A QUANTITATIVE ISOPERIMETRIC INEQUALITY ON THE SPHERE

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ABSTRACT. In this paper we prove a quantitative version of the isoperimetric inequality on the sphere with a constant independent of the volume of the set E.

1. INTRODUCTION

Recent years have seen an increasing interest in quantitative isoperimetric inequalities motivated by classical papers by Bernstein and Bonnesen [4, 6] and further developments in [14, 18, 17]. In the Euclidean case the optimal result (see [16, 13, 8]) states that if E is a set of finite measure in $\mathbb{R}^n$ then

$$\delta(E) \geq c(n)\alpha(E)^2 \tag{1.1}$$

holds true. Here the *Fraenkel asymmetry index* is defined as

$$\alpha(E) := \min \frac{|E\Delta B|}{|B|},$$

where the minimum is taken among all balls $B \subset \mathbb{R}^n$ with $|B| = |E|$, and the *isoperimetric gap* is given by

$$\delta(E) := \frac{P(E) - P(B)}{P(B)}.$$

The stability estimate (1.1) has been generalized in several directions, for example to the case of the Gaussian isoperimetric inequality [7], to the Almgren higher codimension isoperimetric inequality [2, 5] and to several other isoperimetric problems [3, 1, 9]. In the present paper we address the stability of the isoperimetric inequality on the sphere by Schmidt [21] stating that if $E \subset S^n$ is a measurable set having the same measure as a geodesic ball $B_\vartheta \subset S^n$ for some radius $\vartheta \in (0, \pi)$, then

$$\mathbf{P}(E) \geq \mathbf{P}(B_\vartheta),$$

with equality if and only if E is a geodesic ball. Here, $\mathbf{P}(E)$ stands for the perimeter of E, that is $\mathbf{P}(E) = \mathcal{H}^{n-1}(\partial E)$ if E is smooth and $n \geq 2$ throughout the whole paper.

In view of the previously mentioned stability results the natural counterpart of (1.1) would be the inequality

$$\frac{\mathbf{P}(E) - \mathbf{P}(B_\vartheta)}{\mathbf{P}(B_\vartheta)} \geq c(n)\boldsymbol{\alpha}(E)^2, \tag{1.2}$$

where now the Fraenkel asymmetry index is defined by

$$\boldsymbol{\alpha}(E) := \min \frac{|E\Delta B_\vartheta|}{|B_\vartheta|}. \tag{1.3}$$

The minimum is taken over all geodesic balls $B_\vartheta \subset S^n$ with $|E| = |B_\vartheta|$. Notice, that we are denoting the $\mathcal{H}^n$-measure of a set E by $|E|$. When compared with (1.1) inequality (1.2), even if it looks similar, has a completely different nature; in fact (1.1) is scaling invariant (i.e. invariant under homotheties), while there is no scaling at all on S^n. Indeed, it would be quite easy to adapt one of the different arguments in the papers [16, 13, 8] in order to prove (1.2) with a constant depending additionally on the volume of the set E, but

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blowing up as $\vartheta \downarrow 0$. In fact, the difficult case is when the set E has a small volume sparsely distributed over the sphere. In this situation a localization argument aimed to reduce the problem to the flat Euclidean estimate (1.1) cannot work.

To state our main result we introduce the *oscillation index* $\boldsymbol{\beta}(E)$ of a set $E \subset S^n$,

$$\boldsymbol{\beta}^2(E) := \tfrac{1}{2} \min_{p_o \in S^n} \int_{\partial E} \big|\nu_E(x) - \nu_{B_{\vartheta(x)}(p_o)}(x)\big|^2 \, d\mathcal{H}^{n-1}(x)\,, \tag{1.4}$$

where $\nu_E(x)$ is the outer unit normal to E at the point $x \in \partial E$ (contained in the tangent plane to S^n at x, i.e. $\nu_E(x) \cdot x = 0$) and $\nu_{B_{\vartheta(x)}(p_o)}(x)$ is the outer unit normal to the geodesic ball $B_{\vartheta(x)}(p_o)$ centered at p_o whose boundary passes through x. Note, that the Fraenkel asymmetry $\boldsymbol{\alpha}(E)$ measures the L^1-distance between the set E and the optimal geodesic ball of the same volume, while the oscillation index $\boldsymbol{\beta}(E)$ is a measure for the distance between the distributional derivatives of χ_E and of χ_{B_ϑ}, where B_ϑ is an optimal geodesic ball in (1.4) of the same volume as E. Alternatively, the oscillation index can be viewed as an excess functional between ∂E and ∂B_ϑ. The two indices are related by a Poincaré-type inequality (cf. Lemma 2.7), stating that

$$\boldsymbol{\beta}^2(E) \geq c(n)\mathbf{P}(B_\vartheta)\boldsymbol{\alpha}^2(E). \tag{1.5}$$

The main result of the present paper can now be formulated as follows:

Theorem 1.1. *There exists a constant $c(n)$ such that for any set $E \subset S^n$ of finite perimeter with volume $|E| = |B_\vartheta|$ for some $\vartheta \in (0, \pi)$, the following inequality holds*

$$\mathbf{D}(E) := \mathbf{P}(E) - \mathbf{P}(B_\vartheta) \geq c(n)\boldsymbol{\beta}^2(E). \tag{1.6}$$

We mention that (1.6) is the counterpart for the sphere of a similar inequality proved in [15], where a suitable definition of oscillation index in the euclidean case was introduced for the first time. Note that as in the euclidean case by combining (1.6) with (1.5) immediately yields the stability inequality (1.2). On the other hand it is clear that (1.6) is stronger than (1.2), since by Lemma 2.7

$$\mathbf{P}(E) - \mathbf{P}(B_\vartheta) \leq \boldsymbol{\beta}^2(E).$$

As in [8, 15, 5] the starting point for the proof of Theorem 1.1 is a Fuglede-type stability result aimed to establish (1.6) in the special case of sets $E \subset S^n$ whose boundary can be written as a radial graph over the boundary of a ball $B_\vartheta(p_o)$ with the same volume. To establish such a result one could follow in principle the strategy used in the Euclidean case [14]. However, to deduce (1.6) for radial graphs with a constant not depending on the volume needs much more care in the estimations (cf. proof of Theorem 3.1). The main difficulty arises when passing from the special situation of radial graphs to arbitrary sets. To deal with this issue we need to change significantly the strategies developed in [8, 15, 5].

To explain where the major difficulties come from, we observe that the oscillation index can be re-written (see (2.5)) in the form

$$\boldsymbol{\beta}^2(E) = \mathbf{P}(E) - (n-1) \max_{p_o \in S^n} \int_E \frac{x \cdot p_o}{\sqrt{1 - (x \cdot p_o)^2}} \, d\mathcal{H}^{n-1}\,.$$

From this formula it is clear that the core of the proof is to provide estimates for the singular integral

$$\int_E \frac{x \cdot p_o}{\sqrt{1 - (x \cdot p_o)^2}} \, d\mathcal{H}^{n-1} \tag{1.7}$$

and its maximum with respect to p_o, independent of the volume of E. This requires new technically involved ideas and strategies; cf. the proofs of the Continuity Lemma 2.6, the Slicing Lemma 4.1 and Subsections 6.4 and 6.5 from the proof of the Main Theorem 1.1. In fact, in the contradiction argument used to deduce (1.6) for general sets from the case of a radial graph we need to show that all the constants are independent of the volume of E. The arguments become particularly delicate when the volume of E is small. In this case inequality (1.6) shows a completely different nature depending on the size of the ratio $\boldsymbol{\beta}^2(E)/\mathbf{P}(B_\vartheta)$. In fact, if $|E| \to 0$ and also $\boldsymbol{\beta}^2(E)/\mathbf{P}(B_\vartheta) \to 0$, then E behaves asymptotically like a flat set, i.e. a set in $\mathbb{R}^n$ and inequality (1.6) can be proven by reducing to the euclidean case, rescaling and then arguing as when E has large volume. However, the most difficult situation to deal with is when $|E| \to 0$ and $\boldsymbol{\beta}^2(E)/\mathbf{P}(B_\vartheta) \to \eta_o > 0$. This case has to be treated with ad hoc estimates for the singular integral (1.7).

2. Preliminaries

2.1. **Geodesic balls and spheres in S^n.** For a fixed point $p_o \in S^n$ the geodesic distance of a point $x \in S^n$ to p_o is given by

$$\mathrm{dist}_{S^n}(x, p_o) = \arccos(x \cdot p_o).$$

Here $x \cdot p_o$ denotes the usual Euclidean scalar product of the ambient space $\mathbb{R}^{n+1}$. In case that in the context the center p_o is fixed we use the short hand notation $\vartheta(x) \equiv \mathrm{dist}_{S^n}(x, p_o)$. The open geodesic ball $B_{\vartheta_o}(p_o) := \{x \in S^n : \mathrm{dist}_{S^n}(x, p_o) < \vartheta_o\}$ of radius $\vartheta_o \in (0, \pi)$ centered at p_o can be parametrized by

$$B_{\vartheta_o}(p_o) \equiv \big\{\omega \sin\vartheta + p_o \cos\vartheta \colon \vartheta \in [0, \vartheta_o),\ |\omega| = 1,\ \omega \perp p_o\big\}.$$

The boundary of the geodesic ball $B_{\vartheta_o}(p_o)$, i.e. the geodesic sphere $S_{\vartheta_o}(p_o) := \{x \in S^n : \mathrm{dist}_{S^n}(x, p_o) = \vartheta_o\}$ centered at p_o with geodesic radius ϑ_o, is then parametrized by

$$S_{\vartheta_o}(p_o) \equiv \big\{\omega \sin\vartheta_o + p_o \cos\vartheta_o \colon |\omega| = 1,\ \omega \perp p_o\big\}.$$

In case that the center p_o is the north pole e_{n+1} we simply write $B_{\vartheta_o} := B_{\vartheta_o}(e_{n+1})$ and $S_{\vartheta_o} := S_{\vartheta_o}(e_{n+1})$. The volume of geodesic balls and the area of geodesic spheres are given by

$$\mathcal{H}^{n-1}(S_{\vartheta_o}(p_o)) = n\omega_n \sin^{n-1}\vartheta_o \quad \text{and} \quad |B_{\vartheta_o}(p_o)| = n\omega_n \int_0^{\vartheta_o} \sin^{n-1}\sigma\, d\sigma.$$

Here and in the following we denote by $|E|$ the $\mathcal{H}^n$-measure of a subset E of $\mathbb{R}^{n+1}$. In the sequel we shall need several times the following simple lemma.

Lemma 2.1. *For $p_o \in S^n$ and $0 \le \vartheta_1 < \vartheta_2 \le \frac{\pi}{2}$ we have*

$$\tfrac{n\omega_n}{2}\big(\tfrac{\vartheta_2}{\pi}\big)^{n-1}(\vartheta_2 - \vartheta_1) \le |B_{\vartheta_2}(p_o) \setminus B_{\vartheta_1}(p_o)| \le n\omega_n \vartheta_2^{n-1}(\vartheta_2 - \vartheta_1)$$

and

$$\mathcal{H}^{n-1}(S_{\vartheta_2}(p_o)) - \mathcal{H}^{n-1}(S_{\vartheta_1}(p_o)) \le n(n-1)\omega_n \vartheta_2^{n-2}(\vartheta_2 - \vartheta_1)$$

Proof. The upper bound in the first inequality easily follows, since

$$|B_{\vartheta_2}(p_o) \setminus B_{\vartheta_1}(p_o)| = n\omega_n \int_{\vartheta_1}^{\vartheta_2} \sin^{n-1}\sigma\, d\sigma \le n\omega_n \vartheta_2^{n-1}(\vartheta_2 - \vartheta_1).$$

Similarly, we get the lower bound

$$\begin{aligned} |B_{\vartheta_2}(p_o) \setminus B_{\vartheta_1}(p_o)| &\ge n\omega_n \int_{(\vartheta_1+\vartheta_2)/2}^{\vartheta_2} \sin^{n-1}\sigma\, d\sigma \\ &\ge \tfrac{n\omega_n}{2} \sin^{n-1}\big(\tfrac{\vartheta_2}{2}\big)(\vartheta_2 - \vartheta_1) \ge \tfrac{n\omega_n}{2}\big(\tfrac{\vartheta_2}{\pi}\big)^{n-1}(\vartheta_2 - \vartheta_1). \end{aligned}$$

Finally, the second inequality is obtained as follows:

$$\begin{aligned} \mathcal{H}^{n-1}(S_{\vartheta_2}(p_o)) - \mathcal{H}^{n-1}(S_{\vartheta_1}(p_o)) &= n(n-1)\omega_n \int_{\vartheta_1}^{\vartheta_2} \sin^{n-2}\sigma \cos\sigma\, d\sigma \\ &\le n(n-1)\omega_n \vartheta_2^{n-2}(\vartheta_2 - \vartheta_1). \end{aligned}$$

This finishes the proof of the lemma. $\square$

By $\nu_{B_{\vartheta_o}(p_o)}$ we denote the outer unit normal vector-field of $B_{\vartheta_o}(p_o)$ along its boundary sphere $S_{\vartheta_o}(p_o)$. At a point $x \in S_{\vartheta_o}(p_o)$ this vector-field is given by

$$\nu_{B_{\vartheta_o}(p_o)}(x) = \frac{x - (x \cdot p_o)p_o}{\sqrt{1-(x\cdot p_o)^2}} \cos\vartheta_o - p_o \sin\vartheta_o = \frac{(x\cdot p_o)x - p_o}{\sqrt{1-(x\cdot p_o)^2}}.$$

For a fixed geodesic sphere $S_{\vartheta_o}(p_o)$ the nearest point retraction onto $S_{\vartheta_o}(p_o)$ is defined on $S^n \setminus \{p_o, -p_o\}$. It can be computed explicitly by

$$\pi_{B_{\vartheta_o}(p_o)}(x) = \frac{x - (x \cdot p_o)p_o}{\sqrt{1-(x\cdot p_o)^2}} \sin\vartheta_o + p_o \cos\vartheta_o \quad \text{for } x \in S^n \setminus \{p_o, -p_o\}.$$

Finally, for $x \in S^n$ and a fixed center $p_o \in S^n$ we recall that $\vartheta(x) \equiv \operatorname{dist}_{S^n}(x, p_o)$ and therefore $\nu_{B_{\vartheta(x)}(p_o)}(x)$ is the outer unit normal of the boundary of the geodesic ball $B_{\vartheta(x)}(p_o)$ with $x \in S_{\vartheta(x)}(p_o)$.

2.2. **Sets of finite perimeter on S^n.** Throughout the paper we shall freely use concepts and results for sets of finite perimeter in the sphere whenever their adaptation from the Euclidean setting to the sphere setting is straightforward and does not show any particular difficulties. Our notation follows [22, §37]. We consider integer multiplicity rectifiable $(n-1)$-currents $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+1})$ with $\operatorname{spt} T \subset S^n$ and such that

$$T = \partial[\![E]\!]$$

holds true, where E is an $\mathcal{H}^n$-measurable subset of S^n. For such currents T we have that if $\psi\colon \mathbb{R}^{n+1} \supset U \to W \subset \mathbb{R}^{n+1}$ is a C^2 diffeomorphism such that $\psi(U \cap S^n) = W \cap (\mathbb{R}^n \times \{0\}) =: G$, then $\psi(E \cap U)$ has locally finite perimeter in G. Moreover, any T as above satisfying $\mathbf{M}_U(T) = \mathbf{M}(\partial[\![E]\!] \llcorner U) < \infty$ whenever $U \Subset \mathbb{R}^{n+1}$, is automatically an integer multiplicity rectifiable current with $\Theta^n(T, x_o) = 1$ for μ_T-a.e. $x_o \in \mathbb{R}^{n+1}$. Sets E with the above property are termed sets of finite perimeter in S^n. Here μ_T stands for the total variation measure of $\partial[\![E]\!]$. Other abbreviations which are used in the literature are $\|\partial[\![E]\!]\|$ or $\|\partial\chi_E\|$. By the Riesz representation theorem we know that there exists a $\|\partial\chi_E\|$-measurable tangential vector-field $\nu\colon S^n \ni x \mapsto \nu(x) \in T_xS^n$ with $|\nu(x)| = 1$ for $\|\partial\chi_E\|$ almost every x, such that

$$\int_E \operatorname{div}_{S^n} g \, d\mathcal{H}^n = \int_{S^n} g \cdot \nu \|\partial\chi_E\|$$

holds true whenever g is a smooth tangential vector-field on S^n. This allows us to define for sets $E \subset S^n$ of finite perimeter the reduced boundary $\partial^* E$ as the set of those points $x_o \in S^n$ for which the limit

$$\nu_E(x_o) := \lim_{\varrho \downarrow 0} \frac{\int_{B_\varrho(x_o)} \nu d\|\partial\chi_E\|}{\|\partial\chi_E\|(B_\varrho(x_o))}$$

exists and satisfies $|\nu_E(x_o)| = 1$ and $\nu_E(x_o) \in T_{x_o}S^n$. The De Giorgi structure theorem on the sphere then states that $\partial^* E$ is countably $(n-1)$-rectifiable and that the total variation measure $\|\partial\chi_E\|$ is supported on $\partial^* E$, that is $\|\partial\chi_E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$. Moreover, the Gauss-Green theorem holds true in the following form:

$$\int_E \operatorname{div}_{S^n} g \, d\mathcal{H}^n = \int_{\partial^* E} g \cdot \nu_E d\mathcal{H}^{n-1} \tag{2.1}$$

whenever g is a smooth tangential vector-field on S^n (cf. [22, p. 46 (7.6)]).

2.3. **Isoperimetric inequalities on the sphere.** The isoperimetric property of geodesic balls goes back to Schmidt [21] and ensures that for any set $E \subset S^n$ of finite perimeter with $|E| = |B_{\vartheta_o}|$ and $0 < \vartheta_o < \pi$ there holds

$$\mathbf{P}(B_{\vartheta_o}) \le \mathbf{P}(E). \tag{2.2}$$

Equality occurs in (2.2) if and only if E is a geodesic ball with radius ϑ_o. Moreover, for any set E of finite perimeter with $\mathbf{P}(E) = \mathbf{P}(B_{\vartheta_o})$ for some $0 < \vartheta_o < \pi$ there holds

$$|E| \le |B_{\vartheta_o}| \qquad \text{or} \qquad |S^n \setminus E| \le |B_{\vartheta_o}|. \tag{2.3}$$

Also in this case equality occurs if and only if E is a geodesic ball.

From (2.2) we can conclude two kinds of isoperimetric inequalities. The first one states that for any set $E \subset S^n$ of finite perimeter with $|E| \le |B_{\vartheta_o}|$ and $0 < \vartheta_o \le \frac{\pi}{2}$ there holds

$$|E| \le \frac{|B_{\vartheta_o}|}{\mathbf{P}(B_{\vartheta_o})^{\frac{n}{n-1}}} \mathbf{P}(E)^{\frac{n}{n-1}}.$$

We note that the mapping

$$(0, \pi) \ni \vartheta \mapsto c_n(\vartheta) := \frac{|B_\vartheta|}{\mathbf{P}(B_\vartheta)^{\frac{n}{n-1}}}$$

is non-decreasing. For more details we refer to [11, 2.5]. The second isoperimetric inequality following from (2.2) is a linear one. This inequality states that for any set $E \subset S^n$ of finite perimeter with volume $0 < |E| \le |B_{\vartheta_o}| < |S^n|$ there holds

$$|E| \le \frac{|B_{\vartheta_o}|}{\mathbf{P}(B_{\vartheta_o})} \mathbf{P}(E). \tag{2.4}$$

We note that the function

$$(0, \pi) \ni \vartheta \mapsto Q_n(\vartheta) := \frac{|B_\vartheta|}{\mathbf{P}(B_\vartheta)}$$

is non-decreasing with $Q_n(\vartheta) \downarrow 0$ as $\vartheta \downarrow 0$ and $Q_n(\vartheta) \uparrow \infty$ as $\vartheta \uparrow \pi$. Finally, we state the relative isoperimetric inequality on the sphere. The proof goes as in the Euclidean case.

Lemma 2.2 (Relative isoperimetric inequality)**.** *There exists a constant $c = c(n)$ such that for any $p_o \in S^n$ and $r \in (0, \frac{\pi}{2}]$ and any set $G \subset S^n$ of finite perimeter there holds*

$$\min\big\{|B_r(p_o) \cap G|, |B_r(p_o) \setminus G|\big\}^{\frac{n-1}{n}} \le c\, \mathbf{P}(G; B_r(p_o)).$$

Here, $\mathbf{P}(G; B_r(p_o))$ denotes the perimeter of G in $B_r(p_o)$.

2.4. **The oscillation index.** In this section we define for a set $E \subset S^n$ of finite perimeter with volume $|E| \in (0, |S^n|)$ its L^2-oscillation index of the unit outer normal relative to a geodesic ball $B_{\vartheta_o}(p_o)$ of the same volume, i.e. $|E| = |B_{\vartheta_o}(p_o)|$, by the following construction. Since we can not compare the unit outer normal $\nu_E(x)$ of the set E in a point $x \in \partial^* E$, which is contained in the tangent space $T_x S^n$, directly with the unit outer normal $\nu_{B_{\vartheta_o}(p_o)}(\pi(x))$ of the geodesic ball of the same volume at the nearest point $\pi(x)$, which is contained in $T_{\pi(x)} S^n$, we need to use the parallel transport along geodesics in S^n, which are given by great circles. Let $P_{x,y} \colon T_x S^n \to T_y S^n$ denote the parallel transport of tangent vectors. Then $P_{x,\pi(x)} \nu_E(x)$ is tangent to S^n at $\pi(x)$. Since the parallel transport is an isometry between the tangent spaces we have

$$\begin{aligned} |P_{x,\pi(x)} \nu_E(x) - \nu_{B_{\vartheta_o}(p_o)}(\pi(x))| &= |\nu_E(x) - P_{\pi(x),x} \nu_{B_{\vartheta_o}(p_o)}(\pi(x))| \\ &= |\nu_E(x) - \nu_{B_{\vartheta(x)}(p_o)}(x)|, \end{aligned}$$

where we used the short-hand notation $\vartheta(x) = \arccos(x \cdot p_o)$ from Section 2.1. The L^2-oscillation index of E relative to $B_{\vartheta_o}(p_o)$ is then defined by

$$\boldsymbol{\beta}(E; p_o) := \left[\tfrac{1}{2} \int_{\partial^* E} \big|\nu_E(x) - \nu_{B_{\vartheta(x)}(p_o)}(x)\big|^2 \, d\mathcal{H}^{n-1}(x)\right]^{\frac{1}{2}}.$$

Here $\partial^* E$ denotes again the reduced boundary of E and ν_E its outer unit normal. Taking the infimum with respect to all geodesic balls $B_{\vartheta_o}(p_o)$ we finally define the L^2**-oscillation index** of E by

$$\boldsymbol{\beta}(E) := \min_{p_o \in S^n} \boldsymbol{\beta}(E; p_o).$$

To derive an alternative – and sometimes more useful – expression for the oscillation index we firstly re-write the oscillation index as follows:

$$\begin{aligned}
\boldsymbol{\beta}^2(E;p_o) &= \int_{\partial^* E} 1 - \nu_E(x) \cdot \nu_{B_{\vartheta(x)}(p_o)}(x) \, d\mathcal{H}^{n-1}(x) \\
&= \mathbf{P}(E) - \int_{\partial^* E} \nu_E(x) \cdot \nu_{B_{\vartheta(x)}(p_o)}(x) \, d\mathcal{H}^{n-1}(x).
\end{aligned}$$

To proceed further, we recall that

$$Y(x) := \nu_{B_{\vartheta(x)}(p_o)}(x) = \frac{(x \cdot p_o)x - p_o}{\sqrt{1-(x\cdot p_o)^2}}.$$

Note that $\nu_{B_{\vartheta(x)}(p_o)}(x) \cdot x = 0$. By the Gauss-Green theorem (2.1) for sets of finite perimeter on submanifolds we obtain

$$\int_{\partial^* E} \nu_E \cdot Y \, d\mathcal{H}^{n-1} = \int_E \operatorname{div}_{S^n} Y \, d\mathcal{H}^n.$$

Next, we compute the tangential divergence $\operatorname{div}_{S^n} Y$ of the vector-field Y. We obtain

$$\begin{aligned}
\operatorname{div}_{S^n} Y &= \sum_{i=1}^n \tau_i \cdot D_{\tau_i} Y \\
&= \sum_{i=1}^n \tau_i \cdot \left[\frac{(\tau_i \cdot p_o)x + (x \cdot p_o)\tau_i}{\sqrt{1-(x\cdot p_o)^2}} + \big((x\cdot p_o)x - p_o\big) \frac{(x\cdot p_o)(\tau_i \cdot p_o)}{\sqrt{1-(x\cdot p_o)^2}^3} \right] \\
&= \sum_{i=1}^n \left[\frac{x \cdot p_o}{\sqrt{1-(x\cdot p_o)^2}} - \frac{(x\cdot p_o)(\tau_i \cdot p_o)^2}{\sqrt{1-(x\cdot p_o)^2}^3} \right] \\
&= \frac{(n-1)(x\cdot p_o)}{\sqrt{1-(x\cdot p_o)^2}}.
\end{aligned}$$

Inserting this above, we finally arrive at

$$\boldsymbol{\beta}^2(E;p_o) = \mathbf{P}(E) - \boldsymbol{\gamma}(E,p_o),$$

where

$$\boldsymbol{\gamma}(E;p_o) := (n-1) \int_E \frac{x \cdot p_o}{\sqrt{1-(x\cdot p_o)^2}} \, d\mathcal{H}^n.$$

Taking the minimum over all points $p_o \in S^n$ we infer that

$$\boldsymbol{\beta}^2(E) = \mathbf{P}(E) - \boldsymbol{\gamma}(E), \tag{2.5}$$

where we have abbreviated

$$\boldsymbol{\gamma}(E) := \max_{p_o \in S^n} \boldsymbol{\gamma}(E;p_o). \tag{2.6}$$

We note that in the case of a geodesic ball $E \equiv B_{\vartheta_o}(p_o)$ on the sphere we have

$$\boldsymbol{\gamma}(B_{\vartheta_o}(p_o)) = \mathbf{P}(B_{\vartheta_o}(p_o)). \tag{2.7}$$

Definition 2.3. A set of finite perimeter $E \subset S^n$ is centered at a point $p_o \in S^n$ if

$$\boldsymbol{\beta}^2(E) = \boldsymbol{\beta}^2(E;p_o).$$

We note that in general the center of a set is not uniquely defined, since there may be several of those points. In points p_o where the set E is centered, the function $p_o \mapsto \boldsymbol{\gamma}(E; p_o)$ takes its maximum and hence $\boldsymbol{\gamma}(E) = \boldsymbol{\gamma}(E; p_o)$. We continue with two simple auxiliary lemmas.

Lemma 2.4. *If $E \subset B_{\vartheta_o}$ with $\vartheta_o \in (0, \frac{\pi}{2}]$ then every center p_o of E is contained in $B_{2\vartheta_o}$.*

Proof. Assumed that $p_o \notin B_{2\vartheta_o}$, then for all $x \in E$ we have $\operatorname{dist}_{S^n}(x, p_o) > \operatorname{dist}_{S^n}(x, e_{n+1})$. Hence $x \cdot p_o < x \cdot e_{n+1}$ and thus $\gamma(E; p_o) < \gamma(E; e_{n+1})$, a contradiction to the maximality of the center. $\square$

Lemma 2.5. *For any set $G \subset S^n$ of finite perimeter and any $p_o \in S^n$ we have*

$$\int_G \frac{|x \cdot p_o|}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n(x) \le c(n)\, |G|^{\frac{n-1}{n}}$$

Proof. We choose a radius ϑ such that $|B_\vartheta(p_o)| = |G|$. Using the fact that $|x \cdot p_o| / \sqrt{1 - (x \cdot p_o)^2}$ becomes larger when x gets closer to either p_o or $-p_o$ and recalling (2.7) we find that

$$\begin{aligned}\int_G \frac{|x \cdot p_o|}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n(x) &\le 2 \int_{B_\vartheta(p_o)} \frac{|x \cdot p_o|}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n = \tfrac{2}{n-1}\, \mathbf{P}(B_\vartheta(p_o)) \\ &\le c(n)\, |B_\vartheta(p_o)|^{\frac{n-1}{n}} = c(n)\, |G|^{\frac{n-1}{n}}.\end{aligned}$$

This proves the claimed estimate. $\square$

The next lemma shows that the centers depend continuously on the set under consideration with respect to L^1-convergence.

Lemma 2.6 (Continuity of centers with respect to the L^1-topology). *For every $\varepsilon > 0$ there exists $\delta(n, \varepsilon) > 0$ such that there holds: If $G \subset S^n$ is a set of finite perimeter satisfying $|G \Delta B_{\vartheta_o}(p_o)| < \delta \vartheta_o^n$, with $\vartheta_o \in (0, \frac{\pi}{2}]$, then $\operatorname{dist}_{S^n}(y_G, p_o) < \varepsilon \vartheta_o$ holds true for any center y_G of G.*

Proof. We argue by contradiction assuming that there exist $\varepsilon \in (0, 1)$ and sets $G_k \subset S^n$ of finite perimeter and radii $\vartheta_k \in (0, \frac{\pi}{2}]$, $k \in \mathbb{N}$, such that

$$\frac{|G_k \Delta B_{\vartheta_k}(p_o)|}{\vartheta_k^n} \to 0 \quad \text{as } k \to \infty, \qquad \text{and} \qquad \operatorname{dist}_{S^n}(y_k, p_o) \ge 2\varepsilon \vartheta_k \text{ for all } k, \tag{2.8}$$

where y_k is a center of G_k for $k \in \mathbb{N}$. Without loss of generality we can also assume that $\vartheta_k \to \vartheta_o \in [0, \frac{\pi}{2}]$ and $y_k \to y_o$. Since y_k is a center for G_k, by maximality of $\boldsymbol{\gamma}(G_k; \cdot)$ we have that

$$\int_{G_k} \frac{x \cdot p_o}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n \le \int_{G_k} \frac{x \cdot y_k}{\sqrt{1 - (x \cdot y_k)^2}}\, d\mathcal{H}^n. \tag{2.9}$$

We claim that

$$\lim_{k \to \infty} \frac{1}{\vartheta_k^{n-1}} \left[\int_{G_k} \frac{x \cdot p_o}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n - \int_{B_{\vartheta_k}(p_o)} \frac{x \cdot p_o}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n \right] = 0. \tag{2.10}$$

By I_k we denote the quantity in the limit in (2.10). Using Lemma 2.5 we get

$$\begin{aligned}|\mathrm{I}_k| &\le \frac{1}{\vartheta_k^{n-1}} \int_{G_k \Delta B_{\vartheta_k}(p_o)} \frac{|x \cdot p_o|}{\sqrt{1 - (x \cdot p_o)^2}}\, d\mathcal{H}^n \\ &\le \frac{c(n)}{\vartheta_k^{n-1}}\, |G_k \Delta B_{\vartheta_k}(p_o)|^{\frac{n-1}{n}} = c(n) \left(\frac{|G_k \Delta B_{\vartheta_k}(p_o)|}{\vartheta_k^n} \right)^{\frac{n-1}{n}}.\end{aligned}$$

Together with the first convergence in (2.8) this proves the claim (2.10).

Next we consider the right-hand side of (2.9). Similarly to (2.10) from above, we now claim that

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n-1}}\bigg[\int_{G_k}\frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\,d\mathcal{H}^n-\int_{B_{\vartheta_k}(p_o)}\frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\,d\mathcal{H}^n\bigg]=0. \tag{2.11}$$

To this aim we choose rotations $T_k\in S\mathbb{O}(n)$ which take the great circle passing through p_o and y_k into itself and such that $T_k y_k=p_o$. Again, passing to a subsequence we may assume that $T_k\to T\in S\mathbb{O}(n)$. By a change of variable we rewrite the quantity in the square brackets as

$$\int_{T_k(G_k)}\frac{x\cdot p_o}{\sqrt{1-(x\cdot p_o)^2}}\,d\mathcal{H}^n-\int_{T_k(B_{\vartheta_k}(p_o))}\frac{x\cdot p_o}{\sqrt{1-(x\cdot p_o)^2}}\,d\mathcal{H}^n.$$

Arguing almost exactly as in the proof of (2.10) we obtain (2.11).

Thus, putting together (2.9), (2.10) and (2.11), we conclude that

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{\vartheta_k^{n-1}}\int_{B_{\vartheta_k}(p_o)}\frac{x\cdot p_o}{\sqrt{1-(x\cdot p_o)^2}}\,d\mathcal{H}^n\\ &\qquad\le\lim_{k\to\infty}\frac{1}{\vartheta_k^{n-1}}\int_{B_{\vartheta_k}(p_o)}\frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\,d\mathcal{H}^n,\end{aligned} \tag{2.12}$$

holds true. In this step we possibly have to pass to a subsequence such that the limits on both sides exist. We now distinguish between the cases $\vartheta_o>0$ and $\vartheta_o=0$.

The case $\boldsymbol{\vartheta_o>0}$. In this case we get from (2.12) the inequality

$$\int_{B_{\vartheta_o}(p_o)}\frac{x\cdot p_o}{\sqrt{1-(x\cdot p_o)^2}}\,d\mathcal{H}^n\le\int_{B_{\vartheta_o}(p_o)}\frac{x\cdot y_o}{\sqrt{1-(x\cdot y_o)^2}}\,d\mathcal{H}^n.$$

Applying the Gauß-Green theorem to both sides of the previous inequality we infer that

$$\int_{\partial B_{\vartheta_o}(p_o)}1\,d\mathcal{H}^{n-1}\le\int_{\partial B_{\vartheta_o}(p_o)}\nu_{B_{\vartheta_o}(p_o)}(x)\cdot\nu_{B_{\vartheta(x)}(y_o)}(x)\,d\mathcal{H}^{n-1}(x).$$

But this inequality can only hold if $y_o=p_o$, which gives the desired contradiction.

The Case $\boldsymbol{\vartheta_o=0}$. To simplify the notation we assume without loss of generality that $p_o=e_{n+1}$ and set $w_k:=T_k(e_{n+1})$. We write $w_k=\omega_k\sin\psi_k+e_{n+1}\cos\psi_k$, for some $\omega_k\in S^{n-1}$ and some angle $\psi_k>0$. From the second equation in (2.8) we conclude that also $\mathrm{dist}_{S^n}(w_k,e_{n+1})\ge 2\varepsilon\vartheta_k$, hence

$$\psi_k\ge 2\varepsilon\vartheta_k\qquad\text{for all } k\in\mathbb{N}.$$

Then, by performing a rotation around the $\mathbb{R}e_{n+1}$ axis, we may also assume that all the w_k lie on the same great circle through the north pole e_{n+1}, which means that there exist a unique $\omega_o\in S^{n-1}$ such that there holds:

$$w_k=\omega_o\sin\psi_k+e_{n+1}\cos\psi_k\qquad\text{for all } k\in\mathbb{N}.$$

Finally, observe that by a change of variable, we may rewrite

$$\begin{aligned}\int_{B_{\vartheta_k}(e_{n+1})}\frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\,d\mathcal{H}^n&=\int_{T_k(B_{\vartheta_k}(e_{n+1}))}\frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\,d\mathcal{H}^n\\ &=\int_{B_{\vartheta_k}(w_k)}\frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\,d\mathcal{H}^n.\end{aligned}$$

We now claim that

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n-1}}\bigg[\int_{B_{\vartheta_k}}\frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\,d\mathcal{H}^n-\int_{B_{\vartheta_k}(w_k)}\frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\,d\mathcal{H}^n\bigg]>0, \tag{2.13}$$

where we use the short hand notation B_{ϑ_k} instead of $B_{\vartheta_k}(e_{n+1})$. Together with (2.12) this will give the final contradiction.

Note now that $w_k \to e_{n+1}$ as $k \to \infty$. In fact, if this were not true there would exist $\psi_o \in (0, \frac{\pi}{2})$ such that $\psi_k > \psi_o$ for infinitely many k. But this is not possible because in this case we would have

$$\lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}} \frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n = \lim_{k\to\infty} \frac{\mathbf{P}(B_{\vartheta_k})}{(n-1)\vartheta_k^{n-1}} = \frac{n\omega_n}{n-1}$$

while, on the other hand, we would have also

$$\limsup_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}(w_k)} \frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \le \limsup_{k\to\infty} \frac{|B_{\vartheta_k}(w_k)|}{\vartheta_k^{n-1}} \cot(\psi_o - \vartheta_k) = 0,$$

thus contradicting at once (2.12).

Since the function $[0,1) \ni t \mapsto t/\sqrt{1-t^2}$ is increasing, the worst case in which to prove (2.13) is when the angle ψ_k is precisely equal to $\varepsilon\vartheta_k$. Therefore, from now on we assume that

$$w_k = \omega_o \sin(2\varepsilon\vartheta_k) + e_{n+1}\cos(2\varepsilon\vartheta_k) \qquad \text{for all } k \in \mathbb{N}.$$

In order to prove (2.13), we first observe that

$$\lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}(w_k)} \frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n = \lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}(w_k)} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n$$

holds true and moreover, that a similar identity is valid for the first integral in (2.13). To prove the equality above, it is enough to observe that since $w_k \to e_{n+1}$ as $k \to \infty$ we have

$$0 < \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}(w_k)} \frac{1-x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n = \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k}(w_k)} \frac{\sqrt{1-x_{n+1}}}{\sqrt{1+x_{n+1}}}\, d\mathcal{H}^n \le c\,\vartheta_k,$$

for some positive constant c independent of k. In view of this observation, the claim (2.13) reduces to

$$\lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \left[\int_{B_{\vartheta_k}} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n - \int_{B_{\vartheta_k}(w_k)} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \right] > 0,$$

which in turn can be re-written in the form

$$\lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \left[\int_{B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k)} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n - \int_{B_{\vartheta_k}(w_k) \setminus B_{\vartheta_k}} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \right] > 0.$$

Writing the generic point in S^n in the form $x = \omega \sin\vartheta + e_{n+1}\cos\vartheta$ with $\omega \in S^{n-1}$ and $\vartheta \in [0,\pi]$, we observe that if $x \in B_{\vartheta_k}(w_k) \setminus B_{\vartheta_k}$ then $\sqrt{1-x_{n+1}^2} = \sin\vartheta > \sin\vartheta_k$. Therefore, since $|B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k)| = |B_{\vartheta_k}(w_k) \setminus B_{\vartheta_k}|$, we immediately have that

$$\int_{B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k)} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n - \int_{B_{\vartheta_k}(w_k) \setminus B_{\vartheta_k}} \frac{1}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \ge \int_{B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k)} \Big(\frac{1}{\sin\vartheta} - \frac{1}{\sin\vartheta_k} \Big)\, d\mathcal{H}^n.$$

Thus our claim will be proved if we establish that there holds:

$$\lim_{k\to\infty} \frac{1}{\vartheta_k^{n-1}} \int_{B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k)} \Big(\frac{1}{\sin\vartheta} - \frac{1}{\sin\vartheta_k} \Big)\, d\mathcal{H}^n > 0. \tag{2.14}$$

In order to prove (2.14) we set $S_\varepsilon = \{\omega \in S^{n-1} : \omega \cdot \omega_o \le -1+\varepsilon\}$. Next we show that the following implication holds true:

$$\omega \in S_\varepsilon,\ \vartheta \in (\vartheta_k(1-\varepsilon), \vartheta_k) \quad \Rightarrow \quad x = \omega \sin\vartheta + e_{n+1}\cos\vartheta \in B_{\vartheta_k} \setminus B_{\vartheta_k}(w_k) \tag{2.15}$$

In fact, since $\vartheta < \vartheta_k$ we clearly have $x \in B_{\vartheta_k}$. To prove that $x \notin B_{\vartheta_k}(w_k)$ we have to show that $x \cdot w_k \le \cos\vartheta_k$. To this aim, observe that for k large enough we have

$$\begin{aligned} x \cdot w_k &= \omega \cdot \omega_o \sin\vartheta \sin(2\varepsilon\vartheta_k) + \cos\vartheta\cos(2\varepsilon\vartheta_k) \\ &\le (-1+\varepsilon)\sin[\vartheta_k(1-\varepsilon)]\sin(2\varepsilon\vartheta_k) + \cos[\vartheta_k(1-\varepsilon)]\cos(2\varepsilon\vartheta_k) < \cos\vartheta_k. \end{aligned}$$

Here, the last inequality follows by computing

$$\lim_{k\to\infty} \frac{\cos[\vartheta_k(1-\varepsilon)]\cos(2\varepsilon\vartheta_k) - \cos\vartheta_k}{(1-\varepsilon)\sin[\vartheta_k(1-\varepsilon)]\sin(2\varepsilon\vartheta_k)} = \frac{2-5\varepsilon}{4(1-\varepsilon)^2} < 1,$$

since $\varepsilon \in (0,1)$. Therefore, in order to prove (2.14) we are left with establishing that

$$\lim_{k\to\infty} \frac{\mathcal{H}^{n-1}(S_\varepsilon)}{\vartheta_k^{n-1}} \int_{\vartheta_k(1-\varepsilon)}^{\vartheta_k} \Big(\frac{1}{\sin\sigma} - \frac{1}{\sin\vartheta_k}\Big) \sin^{n-1}\sigma\, d\sigma > 0.$$

An elementary calculation shows that the above limit is equal to $\mathcal{H}^{n-1}(S_\varepsilon)$ multiplied by the quantity

$$\frac{1-(1-\varepsilon)^{n-1}}{n-1} - \frac{1-(1-\varepsilon)^n}{n} = \tfrac12\varepsilon^2 + o(\varepsilon^2) > 0,$$

provided ε is small enough. This concludes the proof of the lemma. □

Lemma 2.7. *There exists a constant $c = c(n) > 0$ such that for any set $E \subset S^n$ of finite perimeter with $|E| = |B_{\vartheta_o}|$ for some $\vartheta_o \in (0, \frac{\pi}{2}]$ there holds*

$$\boldsymbol{\beta}(E)^2 \ge \mathbf{D}(E) + c(n)\vartheta_o^{n-1}\boldsymbol{\alpha}(E)^2,$$

where $\mathbf{D}(E)$ is defined in (1.6) and $\boldsymbol{\alpha}(E)$ in (1.3).

Proof. Let $E \subset S^n$ be a set of finite perimeter which is centered at $p_o \in S^n$. Without loss of generality we may assume that p_o is the north pole e_{n+1}. We use $\mathbf{P}(B_{\vartheta_o}) = \boldsymbol{\gamma}(B_{\vartheta_o})$ in order to rewrite (2.5) in the form

$$\boldsymbol{\beta}(E)^2 = \mathbf{P}(E) - \boldsymbol{\gamma}(E) = \mathbf{D}(E) + \boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E). \tag{2.16}$$

In the following we consider the difference $\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E)$. By the definition of $\boldsymbol{\gamma}$ in (2.6) and $x \cdot p_o = x_{n+1}$ we find

$$\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E) = \int_{B_{\vartheta_o}\setminus E} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n - \int_{E\setminus B_{\vartheta_o}} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n.$$

Since $|E| = |B_{\vartheta_o}|$, we have

$$|B_{\vartheta_o} \setminus E| = |E \setminus B_{\vartheta_o}| =: a.$$

Next, we choose radii $0 \le r < \vartheta_o < R \le \pi$ such that $|B_R \setminus B_{\vartheta_o}| = a = |B_{\vartheta_o} \setminus B_r|$; this means that we have

$$\int_{\vartheta_o}^R \sin^{n-1}\sigma\, d\sigma = \frac{a}{n\omega_n} = \int_r^{\vartheta_o} \sin^{n-1}\sigma\, d\sigma.$$

Since $x_{n+1} \in [-1,1]$ decreases if the distance of x from e_{n+1} increases, we can use the following argument to estimate the first integral of the right-hand side in the identity for $\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E)$ as follows: Since by the choice of r we have $|B_{\vartheta_0} \setminus B_r| = |B_{\vartheta_o} \setminus E|$ and moreover since the function $(-1,1) \ni t \mapsto t/\sqrt{1-t^2}$ is strictly increasing we conclude

$$\int_{B_{\vartheta_o}\setminus E} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \ge \int_{B_{\vartheta_o}\setminus B_r} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n.$$

A similar argument yields

$$\int_{E\setminus B_{\vartheta_o}} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \le \int_{B_R\setminus B_{\vartheta_o}} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n.$$

We insert these inequalities in the above expression for $\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E)$ and arrive at

$$\begin{aligned}\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E) &\ge \int_{B_{\vartheta_o}\setminus B_r} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n - \int_{B_R\setminus B_{\vartheta_o}} \frac{(n-1)x_{n+1}}{\sqrt{1-x_{n+1}^2}}\, d\mathcal{H}^n \\ &= n(n-1)\omega_n\Big[\int_r^{\vartheta_o} \cos\sigma \sin^{n-2}\sigma\, d\sigma - \int_{\vartheta_o}^R \cos\sigma\sin^{n-2}\sigma\, d\sigma\Big] \\ &= n\omega_n\big[2\sin^{n-1}\vartheta_o - (\sin^{n-1}R + \sin^{n-1}r)\big].\end{aligned}$$

We now define the function

$$F(\vartheta) := n\omega_n \int_{\vartheta_o}^{\vartheta} \sin^{n-1}\sigma\, d\sigma \qquad \text{for } \vartheta\in(0,\pi).$$

Since $\vartheta_o = F^{-1}(0)$, $R = F^{-1}(a)$ and $r = F^{-1}(-a)$ we can rewrite the last inequality in the following form:

$$\begin{aligned}&\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E) \\ &\quad\ge n\omega_n\big[2\sin^{n-1}(F^{-1}(0)) - \big(\sin^{n-1}(F^{-1}(a)) + \sin^{n-1}(F^{-1}(-a))\big)\big].\end{aligned} \tag{2.17}$$

Since $\sin^{n-1}\circ F^{-1}$ is uniformly concave in the interval $(-|B_{\vartheta_o}|, |S^n| - |B_{\vartheta_o}|)$, for any $t, s\in[-a,a]$ we have that

$$\sin^{n-1}(F^{-1}(t)) + \sin^{n-1}(F^{-1}(s)) \le 2\sin^{n-1}\Big(F^{-1}\Big(\frac{t+s}{2}\Big)\Big) - c(n,\vartheta_o)|t-s|^2,$$

where

$$c(n,\vartheta_o) = \tfrac14 \inf_{t\in[-a,a]} (-\sin^{n-1}\circ F^{-1})''(t) = \frac{n-1}{4n^2\omega_n^2}\inf_{t\in[-a,a]} \frac{1}{\sin^{n+1}\circ F^{-1}(t)}.$$

To conclude the desired estimate we need to control $c(n,\vartheta_o)$ in terms of ϑ_o. This can easily be seen as follows: If $\frac{\pi}{4}\le\vartheta_o\le\frac{\pi}{2}$, we have $c(n,\vartheta_o)\ge\frac{n-1}{4n^2\omega_n^2}$, while, if $0<\vartheta_o<\frac{\pi}{4}$, we have $a\le|B_{\vartheta_o}|\le F(2\vartheta_o)$, hence $\sin^{n+1}\circ F^{-1}(t)\le \sin^{n+1}(2\vartheta_o)$ for all $t\in[-a,a]$. In any case, we have established that there exists a constant $\kappa(n)$ such that

$$c(n,\vartheta_o)\ge\frac{\kappa(n)}{\vartheta_o^{n+1}} \qquad \text{for all } \vartheta_o\in(0,\tfrac{\pi}{2}].$$

We use the preceding lower bound for the constant c in the inequality expressing the uniform concavity of $\sin^{n-1}\circ F^{-1}$ with $t=a$ and $s=-a$. The resulting inequality we afterwards insert into (2.17). In this way we obtain

$$\boldsymbol{\gamma}(B_{\vartheta_o}) - \boldsymbol{\gamma}(E) \ge \frac{c(n)}{\vartheta_o^{n+1}}(2a)^2 = \frac{c(n)}{\vartheta_o^{n+1}}\big(|B_{\vartheta_o}\setminus E| + |E\setminus B_{\vartheta_o}|\big)^2 = \frac{c(n)}{\vartheta_o^{n+1}}|E\Delta B_{\vartheta_o}|^2.$$

Inserting the preceding inequality in (2.16) we find

$$\boldsymbol{\beta}(E)^2 \ge \mathbf{D}(E) + \frac{\kappa(n)}{\vartheta_o^{n+1}}|E\Delta B_{\vartheta_o}|^2 \ge \mathbf{D}(E) + c(n)\vartheta_o^{n-1}\,\boldsymbol{\alpha}^2(E).$$

In the last line we used the assumption that E is centered at e_{n+1} and the definition of the Fraenkel asymmetry (note that $|E| = |B_{\vartheta_o}|$). This proves the claim. $\square$

2.5. **The barycenter.** For a Cacciopoli set $E \subset S^n$ we define the barycenter $p_o \in S^n$ as one of the minimizers of the mapping

$$S^n \ni p \mapsto \int_E \operatorname{dist}^2_{S^n}(x,p)\,d\mathcal{H}^n \equiv \int_E \arccos^2(x\cdot p)\,d\mathcal{H}^n.$$

We note that such minimizing $p_o \in S^n$ always exists, and these minimizers are denoted as barycenters of E. We note that there could be several barycenters. However, if E is a ball centered at p_o, then p_o is the unique barycenter. Moreover, the integral above is differentiable with respect to p. In a minimizing point p_o, i.e. when p_o is a barycenter, we therefore can compute the tangential gradient with respect to p. This leads to the necessary condition:

$$\int_E \big[x-(x\cdot p_o)p_o\big]\frac{\arccos(x\cdot p_o)}{\sqrt{1-(x\cdot p_o)^2}}\,d\mathcal{H}^n(x)=0.$$

In case a barycenter of E is the north pole e_{n+1} the preceding condition reads as

$$\int_E \arccos x_{n+1}\frac{(x_1,\dots,x_n,0)}{\sqrt{x_1^2+\dots+x_n^2}}\,d\mathcal{H}^n(x)=0.$$

Before stating the relevant properties of the barycenter that we shall need in the sequel, we state the following simple

Lemma 2.8. If $E \subset B_{\vartheta_o}$ with $\vartheta_o \in (0,\frac{\pi}{2}]$ then every barycenter p_o of E is contained in $B_{2\vartheta_o}$.

Proof. If $p_o \notin B_{2\vartheta_o}$ then for every $x \in E$ we would have that $\operatorname{dist}_{S^n}(x,p_o) \ge \vartheta_o$, hence

$$\int_E \operatorname{dist}^2_{S^n}(x,p_o)\,d\mathcal{H}^n \ge \vartheta_o^2|E|.$$

On the other hand, since $E \subset B_{\vartheta_o}$, we have

$$\int_E \operatorname{dist}^2_{S^n}(x,e_{n+1})\,d\mathcal{H}^n < \vartheta_o^2|E|,$$

a contradiction to the minimality of p_o. □

The next lemma deals with the continuity of the barycenter in a special situation that will be useful in the proof of Theorem 1.1.

Lemma 2.9 (Continuity of barycenters). *For every $\varepsilon > 0$ there exists $\delta > 0$ such that there holds: If $G \subset S^n$ is a set of finite perimeter satisfying $B_{\vartheta_o(1-\delta)} \subset G \subset B_{\vartheta_o(1+\delta)}$, for some $\vartheta_o \in (0,\frac{\pi}{2}]$, then $\operatorname{dist}_{S^n}(p_o,e_{n+1}) \le \varepsilon\vartheta_o$ for any barycenter p_o of G.*

Proof. As in the proof of Lemma 2.6 we argue by contradiction assuming that there exist $\varepsilon \in (0,1)$, finite perimeter sets $G_k \subset S^n$, radii $\vartheta_k \in (0,\frac{\pi}{2}]$ and $\delta_k \to 0$, such that

$$B_{\vartheta_k(1-\delta_k)} \subset G_k \subset B_{\vartheta_k(1+\delta_k)}, \qquad \text{and} \qquad \operatorname{dist}_{S^n}(p_k,e_{n+1}) \ge 2\varepsilon\vartheta_k \text{ for all } k, \tag{2.18}$$

where p_k is a barycenter for G_k. Passing to a subsequence we can assume without loss of generality that $\vartheta_k \to \vartheta_o \in [0,\frac{\pi}{2}]$ and $p_k \to p_o$ as $k \to \infty$. Since p_k is a barycenter for G_k, by minimality we have that

$$\frac{1}{\vartheta_k^{n+2}}\int_{G_k}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n \le \frac{1}{\vartheta_k^{n+2}}\int_{G_k}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n. \tag{2.19}$$

We claim that

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\left[\int_{G_k}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n - \int_{B_{\vartheta_k}}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n\right]=0. \tag{2.20}$$

To prove the claim, we note that by Lemma 2.8, we have that $p_k \in B_{3\vartheta_k}$ for k large. Therefore, denoting by I_k the quantity in the above limit, we immediately get

$$|\mathrm{I}_k| \le \frac{1}{\vartheta_k^{n+2}}\int_{G_k\Delta B_{\vartheta_k}}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n \le \frac{c}{\vartheta_k^n}|G_k\Delta B_{\vartheta_k}| \le c(n)\delta_k,$$

where we also used Lemma 2.1. For the last inequality we used $(2.18)_1$. This proves (2.20). Exactly in the same way we obtain

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\left[\int_{G_k}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n-\int_{B_{\vartheta_k}}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n\right]=0.$$

Joining the preceding identity with (2.20) and (2.19) we can conclude that

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\int_{B_{\vartheta_k}}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n\le\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\int_{B_{\vartheta_k}}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n. \tag{2.21}$$

Here, we possibly have to pass to a subsequence such that the limits on both sides exist. Having arrived at this stage we distinguish as in the proof of Lemma 2.6 between the cases $\vartheta_o>0$ and $\vartheta_o=0$.

The case $\boldsymbol{\vartheta_o>0}$. In this case the limits in (2.21) can be easily computed and we obtain the inequality

$$\int_{B_{\vartheta_o}}\arccos^2(x\cdot p_o)\,d\mathcal{H}^n\le\int_{B_{\vartheta_o}}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n.$$

The preceding inequality can only hold if $p_o=e_{n+1}$, and this gives the desired contradiction since we must have $\operatorname{dist}(p_o,e_{n+1})\ge 2\varepsilon\vartheta_o>0$.

The case $\boldsymbol{\vartheta_o=0}$. We define $w_k:=T_k(e_{n+1})$. Here the rotations $T_k\in S\mathbb{O}(n)$ are chosen to map the great circle passing through p_k and e_{n+1} into itself and such that $T_kp_k=e_{n+1}$. Again, passing to a further subsequence we can assume that $T_k\to T\in S\mathbb{O}(n)$. Next, we write $w_k=\omega_k\sin\psi_k+e_{n+1}\cos\psi_k$ with some $\omega_k\in S^{n-1}$ and some angle $\psi_k>0$. Notice that from $(2.18)_2$ we have that also $\operatorname{dist}(w_k,e_{n+1})\ge 2\varepsilon\vartheta_k$, hence

$$\psi_k\ge 2\varepsilon\vartheta_k\qquad\text{for all } k\in\mathbb{N}.$$

By performing a rotation around the $\mathbb{R}e_{n+1}$ axis, we may also assume that there exists a unique $\omega_o\in S^{n-1}$ such that

$$w_k=\omega_o\sin\psi_k+e_{n+1}\cos\psi_k\qquad\text{for all } k\in\mathbb{N}.$$

Finally, by a change of variable, we can rewrite the integral appearing on the left-hand side of (2.21) in the form

$$\int_{B_{\vartheta_k}}\arccos^2(x\cdot p_k)\,d\mathcal{H}^n=\int_{B_{\vartheta_k}(w_k)}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n.$$

To obtain the final contradiction we establish now that the following inequality

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\left[\int_{B_{\vartheta_k}}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n-\int_{B_{\vartheta_k}(w_k)}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n\right]<0 \tag{2.22}$$

holds true. Since the second integral increases when the distance of w_k from e_{n+1} increases, the worst case in which we have to prove (2.22) is achieved when the angle ψ_k is precisely equal to $2\varepsilon\vartheta_k$. Therefore, from now on we assume that there holds:

$$w_k=\omega_o\sin(2\varepsilon\vartheta_k)+e_{n+1}\cos(2\varepsilon\vartheta_k)\qquad\text{for all } k\in\mathbb{N}.$$

Writing points in S^n in the form $x=\omega\sin\vartheta+e_{n+1}\cos\vartheta$ with $\omega\in S^{n-1}$, we observe that if $x\in B_{\vartheta_k}(w_k)\setminus B_{\vartheta_k}$ then $\arccos(x\cdot e_{n+1})>\vartheta_k$. Therefore, since $|B_{\vartheta_k}\setminus B_{\vartheta_k}(w_k)|=|B_{\vartheta_k}(w_k)\setminus B_{\vartheta_k}|$, we immediately have that

$$\begin{aligned}\int_{B_{\vartheta_k}\setminus B_{\vartheta_k}(w_k)}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n&-\int_{B_{\vartheta_k}(w_k)\setminus B_{\vartheta_k}}\arccos^2(x\cdot e_{n+1})\,d\mathcal{H}^n\\&\le\int_{B_{\vartheta_k}\setminus B_{\vartheta_k}(w_k)}\left(\vartheta^2-\vartheta_k^2\right)d\mathcal{H}^n.\end{aligned}$$

Therefore, the claim (2.22) will be true if we show that

$$\lim_{k\to\infty}\frac{1}{\vartheta_k^{n+2}}\int_{B_{\vartheta_k}\setminus B_{\vartheta_k}(w_k)}(\vartheta^2-\vartheta_k^2)\,d\mathcal{H}^n<0 \tag{2.23}$$

holds true. To prove the preceding inequality we define as in the proof of Lemma 2.6 the set $S_\varepsilon=\{\omega\in S^{n-1}:\omega\cdot\omega_o\le -1+\varepsilon\}$. From (2.15) we recall that points $x=\omega\sin\vartheta+e_{n+1}\cos\vartheta$ with $\omega\in S_\varepsilon$ and $\vartheta\in(\vartheta_k(1-\varepsilon),\vartheta_k)$ are contained in $B_{\vartheta_k}\setminus B_{\vartheta_k}(w_k)$. Therefore, in order to prove (2.23) we are left with establishing that

$$\lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(S_\varepsilon)}{\vartheta_k^{n+2}}\int_{\vartheta_k(1-\varepsilon)}^{\vartheta_k}(\vartheta^2-\vartheta_k^2)\sin^{n-1}\vartheta\,d\vartheta<0$$

holds true. This is indead true, since a simple calculation shows that the above limit is equal to

$$\mathcal{H}^{n-1}(S_\varepsilon)\bigg[\frac{1-(1-\varepsilon)^{n+2}}{n+2}-\frac{1-(1-\varepsilon)^n}{n}\bigg]=\mathcal{H}^{n-1}(S_\varepsilon)\big[-\varepsilon^2+o(\varepsilon^2)\big]<0,$$

provided ε is small enough. This finally concludes the proof of Lemma 2.9. □

3. Fuglede's result for nearly spherical sets on S^n

In this section we consider sets $E\subseteq S^n$ which are nearly spherical in the sense that the boundary of E is a *radial graph* over S_{ϑ_o}. By this we mean that ∂E admits a global Lipschitz parametrization $X\colon\mathbb{R}^n\supset S^{n-1}\to\partial E$ of the form

$$X(\omega)=(\omega,0)\sin[\vartheta_o(1+u(\omega))]+e_{n+1}\cos[\vartheta_o(1+u(\omega))],\quad \omega\in S^{n-1}.$$

To have $X\in W^{1,\infty}(S^{n-1},S^n)$ we must impose that the scalar-valued function $u\colon S^{n-1}\to[-1,1]$ is of class $W^{1,\infty}$. We note that $X(S^{n-1})=\partial E$, which means that

$$\partial E=\{X(\omega):\omega\in S^{n-1}\}. \tag{3.1}$$

With respect to nearly spherical graphs on S^n we have the following

Theorem 3.1 (Fuglede's theorem on the sphere). *There exist constants $c_1>0$ and $\varepsilon_o\in(0,\frac12]$ only depending on n such that the following holds true: Suppose that $E\subseteq S^n$ is a nearly spherical set with barycenter at the northpole e_{n+1} satisfying the volume constraint*

$$|E|=|B_{\vartheta_o}|, \tag{3.2}$$

for some $\vartheta_o\in(0,\frac{\pi}{2}]$. Furthermore, suppose that the function $u\in W^{1,\infty}(S^{n-1})$ from the global graph representation is Lipschitz continuous. If

$$\|u\|_{W^{1,\infty}(S^{n-1})}\le\varepsilon_o, \tag{3.3}$$

then there holds

$$\frac{\mathcal{H}^{n-1}(\partial E)-\mathcal{H}^{n-1}(S_{\vartheta_o})}{\mathcal{H}^{n-1}(S_{\vartheta_o})}\ge c_1\,\|u\|^2_{W^{1,2}}. \tag{3.4}$$

Proof. The proof is divided into several steps.

Step 1. A parametrization of the set E. We consider the parametrization $\widetilde{X}\colon S^{n-1}\times[0,\vartheta_o]\to S^n$ of the set E defined by

$$\widetilde{X}(\omega,\vartheta):=(\omega,0)\sin[\vartheta(1+u(\omega))]+e_{n+1}\cos[\vartheta(1+u(\omega))]$$

for $\omega\in S^{n-1}$ and $\vartheta\in[0,\vartheta_o]$. For simplicity we use the short hand notation $\mathbf{s}:=\sin[\vartheta(1+u)]$ and $\mathbf{c}:=\cos[\vartheta(1+u)]$ and compute the derivative $D\widetilde{X}$ (with respect to the orthonormal basis $\tau_1, \dots, \tau_{n-1}$ and e_{n+1} in the tangent space to $T_{(\omega,\vartheta)}[S^{n-1} \times \mathbb{R}]$ and the associated orthonormal basis $(\tau_1, 0), \dots, (\tau_{n-1}, 0), (\omega, 0), e_{n+1}$ in $\mathbb{R}^{n+1}$)

$$D\widetilde{X} = \begin{pmatrix} \mathbf{s} & 0 & \dots & 0 & 0 \\ 0 & \mathbf{s} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{s} & 0 \\ \mathbf{c}\,\vartheta\nabla_{\tau_1} u & \mathbf{c}\,\vartheta\nabla_{\tau_2} u & \dots & \mathbf{c}\,\vartheta\nabla_{\tau_{n-1}} u & \mathbf{c}(1+u) \\ -\mathbf{s}\,\vartheta\nabla_{\tau_1} u & -\mathbf{s}\,\vartheta\nabla_{\tau_2} u & \dots & -\mathbf{s}\,\vartheta\nabla_{\tau_{n-1}} u & -\mathbf{s}(1+u) \end{pmatrix}.$$

The last two lines of the matrix are linearly dependent, so that we obtain for the n-Jacobian of $\widetilde{X}$ that

$$[J\widetilde{X}]^2 = \mathbf{s}^{2(n-1)}\mathbf{c}^2(1+u)^2 + \mathbf{s}^{2(n-1)}\mathbf{s}^2(1+u)^2 = \mathbf{s}^{2(n-1)}(1+u)^2,$$

and hence

$$J\widetilde{X} = (1+u)\sin^{n-1}[\vartheta(1+u)]. \tag{3.5}$$

Step 2. Consequence of the volume constraint (3.2)**.** Using the expression for the n-Jacobian of $\widetilde{X}$ from (3.5), the volume constraint (3.2) can be rewritten in the form

$$0 = \int_0^{\vartheta_o} \int_{S^{n-1}} \Big[(1+u)\sin^{n-1}[\vartheta(1+u)] - \sin^{n-1}\vartheta\Big]\, d\mathcal{H}^{n-1}\, d\vartheta.$$

In order to to derive an expansion of this integral in terms of u we define

$$F(\vartheta) := \int_0^{\vartheta} \sin^{n-1} s\, ds \qquad \text{for } \vartheta \in [0, 2\vartheta_o].$$

Then, in terms of F the volume constraint takes the following form:

$$\begin{aligned} 0 &= \int_{S^{n-1}} \big[F\big(\vartheta_o(1+u)\big) - F(\vartheta_o)\big]\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} \Big[F'(\vartheta_o)\,\vartheta_o u + \tfrac{1}{2}F''(\vartheta_o)\,\vartheta_o^2 u^2\Big]\, d\mathcal{H}^{n-1} + R_{11} \\ &= \int_{S^{n-1}} \Big[\vartheta_o u\, \sin^{n-1}\vartheta_o + \tfrac{n-1}{2}\vartheta_o^2 u^2 \sin^{n-2}\vartheta_o \cos\vartheta_o\Big]\, d\mathcal{H}^{n-1} + R_{11} \\ &= \vartheta_o \sin^{n-1}\vartheta_o \int_{S^{n-1}} \Big[u + \tfrac{n-1}{2}\tfrac{\cos\vartheta_o}{\sin\vartheta_o}\,\vartheta_o u^2\Big]\, d\mathcal{H}^{n-1} + R_{11}, \end{aligned}$$

where the remainder R_{11} is given by

$$\begin{aligned} R_{11} &= \frac{(n-1)\vartheta_o^3}{6} \int_{S^{n-1}} u^3 \Big[(n-2)\sin^{n-3}(\vartheta_o + \tau\vartheta_o u)\cos^2(\vartheta_o + \tau\vartheta_o u) \\ &\qquad\qquad - \sin^{n-1}(\vartheta_o + \tau\vartheta_o u)\Big]\, d\mathcal{H}^{n-1} \\ &= \vartheta_o \sin^{n-1}\vartheta_o \int_{S^{n-1}} u^3 \frac{(n-1)\vartheta_o^2}{6\sin^2\vartheta_o} \Bigg[\frac{(n-2)\sin^{n-3}(\vartheta_o + \tau\vartheta_o u)}{\sin^{n-3}\vartheta_o} \\ &\qquad\qquad - \frac{(n-1)\sin^{n-1}(\vartheta_o + \tau\vartheta_o u)}{\sin^{n-3}\vartheta_o}\Bigg]\, d\mathcal{H}^{n-1}, \end{aligned}$$

for some function τ on S^n with values in $(0,1)$. Therefore, we end up with

$$\int_{S^{n-1}} u\, d\mathcal{H}^{n-1} = -\frac{n-1}{2}\,\frac{\vartheta_o \cos\vartheta_o}{\sin\vartheta_o} \int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} + R_1, \tag{3.6}$$

where due to the smallness condition (3.3) the remainder satisfies

$$|R_1| \le c(n)\,\varepsilon_o \|u\|_{L^2}^2$$

for some positive constant $c(n)$ independent of ϑ_o.

Step 3. Consequence of the barycenter constraint. From the barycenter condition, i.e. the condition that the barycenter of E is the north pole e_{n+1}, we conclude that for $i = 1, \dots, n$ there holds:

$$\int_0^{\vartheta_o} \int_{S^{n-1}} \vartheta(1+u)\, J\widetilde{X}\, \omega_i \, d\mathcal{H}^{n-1}\, d\vartheta = 0.$$

Using (3.5) for the n-Jacobian of $\widetilde{X}$ we obtain

$$\int_0^{\vartheta_o} \int_{S^{n-1}} \vartheta(1+u)^2 \sin^{n-1}[\vartheta(1+u)]\, \omega_i \, d\mathcal{H}^{n-1}\, d\vartheta = 0 \tag{3.7}$$

for $i = 1, \dots, n$. We introduce another auxiliary function by setting

$$F(\vartheta) := \int_0^{\vartheta} s \sin^{n-1} s \, ds \qquad \text{for } \vartheta \in [0, 2\vartheta_o].$$

Then, (3.7) can be rewritten as

$$\int_{S^{n-1}} \big[F(\vartheta_o(1+u) - F(\vartheta_o)\big]\omega_i \, d\mathcal{H}^{n-1} = 0 \tag{3.8}$$

for $i = 1, \dots, n$. Here, we also used the fact that $\int_{S^{n-1}} \omega_i \, d\mathcal{H}^{n-1} = 0$. On the other hand, by Taylor expansion we have

$$\begin{aligned}
F(\vartheta_o{+}\vartheta_o u) - F(\vartheta_o) &= \vartheta_o^2 u \sin^{n-1}\vartheta_o + \frac{\vartheta_o^2 u^2}{2}\big[\sin^{n-1}(\vartheta_o{+}\tau\vartheta_o u) \\
&\quad + (n-1)(\vartheta_o{+}\tau\vartheta_o u)\sin^{n-2}(\vartheta_o{+}\tau\vartheta_o u)\cos(\vartheta_o{+}\tau\vartheta_o u)\big] \\
&= \vartheta_o^2 \sin^{n-1}\vartheta_o \bigg[u + \tfrac{1}{2}u^2 \frac{\sin^{n-1}(\vartheta_o{+}\tau\vartheta_o u)}{\sin^{n-1}\vartheta_o} \\
&\quad + \frac{(n-1)u^2\vartheta_o(1+\tau u)}{2\sin\vartheta_o} \cdot \frac{\sin^{n-2}(\vartheta_o{+}\tau\vartheta_o u)\cos(\vartheta_o{+}\tau\vartheta_o u)}{\sin^{n-2}\vartheta_o}\bigg],
\end{aligned}$$

for some function τ on S^{n-1} with $\tau \in (0,1)$. Thus, inserting the right-hand side of the preceding equality into (3.8) and recalling the smallness condition (3.3) we immediately get a bound for the first order Fourier coefficients of u of the following form:

$$\bigg|\int_{S^{n-1}} u\, \omega_i \, d\mathcal{H}^{n-1}\bigg| \le c(n)\, \varepsilon_o \|u\|_{L^2} \quad \forall\, i = 1, \dots, n \tag{3.9}$$

where $c(n)$ is again independent on ϑ_o.

Step 4. The Jacobian of X. We compute the $(n-1)$-Jacobian of X with respect to an orthonormal basis $\tau_1, \dots, \tau_{n-1}$ in the tangent space to $T_\omega S^{n-1}$ and the associated orthonormal basis $(\tau_1, 0), \dots, (\tau_{n-1}, 0), (\omega, 0), e_{n+1}$ in $\mathbb{R}^{n+1}$. We note that the first n vectors are pairwise orthonormal in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. With the abbreviations $\mathbf{s} := \sin[\vartheta_o(1+u)]$ and $\mathbf{c} := \cos[\vartheta_o(1+u)]$, we then have

$$DX = \begin{pmatrix} \mathbf{s} & 0 & \dots & 0 \\ 0 & \mathbf{s} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{s} \\ \mathbf{c}\,\vartheta_o \nabla_{\tau_1} u & \mathbf{c}\,\vartheta_o \nabla_{\tau_2} u & \dots & \mathbf{c}\,\vartheta_o \nabla_{\tau_{n-1}} u \\ -\mathbf{s}\,\vartheta_o \nabla_{\tau_1} u & -\mathbf{s}\,\vartheta_o \nabla_{\tau_2} u & \dots & -\mathbf{s}\,\vartheta_o \nabla_{\tau_{n-1}} u \end{pmatrix}.$$

Since the last two lines of the matrix are linearly dependent, the 2×2 minors of the associated matrix $(\mathbf{c}\,\vartheta_o \nabla_\tau u, \mathbf{s}\,\vartheta_o \nabla_\tau u)$ are zero and therefore we get for the $(n-1)$-Jacobian of X that

$$[JX]^2 = \mathbf{s}^{2(n-1)} + \mathbf{s}^{2(n-2)}\mathbf{s}^2 \vartheta_o^2 |\nabla_\tau u|^2 + \mathbf{s}^{2(n-2)}\mathbf{c}^2 \vartheta_o^2 |\nabla_\tau u|^2$$

$$= \mathbf{s}^{2(n-1)} + \mathbf{s}^{2(n-2)} \vartheta_o^2 |\nabla_\tau u|^2$$

which is the same as

$$JX = \sin^{n-1}[\vartheta_o(1+u)] \sqrt{1 + \frac{\vartheta_o^2 |\nabla_\tau u|^2}{\sin^2[\vartheta_o(1+u)]}}\,. \tag{3.10}$$

At this stage the smallness condition we need to have is that $\vartheta_o(1+u)$ is uniformly bounded below from 0. But this follows, since we impose a smallness condition (3.3) on u which guarantees that $\|u\|_{L^\infty} \le \frac{1}{2}$. By Taylor's formula we have

$$\sqrt{1 + \frac{\vartheta_o^2 |\nabla_\tau u|^2}{\sin^2[\vartheta_o(1+u)]}} = 1 + \frac{1}{2} \frac{\vartheta_o^2 |\nabla_\tau u|^2}{\sin^2[\vartheta_o(1+u)]} + R_{21},$$

where

$$|R_{21}| \le \frac{1}{8} \frac{\vartheta_o^4 |\nabla_\tau u|^4}{\sin^4[\vartheta_o(1+u)]} \le \frac{1}{8} \frac{\vartheta_o^4 |\nabla_\tau u|^4}{\sin^4(\vartheta_o/2)} \le c\, \varepsilon_o^2 |\nabla_\tau u|^2,$$

for some universal constant c independent of ϑ_o and n. Here, we used the inequality $1/\sin^2[\vartheta_o(1+u)] \le 1/\sin^2(\vartheta_o/2)$ which again follows from (3.3) since $|u| \le 1/2$ and the fact that $0 < \vartheta_o \le \frac{\pi}{2}$. In the preceding expansion we replace in the second term of the right-hand side $\sin^2[\vartheta_o(1+u)]$ by $\sin^2 \vartheta_o$. The resulting error can be estimated by (3.3) as follows:

$$\begin{aligned}
\left| \frac{1}{\sin^2[\vartheta_o(1+u)]} - \frac{1}{\sin^2 \vartheta_o} \right| &\le \left| \frac{\sin^2 \vartheta_o - \sin^2[\vartheta_o(1+u)]}{\sin^2(\vartheta_o/2) \sin^2 \vartheta_o} \right| \\
&\le \frac{1}{\sin^4(\vartheta_o/2)} \left| \int_0^1 \frac{d}{d\tau} \sin^2[\vartheta_o(1+\tau u)]\, d\tau \right| \\
&\le \frac{2\vartheta_o |u|}{\sin^4(\vartheta_o/2)} \int_0^1 |\sin[\vartheta_o(1+\tau u)|| \cos[\vartheta_o(1+\tau u)|\, d\tau \\
&\le \frac{c\vartheta_o |u|}{\sin^3(\vartheta_o/2)} \le \frac{c\, \varepsilon_o}{\sin^2 \vartheta_o},
\end{aligned}$$

again with some absolute constant c independent of ϑ_o and n. Inserting this above we obtain the expansion

$$\sqrt{1 + \frac{\vartheta_o^2 |\nabla_\tau u|^2}{\sin^2[\vartheta_o(1+u)]}} = 1 + \frac{1}{2} \frac{\vartheta_o^2 |\nabla_\tau u|^2}{\sin^2 \vartheta_o} + R_{22},$$

with a remainder R_{22} for which the following bound holds true:

$$|R_{22}| \le c\, \varepsilon_o |\nabla_\tau u|^2.$$

Using this expansion in the formula for the Jacobian JX we end up with

$$JX = \sin^{n-1}[\vartheta_o(1+u)] \left[1 + \frac{\vartheta_o^2 |\nabla_\tau u|^2}{2 \sin^2 \vartheta_o} \right] + R_2, \tag{3.11}$$

where the remainder satisfies

$$|R_2| \le c(n) \varepsilon_o \sin^{n-1} \vartheta_o\, |\nabla_\tau u|^2, \tag{3.12}$$

for some positive constant $c(n)$ not depending on ϑ_o.

Step 5. Lower bound for the isoperimetric gap. Using the expansion (3.11) for the $(n-1)$-Jacobian of X we can control the isoperimetric gap of ∂E from below by

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) &= \int_{S^{n-1}} \left(JX - \sin^{n-1} \vartheta_o \right) d\mathcal{H}^{n-1} \\
&= \int_{S^{n-1}} \left[\sin^{n-1}[\vartheta_o(1+u)] \left[1 + \frac{\vartheta_o^2 |\nabla_\tau u|^2}{2 \sin^2 \vartheta_o} \right] - \sin^{n-1} \vartheta_o + R_2 \right] d\mathcal{H}^{n-1},
\end{aligned}$$

where the remainder R_2 fulfills the pointwise bound from (3.12). We use the abbreviations: $\mathbf{s}_o = \sin\vartheta_o$, $\mathbf{c}_o = \cos\vartheta_o$. Using Taylor's formula in order to expand the first factor of the first summand we find that for a suitable $\tau \in (0,1)$

$$\begin{aligned}
\sin^{n-1}[\vartheta_o(1+u)] &= \\
&= \mathbf{s}_o^{n-1} + (n-1)\vartheta_o u\, \mathbf{s}_o^{n-2}\mathbf{c}_o + \tfrac{n-1}{2}\,\vartheta_o^2 u^2\big[(n-2)\mathbf{s}_o^{n-3} - (n-1)\mathbf{s}_o^{n-1}\big] \\
&\quad + \tfrac{n-1}{6}\,\vartheta_o^3 u^3 \cos(\vartheta_o + \tau\vartheta_o u) \\
&\qquad \cdot \big[(n-2)(n-3)\sin^{n-4}(\vartheta_o + \tau\vartheta_o u) - (n-1)^2 \sin^{n-2}(\vartheta_o + \tau\vartheta_o u)\big] \\
&= \mathbf{s}_o^{n-1} + (n-1)\vartheta_o u\, \mathbf{s}_o^{n-2}\mathbf{c}_o + \tfrac{n-1}{2}\,\vartheta_o^2 u^2 \mathbf{s}_o^{n-3}\big[(n-2) - (n-1)\mathbf{s}_o^2\big] \\
&\quad + \tfrac{n-1}{6}\,\vartheta_o^3 u^3 \cos(\vartheta_o + \tau\vartheta_o u)\sin^{n-4}(\vartheta_o + \tau\vartheta_o u) \\
&\qquad \cdot \big[(n-2)(n-3) - (n-1)^2 \sin^2(\vartheta_o + \tau\vartheta_o u)\big] \\
&= \mathbf{s}_o^{n-1}\bigg[1 + \frac{(n-1)\vartheta_o u\,\mathbf{c}_o}{\mathbf{s}_o} + \frac{(n-1)\vartheta_o^2 u^2}{2\mathbf{s}_o^2}\big[(n-1)\mathbf{c}_o^2 - 1\big] \\
&\quad + \frac{(n-1)\vartheta_o^3 u^3}{6\mathbf{s}_o^{n-1}}\sin^{n-4}(\vartheta_o + \tau\vartheta_o u)\cos(\vartheta_o + \tau\vartheta_o u) \\
&\qquad \cdot \big[(n-2)(n-3) - (n-1)^2 \sin^2(\vartheta_o + \tau\vartheta_o u)\big]\bigg] \\
&= \mathbf{s}_o^{n-1}\bigg[1 + \frac{(n-1)\vartheta_o u\,\mathbf{c}_o}{\mathbf{s}_o} + \frac{(n-1)\vartheta_o^2 u^2}{2\mathbf{s}_o^2}\big[(n-1)\mathbf{c}_o^2 - 1\big] + R_{31}\bigg],
\end{aligned}$$

holds true, where the remainder can be bounded by

$$|R_{31}| \le c(n)\,|u|^3 \le c(n)\,\varepsilon_o |u|^2,$$

for a suitable constant $c(n)$ depending only on the dimension. Inserting the preceding identity into the formula for the isoperimetric gap above yields

$$\begin{aligned}
&\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \\
&\quad = \tfrac12 \mathbf{s}_o^{n-3}\vartheta_o^2\bigg[\int_{S^{n-1}} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} + (n-1)\big[(n-1)\mathbf{c}_o^2 - 1\big]\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} \\
&\qquad\qquad + \frac{2(n-1)\mathbf{s}_o\mathbf{c}_o}{\vartheta_o}\int_{S^{n-1}} u\, d\mathcal{H}^{n-1} + R_3\bigg],
\end{aligned}$$

where now the remainder satisfies

$$|R_3| \le c(n)\,\varepsilon_o \|u\|^2_{W^{1,2}},$$

for a suitable constant $c(n)$ independent of ϑ_o. At this point we use (3.6) for the integral involving only u and end up with

$$\begin{aligned}
&\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \\
&\quad \ge \tilde c\bigg[\int_{S^{n-1}} |\nabla_\tau u|^2\, d\mathcal{H}^{n-1} - (n-1)\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} - c\,\varepsilon_o \|u\|^2_{W^{1,2}}\bigg],
\end{aligned} \tag{3.13}$$

for constants $\tilde c = \tilde c(\vartheta_o) = \frac12 \vartheta_o^2 \sin^{n-3}\vartheta_o$ and $c = c(n)$.

Step 6. Fourier expansion of u. By $\{Y_{j,\ell} : j \in \mathbb{N}_o,\ \ell = 1,\dots,m_j\}$ we denote the orthonormal basis of spherical harmonics in $L^2(S^{n-1})$, i.e. we have

$$-\Delta_{S^{n-1}} Y_{j,\ell} = j(j+n-2)Y_{j,\ell} \quad \text{for } j \in \mathbb{N}_o \text{ and } \ell = 1,\dots,m_j,$$

where m_j denotes the dimension of the eigenspace associated to the eigenvalue $j(j+n-2)$. Note that $m_1 = n$ and

$$m_j := \binom{n+j-1}{n-1} - \binom{n+j-3}{n-1} \qquad \text{for } j \ge 2.$$

The orthonormality of $Y_{j,\ell}$ means that there holds

$$\int_{S^{n-1}} Y_{j_1,\ell_1} Y_{j_2,\ell_2}\, d\mathcal{H}^{n-1} = \delta_{j_1,j_2}\delta_{\ell_1,\ell_2}$$

for $j_1, j_2 \in \mathbb{N}_o$, $\ell_1 = 1, \dots, m_{j_1}$ and $\ell_2 = 1, \dots, m_{j_2}$. We now consider the expansion of u via the corresponding Fourier series

$$u = \sum_{j=0}^{\infty} \sum_{\ell=1}^{m_j} a_{j,\ell} Y_{j,\ell},$$

where the Fourier coefficients $a_{j,\ell} \in \mathbb{R}$ of u are given by

$$a_{j,\ell} := \int_{S^{n-1}} u Y_{j,\ell}\, d\mathcal{H}^{n-1}.$$

In terms of the Fourier coefficients the L^2-norms of u and ∇u can be expressed as follows

$$\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} = \sum_{j=0}^{\infty} \sum_{\ell=1}^{m_j} a_{j,\ell}^2, \quad \int_{S^{n-1}} |\nabla u|^2\, d\mathcal{H}^{n-1} = \sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} j(j+n-2) a_{j,\ell}^2.$$

For convenience in notation we abbreviate

$$I(\mu) := \sum_{j=\mu}^{\infty} \sum_{\ell=1}^{m_j} \big[j(j+n-2)+1\big] a_{j,\ell}^2 \qquad \text{for } \mu \in \mathbb{N}_0$$

and note that $I(0) = \|u\|_{W^{1,2}}^2$. At this point we note that $Y_o \equiv 1/\kappa_n$ and $Y_{1,\ell}(x) = \sqrt{n}\omega_\ell/\kappa_n$ for $\ell = 1, \dots, n$, where $\kappa_n = \sqrt{n\omega_n}$, so that the zero order coefficient a_o is given by $a_o = (1/\kappa_n)\int_{S^{n-1}} u\, d\mathcal{H}^{n-1}$ and the first order coefficients by $a_{1,\ell} = (\sqrt{n}/\kappa_n)\int_{S^{n-1}} u\omega_\ell\, d\mathcal{H}^{n-1}$ for $\ell = 1, \dots, n$. These integrals have been estimated before in Steps 2 and 3 and therefore the bounds can be rewritten in terms of the Fourier coefficients. From (3.6) and (3.3) we infer the following estimate for a_o:

$$a_o^2 \le c\,\varepsilon_o \|u\|_{L^2}^2 \le c(n)\,\varepsilon_o I(0), \tag{3.14}$$

while from (3.9) and (3.3) we get the following bound for $a_1 := (a_{1,1}, \dots, a_{1,n})$:

$$|a_1|^2 = \sum_{\ell=1}^{n} (a_{1,\ell})^2 \le c\,\varepsilon_o \|u\|_{L^2}^2 \le c(n)\,\varepsilon_o I(0). \tag{3.15}$$

We now rewrite the bound for the isoperimetric gap from (3.13) in terms of the Fourier-coefficients. With (3.14) and (3.15) we obtain

$$\begin{aligned}
&\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \\
&\quad \ge \tilde{c}\Big[\sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} j(j+n-2) a_{j,\ell}^2 - (n-1)\sum_{j=0}^{\infty} \sum_{\ell=1}^{m_j} a_{j,\ell}^2 - c\,\varepsilon_o I(0)\Big] \\
&\quad \ge \tilde{c}\Big[\frac{1}{2}\sum_{j=2}^{\infty} \sum_{\ell=1}^{m_j} \big[j(j+n-2)+1\big] a_{j,\ell}^2 - c\,\varepsilon_o I(0)\Big] \\
&\quad = \tilde{c}\Big[\tfrac{1}{2} I(2) - c\,\varepsilon_o I(0)\Big] \\
&\quad = \tilde{c}\Big[\big(\tfrac{1}{2} - c\,\varepsilon_o\big) I(0) - \tfrac{1}{2} a_o^2 - \tfrac{n}{2}\sum_{\ell=1}^{n} a_{1,\ell}^2\Big],
\end{aligned}$$

where in the third last line we have used that $j(j+n-2) - (n-1) \ge \frac{1}{2}[j(j+n-2)+1]$ for $j \ge 2$. At this point we use again (3.14) and (3.15) to infer that

$$\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \ge \tilde{c}\big(\tfrac{1}{2} - c\,\varepsilon_o\big) I(0).$$

Here, we choose $\varepsilon_o > 0$ depending only on n but not on ϑ_o small enough to have

$$\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \ge \tfrac14 \tilde c\, I(0) = \tfrac18 \vartheta_o^2 \sin^{n-3}\vartheta_o \|u\|^2_{W^{1,2}}.$$

Dividing the above inequality by the measure of S_{ϑ_o} we finally obtain the desired lower bound for the renormalized isoperimetric gap

$$\frac{\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o})}{\mathcal{H}^{n-1}(S_{\vartheta_o})} \ge \frac{\vartheta_o^2}{8n\omega_n \sin^2\vartheta_o}\|u\|^2_{W^{1,2}} \ge \tfrac{1}{8n\omega_n}\|u\|^2_{W^{1,2}}.$$

But this proves the estimate (3.4) and finishes the proof of the Fuglede type inequality for nearly spherical sets on the sphere S^n. □

In the following we establish that for nearly spherical sets with small Lipschitz-norm of the representing function u the $\|\cdot\|_{W^{1,2}}$-norm of u bounds the L^2-asymmetry of ∂E. To be more precise, we show that the following assertion holds true:

Lemma 3.2. *Suppose that $E \subseteq S^n$ is a nearly spherical set with volume $|E| = |B_{\vartheta_o}|$, for some $\vartheta_o \in (0, \frac{\pi}{2}]$. Furthermore, suppose that the representing function $u \in W^{1,\infty}(S^{n-1})$ from the global graph representation is Lipschitz continuous with $\|u\|_{W^{1,\infty}(S^{n-1})} \le \frac12$. Then there exists a constant $\tilde c > 1$ independent of ϑ_o and n such that*

$$\tfrac{\sin^{n-1}\vartheta_o}{\tilde c}\,\|u\|^2_{W^{1,2}} \le \boldsymbol{\beta}(E, e_{n+1})^2 \le \tilde c\,\sin^{n-1}\vartheta_o\,\|u\|^2_{W^{1,2}}.$$

Proof. As in the proofs before, we set $\mathbf{s} := \sin[\vartheta_o(1+u(\omega))]$, $\mathbf{c} := \cos[\vartheta_o(1+u(\omega))]$ and $\mathbf{s}_o := \sin\vartheta_o$, $\mathbf{c}_o := \cos\vartheta_o$. The unit outer normal vector to ∂E in the point $x = X(\omega)$ is given by

$$\nu_E(x) = \frac{(\omega,0)\mathbf{sc} - (\nabla_\tau u(\omega),0)\vartheta_o - e_{n+1}\mathbf{s}^2}{\sqrt{\mathbf{s}^2 + \vartheta_o^2|\nabla_\tau u(\omega)|^2}}.$$

On the other hand the unit outer normal vector to $B_{\vartheta(x)}$ with $x = X(\omega)$ is

$$\nu_{B_{\vartheta(x)}}(x) = (\omega,0)\mathbf{c} - e_{n+1}\mathbf{s}.$$

These expressions allow us to compute the square of the difference of both normals (recall that $x = X(\omega)$ for some $\omega \in S^{n-1}$)

$$\begin{aligned}
\tfrac12\big|\nu_E(x) - \nu_{B_{\vartheta_o}}(\pi_{B_{\vartheta_o}}(x))\big|^2 &= 1 - \nu_E(x)\cdot\nu_{B_{\vartheta_o}}(\pi_{B_{\vartheta_o}}(x)) \\
&= 1 - \frac{\mathbf{s}}{\sqrt{\mathbf{s}^2 + \vartheta_o^2|\nabla_\tau u(\omega)|^2}} \\
&= \frac{1}{\sqrt{1+\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2}}\Big[\sqrt{1+\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2} - 1\Big] \\
&= \frac{\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2}{\Big[\sqrt{1+\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2}+1\Big]\sqrt{1+\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2}}.
\end{aligned}$$

Since $\|u\|_{W^{1,\infty}(S^{n-1})} \le \frac12$ we have

$$\frac{\vartheta_o}{\mathbf{s}} \le \frac{\vartheta_o}{\sin\frac{\vartheta_o}{2}} \le \frac{\frac{\pi}{2}}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt2} \le \sqrt5$$

and therefore

$$\tfrac12\big|\nu_E(x) - \nu_{B_{\vartheta_o}}(\pi_{B_{\vartheta_o}}(x))\big|^2 \le \frac{\vartheta_o^2}{\mathbf{s}^2}|\nabla_\tau u(\omega)|^2 \le 5|\nabla_\tau u(\omega)|^2.$$

On the other hand, using $\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2 \le \frac18\pi^2 \le 2$ and hence $\sqrt{1+\vartheta_o^2/\mathbf{s}^2|\nabla_\tau u(\omega)|^2} \le \sqrt3 \le 2$ we get the following lower bound:

$$\tfrac12\big|\nu_E(x) - \nu_{B_{\vartheta_o}}(\pi_{B_{\vartheta_o}}(x))\big|^2 \ge \frac{\vartheta_o^2}{6\mathbf{s}^2}|\nabla_\tau u(\omega)|^2 \ge \frac{2|\nabla_\tau u(\omega)|^2}{27}.$$

Integrating the first of the two preceding inequalities with respect to ω over S^{n-1}, recalling the formula for the n-Jacobian of X from (3.10) and that $\|\nabla_\tau u\|_{L^\infty} \le \frac{1}{2}$, we obtain

$$\begin{aligned}\boldsymbol{\beta}^2(E; e_{n+1}) &= \tfrac{1}{2}\int_{\partial^* E} \left|\nu_E(x) - \nu_{B_{\vartheta_o}}(\pi_{B_{\vartheta_o}}(x))\right|^2 d\mathcal{H}^{n-1} \\ &\le 5\int_{S^{n-1}} |\nabla_\tau u(\omega)|^2 JX\, d\mathcal{H}^{n-1} \le c \sin^{n-1}\vartheta_o\, \|\nabla_\tau u\|^2_{L^2},\end{aligned}$$

for some constant c independent of ϑ_o and n. A similar estimate from below also holds true. Hence the lemma is proved. $\square$

Combining Fuglede's Theorem 3.1 and Lemma 3.2 (here we can take instead of e_{n+1} the minimum over all points $p_o \in S^n$ and therefore we are able to replace $\boldsymbol{\beta}^2(E; e_{n+1})$ by $\boldsymbol{\beta}^2(E)$) we deduce the sharp quantitative isoperimetric inequality for nearly sperical sets on the sphere.

Corollary 3.3. *Under the assumptions of Theorem 3.1 there exists a constant $c_2 = c_2(n) > 0$ such that the following inequality holds*

$$\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(S_{\vartheta_o}) \ge c_2 \boldsymbol{\beta}(E)^2. \tag{3.16}$$

4. Reduction to sets with small support

Our aim in this section is to prove Lemma 4.1 whose proof in our case is much more delicate than the one in the Euclidean case, considered for instance in Lemma 5.1 in [16].

Lemma 4.1. *There exist $\widetilde{C} = \widetilde{C}(n) > 1$ and a radius $R = R(n) > 1$ such that if $E \subset S^n$ is a set of finite perimeter with $|E| = |B_{\vartheta_o}|$ for some $\vartheta_o \in (0, \pi)$ one can find another set $E' \subset E$, with $|E'| \ge |B_{\vartheta_o/2}|$, such that $E' \subset B_{R\vartheta_o}(p_o)$ for some $p_o \in S^n$ and satisfying*

$$\boldsymbol{\beta}^2(E) \le \boldsymbol{\beta}^2(E') + \widetilde{C}\,\mathbf{D}(E) + \widetilde{C}\,(\vartheta_o \mathbf{D}(E))^{\frac{n-1}{n}} \quad \textit{and} \quad \mathbf{D}(E') \le \widetilde{C}\,\mathbf{D}(E). \tag{4.1}$$

We start with some definitions and a few technical lemmas, whose assertions will be needed in the proof. First, we define for $\vartheta \in [0, \pi]$ the function

$$F(\vartheta) := n\omega_n \int_0^\vartheta \sin^{n-1}\sigma\, d\sigma\,.$$

Clearly, $F(\vartheta) = |B_\vartheta|$ and F is a strictly increasing function from $[0, \pi]$ into $[0, |S^n|]$. We note that for $\vartheta \in [0, \pi]$ and $t \in [0, |S^n|]$ there holds

$$F(\vartheta) + F(\pi - \vartheta) = |S^n|, \qquad F^{-1}(t) + F^{-1}(|S^n| - t) = \pi. \tag{4.2}$$

Moreover, we have

$$\lim_{\vartheta \downarrow 0} \frac{F(\vartheta)}{\vartheta^n} = \omega_n, \qquad \lim_{t \downarrow 0} \frac{F^{-1}(t)}{t^{\frac{1}{n}}} = \omega_n^{-\frac{1}{n}}. \tag{4.3}$$

Next, we define *the isoperimetric profile of the sphere* by setting for all $t \in [0, |S^n|]$

$$\varphi(t) := n\omega_n \sin^{n-1}(F^{-1}(t))\,. \tag{4.4}$$

The isoperimetric profile associates to every geodesic ball in S^n of measure t its perimeter $\varphi(t)$. We note that

$$\varphi'(t) = \frac{n-1}{\tan(F^{-1}(t))}, \quad \varphi''(t) = -\frac{n-1}{n\omega_n \sin^{n+1}(F^{-1}(t))}, \quad \lim_{t \downarrow 0} \frac{\varphi(t)}{t^{\frac{n-1}{n}}} = n\omega_n^{\frac{1}{n}}. \tag{4.5}$$

The last identity follows from $(4.3)_2$. In particular, φ is a concave function, with a maximum at $\frac{1}{2}|S^n|$. Finally, for any $\vartheta \in (0, \frac{\pi}{2}]$ we define

$$\psi_\vartheta(s) = \varphi(s|B_\vartheta|) + \varphi((1-s)|B_\vartheta|) - \varphi(|B_\vartheta|) \quad \text{for } s \in [0, 1]. \tag{4.6}$$

Clearly, also the function ψ_ϑ is concave and $\psi_\vartheta(s) = \psi_\vartheta(1 - s)$. In the next two technical lemmas we collect some useful properties of ψ_ϑ.

Lemma 4.2. *There exist a constant $c_1(n) > 1$ such that for all $\vartheta \in (0, \frac{\pi}{2}]$ there holds*

$$\frac{1}{c_1(n)} \le \frac{\psi_\vartheta\big(\frac12\big)}{\mathbf{P}(B_\vartheta)} \le c_1(n). \tag{4.7}$$

Proof. To prove (4.7) we only need to show that the ratio has finite and strictly positive limit at zero. To this aim, using $(4.5)_3$, we immediately get

$$\begin{aligned}\lim_{\vartheta\downarrow 0} \frac{\psi_\vartheta\big(\frac12\big)}{\mathbf{P}(B_\vartheta)} &= \lim_{\vartheta\downarrow 0} \frac{2\varphi(\frac12|B_\vartheta|) - \varphi(|B_\vartheta|)}{\varphi(|B_\vartheta|)} \\ &= \lim_{t\downarrow 0} \frac{2\varphi(t) - \varphi(2t)}{\varphi(2t)} = 2^{\frac1n} - 1\,.\end{aligned}$$

□

The next result gives a precise behavior of the function ψ_ϑ in the interval $(0,1)$.

Lemma 4.3. *There exists a positive constant $c_2(n)$ such that for all $\vartheta \in (0, \frac{\pi}{2}]$ holds*

$$|B_\vartheta| \int_0^1 \frac{ds}{\psi_\vartheta(s)} < c_2(n)\vartheta. \tag{4.8}$$

Proof. We are going to show that the quantity

$$\begin{aligned}\frac{|B_\vartheta|}{\vartheta}\int_0^1 \frac{ds}{\psi_\vartheta(s)} &= \frac{|B_\vartheta|}{\vartheta}\int_0^1 \frac{ds}{\varphi(s|B_\vartheta|) + \varphi((1-s)|B_\vartheta|) - \varphi(|B_\vartheta|)} \\ &= \frac{1}{\vartheta}\int_0^{|B_\vartheta|} \frac{dt}{\varphi(t) + \varphi(|B_\vartheta| - t) - \varphi(|B_\vartheta|)} \\ &= \frac{2}{\vartheta}\int_0^{\frac12|B_\vartheta|} \frac{dt}{\varphi(t) + \varphi(|B_\vartheta| - t) - \varphi(|B_\vartheta|)}\end{aligned} \tag{4.9}$$

is bounded for $\vartheta \in (0, \frac{\pi}{2}]$ by a constant depending only on the dimension n. To estimate the last integral we start by observing that there exists a positive constant $\kappa(n)$ such that

$$\varphi(t) + \varphi(b-t) - \varphi(b) \ge \kappa(n) b^{\frac{n-1}{n}} \quad \text{for } b \in [0, \tfrac12|S^n|] \text{ and } t \in [\tfrac14 b, \tfrac12 b]. \tag{4.10}$$

To prove (4.10) we recall (4.5) which yields for $t \in [\frac14 b, \frac12 b]$ that $\varphi'(t) - \varphi'(b-t) \ge 0$ so that $t \mapsto \varphi(t) + \varphi(b-t)$ is non decreasing in this range of t, and therefore

$$\frac{\varphi(t) + \varphi(b-t) - \varphi(b)}{b^{\frac{n-1}{n}}} \ge \frac{\varphi\big(\frac b4\big) + \varphi\big(\frac{3b}{4}\big) - \varphi(b)}{b^{\frac{n-1}{n}}}.$$

Now, the estimate (4.10) follows from the fact that

$$\lim_{b\downarrow 0} \frac{\varphi\big(\frac b4\big) + \varphi\big(\frac{3b}{4}\big) - \varphi(b)}{b^{\frac{n-1}{n}}} = n\omega_n^{\frac1n}\Big[\big(\tfrac14\big)^{\frac{n-1}{n}} + \big(\tfrac34\big)^{\frac{n-1}{n}} - 1\Big] > 0.$$

On the other hand we can show that

$$\varphi(t) + \varphi(b-t) - \varphi(b) \ge \gamma(n) t^{\frac{n-1}{n}} \quad \text{for } b \in [0, \tfrac12|S^n|] \text{ and } t \in (0, \tfrac14 b], \tag{4.11}$$

holds true with a suitable constant $\gamma(n) > 0$. The estimate (4.11) will follow from

$$\varphi'(t) - \varphi'(b-t) \ge \tfrac{(n-1)}{n}\gamma\, t^{-\frac1n} \qquad \text{for any } t \in (0, \tfrac14 b] \tag{4.12}$$

by integration. To prove (4.12) we use $(4.5)_1$ to infer

$$\begin{aligned}\frac{\varphi'(t) - \varphi'(b-t)}{n-1} &= \frac{1}{\tan(F^{-1}(t))} - \frac{1}{\tan(F^{-1}(b-t))} \\ &= \frac{\tan(F^{-1}(b-t)) - \tan(F^{-1}(t))}{\tan(F^{-1}(t))\tan(F^{-1}(b-t))} \\ &\ge \frac{\tan\big(F^{-1}\big(\frac{3b}{4}\big)\big) - \tan(F^{-1}(b/4))}{\tan(F^{-1}(t))\tan(F^{-1}(b))}\,.\end{aligned}$$

In the last line we used the monotonicity of $t \mapsto \tan\circ F^{-1}$. Using $(4.3)_2$ for the term appearing in the last line we deduce that

$$\lim_{b\downarrow 0}\frac{\tan\big(F^{-1}\big(\frac{3b}{4}\big)\big)-\tan(F^{-1}(b/4))}{\tan(F^{-1}(b))}=\big(\tfrac{3}{4}\big)^{\frac{1}{n}}-\big(\tfrac{1}{4}\big)^{\frac{1}{n}}>0$$

holds true. This allows us to conclude that for any $b\in[0,\frac{1}{2}|S^n|]$ and $t\in(0,\frac{1}{4}b]$ there holds

$$\frac{\varphi'(t)-\varphi'(b-t)}{n-1}\geq\frac{c(n)}{\tan(F^{-1}(t))}\geq\frac{c(n)}{F^{-1}(t)}\geq\frac{\gamma(n)}{nt^{\frac{1}{n}}},$$

for a suitable constant $\gamma(n)>0$. This proves (4.12).

Now, the proof of (4.8) follows easily by combining (4.9) with the estimates (4.10) and (4.11). In fact, we have

$$\begin{aligned}\frac{|B_\vartheta|}{\vartheta}\int_0^1\frac{ds}{\psi_\vartheta(s)}&=\frac{2}{\vartheta}\int_0^{\frac{1}{4}|B_\vartheta|}\frac{dt}{\varphi(t)+\varphi(|B_\vartheta|-t)-\varphi(|B_\vartheta|)}\\&\quad+\frac{2}{\vartheta}\int_{\frac{1}{4}|B_\vartheta|}^{\frac{1}{2}|B_\vartheta|}\frac{dt}{\varphi(t)+\varphi(|B_\vartheta|-t)-\varphi(|B_\vartheta|)}\\&\leq\frac{2}{\vartheta}\int_0^{\frac{1}{4}|B_\vartheta|}\frac{dt}{\gamma(n)t^{\frac{n-1}{n}}}+\frac{|B_\vartheta|}{2\vartheta}\frac{1}{\kappa(n)|B_\vartheta|^{\frac{n-1}{n}}}<c_2(n),\end{aligned}$$

for a suitable constant c_2 depending only on n. □

At this stage we have all prerequisites at hand to give the

Proof Lemma 4.1. We start with some simple observations. First, by taking the radius $R=R(n)$ sufficiently large, we may assume that $\vartheta_o\leq\vartheta(n)<\frac{1}{4}\pi$, where $\vartheta(n)>0$ is a fixed angle, which we choose later in the course of the proof in a universal way depending as indicated only on n. Moreover, we may also assume that

$$\mathbf{D}(E)\leq\frac{1}{c_3(n)}\psi_{\vartheta_o}\big(\tfrac{1}{2}\big)\tag{4.13}$$

holds true for a suitable constant $c_3(n)>2$ to be chosen later. Indeed, if (4.13) does not hold, we choose $E'=B_{\vartheta_o}$ and obtain

$$\begin{aligned}\boldsymbol{\beta}(E)^2&\leq 2\mathbf{P}(E)=2\mathbf{P}(B_{\vartheta_o})+2\mathbf{D}(E)=2\mathbf{D}(E)\bigg(\frac{\mathbf{P}(B_{\vartheta_o})}{\mathbf{D}(E)}+1\bigg)\\&\leq 2\mathbf{D}(E)\bigg(c_3\frac{\mathbf{P}(B_{\vartheta_o})}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)}+1\bigg)\leq 2\mathbf{D}(E)(c_3c_1+1),\end{aligned}$$

where c_1 is the constant appearing in Lemma 4.2.

We now fix a point $p_o\in S^n$, for instance the north pole $p_o=e_{n+1}$, and define for all $\vartheta\in(0,\pi)$ the sets

$$E_\vartheta^-:=E\cap B_\vartheta,\qquad E_\vartheta^+:=E\setminus E_\vartheta^-.$$

For a.e. ϑ we have $\mathcal{H}^{n-1}(\partial^*E\cap S_\vartheta)=0$, hence

$$\mathbf{P}(E_\vartheta^-)=\mathbf{P}(E;B_\vartheta)+v_E(\vartheta),\qquad\mathbf{P}(E_\vartheta^+)=\mathbf{P}(E;S^n\setminus\overline{B}_\vartheta)+v_E(\vartheta),$$

where $v_E(\vartheta):=\mathcal{H}^{n-1}(E\cap S_\vartheta)$ denotes the measure of the slice $E\cap S_\vartheta$. Therefore, for a.e. ϑ we have

$$v_E(\vartheta)=\tfrac{1}{2}\big[\mathbf{P}(E_\vartheta^-)+\mathbf{P}(E_\vartheta^+)-\mathbf{P}(E)\big].\tag{4.14}$$

For $\vartheta\in(0,\pi)$ we also define the function

$$g(\vartheta):=\frac{|E_\vartheta^-|}{|B_{\vartheta_o}|}.$$

Since $|E| = |B_{\vartheta_o}|$ the function g is increasing with values in $[0,1]$ and moreover $g'(\vartheta) = v_E(\vartheta)/|B_{\vartheta_o}|$ for a.e. ϑ. Using the isoperimetric inequality (2.2) and recalling the definition of the isoperimetric function φ given in (4.4) we deduce that $\mathbf{P}(E^-_\vartheta) \geq \varphi(|E^-_\vartheta|) = \varphi(g(\vartheta)|B_{\vartheta_o}|)$. Similarly, we also have $\mathbf{P}(E^+_\vartheta) \geq \varphi((1-g(\vartheta))|B_{\vartheta_o}|)$. Therefore, from (4.14), recalling the definition of ψ_{ϑ_o} given in (4.6), we obtain for a.e. ϑ

$$
\begin{aligned}
v_E(\vartheta) &\geq \tfrac{1}{2}\big[\varphi(g(\vartheta)|B_{\vartheta_o}|) + \varphi((1-g(\vartheta))|B_{\vartheta_o}|) - \varphi(|B_{\vartheta_o}|) - \mathbf{D}(E)\big] \\
&= \tfrac{1}{2}\big[\psi_{\vartheta_o}(g(\vartheta)) - \mathbf{D}(E)\big] \\
&= \tfrac{1}{4}\psi_{\vartheta_o}(g(\vartheta)) + \tfrac{1}{4}\big[\psi_{\vartheta_o}(g(\vartheta)) - 2\mathbf{D}(E)\big].
\end{aligned} \tag{4.15}
$$

Since $\psi_{\vartheta_o}(0) = \psi_{\vartheta_o}(1) = 0$ and $\psi_{\vartheta_o}\big(\frac{1}{2}\big) \geq c_3(n)\mathbf{D}(E) > 2\mathbf{D}(E)$ by (4.13) and the choice of $c_3 > 2$, we can find two angles $\vartheta_1 < \vartheta_2$ such that

$$
\begin{cases}
\psi_{\vartheta_o}(g(\vartheta_1)) = 2\mathbf{D}(E), \quad \psi_{\vartheta_o}(g(\vartheta_2)) = 2\mathbf{D}(E); \\
\qquad \psi_{\vartheta_o}(g(\vartheta)) \geq 2\mathbf{D}(E) \text{ for all } \vartheta \in (\vartheta_1, \vartheta_2).
\end{cases} \tag{4.16}
$$

Therefore from (4.15) we get that

$$
v_E(\vartheta) \geq \tfrac{1}{4}\psi_{\vartheta_o}(g(\vartheta)) \qquad \text{for a.e. } \vartheta \in (\vartheta_1, \vartheta_2),
$$

which means that

$$
g'(\vartheta) \geq \frac{\psi_{\vartheta_o}(g(\vartheta))}{4|B_{\vartheta_o}|} \qquad \text{for a.e. } \vartheta \in (\vartheta_1, \vartheta_2)
$$

holds true. Therefore, by integrating the preceding inequality over the interval $(\vartheta_1, \vartheta_2)$ we can conclude by an application of Lemma 4.3 that

$$
\vartheta_2 - \vartheta_1 \leq 4|B_{\vartheta_o}| \int_{\vartheta_1}^{\vartheta_2} \frac{g'(\vartheta)}{\psi_{\vartheta_o}(g(\vartheta))}\, d\vartheta \leq 4|B_{\vartheta_o}| \int_0^1 \frac{ds}{\psi_{\vartheta_o}(s)} < 4c_2(n)\vartheta_o. \tag{4.17}
$$

Here c_2 is of course the constant provided by Lemma 4.3. At this stage we use the smallness condition on ϑ_o by choosing the angle $\vartheta(n)$ such that $4c_2(n)\vartheta(n) \leq \frac{\pi}{4}$. Then, the assumption $\vartheta_o \leq \vartheta(n)$ implies that

$$
\vartheta_2 - \vartheta_1 \leq 4c_2(n)\vartheta_o \leq \tfrac{\pi}{4},
$$

holds true. Next, we use the concavity of ψ_{ϑ_o} and the fact that $\psi_{\vartheta_o}(0) = 0 = \psi_{\vartheta_o}(1)$ to conclude that $\psi_{\vartheta_o}(s) > 2s\psi_{\vartheta_o}(\frac{1}{2})$ for any $s \in (0, \frac{1}{2})$ and $\psi_{\vartheta_o}(s) > 2(1-s)\psi_{\vartheta_o}(\frac{1}{2})$ for any $s \in (\frac{1}{2}, 1)$. In particular, from the first two identities in (4.16), we conclude that

$$
g(\vartheta_1) < \frac{\mathbf{D}(E)}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)}, \qquad g(\vartheta_2) > 1 - \frac{\mathbf{D}(E)}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)}\,. \tag{4.18}
$$

Moreover, enlarging ϑ_1 and reducing ϑ_2 if necessary a bit we may assume without loss of generality that (4.17) and (4.18) still hold and that

$$
\mathcal{H}^{n-1}(\partial^* E \cap S_{\vartheta_1}) = 0 = \mathcal{H}^{n-1}(\partial^* E \cap S_{\vartheta_2}). \tag{4.19}
$$

In view of the preceding inequality three cases are possible (recall that we have assumed at the very beginning that $\vartheta_o < \frac{\pi}{4}$):

(i) $\vartheta_1 \leq \vartheta_o$, hence $\vartheta_2 < \frac{\pi}{2}$;
(ii) $\vartheta_2 \geq \pi - \vartheta_o$, hence $\vartheta_1 > \frac{\pi}{2}$; and finally
(iii) $\vartheta_o < \vartheta_1$ and $\vartheta_2 < \pi - \vartheta_o$.

Accordingly to these cases, we define

$$
\widetilde{E} = \begin{cases}
E \cap B_{\vartheta_2} & \text{if } \vartheta_1 < \vartheta_o, \\
E \setminus B_{\vartheta_1} & \text{if } \vartheta_2 \geq \pi - \vartheta_o, \\
(E \cap B_{\vartheta_2}) \setminus B_{\vartheta_1} & \text{if } \vartheta_o < \vartheta_1 < \vartheta_2 < \pi - \vartheta_o,
\end{cases}
$$

Thus, no matter which of the three possible definitions of $\widetilde{E}$ we choose, we conclude taking also (4.13) into account that

$$|E| - |\widetilde{E}| \leq \frac{2\mathbf{D}(E)|B_{\vartheta_o}|}{\psi_{\vartheta_o}\left(\frac{1}{2}\right)} \leq \frac{2|B_{\vartheta_o}|}{c_3(n)} \tag{4.20}$$

holds true. Therefore, in order to estimate $\mathbf{D}(E)$ we need to control the difference between the perimeters of E and $\widetilde{E}$. We argue according to the three cases from above. We start with the **case (i)**, i.e. $\vartheta_1 < \vartheta_o$. Then, since $\vartheta_2 < \frac{\pi}{2}$, we have recalling (4.19) that

$$\begin{aligned}\mathbf{P}(\widetilde{E}) - \mathbf{P}(E) &= \mathbf{P}(B_{\vartheta_2}) - \mathbf{P}(B_{\vartheta_2} \cup E) \\ &\leq \mathbf{P}(B_{\vartheta_2}) - \mathbf{P}((B_{\vartheta_2} \cup E) \cap \{x_{n+1} \geq 0\}) \leq 0,\end{aligned}$$

where the last inequality follows from the isoperimetric inequality. Next, we consider the **case (ii)**, i.e. $\vartheta_2 \geq \pi - \vartheta_o$. Here, we know that $\vartheta_1 > \frac{\pi}{2}$. By the definition of $\widetilde{E} = E \setminus B_{\vartheta_1}$ we obtain by a similar reasoning as in case (i) that

$$\begin{aligned}\mathbf{P}(\widetilde{E}) - \mathbf{P}(E) &= \mathbf{P}(B_{\vartheta_1}) - \mathbf{P}(B_{\vartheta_1} \setminus E) \\ &\leq \mathbf{P}(B_{\vartheta_1}) - \mathbf{P}((B_{\vartheta_1} \setminus E) \cap \{x_{n+1} \leq 0\}) \leq 0,\end{aligned}$$

where the last inequility again follows by the isoperimetric inequality. Finally, we consider the **case (iii)**, that is $\vartheta_o < \vartheta_1 < \vartheta_2 < \pi - \vartheta_o$. Here we distinguish between three cases: $\vartheta_2 \leq \frac{\pi}{2}$, $\vartheta_1 \geq \frac{\pi}{2}$ and $\vartheta_1 < \frac{\pi}{2} < \vartheta_2$. We shall discuss in detail only the latter case, the other two cases are similar and in fact easier. In this last case we have, using (4.19), the isoperimetric inequality, the concavity of φ, $(4.5)_1$ and the fact that $\varphi'(|B_{\vartheta_2} \cup E|) \leq 0$

$$\begin{aligned}\mathbf{P}(\widetilde{E}) - \mathbf{P}(E) &= \big[\mathbf{P}(B_{\vartheta_1}) - \mathbf{P}(B_{\vartheta_1} \setminus E)\big] + \big[\mathbf{P}(B_{\vartheta_2}) - \mathbf{P}(B_{\vartheta_2} \cup E)\big] \\ &\leq \big[\varphi(|B_{\vartheta_1}|) - \varphi(|B_{\vartheta_1} \setminus E|)\big] + \big[\varphi(|B_{\vartheta_2}|) - \varphi(|B_{\vartheta_2} \cup E|)\big] \\ &\leq (|B_{\vartheta_1}| - |B_{\vartheta_1} \setminus E|)\varphi'(|B_{\vartheta_1} \setminus E|) - (|B_{\vartheta_2} \cup E| - |B_{\vartheta_2}|)\varphi'(|B_{\vartheta_2} \cup E|) \\ &\leq (|E| - |\widetilde{E}|)\left(\frac{(n-1)}{\tan(F^{-1}(|B_{\vartheta_1} \setminus E|))} - \frac{(n-1)}{\tan(F^{-1}(|B_{\vartheta_2} \cup E))|}\right) \\ &= (|E| - |\widetilde{E}|)\left(\frac{(n-1)}{\tan(F^{-1}(|B_{\vartheta_1} \setminus E|))} + \frac{(n-1)}{\tan(F^{-1}(|S^n| - |B_{\vartheta_2} \cup E|))}\right),\end{aligned}$$

where the last equality holds thanks to $(4.2)_2$. Using the fact that $\vartheta_1 > \vartheta_o$ and (4.20) we infer that

$$|B_{\vartheta_1} \setminus E| \geq |B_{\vartheta_1}| - |E \Delta \widetilde{E}| \geq |B_{\vartheta_o}|\left(1 - \frac{2}{c_3(n)}\right) \geq \tfrac{1}{2}|B_{\vartheta_o}|,$$

provided we have $c_3(n) \geq 4$. Similarly, recalling that $\vartheta_2 < \pi - \vartheta_o$ and (4.20) again, we get

$$|S^n| - |B_{\vartheta_2} \cup E| \geq |S^n| - |B_{\vartheta_2}| - |E \Delta \widetilde{E}| \geq |B_{\vartheta_o}| - |E \Delta \widetilde{E}| \geq \tfrac{1}{2}|B_{\vartheta_o}|.$$

These bounds from below allow us to estimate the last two terms appearing in the last line of the preceding estimate for the difference of the perimeters from above (note that $t \mapsto \tan \circ F^{-1}$ is increasing). Taking also into account the first inequality in (4.20) and Lemma 4.2, we can conclude that

$$\begin{aligned}\mathbf{P}(\widetilde{E}) - \mathbf{P}(E) &\leq \frac{4(n-1)\mathbf{D}(E)|B_{\vartheta_o}|}{\psi_{\vartheta_o}\left(\frac{1}{2}\right)\tan(F^{-1}(\frac{1}{2}|B_{\vartheta_o}|))} \leq c(n)\frac{\mathbf{D}(E)|B_{\vartheta_o}|}{\psi_{\vartheta_o}\left(\frac{1}{2}\right)|B_{\vartheta_o}|^{\frac{1}{n}}} \\ &\leq c(n)\frac{\mathbf{D}(E)\mathbf{P}(B_{\vartheta_o})}{\psi_{\vartheta_o}\left(\frac{1}{2}\right)} \leq c_4(n)\mathbf{D}(E),\end{aligned}$$

for a suitable constant $c_4(n)$ depending only on n.

Using the preceding estimates for the cases (i)–(iii), the concavity of φ, (4.20), $(4.5)_1$, the fact that $|\widetilde{E}| \geq \frac{1}{2}|B_{\vartheta_o}|$ (which is a consequence of (4.20) and the volume constraint $|E| = |B_{\vartheta_o}|$) and Lemma 4.2 we find that

$$
\begin{aligned}
\mathbf{D}(\widetilde{E}) &= \mathbf{D}(E) + \big[\mathbf{P}(\widetilde{E}) - \mathbf{P}(E)\big] + \big[\varphi(|E|) - \varphi(|\widetilde{E}|)\big] \\
&\leq (1+c_4)\mathbf{D}(E) + \big(|E| - |\widetilde{E}|\big)\varphi'(|\widetilde{E}|) \\
&\leq (1+c_4)\mathbf{D}(E) + c(n)\frac{\mathbf{D}(E)|B_{\vartheta_o}|}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)|\widetilde{E}|^{\frac{1}{n}}} \\
&\leq (1+c_4)\mathbf{D}(E) + c(n)\frac{\mathbf{D}(E)\mathbf{P}(B_{\vartheta_o})}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)} \leq c_5(n)\mathbf{D}(E),
\end{aligned}
\tag{4.21}
$$

for a suitable constant c_5 depending only on n. Now, arguing as in the proof of (4.21), we see that

$$
\begin{aligned}
\boldsymbol{\beta}(E)^2 &= \boldsymbol{\beta}(\widetilde{E})^2 + \big[\mathbf{P}(E) - \mathbf{P}(\widetilde{E})\big] + \big[\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E)\big] \\
&\leq \boldsymbol{\beta}(\widetilde{E})^2 + \mathbf{D}(E) + \big[\varphi(|E|) - \varphi(|\widetilde{E}|)\big] + \big[\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E)\big] \\
&\leq \boldsymbol{\beta}(\widetilde{E})^2 + c(n)\mathbf{D}(E) + \big[\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E)\big],
\end{aligned}
\tag{4.22}
$$

holds true, and finally it remains to estimate the difference $\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E)$. For this we denote by $y_{\widetilde{E}}$ a center for $\widetilde{E}$. Then, using Lemma 2.5, (4.20) and Lemma 4.2 we find

$$
\begin{aligned}
\boldsymbol{\gamma}(\widetilde{E}) - \boldsymbol{\gamma}(E) &\leq \int_{\widetilde{E}} \frac{(n-1)(x\cdot y_{\widetilde{E}})}{\sqrt{1-(x\cdot y_{\widetilde{E}})^2}}\,d\mathcal{H}^n - \int_{E} \frac{(n-1)(x\cdot y_{\widetilde{E}})}{\sqrt{1-(x\cdot y_{\widetilde{E}})^2}}\,d\mathcal{H}^n \\
&\leq \int_{E\Delta\widetilde{E}} \frac{(n-1)|x\cdot y_{\widetilde{E}}|}{\sqrt{1-(x\cdot y_{\widetilde{E}})^2}}\,d\mathcal{H}^n \leq c(n)|E\Delta\widetilde{E}|^{\frac{n-1}{n}} \\
&\leq c(n)\bigg(\frac{2\mathbf{D}(E)|B_{\vartheta_o}|}{\psi_{\vartheta_o}\big(\frac{1}{2}\big)}\bigg)^{\frac{n-1}{n}} \leq c(n)\big(\vartheta_o\mathbf{D}(E)\big)^{\frac{n-1}{n}}.
\end{aligned}
$$

Inserting the preceding inequality into (4.22) we obtain

$$
\boldsymbol{\beta}(E)^2 \leq \boldsymbol{\beta}(\widetilde{E})^2 + c(n)\mathbf{D}(E) + c(n)(\vartheta_o\mathbf{D}(E))^{\frac{n-1}{n}},
$$

for a suitable constant $c(n)$ depending only on n. From this last inequality, (4.17) and (4.20) we conclude that the set $\widetilde{E}$ satisfies the desired estimate (4.1) and has a volume which is bounded from below by $|\widetilde{E}| \geq (1-2/c_3(n))|B_{\vartheta_o}|$. Moreover, in case (i) $\widetilde{E}$ is contained in B_{ϑ_2} with $\vartheta_2 \leq (4c_2(n)+1)\vartheta_o$ and in case (ii) $\widetilde{E}$ is contained in $S^n \setminus B_{\vartheta_1}$ with $\vartheta_1 \geq \pi - (4c_2(n)+1)\vartheta_o$ so that we can take $E' = \widetilde{E}$. Finally, in case (iii) $\widetilde{E}$ is contained in the slab $\{x : \vartheta_1 \leq \operatorname{dist}_{S^n}(x, e_{n+1}) \leq \vartheta_2\}$ of width $\vartheta_2 - \vartheta_1 \leq 4c_2(n)\vartheta_o$. At this stage we can repeat the above construction starting with E replaced by $\widetilde{E}$ and with e_{n+1} replaced now by $p_o = e_n$. In this way we obtain a new set $\widetilde{E}_1$ which will satisfy (4.1), $|\widetilde{E}_1| \geq (1-2/c_3(n))^2|B_{\vartheta_o}|$ and that will be contained either in a ball of radius smaller than $(4c_2(n)+1)\vartheta_o$, or in two orthogonal slabs of width smaller than $4c_2(n)\vartheta_o$. Iterating the argument $n-2$ times more if necessary, we arrive to the final conclusion of the Lemma. □

5. Relevant perimeter functionals on the sphere

In this section we study certain perimeter functionals on the sphere. These functionals will allow us to derive the quantitative isoperimertic inequality. We start our considerations with perturbed perimeter functionals, whose minimizers will be geodesic balls.

5.1. **Perturbed perimeter functionals.** In this section we consider functionals of the type

$$\mathbf{F}_{\vartheta_o}(E) := \mathbf{P}(E) + \frac{\Lambda}{\vartheta_o}\big||E| - |B_{\vartheta_o}|\big|,$$

for sets $E \subset S^n$ of finite perimeter and for given $\Lambda \geq 1$ and $\vartheta_o \in (0, \frac{\pi}{2}]$. The following Lemma shows that minimizers of the perturbed functionals $\mathbf{F}_{\vartheta_o}$ are geodesic balls with radius ϑ_o provided the constant Λ is chosen large enough. The precise statement is as follows:

Lemma 5.1. *There exists a constant $\Lambda_o(n) \geq 1$ such that any minimizer of the functional $\mathbf{F}_{\vartheta_o}$ for some given $\vartheta_o \in (0, \frac{\pi}{2}]$ is a geodesic ball $B_{\vartheta_o}(p_o)$, provided $\Lambda \geq \Lambda_o(n)$.*

Proof. Let $E \subset S^n$ be a minimizer of the functional $\mathbf{F}_{\vartheta_o}$. Then by minimality, we have

$$\mathbf{P}(E) \leq \mathbf{F}_{\vartheta_o}(E) \leq \mathbf{F}_{\vartheta_o}(B_{\vartheta_o}) = \mathbf{P}(B_{\vartheta_o}).$$

By the isoperimetric property of geodesic balls from (2.3) we have that either $|E| \leq |B_{\vartheta_o}|$ or $|S^n \setminus E| \leq |B_{\vartheta_o}|$. Suppose that $|E| < |B_{\vartheta_o}|$. Then, there would exist $\vartheta < \vartheta_o$ such that $|E| = |B_\vartheta|$. But this implies

$$\mathbf{P}(E) + \frac{\Lambda}{\vartheta_o}\big(|B_{\vartheta_o}| - |B_\vartheta|\big) \leq \mathbf{P}(B_{\vartheta_o}).$$

Now, by the isoperimetric property of spheres (2.2) we know $\mathbf{P}(B_\vartheta) \leq \mathbf{P}(E)$, so that

$$\frac{\Lambda}{\vartheta_o}\big(|B_{\vartheta_o}| - |B_\vartheta|\big) \leq \mathbf{P}(B_{\vartheta_o}) - \mathbf{P}(B_\vartheta).$$

By Lemma 2.1 the last inequality implies

$$\frac{\Lambda}{2\vartheta_o}\Big(\frac{\vartheta_o}{\pi}\Big)^{n-1}(\vartheta_o - \vartheta) \leq (n-1)\vartheta_o^{n-2}(\vartheta_o - \vartheta).$$

This, however, is impossible as long as we choose Λ in dependence on n large enough, i.e.

$$\Lambda > 2(n-1)\pi^{n-1}. \tag{5.1}$$

This proves, that $|E| < |B_{\vartheta_o}|$ cannot hold.

Let us similarly show that $|S^n \setminus E| < |B_{\vartheta_o}|$ cannot hold either. In fact in this case there would exist $\vartheta < \vartheta_o$ such that $|S^n \setminus E| = |B_\vartheta|$. This implies that

$$\mathbf{P}(E) + \frac{\Lambda}{\vartheta_o}\big(|S^n| - |B_\vartheta| - |B_{\vartheta_o}|\big) \leq \mathbf{P}(B_{\vartheta_o})$$

holds true. Now, by the isoperimetric inequality (2.2) we know that $\mathbf{P}(B_\vartheta) \leq \mathbf{P}(S^n \setminus E) = \mathbf{P}(E)$, so that

$$\frac{\Lambda}{\vartheta_o}\big(|S^n| - |B_\vartheta| - |B_{\vartheta_o}|\big) \leq \mathbf{P}(B_{\vartheta_o}) - \mathbf{P}(B_\vartheta),$$

hence

$$\frac{\Lambda}{\vartheta_o}\big(|B_{\frac{\pi}{2}}| - |B_\vartheta| + |B_{\frac{\pi}{2}}| - |B_{\vartheta_o}|\big) \leq \mathbf{P}(B_{\vartheta_o}) - \mathbf{P}(B_\vartheta).$$

By Lemma 2.1 this inequality implies

$$\frac{\Lambda}{2^n\vartheta_o}(\pi - \vartheta_o - \vartheta) \leq (n-1)\vartheta_o^{n-2}(\vartheta_o - \vartheta).$$

Since $\vartheta_o \leq \frac{\pi}{2}$ we know that $\pi - \vartheta_o - \vartheta \geq \vartheta_o - \vartheta$ and thus the above inequality cannot hold if we choose Λ as in (5.1). This shows that either $|E| = |B_{\vartheta_o}|$ or $|S^n \setminus E| = |B_{\vartheta_o}|$. But since $\mathbf{P}(E) \leq \mathbf{P}(B_{\vartheta_o})$ by the isoperimetric inequality E must be a ball. Now, the minimality of E implies that E is a ball of radius ϑ_o. $\square$

5.2. **The penalization procedure.** In this section we deal with functionals of the type

$$\mathbf{F}(G) := \mathbf{P}(G) + \frac{\Lambda}{\vartheta_o}\big||G| - |B_{\vartheta_o}|\big| + C_o|\boldsymbol{\beta}^2(G) - \varepsilon^2|, \tag{5.2}$$

where $\Lambda \geq 1$, $\vartheta_o \in (0, \frac{\pi}{2}]$, $C_o \in [0,1]$ and $\varepsilon \geq 0$. The advantage of these kind of functionals steams from the fact that minimizers should have a volume close to the prescribed volume $|B_{\vartheta_o}|$ and an L^2-asymmetry close to ε. As a first step we prove the lower semi-continuity of these penalized functionals. This can be achieved once the continuity of the functional γ is established.

Lemma 5.2. *Whenever G_k, G are measurable sets in S^n with $G_k \to G$ in $L^1(S^n)$ there holds*

$$\lim_{k\to\infty} \boldsymbol{\gamma}(G_k) = \boldsymbol{\gamma}(G),$$

i.e the functional $\boldsymbol{\gamma}$ defined in (2.6) *is continuous with respect to L^1-convergence.*

Proof. We consider a sequence $G_k \subset S^n$ with $G_k \to G \subset S^n$ in $L^1(S^n)$. Next, we choose centers y_k of G_k and y of G; this means that there hold $\boldsymbol{\gamma}(G_k) = \boldsymbol{\gamma}(G_k; y_k)$ and $\boldsymbol{\gamma}(G) = \boldsymbol{\gamma}(G; y)$. By the maximality of the center we have

$$\boldsymbol{\gamma}(G_k, y) \leq \boldsymbol{\gamma}(G_k, y_k) = \boldsymbol{\gamma}(G_k).$$

Together with $\lim_{k\to\infty} \boldsymbol{\gamma}(G_k, y) = \boldsymbol{\gamma}(G, y)$, which is a consequence of the L^1 convergence and the dominated convergence theorem, we infer the lower semi-continuity, i.e.

$$\liminf_{k\to\infty} \boldsymbol{\gamma}(G_k) \geq \boldsymbol{\gamma}(G).$$

To prove the upper semi-continuity we apply Lemma 2.5 to infer that

$$\begin{aligned}\boldsymbol{\gamma}(G_k) &\leq \boldsymbol{\gamma}(G) + \int_{G_k\setminus G} \frac{(n-1)(x\cdot y_k)}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n - \int_{G\setminus G_k} \frac{(n-1)(x\cdot y_k)}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n \\ &\leq \boldsymbol{\gamma}(G) + \int_{G_k\Delta G} \frac{(n-1)|x\cdot y_k|}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n \leq \boldsymbol{\gamma}(G) + c(n)\,|G_k \Delta G|^{\frac{n-1}{n}}.\end{aligned}$$

Since $G_k \to G$ in $L^1(S^n)$ we conclude the upper semi-continuity, i.e.

$$\limsup_{k\to\infty} \boldsymbol{\gamma}(G_k) \leq \boldsymbol{\gamma}(G).$$

Together with the lower semi-continuity this establishes the continuity of $\boldsymbol{\gamma}$ with respect to L^1-convergence. □

The preceding Lemma allows us to prove the lower semi-continuity of the penalized functionals $\mathbf{F}$.

Lemma 5.3. *The functional* $\mathbf{F}$ *defined in* (5.2) *is lower semi-continuous with respect to L^1-convergence, i.e.*

$$\mathbf{F}(G) \leq \liminf_{k\to\infty} \mathbf{F}(G_k)$$

whenever G_k, G are sets in S^n of finite perimeter with $G_k \to G$ in $L^1(S^n)$.

Proof. Let G_k and $G \subset S^n$ are finite perimeter sets as in the statement of the lemma. Passing to a subsequence we may assume without loss of generality that

$$\liminf_{k\to\infty} \mathbf{F}(G_k) = \lim_{k\to\infty} \mathbf{F}(G_k).$$

Passing to another subsequence we may also assume that $\lim_{k\to\infty} \mathbf{P}(G_k) = \alpha$. By the lower semi-continuity of the perimeter with respect to L^1-convergence we have $\alpha \geq \mathbf{P}(G)$. Using (2.5) and Lemma 5.2 we infer that there holds:

$$\lim_{k\to\infty} \mathbf{F}(G_k) = \lim_{k\to\infty} \Big[\mathbf{P}(G_k) + \frac{\Lambda}{\vartheta_o}\big||G_k| - |B_{\vartheta_o}|\big| + C_o\big|\boldsymbol{\beta}^2(G_k) - \varepsilon\big|\Big]$$

$$= \lim_{k\to\infty} \Big[\mathbf{P}(G_k) + \frac{\Lambda}{\vartheta_o}\big||G_k| - |B_{\vartheta_o}|\big| + C_o\big|\mathbf{P}(G_k) - \boldsymbol{\gamma}(G_k) - \varepsilon\big|\Big]$$
$$= \alpha + \frac{\Lambda}{\vartheta_o}\big||G| - |B_{\vartheta_o}|\big| + C_o\big|\alpha - \boldsymbol{\gamma}(G) - \varepsilon\big|\Big]$$
$$\geq \mathbf{F}(G) + (1 - C_o)(\alpha - \mathbf{P}(G)) \geq \mathbf{F}(G).$$

Here we used the assumption $0 \leq C_o \leq 1$ in the last step. This proves the desired lower semi-continuity of the functionals $\mathbf{F}$ and concludes the proof. □

5.3. **Quasi and almost minimizers of the perimeter.** In this section we recall the notion of (K, r_o)-quasi-minimizers, as well as the one of (K, r_o)-almost minimizers. Both of them play a crucial role in our proof of the quantitative isoperimetric inequality. The first one, i.e. the one of (K, r_o)-quasi-minimizers, allows us to conclude by a metric space version of a result going back to David & Semmes [10] (see [19, Theorem 5.2]) that minimizers G and their complement are locally porous. The second notion, i.e. the one of (K, r_o)-almost minimality allows us to conclude that sequences of (K, r_o)-almost minimizers which converge in L^1 to a geodesic ball must be regular for large indices, more precisely they become spherical graphs over the boundary of the limit geodesic sphere. This reasoning goes back to [24] and uses the regularity theory for (K, r_o)-almost minimizers from [23]. We start with the following

Definition 5.4 (Quasi-minimizers of the perimeter)**.** For a given radius $r_o > 0$ and a given constant $K \geq 1$ we say that a set $G \subset S^n$ of finite perimeter is a perimeter (K, r_o)-*quasi-minimizer* if

$$\mathbf{P}(G; B_r(x)) \leq K\, \mathbf{P}(D; B_r(x))$$

whenever $D \subset S^n$ is a finite perimeter set such that $G\Delta D \Subset B_r(x)$ for some ball $B_r(x)$ with $0 < r \leq r_o$. Here, $\mathbf{P}(G; B_r(x))$ denotes the perimeter of G in $B_r(x)$. □

Lemma 5.5. *Suppose that G is a minimizer of the functional $\mathbf{F}$ defined in* (5.2) *for some $\Lambda \geq 1$, $\vartheta_o \in (0, \frac{\pi}{2}]$, $C_o \in (0, \frac{1}{8}]$ and $\varepsilon \geq 0$. Then, there exists $r_o = r_o(\Lambda) > 0$ such that G is a perimeter $(K, r_o\vartheta_o)$-quasi-minimizer with $K = 3$.*

Proof. We consider $D \subset S^n$ satisfying $G\Delta D \Subset B_{r\vartheta_o}(x)$ for some ball $B_{r\vartheta_o}(x)$, with $r \leq 1$. By the $\mathbf{F}$-minimality of G we find that

$$\mathbf{P}(G) \leq \mathbf{P}(D) + \frac{\Lambda}{\vartheta_o}\big||G| - |D|\big| + C_o\big|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)\big| \tag{5.3}$$

From Section 2.3 we recall the definition

$$Q_n(r\vartheta_o) := \frac{|B_{r\vartheta_o}|}{\mathbf{P}(B_{r\vartheta_o})} \leq r\vartheta_o,$$

where the last inequality holds since $r \leq 1$ and $\vartheta_o \leq \frac{\pi}{2}$. From the linear isoperimetric inequality (2.4) we conclude that there holds:

$$\big||G| - |D|\big| \leq |G\Delta D| \leq Q_n(r\vartheta_o)\, \mathbf{P}(G\Delta D) \leq r\vartheta_o\big[\mathbf{P}(G; B_{r\vartheta_o}(x)) + \mathbf{P}(D; B_{r\vartheta_o}(x))\big].$$

In order to estimate the last term on the right-hand side of (5.3) we first consider the case $\boldsymbol{\beta}(G) \leq \boldsymbol{\beta}(D)$. Denoting by y_G a center of G we obtain (using the definition of the center)

$$\begin{aligned}
&\big|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)\big| \\
&\quad\leq \boldsymbol{\beta}^2(D; y_G) - \boldsymbol{\beta}^2(G) \\
&\quad= \int_{\partial^* D} \big[1 - \nu_D \cdot \nu_{B_{\vartheta(x)}(y_G)}\big]\, d\mathcal{H}^{n-1} - \int_{\partial^* G} \big[1 - \nu_G \cdot \nu_{B_{\vartheta(x)}(y_G)}\big]\, d\mathcal{H}^{n-1} \\
&\quad\leq \mathbf{2}P(D; B_{r\vartheta_o}(x)).
\end{aligned}$$

On the other hand, if $\boldsymbol{\beta}(G) > \boldsymbol{\beta}(D)$ we use the same argument to conclude

$$\big|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)\big| \leq 2\mathbf{P}(G; B_{r\vartheta_o}(x)).$$

Therefore, in any case we have that

$$\left|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)\right| \le 2\big[\mathbf{P}(D; B_{r\vartheta_o}(x)) + \mathbf{P}(G; B_{r\vartheta_o}(x))\big]$$

holds true. Inserting the preceding inequalities into (5.3) and re-absorbing the terms with $\mathbf{P}(G; B_{r\vartheta_o}(x))$ appearing on the right-hand side into the left-hand side we arrive at

$$\big(1 - 2C_o - \Lambda r\big)\,\mathbf{P}(G; B_{r\vartheta_o}(x)) \le \big(1 + 2C_o + \Lambda r\big)\,\mathbf{P}(D; B_{r\vartheta_o}(x))$$

Since $C_o \le \frac{1}{8}$ we can choose $r_o = \min\{1, \frac{1}{4\Lambda}\}$, thus getting

$$\mathbf{P}(G; B_{r\vartheta_o}(x)) \le 3\mathbf{P}(D; B_{r\vartheta_o}(x))$$

whenever D is a finite perimeter set such that $G\Delta D \Subset B_{r\vartheta_o}(x)$ with $0 < r \le r_o$. This proves the assertion of the lemma. □

The following result has been proved in [19, Theorem 5.2] in the context of metric spaces. The Euclidean version is due to David and Semmes [10].

Theorem 5.6. *Suppose that $G \subset S^n$ is an area (K, r_o)-quasi-minimizer. Then, up to modifying G in a set of measure zero, the topological boundary of G coincides with the reduced boundary, i.e. $\partial G = \partial^* G$. Moreover G and $S^n \setminus G$ are locally porous in the sense that there exists a constant $C > 1$, depending only on K and n such that for every $x \in \partial G$ and $0 < r < r_o$ there are points $y, z \in B_r(x)$ for which*

$$B_{r/C}(y) \subset G \qquad \text{and} \qquad B_{r/C}(z) \subset S^n \setminus G$$

hold true.

As already mentioned at the beginning of this section we need a regularity result which ensures that in the contradiction argument our sequence of minimizers to penalized functionals are nearly spherical graphs. Such a result can be deduced once it is shown that the minimizers are almost minimizing in a certain sense, which allows to apply the regularity theory for almost minimizers. The precise notion which is suited for our purposes is the following one:

Definition 5.7 (Almost perimeter minimizing sets)**.** For a given radius $r_o > 0$ and a constant $K > 0$ we say that a finite perimeter set $G \subset S^n$ is a (K, r_o)-*almost minimizer of the perimeter* if

$$\mathbf{P}(G) \le \mathbf{P}(D) + K|G\Delta D|$$

holds true whenever $D \subset S^n$ is set of finite perimeter such that $G\Delta D \Subset B_r(x)$ for some geodesic ball $B_r(x)$ with radius $r \in (0, r_o]$. □

The following Lemma ensures that minimizers of the penalized functionals $\mathbf{F}$ are (K, r_o)-minimizeres for suitable values of K and r_o, provided the set G contains a geodesic ball and is itself contained in larger geodesic ball. The precise statement is as follows:

Lemma 5.8. *Suppose that G minimizes the functional* $\mathbf{F}$ *defined in* (5.2) *where $\Lambda \ge 1$, $\vartheta_o \in (0, \frac{\pi}{2}]$, $C_o \in (0,1)$ and $\varepsilon > 0$. Then, there exist $\delta_o = \delta_o(n) > 0$, $r_1 = r_1(n) > 0$ and $K = K(n, \Lambda, C_o)$ such that the annulus assumption*

$$B_{\vartheta_o(1-\delta_o)}(p_o) \subset G \subset B_{\vartheta_o(1+\delta_o)}(p_o) \tag{5.4}$$

implies that G is a $(\frac{K}{\vartheta_o}, r_1\vartheta_0)$-almost minimizer of the perimeter.

Proof. Since p_o is fixed, we write as usual B_ϑ instead of $B_\vartheta(p_o)$. Let δ_o and $r_1 > 0$ be arbitrary but fixed. We will choose δ_o and r_1 in the course of the proof in dependence on n in a universal way. We consider $D \subset S^n$ satisfying $G\Delta D \Subset B_{r\vartheta_o}(y)$ for some ball $B_{r\vartheta_o}(y)$ with $0 < r \le r_1$. If $B_{r\vartheta_o}(y) \subset B_{\vartheta_o(1-\delta_o)}$ we have by (5.4) that

$$G\Delta D \Subset B_{r\vartheta_o}(y) \subset B_{\vartheta_o(1-\delta_o)} \subset G.$$

But $G\Delta D \Subset G$ implies that $\mathbf{P}(G) \le \mathbf{P}(D)$. In the case $B_{r\vartheta_o}(y) \cap B_{\vartheta_o(1+\delta_o)} = \emptyset$ we have $(G\Delta D) \cap B_{\vartheta_o(1+\delta_o)} = \emptyset$ and therefore also in this case $\mathbf{P}(G) \le \mathbf{P}(D)$ holds true. Therefore, it remains to consider the case in which $B_{r\vartheta_o}(y) \cap (B_{\vartheta_o(1+\delta_o)} \setminus B_{\vartheta_o(1-\delta_o)}) \ne \emptyset$. Due to the minimality of G and (2.5) we get

$$\begin{aligned}\mathbf{P}(G) &\le \mathbf{P}(D) + \frac{\Lambda}{\vartheta_o}\big||G| - |D|\big| + C_o\big|\boldsymbol{\beta}^2(D) - \boldsymbol{\beta}^2(G)\big| \\ &\le \mathbf{P}(D) + C_o|\mathbf{P}(G) - \mathbf{P}(D)| + \frac{\Lambda}{\vartheta_o}|G\Delta D| + C_o|\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D)|.\end{aligned}$$

Distinguishing between the cases $\mathbf{P}(G) \le \mathbf{P}(D)$ and $\mathbf{P}(D) < \mathbf{P}(G)$ to re-absorb the term containing $\mathbf{P}(G)$ from the right into the left hand side we infer that

$$\mathbf{P}(G) \le \mathbf{P}(D) + \tfrac{\Lambda}{\vartheta_o(1-C_o)}|G\Delta D| + \tfrac{C_o}{1-C_o}|\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D)|. \tag{5.5}$$

Now, we observe that the annulus assumption (5.4) and Lemma 2.1 imply that if $\delta_o < 1$

$$|G\Delta B_{\vartheta_o}| \le |B_{\vartheta_o(1+\delta_o)} \setminus B_{\vartheta_o(1-\delta_o)}| \le n\omega_n[\vartheta_o(1+\delta_o)]^{n-1}2\vartheta_o\delta_o \le c(n)\vartheta_o^n\delta_o.$$

For the comparison set D we have

$$|D\Delta B_{\vartheta_o}| \le |B_{r\vartheta_o}(y) \cup (G\Delta B_{\vartheta_o})| \le c(n)\big(r_1^n + \delta_o\big)\vartheta_o^n.$$

Therefore, choosing δ_o and r_1 in dependence on n sufficiently small, from Lemma 2.6, i.e. the continuity of the center with respect to L^1-topology, we deduce that

$$\operatorname{dist}_{S^n}(y_G, p_o) \le \tfrac14\vartheta_o \quad \text{and} \quad \operatorname{dist}_{S^n}(y_D, p_o) \le \tfrac14\vartheta_o.$$

Moreover, since $B_{r\vartheta_o}(y) \cap (B_{\vartheta_o(1+\delta_o)} \setminus B_{\vartheta_o(1-\delta_o)}) \ne \emptyset$ we obtain – after reducing the values of r_1 and δ_o in such a way that $2r_1 + \delta_o < \frac12$ if necessary – for any $x \in G\Delta D$ that there holds

$$\operatorname{dist}_{S^n}(x, y_G) \ge \operatorname{dist}_{S^n}(x, p_o) - \operatorname{dist}_{S^n}(y_G, p_o) \ge \vartheta_o(1-\delta_o) - 2r\vartheta_o - \tfrac14\vartheta_o \ge \tfrac{\vartheta_o}{4}$$

and

$$\operatorname{dist}_{S^n}(x, y_G) \le \operatorname{dist}_{S^n}(x, p_o) + \operatorname{dist}_{S^n}(y_G, p_o) \le \vartheta_o(1+\delta_o) + 2r\vartheta_o + \tfrac14\vartheta_o \le \pi - \tfrac{\vartheta_o}{4}.$$

But this implies that

$$|x \cdot y_G| = \big|\cos\big(\operatorname{dist}_{S^n}(x, y_G)\big)\big| \le \cos\big(\tfrac{\vartheta_o}{4}\big)$$

holds true for those x. Recalling the definition of $\boldsymbol{\gamma}$ in (2.6) – in particular the fact that the maximum is attained in the center of the set – and using the last inequality we find

$$\begin{aligned}\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D) &\le \int_G \frac{(n-1)x \cdot y_G}{\sqrt{1-(x\cdot y_G)^2}}\,d\mathcal{H}^n - \int_D \frac{(n-1)x \cdot y_G}{\sqrt{1-(x\cdot y_G)^2}}\,d\mathcal{H}^n \\ &\le \int_{G\Delta D} \frac{(n-1)|x \cdot y_G|}{\sqrt{1-(x\cdot y_G)^2}}\,d\mathcal{H}^n \le (n-1)\cot\big(\tfrac{\vartheta_o}{4}\big)\,|G\Delta D|.\end{aligned}$$

Similarly, we can show that $|x \cdot y_D| \le \cos(\frac{\vartheta_o}{4})$ for any $x \in G\Delta D$ and therefore

$$\begin{aligned}\boldsymbol{\gamma}(D) - \boldsymbol{\gamma}(G) &\le \int_D \frac{(n-1)x \cdot y_D}{\sqrt{1-(x\cdot y_D)^2}}\,d\mathcal{H}^n - \int_G \frac{(n-1)x \cdot y_D}{\sqrt{1-(x\cdot y_D)^2}}\,d\mathcal{H}^n \\ &\le \int_{G\Delta D} \frac{(n-1)|x \cdot y_D|}{\sqrt{1-(x\cdot y_D)^2}}\,d\mathcal{H}^n \le (n-1)\cot\big(\tfrac{\vartheta_o}{4}\big)\,|G\Delta D|.\end{aligned}$$

Joining the last two inequalities we obtain

$$|\boldsymbol{\gamma}(G) - \boldsymbol{\gamma}(D)| \le (n-1)\cot\big(\tfrac{\vartheta_o}{4}\big)\,|G\Delta D|.$$

We insert the preceding inequality into (5.5) and arrive at

$$\mathbf{P}(G) \le \mathbf{P}(D) + \tfrac{1}{1-C_o}\big[\tfrac{\Lambda}{\vartheta_o} + C_o(n-1)\cot\big(\tfrac{\vartheta_o}{4}\big)\big]|G\Delta D|.$$

But, this proves that G is a $(\frac{K}{\vartheta_o}, r_1\vartheta_o)$-almost minimizer of the perimeter with a constant $K = \frac{1}{1-C_o}\big[\Lambda + 4C_o(n-1)\big]$ and a radius $r_1 = r_1(n) > 0$. $\square$

In the following we establish the prerequisites needed to prove the regularity result for almost minimizers of the perimeter from Theorem 5.11. First, we state density estimates for (K, r_o)-almost minimizers of the perimeter. These estimates are similar to those ones in [23] (see also of [20, Theorem 21.11]). For the metric version we refer to [19].

Lemma 5.9 (Density estimates)**.** *Let $G \subset S^n$ be a (K, r_o)-quasi-minimizer of the perimeter with parameters $K, r_o > 0$. Then*

$$\frac{1}{(K+1)^n} \le \frac{|G \cap B_r(p_o)|}{|B_r|} \le 1 - \frac{1}{(K+1)^n}$$

and

$$\frac{\mathbf{P}(G; B_r(p_o))}{|B_r|^{\frac{n-1}{n}}} \ge c(n, K)$$

whenever $p_o \in \partial G$ and $0 < r < \min\{r_o, \frac{\pi}{2}\}$.

The proof of the following compactness result follows essentially the lines of the proof of [22, Theorem 34.5]. For the Kuratowski convergence we refer to [5, Lemma 5.6].

Lemma 5.10. *Let $G_k \subset S^n$ be a sequence of (K, r_o)-almost minimizers with structural parameters $K, r_o > 0$ satisfying $\sup_{k\in\mathbb{N}} \mathbf{P}(G_k) < \infty$. If G_k converges in $L^1(S^n)$ to some set $G \subset S^n$, then G is a (K, r_o)-almost minimizer of the perimeter. Moreover, denoting by μ_{G_k} and μ_G the perimeter measure on $\partial^* G_k$ and $\partial^* G$, respectively, we have $|\mu_{G_k}| \to |\mu_G|$ in the sense of Radon measures and ∂G_k converges to ∂G in the Kuratowski sense as $k \to \infty$, i.e.*

(i) *if $x_k \in \partial G_k$ for any $k \in \mathbb{N}$, then any limit point x belongs to ∂G.*
(ii) *for every $x \in \partial G$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \partial G_k$ for any $k \in \mathbb{N}$ converging to x.*

Next, we state a regularity result for almost minimizers of the perimeter. Since we already have established the Kuratowski convergence in Lemma 5.10, we can transform to a flat situation and then proceed as in [12] or [20, Chapter 3]. For the sake of brevity we skip the details.

Theorem 5.11. *Suppose that $G_k \subset S^n$ is a sequence of (K, r_o)-almost minimizers of the perimeter for some constant $K > 0$ and some radius $r_o > 0$ satisfying*

$$\sup_{k\in\mathbb{N}} \mathbf{P}(G_k) < \infty \qquad \text{and} \qquad \chi_{G_k} \to \chi_{B_{\vartheta_o}} \text{ in } L^1(S^n),$$

where $\vartheta_o \in (0, \frac{\pi}{2}]$. Then, there exists $k_o \in \mathbb{N}$ such that for any $k \ge k_o$ the set G_k is a nearly spherical set in the sense of definition (3.1) *with a representing function $u_k \in C^{1,\frac{1}{2}}(S^{n-1})$. Furthermore, we have $u_k \to 0$ in $C^{1,\alpha}(S^{n-1})$ for every $\alpha \in (0, \frac{1}{2})$.*

6. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. The proof is divided in quite a few steps and at certain stages we have to distinguish several cases. We start with the observation that without loss of generality we may assume that E is centered at the north pole e_{n+1}, i.e. $\boldsymbol{\beta}(E) = \boldsymbol{\beta}(E; e_{n+1})$ and that $|E| = |B_{\vartheta_o}|$ for some $\vartheta_o \in (0, \frac{\pi}{2}]$. Indeed, if $\vartheta_o \in (\frac{\pi}{2}, \pi)$ we apply (1.6) to $S^n \setminus E$ instead of E to infer that

$$\begin{aligned}\mathbf{D}(E) &= \mathbf{P}(E) - \mathbf{P}(B_{\vartheta_o}) = \mathbf{P}(S^n \setminus E) - \mathbf{P}(B_{\pi-\vartheta_o})\\ &= \mathbf{D}(S^n \setminus E) \ge C\boldsymbol{\beta}(S^n \setminus E)^2 = C\boldsymbol{\beta}(E)^2.\end{aligned}$$

6.1. **Reduction to sets with small isoperimetric gap.** First, we show that it is enough to prove: There exists a constant $\chi_o > 0$ such that whenever $E \subset S^n$ satisfies $|E| = |B_{\vartheta_o}|$, $\boldsymbol{\beta}(E) = \boldsymbol{\beta}(E; e_{n+1})$ and $\mathbf{D}(E) \le \chi_o \vartheta_o^{n-1}$, then the quantitative isoperimetric inequality

$$\mathbf{D}(E) \ge C_1 \boldsymbol{\beta}(E)^2 \tag{6.1}$$

holds true with a constant $C_1 = C_1(n)$. Indeed, for a set $E \subset S^n$ of finite perimeter with $|E| = |B_{\vartheta_o}|$ and $\mathbf{D}(E) > \chi_o \vartheta_o^{n-1}$ we have

$$\begin{aligned}
\boldsymbol{\beta}(E)^2 &\le 2\mathbf{P}(E) = 2\big(\mathbf{D}(E) + \mathbf{P}(B_{\vartheta_o})\big) = 2\big(\mathbf{D}(E) + n\omega_n \sin^{n-1}\vartheta_o\big) \\
&\le 2\bigg(1 + \frac{n\omega_n \sin^{n-1}\vartheta_o}{\chi_o \vartheta_o^{n-1}}\bigg)\mathbf{D}(E) \le 2\bigg(1 + \frac{n\omega_n}{\chi_o}\bigg)\mathbf{D}(E)
\end{aligned}$$

i.e. the quantitative isoperimetric inequality holds with $C = \big[2(1 + n\omega_n \chi_o^{-1})\big]^{-1}$. On the other hand, if $\mathbf{D}(E) \le \delta_o \vartheta_o^{n-1}$ the inequality would follow, if (6.1) would be established.

6.2. **The contradiction assumption.** We argue by contradiction, assuming that (6.1) fails to hold. Then, there exists a sequence of sets of finite perimeter $E_k \subset S^n$ with volumes $|E_k| = |B_{\vartheta_k}|$ satisfying

$$\frac{\mathbf{D}(E_k)}{\vartheta_k^{n-1}} = \frac{\mathbf{P}(E_k) - \mathbf{P}(B_{\vartheta_k})}{\vartheta_k^{n-1}} \to 0 \quad \text{as } k \to \infty, \tag{6.2}$$

and

$$\mathbf{D}(E_k) < C_1 \boldsymbol{\beta}^2(E_k). \tag{6.3}$$

Moreover, we may also assume that $\vartheta_k \to \vartheta_o \in [0, \frac{\pi}{2}]$ and that $\boldsymbol{\beta}(E_k) = \boldsymbol{\beta}(E_k; e_{n+1})$ for any k, which means that the sets E_k are centered at e_{n+1}. At this point we need to distinguish three cases.

6.3. **The case $\boldsymbol{\vartheta_o > 0}$.** This case is divided into several steps, which will guide us also in one of the more involved cases where $\vartheta_o = 0$. This case has to be divided into two sub-cases whether or not the L^2-oscillation index converges fast enough to 0.

6.3.1. *Convergence to B_{ϑ_o}.* From (6.2) we know that the sequence of perimeters $\mathbf{P}(E_k)$ is uniformly bounded and therefore by compactness we infer that up to a (not relabeled) subsequence we have $E_k \to E_\infty$ in $L^1(S^n)$. In particular we conclude that $|E_\infty| = |B_{\vartheta_o}|$. By the optimal isoperimetric inequality from (2.2) we must have $\mathbf{P}(E_\infty) \ge \mathbf{P}(B_{\vartheta_o})$. On the other hand, by the lower semicontinuity of the perimeter with respect to L^1-convergence and by (6.2) we have

$$\mathbf{P}(E_\infty) \le \liminf_{k\to\infty} \mathbf{P}(E_k) = \mathbf{P}(B_{\vartheta_o}).$$

But this implies $\mathbf{P}(E_\infty) = \mathbf{P}(B_{\vartheta_o})$, i.e. in the isoperimetric inequality we have equality, so that $E_\infty = B_{\vartheta_o}(p_o)$ for some $p_o \in S^n$. However, since $E_k \to B_{\vartheta_o}(p_o)$ in $L^1(S^n)$ and that all the E_k are centered at e_{n+1}, by Lemma 2.6 (i.e. the continuity of the centers with respect to L^1-convergence) we conclude that E_∞ is also centered at the north pole e_{n+1}, i.e. $E_\infty = B_{\vartheta_o}$. Moreover, using Lemma 5.2, we have that $\boldsymbol{\gamma}(E_k) \to \boldsymbol{\gamma}(B_{\vartheta_o})$. Therefore, taking (6.2) into account we find that

$$\varepsilon_k^2 := \boldsymbol{\beta}^2(E_k) = \mathbf{P}(E_k) - \boldsymbol{\gamma}(E_k) \to \mathbf{P}(B_{\vartheta_o}) - \boldsymbol{\gamma}(B_{\vartheta_o}) = 0$$

in the limit $k \to \infty$.

6.3.2. *Penalization.* Taking $\Lambda \geq \Lambda_o$, where Λ_o is as in Lemma 5.1, and $C_o \in (0, 1/8)$, we define the penalized functionals

$$\mathbf{F}_k(G) := \mathbf{P}(G) + \frac{\Lambda}{\vartheta_k}\big||G| - |B_{\vartheta_k}|\big| + C_o\big|\boldsymbol{\beta}^2(G) - \varepsilon_k^2\big|,$$

By Lemma 5.3 we know that the functionals $\mathbf{F}_k$ are lower semicontinuous with respect to L^1-convergence of sets in S^n and therefore for any $k \in \mathbb{N}$ there exists a minimizer $G_k \subset S^n$ of $\mathbf{F}_k$. By the minimality of G_k – note that $\boldsymbol{\beta}(B_{\vartheta_o}) = 0$ – we have

$$\mathbf{P}(G_k) \leq \mathbf{F}_k(G_k) \leq \mathbf{F}_k(B_{\vartheta_k}) = \mathbf{P}(B_{\vartheta_k}) + C_o\,\varepsilon_k^2,$$

so that also the sequence $\mathbf{P}(G_k)$ is uniformly bounded; this implies that χ_{G_k} is a uniformly bounded sequence in $BV(S^n)$. Therefore we may extract a (not relabeled) subsequence satisfying $G_k \to G_\infty$ in L^1 for some $G_\infty \subset S^n$. Next, we define the functionals

$$\mathbf{G}_k(G) := \mathbf{P}(G) + \frac{\Lambda}{\vartheta_k}\big||G| - |B_{\vartheta_k}|\big| \qquad \text{for } k \in \mathbb{N}.$$

Having chosen $\Lambda \geq \Lambda_o$, where Λ_o is the constant from Lemma 5.1, we conclude that B_{ϑ_k} is a minimizer of $\mathbf{G}_k$. Hence, by the minimality of G_k and the contradiction assumption (6.3) we obtain

$$\begin{aligned} \mathbf{G}_k(G_k) + C_o|\boldsymbol{\beta}^2(G_k) - \varepsilon_k^2| &= \mathbf{F}_k(G_k) \leq \mathbf{F}_k(E_k) = \mathbf{P}(E_k) \\ &< \mathbf{P}(B_{\vartheta_k}) + C_1\,\varepsilon_k^2 = \mathbf{G}_k(B_{\vartheta_k}) + C_1\,\varepsilon_k^2 \leq \mathbf{G}_k(G_k) + C_1\,\varepsilon_k^2. \end{aligned} \tag{6.4}$$

But this implies $|\boldsymbol{\beta}^2(G_k) - \varepsilon_k^2| \leq C_1\varepsilon_k^2/C_o$ and moreover $\boldsymbol{\beta}(G_k) \to 0$ as $k \to \infty$, since $\varepsilon_k \to 0$. Further, distinguishing between the cases $\beta(G_k) \geq \varepsilon_k$ (in which the following estimate trivially holds true) and $\varepsilon_k \geq \beta(G_k)$ (in which we use the preceding inequality) we conclude that there holds

$$\varepsilon_k^2 \leq \frac{C_o}{C_o - C_1}\,\boldsymbol{\beta}^2(G_k) \tag{6.5}$$

provided we impose that $C_1 < C_o$ holds. Moreover, by the lower semicontinuity of the perimeter with respect to L^1-convergence and the minimizing property of G_k we infer that

$$\begin{aligned} \mathbf{G}_\infty(G_\infty) &:= \mathbf{P}(G_\infty) + \frac{\Lambda}{\vartheta_o}\big||G_\infty| - |B_{\vartheta_o}|\big| \\ &\leq \liminf_{k\to\infty} \mathbf{F}_k(G_k) \leq \liminf_{k\to\infty} \mathbf{F}_k(B_{\vartheta_o}) = \mathbf{P}(B_{\vartheta_o}) = \mathbf{G}_\infty(B_{\vartheta_o}). \end{aligned}$$

Since B_{ϑ_o} is a minimizer of $\mathbf{G}_\infty$, we conclude that also G_∞ is a minimizer of $\mathbf{G}_\infty$. Hence, Lemma 5.1 ensures that G_∞ is a geodesic ball $B_{\vartheta_o}(p_o)$ for some $p_o \in S^n$. Finally, we rotate all the sets G_k in such a way that their barycenter is the north pole e_{n+1}. Thus, we conclude that G_k converge to B_{ϑ_o}, by a continuity argument similar to the one we used in the case $\vartheta_o > 0$ of the proof of Lemma 2.9.

6.3.3. *Almost ball property of G_k.* Denoting by $\delta_o(n) > 0$ the constant from Lemma 5.8 we prove that for any $\delta \in (0, \delta_o)$ there exists $k_o = k_o(\delta) \in \mathbb{N}$ such that the inclusion

$$B_{\vartheta_k(1-\delta)} \subset G_k \subset B_{\vartheta_k(1+\delta)} \tag{6.6}$$

holds true for any $k \geq k_o$. First, we apply Lemma 5.5 to conclude that there exists a radius $r_o = r_o(\Lambda) > 0$ such that G_k is a $(3, r_o\vartheta_k)$-quasi minimizer of the perimeter for any $k \in \mathbb{N}$. By Theorem 5.6 this implies that the topological boundary ∂G_k coincides with the reduced boundary $\partial^* G_k$ after modifying the set G_k on a set of measure 0; that is we assume from now on that $\partial G_k = \partial^* G_k$.

Now, we argue by contradiction. We assume that there exists $0 < \delta < \min\{\delta_o, r_o\}$, such that for infinitely many $k \in \mathbb{N}$ we can find $x_k \in \partial G_k$ with

$$x_k \notin B_{\vartheta_k(1+\delta)} \setminus B_{\vartheta_k(1-\delta)}.$$

By Theorem 5.6 there exist $y_k, z_k \in B_{\frac{\delta\vartheta_k}{2}}(x_k)$ such that the inclusions $B_{\frac{\delta\vartheta_k}{2C}}(y_k) \subset G_k$ and $B_{\frac{\delta\vartheta_k}{2C}}(z_k) \subset S^n \setminus G_k$ hold true. If $x_k \in S^n \setminus B_{\vartheta_k(1+\delta)}$ we would have $B_{\frac{\delta\vartheta_k}{2C}}(y_k) \subset S^n \setminus B_{\vartheta_k}$ and hence $B_{\frac{\delta\vartheta_k}{2C}}(y_k) \subset G_k \setminus B_{\vartheta_k}$. On the other hand, if $x_k \in B_{\vartheta_k(1-\delta)}$ we have $B_{\frac{\delta\vartheta_k}{2C}}(z_k) \subset B_{\vartheta_k}$ and hence $B_{\frac{\delta\vartheta_k}{2C}}(z_k) \subset B_{\vartheta_k} \setminus G_k$. Therefore we either have

$$|G_k \setminus B_{\vartheta_k}| \geq |B_{\frac{\delta\vartheta_k}{2C}}(y_k)| > 0 \qquad \text{or} \qquad |B_{\vartheta_k} \setminus G_k| \geq |B_{\frac{\delta\vartheta_k}{2C}}(z_k)| > 0 \tag{6.7}$$

for infinitely many $k \in \mathbb{N}$. But this contradicts the fact that $|G_k \Delta B_{\vartheta_k}| \to 0$ and therefore proves the claim.

6.3.4. *Regularity.* From (6.6) and Lemma 5.8 we infer that there exists $r_1 = r_1(n) > 0$ and $K = K(n, \Lambda, C_o)$ such that G_k is a $(K/\vartheta_k, r_1\vartheta_k)$-almost minimizer of the perimeter and thus also a $(2K/\vartheta_o, r_1\vartheta_o/2)$-almost minimizer of the perimeter for k large. Therefore, since G_k is converging in L^1 to B_{ϑ_o}, we are allowed to apply Theorem 5.11 which yields for k large enough that the set G_k admits a spherical graph representation of class $C^{1,\frac{1}{2}}(S^n)$. This means that there exist functions $v_k \in C^{1,\frac{1}{2}}(S^{n-1})$ such that

$$\partial G_k = \big\{(\omega, 0)\sin[\vartheta_o(1 + v_k(\omega))] + e_{n+1}\cos[\vartheta_o(1 + v_k(\omega))] : \omega \in S^{n-1}\big\}.$$

with $\|v_k\|_{W^{1,\infty}(S^{n-1})} \to 0$ as $k \to \infty$.

6.3.5. *The final contradiction.* Note that the previous representation shows that G_k is also nearly spherical with respect to the ball $B_{\vartheta'_k}$, where $|B_{\vartheta'_k}| = |G_k|$. In fact, letting $u_k = \frac{\vartheta_o}{\vartheta'_k}(1 + v_k) - 1$ we have

$$\partial G_k = \big\{(\omega, 0)\sin[\vartheta'_k(1 + u_k(\omega))] + e_{n+1}\cos[\vartheta'_k(1 + u_k(\omega))] : \omega \in S^{n-1}\big\},$$

and moreover $u_k \to 0$ in $W^{1,\infty}(S^{n-1})$ as $k \to \infty$. Since $G_k \to B_{\vartheta_o}$ we easily see that $\frac{\vartheta'_k}{\vartheta_k} \to 1$. Therefore, by Lemma 2.1, we get that

$$\big|\mathbf{P}(B_{\vartheta_k}) - \mathbf{P}(B_{\vartheta'_k})\big| \leq \frac{c(n)}{\vartheta_k}\big||B_{\vartheta_k}| - |G_k|\big| \leq \frac{\Lambda}{\vartheta_k}\big||B_{\vartheta_k}| - |G_k|\big|,$$

provided that we take $\Lambda \geq c(n)$. Using this inequality, (6.4) and (6.5) we get that

$$\begin{aligned}
\mathbf{D}(G_k) &= \big[\mathbf{P}(G_k) - \mathbf{P}(B_{\vartheta_k})\big] + \big[\mathbf{P}(B_{\vartheta_k}) - \mathbf{P}(B_{\vartheta'_k})\big] \\
&\leq \mathbf{P}(G_k) + \frac{\Lambda}{\vartheta_k}\big||B_{\vartheta_k}| - |G_k|\big| - \mathbf{P}(B_{\vartheta_k}) \\
&= \boldsymbol{G}_k(G_k) - \mathbf{P}(B_{\vartheta_k}) \leq C_1\varepsilon_k^2 \leq \frac{C_oC_1}{C_o - C_1}\boldsymbol{\beta}^2(G_k),
\end{aligned}$$

a contradiction to (3.16), provided we take C_1 from the contradiction assumption so small that $C_oC_1/(C_o - C_1) < c_2(n)$ holds true. This finishes the proof in the case $\vartheta_o > 0$.

In the sequel we are going to assume that $\vartheta_o = 0$. We shall also assume, without loss of generality that there exists a non relabeled subsequence such that

$$\lim_{k\to\infty} \vartheta_k^{1-n}\boldsymbol{\beta}^2(E_k) = \eta_o \in [0, \infty] \tag{6.8}$$

6.4. **The case $\boldsymbol{\vartheta_o = 0}$ and $\boldsymbol{\eta_o = 0}$.** Most parts of the proof follow the line of proof from the case $\vartheta_o > 0$ and in the following we only indicate the differences.

6.4.1. *Necessary changes with respect to the case $\vartheta_o > 0$.* First, the conclusion from Section 6.3.1, i.e. that $\varepsilon_k = \boldsymbol{\beta}(E_k) \to 0$ as $k \to \infty$ now trivially holds. With that at hand the argument from Section 6.3.2 works as before. In particular equations (6.4) and (6.5) are obtained exactly as in Section 6.3.2. Only the conclusion of this Section is now different. We define ϑ_k' such that $|G_k| = |B_{\vartheta_k'}|$ and observe that

$$\lim_{k\to\infty} \frac{\vartheta_k'}{\vartheta_k} = 1 \tag{6.9}$$

holds true. In fact from (6.4) we get that

$$\mathbf{P}(G_k) - \mathbf{P}(B_{\vartheta_k}) + \frac{\Lambda}{\vartheta_k}\big||G_k| - |B_{\vartheta_k}|\big| \le C_1\varepsilon_k^2 = C_1\boldsymbol{\beta}(E_k)^2. \tag{6.10}$$

If $\vartheta_k' > \vartheta_k$ from the previous equation we get, since $|G_k| = |B_{\vartheta_k'}|$, that

$$\frac{|B_{\vartheta_k'}| - |B_{\vartheta_k}|}{\vartheta_k^n} \le \frac{C_1\boldsymbol{\beta}(E_k)^2}{\Lambda\vartheta_k^{n-1}},$$

holds, from which the claim (6.9) immediately follows. If instead $\vartheta_k' < \vartheta_k$ we estimate the left hand side of (6.10) with the help of Lemma 2.1 taking into account that $\mathbf{P}(G_k) \ge \mathbf{P}(B_{\vartheta_k'})$ and obtain

$$n\omega_n\Big[\frac{\Lambda}{2\pi^{n-1}} - (n-1)\Big]\vartheta_k^{n-2}(\vartheta_k - \vartheta_k') \le C_1\boldsymbol{\beta}(E_k)^2,$$

from which the claim (6.9) follows by choosing $\Lambda > 2\pi^{n-1}(n-1)$. Now, from Lemma 2.7 and (6.5), one gets for k large that

$$\boldsymbol{\alpha}(G_k)^2 \le \frac{c(n)}{\vartheta_k^{n-1}}\boldsymbol{\beta}(G_k)^2 \le \frac{c(n)}{\vartheta_k^{n-1}}\Big(1 + \frac{C_1}{C_o}\Big)\varepsilon_k^2 \le \frac{2c(n)\boldsymbol{\beta}(E_k)^2}{\vartheta_k^{n-1}},$$

provided C_1 is chosen to be smaller than C_o. Therefore, rotating G_k if necessary, we can assume that $\boldsymbol{\alpha}(G_k) = |G_k \Delta B_{\vartheta_k'}|/|B_{\vartheta_k'}|$; note that G_k has the volume $|B_{\vartheta_k'}|$. Taking (6.9) into account, we find that

$$\begin{aligned}\lim_{k\to\infty}\frac{|G_k\Delta B_{\vartheta_k}|}{\vartheta_k^n} &\le c(n)\lim_{k\to\infty}\boldsymbol{\alpha}(G_k) + \lim_{k\to\infty}\frac{|B_{\vartheta_k'}\Delta B_{\vartheta_k}|}{\vartheta_k^n}\\ &\le c(n)\lim_{k\to\infty}\frac{\boldsymbol{\beta}(E_k)}{\vartheta_k^{\frac{n-1}{2}}} + \lim_{k\to\infty}\frac{\big||B_{\vartheta_k'}| - |B_{\vartheta_k}|\big|}{\vartheta_k^n} = 0.\end{aligned} \tag{6.11}$$

Having arrived at this stage we proceed as in Section 6.3.3 until formula (6.7), from which the desired contradiction can be derived. In fact, if the first of the two inequalities holds (the other case is similar), we get for all $0 < \delta < \min\{\delta_o, r_o\}$ that

$$\delta^n\vartheta_k^n \le c(n)C^n|G_k \setminus B_{\vartheta_k}|,$$

and therefore the contradiction follows with (6.11).

Finally, as in Section 6.3.4 an application of Lemma 5.8 implies that there exist $r_1 = r_1(n) > 0$ and $K = K(n, \Lambda, C_o)$ such that the sets G_k are $(K/\vartheta_k, r_1\vartheta_k)$-almost minimizers of the perimeter.

6.4.2. *The final conclusion in the case $\vartheta_o = 0$ and $\eta_o = 0$.* Instead of the sets G_k we consider the sets $\widetilde{G}_k$ obtained by projecting G_k onto the plane $\{x_{n+1} = 0\}$. By $B_r^{(n)}$ we denote the ball in $\mathbb{R}^n$ of radius r with center at the origin. From what we proved in Section 6.3.3 we know that for all $0 < \delta < \min\{\delta_o, r_o\}$ the following inclusion

$$B^{(n)}_{\sin(\vartheta_k(1-\delta))} \subset \widetilde{G}_k \subset B^{(n)}_{\sin(\vartheta_k(1+\delta))},$$

holds true, provided k is sufficiently large. Using these inclusions, the almost minimality property of G_k, the area formula for rectifiable sets and the transformation formula for measurable sets it is immediate to check that for k large enough the sets $\widetilde{G}_k$ are $(2K/\vartheta_k, r_1\vartheta_k/2)$-almost minimizers of the functional

$$G \mapsto \int_{\partial^* G} \sqrt{\frac{1-(z\cdot\nu_G)^2}{1-|z|^2}}\, d\mathcal{H}^{n-1}\,.$$

Here we used the facts that the sets G_k are localized in balls $B_{\vartheta_k(1+\delta)}$ with $\vartheta_k \to 0$ and that the parametrization $z \mapsto z+\sqrt{1-|z|^2}e_{n+1}$ of the sphere is a bi-Lipschitz parametrization with bi-Lipschitz constant converging to 1 on the balls $B^{(n)}_{\sin(\vartheta_k(1+\delta_o))}$. Thus, for $k \gg 1$, the rescaled sets $\widetilde{H}_k = \widetilde{G}_k/\vartheta_k$ are $(2K, r_1/2)$-almost minimizers of the functionals

$$H \mapsto \int_{\partial^* H} \sqrt{\frac{1-\vartheta_k^2(z\cdot\nu_H)^2}{1-\vartheta_k^2|z|^2}}\, d\mathcal{H}^{n-1}.$$

Moreover the sets $\widetilde{H}_k$ are converging in Haudorff distance to the unit ball $B^{(n)}_1$. By applying the analogous of Theorem 5.11 to the above functionals (with estimates uniform in k, since the integrands are approaching 1 in C^2 on $\mathcal{K}\times S^{n-1}$ for any compact set $\mathcal{K}\subset\mathbb{R}^n$), we then conclude that the sets $\widetilde{H}_k$ converge in $C^{1,\alpha}$ to the unit ball $B^{(n)}_1$ for any $0<\alpha<1/2$. In particular, there exist functions w_k converging to zero in $C^{1,\alpha}(S^{n-1})$, such that

$$\partial\widetilde{H}_k = \big\{\omega(1+w_k(\omega)) : \omega\in S^{n-1}\big\},$$

and hence

$$\partial\widetilde{G}_k = \big\{\vartheta_k\omega(1+w_k(\omega)) : \omega\in S^{n-1}\big\}.$$

Setting now $\widetilde{w}_k = \frac{\arcsin[\vartheta_k(1+w_k)]-\vartheta_k}{\vartheta_k}$, also the functions $\widetilde{w}_k$ converge to 0 in $C^{1,\alpha}(S^{n-1})$ and we have

$$\partial G_k = \big\{(\omega,0)\sin[\vartheta_k(1+\widetilde{w}_k(\omega))] + e_{n+1}\cos[\vartheta_k(1+\widetilde{w}_k(\omega))] : \omega\in S^{n-1}\big\}.$$

Moreover, since $B_{\vartheta_k(1-\delta_k)} \subset G_k \subset B_{\vartheta_k(1+\delta_k)}$ for some $\delta_k \downarrow 0$, we may apply Lemma 2.9, thus getting that if p_k is a barycenter of G_k, then $\vartheta_k^{-1}\mathrm{dist}(p_k, e_{n+1}) \to 0$. Therefore, after rotating the sets G_k so that they all have barycenter at e_{n+1}, we may assume that

$$\partial G_k = \big\{(\omega,0)\sin[\vartheta_k(1+v_k(\omega))] + e_{n+1}\cos[\vartheta_k(1+v_k(\omega))] : \omega\in S^{n-1}\big\},$$

for a new sequence of functions v_k still converging to zero on $C^{1,\alpha}(S^{n-1})$. Finally, setting $u_k = \frac{\vartheta_k}{\vartheta'_k}(1+v_k)-1\to 0$, we see that ∂G_k can be written in the form

$$\partial G_k = \big\{(\omega,0)\sin[\vartheta'_k(1+u_k(\omega))] + e_{n+1}\cos[\vartheta'_k(1+u_k(\omega))] : \omega\in S^{n-1}\big\}.$$

Having arrived at this stage the contradiction can be obtained exactly as in Section 6.3.5.

6.5. **The case $\vartheta_o = 0$ and $\eta_o > 0$.** In this case the proof differs completely from those ones of the two previous cases.

6.5.1. *Localization.* Without loss of generality we may assume that

$$E_k \subset B_{R_o\vartheta_k}(p_k) \qquad \text{for all } k, \tag{6.12}$$

for some point $p_k \in S^n$, where $R_o = 2R$ and R is the radius provided by Lemma 4.1. In fact, if (6.12) does not hold we apply Lemma 4.1 and find a sequence of sets $E'_k \subset E_k$ of measure $|E'_k| = |B_{\vartheta'_k}|$ with $\vartheta'_k \in [\vartheta_k/2, \vartheta_k]$. Moreover, the new sets E'_k are contained in a ball of radius $R\vartheta_k$. Then, we have

$$E'_k \subset B_{R\vartheta_k}(p_k) \subset B_{R_o\vartheta'_k}(p_k) \qquad \text{for all } k.$$

Moreover, the estimate $(4.1)_1$ from Lemma 4.1 ensures for $k\gg 1$ that

$$\boldsymbol{\beta}(E_k)^2 \le \boldsymbol{\beta}(E'_k)^2 + \widetilde{C}\mathbf{D}(E_k) + \widetilde{C}(\vartheta_k\mathbf{D}(E_k))^{\frac{n-1}{n}}$$

$$(6.13) \qquad = \boldsymbol{\beta}(E_k')^2 + \widetilde{C}\frac{\mathbf{D}(E_k)}{\vartheta_k^{n-1}}\frac{\vartheta_k^{n-1}}{\boldsymbol{\beta}(E_k)^2}\boldsymbol{\beta}(E_k)^2 + \widetilde{C}\bigg(\frac{\mathbf{D}(E_k)}{\vartheta_k^{n-1}}\bigg)^{\frac{n-1}{n}}\frac{\vartheta_k^{n-1}}{\boldsymbol{\beta}(E_k)^2}\boldsymbol{\beta}(E_k)^2.$$

Recalling (6.2) and the fact that the ratio $\frac{\vartheta_k^{n-1}}{\boldsymbol{\beta}(E_k)^2}$ converges to $1/\eta_o < \infty$ (recall the definition (6.8)) we can re-absorb the second and the third term of the right-hand side on the left-hand side and obtain for $k \gg 1$ that

$$\boldsymbol{\beta}(E_k)^2 \leq 2\boldsymbol{\beta}(E_k')^2.$$

In turn we now apply $(4.1)_2$ from Lemma 4.1, (6.3) and the last inequality to obtain

$$(6.14) \qquad \mathbf{D}(E_k') \leq \widetilde{C}\mathbf{D}(E_k) \leq \widetilde{C}C_1\boldsymbol{\beta}(E_k)^2 \leq 2\widetilde{C}C_1\boldsymbol{\beta}(E_k')^2,$$

where $\widetilde{C}$ is the constant provided by Lemma 4.1. Moreover, combining the first line in (6.13) with (6.2), we get

$$(6.15) \qquad \liminf_{k\to\infty} \vartheta_k^{1-n}\boldsymbol{\beta}(E_k')^2 \geq \eta_o\,.$$

In conclusion, up to writing E_k in place of E_k' and ϑ_k in place of ϑ_k', we may assume that our sets E_k satisfy the conditions (6.12), (6.14) and (6.15).

6.5.2. *The strategy of the proof.* Since the sets E_k are contained in $B_{R_o\vartheta_k}(p_k)$, we may conclude by Lemma 2.4 that all its centers are contained in $B_{2R_o\vartheta_k}(p_k)$. Therefore, by rotating the centers p_k to e_{n+1}, we may assume without loss of generality that E_k has a center at e_{n+1} and that $E_k \subset B_{3R_o\vartheta_k}(e_{n+1})$.

Next, by $\widetilde{E}_k$ we denote the projection of E_k on the plane $\{x_{n+1} = 0\}$. In the following we shall denote by $P(E)$ the standard euclidean perimeter, by $D(E) = P(E) - P(B_r^{(n)})$, where $|B_r^{(n)}| = |E|$, the usual isoperimetric gap, and by $\beta(E)$ the oscillation index in the euclidean case (see [15]). We recall that

$$\beta(E)^2 = P(E) - \max_{w\in\mathbb{R}^n} \gamma^{(n)}(E;w),$$

where

$$\gamma^{(n)}(E;w) = \int_E \frac{n-1}{|z-w|}\,dz\,.$$

In [15] it is proved that for any set $E \subset \mathbb{R}^n$ of finite measure and of finite perimeter the sharp quantitative isoperimetric inequality

$$(6.16) \qquad C_o(n)\beta^2(E) \leq D(E),$$

holds true with some positive constant $C_o(n) < 1$.

Let us now denote by r_k the radius of a ball $B_{r_k}^{(n)} \subset \mathbb{R}^n$ such that $|\widetilde{E}_k| = |B_{r_k}^{(n)}|$. Then, we have

$$\begin{aligned}
\mathbf{D}(E_k) &= \mathbf{P}(E_k) - \mathbf{P}(B_{\vartheta_k}) \\
&= D(\widetilde{E}_k) + \mathbf{P}(E_k) - P(\widetilde{E}_k) + n\omega_n(r_k^{n-1} - \sin^{n-1}\vartheta_k) \\
&\geq C_o\beta(\widetilde{E}_k)^2 + [\mathbf{P}(E_k) - P(\widetilde{E}_k)] + n\omega_n(r_k^{n-1} - \sin^{n-1}\vartheta_k),
\end{aligned}$$

where C_o is the constant appearing in (6.16). Now, by z_k we denote a point in $\mathbb{R}^n$ having the property that

$$(6.17) \qquad \beta(\widetilde{E}_k)^2 = P(\widetilde{E}_k) - \gamma^{(n)}(\widetilde{E}_k;z_k).$$

Then, from the second last inequality and the fact that E_k has a center at the north pole e_{n+1}, we conclude that

$$\begin{aligned}
(6.18) \quad \mathbf{D}(E_k) &\geq C_o\boldsymbol{\beta}(E_k)^2 + (1-C_o)[\mathbf{P}(E_k) - P(\widetilde{E}_k)] + n\omega_n(r_k^{n-1} - \sin^{n-1}\vartheta_k) \\
&\quad + (n-1)C_o\bigg[\int_{E_k}\frac{x\cdot e_{n+1}}{\sqrt{1-(x\cdot e_{n+1})^2}}\,d\mathcal{H}^n - \int_{\widetilde{E}_k}\frac{dz}{|z-z_k|}\bigg].
\end{aligned}$$

$$= C_o\boldsymbol{\beta}(E_k)^2 + \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k,$$

with the obvious labeling of the terms I_k, II_k and III_k. We claim that

$$\mathrm{I}_k \geq 0, \qquad \lim_{k\to\infty} \vartheta_k^{1-n}\mathrm{II}_k = 0, \qquad \liminf_{k\to\infty} \vartheta_k^{1-n}\mathrm{III}_k \geq 0\,. \tag{6.19}$$

Once the claim will be established the proof can be easily completed observing that (6.18), (6.17) and (6.15) imply for k large the lower bound

$$\begin{aligned}\mathbf{D}(E_k) &\geq C_o\boldsymbol{\beta}(E_k)^2 + \mathrm{II}_k + \mathrm{III}_k \geq \tfrac{1}{2}C_o\boldsymbol{\beta}(E_k)^2 + \tfrac{1}{4}C_o\eta_o\vartheta_k^{n-1} + \mathrm{II}_k + \mathrm{III}_k \\ &\geq \tfrac{1}{2}C_o\boldsymbol{\beta}(E_k)^2 + \vartheta_k^{n-1}\Big(\tfrac{1}{4}C_o\eta_o + \vartheta_k^{1-n}\mathrm{II}_k + \vartheta_k^{1-n}\mathrm{III}_k\Big) \geq \tfrac{1}{2}C_o\boldsymbol{\beta}(E_k)^2.\end{aligned}$$

But this contradicts (6.14) provided we choose $C_1 < C_o/(4\widetilde{C})$ and finishes the proof in this case.

6.5.3. *Estimate of I_k and II_k.* The estimate $(6.19)_1$ follows by an application of the area formula. In fact, we have

$$\frac{\mathrm{I}_k}{1-C_o} = \mathbf{P}(E_k) - P(\widetilde{E}_k) = \int_{\partial^*\widetilde{E}_k}\left[\sqrt{\frac{1-(z\cdot\nu_{\widetilde{E}_k})^2}{1-|z|^2}} - 1\right]dz \geq 0.$$

To estimate II_k we use in turn the definition of r_k and the fact that $E_k \subset B_{3R_o\vartheta_k}$, which implies $\widetilde{E}_k \subset B^{(n)}_{3R_o\vartheta_k}$. This leads us to

$$\begin{aligned}\omega_n r_k^n = |\widetilde{E}_k| &= \int_{\widetilde{E}_k}\frac{\sqrt{1-|z|^2}}{\sqrt{1-|z|^2}}\,dz \geq \int_{\widetilde{E}_k}\frac{1-|z|^2}{\sqrt{1-|z|^2}}\,dz \\ &\geq \big(1-9R_o^2\vartheta_k^2\big)\int_{\widetilde{E}_k}\frac{dz}{\sqrt{1-|z|^2}} = \big(1-9R_o^2\vartheta_k^2\big)|E_k|.\end{aligned}$$

Therefore we can conclude that

$$\big(1-9R_o^2\vartheta_k^2\big)|E_k| \leq \omega_n r_k^n = |\widetilde{E}_k| \leq |E_k|,$$

so that

$$n\big(1-9R_o^2\vartheta_k^2\big)\int_0^{\vartheta_k}\sin^{n-1}\sigma\,d\sigma \leq r_k^n \leq n\int_0^{\vartheta_k}\sin^{n-1}\sigma\,d\sigma.$$

In particular, since $\vartheta_k \to 0$, we conclude that

$$\lim_{k\to\infty}\frac{r_k}{\vartheta_k} = 1,$$

which easily implies that

$$\lim_{k\to\infty}\frac{|\sin^{n-1}\vartheta_k - r_k^{n-1}|}{\vartheta_k^{n-1}} = 0,$$

proving the second inequality in (6.19).

6.5.4. *Estimate of III_k.* In this final step we denote by c a generic constant that may change from line to line depending only on R_o and n, but not on k. First, by rotating the sets $\widetilde{E}_k$ around the origin we may assume that there exists $\omega_o \in S^{n-1}$ such that

$$z_k = \omega_o\sin\varphi_k,$$

for a suitable angle $\varphi_k \geq 0$. Since $\widetilde{E}_k \subset B^{(n)}_{3R_0\vartheta_k}$ and $z_k \in B^{(n)}_{3R_0\vartheta_k}$ we get also that $0 \leq \varphi_k \leq 6R_o\vartheta_k$. We now define

$$y_k = (\omega_o, 0)\sin\varphi_k + e_{n+1}\cos\varphi_k.$$

Let $\varepsilon \in (0,1)$ fixed to be chosen later in the proof. By $\widetilde{Q}_\varepsilon(\omega_o)$ we denote the cone $\{z \in \mathbb{R}^n_{\neq 0} : \frac{z}{|z|}\cdot\omega_o > 1-\varepsilon\}$ and by $\widetilde{Q}^c_\varepsilon(\omega_o)$ its complement. Similarly, writing points $x \in S^n$ in the form $x = (\omega_x, 0)\sin\varphi_x + e_{n+1}\cos\varphi_x$ we define $Q_\varepsilon(\omega_o) := \{x \in S^n :$

$\omega_x \cdot \omega_o > 1-\varepsilon\}$. By $Q^c_\varepsilon(\omega_o)$ we denote its complement in S^n. These sets can be considered as cones in the sphere. Recalling that E_k is centered at e_{n+1} we find that

$$
\begin{aligned}
\frac{\mathrm{III}_k}{C_o(n-1)} &\geq \int_{E_k} \frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n - \int_{\widetilde{E}_k} \frac{dz}{|z-z_k|} \\
&= \int_{E_k\cap Q_\varepsilon(\omega_o)} \frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n - \int_{\widetilde{E}_k\cap \widetilde{Q}_\varepsilon(\omega_o)} \frac{dz}{|z-z_k|} \\
&\quad + \int_{E_k\cap Q^c_\varepsilon(\omega_o)} \frac{x\cdot y_k}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n - \int_{\widetilde{E}_k\cap \widetilde{Q}^c_\varepsilon(\omega_o)} \frac{dz}{|z-z_k|} \\
&= \mathrm{III}_{k,1} + \mathrm{III}_{k,2}.
\end{aligned}
\tag{6.20}
$$

Now, since $z_k \in B^{(n)}_{3R_o\vartheta_k}$, $y_k \in B_{6R_o\vartheta_k}$ and $\widetilde{E}_k \subset B^{(n)}_{3R_o\vartheta_k}$ and $x\cdot y_k \geq 0$ for any $x \in E_k$ we have

$$
\mathrm{III}_{k,1} \geq -\int_{B^{(n)}_{6R_o\vartheta_k}(z_k)\cap \widetilde{Q}_\varepsilon(\omega_o)} \frac{dz}{|z-z_k|}. \tag{6.21}
$$

To estimate this integral, we denote by π_o the orthogonal projection of $\mathbb{R}^n$ onto the hyperplane in $\mathbb{R}^n$ passing through the origin and orthogonal to ω_o. For $r > 0$ we set $C_r := \{z \in \mathbb{R}^n : |\pi_o(z)| \leq r\}$. Then, C_r is the cylinder whose axis of symmetry is given by $\omega_o\mathbb{R}$ and whose cross section is an $(n-1)$-dimensional disk of radius r. Note that if $z \in B^{(n)}_{6R_o\vartheta_k}(z_k) \cap \widetilde{Q}_\varepsilon(\omega_o)$, then

$$
|\pi_o(z)|^2 = |z|^2 - (\omega_o\cdot z)^2 \leq |z|^2 - (1-\varepsilon)^2|z|^2 \leq 2\varepsilon|z|^2 \leq 13^2R_o^2\vartheta_k^2\varepsilon.
$$

This however implies that $B^{(n)}_{6R_o\vartheta_k}(z_k) \cap \widetilde{Q}_\varepsilon(\omega_o) \subset C_{13R_o\vartheta_k\sqrt{\varepsilon}}$. Since z_k lies in the symmetry axis of the cylinders C_r by construction we have $C_r - z_k = C_r$ so that (6.21) can be estimated from below in the form

$$
\begin{aligned}
\mathrm{III}_{k,1} &\geq -\int_{B^{(n)}_{6R_o\vartheta_k}(z_k)\cap C_{13R_o\vartheta_k\sqrt{\varepsilon}}} \frac{dz}{|z-z_k|} \\
&= -\int_{B^{(n)}_{6R_o\vartheta_k}\cap C_{13R_o\vartheta_k\sqrt{\varepsilon}}} \frac{dz}{|z|} = -\vartheta_k^{n-1}\int_{B^{(n)}_{6R_o}\cap C_{13R_o\sqrt{\varepsilon}}} \frac{dw}{|w|}.
\end{aligned}
\tag{6.22}
$$

Now, we turn our attention to the term $\mathrm{III}_{k,2}$ for which we have

$$
\begin{aligned}
\mathrm{III}_{k,2} &\geq \left[\int_{E_k\cap Q^c_\varepsilon(\omega_o)} \frac{1}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n - \int_{\widetilde{E}_k\cap \widetilde{Q}^c_\varepsilon(\omega_o)} \frac{dz}{|z-z_k|\sqrt{1-|z|^2}}\right] \qquad (6.23)\\
&\quad + \int_{E_k\cap Q^c_\varepsilon(\omega_o)} \frac{x\cdot y_k - 1}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n \\
&= \mathrm{III}_{k,3} + \mathrm{III}_{k,4}, \qquad (6.24)
\end{aligned}
$$

with the obvious meaning for $\mathrm{III}_{k,4}$ and $\mathrm{III}_{k,3}$. We first treat the term $\mathrm{III}_{k,4}$. Recalling the inclusion $E_k \subset B_{3R_o\vartheta_k}$ and the fact that $\varphi_k \leq 6R_o\vartheta_k$, we estimate for k large enough the term $\mathrm{III}_{k,4}$ as follows:

$$
\begin{aligned}
|\mathrm{III}_{k,4}| &\leq \int_{E_k} \frac{|\sin\varphi_x\sin\varphi_k| + |\cos\varphi_x\cos\varphi_k - 1|}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n \\
&\leq cR_o^2\vartheta_k^2\int_{E_k} \frac{d\mathcal{H}^n}{\sqrt{1-(x\cdot y_k)^2}} \leq cR_o^2\vartheta_k^2\boldsymbol{\gamma}(E_k;y_k) \\
&\leq cR_o^2\vartheta_k^2\mathbf{P}(E_k) = cR_o^2\vartheta_k^2\big[\mathbf{D}(E_k) + \mathbf{P}(B_{\vartheta_k})\big] \leq cR_o^2\vartheta_k^{n+1}.
\end{aligned}
\tag{6.25}
$$

In the last inequality we used (6.2); moreover in the second last line we used the fact that $\boldsymbol{\gamma} \leq \boldsymbol{P}$ which is a consequence of the very definition of $\boldsymbol{\gamma}$, cf. (2.5). At this stage it remains to estimate the term $\mathrm{III}_{k,3}$. In the sequel we use the following short hand notation:

$$
\mathbf{s} = \sin\varphi_x,\ \mathbf{s}_k = \sin\varphi_k,\ \mathbf{c} = \cos\varphi_x,\ \mathbf{c}_k = \cos\varphi_k,\ \mathbf{o} = \omega_x\cdot\omega_0.
$$

Using the area formula and noting that $Q^c_\varepsilon(\omega_o) = \{\mathbf{o} < 1-\varepsilon\}$ in our short hand notion we find that

$$\mathrm{III}_{k,3} = \int_{E_k\cap\{\mathbf{o}<1-\varepsilon\}} \left[\frac{1}{\sqrt{1-(\mathbf{oss}_k+\mathbf{cc}_k)^2}} - \frac{1}{\sqrt{\mathbf{s}^2+\mathbf{s}_k^2-2\mathbf{oss}_k}} \right] d\mathcal{H}^n$$

Then, using the elementary inequality

$$\left| \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right| \leq \frac{|b-a|}{b\sqrt{a}},$$

we obtain, after some calculations,

$$\begin{aligned}
\mathrm{III}_{k,3} &\geq -\int_{E_k\cap\{\mathbf{o}<1-\varepsilon\}} \frac{|\mathbf{s}^2\mathbf{s}_k^2(1+\mathbf{o}^2)+2\mathbf{oss}_k(\mathbf{cc}_k-1)|}{\sqrt{1-(\mathbf{oss}_k+\mathbf{cc}_k)^2}(\mathbf{s}^2+\mathbf{s}_k^2-2\mathbf{oss}_k)}\, d\mathcal{H}^n \\
&\geq -\frac{1}{\varepsilon}\int_{E_k\cap\{\mathbf{o}<1-\varepsilon\}} \frac{|\mathbf{s}^2\mathbf{s}_k^2(1+\mathbf{o}^2)+2\mathbf{oss}_k(\mathbf{cc}_k-1)|}{\sqrt{1-(\mathbf{oss}_k+\mathbf{cc}_k)^2}(\mathbf{s}^2+\mathbf{s}_k^2)}\, d\mathcal{H}^n \\
&\geq -\frac{1}{2\varepsilon}\int_{E_k\cap\{\mathbf{o}<1-\varepsilon\}} \frac{|\mathbf{ss}_k(1+\mathbf{o}^2)+2\mathbf{o}(\mathbf{cc}_k-1)|}{\sqrt{1-(\mathbf{oss}_k+\mathbf{cc}_k)^2}}\, d\mathcal{H}^n \\
&\geq -\frac{cR_o^2\vartheta_k^2}{\varepsilon}\int_{E_k} \frac{1}{\sqrt{1-(x\cdot y_k)^2}}\, d\mathcal{H}^n \geq -\frac{cR_o^2\vartheta_k^2\mathbf{P}(E_k)}{\varepsilon} \geq -\frac{cR_o^2\vartheta_k^{n+1}}{\varepsilon}.
\end{aligned}$$

Joining this last inequality with the estimates (6.20) – (6.25) we see that also the third inequality in (6.19) holds true by letting first $k\to\infty$ and then $\varepsilon\to 0^+$. This finishes the proof in the case $\vartheta_o = 0$ and $\eta_o > 0$. □

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