ON A CLASS OF SEMILINEAR EVOLUTION EQUATIONS FOR VECTOR POTENTIALS ASSOCIATED WITH MAXWELL'S EQUATIONS IN CARNOT GROUPS

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Abstract. In this paper we prove existence and regularity results for a class of semilinear evolution equations that are satisfied by vector potentials associated with Maxwell's equations in Carnot groups (connected, simply connected, stratified nilpotent Lie groups). The natural setting for these equations is provided by the so-called Rumin's complex of intrinsic differential forms.

1. Introduction

The aim of this paper is to prove existence of strong solutions for a class of higher order semilinear evolution equations in Carnot groups (i.e. connected simply connected stratified nilpotent Lie groups) satisfied by vector potentials associated with "intrinsic" Maxwell's equations in the group. These equations, though not hyperbolic, can still be called "wave equations" because of their origin, as "equations for a vector potential", from a class of intrinsic Maxwell's equations, precisely as it holds in the Euclidean setting, where the vector potential associated with classical Maxwell's equations satisfies d'Alembert's wave equation. Let us remind this procedure in the Euclidean setting.

Consider the space-time $\mathbb{R} \times \mathbb{R}^3$ of special relativity, end we denote by $s \in \mathbb{R}$ the time variable and by $x \in \mathbb{R}^3$ the space variable. If (Ω^*, d) is the de Rham complex of differential forms in $\mathbb{R} \times \mathbb{R}^3$, classical Maxwell's equations can be formulated in their simplest form as follows: we fix the standard volume form dV in $\mathbb{R}^3$, and we consider a 2-form $F \in \Omega^2$ (Faraday's form), that can be always written as $F = ds \wedge E + B$, where E is the electric field 1-form and B is the magnetic induction 2-form. Then, if we assume for sake of simplicity all "physical" constants (i.e. magnetic permeability and electric permittivity) equal to 1, classical Maxwell's equations become

$$dF = 0 \quad \text{and} \quad d(*_M F) = \mathcal{J}. \tag{1}$$

1991 *Mathematics Subject Classification.* 35Q61, 35R03, 58A10, 49J45 .

Key words and phrases. Carnot groups, differential forms, semilinear Maxwell's equations.

B.F. is supported by MIUR, Italy, by University of Bologna, Italy, funds for selected research topics and by EC project CG-DICE. E.O. is supported by University of Bologna, Italy, funds for selected research topics. E.V. is supported by Swiss National Found and by EC project CG-DICE..

Here $*_M$ is the Hodge-star operator associated with the space-time Minkowskian metric and the volume form $ds \wedge dV$ in $\mathbb{R} \times \mathbb{R}^3$, and $\mathcal{J} = ds \wedge *J - \rho$ is a closed 3-form in $\mathbb{R} \times \mathbb{R}^3$, where $*J$ and $\rho = \rho_0 dV$ are respectively the current density 2-form and the charge density 3-form (here $*$ is the standard Hodge-star operator in $\mathbb{R}^3$ associated with the Euclidean metric and the volume form dV). Since $dF = 0$, we can always assume that $F = dA$, where A (the electromagnetic potential 1-form) can be written as $A = A_\Sigma + \varphi ds$. If, in addition, A_Σ and φ satisfy suitable gauge conditions, then they satisfy the wave equations

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta A_\Sigma - J \tag{2}$$

$$\frac{\partial^2 \varphi}{\partial s^2} = \Delta\varphi + \rho_0, \tag{3}$$

where ΔA_Σ in (2) is the positive Hodge Laplacian on 1-forms

$$\Delta A_\Sigma = (d^*d + dd^*)A_\Sigma,$$

whereas $\Delta\varphi$ in (3) is the usual negative Laplacian on functions. We remind that, in the Euclidean space, the Hodge Laplace operator Δ acts diagonally on 1-forms, i.e.

$$\Delta A_\Sigma := \Delta\Big(\sum_i A_{\Sigma,i}\, dx^i\Big) = \sum_i (\Delta A_{\Sigma,i})\, dx^i,$$

so that equation (2) reduces to a system of uncoupled wave equations.

An extensive literature is dedicated to the semilinear counterpart of the equations (2). These equations can therefore be reduced to scalar equations of the form

$$\frac{\partial^2 u}{\partial s^2} = \Delta u + f(u), \tag{4}$$

where, typically, $f(u) = -\mu u + |u|^{p-1}u$, with $p > 1$. We refer for instance to [17], [18], [29], [1].

To state our results, we need to sketch preliminarily the basic notions and the main results of Maxwell's theory in Carnot groups.

A connected and simply connected Lie group $(\mathbb{G}, \cdot)$ (in general non-commutative) is said a *Carnot group of step* κ if its Lie algebra $\mathfrak{g}$ admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. The first layer V_1, the so-called *horizontal layer*, plays a key role in the theory, since it generates $\mathfrak{g}$ by commutation.

We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

As we shall see in Theorem 2.5, the homogeneous dimension plays a crucial role in imbedding theorems for Sobolev spaces associated with Carnot groups.

For our purposes, it is important to remind the following definition: the Carnot group $\mathbb{G}$ is said to be *free* if its Lie algebra is free, i.e, if the commutators satisfy no linear relations other than antisymmetry and Jacobi identity.

A Carnot group $\mathbb{G}$ can be always identified, through exponential coordinates, with the Euclidean space $(\mathbb{R}^n, \cdot)$, where n is the dimension of $\mathfrak{g}$, and $(x, y) \mapsto x \cdot y$ is a suitable group operation in $\mathbb{R}^n$. The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

Obviously, Euclidean spaces $(\mathbb{R}^n, +)$ are commutative Carnot groups, and, more precisely, the only commutative Carnot groups. Indeed, in this case the stratification of the algebra consists of only one layer, i.e. the Lie algebra reduces to the horizontal layer.

In addition, throughout this paper we assume that $n > 2$.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [13] Chapter 1). The dilations δ_λ are group automorphisms, since $\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda(x \cdot y)$.

From now on, we use the word "intrinsic" when we want to stress a privileged role played by the horizontal layer and by group translations and dilations in $(\mathbb{R}^n, \cdot)$. In particular, we remind that the natural "intrinsic" counterpart of the linear transformations in $\mathbb{R}^n$ is provided by the class of the homogeneous endomorfisms of $\mathbb{G}$, where, coherently, homogeneity must be meant with respect to group translations. In exponential coordinates, homogeneous endomorfisms are linear contact maps, i.e. linear maps that preserve the stratification.

The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, called the horizontal bundle. A section of $H\mathbb{G}$ is called a horizontal vector field.

Among Carnot groups, the simplest but, at the same time, non-trivial (since non-Abelian) instance is provided by Heisenberg groups $\mathbb{H}^N$, and in particular by the first Heisenberg group $\mathbb{H}^1$. Precise definitions will be given in a moment; let us remind that $\mathbb{H}^1$ is a free group of step 2 with 2 generators, and that it is in some sense the "model" of all topologically 3-dimensional contact structures.

The Heisenberg group $\mathbb{H}^N$ can be identified with $\mathbb{R}^{2N+1}$, with variables (x, y, t), $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Set $X_i := \partial_{x_i} + 2y_i\partial_t$, $Y_i := \partial_{y_i} - 2x_i\partial_t$, $T := \partial_t$. The stratification of its algebra $\mathfrak{h}$ is given by $\mathfrak{h} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X_1, \dots, X_N, Y_1, \dots, Y_N\}$ and $V_2 = \text{span}\,\{T\}$. For sake of simplicity, if $N = 1$, we write $X := X_1$ and $Y = Y_1$.

Recently, a notion of "intrinsic" Maxwell's equations in Carnot groups has been introduced in a series of papers ([8], [15], [16], [3]). The setting for these equations is provided by a subcomplex of "intrinsic" differential forms

(E_0^*, d_c) – homotopic to de Rham's complex (Ω^*, d) – introduced by Rumin in [28], [27], (see also [5]). The main features of this theory are sketched in Section 4. To keep this introduction as simple as possible, from now on we refer only to the case of free groups.

Here is important to stress that a differential 1-form α belongs to E_0^1 if and only if is horizontal, i.e, if it is dual of an horizontal vector field. In addition, when acting on intrinsic 1-forms in free Carnot groups, the "exterior differential" d_c is an operator of order κ (the step of the group) in the horizontal derivatives. In particular, the "Hodge Laplacian" associated with d_c, i.e.

$$\delta_c d_c + d_c \delta_c, \tag{5}$$

generally is not homogeneous (and therefore, as long as we know, we lack Rockland type hypoellipticity results (see, e.g. [21]) and sharp a priori estimates in a natural scale of Sobolev spaces, the so-called Folland-Stein spaces - see Section 2). Since $\mathbb{G}$ is *free*, then d_c is an homogeneous differential operator of order κ (the step of the group) in the horizontal derivatives when acting on 1-forms (see [16]), but the "Hodge Laplacian" fails to be homogeneous. Indeed, on 1–forms, $\delta_c d_c$ is an operator of order 2κ, while $d_c \delta_c$ is a 2nd order one.

To overcome this difficulty, we remind that in $\mathbb{H}^1$ (where $\kappa = 2$), M. Rumin in [26] introduces a new homogeneous 4th order operator $\delta_c d_c + (d_c \delta_c)^2$ that satisfies sharp a priori estimates in intrinsic Folland-Stein spaces of order 4. We apply the same idea in free groups of arbitrary step κ and we obtain an homogeneous operator of order 2κ in the horizontal derivatives acting on intrinsic 1-forms

$$\Delta_{\mathbb{G},1} = \delta_c d_c + (d_c \delta_c)^\kappa.$$

In [16] it is proved that $\Delta_{\mathbb{G},1}$ satisfies sharp a priori estimates of order 2κ and is self-adjoint (see Theorem 3.12).

Consider for instance the case $\mathbb{G} = \mathbb{H}^1$. We denote by $\theta := dt + \frac{1}{2}(y\,dx - x\,dy)$ the contact form of $\mathbb{H}^1$. Then

$$\begin{aligned} E^1_{0,\mathbb{H}^1} &= \text{span}\,\{dx, dy\}; \\ E^2_{0,\mathbb{H}^1} &= \text{span}\,\{dx \wedge \theta, dy \wedge \theta\}; \\ E^3_{0,\mathbb{H}^1} &= \text{span}\,\{dx \wedge dy \wedge \theta\}. \end{aligned}$$

The action of d_c on $E^1_{0,\mathbb{H}^1}$ is the following ([26], [14], [7]): let $\alpha = \alpha_1 dx + \alpha_2 dy \in E^1_{0,\mathbb{H}^1}$ be given. Then

$$\begin{aligned} d_{c,\mathbb{H}^1}\alpha &= (X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1) dx \wedge \theta \\ &\quad + (2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2) dy \wedge \theta \\ &:= P_1(\alpha_1, \alpha_2) dx \wedge \theta + P_2(\alpha_1, \alpha_2) dy \wedge \theta. \end{aligned}$$

Let us go back to general free groups of arbitrary step κ (though the following arguments can be carried on in more general geometric settings, like, for instance, Heisenberg groups of higher order: see Remark 3.10 below). Keeping in mind (1), it is natural to define Maxwell's equations in $\mathbb{G}$ as follows: if $F \in \hat{E}_0^2$, we call *Maxwell's equations in* $\mathbb{G}$ the system

$$\hat{d}_c F = 0 \quad \text{and} \quad \hat{d}_c(*_M F) = \mathcal{J}, \tag{6}$$

where $\hat{d}_c$ denotes the Rumin's differential of the complex $(\hat{E}_0^*, \hat{d}_c)$ in the space-time product group $\mathbb{R} \times \mathbb{G}$, and $\hat{d}_c \mathcal{J} = 0$. So far, the argument may appear as purely formal, but we notice first that the equations (6) are invariant under the action of a class of suitable contact Lorentz transformations. Moreover, it is possible to show ([3], [4]) that the equations in (6) are limits of usual equations in a very strongly anisotropic matter.

If F is a solution of (6), then it is a closed form. Therefore it admits a vector potential

$$A := A_\Sigma + \varphi\, ds \in \hat{E}_0^1 \quad \text{such that} \quad \hat{d}_c A = F, \tag{7}$$

where

$$\partial_s^2 A_\Sigma = -\Delta_{\mathbb{G},1} A_\Sigma - J \tag{8}$$

$$\partial_s^2 \varphi = -(-\Delta_{\mathbb{G}})^\kappa \varphi + (-\Delta_{\mathbb{G}})^{\kappa-1} \rho_0. \tag{9}$$

Here $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2 (= -\Delta_{\mathbb{G},0})$ is the usual subelliptic Laplacian in $\mathbb{G}$, provided the following gauge condition holds:

$$\delta_c (d_c \delta_c)^{\kappa-1} A_\Sigma + \frac{\partial \varphi}{\partial s} = 0. \tag{10}$$

It is important to notice that the equation for A_Σ cannot be diagonalized, unlike the Euclidean case. But the main new phenomenon is that the "wave equations" we obtain utterly differ even in the scalar case from what one could imagine as "wave equations in the group", i.e.

$$\frac{\partial^2 \varphi}{\partial s^2} - \Delta_{\mathbb{G}} \varphi = 0. \tag{11}$$

Indeed, in spite of the energy estimate we can obtain for the equation (8) (see Lemma 4.7), the equations we obtain are by no means hyperbolic, by [22], Theorem 5.5.2, since they contain second order derivatives in s and 2κ-th order derivatives in x, so that their principal parts are (degenerate) elliptic. Thus, we should not expect any hyperbolic behavior, as, for instance, finite speed of propagation like in (3) (see, e.g., [25], [19]).

Existence and uniqueness results for equations (8) and (9) can be easily deduced by means of abstract arguments (see Theorem 4.6). The aim of this paper is to prove local existence results for the semilinear counterparts of (8) and (9). More precisely, we prove a local existence result for the Cauchy problem

$$\begin{cases} \partial_s^2 \alpha + \Delta_{\mathbb{G},1} \alpha & = g(\alpha)\alpha\,, \text{ in } I\,, \\ \alpha_{|s=0} & = \alpha_0, \\ \alpha_{s|s=0} & = \alpha_1\,. \end{cases} \tag{12}$$

Here $h(\alpha) = g(\alpha)\alpha$, where $g : E_0^1 \to \mathbb{R}$ is a locally Lipschitz continuous function such that $g(0) = 0$ and

$$|g(\alpha) - g(\beta)| \le C\|\alpha - \beta\| \left(1 + \|\alpha\|^{p-2} + \|\beta\|^{p-2}\right), \tag{13}$$

for some $p \ge 1$.

Our proof relies on a classical fixed point argument, following the guidelines of [20]. The main theorem (see Theorem 5.3) reads as follows:

Let $T \in \mathbb{R}^+$ *and set* $I = [0,T]$. *In addition, let* $\alpha_0 \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$ *be given, and let* g *satisfy* (13).

Assume $\mathbb{G}$ *is a Carnot group satisfying one of the following assumptions:*

- $\mathbb{G} = \mathbb{H}^1$ *and* $p \geq 1$;
- $\mathbb{G} = \mathbb{H}^N$ *with* $N > 1$ *and* $1 \leq p \leq 1 + 1/N$;
- $\mathbb{G}$ *is a free group of step* κ, $2\kappa < Q$ *and* $1 \leq p \leq Q/(Q - 2\kappa)$.

Then there exists $\delta \in (0,T]$, *such that problem* (12) *has a strong solution in* $[0,\delta]$.

A couple of comments are now in order, since the range of p in cases ii) and iii) may appear very restrictive. However, it relies on deep features of the group. First of all, in the Sobolev imbedding theorem (see 2.5), the homogeneous dimension Q of $\mathbb{G}$ plays the role of the dimension n of $\mathbb{G}$ as a manifold. In general, $Q >> n$ (in fact, $Q = n$ only for the commutative group $(\mathbb{R}^n, +)$). But this is a well known feature of Carnot groups, affecting, for instance, geometric measure theory in groups and nonlinear partial differential equations. On the contrary, here there is another more subtle consequence of the non-commutativity of the group that interferes with our problem, yielding weaker results (i.e. requiring a narrower range for the exponent p in (13)) when compared with their Euclidean counterpart: basically, we cannot expect local Strichartz-type estimates for our higher order equations (for the Euclidean setting we refer, for instance, to [18], [23], and to [20] and the reference therein for higher order equations). To illustrate this phenomenon, let us restrict ourselves to the simplest case $\mathbb{G} = \mathbb{H}^1$ and consider the scalar equation (9) for the time-component of the vector potential. In addition, choose $\rho_0 \equiv 0$.

Thus, equation (9) becomes

$$\partial_s^2 \varphi = -(-\Delta_{\mathbb{G}})^2 \varphi, \tag{14}$$

that can be written as the action of the product of two Schrödinger operators

$$(\partial_s + i\Delta_{\mathbb{G}})(\partial_s - i\Delta_{\mathbb{G}})\varphi = 0. \tag{15}$$

Now, it follows from [2] that there exist traveling wave solutions of the Schrödinger's equation in $\mathbb{H}^1$

$$(\partial_s + i\Delta_{\mathbb{G}})\varphi = 0 \tag{16}$$

of the form (with our notations)

$$\phi_\lambda(s; x, y, t) := f(\lambda x, \lambda y, \lambda^2 t - \lambda^2 s) \quad \lambda > 0 \tag{17}$$

for a suitable function $f \in \mathcal{S}(\mathbb{H}^1)$. Clearly, the solution ϕ_λ has L^r-norm in $\mathbb{H}^1$ independent of $s \in \mathbb{R}$ for $1 \leq r \leq \infty$. Moreover, if $\lambda > 0$, then $\|\phi_\lambda\|_{L^r(\mathbb{H}^1)} = c\lambda^{4/r}$, and necessarily Strichartz-type estimates fail to hold even for short times.

On the other hand, the existence of the traveling waves (17) for Schrödinger's equation in $\mathbb{H}^1$ comes from another peculiar feature of Heisenberg groups: the existence of the so-called *forbidden values* $\alpha \in \mathbb{R}$ for the differential operator

$$\mathcal{L}_\alpha = X^2 + Y^2 - i\alpha T$$

for which $\mathcal{L}_\alpha$ has a non-trivial kernel (see [30], Chapter XIII, 2.3.2). Basically, this is due to the fact that T is at the same time a *differential operator of order 1* and a second order operator with respect to group dilations. Clearly, this phenomenon fails to hold in the Euclidean space.

Finally, we stress that, on the other side, Strichartz-type estimates hold for the equation (11), the usual homogeneous *second order* wave equation in $\mathbb{H}^1$ (see [2]). Thus, the lack of these estimates is originated in our case by the combination of the structure of the group and of the order of the equation.

2. Carnot groups

Definition 2.1. Let $e_1, \dots, e_n$ be a basis of $\mathfrak{g} = V_1 \oplus \cdots \oplus V_\kappa$ adapted to the stratification of $\mathfrak{g}$, and let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \dots, n$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$, making $\{X_1, \dots, X_n\}$ an orthonormal basis.

From now on, we set $m_i := \dim V_i$, $i = 1, \dots, \kappa$. We write also m instead of m_1.

Following [13], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [9], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations.

Let now $\{X_1, \dots, X_m\}$ be a basis of the first layer of $\mathfrak{g}$, and denote by $\mathcal{L}$ the associated positive sub-Laplacian

$$\mathcal{L} := -\sum_{j=1}^{m} X_j^2.$$

If $1 < p < \infty$ and $k \geq 0$, we denote by $W^{k,p}_{\mathbb{G}}(\mathbb{G})$ (the Folland-Stein Sobolev space) the domain of $\mathcal{L}^{k/2}(\mathbb{G})$ in $L^p(\mathbb{G})$ endowed with the graph norm (see [12] or [13]).

We remind that

Proposition 2.2. *If $1 < p < \infty$ and $k \geq 0$, then the space $W^{k,p}_{\mathbb{G}}(\mathbb{G})$ is independent of the choice of $X_1, \dots, X_m$.*

Proposition 2.3. *If $1 < p < \infty$ and $k \geq 0$, then $\mathcal{D}(\mathbb{G})$ is dense in $W^{k,p}_{\mathbb{G}}(\mathbb{G})$.*

Proposition 2.4. *Let k be a positive integer, $1 \leq p < \infty$. Then $W^{k,p}_{\mathbb{G}}(\mathbb{G})$ consists of all functions $f \in L^p(\mathbb{G})$ with distributional derivatives $X^I f \in L^p(\mathbb{G})$ for any X^I with $d(I) \leq k$, endowed with the natural norm.*

Theorem 2.5 ([12], Theorem 4.17 and Proposition 4.2)**.** *If* $1 < p < q < \infty$ *and* $\alpha, \beta \geq 0$ *are such that*

$$\beta = \alpha - Q\left(\frac{1}{p} - \frac{1}{q}\right),$$

or

$$p = q \quad \textit{and} \quad \beta < \alpha,$$

then $W^{\alpha,p}_{\mathbb{G}}(\mathbb{G})$ *is continuously embedded in* $W^{\beta,q}_{\mathbb{G}}(\mathbb{G})$*.*

3. Differential forms in Carnot groups

In order to write our Maxwell's equation in $\mathbb{G}$, we need to use the language of differential forms. More precisely, we need to present a class of intrinsic differential forms, that has been introduced by Rumin in [28], [27], (see also [5]).

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $\{X_1, \dots, X_n\}$, is the family of covectors $\{\theta_1, \dots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\{\theta_1, \dots, \theta_n\}$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \dots, \theta_n$ may have polynomial (hence variable) coefficients. Following Federer (see [11], 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leq h \leq n$,

$$\bigwedge_h \mathfrak{g} := \operatorname{span}\{X_{i_1} \wedge \dots \wedge X_{i_h} : 1 \leq i_1 < \dots < i_h \leq n\},$$
$$\bigwedge^h \mathfrak{g} := \operatorname{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_h} : 1 \leq i_1 < \dots < i_h \leq n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*, respectively. We denote by Ω_h and Ω^h the spaces of all sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively, for $h = 0, 1, \dots, n$. We refer to elements of Ω_h as fields of h-vectors and to elements of Ω^h as h-forms and to (Ω^*, d) as the de Rham complex.

We denote by Θ^h the basis $\{\theta_{i_1} \wedge \dots \wedge \theta_{i_h} : 1 \leq i_1 < \dots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \dots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \dots \wedge \theta_{i_h}$ orthonormal.

Definition 3.1. We define linear isomorphisms (Hodge duality: see [11] 1.7.8)

$$* : \bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \leq h \leq n$, putting, for $v, w \in \bigwedge_h \mathfrak{g}$ and $\varphi, \psi \in \bigwedge^h \mathfrak{g}$

$$v \wedge *w = \langle v, w \rangle X_1 \wedge \dots \wedge X_n, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_1 \wedge \dots \wedge \theta_n.$$

From now on, we refer to the n-form

$$dV := \theta_1 \wedge \cdots \wedge \theta_n$$

as the canonical volume form in $\mathbb{G}$.

If d is the usual De Rham exterior differential, we denote by $\delta = d^*$ its formal adjoint in $L^2(\mathbb{G}, \Omega^*)$.

Definition 3.2. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has weight k, and we write $w(\alpha) = k$, if its dual vector $\alpha^\natural$ is in V_k. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Remark 3.3. As shown in [5], Remark 2.4, if $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$, and we have

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{18}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are the smallest and the largest weight of left-invariant h-covectors, respectively. Since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p} \mathfrak{g}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p} \mathfrak{g}$.

Keeping in mind the decomposition (18), we can define in the same way several left invariant fiber bundles over $\mathbb{G}$, that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$. Notice also that the fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$.

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We set

$$\Omega^h := \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{19}$$

The following crucial property of the weight follows from Cartan identity: see [28], Section 2.1:

Lemma 3.4. *We have $d(\bigwedge^{h,p} \mathfrak{g}) \subset \bigwedge^{h+1,p} \mathfrak{g}$, i.e., if $\alpha \in \bigwedge^{h,p} \mathfrak{g}$ is a left invariant h-form of weight p with $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$.*

Definition 3.5. If $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \theta_i^h \in \Omega^{h,p}$ is a smooth form of pure weight p, then we have

$$d\alpha = d_0\alpha + \sum_{\ell=1}^{\kappa} d_\ell \alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight, and

$$d_\ell \alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{X_j \in V_\ell} (X_j \alpha_i)\, \theta_j \wedge \theta_i^h,$$

increases the weight by ℓ units ($\ell = 1, \dots \kappa$); here κ is the step of the group. In particular, d_0 is an algebraic operator.

Definition 3.6. If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap (\operatorname{Im} d_0)^\perp \subset \Omega^h$$

The elements of E_0^h are *intrinsic h-forms on* $\mathbb{G}$. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (18).

Since it is easy to see that $E_0^1 = \operatorname{span}\{\theta_1, \dots, \theta_m\}$, without loss of generality we can take $\xi_j = \theta_j$ for $j = 1, \dots, m$.

If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_p E_0^{h,p}.$$

We define now a (pseudo) inverse of d_0 as follows:

Lemma 3.7. *If $\beta \in \bigwedge^{h+1} \mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{g} \cap (\ker d_0)^\perp$ such that $d_0\alpha - \beta \in (\operatorname{Im} d_0)^\perp$. We set $\alpha := d_0^{-1}\beta$. Notice that d_0^{-1} preserves the weights.*

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [28] and [5], Section 2) .

Theorem 3.8. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \text{and} \quad F := \operatorname{Im}(d_0^{-1}) + \operatorname{Im}(d d_0^{-1}).$$

We have

i) *Let Π_E be the projection on E along F (that is not an orthogonal projection). Then for any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E\alpha)_j$ the component of $\Pi_E\alpha$ of weight j, then*

$$(20) \qquad \begin{aligned} &(\Pi_E\alpha)_p = \alpha \\ &(\Pi_E\alpha)_{p+k+1} = -d_0^{-1}\Big(\sum_{1\le\ell\le k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\Big). \end{aligned}$$

ii) *Π_E is a chain map, i.e.*

$$d\Pi_E = \Pi_E d.$$

iii) *Let Π_{E_0} be the orthogonal projection from Ω^* on E_0^*, then*

$$(21) \qquad \Pi_{E_0} = Id - d_0^{-1}d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0}^\perp = d_0^{-1}d_0 + d_0 d_0^{-1}.$$

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.$$

We have:

iv) $d_c^2 = 0$;

v) *the complex $E_0 := (E_0^*, d_c)$ is exact;*

vi) *with respect to the bases* Ξ^*, *the intrinsic differential* d_c *can be seen as a matrix-valued operator such that, if* α *has weight* p, *then the component of weight* q *of* $d_c\alpha$ *is given by an homogeneous differential operator in the horizontal derivatives of order* $q-p\geq 1$, *acting on the components of* α.

If we need to stress that the complex is built on a specific group $\mathbb{G}$, *we shall denote it by* $(E^*_{0,\mathbb{G}}, d_{c,\mathbb{G}})$, *to avoid misunderstandings.*

From now on, we restrict ourselves to assume $\mathbb{G}$ is a free group of step κ. The technical reason for this choice relies in the following property.

Theorem 3.9 ([16], Theorem 5.9). *Let* $\mathbb{G}$ *be a free group of step* κ. *Then all forms in* E_0^1 *have weight* 1 *and all forms in* E_0^2 *have weight* $\kappa+1$.

In particular, the differential $d_c : E_0^1 \to E_0^2$ *can be identified, with respect to the adapted bases* Ξ_0^1 *and* Ξ_0^2, *with a homogeneous matrix-valued differential operator of degree* κ *in the horizontal derivatives.*

Remark 3.10. When $N>1$, the N-th Heisenberg group $\mathbb{H}^N$ is not a free group. Nevertheless, all 2-forms in E_0^2 have the same weight 2.

We denote by $\delta_c = \delta_{c,\mathbb{G}} = d_c^* = d^*_{c,\mathbb{G}}$ the formal adjoint of d_c in $L^2(\mathbb{G}, E_0^*)$. The following assertion holds.

Definition 3.11. If $0\leq h\leq n$, $k\geq 0$ and $1<p\leq\infty$, then we denote by $W^{k,p}_{\mathbb{G}}(\mathbb{G}, E_0^h)$ the space of all forms in E_0^h with coefficients in $W^{k,p}_{\mathbb{G}}(\mathbb{G})$, endowed with its natural norm. It is easy to see that this definition is independent of the chosen basis of $\bigwedge^h \mathfrak{g}$.

Theorem 3.12 ([16]). *Let* $\mathbb{G}$ *be a Carnot group. Suppose all intrinsic 2-forms have the same weight* N_2 *(by Theorem 3.9 this holds true for any free group* $\mathbb{G}$ *of step* κ, *with* $N_2=\kappa+1$, *but also for all Heisenberg groups* $\mathbb{H}^N$ *with* $N>1$ *with* $N_2=2$, *by Remark 3.10). Set* $N_2-1:=r$. *We denote by* $\Delta_{\mathbb{G},1} := \delta_c d_c + (d_c\delta_c)^r$ *Rumin's homogeneous Hodge Laplacian on intrinsic 1-forms in* $\mathbb{G}$.

i) $\Delta_{\mathbb{G},1}$ *is maximal subelliptic, i.e. there exists* $C>0$ *such that for any multi–index* I *with* $d(I)=r$

$$\|X^I\alpha\|_{L^2(\mathbb{G},E_0^1)} \leq C\left(\langle\Delta_{\mathbb{G},1}\alpha,\alpha\rangle_{L^2(\mathbb{G},E_0^1)} + \|\alpha\|_{L^2(\mathbb{G},E_0^1)}\right) \tag{22}$$

for any $\alpha\in\mathcal{D}(\mathbb{G},E_0^1)$.

ii) *If* $1<p<\infty$ *is fixed, then there exists* $C>0$ *such that for any multi–index* I *with* $d(I)=2r$ *we have*

$$\|X^I\alpha\|_{L^p(\mathbb{G},E_0^1)} \leq C\left(\|\Delta_{\mathbb{G},1}\alpha\|_{L^p(\mathbb{G},E_0^1)} + \|\alpha\|_{L^p(\mathbb{G},E_0^1)}\right) \tag{23}$$

for any $\alpha\in\mathcal{D}(\mathbb{G},E_0^1)$ *(if* $p=2$ *this means that* $\Delta_{\mathbb{G},1}$ *is maximal hypoelliptic in the sense of* [21]*);*

iii) *the unbounded operator in* $L^2(\mathbb{G},E_0^1)$

$$\Delta_{\mathbb{G},1} \quad \textit{with domain} \quad W^{2r,2}_{\mathbb{G}}(\mathbb{G},E_0^1)$$

is self-adjoint and nonnegative.

Definition 3.13. If $1 \leq h \leq n$, we say that T is a h-current on $\mathbb{G}$ if T is a continuous linear functional on $\mathcal{D}(\mathbb{G}, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\mathbb{G}, E_0^h)$.

Any (usual) distribution $T \in \mathcal{D}'(\mathbb{G})$ can be identified canonically with an n-current $\tilde{T} \in \mathcal{D}'(\mathbb{G}, E_0^n)$ through the formula

$$\langle \tilde{T} | \alpha \rangle := \langle T | * \alpha \rangle \tag{24}$$

for any $\alpha \in \mathcal{D}(\mathbb{G}, E_0^n)$. Reciprocally, by (24), any n-current $\tilde{T}$ can be identified with an usual distribution $T \in \mathcal{D}'(\mathbb{G})$.

Following [11], 4.1.7, we give the following definition.

Definition 3.14. If $T \in \mathcal{D}'(\mathbb{G}, E_0^n)$, and $\varphi \in \mathcal{E}(\mathbb{G}, E_0^k)$, with $0 \leq k \leq n$, we define $T \llcorner \varphi \in D'(\mathbb{G}, E_0^{n-k})$ by the identity

$$\langle T \llcorner \varphi | \alpha \rangle := \langle T | \alpha \wedge \varphi \rangle$$

for any $\alpha \in \mathcal{D}(\mathbb{G}, E_0^{n-k})$.

We notice that, when $\varphi \in \mathcal{E}(\mathbb{G}, E_0^k)$ and $\alpha \in \mathcal{D}(\mathbb{G}, E_0^{n-k})$, then the wedge product $\alpha \wedge \varphi$ belongs to $\mathcal{D}(\mathbb{G}, E_0^n)$, since $E_0^n = \Omega^n$.

The following result is taken from [6], Propositions 5 and 6, and Definition 10, but we refer also to [10], Sections 17.3, 17.4 and 17.5.

If $1 \leq h \leq n$ and $\Xi_0^h = \{\xi_1^h, \dots, \xi_{\dim E_0^h}^h\}$ is a left invariant basis of E_0^h and $T \in \mathcal{D}'(\mathbb{G}, E_0^h)$, then there exist (uniquely determined) $T_1, \dots, T_{\dim E_0^h} \in \mathcal{D}'(\mathbb{G})$ such that

$$T = \sum_j \tilde{T}_j \llcorner (* \xi_j^h).$$

It is well known that currents can be seen as forms with distributional coefficients in the following sense: if $\alpha \in E(\mathbb{G}, E_0^h)$, then α can be identified canonically with a h-current T_α through the formula

$$\langle T_\alpha | \varphi \rangle := \int_{\mathbb{G}} * \alpha \wedge \varphi \tag{25}$$

for any $\varphi \in \mathcal{D}(\mathbb{G}, E_0^h)$. Moreover, if $\alpha = \sum_j \alpha_j \xi_j^h$ then

$$T_\alpha = \sum_j \tilde{\alpha}_j \llcorner (* \xi_j^h)$$

Using the language of intrinsic currents, we can now characterize the dual Folland-Stein Sobolev space in (E_0^*, d_c) (see [5], Proposition 3.14). We state the result only for intrinsic 1-forms and $p = 2$.

Proposition 3.15. *We remind that E_0^1 is the dual space of the first layer of the stratification of $\mathfrak{g}$, and that therefore $\dim E_0^1 = m$. Moreover $\{\theta_j\}_{j=1}^m$ is a left invariant orthonormal basis of E_0^1. Then, if $k \geq 0$, the dual space $\big(W_{\mathbb{G}}^{k,2}(\mathbb{G}, E_0^1)\big)^*$ coincides with the set of all currents $T \in \mathcal{D}'(\mathbb{G}, E_0^1)$ of the form*

$$T = \sum_{j=1}^m \tilde{T}_j \llcorner (* \theta_j), \tag{26}$$

where $\tilde{T}_j \in W^{-k,2}_{\mathbb{G}}(\mathbb{G})$ *for all* $j \in \{1, \dots, m\}$*. Moreover, if* T *is as in (26)*

$$\|T\|_{W^{-k,2}_{\mathbb{G}}(\mathbb{G},E_0^1)} \approx \sum_{j=1}^{m} \|\tilde{T}_j\|_{W^{-k,2}_{\mathbb{G}}(\mathbb{G})}.$$

4. Space-time Carnot groups and Maxwell's equations

From now on, we denote by x a "space" point in the Carnot group $\mathbb{G}$, and by $s \in \mathbb{R}$ the "time", and we choose in $\hat{\mathbb{G}} := \mathbb{R} \times \mathbb{G}$ the canonical volume form $ds \wedge dV$, where, as above, $dV = \theta_1 \wedge \cdots \wedge \theta_n$ is the canonical volume form in $\mathbb{G}$.

Denote by S the vector field $\frac{\partial}{\partial s}$ in $\mathbb{R} \times \mathbb{G}$. The Lie group $\hat{\mathbb{G}}$ is a Carnot group; its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$\hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \tag{27}$$

where $\hat{V}_1 = \operatorname{span}\{S, V_1\}$. Since the adapted basis $\{X_1, \dots, X_n\}$ has been already fixed once and for all, the associated orthonormal fixed basis for $\hat{\mathfrak{g}}$ will be $\{S, X_1, \dots, X_n\}$. Consider the Lie derivative $\mathcal{L}_S$ along S. When acting on h-forms α in $\mathbb{G}$, without risk of misunderstandings, we write $S\alpha$ for $\mathcal{L}_S\alpha$.

The following structure lemma for intrinsic forms in $\hat{\mathbb{G}}$ was proved in [8] and also in [16].

Lemma 4.1. *If* $1 \le h \le n$*, then a* h*-form* α *belongs to* $\hat{E}_0^h$ *if and only if it can be written as*

$$\alpha = ds \wedge \beta + \gamma, \tag{28}$$

where $\beta \in E_0^{h-1}$ *and* $\gamma \in E_0^h$ *are respectively intrinsic* $(h-1)$*-forms and* h*-forms in* $\mathbb{G}$ *with coefficients depending on* x *and* s*.*

As in special relativity, the space-time $\mathbb{R} \times \mathbb{G}$ can be endowed with a Minkowskian scalar product $\langle \cdot, \cdot \rangle_M$ in $\bigwedge_* \hat{\mathfrak{g}}$ and $\bigwedge^* \hat{\mathfrak{g}}$ (see [16], Definition 4.1).

Definition 4.2. If $1 \le h \le n$, we set

$$\langle ds \wedge \beta + \gamma, ds \wedge \beta' + \gamma' \rangle_M := \langle \gamma, \gamma' \rangle - \langle \beta, \beta' \rangle,$$

for $\beta, \beta' \in E_0^{h-1}$ and $\gamma, \gamma' \in E_0^h$. In addition, we denote by $*_M$ the Hodge operator $*_M : \bigwedge^h \hat{\mathfrak{g}} \to \bigwedge^{n-h} \hat{\mathfrak{g}}$ associated with the Minkowskian scalar product in $\bigwedge^* \hat{\mathfrak{g}}$, with respect to the volume form $ds \wedge dV$, by

$$\alpha \wedge *_M\beta = \langle \alpha, \beta \rangle_M ds \wedge dV.$$

Let $\mathcal{J}$ be a fixed closed intrinsic n-form in $\mathbb{R} \times \mathbb{G}$ (a source form). We can write $\mathcal{J} = ds \wedge *J - \rho$, where $J = J(s, \cdot)$ is an intrinsic 1-form on $\mathbb{G}$ and $\rho(s, \cdot) = \rho_0(s, \cdot)\, dV$ is a volume form on $\mathbb{G}$ for any fixed $s \in \mathbb{R}$.

If $F \in \hat{E}_0^2$, we call *Maxwell's equations in* $\mathbb{G}$ the system

$$\hat{d}_c F = 0 \quad \text{and} \quad \hat{d}_c(*_M F) = \mathcal{J} \tag{29}$$

(for sake of simplicity, we assume all "physical" constants to be 1). This system corresponds to a particular choice of the so-called constitutive relations. We refer to [8], [16] for further comments (in particular for invariance under suitable contact Lorentz transformation).

If F is a solution of (29), then it is a closed form. Therefore it admits a vector potential

$$A := A_\Sigma + \varphi\, ds \in \hat{E}_0^1 \quad \text{such that} \quad \hat{d}_c A = F. \tag{30}$$

Now we can define our intrinsic "wave equations" for Carnot groups satisfying the assumptions of Theorem 3.12.

Theorem 4.3 ([16], Theorem 5.12)**.** *Let $\mathbb{G}$ be a Carnot group satisfying the assumption of Theorem 3.12. Suppose $F \in \hat{E}_0^2$ satisfies* (29)*. Then $F = \hat{d}_c A$ with $A = \sum_{j=1}^m A_j\theta_j + \varphi\, ds := A_\Sigma + \varphi\, ds \in \hat{E}_0^1$, where*

$$\partial_s^2 A_\Sigma \;=\; -\Delta_{\mathbb{G},1} A_\Sigma - J \tag{31}$$

$$\partial_s^2 \varphi \;=\; -(-\Delta_{\mathbb{G}})^r \varphi \;+ (-\Delta_{\mathbb{G}})^{r-1}\rho_0, \tag{32}$$

where $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2 (= -\Delta_{\mathbb{G},0})$ is the usual subelliptic Laplacian in $\mathbb{G}$, provided the following gauge condition holds:

$$\delta_c (d_c\delta_c)^{r-1} A_\Sigma + \frac{\partial\varphi}{\partial s} = 0. \tag{33}$$

Notice condition (33) *can also be written as*

$$(-\Delta_{\mathbb{G}})^{r-1}\delta_c A_\Sigma + \frac{\partial\varphi}{\partial s} = 0. \tag{34}$$

Remark 4.4. The gauge condition (33) is always satisfied if we replace A by $A + \hat{d}_c f$, with f satisfying

$$\partial_s^2 f = -(-\Delta_{\mathbb{G}})^r f - \Big(\delta_c(d_c\delta_c)^{r-1}A_\Sigma + \frac{\partial\varphi}{\partial s}\Big).$$

Remark 4.5. For sake of simplicity, suppose now $\rho \equiv 0$ (no charges). If, for instance, we associate with equations (31) and (32) sufficiently regular Cauchy data

$$\begin{cases} A_{\Sigma|s=0} = A_{\Sigma,0}, & \frac{\partial A_\Sigma}{\partial s}{}_{|s=0} = A_{\Sigma,1}; \\ \varphi_{|s=0} = \varphi_0, & \frac{\partial\varphi}{\partial s}{}_{|s=0} = \varphi_1, \end{cases} \tag{35}$$

then the statement of Theorem 4.3 can be reversed. More precisely (see [16], Section 5 for a detailed discussion), let $A_\Sigma \in E^1_{0,\mathbb{H}^1}$ and φ satisfy (31), (32) and (33). Assume also that they have L^2-traces on $s = 0$ satisfying (35), with

$$A_{\Sigma,1} \in W^{1,2}_{\mathbb{G}}(\mathbb{G}, E_0^1) \text{ and } \varphi_0 \in W^{2,2}_{\mathbb{G}}(\mathbb{G},) \text{ if } r > 1.$$

Then

$$F = d_c A := d_c(A_\Sigma + \varphi\, ds)$$

satisfies Maxwell's system (29).

In view of the self-adjointness of $\Delta_{\mathbb{G},1}$ (see Theorem 3.12, iii)), next theorem follows from [24], Ch. 3, Theorem 8.2.

Theorem 4.6. *Let $\mathbb{G}$ be a Carnot group satisfying the assumptions of Theorem 3.12, $T \in \mathbb{R}^+$, and set $I = [0, T]$. Let $\alpha_0 \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$ and $J \in L^2\big(I, L^2(\mathbb{G}, E_0^1)\big)$ be given. Then, there exists a unique strong solution*

$$\alpha \in C^0(I, W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)) \cap C^1(I, L^2(\mathbb{G}, E_0^1))$$

of the linear equation

$$\begin{cases} \partial_s^2\alpha + \Delta_{\mathbb{G},1}\alpha = J, & \text{if } s \in [0,T]\,, \\ \alpha_{|s=0} = \alpha_0, & \alpha_{s|s=0} = \alpha_1\,. \end{cases} \tag{36}$$

If, in addition, $J \in C^0\left(I, W_{\mathbb{G}}^{-r,2}(\mathbb{G}, E_0^1)\right)$*, then* $\alpha \in C^2\left(I, W_{\mathbb{G}}^{-r,2}(\mathbb{G}, E_0^1)\right)$.

Lemma 4.7. *Let α be a solution of* (36)*. Then*

$$\|\alpha\|_{C^0\left(I, W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)\right)} + \|\partial_s\alpha\|_{C^0\left(I,L^2(\mathbb{G},E_0^1)\right)} \le \\ \le K\left(\sqrt{E_0(\alpha_0,\alpha_1)} + T^{1/2}\,\|J\|_{L^2(I,L^2(\mathbb{G},E_0^1))}\right), \tag{37}$$

where the kinetic energy $E\,(\alpha_0,\alpha_1)$ is defined as

$$E\,(\alpha_0,\alpha_1) = \frac{1}{2}\left(\|\alpha_0\|^2_{W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)} + \|\alpha_1\|^2_{L^2(\mathbb{G},E_0^1)}\right)$$

and K does not depend on α_0, α_1 and J.

Proof. By the *energy equality* in [24], Ch. 3, Lemma 8.3, and elementary computations, we get, for any $s \in (0,T]$,

$$\|\alpha(s)\|^2_{W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)} + \|\partial_s\alpha(s)\|^2_{L^2(\mathbb{G},E_0^1)} \le$$

$$\le \|\alpha_0\|^2_{W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)} + \|\alpha_1\|^2_{L^2(\mathbb{G},E_0^1)} + 2\int_0^s \|J(\sigma)\|_{L^2(\mathbb{G},E_0^1)}\|\partial_s\alpha(\sigma)\|_{L^2(\mathbb{G},E_0^1)}d\sigma \le$$

$$\le \|\alpha_0\|^2_{W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)} + \|\alpha_1\|^2_{L^2(\mathbb{G},E_0^1)} + 2\|\partial_s\alpha\|_{C^0(I,L^2(\mathbb{G},E_0^1))}\int_0^T \|J(\sigma)\|_{L^2(\mathbb{G},E_0^1)}d\sigma \le$$

$$\le \|\alpha_0\|^2_{W_{\mathbb{G}}^{r,2}(\mathbb{G},E_0^1)} + \|\alpha_1\|^2_{L^2(\mathbb{G},E_0^1)} + \frac{1}{2}\|\partial_s\alpha\|^2_{C^0(I,L^2(\mathbb{G},E_0^1))} + 2T\|J\|^2_{L^2(I,L^2(\mathbb{G},E_0^1))}\,.$$

Taking the supremum in the s variable, we get Eq. (37). □

5. Nonlinear equations

Our aim is to prove the existence of a local strong solution of the Cauchy problem for the equation

$$\partial_s^2\alpha + \Delta_{\mathbb{G},1}\alpha = h(\alpha). \tag{38}$$

Here $h(\alpha) = g(\alpha)\alpha$, where $g : E_0^1 \to \mathbb{R}$ is a locally Lipschitz continuous function.

Definition 5.1. Let I an interval in $\mathbb{R}$, such that $0 \in I$, $\alpha_0 \in W_{\mathbb{G}}^{r,2}(\mathbb{G}, E_0^1)$ and $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$. We say that α is a strong solution in I of the Cauchy problem

$$\begin{cases} \partial_s^2\alpha + \Delta_{\mathbb{G},1}\alpha &= g(\alpha)\alpha\,, \text{ in } I\,, \\ \alpha_{|s=0} = \alpha_0, & \alpha_{s|s=0} = \alpha_1\,, \end{cases} \tag{39}$$

if $\alpha \in C^0(I, W_{\mathbb{G}}^{r,2}(\mathbb{G}, E_0^1)) \cap C^1(I, L^2(\mathbb{G}, E_0^1))$ and satisfies (39).

In what follows we always suppose $g(0) = 0$ and

$$|g(\alpha) - g(\beta)| \le C\|\alpha - \beta\|\big(1 + \|\alpha\|^{p-2} + \|\beta\|^{p-2}\big)\,, \tag{40}$$

for some $p \ge 1$. In particular,

$$|g(\alpha)| \le C\big(1 + \|\alpha\|^{p-1}\big)\,, \qquad \|h(\alpha)\| \le C\big(1 + \|\alpha\|^{p}\big)\,.$$

Lemma 5.2. *Let $\mathbb{G}$ be a Carnot group of step κ. Then $Q \ge 2\kappa$ and equality holds if only if $\mathbb{G} = \mathbb{H}^1$ (in this case $\kappa = 2$ and $Q = 4$).*

Proof. By definition,

$$Q = \sum_{i=1}^{\kappa} i m_i\,.$$

If $\kappa = 1$, the group becomes a Euclidean space of dimension greater than 2 and the statement is verified.

Suppose now $\kappa > 1$ and $Q \le 2\kappa$.

If $\kappa = 2$, we have

$$Q = m_1 + 2m_2 \le 4\,,$$

which is possible (since dimension is greater than 2) iff $m_1 = 2$ and $m_2 = 1$; so we get $Q = 2\kappa$ and $\mathbb{G} = \mathbb{H}^1$.

If $\kappa > 2$, we get

$$m_1 + (\kappa - 1)m_{\kappa-1} + \kappa m_\kappa \le 2\kappa\,;$$

since $m_1 \ge 2$ and $m_{\kappa-1} \ge 1$, then $1+\kappa\,(m_\kappa + 1) \le 2\kappa$; hence $1 \le \kappa\,(1 - m_\kappa)$, which is impossible, since $m_\kappa \ge 1$. □

Theorem 5.3. *Let $\mathbb{G}$ be a Carnot group satisfying the assumptions of Theorem 3.12. Let $T \in \mathbb{R}^+$ and set $I = [0, T]$. In addition, let $\alpha_0 \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$ be given, and let g satisfy (40).*

Assume one of the following assumptions holds

- $\mathbb{G} = \mathbb{H}^1$ *and* $p \ge 1$*;*
- $\mathbb{G} = \mathbb{H}^N$ *with* $N > 1$ *and* $1 \le p \le 1 + 1/N$*;*
- $\mathbb{G}$ *is a free group of step* κ*,* $2\kappa < Q$ *and* $1 \le p \le Q/(Q - 2\kappa)$*.*

Then there exists $\delta \in (0, T]$, such that problem (39) has a strong solution in $[0, \delta]$.

Proof. First of all, set $\mathcal{H} := C^0(I, W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)) \cap C^1(I, L^2(\mathbb{G}, E_0^1))$. Our proof follows the ideas in [20]: we consider the associated linear problem and then we apply the Banach fixed point theorem combined with the previous linear results.

Consider a compactly supported C^∞ function $\eta : \mathbb{R} \to [0, 1]$, such that $\eta \equiv 1$ on $[-1, 1]$, and define

$$h_1(\alpha) = \eta(\|\alpha\|)h(\alpha)\,, \qquad h_2(\alpha) = \big(1 - \eta(\|\alpha\|)\big)h(\alpha)\,.$$

We observe that h_1 is globally Lipschitz continuous, while h_2 satisfies the estimate

$$\|h_2(\alpha)\| \le C\,(1 - \eta(\|\alpha\|))\,(1 + \|\alpha\|^p) \le C_1\,(1 - \eta(\|\alpha\|))\,\|\alpha\|^p\,,$$

since $\|\alpha\| > 1$ on the support of $1 - \eta$. Moreover, $h_1(0) = h_2(0) = 0$.

Now, take $\alpha \in \mathcal{H}$ and consider the following linear Cauchy problems

$$\begin{cases} \partial_s^2 \beta_0 + \Delta_{\mathbb{G},1} \beta_0 = 0 \,, \quad s \in I \,, \\ \beta_0(0) = \alpha_0 \,, \qquad \partial_s \beta_0(0) = \alpha_1 \,. \end{cases} \tag{41}$$

$$\begin{cases} \partial_s^2 \beta_1 + \Delta_{\mathbb{G},1} \beta_1 = h_1(\alpha) \,, \quad s \in I \,, \\ \beta_1(0) = 0 \,, \qquad \partial_s \beta_1(0) = 0 \,. \end{cases} \tag{42}$$

$$\begin{cases} \partial_s^2 \beta_2 + \Delta_{\mathbb{G},1} \beta_2 = h_2(\alpha) \,, \quad s \in I \,, \\ \beta_2(0) = 0 \,, \qquad \partial_s \beta_2(0) = 0 \,. \end{cases} \tag{43}$$

By Theorem 4.6, problem (41) has a unique strong solution in $\mathcal{H}$: call it β_0.

Furthermore, keeping in mind that $\|\alpha\|$ is bounded on the support of η, we have

$$\begin{aligned} \|h_1(\alpha)\|^2_{L^2(I,L^2(\mathbb{G},E_0^1))} &= \int_I \|h_1(\alpha(s))\|^2_{L^2(\mathbb{G},E_0^1)} ds \le \\ &\le C \int_I \|\alpha(s)\|^2_{L^2(\mathbb{G},E_0^1)} ds \le C\, T\, \|\alpha\|^2_{C(I,L^2(\mathbb{G},E_0^1))} \,. \end{aligned} \tag{44}$$

Thus, problem (42) has a unique strong solution in $\mathcal{H}$: call it $\beta_1(\alpha)$.

To prove the existence of a strong solution of (43), we must distinguish the case $\mathbb{G} = \mathbb{H}^1$ from cases ii) and iii). Indeed, suppose first $\mathbb{G} = \mathbb{H}^1$. Then, keeping in mind that $Q = 4$, by Theorem 2.5, if $s \in I$ we have

$$\|\alpha(s)\|_{L^{2p}(\mathbb{G},E_0^1)} \le C \|\alpha(s)\|_{W^{2(1-1/p),2}_{\mathbb{G}}(\mathbb{G},E_0^1)} \le C \|\alpha(s)\|_{W^{2,2}_{\mathbb{G}}(\mathbb{G},E_0^1)}.$$

Thus, since $r = \kappa = 2$, we obtain

$$\begin{aligned} \|h_2(\alpha)\|^2_{L^2(I,L^2(\mathbb{G},E_0^1))} &= \int_I \|h_2\left(\alpha(s)\right)\|^2_{L^2(\mathbb{G},E_0^1)} ds \le \\ &\le C \int_I \left(\int_{\mathbb{G}} \left|1 - \eta(\|\alpha(s,x)\|)\right|^2 \|\alpha(s,x)\|^{2p} dx \right) ds \le \\ &\le C \int_I \left(\int_{\mathbb{G}} \|\alpha(s,x)\|^{2p} dx \right) ds \le C \int_I \|\alpha(s)\|^{2p}_{W^{2,2}_{\mathbb{G}}(\mathbb{G},E_0^1)} ds \le \\ &\le CT\, \|\alpha\|^{2p}_{C^0(I,W^{2,2}_{\mathbb{G}}(\mathbb{G},E_0^1))} \,. \end{aligned} \tag{45}$$

On the other hand, in case ii), by definition $r = 1$, and hence $Q = 2n+1 > 2r$, whereas, in case iii), then $r = \kappa$, so that again $Q > 2r$, by Lemma 5.2. Thus, we can apply Theorem 2.5 and we obtain

$$W^{r,2}_{\mathbb{G}}(\mathbb{G}) \text{ is continuously embedded in } L^{2Q/(Q-2r)}_{\mathbb{G}}(\mathbb{G}).$$

Therefore, keeping in mind that $\|\alpha\| > 1$ on the support of $1-\eta$, we get

$$\begin{aligned}
\|h_2(\alpha)\|^2_{L^2(I,L^2(\mathbb{G},E_0^1))} &= \int_I \|h_2(\alpha(s))\|^2_{L^2(\mathbb{G},E_0^1)} ds \le \\
&\le C\int_I \left(\int_{\mathbb{G}} \left|1-\eta(\|\alpha(s,x)\|)\right|^2 \|\alpha(s,x)\|^{2p} dx\right) ds \le \\
&\le C\int_I \left(\int_{\mathbb{G}} \|\alpha(s,x)\|^{2p} dx\right) ds \le \\
&\le C\int_I \left(\int_{\mathbb{G}} \|\alpha(s,x)\|^{2Q/(Q-2r)} dx\right) ds \le \\
&\le C\int_I \|\alpha(s)\|^{2Q/(Q-2r)}_{W^{r,2}_{\mathbb{G}}(\mathbb{G},E_0^1)} ds \le C_2\, T\, \|\alpha\|^{2Q/(Q-2r)}_{C^0(I,W^{r,2}_{\mathbb{G}}(\mathbb{G},E_0^1))} .
\end{aligned} \tag{46}$$

Thus, in any case, we obtain that, if $\alpha \in \mathcal{H}$, then problem (43) has a unique strong solution in $\mathcal{H}$: call it $\beta_2(\alpha)$.

Set

$$\chi : \mathcal{H} \to \mathcal{H}, \qquad \chi(\alpha) = \beta_0 + \beta_1(\alpha) + \beta_2(\alpha).$$

We want to prove that χ maps a closed ball in $\mathcal{H}$ into itself and that it is a contraction; so, by Banach fixed point theorem, the required result will follow.

In case ii) and iii), combining (44) and (46), we get the following estimate

$$\begin{aligned}
\|h(\alpha)\|_{L^2(I,L^2(\mathbb{G},E_0^1))} &\le \\
&\le T^{1/2}\left(C_1\|\alpha\|_{C^1(I,L^2(\mathbb{G},E_0^1))} + C_2\|\alpha\|^{Q/(Q-2r)}_{C^0(I,W^{r,2}_{\mathbb{G}}(\mathbb{G},E_0^1))}\right).
\end{aligned} \tag{47}$$

Now choose $M \ge 2K\sqrt{E_0(\alpha_0,\alpha_1)}$, where K is the constant appearing in Lemma 4.7. Then, if $\|\alpha\|_{\mathcal{H}} \le M$, we have

$$\|\chi(\alpha)\|_{\mathcal{H}} \le K\left(\sqrt{E_0(\alpha_0,\alpha_1)} + (C_1+C_2)T^{1/2}M^{Q/(Q-2r)}\right). \tag{48}$$

If, for example, we choose T sufficiently small such that

$$(C_1+C_2)T^{1/2}M^{Q/(Q-2r)} < \sqrt{E_0(\alpha_0,\alpha_1)},$$

we get

$$\|\chi(\alpha)\|_{\mathcal{H}} \le 2K\sqrt{E_0(\alpha_0,\alpha_1)} \le M.$$

On the other hand, in case i), equation (48) becomes

$$\|\chi(\alpha)\|_{\mathcal{H}} \le K\left(\sqrt{E_0(\alpha_0,\alpha_1)} + (C_1+C_2)T^{1/2}M^{p}\right). \tag{49}$$

Choosing T sufficiently small, we get again

$$\|\chi(\alpha)\|_{\mathcal{H}} \le M.$$

Thus, it remains only to prove that there exists $\delta \in (0,T]$, such that χ is a contraction in the space $\mathcal{H}$ defined on $[0,\delta]$. Let $\alpha,\beta \in C^0(J,W^{r,2}_{\mathbb{G}}(\mathbb{G},E_0^1)) \cap C^1(J,L^2(\mathbb{G},E_0^1))$, where $J=[0,\delta]$; then,

$$\begin{aligned}
\|h_1(\alpha) - h_1(\beta)\|^2_{L^2(J,L^2(\mathbb{G},E_0^1))} &\le \\
&\le C\|\alpha-\beta\|^2_{L^2(J,L^2(\mathbb{G},E_0^1))} \le C\,\delta\,\|\alpha-\beta\|^2_{\mathcal{H}},
\end{aligned} \tag{50}$$

and

$$\|h_2(\alpha)-h_2(\beta)\|^2_{L^2(J,L^2(\mathbb{G},E_0^1))}\leq$$

(51)
$$\leq C\int_J\left(\int_{\mathbb{G}}\|\alpha(s,x)-\beta)s,x)\|^2\big(1+\|\alpha\|^{p-1}+\|\beta\|^{p-1}\big)^2dx\right)ds\leq$$
$$\leq C_{M,p}\,\delta\,\|\alpha-\beta\|^2_{\mathcal{H}}\,.$$

Hence, remembering the definition of χ, we get

$$\|\chi(\alpha)-\chi(\beta)\|_{\mathcal{H}}\leq\|h_1(\alpha)-h_1(\beta)\|_{L^2(J,L^2(\mathbb{G},E_0^1))}$$
$$+\|h_2(\alpha)-h_2(\beta)\|_{L^2(J,L^2(\mathbb{G},E_0^1))}\leq C_{M,p}\,\delta^{1/2}\,\|\alpha-\beta\|_{\mathcal{H}}\,.$$

Now, if we choose δ sufficiently small, it is proved that χ is a contraction from $\overline{B_M(0)}\subseteq\mathcal{H}$ into itself and the assertion follows. □

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