STABILITY OF PÓLYA-SZEGŐ INEQUALITY FOR LOG-CONCAVE FUNCTIONS

M. BARCHIESI, G. M. CAPRIANI, N. FUSCO, G. PISANTE

Abstract. A quantitative version of Pólya-Szegő inequality is proven for log-concave functions in the case of Steiner and Schwarz rearrangements.

1. Introduction

The Pólya-Szegő principle states that, given a non-negative function $u : \mathbb{R}^n \to \mathbb{R}$, the Dirichlet integral $\int_{\mathbb{R}^n} |\nabla u|^p$ decreases under suitable rearrangements, the two most common of which are the Schwarz spherical symmetrization about a point and Steiner symmetrization about a hyperplane. Their corresponding Pólya-Szegő inequalities are a powerful tool to approach a wide number of variational problems of geometric and functional nature.

Although the Pólya-Szegő inequality is known from long time, the issue of characterizing the extremals has been studied only more recently. In particular the first characterization of equality cases in the Pólya-Szegő inequality for spherical rearrangements has been provided by Brothers and Ziemer in [7] (see also [15] for an alternative proof). Instead, for Steiner symmetrization the characterization has been obtained in [13]. Both these results have been extended to the intermediate codimensions in [9].

When compared with these results, the natural issue of proving quantitative versions of the Pólya-Szegő inequality is a much more delicate task. The reason is that, even when the Dirichlet integral of a function u and of its symmetral coincide, u can be very different from its symmetral. In the case of Schwarz symmetrization this phenomenon may appear when the gradient of u is zero on sets of positive measure. Similarly, in the case of Steiner symmetrization the phenomenon could appear as soon as the derivative of u in the direction orthogonal to the symmetrization hyperplane is zero on sets of positive measure. Therefore any stability result for the Pólya-Szegő inequality must require a control of the measure of the set where the gradient or some of the derivatives are small (see [13, 11] and the examples therein).

In this paper we deal with the stability of the Pólya-Szegő inequality for the symmetrization of functions having at least a mild form of concavity that is a natural geometric compromise to avoid the phenomena described above. At the same time the class of functions to which our stability results apply is large enough to include the solutions of the torsion problem and the first eigenfunction of the Laplacian operator with Dirichlet boundary conditions in smooth convex domains (see [5, 21, 20]).

To describe our main results let us recall the definition of Steiner symmetrization for a measurable function $u : \mathbb{R}^n \to [0, \infty)$ with compact support. For simplicity we write a point $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The Steiner symmetral u^s of u with respect to the hyperplane $\{x_n = 0\}$ is defined as

$$u^s(x) := \inf \left\{ t > 0 : \mathcal{L}^1(\{s : u(x', s) > t\}) \leq 2|x_n| \right\}$$

Date: April 25, 2014.

for any $x \in \mathbb{R}^n$. If the function u belongs to $W^{1,p}(\mathbb{R}^n)$, then also u^s belongs to the same space and the Pólya-Szegő inequality states that

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^n} |\nabla u^s|^p \, dx.$$

Let us denote by

$$\Delta(u, u^s) := \int_{\mathbb{R}^n} |\nabla u|^p \, dx - \int_{\mathbb{R}^n} |\nabla u^s|^p \, dx$$

the gap between the two sides of the Pólya-Szegő inequality. Our main result shows that for a log-concave function u this gap controls the L^1 distance between u^s and (a suitable translation of) u. Here and throughout the paper, given a function $u : D \to \mathbb{R}$, it will always be understood that the function is extended to the whole $\mathbb{R}^n$ by setting $u(x) = 0$ outside D. We recall that for a bounded open convex set $D \subset \mathbb{R}^n$ the *outer radius* is the radius of the smallest ball containing D while the *inner radius* is the radius of the largest ball contained in D.

Theorem 1.1. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and let $u \in W_0^{1,p}(D)$ be a non-negative and log-concave function. Assume that the subgraph of u is star-shaped with respect to a ball of radius m. Then*

$$\inf_{h \in \mathbb{R}} \int_{\mathbb{R}^n} |u(x', x_n + h) - u^s(x)| \, dx \leq \begin{cases} c \, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u^s\|_{L^p}^{\frac{2-p}{2}} \Delta(u, u^s)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c \, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \Delta(u, u^s)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases} \tag{1.1}$$

where $c = c(n, p)$ and M is the maximum between $\|u\|_{L^\infty}$ and the outer radius of D.

Recall that, given $C \subset E \subset \mathbb{R}^n$, the set E is said to be *star-shaped* with respect to C if for any $x \in E$ and $y \in C$ the segment joining x and y is contained in E. The star-shapedness condition on the subgraph ensures that the function u is not too flat near the boundary of D. This condition is automatically satisfied when $|\nabla u|$ is bounded from below in a neighborhood of ∂D by some positive constant, as proved in Theorem 4.1. For instance, this happens when u is the first eigenfunction of the Laplacian in a smooth convex domain. It can be shown that if the subgraph of u is star-shaped with respect to a ball of radius m then also the subgraph of u^s is star-shaped with respect to a ball of the same radius (see the proof of Theorem 1.3).

Note that estimate (1.1) is scaling invariant in the sense that it does not change by replacing u with the function $\lambda u(x/\lambda)$. Note also that the exponent $1/2$ of the gap in the case $p \leq 2$ is optimal as shown by a simple one dimensional example (see Example 3.3). We conjecture that also in the case $p > 2$ the optimal exponent should be $1/2$.

It should be interesting, when u is the first eigenfunction of the Laplacian in a smooth convex domain D, to have a quantitative Faber-Krahn type inequality under Steiner symmetrization, i.e., to control through the gap $\Delta(u, u^s)$ the symmetric difference between D and its Steiner symmetral D^s. We refer to the recent work [6] for a sharp quantitative version of the Faber-Krahn principle.

In the case u is concave or, more generally, α-concave the previous theorem can be further improved. Recall that a positive function u is called *α-concave*, for a given $\alpha \in (0, 1]$, if $\tilde{u} := u^\alpha$ is concave.

Theorem 1.2. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and let $u \in W_0^{1,p}(D)$ be a non-negative and α-concave function, $\alpha \in (0, 1]$. Then inequality* (1.1) *holds with $c = c(n, p, \alpha)$ and the ratio*

M^{n+2}/m^{n+1} replaced by $\tilde{M}^{n+2}/\|\tilde{u}\|_{L^1}$, where $\tilde{M}$ is the maximum between $\|\tilde{u}\|_{L^\infty}$ and the outer radius of D.

This theorem applies to the solutions of the torsional rigidity problem in smooth convex domains which are known to be 1/2-concave. Note also that in the theorem no assumption is made on the star-shapedness of the subgraph of u.

An interesting features of the estimate (1.1) is that it retains the same structure when iterated along different directions. This allows us to prove a similar estimate for the Schwarz rearrangement of log-concave and α-concave functions. To this aim, we recall that, given a measurable function $u : \mathbb{R}^n \to [0, \infty)$ with compact support, the Schwarz symmetral u^* of u with respect to the origin is defined as

$$u^*(x) := \inf \{t > 0 : \mathcal{L}^n(\{y : u(y) > t\}) \leq \omega_n |x|^n\},$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. We denote by

$$\Delta(u, u^*) := \int_{\mathbb{R}^n} |\nabla u|^p \, dx - \int_{\mathbb{R}^n} |\nabla u^*|^p \, dx$$

the gap between the two sides of the Pólya-Szegő inequality in the case of Schwarz rearrangement.

Theorem 1.3. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and let $u \in W_0^{1,p}(D)$ be a non-negative and log-concave function. Assume that the subgraph of u is star-shaped with respect to a ball of radius m. Then*

$$\inf_{h \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u(x+h) - u^*(x)| \, dx \leq \begin{cases} c \, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u, u^*)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c \, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \Delta(u, u^*)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases} \tag{1.2}$$

where $c = c(n, p)$ and M is as in Theorem 1.1.

The counterpart of the above theorem in the α-concave case has a similar statement, with the only difference that $c = c(n, p, \alpha)$ and the ratio M^{n+2}/m^{n+1} has to be replaced by $\tilde{M}^{n+2}/\|\tilde{u}\|_{L^1}$.

In the last section of the paper we prove similar stability results for *quasiconcave* functions, i.e., functions having convex level sets. In order to extend the estimate (1.1) to this more general setting, we have not only to prevent u from having a plateau but also to control in a quantitative way how flat is the subgraph, i.e., how much ∇u is far from being null. From this point of view, a reasonable condition is to require that $|\nabla u|$ is bounded from below by some positive constant. The precise statement is given in Theorem 4.1.

2. Preliminary results

We start by recalling some definitions and basic results that will be instrumental in the proof of the quantitative estimate (1.1). In the following, a point $x \in \mathbb{R}^n$, when $n \geq 2$, will be usually labeled by (x', x_n), where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$; similarly, when $\xi \in \mathbb{R}^{n+1}$ we shall write it as (x', x_n, t). To emphasize the different roles of the variables x_n and t we shall also write $\mathbb{R}^{n+1} = \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}_t$ with the obvious meaning. Consistently, we shall denote by $\nabla u = (\nabla_{x'} u, \partial_{x_n} u)$ the gradient of a function u depending on the variables (x', x_n) and by $\nabla f = (\nabla_{x'} f, \partial_t f)$ the gradient of a function f depending on (x', t). A vector $\nu \in \mathbb{R}^{n+1}$ will be also written as $(\nu_{x'}, \nu_{x_n}, \nu_t) \in \mathbb{R}^{n+1} = \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}_t$. Finally, for any integer $k \geq 1$, $\mathcal{L}^k$ stands for the outer Lebesgue measure in $\mathbb{R}^k$, while $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure.

Given any measurable set $E \subset \mathbb{R}^{n+1}$, for every $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}_t$ we set

$$E_{x',t} := \{x_n \in \mathbb{R} : (x', x_n, t) \in E\} \quad \text{and} \quad l(x', t) = \frac{1}{2}\mathcal{L}^1(E_{x',t}). \tag{2.1}$$

Then, we define the *Steiner symmetral* E^s of E about the hyperplane $\{x_n = 0\}$ as

$$E^s := \{(x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}_t : |x_n| < l(x', t)\}.$$

We define also the *barycenter* of the slice $E_{x',t}$ as

$$b(x', t) := \begin{cases} \dfrac{1}{2l(x', t)} \displaystyle\int_{E_{x',t}} x_n \, dx_n & \text{if } 0 < l(x', t) < \infty \text{ and } |x_n| \in L^1(E_{x',t}); \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Finally, if E is a measurable set in $\mathbb{R}^n$, its Steiner symmetral about the hyperplane $\{x_n = 0\}$ is defined similarly, after replacing (2.1) by parallel definitions of $E_{x'}$ and $l(x')$.

If $E \subset \mathbb{R}^{n+1}$ is a measurable set and U is an open set in $\mathbb{R}^{n+1}$, the *perimeter* of E in U will be denoted by $P(E; U)$, while $P(E)$ will be a short-hand notation for $P(E; \mathbb{R}^{n+1})$. If E has finite perimeter we denote by $\partial^* E$ its *reduced boundary* and by ν^E the (generalized) *inner normal*. For the precise definitions of these quantities and the main properties of sets of finite perimeter we refer to [1, Chapter 3]. Let Ω be an open set in $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Here and in the following we will denote, with a slight abuse of notation, by $\Omega \times \mathbb{R}$ the cylinder $\{(x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}_t : (x', t) \in \Omega\}$. If $E \subset \mathbb{R}^{n+1}$ is a set of finite perimeter, we say that E does not have *plateaus* in the direction x_n inside the cylinder $\Omega \times \mathbb{R}$ if

$$\mathcal{H}^n(\{\xi \in \partial^* E : \nu^E_{x_n}(\xi) = 0\} \cap (\Omega \times \mathbb{R})) = 0. \tag{2.3}$$

We say that E has the *segment property* in the direction x_n inside $\Omega \times \mathbb{R}$ if for a.e. $(x', t) \in \Omega$

$$E_{x',t} \text{ is } \mathcal{L}^1\text{-equivalent to a segment.} \tag{2.4}$$

The following result, which is contained in [10, Proposition 1.2] and [2, Theorem 4.3], relates the segment property and the absence of plateaus to the regularity properties of the functions l and b.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R}_t$ be an open set and let $E \subset \mathbb{R}^{n+1}$ be a set of finite perimeter in $\Omega \times \mathbb{R}$ such that $\mathcal{L}^{n+1}(E \cap (\Omega \times \mathbb{R})) < \infty$. If E satisfies* (2.3)*, then $l \in W^{1,1}(\Omega)$ and $\|\nabla l\|_{L^1(\Omega)} \leq P(E; \Omega \times \mathbb{R})/2$. Conversely, if $l \in W^{1,1}(\Omega)$ then the Steiner symmetral E^s of E satisfies* (2.3)*. Moreover, if E has also the segment property* (2.4) *and if the Lebesgue representative l^* of l is strictly positive $\mathcal{H}^{n-1}$-a.e. in Ω, then $b \in W^{1,1}_{\text{loc}}(\Omega)$ and $\|\nabla b\|_{L^1(\Omega)} \leq P(E; \Omega \times \mathbb{R})/2$.*

Note that if E is bounded then also b is bounded, hence $b \in W^{1,1}(\Omega)$.

Let $u : \mathbb{R}^n \to [0, \infty)$ be a measurable function such that for a.e. $x' \in \mathbb{R}^{n-1}$

$$l_u(x', t) := \frac{1}{2}\mathcal{L}^1\left(\{x_n \in \mathbb{R} : u(x', x_n) > t\}\right) < \infty \text{ for all } t > 0.$$

Then, the *Steiner rearrangement* of u with respect to the hyperplane $\{x_n = 0\}$ is the function $u^s : \mathbb{R}^n \to [0, \infty)$ defined for any $(x', x_n) \in \mathbb{R}^n$ by

$$u^s(x) := \inf\left\{t > 0 : l_u(x', t) \leq |x_n|\right\}.$$

Note that if we denote by E_u the *subgraph* of u, i.e.,

$$E_u := \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_t : 0 < t < u(x)\},$$

we have $l_u(x', t) = \mathcal{L}^1((E_u)_{x',t})/2$ for all $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}_t$.

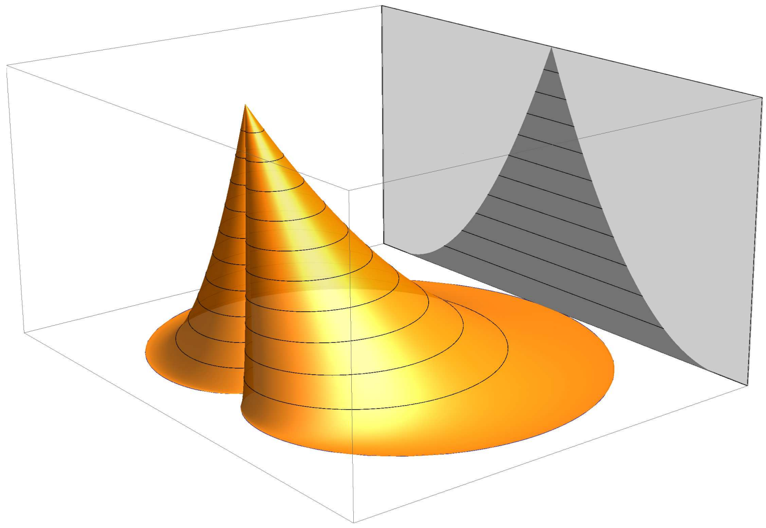

FIGURE 1. In dark grey Ω_u and in grey the hyperplane $\{x_n = 0\}$.

Lemma 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded open set and let $u : D \to [0, \infty)$ be a continuous function. Then*

$$(E_u)^s = E_{u^s}.$$

Proof. Being D bounded, l_u is finite everywhere in $\mathbb{R}^{n-1} \times (0, \infty)$. By freezing the variable x', we reduce to the case $n = 1$. First of all we prove that

$$\lambda(t) := \mathcal{L}^1(\{y \in \mathbb{R} : u^s(y) > t\}) = 2l_u(t) \quad \text{for any } t > 0. \tag{2.5}$$

Indeed, given $y \in \mathbb{R}$, if $t < u^s(y) = \inf\{s > 0 : l_u(s) \leq |y|\}$ then $|y| < l_u(t)$ and so $\lambda(t) \leq 2l_u(t)$. On the other hand, by the continuity of u, for any $\varepsilon > 0$

$$u^s(l_u(t+\varepsilon)) = \inf\{s > 0 : l_u(s) \leq l_u(t+\varepsilon)\} = t + \varepsilon > t.$$

Since $u^s(|\cdot|)$ is non-increasing, $\{y > 0 : u^s(y) > t\} \supset (0, l_u(t+\varepsilon))$ and so

$$\lambda(t) \geq 2l_u(t+\varepsilon) = 2l_u(t) - \mathcal{L}^1(\{y \in \mathbb{R} : t + \varepsilon \geq u(y) > t\}).$$

For $\varepsilon \to 0^+$ we have $\lambda(t) \geq 2l_u(t)$.

Now, $(x, t) \in E_{u^s}$ means that $t < \inf\{s > 0 : l_u(s) \leq |x|\}$, and this implies $|x| < l_u(t)$, i.e., $(x, t) \in (E_u)^s$. On the other hand, $(x, t) \in (E_u)^s$ means that $|x| < l_u(t)$, and this implies $l_u(s) < l_u(t)$ for all $s > 0$ such that $l_u(s) \leq |x|$. Since l_u is non-increasing, $t < s$ and so $t \leq u^s(x)$. We use (2.5) to prove that $t \neq u^s(x)$. In fact, if $t = u^s(x)$ we have the contradiction

$$\lambda(t) = \mathcal{L}^1(\{y \in \mathbb{R} : u^s(y) > u^s(x)\}) \leq \mathcal{L}^1(\{y \in \mathbb{R} : |y| < |x|\}) = 2|x| < 2l_u(t).$$

□

We shall denote by Ω_u the projection of E_u on $\mathbb{R}^{n-1} \times \mathbb{R}_t$ (see Figure 1):

$$\Omega_u := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}_t : 0 < t < u(x', x_n) \text{ for some } x_n \in \mathbb{R}\}.$$

Note that $\Omega_u = \Omega_{u^s}$ up to a set of zero $\mathcal{H}^n$ measure, and that Ω_u is open if u is continuous. Finally, we shall denote by $b_u : \Omega_u \to \mathbb{R}$ the function giving the barycenter of the sections of E_u defined as in (2.2).

In what follows the function u will always be fixed. Therefore, when no confusion arises we shall often write E, Ω, l and b in place of E_u, Ω_u, l_u and b_u. Note that if $u \in W_0^{1,p}(D)$ for some $p \geq 1$, then its extension to $\mathbb{R}^n$ (obtained setting $u = 0$ outside D, and still denoted by u) belongs to $W^{1,p}(\mathbb{R}^n)$ and thus also $u^s \in W^{1,p}(\mathbb{R}^n)$. Moreover, using the $W^{1,p}$ continuity of the Steiner rearrangement (see [8]) it turns out that $u^s|_{D^s} \in W_0^{1,p}(D^s)$.

Our strategy to prove Theorem 1.1 is based on a "change of perspective". Using the segment property we are going to look at E_u as a domain bounded by the graphs of the functions $b_u - l_u$ and $b_u + l_u$. This will allow us to rewrite the integral $\int |\nabla u|^p$ in terms of l_u and b_u. To this aim we need the following result on the representation of the normals to a subgraph (see [18, Part I, Ch. 4, Sect. 1.5, Theorems 4 and 5]).

Theorem 2.3. *Let $D \subset \mathbb{R}^n$ be an open set and $u \in W^{1,1}_{\mathrm{loc}}(D)$ a non-negative function. Then E_u has locally finite perimeter in $D \times \mathbb{R}$ and for $\mathcal{H}^n$-a.e. $\xi \in \partial^* E_u \cap (D \times (0,\infty))$*

$$\nu^{E_u}(\xi) = \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \frac{-1}{\sqrt{1 + |\nabla u|^2}} \right). \tag{2.6}$$

The next lemma is based on the representation formula (2.6).

Lemma 2.4. *Let $D \subset \mathbb{R}^n$ be a bounded open set, $u \in W^{1,p}(D)$, $p \geq 1$, be a non-negative and continuous function whose subgraph E_u satisfies* (2.3) *and* (2.4). *Then*

$$\partial_t b_u + \partial_t l_u < 0 \quad \textit{and} \quad \partial_t b_u - \partial_t l_u > 0 \quad \textit{a.e. in } \Omega_u \tag{2.7}$$

and

$$\int_D |\nabla u|^p \, dx = \int_{\Omega_u} \frac{(1 + |\nabla_{x'} b_u + \nabla_{x'} l_u|^2)^{\frac{p}{2}}}{|\partial_t b_u + \partial_t l_u|^{p-1}} \, dx' dt + \int_{\Omega_u} \frac{(1 + |\nabla_{x'} b_u - \nabla_{x'} l_u|^2)^{\frac{p}{2}}}{|\partial_t b_u - \partial_t l_u|^{p-1}} \, dx' dt. \tag{2.8}$$

Proof. We shall drop the subscript in E_u and in all the other quantities involving u. Note that by Theorem 2.1 both l and b belong to $W^{1,1}(\Omega)$. Since E has the segment property in the direction x_n, we have $E = E^+ \cap E^-$, where

$$\begin{aligned} E^+ &:= \left\{ (x,t) \in \mathbb{R}^{n+1} : (x',t) \in \Omega,\, x_n < b(x',t) + l(x',t) \right\}, \\ E^- &:= \left\{ (x,t) \in \mathbb{R}^{n+1} : (x',t) \in \Omega,\, x_n > b(x',t) - l(x',t) \right\}. \end{aligned}$$

Note that, since u is continuous, Ω is an open set. Since almost every point of $\partial^* E^+$ lies on the graph of both u and $b + l$, by applying (2.6) twice we get that for $\mathcal{H}^n$-a.e. $\xi \in \partial^* E^+$, $\xi = (x', b(x',t) + l(x',t), t)$,

$$\begin{aligned} \nu(\xi) &= \left(\frac{\nabla_{x'} b + \nabla_{x'} l}{\sqrt{1 + |\nabla b + \nabla l|^2}}, \frac{-1}{\sqrt{1 + |\nabla b + \nabla l|^2}}, \frac{\partial_t b + \partial_t l}{\sqrt{1 + |\nabla b + \nabla l|^2}} \right) \\ &= \left(\frac{\nabla_{x'} u}{\sqrt{1 + |\nabla u|^2}}, \frac{\partial_{x_n} u}{\sqrt{1 + |\nabla u|^2}}, \frac{-1}{\sqrt{1 + |\nabla u|^2}} \right), \end{aligned} \tag{2.9}$$

where ν stands for the generalized inner normal to E and the functions b, l are evaluated at the point (x', t), while u is evaluated at $x = (x', b(x',t) + l(x',t))$. The corresponding formula for a point $\xi = (x', b(x',t) - l(x',t), t) \in \partial^* E^-$ gives for the inner normal

$$\begin{aligned} \nu(\xi) &= \left(\frac{\nabla_{x'} l - \nabla_{x'} b}{\sqrt{1+|\nabla l - \nabla b|^2}}, \frac{1}{\sqrt{1+|\nabla l - \nabla b|^2}}, \frac{\partial_t l - \partial_t b}{\sqrt{1+|\nabla l - \nabla b|^2}} \right) \\ &= \left(\frac{\nabla_{x'} u}{\sqrt{1+|\nabla u|^2}}, \frac{\partial_{x_n} u}{\sqrt{1+|\nabla u|^2}}, \frac{-1}{\sqrt{1+|\nabla u|^2}} \right). \end{aligned} \tag{2.10}$$

Note that by comparing the last components of the vectors representing ν in (2.9) and (2.10) we deduce in particular the first and the second inequality in (2.7), respectively.

By the second equalities in (2.9) and (2.10) we have, using the area formula,

$$\int_D |\nabla u|^p \, dx = \int_{\partial^* E \cap (D \times (0,\infty))} \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^2}} d\mathcal{H}^n = \int_{\partial^* E \cap (D \times (0,\infty))} \frac{|\nu_x|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n .$$

Setting $A := \{(x,t) \in \mathbb{R}^{n+1} : (x',t) \in \Omega\}$ and splitting the last integral over the boundaries of E^+ and E^-, by the first equalities in (2.9) and (2.10), we get

$$\begin{aligned} \int_D |\nabla u|^p \, dx &= \int_{A \cap \partial^* E^+} \frac{|\nu_x|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n + \int_{A \cap \partial^* E^-} \frac{|\nu_x|^p}{|\nu_t|^{p-1}} d\mathcal{H}^n \\ &= \int_{A \cap \partial^* E^+} \frac{(1+|\nabla_{x'} b + \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}} \frac{1}{\sqrt{1+|\nabla b + \nabla l|^2}} d\mathcal{H}^n \\ &\quad + \int_{A \cap \partial^* E^-} \frac{(1+|\nabla_{x'} b - \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}} \frac{1}{\sqrt{1+|\nabla b - \nabla l|^2}} d\mathcal{H}^n \\ &= \int_\Omega \frac{(1+|\nabla_{x'} b + \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}} \, dx' dt + \int_\Omega \frac{(1+|\nabla_{x'} b - \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}} \, dx' dt. \end{aligned}$$

□

Remark 2.5. If u satisfies the assumptions of Lemma 2.4, using the equality $(E_u)^s = E_{u^s}$ stated in Lemma 2.2 and applying Theorem 2.1, we get that also E_{u^s} satisfies condition (2.3). On the other hand, E_{u^s} trivially satisfies the segment property (2.4), therefore (2.8) holds also for u^s and becomes

$$\int_{D^s} |\nabla u^s|^p \, dx = 2 \int_\Omega \frac{(1+|\nabla_{x'} l_u|^2)^{\frac{p}{2}}}{|\partial_t l_u|^{p-1}} \, dx' dt \,.$$

Concave functions, and more generally log-concave and the α-concave functions, satisfy condition (2.3), as shown in the next two lemmas. Therefore, we may apply to these functions the results of Lemma 2.4.

Lemma 2.6. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and $u : D \to (0,\infty)$ a concave function. Then u^s is concave in D^s and E_u satisfies condition* (2.3).

Proof. Since u is concave, E_u is convex. Therefore, since the convexity is preserved by the Steiner symmetrization, $(E_u)^s$ is convex. By the equality $(E_u)^s = E_{u^s}$ proved in Lemma 2.2, it follows that u^s is concave. Moreover, since E_u is convex, there exist two functions f_1, $f_2 : \Omega_u \to \mathbb{R}$, f_1 concave and f_2 convex, such that

$$E_u = \{(x', x_n, t) \in \mathbb{R}^{n+1} : (x',t) \in \Omega_u \text{ and } f_2(x',t) < x_n < f_1(x',t)\}.$$

From (2.9) and (2.10) we get that for $\mathcal{H}^n$-a.e. $(x', t) \in \Omega_u$

$$\nu_{x_n}^{E_u}(x', f_1(x',t), t) = -\frac{1}{\sqrt{1+|\nabla f_1(x',t)|^2}}, \qquad \nu_{x_n}^{E_u}(x', f_2(x',t), t) = \frac{1}{\sqrt{1+|\nabla f_2(x',t)|^2}},$$

and so condition (2.3) follows. □

Next lemma deals with the case of log-concave and α-concave functions.

Lemma 2.7. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and $u : D \to (0, \infty)$ a log-concave function. Then u^s is log-concave in D^s and E_u satisfies condition* (2.3)*. Similarly, if u is α-concave for some $\alpha \in (0, 1]$, then also u^s is α-concave and E_u satisfies* (2.3)*.*

Proof. We shall only give the proof in the log-concave case, the other case being similar. Set $\tilde{u} := \ln u$. Observing that $\ln u^s = (\tilde{u})^s$, by Lemma 2.6 we can conclude that $\ln u^s$ is concave in D^s. Moreover, arguing as in the proof of Lemma 2.6, setting $\tilde{E} := \{(x', x_n, t) \in \mathbb{R}^{n+1} : (x', x_n) \in D,\ \tilde{u}(x', x_n) > t\}$, we have that $\nu_{x_n}^{\tilde{E}} \neq 0$ $\mathcal{H}^n$-a.e. in $\partial\tilde{E}$, i.e., $\tilde{E}$ satisfies condition (2.3). Since $E = \phi(\tilde{E})$, where $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the map $\phi(x, t) := (x, e^t)$, we get that also E satisfies (2.3). □

Once the integral $\int_D |\nabla u|^p$ is rewritten in terms of b and l, we will evaluate $\Delta(u, u^s)$ only in terms of the integral of the product of $|\nabla b|$ and a suitable weight function. In order to conclude, we will need the following weighted Poincaré inequalities. The first one is an ad hoc extension of [2, Corollary 5.2] and will be used in the proof of Theorem 1.2, with $w = \tilde{u}^s(\cdot, 0)$, where $\tilde{u} = u^\alpha$. The second one has been proved in [2] and will be used in the proof of Theorem 1.1 to deal with the log-concave case.

Proposition 2.8. *Let $C \subset \mathbb{R}^{n-1}$, $n \geq 2$, be a bounded open convex set, let $w : C \to (0, \infty)$ be a concave function, and let Ω be the subgraph of w, i.e.,*

$$\Omega := \left\{x \in \mathbb{R}^n : x' \in C \text{ and } 0 < x_n < w(x')\right\}.$$

Denote by r and R the inner and the outer radius of Ω, respectively. Given $\beta \geq 0$, the following weighted Poincaré inequality holds:

$$\int_\Omega |f(x) - \overline{f}| x_n^\beta \, dx \leq c\, \frac{R}{r} \int_\Omega |\nabla f(x)| x_n^\beta \operatorname{dist}(x, \partial\Omega)\, dx \qquad \text{for all } f \in W^{1,1}(\Omega), \tag{2.11}$$

where $\overline{f} \in \mathbb{R}$ is a suitably defined average of f and c is a constant depending only on n and β.

Proof. By a density argument we may assume that $f \in C^1$. Define

$$U := \left\{x \in \mathbb{R}^n : x' \in C \text{ and } |x_n| < w(x')\right\}.$$

The set U is a convex set with outer radius smaller than $2R$ and inner radius larger than r. By John's Theorem (see [3, Ch.V, Theorem 2.4] and [19]), the maximum volume ellipsoid S included in U is unique and the inclusions $S \subset U \subset nS$ hold. Note that, since U is symmetric with respect to the hyperplane $\{x_n = 0\}$, also S has to be symmetric: if not, the Steiner symmetral of S would be an ellipsoid different from S, but with the same volume (see [4, Lemma 2]).

Up to a translation and a rotation around the x_n-axis, we may assume that $S = \{x \in \mathbb{R}^n : \sum_{i=1}^n (x_i/a_i)^2 < 1\}$ for some $a_1, \ldots, a_n > 0$. We have that $\max\{a_i\} \leq 2R$ and $\min\{a_i\} \geq r/n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation mapping the unit ball B_1 on S represented by

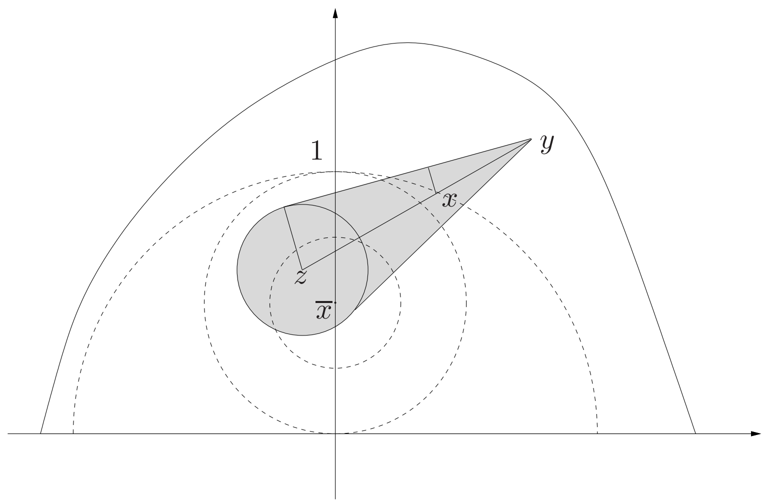

FIGURE 2. In grey the convex set K.

the matrix $(\delta_{ij}a_i)$. Set $\hat{f} = f \circ \phi$, $\hat{U} = \phi^{-1}(U)$ and $\hat{\Omega} = \phi^{-1}(\Omega)$. The following estimates hold:

$$|\nabla f(\phi(y))| = |\nabla \hat{f}(y)(\nabla\phi(y))^{-1}| = \sqrt{\sum_i a_i^{-2}(\partial_i \hat{f}(y))^2} \geq \frac{|\nabla \hat{f}(y)|}{\max\{a_i\}} \geq \frac{|\nabla \hat{f}(y)|}{2R};$$

$$\operatorname{dist}(\phi(y), \partial\Omega) = \inf_{z\in\partial\hat{\Omega}} \sqrt{\sum_i a_i^2(y_i - z_i)^2} \geq \min\{a_i\}\operatorname{dist}(y, \partial\hat{\Omega}) \geq \frac{r}{n}\operatorname{dist}(y, \partial\hat{\Omega}).$$

Since $\det \nabla\phi = \mathcal{L}^n(S)/\mathcal{L}^n(B_1)$ is constant, we have for any $\overline{f} \in \mathbb{R}$

$$\int_\Omega |f(x) - \overline{f}|x_n^\beta \, dx = a_n^\beta \det \nabla\phi \int_{\hat{\Omega}} |\hat{f}(y) - \overline{f}|y_n^\beta \, dy;$$

$$\int_\Omega |\nabla f(x)|x_n^\beta \operatorname{dist}(x, \partial\Omega) \, dx \geq \frac{r}{2nR} a_n^\beta \det \nabla\phi \int_{\hat{\Omega}} |\nabla \hat{f}(y)|y_n^\beta \operatorname{dist}(y, \partial\hat{\Omega}) \, dy.$$

Therefore, it will be enough to show that for some $\overline{f} \in \mathbb{R}$ and some c depending only on n and β

$$\int_{\hat{\Omega}} |\hat{f}(y) - \overline{f}|y_n^\beta \, dy \leq c \int_{\hat{\Omega}} |\nabla \hat{f}(y)|y_n^\beta \operatorname{dist}(y, \partial\hat{\Omega}) \, dy. \tag{2.12}$$

Note that $B_1 \subset \hat{U} \subset B_n$, and that in particular $\hat{\Omega}$ includes the ball of radius $1/2$ centered in $\overline{x} := (0, \ldots, 0, 1/2)$. Let $\overline{f}$ denote the average of $\hat{f}$ on $B := B_{1/4}(\overline{x})$. For every $y \in \hat{\Omega}$ and $z \in B$ we have

$$\hat{f}(z) - \hat{f}(y) = \int_0^1 \nabla \hat{f}((1-s)y + sz) \cdot (z - y) \, ds.$$

Multiplying by $\frac{1}{\mathcal{L}^n(B)}\chi_B(z)$ and integrating over z

$$\overline{f} - \hat{f}(y) = \frac{1}{\mathcal{L}^n(B)} \int_B \int_0^1 \nabla \hat{f}((1-s)y + sz) \cdot (z - y) \, dsdz.$$

Passing to the absolute value, multiplying by y_n^β, and integrating with respect to y, we get

$$\int_{\hat{\Omega}} |\hat{f}(y) - \overline{f}|y_n^\beta \, dy \leq \frac{1}{\mathcal{L}^n(B)} \int_{\hat{\Omega}} \int_B \int_0^1 |\nabla \hat{f}((1-s)y + sz)||z - y|y_n^\beta \, dsdzdy.$$

Setting $x=(1-s)y+sz$ and changing variables from z to x, we get

$$\int_{\hat\Omega}|\hat f(y)-\overline{f}|y_n^\beta\,dy\le\frac{1}{\mathcal{L}^n(B)}\int_{\hat\Omega}\int_{\hat\Omega}\int_0^1|\nabla\hat f(x)|\frac{|x-y|}{s^{n+1}}y_n^\beta\chi_B\left(\frac{x-(1-s)y}{s}\right)dsdxdy.$$

In order to estimate the integral on the right hand side we start by noticing that

$$|x-y|\le 4(n+1)\mathrm{dist}(x,\partial\hat\Omega).$$

In fact, if $y\notin B_{1/4}(z)$, indicating by K the convex hull of $B_{1/4}(z)\cup\{y\}$, we have (see Figure 2)

$$|x-y|=4|z-y|\mathrm{dist}(x,\partial K)\le 4(n+1)\mathrm{dist}(x,\partial\hat\Omega)$$

because $K\subset\hat\Omega$. Conversely, if $y\in B_{1/4}(z)$, we have $|x-y|\le\mathrm{dist}(x,\partial\hat\Omega)$. Moreover, for x and y fixed, the values of s for which $z\in B$ are such that

$$\frac{|x-y|}{s}-|y-\overline{x}|\le\left|\frac{x-(1-s)y}{s}-\overline{x}\right|\le\frac14,\quad\text{and so } s\ge\frac{4|x-y|}{1+4|y-\overline{x}|}\ge\frac{4|x-y|}{5(n+1)}.$$

Observe also that one has always $y_n\le 4nx_n$. Indeed, this is trivial if $y_n\le x_n$. On the other hand, if $x_n\le y_n$, then $x_n\ge z_n\ge 1/4$ and so $y_n\le n\le 4nx_n$.

Finally, setting $A(x):=\{y\in\hat\Omega:|x-y|\le 4(n+1)\mathrm{dist}(x,\partial\hat\Omega)\}$ and interchanging the order of integration, we have

$$\begin{aligned}
\int_{\hat\Omega}|\hat f(y)-\overline{f}|y_n^\beta\,dy&\le\frac{(4n)^\beta}{\mathcal{L}^n(B)}\int_{\hat\Omega}|\nabla\hat f(x)|x_n^\beta\int_{A(x)}|x-y|\int_{\frac{4|x-y|}{5(n+1)}}^1\frac{1}{s^{n+1}}\,dsdydx\\
&=c\int_{\hat\Omega}|\nabla\hat f(x)|x_n^\beta\int_{A(x)}\frac{(5(n+1))^n-4^n|x-y|^n}{4^n|x-y|^{n-1}}\,dydx\\
&\le c\int_{\hat\Omega}|\nabla\hat f(x)|x_n^\beta\int_{A(x)}\frac{1}{|x-y|^{n-1}}\,dydx\\
&=c\int_{\hat\Omega}|\nabla\hat f(x)|x_n^\beta\int_0^{4(n+1)\mathrm{dist}(x,\partial\hat\Omega)}n\omega_n\,d\rho\,dx\\
&=c\int_{\hat\Omega}|\nabla\hat f(x)|x_n^\beta\mathrm{dist}(x,\partial\hat\Omega)\,dx,
\end{aligned}$$

thus proving (2.12). □

The next result has been proved in [2, Proposition 5.1].

Proposition 2.9. *Let $\Omega\subset\mathbb{R}^n$, $n\ge1$, be an open set, and let $\overline{x},\widetilde{x}\in\Omega$ and $0<r\le R<\infty$ be such that $B_r(\overline{x})\subset\Omega\subset B_R(\widetilde{x})$. Assume that Ω is star-shaped with respect to $B_r(\overline{x})$. Then,*

$$\int_\Omega|f(x)-f_{B_{\frac r2}(\overline{x})}|\,dx\le\left(\frac{4R}{r}\right)^{n+1}\int_\Omega|\nabla f(x)|\,\mathrm{dist}(x,\partial\Omega)\,dx\qquad\text{for all } f\in W^{1,1}(\Omega),$$

where $f_{B_{\frac r2}(\overline{x})}$ denotes the average of f on $B_{\frac r2}(\overline{x})$.

To conclude this section, we provide three simple lemmas of geometric nature.

Lemma 2.10. *Let $C\subset\mathbb{R}^n$ be a bounded open convex set with outer radius R. Then its inner radius r can be estimated by*

$$r\ge\frac{\mathcal{L}^n(C)}{\omega_n n^n R^{n-1}}.$$

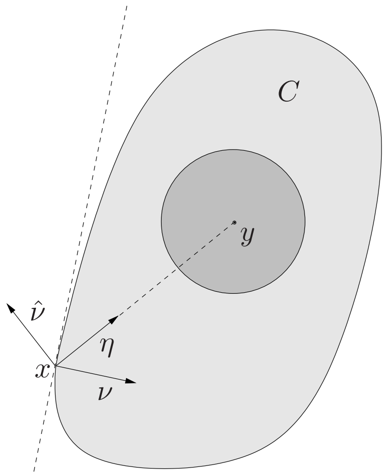

FIGURE 3. In dark grey the ball $B_r(y)$.

Proof. Let S be the maximum ellipsoid included in C. Up to a rigid motion, we may assume that $S = \{x \in \mathbb{R}^n : \sum_{i=1}^n (x_i/l_i)^2 < 1\}$ with $l_1 \leq \ldots \leq l_n$. Clearly, $R \geq l_n$ and $r \geq l_1$. By John's ellipsoid theorem (see [3, Ch.V, Theorem 2.4]), the inclusion $C \subset nS$ holds and therefore $\omega_n r R^{n-1} \geq \mathcal{L}^n(S) \geq \mathcal{L}^n(C)/n^n$. $\square$

Lemma 2.11. *Let $C \subset \mathbb{R}^n$ be a bounded open convex set, and let $B_r(y) \subset C$. Given a point $x \in \partial C$ where the inner normal ν is defined, and set $\eta = (y-x)/|y-x|$, we have*

$$\langle \nu, \eta \rangle \geq \frac{r}{2|y-x|}. \tag{2.13}$$

Proof. If $\langle \nu, \eta \rangle \geq 1/2$, then (2.13) is trivial, since $R := |y-x| \geq r$. So, assume $\langle \nu, \eta \rangle < 1/2$, and consider the vector $\hat{\nu}$ obtained by normalizing $\langle \nu, \eta \rangle \eta - \nu$. Note that $|\langle \nu, \eta \rangle \eta - \nu| \geq 1/2$. Since $B_r(y)$ is included in the half-space $\{z \in \mathbb{R}^n : \langle z - x, \nu \rangle > 0\}$ (see Figure 3), one has

$$0 \leq \langle y + r\hat{\nu} - x, \nu \rangle = \langle y - x, \nu \rangle + r\langle \hat{\nu}, \nu \rangle \leq 2R\langle \nu, \eta \rangle + 2r \left(\langle \nu, \eta \rangle^2 - 1 \right),$$

and therefore

$$\langle \nu, \eta \rangle \geq \frac{-R + \sqrt{R^2 + 4r^2}}{2r} = \frac{2r}{R + \sqrt{R^2 + 4r^2}} \geq \frac{r}{2R}.$$

$\square$

Lemma 2.12. *Let μ and η be unitary vectors in $\mathbb{R}^n$, and let*

$$F_1 := \{z \in B_R : \langle z, \mu \rangle > 0\} + (0, x_n) \quad \text{and} \quad F_2 := \{z \in B_R : \langle z, \eta \rangle > 0\} + (0, y_n)$$

be two half balls with centers on the x_n coordinate axis, whose intersection contains a ball of radius r. Then there exists a positive constant c depending only on n and on the ratio r/R such that

$$|\mu_n||\eta_n| + \langle \mu', \eta' \rangle \geq c - 1.$$

Proof. The intersection $\mathcal{L}^n(F_1 \cap F_2)$ is a continuous function of μ, η, $|x_n - y_n|$, and R, and it is null for $|x_n - y_n| > 2R$. If $|\mu_n||\eta_n| + \langle \mu', \eta' \rangle = -1$, then $\mathcal{L}^n(F_1 \cap F_2) = 0$ (see Figure 4). Therefore, if $\mathcal{L}^n(F_1 \cap F_2) \geq \omega_n r^n > 0$, by compactness $|\mu_n||\eta_n| + \langle \mu', \eta' \rangle \geq c - 1$ for some constant $c = c(n, r, R) > 0$. By a rescaling argument, $c = c(n, r/R)$. $\square$

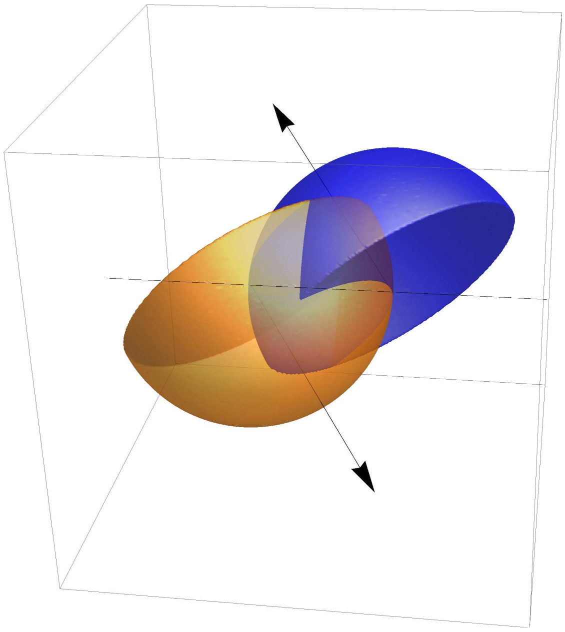

FIGURE 4. The intersection of the two half balls F_1 and F_2.

3. Quantitative estimates

The present section focuses on the proof of our main results. We start with a technical lemma.

Lemma 3.1. *Consider the function $f_p : \mathbb{R}^{n-1} \times (0,\infty) \to \mathbb{R}$ defined as*

$$f_p(x) := \frac{(1+|x'|^2)^{\frac{p}{2}}}{x_n^{p-1}}. \tag{3.1}$$

For a fixed x, given $y \in \mathbb{R}^n$, let $\Phi(x,y)$ be the second order increment of f in the direction y, i.e.,

$$\Phi(x,y) := f_p(x+y) + f_p(x-y) - 2f_p(x).$$

If $|y_n| \le x_n$, then there exists a positive constant $c = c(p)$ such that, for $1 < p \le 2$ we have

$$\Phi(x,y) \ge c\, \frac{(1+|x'|^2+|y'|^2)^{\frac{p-2}{2}}}{x_n^{p-1}} \left(\frac{|y'|^2}{1+|x'|^2} + \frac{|y_n|^2}{x_n^2} \right), \tag{3.2}$$

while for $p > 2$ we have

$$\Phi(x,y) \ge c\, \frac{1}{x_n^{p-1}} \left(\frac{|y'|^p}{(1+|x'|^2)^{\frac{p}{2}}} + \frac{|y_n|^p}{x_n^p} \right).$$

Proof. The proof is based on some careful estimates of a suitable expansion of $\Phi(x,y)$. First, observe that the function $(s,t) \in \mathbb{R} \times (0,\infty) \to t^{1-p}(1+s^2)^{p/2}$ is strictly convex. Hence, the strict convexity of f_p easily follows. Then, a straightforward second order expansion of $\Phi(x,\cdot)$ yields

$$\Phi(x,y) = p \int_{-1}^{1} (1-|\tau|) J_p(\tau)\, d\tau\,,$$

where

$$J_p(\tau) = \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}} \left[|y'|^2 + (p-2)\frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} \right.$$
$$\left. -2(p-1)\frac{\langle x'+\tau y', y'\rangle y_n}{x_n+\tau y_n} + (p-1)\frac{(1+|x'+\tau y'|^2)y_n^2}{(x_n+\tau y_n)^2} \right] \geq 0,$$

thanks to the convexity of Φ.

First case. Let us start treating the case $1 < p \leq 2$. By means of easily verifiable computations, we can write $J_p(\tau)$ as

$$\begin{aligned} J_p(\tau) =& \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}} \left[(p-1)\left(|y'|^2 - 2y_n\frac{\langle x'+\tau y', y'\rangle}{x_n+\tau y_n} + y_n^2 \frac{|x'+\tau y'|^2}{(x_n+\tau y_n)^2}\right) \right. \\ & \left. + (p-1)\frac{y_n^2}{(x_n+\tau y_n)^2} + (2-p)|y'|^2 + (p-2)\frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} \right] \\ =& \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{|x_n+\tau y_n|^{p-1}} \left[(p-1)\frac{|y_n x' - x_n y'|^2 + |y_n|^2}{|x_n+\tau y_n|^2} + (2-p)\left(|y'|^2 - \frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} \right) \right]. \end{aligned} \tag{3.3}$$

By Schwarz inequality

$$(2-p)\left(|y'|^2 - \frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} \right) \geq 0.$$

Moreover, taking into account that $|\tau| \leq 1$, the assumption $|y_n| \leq x_n$ implies $0 \leq x_n + \tau y_n \leq 2x_n$. Thus, we have

$$J_p(\tau) \geq c\, \frac{\left(1+|x'|^2+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2}. \tag{3.4}$$

Finally a dichotomy argument will give us the result. Indeed, if $2|y_n||x'| \geq x_n|y'|$, we can write

$$\frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2} \geq \frac{y_n^2}{x_n^2} \geq \frac{y_n^2}{2x_n^2} + \frac{|y'|^2}{8|x'|^2}.$$

Otherwise, if $2|y_n||x'| < x_n|y'|$, or equivalently

$$2(x_n|y'| - |y_n||x'|) > x_n|y'|,$$

we have

$$\frac{|y_n x' - x_n y'|^2 + y_n^2}{x_n^2} \geq \frac{(x_n|y'| - |y_n||x'|)^2}{x_n^2} + \frac{y_n^2}{x_n^2} \geq \frac{|y'|^2}{4} + \frac{y_n^2}{x_n^2}. \tag{3.5}$$

Combining (3.4) and (3.5), we get (3.2).

Second case. The case $p > 2$ is more involved. While in the previous case we could get the result just performing pointwise estimates, here we will need to exploit the integral form of $\Phi(x, y)$. We use again a dichotomy argument. Let γ be sufficiently large so that $(p-1)\frac{(\gamma-1)^2}{(\gamma+3)^2} + 2 - p \geq \frac{1}{2}$ (note that this condition implies in particular $\gamma > 6$) and suppose first that $|y'| > \gamma|x'|$. Then we have

$$|x'+\tau y'| \geq |\tau||y'| - |x'| \geq \left(\tau - \frac{1}{\gamma}\right)|y'| \tag{3.6}$$

and, for $2/\gamma \le \tau \le 3/\gamma$,

$$\frac{|y_n x' - x_n y'|}{|x_n + \tau y_n|} \ge \frac{|y'||x_n| - |x'||y_n|}{|x_n| + \tau|y_n|} \ge \frac{|y'| - |x'|}{1+\tau} \ge |y'|\frac{1-1/\gamma}{1+3/\gamma} = |y'|\frac{\gamma-1}{\gamma+3}\,. \tag{3.7}$$

Using (3.3), (3.6) and (3.7), recalling the choice of γ and that $J_p \ge 0$, we estimate

$$\begin{aligned}
\Phi(x,y) =& p\int_{-1}^{1}(1-|\tau|)J_p(\tau)\,d\tau \ge \int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}}(1-|\tau|)J_p(\tau)\,d\tau \ge \frac{\gamma-3}{\gamma}\int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} J_p(\tau)d\tau \\
\ge& c\int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}}\left[(p-1)\frac{|y_n x' - x_n y'|^2 + y_n^2}{(x_n+\tau y_n)^2} + (2-p)|y'|^2\right]d\tau \\
\ge& c\int_{\frac{2}{\gamma}}^{\frac{3}{\gamma}} \frac{\left(1+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}}\left[(p-1)\left(\frac{(\gamma-1)^2}{(\gamma+3)^2}|y'|^2 + \frac{y_n^2}{4x_n^2}\right) + (2-p)|y'|^2\right]d\tau \\
\ge& c\frac{\left(1+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}}\left(|y'|^2 + \frac{y_n^2}{x_n^2}\right) \ge c\frac{1}{x_n^{p-1}}\left(|y'|^p + \frac{|y_n|^p}{x_n^p}\right).
\end{aligned}$$

On the other side, when $|y'| \le \gamma|x'|$ and $|\tau| \le 1/2\gamma$,

$$|x'+\tau y'| \ge |x'| - |\tau||y'| \ge (1-|\tau|\gamma)|x'| \ge \frac{1}{2}|x'|.$$

Let $\beta = \frac{1}{2(p-1)}$. By rearranging the expression of J_p we have

$$\begin{aligned}
J_p(\tau) =& \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}}\Bigg[|y'|^2 + (p-2)\frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} \\
&\quad - 2(p-1)\frac{\langle x'+\tau y', y'\rangle y_n}{x_n+\tau y_n} + (p-1)\frac{(1+|x'+\tau y'|^2)y_n^2}{(x_n+\tau y_n)^2}\Bigg] \\
=& \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}}\Bigg[|y'|^2 - \frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} + (p-1)\frac{\langle x'+\tau y', y'\rangle^2}{1-\beta+|x'+\tau y'|^2} \\
&\quad + (p-1)\langle x'+\tau y', y'\rangle^2\left(\frac{1}{1+|x'+\tau y'|^2} - \frac{1}{1-\beta+|x'+\tau y'|^2}\right) \\
&\quad - 2(p-1)\frac{\langle x'+\tau y', y'\rangle y_n}{x_n+\tau y_n} + (p-1)\frac{\beta y_n^2}{(x_n+\tau y_n)^2} \\
&\quad + (p-1)\frac{(1-\beta+|x'+\tau y'|^2)y_n^2}{(x_n+\tau y_n)^2}\Bigg] \\
=& \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}}\Bigg[|y'|^2 - \frac{\langle x'+\tau y', y'\rangle^2}{1+|x'+\tau y'|^2} + \beta\frac{(p-1)y_n^2}{(x_n+\tau y_n)^2} \\
&\quad - \beta(p-1)\frac{\langle x'+\tau y', y'\rangle^2}{(1+|x'+\tau y'|^2)(1-\beta+|x'+\tau y'|^2)} \\
&\quad + (p-1)\left(\frac{\langle x'+\tau y', y'\rangle}{\sqrt{1-\beta+|x'+\tau y'|^2}} - \frac{y_n\sqrt{1-\beta+|x'+\tau y'|^2}}{x_n+\tau y_n}\right)^2\Bigg].
\end{aligned}$$

Therefore, by using Schwarz inequality and removing the square term (recalling that $|y_n| \leq x_n$)

$$J_p(\tau) \geq \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{(x_n+\tau y_n)^{p-1}} \left[\frac{|y'|^2}{1+|x'+\tau y'|^2} + \beta\frac{(p-1)y_n^2}{(x_n+\tau y_n)^2} - \beta\frac{(p-1)|y'|^2}{1+|x'+\tau y'|^2}\right]$$
$$\geq \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{2^p x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'+\tau y'|^2} + \frac{y_n^2}{4x_n^2}\right].$$

This allows us to estimate $\Phi(x,y)$ as follows

$$\Phi(x,y) \geq c\int_{-\frac{1}{2\gamma}}^{\frac{1}{2\gamma}} \frac{\left(1+|x'+\tau y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'+\tau y'|^2} + \frac{y_n^2}{x_n^2}\right] d\tau$$
$$\geq c\int_{-\frac{1}{2\gamma}}^{\frac{1}{2\gamma}} \frac{\left(1+|y'|^2\right)^{\frac{p-2}{2}}}{x_n^{p-1}} \left[\frac{|y'|^2}{1+|x'|^2} + \frac{y_n^2}{x_n^2}\right] d\tau \geq c\,\frac{1}{x_n^{p-1}}\left(\frac{|y'|^p}{1+|x'|^2} + \frac{|y_n|^p}{x_n^p}\right).$$

□

We are now in position to give the proofs of our main results.

Proof of Theorem 1.1. From Lemma 2.7 it follows that the subgraph E_u of u satisfies the assumptions of Lemma 2.4. Therefore, we may rewrite $\int_D |\nabla u|^p$ and $\int_{D^s} |\nabla u^s|^p$ in terms of b_u and l_u. Hence, dropping the subscript in the notation of these quantities and in Ω_u, we have

$$\Delta(u,u^s) = \int_\Omega \frac{(1+|\nabla_{x'}b+\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t b+\partial_t l|^{p-1}} + \frac{(1+|\nabla_{x'}b-\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t b-\partial_t l|^{p-1}} - 2\frac{(1+|\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} dx'dt. \tag{3.8}$$

Recall that by (2.7) one has $\partial_t b+\partial_t l<0$ and $\partial_t b-\partial_t l>0$, and therefore $\partial_t l<0$ and $|\partial_t b|<|\partial_t l|$. The integrand in (3.8) can be written as $f_p(-\nabla l-\nabla b)+f_p(-\nabla l+\nabla b)-2f_p(-\nabla l)$, where f_p is the function defined in (3.1). Therefore, using Lemma 3.1 we have

$$\Delta(u,u^s) \geq \begin{cases} c\displaystyle\int_\Omega \frac{\left(1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2\right)^{\frac{p-2}{2}}}{|\partial_t l|^{p-1}} \left[\frac{|\nabla_{x'}b|^2}{1+|\nabla_{x'}l|^2} + \frac{|\partial_t b|^2}{|\partial_t l|^2}\right] dx'dt & \text{when } 1<p<2; \\ c\displaystyle\int_\Omega \frac{1}{|\partial_t l|^{p-1}} \left[\frac{|\nabla_{x'}b|^p}{(1+|\nabla_{x'}l|^2)^{\frac{p}{2}}} + \frac{|\partial_t b|^p}{|\partial_t l|^p}\right] dx'dt & \text{when } p\geq 2. \end{cases} \tag{3.9}$$

Using twice Hölder's inequality we get in the case $1<p<2$

$$\int_\Omega \frac{|\nabla_{x'}b|}{\sqrt{1+|\nabla_{x'}l|^2}} + \frac{|\partial_t b|}{|\partial_t l|}\, dx'dt \leq c\Delta(u,u^s)^{\frac{1}{2}} \left(\int_\Omega \frac{|\partial_t l|^{p-1}}{\left(1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2\right)^{\frac{p-2}{2}}}\, dx'dt\right)^{\frac{1}{2}}$$
$$\leq c\Delta(u,u^s)^{\frac{1}{2}} \left(\int_\Omega |\partial_t l|\, dx'dt\right)^{\frac{p-1}{2}} \left(\int_\Omega \sqrt{1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2}\, dx'dt\right)^{\frac{2-p}{2}}.$$

Using (2.8) with $p=1$, we obtain

$$\int_\Omega \sqrt{1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2}\, dx'dt \leq \int_\Omega \sqrt{1+|\nabla_{x'}b+\nabla_{x'}l|^2}\, dx'dt + \int_\Omega \sqrt{1+|\nabla_{x'}b-\nabla_{x'}l|^2}\, dx'dt$$
$$= \int_D |\nabla u| dx \leq \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u\|_{L^p},$$

while, denoting by ν^s the inner normal to ∂E^s and using (2.9) and (2.10) with $b \equiv 0$, we have

$$2\int_\Omega |\partial_t l|\, dx'dt = \int_{\partial E^s \cap (D^s \times \mathbb{R})} |\nu_t^s| d\mathcal{H}^n = \mathcal{L}^n(D^s) = \mathcal{L}^n(D).$$

Collecting all the previous estimates, we conclude that

$$\underbrace{\int_\Omega \frac{|\nabla_{x'} b|}{\sqrt{1+|\nabla_{x'} l|^2}}\, dx'dt}_{=:\mathrm{I}} + \underbrace{\int_\Omega \frac{|\partial_t b|}{|\partial_t l|}\, dx'dt}_{=:\mathrm{II}} \le c\mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u\|_{L^p}^{\frac{2-p}{2}} \Delta(u,u^s)^{\frac{1}{2}}. \tag{3.10}$$

Similarly, in the case $p \ge 2$ we have

$$\int_\Omega \frac{|\nabla_{x'} b|}{\sqrt{1+|\nabla_{x'} l|^2}}\, dx'dt + \int_\Omega \frac{|\partial_t b|}{|\partial_t l|}\, dx'dt \le c\Delta(u,u^s)^{\frac{1}{p}} \left(\int_\Omega |\partial_t l|\, dx'dt\right)^{\frac{1}{p'}} \le c\mathcal{L}^n(D)^{\frac{1}{p'}} \Delta(u,u^s)^{\frac{1}{p}}. \tag{3.11}$$

We now turn to the estimate of the left hand side of both (3.10) and (3.11).

Estimate of term I. Note that for all t the section E_t is convex and, by Lemma 2.7, also E_t^s is convex. Let us denote with $\mu(x')$ the inner normal to ∂E_t^s at the point $(x', l(x',t), t)$. Recalling the expression of ν^s given by the first equations in (2.9) and (2.10) with $b \equiv 0$, we have

$$\frac{1}{\sqrt{1+|\nabla_{x'} l(x',t)|^2}} = \frac{|\nu_{x_n}^s|}{|\nu_x^s|} = |\mu_{x_n}(x')|. \tag{3.12}$$

We want to show that

$$|\mu_{x_n}(x')| \ge \frac{\mathrm{dist}(x', \partial\Omega_t)}{\sqrt{2}M} \quad \forall x' \in \Omega_t, \tag{3.13}$$

where M is the maximum between $\|u\|_{L^\infty}$ and the outer radius of D. If $|\mu_{x_n}(x')| > 1/\sqrt{2}$, then (3.13) is trivial because $\mathrm{dist}(x', \partial\Omega_t) \le M$. On the other hand, if $|\mu_{x_n}(x')| \le 1/\sqrt{2}$ then $|\mu_{x'}(x')| > 1/\sqrt{2}$. Let w be the restriction of $l(\cdot, t)$ to the line through x' parallel to $\mu_{x'}$, i.e., $w(s) = l(x' + s\mu_{x'}(x')/|\mu_{x'}(x')|, t)$. Since

$$|w'(0)| = \left|\langle \nabla_{x'} l(x',t), \frac{\mu_{x'}(x')}{|\mu_{x'}(x')|}\rangle\right| = \left|\langle \frac{\mu_{x'}(x')}{|\mu_{x_n}(x')|}, \frac{\mu_{x'}(x')}{|\mu_{x'}(x')|}\rangle\right| = \frac{|\mu_{x'}(x')|}{|\mu_{x_n}(x')|},$$

the inner normal τ to the graph of w at 0 has horizontal and vertical components τ_s and τ_{x_n} whose lengths are proportional to $|\mu_{x'}|$ and $|\mu_{x_n}|$, respectively. Let $s_0 > 0$ be such that $x + s_0\mu_{x'}(x')/|\mu_{x'}(x')| \in \partial\Omega_t$. Since w is concave, the tangent line to the graph of w in 0 reaches the s-axis in a point s_1 beyond s_0 (see Figure 5). Then, by a simple comparison between triangles we get

$$\frac{|\mu_{x_n}(x')|}{|\mu_{x'}(x')|} = \frac{|\tau_{x_n}|}{|\tau_s|} = \frac{s_1}{l(x',t)} \ge \frac{s_0}{l(x',t)} \ge \frac{\mathrm{dist}(x', \partial\Omega_t)}{M},$$

thus proving (3.13) since $|\mu_{x'}(x')| > 1/\sqrt{2}$. A similar argument proves (3.13) also at boundary points of the type $(x', -l(x',t), t)$.

Estimate of term II and conclusion of the proof. From Lemma 2.7 we have that $\ln u^s$ is concave, hence the function $(x', s) \mapsto l(x', e^s)$ is concave. Given x', we set $v(s) = l(x', e^s)$. Note that v is a decreasing function defined for $s \le s_0 := \ln(u^s(x', 0))$ and that $e^{s_0} \in \partial\Omega_{x'}$. Let $\rho(s)$ be the inner normal to the graph of v in s. Arguing as in the previous estimate, we get

$$\frac{1}{|\partial_t l(x', e^s)e^s|} = \frac{1}{|v'(s)|} = \frac{|\rho_{x_n}(s)|}{|\rho_s(s)|} \ge \frac{s_0 - s}{l(x', e^s)} \ge \frac{s_0 - s}{M}.$$

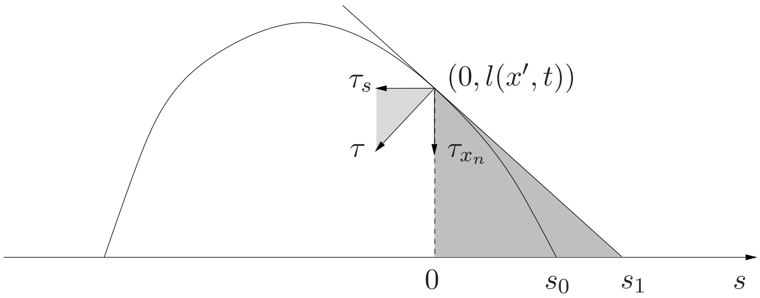

FIGURE 5. The graph of the function w.

By the change of variables $t = e^s$, we have for any $t \in (0, u^s(x, 0)]$

$$\frac{1}{|\partial_t l(x', t)|} \geq \frac{t}{M} \ln \left(\frac{u^s(x', 0)}{t} \right) \geq \frac{\ln 2}{M} \min\{u^s(x', 0) - t, t\} = \frac{\ln 2}{M} \mathrm{dist}(t, \partial\Omega_{x'}). \tag{3.14}$$

Indeed, if $t \leq u^s(x', 0)/2$, then $\ln(u^s(x', 0)/t) \geq \ln 2$ and $\min\{u^s(x', 0) - t, t\} = t$. On the other hand, $\ln \alpha \geq (\alpha - 1) \ln 2$ for any $\alpha \in [1, 2]$. Therefore, if $t > u^s(x', 0)/2$, $t \ln(u^s(x', 0)/t) \geq (u^s(x', 0) - t) \ln 2$. Moreover, $\min\{u^s(x', 0) - t, t\} = u^s(x', 0) - t$.

From (3.12), (3.13) and (3.14) it follows that

$$\int_\Omega \frac{|\nabla_{x'} b(x', t)|}{\sqrt{1 + |\nabla_{x'} l(x', t)|^2}} \, dx'dt + \int_\Omega \frac{|\partial_t b(x', t)|}{|\partial_t l(x', t)|} \, dx'dt \geq \frac{2 \ln 2}{M} \int_\Omega |\nabla b(x', t)| \mathrm{dist}((x', t), \partial\Omega) \, dx'dt. \tag{3.15}$$

Being defined as a projection of E, the set Ω has outer radius $R \leq M$ and it is star-shaped with respect to a ball of radius $r \geq m$. Therefore, by the Poincaré inequality stated in Proposition 2.9,

$$\begin{aligned}
\int_\Omega |\nabla b| \, \mathrm{dist}((x', t), \partial\Omega) \, dx'dt &\geq c \left(\frac{m}{M} \right)^{n+1} \int_\Omega |b - b_0| \, dx'dt \\
&\geq c \left(\frac{m}{M} \right)^{n+1} \int_\Omega \mathcal{L}^1 \big(E^s_{x',t} \triangle (E_{x',t} - b_0) \big) dx'dt \\
&= c \left(\frac{m}{M} \right)^{n+1} \mathcal{L}^n \big(E^s \triangle (E - (0, b_0, 0)) \big) \\
&= c \left(\frac{m}{M} \right)^{n+1} \int_{R^n} |u(x', x_n + b_0) - u^s(x)| dx,
\end{aligned}$$

for a suitable $b_0 \in \mathbb{R}$. Finally, estimate (1.1) follows from this last inequality and (3.10), (3.11), (3.15). $\square$

Proof of Theorem 1.2. The proof of the α-concave case follows the same path of the log-concave one, with some minor changes. If $n = 1$ the set Ω is simply a segment and thus to estimate the left-hand side of (3.10) and (3.11) it is enough to use (3.15) and Proposition 2.9. Let $\tilde{u} := u^\alpha$. When $n \geq 2$, we write $\tilde{b}$, $\tilde{l}$, and $\tilde{\Omega}$ instead of $b_{\tilde{u}}$, $l_{\tilde{u}}$, and $\Omega_{\tilde{u}}$, respectively. One has

$$\tilde{l}(x', s) = l(x', \sqrt[\alpha]{s}) \ \text{ and } \ \tilde{b}(x', s) = b(x', \sqrt[\alpha]{s}),$$

so that, by a change of variables, the left-hand side of (3.10) and (3.11) becomes

$$\frac{1}{\alpha}\int_{\tilde\Omega}\left[\frac{|\nabla_{x'}\tilde b|}{\sqrt{1+|\nabla_{x'}\tilde l|^2}}+\frac{|\partial_s\tilde b|}{|\partial_s\tilde l|}\right]s^{\frac{1-\alpha}{\alpha}}\,dx'ds. \tag{3.16}$$

Note that the subgraph of $\tilde u$ is convex and, by Lemma 2.10, it has inner radius larger than $c\|\tilde u\|_{L^1}/\tilde M^n$, where $\tilde M$ is the maximum between $\|\tilde u\|_\infty$ and the outer radius of D. Moreover, we can estimate, as for (3.12) and (3.13), $\sqrt{1+|\nabla\tilde l|^2}\le\sqrt{2}\tilde M/\mathrm{dist}((x',s),\partial\tilde\Omega)$. Therefore, (3.16) can be estimated from below via Poincaré inequality (2.11) (with $w=\tilde u^s(\cdot\,,0)$):

$$\begin{aligned}\int_{\tilde\Omega}\left[\frac{|\nabla_{x'}\tilde b|}{\sqrt{1+|\nabla_{x'}\tilde l|^2}}+\frac{|\partial_s\tilde b|}{|\partial_s\tilde l|}\right]s^{\frac{1-\alpha}{\alpha}}\,dx'ds &\ge \frac{c}{\tilde M}\int_{\tilde\Omega}|\nabla\tilde b|s^{\frac{1-\alpha}{\alpha}}\mathrm{dist}((x',s),\partial\tilde\Omega)\,dx'ds\\ &\ge c\frac{\|\tilde u\|_{L^1}}{\tilde M^{n+2}}\int_{\tilde\Omega}|\tilde b-b_0|s^{\frac{1-\alpha}{\alpha}}\,dx'ds = c\frac{\|\tilde u\|_{L^1}}{\tilde M^{n+2}}\int_{\Omega}|b-b_0|\,dx'dt.\end{aligned}$$

This last estimate completes the proof. □

We now pass to the proof of Theorem 1.3. Before that we need one more definition. We say that a function $u:\mathbb{R}^n\to\mathbb{R}$ is *n-symmetric* if for any $x\in\mathbb{R}^n$ and for all $i\in\{1,\dots n\}$ we have $u(x)=u(R_ix)$, where R_i is the reflection across the hyperplane $\{x_i=0\}$, i.e., $R_ix=x-2\langle x,e_i\rangle e_i$. Similarly, we say that a set $G\subset\mathbb{R}^n$ is n-symmetric if $R_i(G)=G$ for all $i\in\{1,\dots n\}$. To prove Theorem 1.3, the idea is to apply Steiner symmetrization n times along the n coordinate directions so to transform u in a n-symmetric function, and then to use the following stability result that generalizes [17, Proposition 2.4] (see also [14, Theorem 3]).

Lemma 3.2. *Let $D\subset\mathbb{R}^n$ be a bounded open n-symmetric set and $w\in W_0^{1,p}(D)$ be a non-negative and n-symmetric function, $1<p<\infty$. Then*

$$\int_{\mathbb{R}^n}|w-w^*|\,dx\le\begin{cases}c\,\mathcal{L}^n(D)^{\frac{1}{p'}+\frac1n}\|\nabla w^*\|_{L^p(\mathbb{R}^n)}^{\frac{2-p}{2}}\Delta(w,w^*)^{\frac12} & \text{if } 1<p<2;\\ c\,\mathcal{L}^n(D)^{\frac{1}{p'}+\frac1n}\Delta(w,w^*)^{\frac1p} & \text{if } p\ge2,\end{cases} \tag{3.17}$$

where $c=c(n,p)$.

Proof. By the coarea formula, for $t>0$ we have

$$\mu(t):=\mathcal{L}^n(\{w>t\})=\mathcal{L}^n(\{w>t\}\cap\{\nabla w=0\})+\int_t^\infty\int_{\{w=s\}}\frac{d\mathcal{H}^{n-1}}{|\nabla w|}ds\,.$$

Therefore, for a.e. $t>0$

$$-\mu'(t)\ge\int_{\{w=t\}}\frac{1}{|\nabla w|}d\mathcal{H}^{n-1}\,. \tag{3.18}$$

Moreover, we have that for a.e. $t>0$

$$\mathcal{H}^{n-1}(\{w=t\})=P(\{w>t\})\,.$$

Applying the coarea formula, Hölder's inequality and (3.18), we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla w|^p dx &= \int_0^\infty \int_{\{w=t\}} |\nabla w|^{p-1}\, d\mathcal{H}^{n-1}\, dt \geq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{w=t\})^p}{\left(\int_{\{w=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla w|}\right)^{p-1}} dt \\ &\geq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{w=t\})^p}{(-\mu'(t))^{p-1}} dt = \int_0^\infty \frac{P(\{w>t\})^p}{(-\mu'(t))^{p-1}} dt\,.\end{aligned} \tag{3.19}$$

Given a measurable set $E \subset \mathbb{R}^n$, we define the *Fraenkel asymmetry* of E as

$$A(E) := \inf\left\{\frac{\mathcal{L}^n(E\triangle B)}{\mathcal{L}^n(E)} : B \text{ ball}, \mathcal{L}^n(B) = \mathcal{L}^n(E)\right\}.$$

The quantitative isoperimetric inequality (see, e.g. [16] or [22]) ensures that there exists a constant γ_0, depending only on n, such that

$$n\,\omega_n^{1/n} \mathcal{L}^n(E)^{1/n'} \left(1 + \gamma_0 A(E)^2\right) \leq P(E)\,,$$

for every measurable set $E \subset \mathbb{R}^n$ having finite measure and finite perimeter, where n' is the Hölder's conjugate of n. Moreover (see [22, Lemma 5.2]), if E is n-symmetric, then

$$A(E) \geq \frac{1}{3}\frac{\mathcal{L}^n(E\triangle E^*)}{\mathcal{L}^n(E)}\,, \tag{3.20}$$

where E^* is the ball centered in the origin having the same volume of E. Since w is n-symmetric, so are its level sets $\{w > t\}$ for $t > 0$. Hence, combining (3.19) with (3.20) applied to $E = \{w > t\}$, we get

$$\int_{\mathbb{R}^n} |\nabla w|^p dx \geq (n\omega_n^{1/n})^p \int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} \left(1 + \frac{\gamma_0}{9}\left(\frac{F(t)}{\mu(t)}\right)^2\right)^p dt\,, \tag{3.21}$$

where $F(t) := \mathcal{L}^n(\{w > t\}\triangle\{w^* > t\})$ for $t > 0$. Let us observe that, if we replace w by w^* in (3.19), we have then all equalities, because $|\nabla w^*|$ is constant on the sphere $\{w^* = t\}$ for a.e. $t > 0$ and because (see [12, Lemma 3.2]) also (3.18) turns into an equality. Thus, $P(\{w^* > t\}) = n\omega_n^{1/n}\mu(t)^{1/n'}$ for a.e. $t > 0$ and

$$\int_{\mathbb{R}^n} |\nabla w^*|^p dx = (n\omega_n^{1/n})^p \int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}}\, dt. \tag{3.22}$$

Since $(1+s)^p \geq 1 + ps$ for $s \geq 0$, we deduce from (3.21) and (3.22) that

$$\int_{\mathbb{R}^n} |\nabla w|^p - |\nabla w^*|^p dx \geq \gamma \int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\,, \tag{3.23}$$

for some constant $\gamma > 0$, depending only on n and p.

By layer-cake representation formula and Hölder's inequality we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |w - w^*|\, dx\right)^p &\leq \left(\int_0^\infty F(t)\, dt\right)^p \\ &\leq \left(\int_0^\infty (-\mu'(t))(\mu)^{\frac{p'}{n}}\, dt\right)^{p-1} \underbrace{\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^p \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt.}_{=:I}\end{aligned} \tag{3.24}$$

Note that

$$\int_0^\infty \mu(t)^\alpha \cdot (-\mu'(t))\, dt \leq \frac{\mathcal{L}^n(D)^{\alpha+1}}{\alpha+1} \qquad \text{for every } \alpha > -1\,.$$

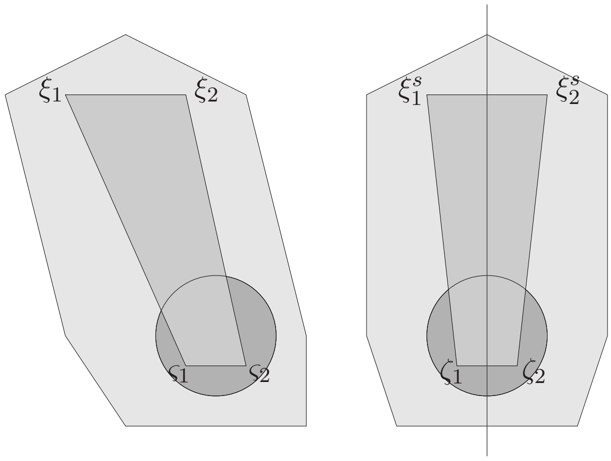

FIGURE 6. The two points ζ_1 and ζ_2 in $B_m((\overline{x}', 0, \overline{t}))$, and the corresponding points ς_1 and ς_2 in $B_m(\overline{\xi})$.

As $0 \le F/\mu \le 2$ we have for $p \ge 2$ that

$$\left(\frac{F(t)}{\mu(t)}\right)^p \le 2^{p-2}\left(\frac{F(t)}{\mu(t)}\right)^2 .$$

Therefore,

$$\left(\int_{\mathbb{R}^n} |w - w^*|\, dx\right)^p \le \gamma \mathcal{L}^n(D)^{p-1+\frac{p}{n}} \left(\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}} dt\right)$$

and using (3.23) we obtain (3.17).

On the other hand, if $1 < p < 2$, by Hölder's inequality we have

$$I \le \left(\int_0^\infty \left(\frac{F(t)}{\mu(t)}\right)^2 \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\right)^{\frac{p}{2}} \left(\int_0^\infty \frac{\mu(t)^{\frac{p}{n'}}}{(-\mu'(t))^{p-1}} dt\right)^{\frac{2-p}{2}} .$$

Hence, using (3.24) and (3.22), we obtain again (3.17). □

Proof of Theorem 1.3. We give the proof only for the case $1 < p < 2$, the other one being similar. First of all note that the property of the subgraph E of u of being star-shaped with respect to a ball of radius m is inherited by the subgraph E^s of u^s. More precisely, if E is star-shaped with respect to $B_m(\overline{\xi})$, for some $\overline{\xi} = (\overline{x}, \overline{t})$, then E^s is star-shaped with respect to $B_m((\overline{x}', 0, \overline{t}))$. Indeed, let ξ_1^s be a point in E^s and ξ_2^s its symmetral with respect to the hyperplane $\{x_n = 0\}$. There are exactly two points $\xi_1 := (x', x_n, t)$ and $\xi_2 := (x', y_n, t)$ in E such that $\xi_1^s := (x', (x_n - y_n)/2, t)$ and $\xi_2^s := (x', (y_n - x_n)/2, t)$. Without loss of generality we can assume $x_n < y_n$. Let also ζ_1 be a point in $B_m((\overline{x}', 0, \overline{t}))$ (on the same side of ξ_1^s with respect to $\{x_n = 0\}$) and ζ_2 its symmetral. We consider the "corresponding" points in $B_m(\overline{\xi})$, i.e., $\varsigma_1 = \zeta_1 + (0, \overline{x}_n, 0)$ and $\varsigma_2 = \zeta_2 + (0, \overline{x}_n, 0)$ (see Figure 6). Being E star-shaped with respect to $B_m(\overline{\xi})$, the segments $\overline{\xi_1\varsigma_1}$ and $\overline{\xi_2\varsigma_2}$ are included in E. Then, since E has the segment property, also the convex hull C of $\{\xi_1, \varsigma_1, \xi_2, \varsigma_2\}$ is included in E. Finally, since ξ_1^s, ζ_1 and ζ_2 belong to C^s, and $C^s \subset E^s$, we have that both the segments $\overline{\xi_1^s\zeta_1}$ and $\overline{\xi_1^s\zeta_2}$ are included in E^s.

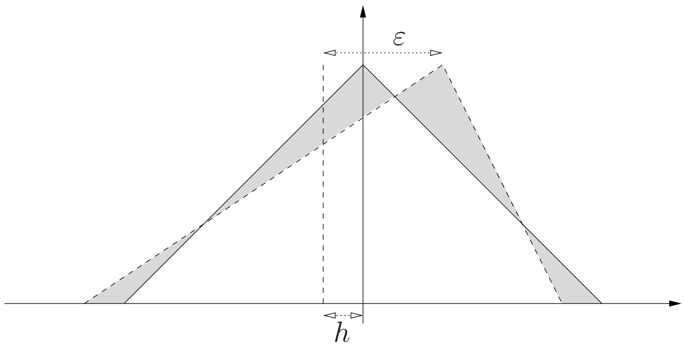

FIGURE 7. The grey area represents $\int |u(x+h) - u^s(x)|$.

We set now $u^{s_0} = u$ and, for every $i = 1, \dots, n$, we indicate by u^{s_i} the Steiner symmetral of $u^{s_{i-1}}$ with respect to the hyperplane $\{x_i = 0\}$. Note that u^{s_n} is n-symmetric. Since the Steiner symmetrization keeps the property of being star-shaped with respect to a ball with a given radius, and since it decreases the outer radius, from inequality (1.1) we get for every $i = 1, \dots, n$

$$\int_{\mathbb{R}^n} |u^{s_{i-1}} - u^{s_i}|\, dx \le c\, \frac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u\|_{L^p}^{\frac{2-p}{2}} \Delta(u^{s_{i-1}}, u^{s_i})^{\frac{1}{2}}, \tag{3.25}$$

up to a suitable translation of $u^{s_{i-1}}$ along the x_i-axis. Moreover, since $\mathcal{L}^n(D) \le M^n \omega_n$ and $m \le M$, by formula (3.17) we get

$$\int_{\mathbb{R}^n} |u^{s_n} - u^*|\, dx \le c\, \frac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u^{s_n}, u^*)^{\frac{1}{2}}. \tag{3.26}$$

If $\|\nabla u\|_{L^p}^p \le 2\|\nabla u^*\|_{L^p}^p$, (1.2) follows from (3.25), (3.26) and the triangle inequality. Otherwise, since the support of u^* is a ball of volume $\mathcal{L}^n(D)$,

$$\begin{aligned}
\inf_{h \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u(x+h) - u^*(x)|\, dx &\le 2\|u^*\|_{L^1} \le c\mathcal{L}^n(D)^{\frac{1}{n}} \|\nabla u^*\|_{L^1} \\
&\le c\mathcal{L}^n(D)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla u^*\|_{L^p} = c\mathcal{L}^n(D)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \left(\int_{\mathbb{R}^n} |\nabla u^*|^p dx \right)^{\frac{1}{2}} \\
&\le c\, \frac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u, u^*)^{\frac{1}{2}}.
\end{aligned}$$

□

To conclude, we show with two examples that for $1 < p \le 2$ the power $1/2$ of the gap in the estimates (1.1) and (3.17) is sharp.

Example 3.3. Let $\varepsilon \in (0,1)$ and let $u : \mathbb{R} \to \mathbb{R}$ be defined as

$$u(x) = \begin{cases} (x+1)/(\varepsilon+1) & \text{if } -1 \le x \le \varepsilon\,; \\ (x-1)/(\varepsilon-1) & \text{if } \varepsilon \le x \le 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then its Steiner rearrangement is

$$u^s(x) = \begin{cases} 1 - |x| & \text{if } -1 \le x \le 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In order to calculate
$$\inf_{h\in\mathbb{R}}\int_{\mathbb{R}}|u(x+h)-u^s(x)|\,dx$$
it is clear (see Figure 7) that we may assume $h\in[0,\varepsilon]$. Then, a straightforward computation shows that for any such h
$$\int_{\mathbb{R}}|u(x+h)-u^s(x)|\,dx=\frac{h^2(4+\varepsilon)-4\varepsilon h+2\varepsilon^2}{\varepsilon(2+\varepsilon)}\,.$$
Hence the infimum is attained at $h=2\varepsilon/(\varepsilon+4)$ and is equal to
$$\frac{2\varepsilon}{4+\varepsilon}\sim\frac{1}{2}\varepsilon\qquad\text{as }\varepsilon\to 0^+\,.$$
On the other hand, a direct computation shows that $\Delta(u,u^s)=(1+\varepsilon)^{1-p}+(1-\varepsilon)^{1-p}-2\sim p(p-1)\varepsilon^2$, hence showing the sharpness of the power $1/2$ in the estimate (1.1).

Example 3.4. Let $\varepsilon\in(0,1)$ and denote by D the ellipse $\{x\in\mathbb{R}^2:(1+\varepsilon)^2x_1^2+x_2^2/(1+\varepsilon)^2<1\}$. Let $u:\mathbb{R}^2\to\mathbb{R}$ defined as $u(x)=1-(1+\varepsilon)^2x_1^2-x_2^2/(1+\varepsilon)^2$ if $x\in D$ and extended by 0 outside D. Then, its Schwarz rearrangement is $u^*(x)=1-x_1^2-x_2^2$ for $x\in B$ and extended by 0 outside B, where B is the ball of radius 1 centered in 0. As the function u is 2-symmetric, denoting by E its subgraph, the set E_t is a convex 2-symmetric set and thus (see [2, Lemma 5.9]) $\mathcal{L}^3(E_t\triangle B_{r_t})=\inf_{h\in\mathbb{R}^2}\mathcal{L}^3((h+E_t)\triangle B_{r_t})$, where B_{r_t} is the disk with the same measure of E_t. Therefore, the infimum
$$\inf_{h\in\mathbb{R}^2}\int_{\mathbb{R}^2}|u(x+h)-u^s(x)|\,dx$$
is attained at $h=0$ and it is equal to $c\mathcal{L}^3(B\triangle D)\sim c\varepsilon$ as $\varepsilon\to 0^+$. By a direct computation we obtain
$$\begin{aligned}
2^{-p}\Delta(u,u^*)&=\int_B\left[\left(x_1^2(1+\varepsilon)^2+\frac{x_2^2}{(1+\varepsilon)^2}\right)^{p/2}-\left(x_1^2+x_2^2\right)^{p/2}\right]dx\\
&=\int_B\left[\left((x_1^2+x_2^2)+2\varepsilon(x_1^2-x_2^2)+\varepsilon^2(x_1^2+3x_2^2)+O(\varepsilon^3)x_2^2\right)^{p/2}-\left(x_1^2+x_2^2\right)^{p/2}\right]dx\\
&=\int_B(x_1^2+x_2^2)^{p/2}\left[\left(1+\frac{2\varepsilon(x_1^2-x_2^2)}{x_1^2+x_2^2}+\frac{\varepsilon^2(x_1^2+3x_2^2)}{x_1^2+x_2^2}+O(\varepsilon^3)\right)^{p/2}-1\right]dx\\
&=p\varepsilon\int_B(x_1^2+x_2^2)^{p/2}\frac{(x_1^2-x_2^2)}{x_1^2+x_2^2}dx\\
&\quad+\frac{p}{2}\varepsilon^2\int_B(x_1^2+x_2^2)^{p/2}\left(\frac{x_1^2+3x_2^2}{x_1^2+x_2^2}+\frac{(p-2)(x_1^2-x_2^2)^2}{(x_1^2+x_2^2)^2}\right)dx+O(\varepsilon^3)\\
&=\frac{p}{2}\varepsilon^2\int_B(x_1^2+x_2^2)^{(p-4)/2}\left[2x_2^2(3x_1^2+x_2^2)+(p-1)(x_1^2-x_2^2)^2\right]dx+O(\varepsilon^3)\sim c(p)\varepsilon^2\,,
\end{aligned}$$
for some positive constant $c(p)$ and where in the second and fourth line we have used the Taylor expansion of $(1+\cdot)^{-2}$ and $(1+\cdot)^{p/2}$, respectively. This proves the sharpness for $1<p\le 2$ of the power $1/2$ in the estimate (3.17).

4. The quasiconcave case

In this section we extend the stability results stated in the Introduction to quasiconcave functions.

Theorem 4.1. *Let $D \subset \mathbb{R}^n$ be a bounded open convex set and let $u \in W_0^{1,p}(D)$ be a non-negative, continuous and quasiconcave function. Assume that there exists $t_0 > 0$ such that, setting $D_+ = \{x \in D : u(x) \geq t_0\}$ and $D_- = \{x \in D : u(x) < t_0\}$,*

- *$|\nabla u| \geq \beta$ in D_- for some $\beta > 0$,*
- *u is concave in D_+,*

and denote by M the maximum between $\|u\|_{L^\infty}$ and the outer radius R of D and by r the inner radius of D_+.

Then, the subgraph of u is star-shaped with respect to a ball of radius $m = \min\{t_0/2, r/2, r\beta/8\}$, $|\nabla u^s| \geq c\beta$ in $D_-^s = \{x \in D^s : u^s(x) < t_0\}$ and

$$\inf_{h\in\mathbb{R}} \int_{\mathbb{R}^n} |u(x', x_n+h) - u^s(x)|\, dx \leq \begin{cases} c\,\dfrac{M^{n+2}}{m^{n+1}}\left(1+\dfrac{1}{\beta}\right)\mathcal{L}^n(D)^{\frac{1}{p'}}\|\nabla u^s\|_{L^p}^{\frac{2-p}{2}}\Delta(u,u^s)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c\,\dfrac{M^{n+2}}{m^{n+1}}\left(1+\dfrac{1}{\beta}\right)\mathcal{L}^n(D)^{\frac{1}{p'}}\Delta(u,u^s)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases}$$

where $c = c(n, p, r/R)$.

Remark 4.2. The class of functions to which the above theorem applies could sound exotic. Yet, the first eigenfunction v of the Dirichlet Laplacian on a smooth bounded open convex set D belongs to this class. Indeed, if D is convex a well-known result of Brascamp and Lieb states that $\ln v$ is concave and this, together with the analyticity of v, implies that there is a unique point x_0 where the gradient of v vanishes (see [5] and also [21, 20]). In particular, v is concave in a neighborhood of x_0, and by the Hopf boundary point lemma, $|\nabla v|$ is larger than a positive constant outside this neighborhood. Note also that since by Theorem 4.1 the subgraph of v is star-shaped with respect to a ball, we may also directly apply to v the conclusions of Theorems 1.1 and 1.3.

Remark 4.3. Since the lower bound condition on the gradient is stable under Steiner symmetrization, Theorem 4.1 can be easily generalized to the Schwarz symmetrization by using the same technique developed for the proof of Theorem 1.3: performing n Steiner symmetrizations along n perpendicular directions so to transform u in an n-symmetric function, and then applying estimate (3.17).

Proof of Theorem 4.1. We start by showing that the subgraph E of u satisfies condition (2.3). Let us consider, for $t \in (0, t_0)$, the level set $E_t := \{x \in \mathbb{R}^n : (x,t) \in E\} = \{x \in D : t < u(x)\}$. Given $x \in \partial E_t$, denote by $\mu(x)$ the inner normal to ∂E_t in x (when it exists). Observe that, since $\nabla u(x) \neq 0$, $\mu(x) = \nu_x(x,t)/|\nu_x(x,t)|$. In particular $\mu_{x_n}(x) = 0$ if $\nu_{x_n}(x,t) = 0$. On the other hand, since E_t is convex, it is included in the hyperplane $\{y \in \mathbb{R}^n : \langle y - x, \mu(x)\rangle > 0\}$. Therefore, if $\mu_{x_n}(x) = 0$, then the projection of (x,t) on $\{x_n = 0\}$ belongs to $\partial\Omega$.

Let us now show that the subgraph of u is star-shaped and that the lower bound condition on the gradient is stable under Steiner symmetrization. Let $B_r(\overline{z})$ be a ball included in D_+. We are going to show that E is star-shaped with respect to the cylinder of base $B_{r/2}(\overline{z}) \times \{0\}$ and height $m' := \min\{t_0, r\beta/4\}$. Of course this cylinder includes the desired ball of radius m. Observe that $E \cap (D_+ \times \mathbb{R}_t)$ is convex and therefore star-shaped with respect to the larger cylinder $B_r(\overline{z}) \times (0, t_0]$. Therefore, it remains to prove that, given $(y,t) \in E \cap (D_- \times \mathbb{R}_t)$ and $(z,h) \in B_{r/2}(\overline{z}) \times (0, m']$, the segment with endpoints (y,t) and (z,h) is included in E. Since E is the subgraph of a function, it is enough to show this fact when $h = m'$. Observe also that if $(y,t) \in E \cap (D_- \times \mathbb{R}_t)$ then $t \in (0, t_0)$.

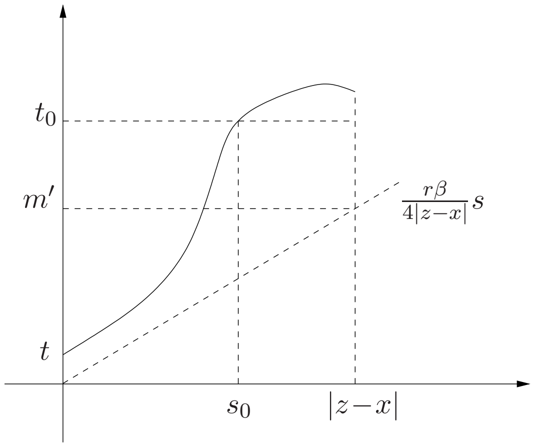

FIGURE 8. The graph of the function w.

Let $t \in (0, t_0)$ and $x \in \partial E_t$. Since E_t includes $B_{r/2}(z)$, by Lemma 2.11 the component of $\mu(x)$ in the direction $\eta(x) := (z-x)/|z-x|$ is bounded from below by $r/(4|z-x|)$. Therefore we have the estimate $\partial_\eta u(x) = |\nabla u(x)|\langle \mu(x), \eta(x)\rangle \geq r\beta/(4|z-x|)$. Consider now, for any fixed $x \in \partial E_t$, the function

$$w(s) := u(x + s\eta(x)) \qquad \text{for } s \in (0, |z-x|].$$

Note that, since u is continuous, also w is continuous. Let $s_0 := \min\{s : w(s) = t_0\}$. Since u is concave in D_+, w is concave in $[s_0, |z-x|]$. On the other hand we have that $w'(s) = \partial_\eta u(x+s\eta(x)) \geq r\beta/(4|z-x|)$ for a.e. $s \in (0, s_0)$. Therefore $w(s) - r\beta/(4|z-x|)s - t$ is increasing up to s_0 and the subgraph of w is star-shaped with respect to the point $(|z-x|, m')$ (see Figure 8). Since any point $(y,t) \in E$ with $y \in D_-$ and $t < t_0$ can be written as $y = x + s\eta(x)$ for some $x \in \partial E_t$ and $s \in (0, s_0)$, we have proved that E is star-shaped with respect to $B_{r/2}(\overline{z}) \times \{m'\}$ and thus also with respect to the cylinder of base $B_{r/2}(\overline{z}) \times \{0\}$ and height m'.

Since the level set E_t is convex, every parallel line to the direction x_n passing at height t through Ω intersects the boundary of E in exactly two points, say (x,t) and (y,t), with $x' = y'$ and $x_n < y_n$. They are transformed by the Steiner symmetrization in the two points $(x^s, t) := (x', (x_n - y_n)/2, t)$ and $(y^s, t) := (x', (y_n - x_n)/2, t)$. Let ν and ν^s be the inner normals to E and E^s, respectively. Adding up the components of $\nu(y,t)$ and $\nu(x,t)$ given in (2.9) and (2.10), respectively, and adding also the corresponding components of $\nu^s(y^s,t)$ and $\nu^s(x^s,t)$ (which are provided by the same formulas with $b = 0$), we easily get

$$\frac{\nu^s_{x',t}(x^s,t)}{|\nu^s_{x_n}(x^s,t)|} + \frac{\nu^s_{x',t}(y^s,t)}{|\nu^s_{x_n}(y^s,t)|} = 2\nabla l = \frac{\nu_{x',t}(x,t)}{|\nu_{x_n}(x,t)|} + \frac{\nu_{x',t}(y,t)}{|\nu_{x_n}(y,t)|}$$

(see also [10, Lemma 3.2]). Then, from these equalities, recalling (2.6), we get

$$2\frac{1}{|\partial_{x_n} u^s(x^s)|} = \frac{1}{|\partial_{x_n} u(x)|} + \frac{1}{|\partial_{x_n} u(y)|}$$

$$2\frac{\nabla_{x'} u^s(x^s)}{|\partial_{x_n} u^s(x^s)|} = \frac{\nabla_{x'} u(x)}{|\partial_{x_n} u(x)|} + \frac{\nabla_{x'} u(y)}{|\partial_{x_n} u(y)|},$$

and since the gradient is normal to the boundary of the level sets

$$
\begin{aligned}
&|\nabla u^s(x^s)|^2\big[|\partial_{x_n}u(x)|+|\partial_{x_n}u(y)|\big]^2\\
&=|\partial_{x_n}u(y)|^2|\nabla_{x'}u(x)|^2+|\partial_{x_n}u(x)|^2|\nabla_{x'}u(y)|^2\\
&\quad+2|\partial_{x_n}u(x)||\partial_{x_n}u(y)|\langle\nabla_{x'}u(x),\nabla_{x'}u(y)\rangle+4|\partial_{x_n}u(x)|^2|\partial_{x_n}u(y)|^2\\
&=|\partial_{x_n}u(y)|^2|\nabla u(x)|^2+|\partial_{x_n}u(x)|^2|\nabla u(y)|^2\\
&\quad+2|\partial_{x_n}u(x)||\partial_{x_n}u(y)|\big[|\partial_{x_n}u(x)||\partial_{x_n}u(y)|+\langle\nabla_{x'}u(x),\nabla_{x'}u(y)\rangle\big]\\
&=|\nabla u(x)|^2|\nabla u(y)|^2\Big[\mu_{x_n}^2(x)+\mu_{x_n}^2(y)+2|\mu_{x_n}(x)||\mu_{x_n}(y)|\big(|\mu_{x_n}(x)||\mu_{x_n}(y)|+\langle\mu_{x'}(x),\mu_{x'}(y)\rangle\big)\Big].
\end{aligned}
$$

By convexity every set E_t is included in the intersection of the two half balls

$$\{z\in B_{2R}:\langle z,\mu(x)\rangle>0\}+(x',x_n)\quad\text{and}\quad\{z\in B_{2R}:\langle z,\mu(y)\rangle>0\}+(x',y_n).$$

Moreover, every set E_t, with $t\in(0,t_0)$, contains $B_r(\overline{z})$. Therefore by Lemma 2.12, there exists a constant $c_1(n,r/R)$ such that $|\mu_{x_n}(x)||\mu_{x_n}(y)|+\langle\mu_{x'}(x),\mu_{x'}(y)\rangle\geq c_1-1$. Going back to the gradient estimate, we have

$$
\begin{aligned}
|\nabla u^s(x^s)|^2&\geq\frac{|\nabla u(x)|^2|\nabla u(y)|^2\Big[c_1\big(\mu_{x_n}^2(x)+\mu_{x_n}^2(y)\big)+(1-c_1)\big(\mu_{x_n}(x)-\mu_{x_n}(y)\big)^2\Big]}{\big[|\nabla u(x)||\mu_{x_n}(x)|+|\nabla u(y)||\mu_{x_n}(y)|\big]^2}\\
&\geq\frac{c_1|\nabla u(x)|^2|\nabla u(y)|^2\big(\mu_{x_n}^2(x)+\mu_{x_n}^2(y)\big)}{\max\{|\nabla u(x)|,|\nabla u(y)|\}^2\big[|\mu_{x_n}(x)|+|\mu_{x_n}(y)|\big]^2}\\
&\geq\frac{c_1}{2}\min\{|\nabla u(x)|,|\nabla u(y)|\}^2\geq\frac{c_1}{2}\beta^2.
\end{aligned}
$$

Finally, let us prove the quantitative estimate. The lower bound condition on ∇u^s can be converted in a condition on ∇l via the formulas (2.9) and (2.10) providing the inner normal to the graph of u^s. Setting $\Omega_-:=\{(x',t)\in\Omega:t<t_0\}$ and recalling (2.6), we have for $q\geq1$

$$
\begin{aligned}
\left(\int_{\Omega_-}\frac{|\partial_t l|^{q+1}}{(1+|\nabla_{x'}l|^2)^{\frac{q}{2}}}dx'dt\right)^{\frac1q}&=\left(\frac12\int_{\partial E^s\cap(\mathbb{R}^n\times(0,t_0))}\frac{|\nu_t^s|^{q+1}}{|\nu_x^s|^q}d\mathcal{H}^n\right)^{\frac1q}\\
&=\left(\frac12\int_{D_-^s}\frac{1}{|\nabla u^s|^q}dx\right)^{\frac1q}\leq\frac{c}{\beta}\mathcal{L}^n(D)^{\frac1q}.
\end{aligned}\tag{4.1}
$$

The difficulty in improving Theorem 1.1 is that in D_- we cannot argue as in the proof of (3.10) and (3.11), since the lack of the concavity does not allow us to control $\sqrt{1+|\nabla l|^2}$ with the distance from $\partial\Omega$. Yet, using Hölder's inequality and estimates (3.9) and (4.1), we get in the case $1<p<2$

$$
\begin{aligned}
\int_{\Omega_-}\frac{|\partial_t b|}{\sqrt{1+|\nabla_{x'}l|^2}}\,dx'dt&\leq c\Delta(u,u^s)^{\frac12}\left(\int_{\Omega_-}\frac{|\partial_t l|^{p+1}}{(1+|\nabla_{x'}l|^2)\,(1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2)^{\frac{p-2}{2}}}\,dx'dt\right)^{\frac12}\\
&\leq c\Delta(u,u^s)^{\frac12}\left(\int_{\Omega_-}\frac{|\partial_t l|^{\frac{p+1}{p-1}}}{(1+|\nabla_{x'}l|^2)^{\frac{1}{p-1}}}\,dx'dt\right)^{\frac{p-1}{2}}\left(\int_{\Omega_-}\sqrt{1+|\nabla_{x'}l|^2+|\nabla_{x'}b|^2}\,dx'dt\right)^{\frac{2-p}{2}}\\
&\leq\frac{c}{\beta}\mathcal{L}^n(D)^{\frac{1}{p'}}\,\|\nabla u\|_{L^p}^{\frac{2-p}{2}}\,\Delta(u,u^s)^{\frac12},
\end{aligned}\tag{4.2}
$$

where in the last inequality we have used (2.8) with $p = 1$ and Hölder's inequality again. If $p \geq 2$, we have instead

$$\begin{aligned}\int_{\Omega_-} \frac{|\partial_t b|}{\sqrt{1+|\nabla_{x'} l|^2}}\,dx'dt &\leq c\Delta(u,u^s)^{\frac{1}{p}} \left(\int_{\Omega_-} \frac{|\partial_t l|^{\frac{2p-1}{p-1}}}{(1+|\nabla_{x'} l|^2)^{\frac{p}{2(p-1)}}}\,dx'dt\right)^{\frac{1}{p'}} \\ &\leq \frac{c}{\beta}\mathcal{L}^n(D)^{\frac{1}{p'}}\Delta(u,u^s)^{\frac{1}{p}}.\end{aligned} \tag{4.3}$$

Let $D^s_+ = \{x \in \mathbb{R}^n : u^s(x) \geq t_0\}$ and $\Omega_+ = \{(x',t) \in \Omega : t \geq t_0\}$. Together with (3.10) and (3.11), estimates (4.2) and (4.3) give us

$$\begin{aligned}&\int_{\Omega_-} \frac{|\nabla b|}{\sqrt{1+|\nabla_{x'} l|^2}}\,dx'dt + \int_{\Omega_+} \frac{|\nabla b|}{\sqrt{1+|\nabla l|^2}}\,dx'dt \\ &\qquad\leq \begin{cases} c\left(1+\dfrac{1}{\beta}\right)\mathcal{L}^n(D)^{\frac{1}{p'}}\|\nabla u\|_{L^p}^{\frac{2-p}{2}}\Delta(u,u^s)^{\frac{1}{2}} & \text{if } 1<p<2; \\ c\left(1+\dfrac{1}{\beta}\right)\mathcal{L}^n(D)^{\frac{1}{p'}}\Delta(u,u^s)^{\frac{1}{p}} & \text{if } p \geq 2.\end{cases}\end{aligned}$$

Thanks to the concavity of u in D_+ and therefore of u^s in D^s_+, we may estimate $\sqrt{1+|\nabla l|^2}$ in Ω_+ in terms of $\operatorname{dist}((x,t), \partial\Omega_+ \cap (\mathbb{R}^n \times (t_0,+\infty)))$ similarly to what we did for (3.12) and (3.13). Thus, since $\partial\Omega_+ \cap (\mathbb{R}^n \times (t_0,+\infty)) \subset \partial\Omega$,

$$\int_{\Omega_+} \frac{|\nabla b|}{\sqrt{1+|\nabla l|^2}}\,dx'dt \geq \frac{1}{\sqrt{2}M}\int_{\Omega_+} |\nabla b|\operatorname{dist}((x',t),\partial\Omega)\,dx'dt.$$

Let μ^s be the inner normal of E^s_t. By the convexity of E_t (and therefore of E^s_t) we have

$$\frac{1}{\sqrt{1+|\nabla_{x'} l|^2}} = \frac{|\nu^s_{x_n}|}{|\nu^s_x|} = |\mu^s_{x_n}| \geq \frac{\operatorname{dist}((x',t),\partial\Omega_t)}{\sqrt{2}M} \quad \forall (x',t) \in \Omega,$$

so that we also have

$$\int_{\Omega_-} \frac{|\nabla b|}{\sqrt{1+|\nabla_{x'} l|^2}}\,dx'dt \geq \frac{1}{\sqrt{2}M}\int_{\Omega_-} |\nabla b|\operatorname{dist}((x',t),\partial\Omega)\,dx'dt.$$

Finally, by means of the weighted Poincaré inequality stated in Proposition 2.9,

$$\begin{aligned}\int_{\Omega} |\nabla b|\operatorname{dist}((x',t),\partial\Omega)\,dx'dt &\geq c_2\left(\frac{m}{M}\right)^{n+1}\int_{\Omega} |b-b_0|\,dx'dt \\ &= c_2\left(\frac{m}{M}\right)^{n+1}\int_{\mathbb{R}^n} |u(x',x_n+b_0)-u^s(x)|dx,\end{aligned}$$

for suitable $c_2 = c_2(n) > 0$ and $b_0 \in \mathbb{R}$. □

Acknowledgments

This work was supported by the ERC Advanced Grant 226234–*AnTeGeFI*. The work of N. Fusco was partially supported by the FiDiPro program. The friendly atmosphere of the University of Jyväskylä is warmly acknowledged. The work of G. Pisante was partially supported by the Marie Curie project IRSES-2009-247486 of the Seventh Framework Programme.

M. Barchiesi gratefully thanks Lorenzo Brasco for stimulating discussions. The authors want also to thank the warm hospitaly of the De Giorgi Center where in July 2012 this work started.

References

[1] L. Ambrosio, N. Fusco & D. Pallara. *Functions of bounded variation and free discontinuity problems*, in the *Oxford Mathematical Monographs*. The Clarendon Press Oxford University Press, New York (2000).

[2] M. Barchiesi, F. Cagnetti & N. Fusco. Stability of the Steiner symmetrization of convex sets. *J. Eur. Math. Soc.*, 15 (2013), 1245–1278.

[3] A.Barvinok. *A course in convexity*, in the *Graduate Studies in Mathematics*, vol. 54. American Mathematical Society, Providence (2002).

[4] J. Bourgain, J. Lindenstrauss & V.D. Milman. Estimates related to Steiner symmetrizations, in *Geometric aspects of functional analysis (1987–88). Lect. Notes Math.*, vol. 1376, pp. 264–273. Springer, Berlin (1989).

[5] H. L. Brascamp & E. H. Lieb. On extensions of Brunn-Minkowski and Prèekopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* 22, 366–389 (1976).

[6] L. Brasco, G. De Philippis & B. Velichkov. Faber-Krahn inequalities in sharp quantitative form. Preprint (2013).

[7] J. E. Brothers & W. P. Ziemer. Minimal rearrangements of Sobolev functions. *J. Reine Angew. Math.* 384, 153–179 (1988).

[8] A. Burchard. Steiner symmetrization is continuous in $W^{1,p}$. *Geom. Funct. Anal.* 7, 823–860 (1997).

[9] G. M. Capriani. The Steiner rearrangement in any codimension. *Calc. Var. Partial Differential Equations* 49, 517–548 (2014).

[10] M. Chlebík, A. Cianchi & N. Fusco. The perimeter inequality under Steiner symmetrization: cases of equality. *Ann. of Math.* 162, 525–555 (2005).

[11] A. Cianchi, L. Esposito, N. Fusco & C. Trombetti. A quantitative Pólya-Szegő principle. *J. Reine Angew. Math.* 614, 153–189 (2008).

[12] A. Cianchi & N. Fusco. Functions of bounded variation and rearrangements. *Arch. for Rat. Mech. and Anal.* 165, 1–40 (2002).

[13] A. Cianchi & N. Fusco. Steiner symmetric extremals in Pólya-Szegő type inequalities. *Adv. Math.* 203, 673–728 (2006).

[14] A. Cianchi, N. Fusco, F. Maggi & A. Pratelli. The sharp Sobolev inequality in quantitative form. *J. Eur. Math. Soc.* 11, 1105–1139 (2009).

[15] A. Ferone & R. Volpicelli. Minimal rearrangements of Sobolev functions: a new proof. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20, 333-339 (2003).

[16] N. Fusco, F. Maggi & A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math.* 168, 941–980 (2008).

[17] N. Fusco, V. Millot & M. Morini. A quantitative isoperimetric inequality for fractional perimeters. *J. Funct. Anal.* 261, 697–715 (2011).

[18] M. Giaquinta, G. Modica, J. Soucek. *Cartesian Currents in the Calculus of Variations, Vol. I*, A Series of Modern Surveys in Mathematics, Vol. 37, Springer-Verlag (1998).

[19] R. Howard. The John ellipsoid theorem. Lecture note (1997).

[20] A. Kennington. Power concavity and boundary value problems. *Indiana Univ. Math. J.* 34, 687–704 (1985).

[21] N. Korevaar. Convex solutions to nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.* 32, 603–614 (1983).

[22] F. Maggi. Some methods for studying stability in isoperimetric type problems. *Bull. Amer. Math. Soc. (N.S.)* 45, 367–408 (2008).

(M. Barchiesi) Dipartimento di Matematica ed Applicazioni, Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

E-mail address: barchies@gmail.com

(G. M. Capriani) Applied Mathematics Münster, Westfälische Wilhelms–Universtität Münster, Einsteinstr 62, D-48149 Münster, Germany

E-mail address: giuseppe.capriani@gmail.com

(N. Fusco) Dipartimento di Matematica ed Applicazioni, Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

E-mail address: n.fusco@unina.it

(G. Pisante) Dipartimento di Matematica e Fisica, Seconda Università degli Studi di Napoli, Viale Lincoln 5, 81100 Caserta, Italy
E-mail address: giovanni.pisante@unina2.it