathbb{R}^n)$ and let $\{\xi_1, ..., \xi_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then the following two conditions are equivalent:*

(i) *For every $\xi = \xi_i + \xi_j$, $1 \leq i, j \leq n$, $u^\xi_y \in SBV(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and*

$$\int_{\Pi^\xi} \big| Du^\xi_y \big| (\Omega^\xi_y) \, d\mathcal{H}^{n-1}(y) < \infty;$$

(ii) *$u \in SBD(\Omega)$.*

Moreover, if $u \in SBD(\Omega)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ the following properties hold:

(a) *$\nabla(u^\xi_y)(t) = e(u)\,(y + t\xi)\,\xi \cdot \xi$ for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$;*

(b) $J_{u^\xi_y} = \big(J^\xi_u\big)^\xi_y$ *for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi^\xi$, *where*

$$J^\xi_u := \{x \in J_u : [u](x)\cdot\xi \neq 0\};$$

(c) *for* $\mathcal{H}^{n-1}$*-a.e.* $\xi \in \mathbb{S}^{n-1}$

$$\mathcal{H}^{n-1}(J_u \setminus J^\xi_u) = 0. \tag{2.3}$$

Note that, if $u_k, u \in L^1(\Omega,\mathbb{R}^n)$ and $u_k \to u$ in $L^1(\Omega,\mathbb{R}^n)$, then for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence (u_{k_j}) such that

$$(u_{k_j})^\xi_y \to u^\xi_y \text{ in } L^1(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_\xi(\Omega).$$

Finally, for the definitions and the main properties of Γ-convergence we refer to [20].

3. Statement of the Main Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 < p < \infty$, $q := \frac{p}{p-1}$ and let $\varepsilon_k > 0$ be an infinitesimal sequence.

Consider the sequence of functionals $F_k\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega) \to [0,+\infty]$ defined by

$$F_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(\mathscr{Q}(v,e(u)) + \frac{\psi(v)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v|^p\Big)dx & \text{if } (u,v) \in H^1(\Omega,\mathbb{R}^n)\times V_{\varepsilon_k},\\ +\infty & \text{otherwise,}\end{cases} \tag{3.1}$$

where $0 < \gamma < \infty$ and

$$\psi \in C^0([0,1]) \text{ is strictly decreasing with } \psi(1) = 0, \tag{3.2}$$

$$V_{\varepsilon_k} := \left\{v \in W^{1,p}(\Omega) : \varepsilon_k \le v \le 1\ \mathcal{L}^n\text{-a.e. in } \Omega\right\}. \tag{3.3}$$

Moreover, the function $\mathscr{Q} : (0,1]\times\mathbb{M}^{n\times n}_{sym} \to \mathbb{R}^+$ satisfies

(H1) $\mathscr{Q}$ is lower semicontinuous and for every $\mathbb{A} \in \mathbb{M}^{n\times n}_{sym}$ the function $\mathscr{Q}(\cdot,\mathbb{A})$ is continuous at $s = 1$;

(H2) for every $s \in (0,1]$, the function $\mathscr{Q}(s,\cdot)$ is a positive definite quadratic form;

(H3) for every $s \in (0,1]$ and $\mathbb{A} \in \mathbb{M}^{n\times n}_{sym}$, the following inequalities hold

$$c_1 s|\mathbb{A}|^2 \le \mathscr{Q}(s,\mathbb{A}) \le c_2 s|\mathbb{A}|^2, \tag{3.4}$$

for suitable positive constants c_1 and c_2;

(H4) the quadratic forms $s^{-1}\mathscr{Q}(s,\cdot)$ converge uniformly on compact sets of $\mathbb{M}^{n\times n}_{sym}$ to some function $\mathscr{Q}_0$ as $s \downarrow 0^+$.

Note that by items (H3) and (H4) above $\mathscr{Q}_0$ is a quadratic form satisfying

$$c_1|\mathbb{A}|^2 \le \mathscr{Q}_0(\mathbb{A}) \le c_2|\mathbb{A}|^2 \quad \text{for every } \mathbb{A} \in \mathbb{M}^{n\times n}_{sym}.$$

In particular, $\mathscr{Q}_0^{1/2}$ is a norm on $\mathbb{M}^{n\times n}_{sym}$, and

$$c_3^{-1}s\,\mathscr{Q}_0(\mathbb{A}) \le \mathscr{Q}(s,\mathbb{A}) \le c_3\,s\,\mathscr{Q}_0(\mathbb{A}) \quad \text{for all } (s,\mathbb{A}) \in (0,1]\times\mathbb{M}^{n\times n}_{sym}, \tag{3.5}$$

with $c_3 := c_2\,c_1^{-1} \ge 1$.

Remark 3.1. *Let us stress that thanks to (H2) and (H3), assumption (H4) is rather natural as it is satisfied by families* $\varepsilon_k^{-1}\mathscr{Q}(\varepsilon_k,\cdot)$, $\varepsilon_k \downarrow 0^+$, *up to the extraction of subsequences.*

For instance, given $\mathscr{Q}_0$ *and* $\mathscr{Q}_1$ *two coercive quadratic forms on* $\mathbb{M}^{n\times n}_{sym}$, *the family* $\mathscr{Q}(s,\mathbb{A}) := s(s\mathscr{Q}_1(\mathbb{A}) + (1-s)\mathscr{Q}_0(\mathbb{A}))$ *satisfies all the assumptions (H1)-(H4) above.*

The asymptotic behaviour of the family (F_k) is described in terms of the functional $\Phi\colon L^1(\Omega,\mathbb{R}^n) \to [0,+\infty]$ given by

$$\Phi(u) := \begin{cases} \displaystyle\int_\Omega \mathscr{Q}_1(e(u))dx + a\mathcal{H}^{n-1}(J_u) + b\int_{J_u}\mathscr{Q}_0^{1/2}([u]\odot\nu_u)d\mathcal{H}^{n-1} & \text{if } u \in SBD^2(\Omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{3.6}$$

where we have set $\mathscr{Q}_1(\mathbb{A}) := \mathscr{Q}(1,\mathbb{A})$ for all $\mathbb{A} \in \mathbb{M}^{n\times n}_{sym}$, and

$$a := 2q^{1/q}(\gamma p)^{1/p}\int_0^1 \psi^{1/q}(s)\,ds, \qquad b := 2\psi^{1/2}(0). \tag{3.7}$$

The Γ-limit of the sequence F_k is identified in suitable subspaces of $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ (cp. with Theorem 3.2 and Remark 4.5 below).

Theorem 3.2. *Assume the conditions in (3.1)-(3.7) are satisfied, and let Ω be a bounded open set with Lipschitz boundary. The Γ-limit of (F_k) in the strong $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ topology is given on the subspace $L^\infty(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ by*

$$F(u,v) := \begin{cases} \Phi(u) & \text{if } v = 1\ \mathcal{L}^n\text{-a.e. in } \Omega,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.8}$$

As usual, we shall prove the previous result by showing separately a lower bound inequality and an upper bound inequality. To this aim we define

$$F' := \Gamma\text{-}\liminf_{k\to\infty} F_k \qquad \text{and} \qquad F'' := \Gamma\text{-}\limsup_{k\to\infty} F_k. \tag{3.9}$$

Then, Theorem 3.2 follows from the ensuing two statements. In the first we establish the lower bound inequality in full generality and identify the domain of the (inferior) Γ-limit; in the second instead we prove the upper bound inequality on L^∞ due to a difficulty probably of technical nature. In addition, in Remark 4.5 we extend the upper bound inequality to all maps in the space SBV.

Theorem 3.3. *Assume (3.1)-(3.7). Let $(u,v) \in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ be such that $F'(u,v)$ is finite. Then, $v = 1$ $\mathcal{L}^n$-a.e. in Ω and*

$$\Phi(u) \le F'(u,1). \tag{3.10}$$

Theorem 3.4. *Assume (3.1)-(3.7) and assume that Ω is a bounded open set with Lipschitz boundary. Then, for every $u \in L^\infty(\Omega,\mathbb{R}^n)$ we have*

$$F''(u,1) \le \Phi(u). \tag{3.11}$$

4. Proof of the Main Results

We start off by establishing the lower bound estimate. We need to introduce further notation: we consider the strictly increasing map $\phi\colon [0,1] \to [0,\infty)$ defined by

$$\phi(t) := \int_0^t \psi^{1/q}(s)\,ds \qquad \text{for every } t \in [0,1]. \tag{4.1}$$

Proof of Theorem 3.3. By the definition of Γ-$\liminf$ it is enough to prove that if (u,v) belongs to $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ and if $(u_k,v_k) \in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ is a sequence such that

$$(u_k,v_k) \to (u,v) \text{ in } L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega), \tag{4.2}$$

$$\sup_k F_k(u_k,v_k) \le L < \infty, \tag{4.3}$$

then $u \in SBD^2(\Omega)$, $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and the ensuing estimates hold true with $\lambda \in (0,1)$

$$\liminf_{k\to\infty} \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))dx \ge \int_\Omega \mathscr{Q}_1(e(u))dx, \tag{4.4}$$

$$\liminf_{k\to\infty} \int_{\Omega\setminus\Omega_k^\lambda} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \ge 2q^{1/q}(\gamma\,p)^{1/p}(\phi(1)-\phi(\lambda))\mathcal{H}^{n-1}(J_u), \tag{4.5}$$

and with fixed $\delta > 0$ there is $\lambda_\delta > 0$ such that for all $\lambda \in (0,\lambda_\delta)$

$$\liminf_{k\to\infty} \int_{\Omega_k^\lambda} \Big(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\Big)dx \ge 2\psi^{1/2}(\lambda)\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot\nu_u)d\mathcal{H}^{n-1} + O(\delta), \tag{4.6}$$

where we have set $\Omega_k^\lambda := \{v_k \le \lambda\}$. Given (4.4)-(4.6) for granted, we conclude (3.10) by letting first $\lambda \downarrow 0$ and then $\delta \downarrow 0$.

In order to simplify the notation, we set

$$I_k^1 := \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))\, dx,$$
$$I_k^2 := \int_{\Omega\setminus\Omega_k^\lambda} \left(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\right) dx,$$
$$I_k^3 := \int_{\Omega_k^\lambda} \left(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\right) dx.$$

Clearly, if (u_k, v_k) satisfies (4.2) and (4.3), then $v_k \to v = 1$ in $L^1(\Omega)$. The fact that u belongs to $SBD^2(\Omega)$ and inequalities (4.4) and (4.5) can be obtained as a by-product of a slicing argument, following the lines of [29, Theorem 4.3]. Here, we pursue a global approach, arguing as in [25, Lemma 3.2.1] (see also [24]).

We first notice that (u_k) is pre-compact in the weak* topology of $BD(\Omega)$. To verify this it is sufficient to prove that

$$\sup_k \int_\Omega |e(u_k)| dx < \infty. \tag{4.7}$$

More precisely we show that

$$F_k(u_k, v_k) \geq \kappa_1 \int_\Omega |e(u_k)|\, dx - \kappa_2, \tag{4.8}$$

with $\kappa_1 := \max_{\lambda\in[0,1]} \big(2(c_1\, \psi(\lambda))^{1/2} \wedge \frac{c_1\,\lambda}{\mathcal{L}^n(\Omega)}\big)$ and $\kappa_2 := 2(c_1\, \psi(0))^{1/2}$. Indeed, on one hand by (3.4) and the Jensen inequality we have

$$I_k^1 = \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k)) dx \geq c_1\, \lambda \int_{\Omega\setminus\Omega_k^\lambda} |e(u_k)|^2 dx \geq \frac{c_1\,\lambda}{\mathcal{L}^n(\Omega)} \Big(\int_{\Omega\setminus\Omega_k^\lambda} |e(u_k)| dx\Big)^2, \tag{4.9}$$

and on the other hand by the Cauchy-Schwartz inequality we find

$$I_k^3 = \int_{\Omega_k^\lambda} \Big(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\Big) dx \geq c_1\, \varepsilon_k \int_{\Omega_k^\lambda} |e(u_k)|^2 dx + \frac{\psi(\lambda)}{\varepsilon_k} \mathcal{L}^n(\Omega_k^\lambda)$$
$$\geq 2(c_1\, \psi(\lambda))^{1/2} \int_{\Omega_k^\lambda} |e(u_k)| dx. \tag{4.10}$$

Adding up estimates (4.9) and (4.10) eventually we get

$$F_k(u_k, v_k) \geq \frac{c_1\,\lambda}{\mathcal{L}^n(\Omega)} \left(\int_{\Omega\setminus\Omega_k^\lambda} |e(u_k)|\, dx\right)^2 + 2(c_1 \psi(\lambda))^{1/2} \int_{\Omega_k^\lambda} |e(u_k)|\, dx,$$

from which it is then easy to obtain inequality (4.8). In conclusion, (4.7) follows directly from (4.3) and (4.8).

Therefore, from (4.7), as u_k converges to u in $L^1(\Omega, \mathbb{R}^n)$, we deduce that $u \in BD(\Omega)$ and that actually $u_k \rightharpoonup u$ weakly*-$BD(\Omega)$.

Proof of estimate (4.4) *and that* $u \in SBD^2(\Omega)$. We construct a function $\tilde{u}_k$ in a way that it is null near the jump set J_u of u and coincides with u_k elsewhere.

Recalling the very definition of ϕ in (4.1) we have that $\phi(v_k) \in W^{1,p}(\Omega)$, and moreover, Young inequality and the BV Coarea Formula yield

$$I_k^2 \geq q^{1/q}(\gamma\, p)^{1/p} \int_{\Omega\setminus\Omega_k^\lambda} \psi^{1/q}(v_k)|\nabla v_k| dx$$
$$= q^{1/q}(\gamma\, p)^{1/p} \int_{\Omega\setminus\Omega_k^\lambda} |\nabla(\phi(v_k))| dx = q^{1/q}(\gamma\, p)^{1/p} \int_{\phi(\lambda)}^{\phi(1)} \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega) dt. \tag{4.11}$$

Fix $\lambda' \in (\lambda, 1)$, the Mean Value theorem ensures for every $k \in \mathbb{N}$ the existence of $t_k \in (\phi(\lambda), \phi(\lambda'))$ such that

$$\int_{\phi(\lambda)}^{\phi(1)} \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega)dt \geq (\phi(\lambda') - \phi(\lambda))\mathrm{Per}\,(\{\phi(v_k) > t_k\}, \Omega). \tag{4.12}$$

Set $\lambda_k := \phi^{-1}(t_k)$, then note that $\Omega \setminus \Omega_k^{\lambda_k} = \{\phi(v_k) > t_k\}$ is a set of finite perimeter satisfying by the latter inequality and (4.3)

$$\mathrm{Per}\,(\Omega \setminus \Omega_k^{\lambda_k}, \Omega) \leq c \tag{4.13}$$

for some $c = c(\lambda, \lambda', \phi, L)$. Let now $\tilde{u}_k := \chi_{\Omega\setminus\Omega_k^{\lambda_k}} u_k$, then the Chain Rule Formula in BV [6, Theorem 3.96] yields that $\tilde{u}_k \in SBV(\Omega, \mathbb{R}^n)$ with

$$D\tilde{u}_k = \chi_{\Omega\setminus\Omega_k^{\lambda_k}} \nabla u_k \mathcal{L}^n \llcorner \Omega + u_k \otimes \nu_{\partial^*\Omega_k^{\lambda_k}} \mathcal{H}^{n-1} \llcorner \partial^*\Omega_k^{\lambda_k}.$$

In particular, $\mathcal{H}^{n-1}(J_{\tilde{u}_k} \setminus \partial^*\Omega_k^{\lambda_k}) = 0$, then by (4.9), (4.11) and (4.13) the functions $\tilde{u}_k$ satisfy

$$\int_\Omega |e(\tilde{u}_k)|^2 dx + \mathcal{H}^{n-1}(J_{\tilde{u}_k}) \leq c \tag{4.14}$$

for some $c = c(\lambda, \lambda', \phi, L, c_1) < \infty$, and in addition

$$\|\tilde{u}_k - u\|_{L^1(\Omega,\mathbb{R}^n)} \leq \|u_k - u\|_{L^1(\Omega,\mathbb{R}^n)} + \int_{\Omega_k^\lambda} |u| dx. \tag{4.15}$$

As $v_k \to 1$ in $L^1(\Omega)$ we find $\mathcal{L}^n(\Omega_k^\lambda) \downarrow 0$, thus (4.15) implies that $\tilde{u}_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$. Since we have established that $u \in BD(\Omega)$, it is easy to deduce from the SBD Compactness Theorem [12, Theorem 1.1] (see also [16, Lemma 5.1]) and from inequality (4.14) that actually $u \in SBD^2(\Omega)$, with

$$e(\tilde{u}_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega, \mathbb{M}^{n\times n}_{sym}), \tag{4.16}$$

and

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{\tilde{u}_k}). \tag{4.17}$$

Eventually, by taking into account that

$$\liminf_{k\to\infty} \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))dx = \liminf_{k\to\infty} \int_\Omega \mathscr{Q}(v_k, e(\tilde{u}_k))dx,$$

(4.4) follows from (4.16), from the convergence $v_k \to 1$ in $L^1(\Omega)$, and from [15, Theorem 2.3.1].

Proof of estimate (4.5). Regrettably, inequality (4.5) is not a straightforward consequence of the previous arguments. Indeed, (4.11), (4.12), (4.17) and $\mathcal{H}^{n-1}(J_{\tilde{u}_k} \setminus \partial^*\Omega_k^{\lambda_k}) = 0$ lead to an estimate differing from (4.5) by a multiplicative factor 2 on the left-hand side. Therefore, we need a more accurate argument. To this aim, we note that by (4.11) and the Fatou Lemma we have

$$\liminf_{k\to\infty} I_k^2 \geq q^{1/q}(\gamma\, p)^{1/p} \int_{\phi(\lambda)}^{\phi(1)} \liminf_{k\to\infty} \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega)\, dt,$$

then in order to conclude (4.5) it suffices to prove that

$$\liminf_k \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega) \geq 2\mathcal{H}^{n-1}(J_u) \quad \text{ for all } t \in (\phi(\lambda), \phi(1)). \tag{4.18}$$

This follows via a slicing argument as established in [25, Lemma 3.2.1] (see also [13, Lemma 2] where the proof is given in a slightly less general setting). We report in what follows the proof of estimate (4.18) for the sake of completeness.

Fixed $t \in (\phi(\lambda), \phi(1))$ for which the right-hand side of (4.18) is finite, we define $\tau := \phi^{-1}(t)$ and $U_k^\tau := \Omega \setminus \Omega_k^\tau$. For every open subset $A \subset \Omega$ and vector $\xi \in \mathbb{S}^{n-1}$, we claim that

$$\liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2 \int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A) d\mathcal{H}^{n-1}, \tag{4.19}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi(A)$ (recall the notations and the results in Theorem 2.1). Given (4.19) for granted, the Coarea Formula for rectifiable sets and the Fatou lemma yield the following lower semicontinuity estimate

$$\liminf_k \operatorname{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right)$$
$$= \liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2\int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A)d\mathcal{H}^{n-1} = 2\int_{J_u^\xi \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}. \quad (4.20)$$

Since $\mathcal{H}^{n-1}(J_u \setminus J_u^\xi) = 0$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ (see (2.3)), we infer from (4.20)

$$\liminf_k \operatorname{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right) \geq 2\int_{J_u \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}. \quad (4.21)$$

In conclusion, inequality (4.18) follows from (4.21) by passing to the supremum on a sequence (ξ_r) dense in $\mathbb{S}^{n-1}$ and applying [6, Lemma 2.35], since the function

$$A \to \liminf_k \operatorname{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right)$$

is superadditive on disjoint open subsets of Ω.

Let us finally prove (4.19). Note that there exists a subsequence (u_r, v_r) of (u_k, v_k) such that

$$\liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) = \lim_r \mathcal{H}^{n-1}(J_{\chi_{U_r^\tau}} \cap A), \quad (4.22)$$
$$\left((u_r)_y^\xi, (v_r)_y^\xi\right) \to \left(u_y^\xi, 1\right) \text{ in } L^1(\Omega_y^\xi) \times L^1(\Omega_y^\xi), \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_\xi(\Omega), \quad (4.23)$$

and with fixed $\eta > 0$, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi(\Omega)$ we find

$$\liminf_r \left(\eta \int_{A_y^\xi} \left((v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 + \frac{\psi\left((v_r)_y^\xi\right)}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} \left|\nabla((v_r)_y^\xi)\right|^p\right) dt + \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A)\right) < \infty, \quad (4.24)$$

by (3.4), (4.3), our choice of τ, and the Fatou lemma.

Fix $y \in \pi_\xi(\Omega)$ be satisfying (4.23), (4.24), and assume also that $\mathcal{H}^0\left(J_{u_y^\xi} \cap A\right) > 0$. Moreover, up to extracting a further subsequence (depending on y and not relabeled for convenience), we may suppose that the lower limit in (4.24) is actually a limit.

Let $\{t_1, ..., t_l\}$ be an arbitrary subset of $J_{u_y^\xi} \cap A$, and let $(I_i)_{1 \leq i \leq l}$ be a family of pairwise disjoint open intervals such that $t_i \in I_i$, $I_i \subset\subset A_y^\xi$. Then, for every $1 \leq i \leq l$, we claim that

$$s_i := \limsup_r \inf_{I_i} (v_r)_y^\xi = 0.$$

Indeed, if s_h was strictly positive for some $h \in \{1, ..., l\}$, then

$$\inf_{I_h} (v_j)_y^\xi \geq \frac{s_h}{2}$$

for a suitable subsequence (v_j) of (v_r), and thus (4.24) would give

$$\int_{I_h} \left|\nabla((u_j)_y^\xi)\right|^2 dt \leq c,$$

for some constant c. Hence, Rellich-Kondrakov's theorem and (4.23) would imply the slice u_y^ξ to be in $W^{1,1}(I_h, \mathbb{R}^n)$, which is a contradiction since by assumption $\mathcal{H}^0\left(J_{u_y^\xi} \cap I_h\right) > 0$. So let $t_r^i \in I_i$ be such that

$$\lim_r (v_r)_y^\xi (t_r^i) = 0,$$

and α_i, $\beta_i \in I_i$, with $\alpha_i < t_r^i < \beta_i$, be such that

$$\lim_r (v_r)_y^\xi (\alpha_i) = \lim_r (v_r)_y^\xi (\beta_i) = 1.$$

Then, there follows

$$\liminf_r \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap I_i) \geq 2.$$

Hence, the subadditivity of the inferior limit and the arbitrariness of l yield

$$\liminf_r \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A) \geq 2\mathcal{H}^0(J_{u_y^\xi} \cap A).$$

Therefore, we obtain

$$\liminf_r \left(\eta \int_{A_y^\xi} \left((v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 + \frac{\psi\left((v_r)_y^\xi\right)}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} \left|\nabla((v_r)_y^\xi)\right|^p\right) dt + \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A)\right) \geq 2\mathcal{H}^0(J_{u_y^\xi} \cap A),$$

which integrated on $\pi_\xi(A)$ gives

$$\liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2\int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A)d\mathcal{H}^{n-1} - \eta c$$

for some positive constant $c = c(L)$. As $\eta \downarrow 0$ we find (4.19).

Proof of estimate (4.6). We employ the blow-up technique introduced by Fonseca and Müller in [27]. First, we observe that by the Cauchy-Schwartz inequality we have

$$I_k^3 \geq \varepsilon_k \int_{\Omega_k^\lambda} \frac{\mathscr{Q}(v_k, e(u_k))}{v_k} dx + \frac{\psi(\lambda)}{\varepsilon_k}\mathcal{L}^n(\Omega_k^\lambda) \geq 2\psi^{1/2}(\lambda) \int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k}\right)^{1/2} dx, \tag{4.25}$$

thus in order to get (4.6) it suffices to show that for all $\delta > 0$ there is $\lambda_\delta > 0$ such that for $\lambda \in (0, \lambda_\delta)$ we have

$$\liminf_k \int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k}\right)^{1/2} dx \geq \int_{J_u} \mathscr{Q}_0^{1/2}([u] \odot \nu)d\mathcal{H}^{n-1} + O(\delta). \tag{4.26}$$

Actually the uniform convergence on compact sets of $\mathbb{M}^{n\times n}_{sym}$ assumed in (H4) above implies that, with fixed $\delta > 0$, for some $\lambda_\delta > 0$ and all $\lambda \in (0, \lambda_\delta)$ we have

$$\begin{aligned}\int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k}\right)^{1/2} dx &= \int_{\Omega_k^\lambda} \mathscr{Q}_{v_k(x)}^{1/2}\left(\frac{e(u_k)}{|e(u_k)|}\right)|e(u_k)|dx \\ &\geq \int_{\Omega_k^\lambda} \left(\mathscr{Q}_0^{1/2}\left(\frac{e(u_k)}{|e(u_k)|}\right) - \delta\right) |e(u_k)|dx \geq \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k))dx - \delta\, |Eu_k|(\Omega),\end{aligned}$$

where we have set $\mathscr{Q}_s(\mathbb{A}) := s^{-1}\mathscr{Q}(s, \mathbb{A})$. Thus, inequality (4.26) is reduced to prove

$$\liminf_k \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k))dx \geq \int_{J_u} \mathscr{Q}_0^{1/2}([u] \odot \nu)d\mathcal{H}^{n-1}, \tag{4.27}$$

being $\delta > 0$ arbitrary and $(|Eu_k|(\Omega))$ being bounded as shown in (4.7).

Let (u_r) be a subsequence of (u_k) such that

$$\liminf_k \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k))dx = \lim_r \int_{\Omega_r^\lambda} \mathscr{Q}_0^{1/2}(e(u_r))dx.$$

In order to prove (4.27), for every Borel set $A \subseteq \Omega$ we introduce

$$\mu_r(A) := \int_{\Omega_r^\lambda \cap A} \mathscr{Q}_0^{1/2}(e(u_r))dx,$$

$$\theta_r(A) := \int_A \mathscr{Q}_0^{1/2}(e(u_r))dx,$$

and

$$\zeta_r(A) := F_r(u_r, v_r, A),$$

where $F_r(\cdot, \cdot, A)$ denotes the functional defined in (3.1) with the set of integration Ω replaced by A.

It is evident that the former set functions are finite Borel measures, with (μ_r), (θ_r) and (ζ_r) actually equi-bounded in mass thanks to inequalities (4.3) and (4.7). Hence, up to subsequences not relabelled for convenience, we may suppose that

$$\mu_r \rightharpoonup \mu, \quad \theta_r \rightharpoonup \theta, \quad \text{and} \quad \zeta_r \rightharpoonup \zeta \quad \text{weakly}^* \text{ in } \mathcal{M}_b^+(\Omega), \tag{4.28}$$

for some μ, θ and $\zeta \in \mathcal{M}_b^+(\Omega)$, respectively.

Being

$$\lim_r \mu_r(\Omega) \geq \mu(\Omega),$$

to infer (4.27) we need only to show that

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u} \geq \mathscr{Q}_0^{1/2}([u] \odot \nu_u) \quad \mathcal{H}^{n-1}\text{-a.e. in } J_u, \tag{4.29}$$

where $\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}$ is the Radon-Nikodým derivative of μ with respect to $\mathcal{H}^{n-1} \llcorner J_u$.

We shall prove the latter inequality for the subset of points x_0 in J_u for which the Radon-Nikodým derivatives

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \ \frac{d\theta}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \ \frac{d\zeta}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \tag{4.30}$$

exist finite,

$$\frac{d\mathscr{Q}_0^{1/2}(\frac{dEu}{d|Eu|})|Eu|}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \mathscr{Q}_0^{1/2}([u] \odot \nu_u)(x_0) \tag{4.31}$$

and

$$\lim_{\rho \to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q_\nu(x_0, \rho))}{\rho^{n-1}} = 1, \tag{4.32}$$

where $\nu := \nu_u(x_0)$, Q_ν is any unitary cube centred in the origin with one face orthogonal to ν, and $Q_\nu(x_0, \rho) := x_0 + \rho\, Q_\nu$. Formula (4.32) is a consequence of the $(\mathcal{H}^{n-1}, n-1)$ rectifiability of J_u (see [6, Theorem 2.83]). Note that all the conditions above define a set of full measure in J_u.

By selecting one of such points $x_0 \in J_u$, we get

$$\begin{aligned}
\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0) &= \lim_{\rho \to 0} \frac{\mu(Q_\nu(x_0, \rho))}{\rho^{n-1}} = \lim_{\substack{\rho \in I \\ \rho \to 0}} \lim_{r \to \infty} \frac{\mu_r(Q_\nu(x_0, \rho))}{\rho^{n-1}} \\
&= \lim_{\substack{\rho \in I \\ \rho \to 0}} \lim_{r \to \infty} \frac{1}{\rho^{n-1}} \left(\theta_r(Q_\nu(x_0, \rho)) - \theta_r(Q_\nu(x_0, \rho) \setminus \Omega_r^\lambda)\right),
\end{aligned} \tag{4.33}$$

where

$$I := \Big\{\rho \in (0, \frac{2}{\sqrt{n}} \mathrm{dist}(x_0, \partial\Omega)) : \mu(\partial Q_\nu(x_0, \rho)) = \theta(\partial Q_\nu(x_0, \rho)) = \zeta(\partial Q_\nu(x_0, \rho)) = 0\Big\}.$$

Note that I is a subset of radii of full measure in $(0, \frac{2}{\sqrt{n}} \mathrm{dist}(x_0, \partial\Omega))$, and that the second equality in (4.33) easily follows from the convergence $\mu_r \rightharpoonup \mu$ weakly* in $\mathcal{M}_b^+(\Omega)$.

Further, we claim that

$$\lim_{\substack{\rho \in I \\ \rho \to 0}} \lim_{r \to \infty} \frac{\theta_r(Q_\nu(x_0, \rho) \setminus \Omega_r^\lambda)}{\rho^{n-1}} = 0. \tag{4.34}$$

Indeed, the Hölder inequality, the very definition of F_k in (3.1), and (3.5) imply

$$\begin{aligned}
\frac{\theta_r(Q_\nu(x_0, \rho) \setminus \Omega_r^\lambda)}{\rho^{n-1}} &= \frac{1}{\rho^{n-1}} \int_{Q_\nu(x_0,\rho) \setminus \Omega_r^\lambda} \mathscr{Q}_0^{1/2}(e(u_r))dx \leq \frac{c_3^{1/2}}{\rho^{n-1}} \int_{Q_\nu(x_0,\rho) \setminus \Omega_k^\lambda} \mathscr{Q}_{v_r(x)}^{1/2}(e(u_r))dx \\
&\leq \Big(c_3 \frac{\mathcal{L}^n(Q_\nu(x_0, \rho) \setminus \Omega_r^\lambda)}{\rho^{n-1}}\Big)^{1/2} \Big(\frac{1}{\rho^{n-1}} \int_{Q_\nu(x_0,\rho) \setminus \Omega_r^\lambda} \mathscr{Q}_{v_r(x)}(e(u_r))dx\Big)^{1/2} \\
&\leq (c_3\rho)^{1/2} \lambda^{-1/2} \Big(\frac{F_r(u_r, v_r, Q_\nu(x_0, \rho))}{\rho^{n-1}}\Big)^{1/2} = (c_3\rho)^{1/2} \lambda^{-1/2} \Big(\frac{\zeta_r(Q_\nu(x_0, \rho))}{\rho^{n-1}}\Big)^{1/2}.
\end{aligned}$$

Finally, equality (4.34) is a consequence of the latter estimate and condition (4.30).

By taking (4.34) into account, (4.33) rewrites as

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \frac{d\theta}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0). \tag{4.35}$$

The convergence of the symmetrized distributional derivatives, i.e.

$$Eu_r \rightharpoonup Eu \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega, \mathbb{M}^{n \times n}_{sym})$$

is a result of (4.2) and (4.7), in turn implying that

$$\theta(Q_\nu(x_0,\rho)) \geq \int_{Q_\nu(x_0,\rho)} \mathscr{Q}_0^{1/2}\Big(\frac{dEu}{d|Eu|}\Big)d|Eu| \tag{4.36}$$

by the convexity of $\mathscr{Q}_0^{1/2}$ and the stated convergence. Thus, by (4.31) and (4.36) we get

$$\frac{d\theta}{d\mathcal{H}^{n-1}\llcorner J_u}(x_0) \geq \liminf_{\rho\to 0}\frac{1}{\rho^{n-1}}\int_{Q_\nu(x_0,\rho)} \mathscr{Q}_0^{1/2}\Big(\frac{dEu}{d|Eu|}\Big)d|Eu| = \mathscr{Q}_0^{1/2}([u]\odot\nu_u)(x_0). \tag{4.37}$$

Eventually, (4.35) and (4.37) conclude the proof of (4.29), and then of (4.27). □

The proof of the Γ-lim sup inequality in Theorem 3.4 takes advantage of the density theorems for $GSBD(\Omega)$ [28, Theorem 3.1] and for $SBV(\Omega,\mathbb{R}^n)$ [19, Theorem 3.1] stated below for convenience of the reader.

Theorem 4.1. *Assume that Ω has Lipschitz boundary, and let $u \in GSBD^2(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$. Then there exists a sequence $(u_k) \subset SBV^2\cap L^\infty(\Omega,\mathbb{R}^n)$ such that each J_{u_k} is contained in the union S_k of a finite number of closed connected pieces of C^1-hypersurfaces, each u_k belongs to $W^{1,\infty}(\Omega\setminus S_k,\mathbb{R}^n)$, and the following properties hold:*

(1) $||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2) $||e(u_k) - e(u)||_{L^2(\Omega,\mathbb{M}^{n\times n}_{sym})} \to 0$,

(3) $\mathcal{H}^{n-1}(J_{u_k}\triangle J_u) \to 0$,

(4) $\displaystyle\int_{J_{u_k}\cup J_u} |u_k^\pm - u^\pm|\wedge M\, d\mathcal{H}^{n-1} \to 0$, *for every $M > 0$.*

Remark 4.2. *Note that the expression in (4) makes sense by* [21, Theorem 5.2], *since one can define the traces $u^\pm$ of a function $u \in GBD(\Omega)$ on any C^1 submanifold of dimension $n-1$.*

We recall next a density result in SBV, for which we need to introduce further terminology. We say that $u \in SBV(\Omega,\mathbb{R}^n)$ is a piecewise smooth SBV-function if $u \in W^{m,\infty}(\Omega\setminus\overline{J_u},\mathbb{R}^n)$ for every m, $\mathcal{H}^{n-1}((\overline{J_u}\cap\Omega)\setminus J_u) = 0$, and the set $\overline{J_u}\cap\Omega$ is a finite union of closed pairwise disjoint $(n-1)$-simplexes intersected with Ω.

Theorem 4.3. *Assume that Ω has Lipschitz boundary. Let $u \in SBV^2\cap L^\infty(\Omega,\mathbb{R}^n)$. Then there exists a sequence (u_k) of piecewise smooth SBV-functions such that*

(1) $||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2) $||\nabla u_k - \nabla u||_{L^2(\Omega,\mathbb{M}^{n\times n})} \to 0$,

(3) $\displaystyle\limsup_k \int_{\overline{A}\cap J_{u_k}} \varphi(x,u_k^+,u_k^-,\nu_{u_k})d\mathcal{H}^{n-1} \leq \int_{\overline{A}\cap J_u}\varphi(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}$,

for every open set $A\subset\Omega$ and for every function $\varphi : \Omega\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1}\to[0,+\infty)$ upper semicontinuous and such that

$$\varphi(x,a,b,\nu) = \varphi(x,b,a,-\nu) \quad \textit{for } x\in\Omega,$$
$$\limsup_{\substack{(y,a',b',\mu)\to(x,a,b,\nu)\\ y\in\Omega}} \varphi(y,a',b',\mu) < +\infty \quad \textit{for } x\in\partial\Omega,$$

for every $a,b\in\mathbb{R}^n$, and $\nu\in\mathbb{S}^{n-1}$.

Remark 4.4. *Note that if $\Omega\subset\mathbb{R}^n$ is an open cube, then the intersection $\overline{J_{u_k}}\cap\Omega$ is a polyhedron. Therefore, adapting the arguments in* [19, Remark 3.5] *and* [18, Corollary 3.11] *we can construct a new approximating sequence $(\tilde{u}_k)$ satisfying all requirements of Theorem 4.3 and such that $J_{\tilde{u}_k}\subset\subset\Omega$.*

Remark 4.5. *The* Γ-$\limsup$ *inequality in Theorem 3.4 is stated only for fields in the subspace* $L^\infty(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *of* $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *since Theorem 4.1 does not guarantee the convergence*

$$\int_{J_{u_k}\cup J_u} |[u_k]-[u]|\,d\mathcal{H}^{n-1}\to 0 \tag{4.38}$$

for every u *in* $SBD^2(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$*. If (4.38) was true, then Theorem 4.1 combined with Theorem 4.3 would allow us to prove the* Γ-$\limsup$ *inequality for those fields* u *that are piecewise smooth. In such a case, the construction of recovery sequences follows quite classical lines, and by density the* Γ-$\limsup$ *inequality in* $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *would be completely proved.*

Nevertheless, this argument applies to fields in $L^\infty(\Omega,\mathbb{R}^n)$ *since the approximating sequence* (u_k) *in Theorem 4.1 is constructed in a way that* $\|u_k\|_{L^\infty(\Omega,\mathbb{R}^n)}\le\|u\|_{L^\infty(\Omega,\mathbb{R}^n)}$*.*

The same conclusion of Theorem 3.4 can be drawn for all fields in $SBV^2(\Omega,\mathbb{R}^n)$*. Indeed, the functional in (3.6) is continuous on sequences of truncations, therefore the conclusion follows by Theorem 4.3 and a diagonal argument. In this respect, take also into account the equality* $GSBV^2(\Omega,\mathbb{R}^n)\cap BD(\Omega)=SBV^2(\Omega,\mathbb{R}^n)$*.*

Finally let us prove the upper bound estimate.

Proof of Theorem 3.4. Let $u\in SBD^2(\Omega)\cap L^\infty(\Omega,\mathbb{R}^n)$, then by the lower semicontinuity of F'' and Theorem 4.1 it is not restrictive to assume that u belongs to $SBV^2\cap L^\infty(\Omega,\mathbb{R}^n)$. By a local reflection argument we can also assume that $\Omega\subset\mathbb{R}^n$ is a open cube and again by the lower semicontinuity of F'', by Theorem 4.3, and by Remark 4.4 we can reduce ourselves to prove (3.11) for a piecewise smooth SBV-function u with $\overline{J_u}\subset\Omega$. Finally, up to a truncation argument, condition $u\in L^\infty(\Omega,\mathbb{R}^n)$ is preserved.

For the construction of the recovery sequence we shall follow the lines of [28, Theorem 3.3] (see also [22, Theorem 3.3]).

Since $\overline{J_u}$ is a finite union of closed pairwise disjoint $(n-1)$-simplexes well-contained in Ω, we reduce to study the case when $S:=\overline{J_u}$ is a $(n-1)$-simplex. In order to simplify the computation we also assume $S\subset\{x_n=0\}$, we denote the generic point $x\in\mathbb{R}^n$ by $x=(\overline{x},x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$, and we orient J_u so that $\nu_u=(0,1)$.

Let

$$\Omega^\pm:=\big\{x\in\Omega:\pm x_n>0\big\}$$

and let L be the maximum between the Lipschitz constants of u in Ω^+ and Ω^-. Let also

$$\sigma_k(\overline{x}):=\frac{\varepsilon_k}{2\psi(0)^{1/2}}\mathscr{Q}_0^{1/2}([u(\overline{x},0)]\odot e_n),\quad\text{for every }\overline{x}\in S. \tag{4.39}$$

Being u^+ and u^- Lipschitz functions, we deduce that σ_k is in turn a Lipschitz function and that

$$|\nabla\sigma_k(\overline{x})|\le c\,\varepsilon_k, \tag{4.40}$$

for $\mathcal{H}^{n-1}$-a.e. $\overline{x}\in S$ and for a suitable constant $c=c(\psi,L,\mathscr{Q}_0)>0$. Moreover, $\sigma_k=0$ on ∂S, where ∂S is the boundary of S in the relative topology of $\mathbb{R}^{n-1}\times\{0\}$.

We set for $\rho\in(0,1)$

$$f(\rho):=\psi(1-\rho),\quad g(\rho):=\Big(\int_0^{1-\rho}\psi^{-1/p}(s)\,ds\Big)^{-1},\quad\text{and}\quad h(\rho):=(f\cdot g)^{1/2}(\rho),$$

and we introduce the infinitesimal sequence $\rho_k:=h^{-1}(\varepsilon_k)$ having the property that

$$\frac{f(\rho_k)}{\varepsilon_k}=\frac{\varepsilon_k}{g(\rho_k)}\to 0\quad\text{as }k\uparrow\infty. \tag{4.41}$$

Denote by w_k the only solution of the following Cauchy problem in the interval $[0,T_k)$ (uniqueness on such an interval follows from (3.2))

$$\begin{cases} w_k'=\Big(\dfrac{q}{\gamma p}\Big)^{1/p}\varepsilon_k^{-1}\psi^{1/p}(w_k)\\ w_k(0)=\varepsilon_k,\end{cases} \tag{4.42}$$

where $T_k \in (0,\infty]$ is given by

$$T_k := \Big(\frac{\gamma p}{q}\Big)^{1/p}\varepsilon_k \int_{\varepsilon_k}^{1} \psi^{-1/p}(s)\,ds.$$

Furthermore, define $\mu_k \in (0, T_k)$

$$\mu_k := \Big(\frac{\gamma p}{q}\Big)^{1/p}\varepsilon_k \int_{\varepsilon_k}^{1-\rho_k} \psi^{-1/p}(s)\,ds, \tag{4.43}$$

thus μ_k is infinitesimal by (4.41).

We are now in a position to introduce the sets

$$A_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ |x_n| < \sigma_k(\overline{x})\Big\},$$
$$B_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ 0 \le |x_n| - \sigma_k(\overline{x}) \le \mu_k\Big\},$$
$$C_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ d(x, \partial S) \le \mu_k\Big\},$$

where $d(x, \partial S)$ is the distance of the point x from the set ∂S.

Consider the sequence (u_k, v_k) defined by

$$u_k(\overline{x}, x_n) := \begin{cases} \dfrac{x_n + \sigma_k(\overline{x})}{2\sigma_k(\overline{x})}\big(u(\overline{x}, \sigma_k(\overline{x})) - u(\overline{x}, -\sigma_k(\overline{x}))\big) + u(\overline{x}, -\sigma_k(\overline{x})) & \text{if } x \in A_k, \\ u(x) & \text{if } x \in \Omega \setminus A_k, \end{cases}$$

and

$$v_k(x) := \begin{cases} \varepsilon_k & \text{if } x \in A_k, \\ w_k(|x_n| - \sigma_k(\overline{x})) & \text{if } x \in B_k, \\ w_k(d(x, \partial S)) & \text{if } x \in C_k, \\ 1 - \rho_k & \text{otherwise.} \end{cases}$$

Then, $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$, moreover we shall show that it provides a recovery sequence following the arguments used in [22, Theorem 3.3, inequalities (71)-(78)]. First note that, for every component u_k^i of u_k for $\mathcal{L}^n$-a.e. $(\overline{x}, x_n) \in A_k$ we have that

$$\begin{aligned} &|D_j u_k^i(\overline{x}, x_n)| \\ &\le \left|\frac{x_n}{\sigma_k(\overline{x})} D_j\sigma_k(\overline{x}) \frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})}\right| + \left|D_j u^i(\overline{x}, -\sigma_k(\overline{x})) - D_n u^i(\overline{x}, -\sigma_k(\overline{x})) D_j\sigma_k(\overline{x})\right| \\ &\quad + \left|D_j u^i(\overline{x}, \sigma_k(\overline{x})) + D_n u^i(\overline{x}, \sigma_k(\overline{x})) D_j\sigma_k(\overline{x}) - D_j u^i(\overline{x}, -\sigma_k(\overline{x})) + D_n u^i(\overline{x}, -\sigma_k(\overline{x})) D_j\sigma_k(\overline{x})\right| \\ &\le |D_j\sigma_k(\overline{x})|\Big(\frac{|[u^i(\overline{x}, 0)]|}{2\sigma_k(\overline{x})} + 4L\Big) + 3L \le c, \end{aligned} \tag{4.44}$$

where $j = 1, \dots, n-1$, and

$$\begin{aligned} |D_n u_k^i(\overline{x}, x_n)| &= \left|\frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})}\right| \\ &= \left|\frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^{i+}(\overline{x}, 0)}{2\sigma_k(\overline{x})} + \frac{u^{i+}(\overline{x}, 0) - u^{i-}(\overline{x}, 0)}{2\sigma_k(\overline{x})} + \frac{u^{i-}(\overline{x}, 0) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})}\right| \\ &\le L + \frac{|[u^i(\overline{x}, 0)]|}{2\sigma_k(\overline{x})} \le \frac{c}{\varepsilon_k}; \end{aligned} \tag{4.45}$$

in the previous estimates $c = c(L)$ and we have used (4.40). In particular, we deduce that u_k is a Lipschitz function.

As far as the computation of the energy $F_k(u_k, v_k)$ is concerned we shall mainly focus on the term

$$\int_{A_k} \mathscr{Q}(v_k, e(u_k))dx.$$

The others are estimated in an elementary way following [28, Theorem 3.3]. More precisely, we have

$$\limsup_k \int_{\Omega\setminus A_k} \mathscr{Q}(v_k, e(u_k))dx = \limsup_k \int_{\Omega\setminus A_k} \mathscr{Q}(v_k, e(u))dx \leq \int_\Omega \mathscr{Q}_1(e(u))dx \tag{4.46}$$

by dominated convergence thanks to assumptions (H1) and (H3); then as a result of a straightforward calculation we infer

$$\limsup_k \int_{A_k} \frac{\psi(v_k)}{\varepsilon_k}dx \leq \lim_k \frac{\psi(\varepsilon_k)}{\psi(0)^{1/2}} \int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)\, d\mathcal{H}^{n-1} = \frac{b}{2}\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)\, d\mathcal{H}^{n-1}; \tag{4.47}$$

furthermore from the very definition of w_k and (4.43) we find

$$\int_{B_k} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq 2(1+O(\varepsilon_k))(\gamma p)^{1/p} q^{1/q}\Big(\int_{\varepsilon_k}^{1-\rho_k} \psi^{1/q}(s)\, ds\Big)\mathcal{H}^{n-1}(J_u); \tag{4.48}$$

finally by the Coarea formula and again by the definition of w_k it follows that

$$\int_{C_k} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq c\,\mu_k \int_{\varepsilon_k}^{1-\rho_k} \psi^{1/q}(s)\ ds \leq c\,\mu_k, \tag{4.49}$$

where $c < \infty$. Therefore, by collecting (4.46)-(4.49), to conclude we need only to verify that

$$\lim_k \int_{A_k} \mathscr{Q}(v_k, e(u_k))dx = \frac{b}{2}\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)d\mathcal{H}^{n-1}.$$

To this aim, observe first that assumption (H3), the very definition of u_k, v_k and estimates (4.44), (4.45) imply

$$\int_{A_k} \mathscr{Q}(v_k, e(u_k))dx = \int_{A_k} \mathscr{Q}\Big(\varepsilon_k, \frac{1}{2}\Lambda(D_n u_k^1, \ldots, D_n u_k^{n-1}, 2D_n u_k^n)\Big)dx + o(1), \quad \text{as } k \uparrow \infty,$$

where $\Lambda : \mathbb{R}^n \to \mathbb{M}^{n\times n}_{sym}$ is defined by

$$(\Lambda(x_1, \ldots, x_n))_{i\,j} := 0 \quad \text{if } i, j < n, \quad (\Lambda(x_1, \ldots, x_n))_{i\,n} := x_i \quad \text{if } i \leq n. \tag{4.50}$$

In addition, the definitions of u_k, of σ_k in (4.39), of A_k, and the 2-homogeneity of $\mathscr{Q}$ yield

$$\int_{A_k} \mathscr{Q}\Big(\varepsilon_k, \frac{1}{2}\Lambda(D_n u_k^1, \ldots, D_n u_k^{n-1}, 2D_n u_k^n)\Big)dx = \frac{b}{2}\int_{J_u} \mathscr{Q}_{\varepsilon_k}(\zeta_k(\overline{x}))\cdot \mathscr{Q}_0^{-1/2}([u](\overline{x}, 0)\odot e_n)d\mathcal{H}^{n-1},$$

where

$$\zeta_k(\overline{x}) := \frac{1}{2}\Lambda\Big(u^1(\overline{x}, \sigma_k(\overline{x})) - u^1(\overline{x}, -\sigma_k(\overline{x})), \ldots, u^{n-1}(\overline{x}, \sigma_k(\overline{x})) - u^{n-1}(\overline{x}, -\sigma_k(\overline{x})), 2(u^n(\overline{x}, \sigma_k(\overline{x})) - u^n(\overline{x}, -\sigma_k(\overline{x})))\Big).$$

Eventually, the conclusion follows by (4.50), by (H4), and by the dominated convergence theorem as (ζ_k) converges uniformly to $[u](\cdot, 0)\odot e_n$ on S as $k \uparrow \infty$. □

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