INTRINSIC LIPSCHITZ GRAPHS WITHIN CARNOT GROUPS

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Abstract. A Carnot group is a connected, simply connected, nilpotent Lie group with stratified Lie algebra. We study the notions of intrinsic graphs and of intrinsic Lipschitz graphs within Carnot groups. Intrinsic Lipschitz graphs are the natural local analogue inside Carnot groups of Lipschitz submanifolds in Euclidean spaces, where 'natural'is meant to stress the fact that these notions depend only on the structure of the algebra. This notion provides a general view on the problem unifying different alternative approaches through Lipschitz parametrizations or level sets. We provide both geometric and analytic characterizations and a clarifying relation between these graphs and Rumin's complex of differential forms. Finally a Rademacher type theorem for one codimensional graphs is proved in a general class of groups.

1. Introduction

A Carnot group $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group with stratified Lie algebra $\mathfrak{g}$. More precisely, this means that the Lie algebra $\mathfrak{g}$ of the left-invariant vector fields on $\mathbb{G}$ has finite dimension n, and admits a *step κ stratification*, i.e. there exist linear subspaces (so-called layers) $\mathfrak{g}_1, \dots, \mathfrak{g}_\kappa$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_\kappa \neq \{0\}, \tag{1}$$

where $\mathfrak{g}_i = \{0\}$ if $i > \kappa$, and $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product that makes the decomposition (1) orthogonal. We refer to the first layer $\mathfrak{g}_1$ as to the *horizontal layer*. It plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutations.

Through exponential coordinates, the group $\mathbb{G}$ can be identified with $(\mathbb{R}^n, \cdot)$, the Euclidean space $\mathbb{R}^n$ endowed with a (generally non-commutative) group law.

Carnot groups are endowed with two families of transformations: the *(left) translation* $\tau_p : \mathbb{G} \to \mathbb{G}$ defined as $q \mapsto \tau_p q := p \cdot q$, and the non-isotropic *group dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$.

The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, called the *horizontal bundle*. A section of $H\mathbb{G}$ is called a horizontal vector field and curves tangent to $H\mathbb{G}$ are called horizontal curves.

Euclidean spaces are commutative Carnot groups, and, more precisely, the only commutative Carnot groups. The simplest but, at the same time, non-trivial instance of non-Abelian Carnot groups is provided by Heisenberg groups $\mathbb{H}^N$, and in particular by the first Heisenberg group $\mathbb{H}^1$. Precise definitions will be given in a moment; let us remind that $\mathbb{H}^1$ is a free group of step 2 with 2 generators, and that it is in some sense the "model" of all topologically 3-dimensional contact structures. In exponential coordinates, the Heisenberg group $\mathbb{H}^N$ can be identified with $\mathbb{R}^{2N+1}$, with variables (x, y, t), $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Set $X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t$, $Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t$, $T := \partial_t$. The stratification of its algebra $\mathfrak{h}$ is given by $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 = \text{span}\,\{X_1, \dots, X_N, Y_1, \dots, Y_N\}$ and $\mathfrak{h}_2 = \text{span}\,\{T\}$.

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Carnot groups are endowed with an intrinsic geometry, the so-called Carnot-Carathéodory geometry (see for instance, choosing in a wide literature, [13], [34] [28]). From now on, the adjective "intrinsic" is meant to emphasize a privileged role played by the horizontal layer and by group translations and dilations. In other words, "intrinsic" notions or properties in the group $\mathbb{G}$ are those depending only on the structure of its Lie algebra $\mathfrak{g}$.

Non commutative Carnot groups, endowed with their Carnot-Carathéodory distance (briefly, cc-distance), are not Riemannian manifolds. In fact, when $\mathbb{G}$ is non commutative, the Carnot–Carathédory distance makes it a metric space that is not Riemannian at any scale ([49]), though furnished, as we already pointed out, with a rich structure of translations and (non-isotropic) dilations. Because of that, Carnot groups are, at the same time, more general structures when compared with Riemannian manifolds, and "rigid" structures akin to Euclidean spaces. This combination of different properties makes Carnot group a natural setting of study when we aim to attach geometric analysis in metric spaces.

Indeed, recently there has been a large amount of work dedicated to the study of Geometric Measure Theory on metric spaces and in particular on Carnot groups. For example rectifiable sets, finite perimeter sets, minimal surfaces and various notions of convex surfaces have been studied.

The notion of rectifiable set, in particular, is a keystone both in the calculus of variations and in geometric measure theory. In the last few years, there has been a general attempt aimed to develop a notion of rectifiable set in metric spaces, and, in particular, in sub-Riemannian metric structures and in Carnot groups (for different notions of rectifiability we refer the reader to [3], [4], [26], [30], [46], [38] and to the references therin). It is worth noticing that, besides their own geometric interest, rectifiable sets in Lie groups appear in several applications, as theoretical computer science, geometry of Banach spaces, mathematical models in neurosciences (see e.g. [17], [16]).

In Euclidean spaces, rectifiable sets are obtained, up to a negligible subset, by "gluing up" countable families of C^1 or of Lipschitz submanifolds. Hence, understanding the objects that, within Carnot groups, naturally take the role of C^1 or of Lipschitz submanifolds is preliminary in order to develop a satisfactory theory of *intrinsically* rectifiable sets.

In the Euclidean setting, implicit function theorem yields that, locally, C^1 submanifolds can be viewed equivalently as (i) *C^1 (or bi-Lipschitz) images of a fixed "parameter space" (usually an open subset of a flat space or of a linear space);* (ii) *non-critical level sets of C^1-functions;* (iii) *graphs of C^1 (or Lipschitz) maps between complementary linear subspaces.*

When looking at regular submanifolds in groups it appears clearly that considering Euclidean regular submanifolds is at the same time too general and too restrictive. More intrinsic definitions and approaches are necessary.

Notion (i) has been the first one to be extended to the setting of general metric spaces (see e.g. [24], [3] where the parameter spaces are taken to be open subsets of Euclidean spaces). When working with a Carnot group $\mathbb{G}$, it is natural to think of using more general spaces of parameters, i.e. open subsets of homogeneous subgroups of $\mathbb{G}$ (see [46] and [38]). A very special instance are horizontal curves, usually defined as images of Lipschitz maps $\mathbb{R} \to \mathbb{G}$.

Also notion (ii) has been largely studied in the recent literature, starting from the implicit function theorem in Carnot groups proved in [26] and [27].

Nevertheless, both approach (i) and (ii) have intrinsic limitations that appeared already inside the Heisenberg groups $\mathbb{H}^n$. Indeed, differently from $\mathbb{R}^n$ - where embedded submanifolds are *equivalently* defined as non-critical level sets or as images of injective differentiable maps - in $\mathbb{H}^n$, low dimensional regular surfaces cannot be seen as non critical level sets and low codimensional ones cannot be seen as (bilipschitz) images of open sets of $\mathbb{R}^n$. The reasons are rooted in the algebraic structure of $\mathbb{H}^n$; indeed, low dimensional horizontal subgroups of $\mathbb{H}^n$ are not normal subgroups, hence they cannot appear as kernels of homogeneous homomorphisms $\mathbb{H}^n \to \mathbb{R}^{n-d}$; on the other side, injective homogeneous homomorphism $\mathbb{R}^d \to \mathbb{H}^n$ do not exist, if $d \geq n+1$ (see [3] and [37]). One could object that this difficulty might be overcome using subgroups of $\mathbb{G}$ instead of $\mathbb{R}^d$ as a parameter space. Indeed this is not the case as examples in [10] and in [7] show.

On the other hand, the notion of graphs *within* Carnot groups is definitely more delicate, since Carnot groups in general cannot be viewed as cartesian products of subgroups (unlike Euclidean spaces). Therefore we need a notion of *intrinsic graph* fitting the structure of the group $\mathbb{G}$. Such a notion appears for the first time in [29], [8], [31], and is associated with a decomposition of $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ as product of two homogeneous *complementary subgroups* $\mathbb{M}$, $\mathbb{H}$ (a Lie subgroup of $\mathbb{G}$ is said to be an homogeneous subgroup if it is invariant under group dilations). We refer to Definition 2.2.1 below for details.

Now, describing regular submanifolds as (intrinsic differentiable) graphs is more general and flexible than using parametrizations or level sets. For instance, already in Heisenberg groups, both non critical level sets and images of regular maps are locally intrinsic differentiable graphs. The same happens for one dimensional and for one codimensional submanifolds in general Carnot groups.

The aim of the first part of the present paper is to provide an introduction to the theory of intrinsic graphs, and, in particular, to that of intrinsic Lipschitz graphs.

The simple idea of intrinsic graph is the following one: let $\mathbb{M}$, $\mathbb{H}$ be complementary homogeneous subgroups of a group $\mathbb{G}$, then the intrinsic (left) graph of $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ is the set

$$\mathrm{graph}\,(f) = \{g \cdot f(g) : g \in \mathcal{A}\}.$$

This notion deserves the adjective "intrinsic" since it is invariant under left translations or homogeneous automorphisms of the group (dilations in particular): see Proposition 2.2.21 below. We stress that neither Euclidean graphs are necessarily intrinsic graphs nor the opposite.

Intrinsic graphs appeared naturally while studying non critical level sets of differentiable functions from $\mathbb{G}$ to $\mathbb{R}^k$. Indeed, implicit function theorems for groups ([26], [30], [27], [19], [20]) can be rephrased stating precisely that these level sets are always, locally, intrinsic graphs (see Theorem 4.5.2 below).

More generally, we say that a subset S of a Carnot group $\mathbb{G}$, is a *(left) intrinsic graph*, in direction of a homogeneous subgroup $\mathbb{H}$, if S intersects each left coset of $\mathbb{H}$ in at most a single point. This weaker notion does not rely on a decomposition of the group as a product of homogeneous complementary subgroups and Riemannian graphs are their possible counterparts in Euclidean setting. As we can expect, an intrinsic graph in direction of a homogeneous subgroup $\mathbb{H}$ that is not complemented lacks several properties of the intrinsic graphs associated with a group decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. Nevertheless, for instance in the Heisenberg group $\mathbb{H}^1$, the so called $\mathbb{T}$-graphs, i.e, graphs in direction of the center of the group $\mathbb{T}$, that is not complemented, have been studied in the last few years.

There is also a remarkably deep relationship between intrinsic graphs associated with a group decomposition and the so-called Rumin's complex (E_0^*, d_c) of differential forms in a Carnot group $\mathbb{G}$. Rumin's theory would need a quite long technical introduction, and hence we refer to the original Rumin's papers [47] and [48], as well as to [14] and [32] for an exhaustive presentation. Here we restrict ourselves to stress that there is a canonical explicit biunivocal correspondence between simple h-covectors in E_0^h and group decompositions, akin to the correspondence in Euclidean spaces between linear manifolds and covectors (see Theorem 2.2.10 below).

If the group $\mathbb{G}$ admits a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, then two canonical projections $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ are naturally defined by the identity $\mathbf{P}_{\mathbb{M}}g \cdot \mathbf{P}_{\mathbb{H}}g \equiv g$ for $g \in \mathbb{G}$. Unlike the Euclidean case, in general $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ are not Lipschitz maps with respect to the cc-distance in $\mathbb{G}$. Nevertheless, the canonical projections yield a notion of *intrinsic cone*: if $g \in \mathbb{G}$ and $\beta \geq 0$ the cones $C_{\mathbb{M},\mathbb{H}}(g, \beta)$, with basis $\mathbb{M}$, axis $\mathbb{H}$, vertex g, opening β are defined as

$$C_{\mathbb{M},\mathbb{H}}(e, \beta) = \{p : \|\mathbf{P}_{\mathbb{M}}p\| \leq \beta \,\|\mathbf{P}_{\mathbb{H}}p\|\}\,, \quad C_{\mathbb{M},\mathbb{H}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{H}}(e, \beta).$$

Thus we can say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic L-Lipschitz* in $\mathcal{E}$ if there is $L > 0$ such that

$$C_{\mathbb{M},\mathbb{H}}(p, 1/L) \cap \operatorname{graph}(f) = \{p\}, \qquad \text{for all } p \in \operatorname{graph}(f).$$

The graph of an intrinsic L-Lipschitz function will be said an intrinsic Lipschitz graph. By construction, this notion satisfies our original request of being invariant under group translations. It is a natural question to ask whether intrinsic Lipschitz functions are metric Lipschitz functions with respect to the cc-distance of $\mathbb{G}$, or, at least, if they are Lipschitz provided appropriate choices of the metrics in the domain or in the target spaces are made. The answer to this question is almost always negative. We refer to Remark 3.1.8 for further comments. In particular, all our theory of intrinsically Lipschitz maps *within* Carnot groups does not fit (with the exception of trivial cases) in Pansu's theory of Lipschitz maps *between* Carnot groups (see [45]).

Finally, we say that a function $f : \mathbb{M} \to \mathbb{H}$, acting between complementary subgroups of $\mathbb{G}$, is *intrinsically differentiable* at a point $m \in \mathbb{M}$ if the graph of f has a tangent homogeneous subgroup in $mf(m) \in \operatorname{graph}(f)$. This notion can be stated also in terms existence of an approximating intrinsic linear function. Intrinsic linear functions, acting between complementary subgroups, are functions whose graphs are homogeneous subgroups. An extensive study of the notion of intrinsic differentiability following an alternative but (likely) equivalent approach is been carried on in [6], [11], [12] [44].

At this point, it is natural to ask whether a Rademacher type theorem holds within the setting of intrinsic graphs in Carnot groups, at least for 1-codimensional graphs. In other words, given $V \in \mathfrak{g}_1$, if $\mathbb{N} = \{\exp(tV), t \in \mathbb{R}\}$ and $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$, then an intrinsic Lipschitz map $f : \mathcal{E}(\subset \mathbb{M}) \to \mathbb{N}$ is intrinsically differentiable almost everywhere with respect to Lebesgue measure in $\mathbb{M}$ (that, in turn, coincides with its Haar measure)? Unfortunately, the answer we can give is only partial. Basically, this can be explained by the fact that Rademacher's theorem in codimesion 1 is in some sense the analytic counterpart of the rectifiability property of the boundaries of sets of finite perimeter (the so-called De Giorgi's theorem). In spite of several sophisticated attempts (see e.g. [5]), so far De Giorgi's theorem in Carnot groups has been proved for Heisenberg groups ([26]) and, more generally, for step 2 Carnot groups ([28]). Only recently, De Giorgi's theorem has been generalized to a much larger class of Carnot groups, the so called groups of type $\star$ (see [39]). The definition of these groups is given in Definition 4.4.1 and reads as follows:
We say that a Carnot group $\mathbb{G}$ is of type $\star$ if its stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$ has the following property: there exists a basis $(X_1, \dots, X_{m_1})$ of $\mathfrak{g}_1$ such that

$$[X_j, [X_j, X_i]] = 0, \qquad \text{for all } i, j = 1, \dots, m_1.$$

Notice that step 2 Carnot groups are of type $\star$, but, for instance, also Iwasawa groups enjoy the same property. Up to now Rademacher's theorem for 1-codimensional intrinsic graphs has been proved only in the setting of Heisenberg groups in [31]. In the present paper, we prove the following Rademacher type theorem for the general class of groups of $\star$ type (see Theorem 4.4.4):
Let $\mathbb{M}$ and $\mathbb{N}$ be complementary subgroups of a Carnot group $\mathbb{G}$ of type $\star$, with $\mathbb{N}$ one-dimensional and horizontal. Let $\mathcal{U} \subset \mathbb{M}$ be relatively open in $\mathbb{M}$ and $\varphi : \mathcal{U} \to \mathbb{N}$ be intrinsic Lipschitz. Then φ is intrinsic differentiable $(\mathcal{L}^{n-1} \llcorner \mathbb{M})$-almost everywhere in $\mathcal{U}$.

In particular, Theorem 4.4.4 yields that the general De Giorgi's theorem of [39] can be equivalently stated in terms of Lipschitz graphs. Roughly speaking, if $\mathbb{G}$ is a Carnot group of type $\star$, and $E \subset \mathbb{G}$ is a set of locally finite $\mathbb{G}$-perimeter, then the associated perimeter measure is concentrated on a portion of the boundary, the so-called reduced boundary $\partial^*_{\mathbb{G}} E \subset \partial E$, and $\partial^*_{\mathbb{G}} E$, up to a set of $(Q-1)$-Hausdorff measure zero, is a countable union of compact subsets of intrinsically Lipschitz graphs.

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The plan of the work is the following:

Section 2 contains the basic notions of the theory of Carnot groups, the notions of complementary subgroups and of intrinsic graphs, as well as the study of the projection operators associated with a group decomposition.

Section 3 is dedicated to a systematic study of intrinsic Lipschitz graphs and to some questions about intrinsic linear functions and intrinsic differentiability.

Section 4 is specialized to functions with values in one dimensional subgroups. It contains an extension theorem for intrinsic Lipschitz functions and a Rademacher's type theorem, as well as the rectifiability theorem for Carnot groups of type $\star$.

2. Notations and definitions

2.1. **Carnot groups.** For a general account, see e.g. [13, 25, 34]. A *graded group* of step κ is a connected, simply connected Lie group $\mathbb{G}$ whose finite dimensional Lie algebra $\mathfrak{g}$ is the direct sum of k subspaces $\mathfrak{g}_i$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \text{for } 1 \le i, j \le \kappa,$$

where $\mathfrak{g}_i = 0$ for $i > \kappa$. We denote as n the dimension of $\mathfrak{g}$ and as n_j the dimension of $\mathfrak{g}_j$, for $1 \le j \le \kappa$.
A *Carnot group* $\mathbb{G}$ of step κ is a graded group of step κ, where $\mathfrak{g}_1$ generates all of $\mathfrak{g}$. That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, \kappa$.
Let $X_1, \dots, X_n$ be a base for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a base for $\mathfrak{g}_1$ and, for $1 < j \le \kappa$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for $\mathfrak{g}_j$. Here we have $m_0 = 0$ and $m_j - m_{j-1} = n_j$, for $1 \le j \le \kappa$.
Because the exponential map is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1X_1 + \cdots + p_nX_n)$ and we identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, i.e. $\mathbb{R}^n$ endowed with the product $\cdot$. The identity of $\mathbb{G}$ is denoted as $e = (0, \dots, 0)$.
If $\mathbb{G}$ is a graded group, for all $\lambda > 0$, the *(non isotropic) dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorphisms of $\mathbb{G}$ defined as

$$\delta_\lambda(p_1, ..., p_n) = (\lambda^{\alpha_1} p_1, \lambda^{\alpha_2} p_2, ..., \lambda^{\alpha_n} p_n),$$

where $\alpha_i = j$, if $m_{j-1} < i \le m_j$. Dilations are defined also for $\lambda \in \mathbb{R}$ setting

$$\delta_\lambda p := \delta_{|\lambda|} p^{-1} = \left(\delta_{|\lambda|} p\right)^{-1}, \text{ when } \lambda < 0.$$

We denote the product of p and $q \in \mathbb{G}$ as $p \cdot q$ or more frequently as pq. The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. It has the form

$$p \cdot q = p + q + \mathcal{Q}(p, q), \qquad \text{for all } p, q \in \mathbb{R}^n, \tag{2}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$. That is

$$\mathcal{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^{\alpha_i} \mathcal{Q}_i(p, q), \qquad \text{for all } p, q \in \mathbb{G} \text{ and } \lambda > 0.$$

We collect now further properties of $\mathcal{Q}$ following from Campbell-Hausdorff formula. First of all $\mathcal{Q}$ is antisimmetric, that is

$$\mathcal{Q}_i(p, q) = -\mathcal{Q}_i(-q, -p), \qquad \text{for all } p, q \in \mathbb{G}.$$

Each $\mathcal{Q}_i(p, q)$ depends indeed only on a section of the components of p and q. Precisely

$$\begin{aligned} &\mathcal{Q}_1(p, q) = ... = \mathcal{Q}_{m_1}(p, q) = 0 \\ &\mathcal{Q}_j(p, q) = \mathcal{Q}_j(p_1, \dots, p_{m_{i-1}}, q_1, \dots, q_{m_{i-1}}), \end{aligned} \tag{3}$$

if $m_{i-1} < j \le m_i$ and $2 \le i$. By Proposition 2.2.22 (4) in [13], for $m_1 < i \le n$ we can write

$$\mathcal{Q}_i(p, q) = \sum_{k,h} \mathcal{R}^i_{k,h}(p, q)(p_k q_h - p_h q_k), \tag{4}$$

where the functions $\mathcal{R}^i_{k,h}$ are polynomials, homogenous of degree $\alpha_i - \alpha_k - \alpha_h$ with respect to group dilations, and the sum is extended to all h, k such that $\alpha_h + \alpha_k \le \alpha_i$. From (4) it follows in particular that

$$\mathcal{Q}_i(p, 0) = \mathcal{Q}_i(0, q) = 0 \quad \text{and} \quad \mathcal{Q}_i(p, p) = \mathcal{Q}_i(p, -p) = 0. \tag{5}$$

Finally, it is useful to think $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$, where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{n_i}$ is the i^{th} layer of $\mathbb{G}$ and to write $p \in \mathbb{G}$ as $(p^1, \dots, p^\kappa)$, with $p^i \in \mathbb{G}^i$. According to this

$$p \cdot q = \big(p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p, q), \dots, p^\kappa + q^\kappa + \mathcal{Q}^\kappa(p, q)\big), \qquad \text{for all } p, q \in \mathbb{G}. \tag{6}$$

Definition 2.1.1. An absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is a sub-unit curve with respect to $X_1, \dots, X_{m_1}$ if there exist measurable real functions $c_1(s), \dots, c_{m_1}(s)$, $s \in [0, T]$ such that $\sum_j c_j^2 \le 1$ and

$$\dot\gamma(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \quad \text{for a.e. } s \in [0, T].$$

Definition 2.1.2. If $p, q \in \mathbb{G}$, we define their Carnot-Carathéodory distance as

$$d_c(p, q) := \inf \{T > 0 : \text{ there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = p,\ \gamma(T) = q\}.$$

By Chow's Theorem, the set of sub-unit curves joining p and q is not empty, furthermore d_c is a distance on $\mathbb{G}$ that induces the Euclidean topology (see chapter 19 in [13] or Theorem 1.6.2 in [43]).

More generally, given any homogeneous norm $\|\cdot\|$, it is possible to define a distance in $\mathbb{G}$ as

$$d(p, q) = d(q^{-1} \cdot p, 0) = \left\|q^{-1} \cdot p\right\|, \qquad \text{for all } p, q \in \mathbb{G}. \tag{7}$$

The distance d in (7) is comparable with the Carnot-Carathéodory distance of $\mathbb{G}$ and

$$d(g \cdot p, g \cdot q) = d(p, q) \quad , \quad d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q) \tag{8}$$

for all $p, q, g \in \mathbb{G}$ and all $\lambda > 0$.
A convenient homogeneous norm, and the one used here, is described in [28, Theorem 5.1]. It is defined as

$$d_\infty(p, 0) := \|p\| := \max_{j=1,\dots,\kappa} \{\varepsilon_j \left\|p^j\right\|^{1/j}_{\mathbb{R}^{n_j}}\}, \qquad \text{for all } p = (p^1, \dots, p^\kappa) \in \mathbb{G}, \tag{9}$$

where $\varepsilon_1 = 1$, and $\varepsilon_2, \dots \varepsilon_\kappa \in (0, 1]$ are suitable positive constants depending on $\mathbb{G}$.
For $r > 0$ and $p \in \mathbb{G}$, we denote by $U_c(p, r)$ and $B_c(p, r)$, respectively, the open and closed balls associated with the Carnot-Carathéodory distance d_c, and with $U(p, r)$, $B(p, r)$ the ones associated with d as in (7).

Definition 2.1.3. The integer $Q = \sum_{j=1}^n \alpha_j = \sum_{i=1}^\kappa i \dim V_i$ is the *homogeneous dimension* of $\mathbb{G}$. We stress that Q is also the Hausdorff dimension of $\mathbb{R}^n$ with respect to d_c (see [42]).

Proposition 2.1.4. *The n-dimensional Lebesgue measure $\mathcal{L}^n$ is the Haar measure of the group $\mathbb{G}$ (see* [51]*). Therefore if $E \subset \mathbb{R}^n$ is measurable, then $\mathcal{L}^n(g \cdot E) = \mathcal{L}^n(E)$ for every $g \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$. We note that*

$$\mathcal{L}^n(U_c(p, r)) = r^Q \mathcal{L}^n(U_c(p, 1)) = r^Q \mathcal{L}^n(U_c(0, 1)). \tag{10}$$

Using the distances d_c or d the family of Hausdorff measures or spherical Hausdorff measures are obtained following Carathéodory's construction (see [24, Section 2.10.2.]). Precisely, if $m \ge 0$ and $\mathcal{A} \subset \mathbb{G}$, the *(intrinsic spherical) Hausdorff measure* $\mathcal{S}^m_d$ is,

$$\mathcal{S}^m_d(\mathcal{A}) := \lim_{\delta \to 0} \mathcal{S}^m_{d,\delta}(\mathcal{A}),$$

where $\mathcal{S}^m_{d,\delta}(\mathcal{A}) = \inf\left\{\sum_i r_i^m : \mathcal{A} \subset \bigcup_i B(p_i, r_i),\ r_i \le \delta\right\}$. The Hausdorff measures $\mathcal{H}^m_d$ are defined analogously.
Translation invariance and dilation homogeneity of Hausdorff measures follow from (8) and, for $\mathcal{A} \subseteq \mathbb{G}$, $p \in \mathbb{G}$ and $r \in [0, \infty)$,

$$\mathcal{S}^m_d(p \cdot \mathcal{A}) = \mathcal{S}^m_d(\mathcal{A}) \quad \text{and} \quad \mathcal{S}^m_d(\delta_r \mathcal{A}) = r^m \mathcal{S}^m_d(\mathcal{A}).$$

Remark 2.1.5. An elementary covering argument yields that

$$\mathcal{S}^{Q-1}_d(\partial B_c(p,r)) = r^{Q-1}\mathcal{S}^{Q-1}_d(\partial B_c(p,1)) < \infty, \quad \text{for all } p \in \mathbb{G}.$$

A *homogeneous subgroup* of a Carnot group $\mathbb{G}$ (see [50, 5.2.4]) is a Lie subgroup $\mathbb{H}$ such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda > 0$. Homogeneous subgroups are linear subspaces of $\mathbb{G}$, when $\mathbb{G}$ is identified with $\mathbb{R}^n$ with exponential coordinates.

Remark 2.1.6. An homogeneous subgroup $\mathbb{H}$ is stratified, that is $\mathbb{H} = \mathbb{H}^1 \oplus \cdots \oplus \mathbb{H}^\kappa$, where $\mathbb{H}^i \subset \mathbb{G}^i$ and $\mathbb{H}^i$ is a linear subspace of $\mathbb{G}^i$. If we denote by $\mathfrak{h}$ the Lie algebra of $\mathbb{H}$, this follows once we prove that

$$\mathfrak{h} = \oplus_{p=1}^\kappa \mathfrak{h}_p, \tag{11}$$

where $\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{g}_p$. Indeed, if $v \in \mathfrak{h}$, we can write $v = \sum_p v_p$, with $v_p \in \mathfrak{g}_p$, $p = 1, \dots, \kappa$. Thus (11) follows if we show that

$$v_p \in \mathfrak{h} \quad \text{for all } p = 1, \dots, \kappa. \tag{12}$$

To this end, we remind that $\mathfrak{h}$ is a vector space and, in addition, it is homogeneous with respect to group dilations. Hence, for $\lambda > 0$,

$$\frac{1}{\lambda}\delta_\lambda v := \frac{1}{\lambda}\sum_p \lambda^p v_p = v_1 + \sum_{p \ge 2} \lambda^{p-1} v_p \in \mathfrak{h}.$$

But $\frac{1}{\lambda}\delta_\lambda v$ is bounded, and hence, if we choose $\lambda = \lambda_n$, with $\lambda_n \to 0$ as $n \to \infty$, we can assume $\left(\frac{1}{\lambda_n}\delta_{\lambda_n} v\right)_n$ has a limit in $\mathfrak{h}$. Thus, we can conclude that $v_1 \in \mathfrak{h}$. We can repeat now the argument replacing v by $v - v_1 \in \mathfrak{h}$, and we write

$$\frac{1}{\lambda^2}\delta_\lambda(v - v_1) = v_2 + \sum_{p \ge 3} \lambda^{p-2} v_p \in \mathfrak{h},$$

obtaining eventually that $v_2 \in \mathfrak{h}$. Iterating this argument, we get (12) and therefore (11).

The *topological dimension* of a (sub)group is the dimension of its Lie algebra. The *metric dimension* of a subset is its Hausdorff dimension, with respect to the Hausdorff measures $\mathcal{S}^k_d$. The metric dimension of a homogeneous subgroup is an integer usually larger than its topological dimension (see [42]).

Definition 2.1.7. $\mathbb{M}$ is a (d_t, d_m)*-subgroup of* $\mathbb{G}$ if $\mathbb{M}$ is a homogeneous subgroup of $\mathbb{G}$ with linear or topological dimension d_t and metric dimension $d_m \ge d_t$.

The notion of *P-differentiability* for functions acting between graded groups was introduced by Pansu in [45].

Definition 2.1.8. Let $\mathbb{G}_1, \mathbb{G}_2$ be graded groups, with homogeneous norms $\|\cdot\|_1$, $\|\cdot\|_2$ and dilations δ^1_λ, δ^2_λ. We say that $L : \mathbb{G}_1 \to \mathbb{G}_2$ is *H-linear*, or is a *homogeneous homomorphism*, if L is a group homomorphism such that

$$L(\delta^1_\lambda g) = \delta^2_\lambda L(g), \quad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0.$$

We say that $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is *P-differentiable* in $g_0 \in \mathcal{A}$ if there is a H-linear function $df_{g_0} : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\left\| \left(df_{g_0}(g_0^{-1} \cdot g)\right)^{-1} \cdot f(g_0)^{-1} \cdot f(g) \right\|_2 = o\big(\|g_0^{-1} \cdot g\|_1 \big), \quad \text{as } \|g_0^{-1} \cdot g\|_1 \to 0,$$

where $o(t)/t \to 0$ as $t \to 0^+$. The H-linear function df_{g_0} is called the *P-differential* of f in g_0.

We denote as $\mathcal{L}_{\mathbb{G}}$ the set of H-linear functions $L : \mathbb{G} \to \mathbb{G}$ endowed with the natural topology. If Ω is an open set in $\mathbb{G}$, we denote as $C^1_{\mathbb{G}}(\Omega)$ the set of continuous real valued functions in Ω such that $d_{\mathbb{G}} f : \Omega \to \mathcal{L}_{\mathbb{G}}$ is continuous. If $X_1, \dots, X_{m_1}$ is a generating family of vector fields, we denote as $C^1_{\mathbb{G}}(\Omega, H\mathbb{G})$ the set of all sections $\phi := \sum_{i=1}^{m_1} \phi_i X_i$ of $H\mathbb{G}$ whose canonical coordinates $\phi_j \in C^1_{\mathbb{G}}(\Omega)$, for $j = 1, \dots, m_1$.

Definition 2.1.9. If $X_1, \dots, X_{m_1}$ is a generating family of vector fields, we define the *horizontal gradient* of a regular function $f : \mathbb{G} \to \mathbb{R}$, as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m_1} (X_i f) X_i.$$

whose canonical coordinates are $(X_1 f, ..., X_{m_1} f)$.
If $\phi = \sum_{i=1}^{m_1} \phi_i X_i$ is a regular horizontal section we define the *horizontal divergence* of ϕ as the real valued function

$$\mathrm{div}_{\mathbb{G}}(\phi) := \sum_{j=1}^{m_1} X_j \phi_j.$$

2.2. **Complementary subgroups and graphs.**

2.2.1. *Complementary subgroups.* From now on $\mathbb{G}$ will always be a homogeneous stratified group, identified with $\mathbb{R}^n$ with exponential coordinates.

Definition 2.2.1. Let $\mathbb{M}, \mathbb{H}$ be homogeneous subgroups of $\mathbb{G}$. We say that $\mathbb{M}, \mathbb{H}$ are *complementary subgroups* in $\mathbb{G}$, if $\mathbb{M} \cap \mathbb{H} = \{e\}$ and if

$$\mathbb{G} = \mathbb{M} \cdot \mathbb{H},$$

that is for each $g \in \mathbb{G}$, there are $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = m \cdot h$.

If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G}$ and one of them is a normal subgroup then $\mathbb{G}$ is said to be the *semi-direct product* of $\mathbb{M}$ and $\mathbb{H}$. If both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $\mathbb{G}$ is said to be the *direct product* of $\mathbb{M}$ and $\mathbb{H}$.

By elementary facts in group theory (see e.g. [36, Lemma 2.8]) if $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, so that $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, then it is also true that

$$\mathbb{G} = \mathbb{H} \cdot \mathbb{M},$$

that is, each $g \in \mathbb{G}$ can be written – in a unique way – as $g = \bar{h}\bar{m}$, with $\bar{m} \in \mathbb{M}$, $\bar{h} \in \mathbb{H}$. Rephrased differently, if $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, also $\mathbb{H}, \mathbb{M}$ are complementary subgroups in $\mathbb{G}$.

Remark 2.2.2. If $\mathbb{G}$ is stratified and $\mathbb{M}$, $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ then they are stratified and also $\mathbb{G}^i = \mathbb{M}^i \oplus \mathbb{H}^i$, for $i = 1, \dots, \kappa$.

Example 2.2.3. Let $\mathbb{G}$ be the Heisenberg group $\mathbb{H}^n$. Then all the possible couples of complementary subgroups of $\mathbb{H}^n$ contain a horizontal subgroup $\mathbb{V}$ of dimension $k \leq n$, isomorphic and isometric to $\mathbb{R}^k$ and a normal subgroup $\mathbb{W}$ of dimension $2n+1-k$, containing the centre $\mathbb{T}$. Moreover $\mathbb{W}^1 \oplus \mathbb{V} = \mathbb{G}^1$.
Similar splittings exist in a general Carnot group $\mathbb{G}$. Indeed, choose any horizontal homogeneous subgroup $\mathbb{H} = \mathbb{H}^1 \subset \mathbb{G}^1$ and a subgroup $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^\kappa$ such that: $\mathbb{H} \oplus \mathbb{M}^1 = \mathbb{G}^1$, and $\mathbb{G}^j = \mathbb{M}^j$ for all $2 \leq j \leq \kappa$. Then $\mathbb{M}$ and $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ and the product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ is semidirect because $\mathbb{M}$ is a normal subgroup.

Example 2.2.4. The Engels group is $\mathbb{E} = (\mathbb{R}^4, \cdot, \delta_\lambda)$, were the group law is defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + (x_1 y_2 - x_2 y_1)/2 \\ x_4 + y_4 + [(x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2)]/2 \\ +(x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1)/12 \end{pmatrix}$$

and the family of dilation is

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4).$$

A basis of left invariant vector fields is X_1, X_2, X_3, X_4 defined as

$$\begin{aligned} X_1(x) &:= \partial_{x_1} - (x_2/2)\, \partial_{x_3} + \big(-x_3/2 - (x_1 x_2 + x_2^2)/12\big)\, \partial_{x_4} \\ X_2(x) &:= \partial_{x_2} + (x_1/2)\, \partial_{x_3} + \big(-x_3/2 + (x_1^2 + x_1 x_2)/12\big)\, \partial_{x_4} \\ X_3(x) &:= \partial_{x_3} - ((x_1 + x_2)/2)\, \partial_{x_4} \\ X_4(x) &:= \partial_{x_4}. \end{aligned}$$

The commutation relations are $[X_1, X_2] = X_3$, $\quad [X_1, X_3] = [X_2, X_3] = X_4$ and all the others commutators are zero.

Inside Engels group there are two families of complementary subgroups. The first one is formed by 1-dimensional horizontal subgroups and 3-dimensional subgroups containing all the vertical directions. This family gives a semidirect splitting of $\mathbb{E}$. The second family is formed by 2-dimensional subgroups that are not normal subgroups. All the computations can be easily done directly, better using a symbolic computation program.

The homogeneous subgroups

$$\mathbb{M}_{\alpha,\beta} := \{(\alpha t, \beta t, 0, 0) : t \in \mathbb{R}\}, \qquad \mathbb{N}_{\gamma,\delta} := \{(\gamma t, \delta t, x_3, x_4) : t, x_3, x_4 \in \mathbb{R}\},$$

are complementary subgroups in $\mathbb{E}$, provided that $\alpha\delta - \beta\gamma \neq 0$. Moreover $\mathbb{N}_{\gamma,\delta}$ is a normal subgroup, hence $\mathbb{E}$ is the semidirect product of $\mathbb{M}_{\alpha,\beta}$ and $\mathbb{N}_{\gamma,\delta}$.

The second family, for $\alpha + \beta \neq 0$, is given by

$$\mathbb{K} := \{(x_1, -x_1, x_3, 0) : x_1, x_3 \in \mathbb{R}\} \qquad \mathbb{H}_{\alpha,\beta} := \{(\alpha t, \beta t, 0, x_4) : t, x_4 \in \mathbb{R}\}.$$

One can compute directly that $\mathbb{K}$ and $\mathbb{H}_{\alpha,\beta}$ are complementary subgroups in $\mathbb{E}$ and that neither $\mathbb{K}$ nor $\mathbb{H}_{\alpha,\beta}$ are normal subgroups. Hence $\mathbb{E} = \mathbb{K} \cdot \mathbb{H}_{\alpha,\beta}$, but the product is not a semidirect product.

Example 2.2.5. Let us hint here some relations between the Rumin's complex of intrinsic differential forms in a Carnot group $\mathbb{G}$ and the existence of complementary subgroups in $\mathbb{G}$. Necessarily, we will be very sketchy here. For further details we refer the reader to [47], [48], [14], [32].

Let $\mathbb{G}$ be a Carnot group, and let $\mathfrak{g}$ be its Lie algebra. The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [24] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \le h \le n$,

$$\bigwedge_h \mathfrak{g} := \mathrm{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \le i_1 < \cdots < i_h \le n\},$$
$$\bigwedge^h \mathfrak{g} := \mathrm{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \le i_1 < \cdots < i_h \le n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*, respectively. As usual $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ define a family of fiber bundle over $\mathbb{G}$ that we still denote as $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$. We denote by Ω_h and Ω^h the spaces of sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$. We refer to elements of Ω_h as fields of h-vectors and to elements of Ω^h as h-forms and to (Ω^*, d) as to the De Rham complex.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$. If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.
The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $\{X_{i_1} \wedge \cdots \wedge X_{i_h}\}$ and $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}\}$ orthonormal.

Definition 2.2.6. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* k, and we write $w(\alpha) = k$, if its dual vector $\alpha^\natural$ is in $\mathfrak{g}_k$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$, and we have (see [14], formula (16))

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g},$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of h-covectors.

We denote also by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{13}$$

The filtration (13) induces a decomposition

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

of the exterior differential $d : \Omega^h \to \Omega^{h+1}$, where d_0 does not increase the weight, and d_i increases the weight by i for $i = 1, \dots, \kappa$. In particular, d_0 is an algebraic operator.

Lemma 2.2.7. *$d_0^2 = 0$, i.e. (Ω^*, d_0) is a complex. Moreover, if $\alpha \in \Omega^h$ is left-invariant, then*

(i) $$d\alpha = d_0\alpha;$$

(ii) *$d_0\alpha$ is left-invariant;*

(iii) *if $d\alpha = d_0\alpha \neq 0$, then the weight of $d_0\alpha$ equals the weight of α.*

The following definition of intrinsic covectors and of intrinsic forms is due to M. Rumin ([47], [48]).

Definition 2.2.8. If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap (\mathrm{Im}\ d_0)^\perp = \ker d_0 \cap \ker(d_0 *),$$

where $*$ denotes the Hodge duality operator associated with the scalar product in $\mathfrak{g}$ and the volume form $dV := \theta_1 \wedge \cdots \wedge \theta_n$.

The elements of E_0^h are denoted as *intrinsic h-forms on $\mathbb{G}$*. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h (the bundle of the intrinsic covectors). In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

We denote by $N_h^{\min}$ and $N_h^{\max}$ the minimum and the maximum, respectively, of the weights of forms in E_0^h. If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} E_0^{h,p}.$$

Indeed, if $\alpha \in E_0^h$, by (13), we can write $\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} \alpha_p$, with $\alpha_p \in \Omega^{h,p}$ for all p. The assertion follows by proving that $\alpha_p \in E_0^h$. Indeed, by definition, $0 = d_0\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} d_0\alpha_p$. But the weight of $d_0\alpha_p$ is different from that of $d_0\alpha_q$ for $p \neq q$, and hence the $d_0\alpha_p$'s are linear independent and therefore they are all 0. The same argument can be repeated for $*\alpha$, and the assertion follows.

The following result shows that a pair of non-parallel intrinsic simple covectors $\xi \in E_0^h$ and $\omega \in E_0^{n-h}$ naturally define a couple of complementary subgroups as in Definition 2.2.1. Following the notations of [35], p.90, if X is a vector field, we denote by $i(X)$ the exterior product.

Proposition 2.2.9. *Let $\mathbb{G}$ be a Lie group of dimension n, and denote by $\mathfrak{g}$ the Lie algebra of the left invariant vector fields on $\mathbb{G}$. Without loss of generality, we may assume that a scalar product is fixed in $\mathfrak{g}$.*

Let now $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ of dimension h, and let $Z_1, \dots, Z_{n-h}$ be a basis of $\mathfrak{h}^\perp$. If we set $\omega_i := Z_i^\natural$ for $i = 1, \dots, n-h$ and $\omega := \omega_1 \wedge \cdots \wedge \omega_{n-h}$, then

i) $\mathfrak{h} = \{X \in \mathfrak{g} : i(X)\omega = 0\} = \{X \in \mathfrak{g} : *\omega \wedge X^\natural = 0\}$.

ii) *there exists $\beta \in \bigwedge^1 \mathfrak{g}$ such that $d\omega = \beta \wedge \omega$.*

Reciprocally, if $\omega := \omega_1 \wedge \cdots \wedge \omega_{n-h} \in \bigwedge^{n-h} \mathfrak{g}$ is a simple left invariant form such that $d\omega = \beta \wedge \omega$ for some $\beta \in \bigwedge^1 \mathfrak{g}$, then

iii) $\mathfrak{h} := \{X \in \mathfrak{g} : i(X)\omega = 0\}$ *is a Lie subalgebra of $\mathfrak{h}$.*

Proof. Assertion i) is well known (see e.g. [24], Section 1.6, [22], Section 2.3). On the other hand, assertions ii) and iii) follow by Frobenius theorem (see, e.g., [1], Theorem 7.4.24). □

Theorem 2.2.10. *If $1 \le h < n$, $\xi \in E_0^h$ and $\omega \in E_0^{n-h}$ are simple covectors such that*

$$\xi \wedge \omega \neq 0,$$

we set

$$\mathfrak{m} := \{X \in \mathfrak{g} : i(X)\xi = 0\}, \qquad \mathfrak{h} := \{X \in \mathfrak{g} : i(X)\omega = 0\}.$$

Then both $\mathfrak{m}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g}$. Moreover $\dim \mathfrak{m} = n-h$, $\dim \mathfrak{h} = h$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. If, in addition, $\xi = \xi_1 \wedge \cdots \wedge \xi_h$, $\omega = \omega_1 \wedge \cdots \wedge \omega_{n-h}$, where all the ξ_i's and the ω_i have pure weights p_i and q_i, respectively, then both $\mathfrak{m}$ and $\mathfrak{h}$ are homogeneous Lie subalgebras of $\mathfrak{g}$. Thus, if we set

$$\mathbb{M} := \exp(\mathfrak{m}) \quad \text{and} \quad \mathbb{H} := \exp(\mathfrak{h}),$$

then $\mathbb{M}$ and $\mathbb{G}$ are complementary subgroups. In particular, since $*E_0^h = E_0^{n-h}$, if $\xi \in E_0^h$, we can choose $\omega := *\xi$. In this case, $\mathfrak{m}$ and $\mathfrak{h}$ are orthogonal.

Reciprocally, suppose $\mathfrak{m}$ *and* $\mathfrak{h}$ *are homogeneous Lie subalgebras of* $\mathfrak{g}$ *such that* $\dim \mathfrak{m} = n-h$, $\dim \mathfrak{h} = h$, *and* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. *Then there exist a scalar product* $\langle \cdot,\cdot\rangle_0$ *in* $\mathfrak{g}$, $\xi \in E_0^h$ *and* $\omega \in E_0^{n-h}$ *such that* $\xi \wedge \omega \neq 0$ *and*

$$\mathfrak{m} := \{X \in \mathfrak{g} \,:\, i(X)\xi = 0\}, \qquad \mathfrak{h} := \{X \in \mathfrak{g} \,:\, i(X)\omega = 0\}.$$

We remind that, as discussed in [47] *and* [32], *Remark 3.13, our definition of Rumin's classes depends on the scalar product in* $\mathfrak{g}$.

Proof. By Proposition 2.2.9, both $\mathfrak{m}$ and $\mathfrak{h}$ are Lie subalgebras, since $d\xi = 0$ and $d\omega = 0$. In addition, $\dim \mathfrak{m} = n-h$, $\dim \mathfrak{h} = h$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and hence $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, since $\mathfrak{m} \cap \mathfrak{h} = \{0\}$.

Suppose now, for instance, $\xi = \xi_1 \wedge \cdots \wedge \xi_h$, where all the ξ_i's have pure weight p_i, $i = 1,\dots,h$. Then ξ has weight $p := p_1 + \cdots + p_h$. Take $X \in \mathfrak{m}$; we can write

$$X = \sum_{\ell=1}^{k} \mu_\ell v_\ell, \quad \text{with } v_\ell \in \mathfrak{g}_\ell.$$

By identity 6 (ii) of [35], p.90, if we set $\hat{\xi}_j := \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_h$, we can write

$$\begin{aligned} 0 = i(X)\xi &= \sum_{1\le j\le h} (-1)^{j+1} \big(i(X)\xi_j\big)\hat{\xi}_j = \sum_{1\le j\le h} (-1)^{j+1} \langle \xi_j | X\rangle \hat{\xi}_j \\ &= \sum_{\ell=1}^{k} \sum_{1\le j\le h} (-1)^{j+1} \mu_\ell \langle \xi_j | v_\ell\rangle \hat{\xi}_j = \sum_{i=1}^{k}\sum_{\ell=1}^{k}\sum_{p_j=i} (-1)^{j+1} \mu_\ell \langle \xi_j | v_\ell\rangle \hat{\xi}_j \\ &= \sum_{i=1}^{k}\sum_{p_j=i} (-1)^{j+1} \mu_i \langle \xi_j | v_i\rangle \hat{\xi}_j, \end{aligned} \tag{14}$$

since $\langle \xi_j | v_\ell\rangle = 0$ if ℓ does not equal the weight of ξ_j. Notice now that, if $p_j = i$, then the weight of $\hat{\xi}_j = p - i$, and then the $\hat{\xi}_j$'s are orthogonal when the p_j's are different. It follows that

$$\sum_{p_j=i} (-1)^{j+1} \mu_i \langle \xi_j | v_i\rangle \hat{\xi}_j = 0 \quad \text{for } i = 1,\dots,k. \tag{15}$$

If now $\lambda > 0$, arguing as in (14), we get

$$\begin{aligned} i(\delta_\lambda X)\xi &= \sum_{1\le j\le h} (-1)^{j+1} \big(i(\delta_\lambda X)\xi_j\big)\hat{\xi}_j = \sum_{1\le j\le h} (-1)^{j+1} \langle \xi_j | \delta_\lambda X\rangle \hat{\xi}_j \\ &= \sum_{\ell=1}^{k} \sum_{1\le j\le h} (-1)^{j+1} \mu_\ell \lambda^\ell \langle \xi_j | v_\ell\rangle \hat{\xi}_j = \sum_{i=1}^{k}\sum_{\ell=1}^{k}\sum_{p_j=i} (-1)^{j+1} \mu_\ell \lambda^\ell \langle \xi_j | v_\ell\rangle \hat{\xi}_j \\ &= \sum_{i=1}^{k} \mu_i \lambda^i \sum_{p_j=i} (-1)^{j+1} \langle \xi_j | v_i\rangle \hat{\xi}_j = 0, \end{aligned}$$

by (15). Then $i(\delta_\lambda X)\xi \in \mathfrak{m}$ that is therefore homogeneous.

Finally, if $\omega = *\xi$, then $\mathfrak{g} \perp \mathfrak{m}$ by [24], Section 1.6.2., p.25. This achieves the proof of the first part of the theorem.

To prove the second part of the theorem, we notice that, by Remark 2.1.6, we can find two bases $\{w_1,\dots,w_{n-h}\}$ of $\mathfrak{m}$ and $\{v_1,\dots,v_h\}$ of $\mathfrak{h}$ such that all the w_j's and the v_j's have pure weights, i.e. such that

$$w_j \in \mathfrak{m}_{p_j}, \text{ for } j = 1,\dots,n-h \quad \text{and} \quad v_j \in \mathfrak{h}_{q_j}, \text{ for } j = 1,\dots,h.$$

Since $\{w_1,\dots,w_{n-h},v_1,\dots,v_h\}$ is a basis of $\mathfrak{g}$, we can take the dual basis $\{\omega_1,\dots,\omega_{n-h},\xi_1,\dots,\xi_h\}$ such that

$$\langle\omega_i|w_j\rangle=\delta_{ij},\quad \langle\xi_i|v_j\rangle=\delta_{ij},\quad \langle\omega_i|v_j\rangle=\langle\xi_i|w_j\rangle=0. \tag{16}$$

Since the vectors $w_1,\dots,w_{n-h},v_1,\dots,v_h$ are linearly independent, we can always assume that $\omega_1\wedge\cdots\wedge\omega_{n-h}\wedge\xi_1\wedge\cdots\wedge\xi_h=dV$. Then we put

$$\omega:=\omega_1\wedge\cdots\wedge\omega_{n-h}\quad\text{and}\quad \xi:=\xi_1\wedge\cdots\wedge\xi_h.$$

We have

$$\mathfrak{m}:=\{X\in\mathfrak{g}\,:\,i(X)\xi=0\},\qquad \mathfrak{h}:=\{X\in\mathfrak{g}\,:\,i(X)\omega=0\}. \tag{17}$$

Indeed, if $X=\sum_\ell\lambda_\ell w_\ell\in\mathfrak{m}$, then, again by identity 6 (ii) of [35], p.90,

$$i(X)\xi=\sum_{1\le j\le h}(-1)^{j+1}\langle\xi_j|X\rangle\hat{\xi}_j=0,$$

so that $\mathfrak{m}\subset\{X\in\mathfrak{g}\,:\,i(X)\xi=0\}$. On the other hand, $\dim\mathfrak{m}=n-h=\dim\{X\in\mathfrak{g}\,:\,i(X)\xi=0\}$, and the first identity in (17) follows. The proof of the second identity is identical.

We want to prove now that $d_0\xi=0$ and $d_0\omega=0$. To this end, we remind that, by Proposition 2.2.9, $d_0\xi=\beta\wedge\xi$. But this is possible only if $d_0\xi=0$, since otherwise $d_0\xi$ and ξ would have the same weight. Analogously we can prove that $d_0\omega=0$.

We define now a new (equivalent) scalar product $\langle\cdot,\cdot\rangle_0$ in $\mathfrak{g}$ making the basis $\{w_1,\dots,w_{n-h},v_1,\dots,v_h\}$ (and therefore also the dual basis $\{\omega_1,\dots,\omega_{n-h},\xi_1,\dots,\xi_h\}$) orthonormal. If we denote by $*_0$ the Hodge duality operator associated with the scalar product $\langle\cdot,\cdot\rangle_0$ and with dV the volume form, we want to show that $*_0\xi=\omega$, i.e. that

$$\alpha\wedge\xi=\langle\alpha,\omega\rangle_0\,dV\quad\text{for all }\alpha\in{\textstyle\bigwedge^{n-h}}\mathfrak{g}. \tag{18}$$

An orthonormal basis of $\bigwedge^{n-h}\mathfrak{g}$ is given by

$$\Sigma^{n-h}:=\{\omega_{i_1}\wedge\cdots\wedge\omega_{i_\ell}\wedge\xi_{j_1}\wedge\cdots\wedge\xi_{j_{n-h-\ell}},\ i_1<i_2<\cdots<i_\ell,j_1<j_2<\cdots<j_{n-h-\ell}\},$$

where the cases $\ell=0$ and $\ell=n-h$ are allowed. Thus, it is enough to test (18) for $\alpha\in\Sigma^{n-h}$. If now $0\le\ell<n-h$ (i.e. if there is at least one factor ξ_k), clearly both terms in (18) are zero. Thus, it is enough to take $\alpha=\omega_1\wedge\cdots\wedge\omega_{i_{n-h}}=\omega$, so that

$$\alpha\wedge\xi=dV\quad\text{and}\quad\langle\alpha,\omega\rangle_0=1,$$

yielding (18).

Thus we obtain $d_0{*_0}\xi=d_0\omega=0$ and $d_0{*_0}\omega=\pm d_0\xi=0$, achieving the proof of the theorem. □

2.2.2. *Components along complementary subgroups.* Given $\mathbb{M},\mathbb{H}$, complementary subgroups of $\mathbb{G}$, the elements $m\in\mathbb{M}$ and $h\in\mathbb{H}$ such that $g=mh$ are unique because of $\mathbb{M}\cap\mathbb{H}=\{e\}$ and are denoted as *components* of g along $\mathbb{M}$ and $\mathbb{H}$ or as *projections* of g on $\mathbb{M}$ and $\mathbb{H}$.

Proposition 2.2.11. *If $\mathbb{M},\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ there is $c_0=c_0(\mathbb{M},\mathbb{H})>0$ such that for all $g=mh$*

$$c_0\left(\|m\|+\|h\|\right)\le\|g\|\le\|m\|+\|h\|\,. \tag{19}$$

Proof. The right hand side is triangular inequality. For the left hand side observe that $\mathbb{M}\cap\partial B(0,1)$ and $\mathbb{H}\cap\partial B(0,1)$ are disjoint compact sets, hence have positive distance. The general statement follows by dilation. □

The following Lemma will be useful many times along this paper (see also Lemma 3.9 and Remark 3.10 in [38]).

Lemma 2.2.12. *Let $\mathbb{G}$ be a step κ group. There is $C = C(\mathbb{G}) > 0$ such that*

$$\left\|p^{-1}q^{-1}pq\right\| \le C\big(\|p\|^{\frac{1}{\kappa}}\|q\|^{\frac{\kappa-1}{\kappa}} + \|q\|^{\frac{1}{\kappa}}\|p\|^{\frac{\kappa-1}{\kappa}}\big), \qquad \text{for all } p,q\in\mathbb{G}, \tag{20}$$

and consequently

$$\left\|q^{-1}pq\right\| \le \|p\| + C\big(\|p\|^{\frac{1}{\kappa}}\|q\|^{\frac{\kappa-1}{\kappa}} + \|q\|^{\frac{1}{\kappa}}\|p\|^{\frac{\kappa-1}{\kappa}}\big), \qquad \text{for all } p,q\in\mathbb{G}. \tag{21}$$

Proof. First we prove (20). Observe that

$$p^{-1}q^{-1}pq = \mathcal{Q}(-p,-q) + \mathcal{Q}(p,q) + \mathcal{Q}(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q))$$

We use now the notation in (6). From (3) it follows that $(p^{-1}q^{-1}pq)^1 = 0$ and

$$(p^{-1}q^{-1}pq)^2 = \mathcal{Q}^2(-p,-q) + \mathcal{Q}^2(p,q) + \mathcal{Q}^2(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q)).$$

But $\mathcal{Q}^2(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q)) = \mathcal{Q}^2(-p-q, p+q) = 0$, since the polynomial $\mathcal{Q}^2$ depends only on the components of the first layer, $\mathcal{Q}^1(p,q) = 0$ and $\mathcal{Q}(-g,g) = 0$ for all $g \in \mathbb{G}$. Hence

$$(p^{-1}q^{-1}pq)^2 = \mathcal{Q}^2(-p,-q) + \mathcal{Q}^2(p,q).$$

In particular $\left|(p^{-1}q^{-1}pq)^2\right|^{\frac{1}{2}}$ can be estimated by $\|p\|^{\frac{1}{2}}\|q\|^{\frac{1}{2}}$ up to a constant factor depending only on $\mathbb{G}$. Now consider the case $2 < l \le k$.

$$(p^{-1}q^{-1}pq)^l = \mathcal{Q}^l(-p,-q) + \mathcal{Q}^l(p,q) + \mathcal{Q}^l(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q)).$$

Remember that $\mathcal{Q}^l(-p,-q)$, $\mathcal{Q}^l(p,q)$ and $\mathcal{Q}^l(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q))$ are homogeneous l-degree polynomials in the variables p and q. Since $\mathcal{Q}(p,0) = \mathcal{Q}(0,q) = 0$,

$$\mathcal{Q}(-p+\mathcal{Q}(-p,0), p+\mathcal{Q}(p,0)) = \mathcal{Q}(-p,p) = 0 = \mathcal{Q}(-q+\mathcal{Q}(0,q), q+\mathcal{Q}(0,q)).$$

Hence the previous homogeneous l-degree polynomials contain only mixed monomials in the variables p and q. Therefore

$$|\mathcal{Q}^l(p,q)|,\ |\mathcal{Q}^l(-p,-q)| \ \text{ and } \ |\mathcal{Q}^l(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q))|$$

can be estimated by $\sum_{i=1}^{l-1}\|p\|^i\|q\|^{l-i}$ up to a constant factor. Moreover

$$\|p\|^i\|q\|^{l-i} \le \|p\|\|q\|^{l-1} + \|p\|^{l-1}\|q\|, \qquad \text{for all } i = 1,\dots,l-1$$

since $\|p\|^i\|q\|^{l-i}$ is estimated by $\|p\|\|q\|^{l-1}$ when $\|p\| \le \|q\|$ and by $\|p\|^{l-1}\|q\|$ when $\|p\| \ge \|q\|$. Hence there is a constant $C_l > 0$ such that

$$\begin{aligned}
&\left|(p^{-1}q^{-1}pq)^l\right|^{\frac{1}{l}} \\
&\le |\mathcal{Q}^l(-p,-q)|^{\frac{1}{l}} + |\mathcal{Q}^l(p,q)|^{\frac{1}{l}} + |\mathcal{Q}^l(-p-q+\mathcal{Q}(-p,-q), p+q+\mathcal{Q}(p,q))|^{\frac{1}{l}} \\
&\le C_l\left(\|p\|^{\frac{1}{l}}\|q\|^{\frac{l-1}{l}} + \|p\|^{\frac{l-1}{l}}\|q\|^{\frac{1}{l}}\right).
\end{aligned}$$

Now (20) follows and, in turn, (21) can be derived from $\left\|q^{-1}pq\right\| \le \|p\| + \left\|p^{-1}q^{-1}pq\right\|$. □

Corollary 2.2.13. *Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups of a step κ group $\mathbb{G}$. If $g \in \mathbb{G}$ let $m, \bar{m} \in \mathbb{M}$ and $h, \bar{h} \in \mathbb{H}$ be the unique elements such that*

$$g = mh = \bar{h}\bar{m}.$$

Then,

$$\|m\| \le \frac{1}{c_0}\|\bar{m}\| + \frac{C}{c_0}\left(\|\bar{m}\|^{\frac{1}{\kappa}}\left\|\bar{h}\right\|^{\frac{\kappa-1}{\kappa}} + \left\|\bar{h}\right\|^{\frac{1}{\kappa}}\|\bar{m}\|^{\frac{\kappa-1}{\kappa}}\right),$$

$$\|h\| \le \frac{1}{c_0}\left\|\bar{h}\right\| + \frac{C}{c_0}\left(\|\bar{m}\|^{\frac{1}{\kappa}}\left\|\bar{h}\right\|^{\frac{\kappa-1}{\kappa}} + \left\|\bar{h}\right\|^{\frac{1}{\kappa}}\|\bar{m}\|^{\frac{\kappa-1}{\kappa}}\right).$$

Consequently, for all $\delta > 0$ there is $c(\delta) = c(\delta, \mathbb{M}, \mathbb{H}) > 0$ such that for all g with $\|g\| \le \delta$

$$\|m\| \le c(\delta) \|\bar{m}\|^{1/\kappa}, \quad \|h\| \le c(\delta) \|\bar{h}\|^{1/\kappa}. \tag{22}$$

Proof. We denote as $p_{\mathbb{M}} \in \mathbb{M}$ and $p_{\mathbb{H}} \in \mathbb{H}$ the unique 'components' of a generic $p \in \mathbb{G}$ such that

$$p = p_{\mathbb{M}} p_{\mathbb{H}}.$$

With this notation,

$$mh = (\bar{h}\bar{m})_{\mathbb{M}} (\bar{h}\bar{m})_{\mathbb{H}};$$

and, by uniqueness of the components,

$$m = (\bar{h}\bar{m})_{\mathbb{M}} = (\bar{h}\bar{m}\bar{h}^{-1})_{\mathbb{M}}, \qquad h = (\bar{h}\bar{m})_{\mathbb{H}} = (\bar{m}^{-1}\bar{h}\bar{m})_{\mathbb{H}}.$$

Hence, by (19) and (21),

$$\|m\| = \left\|(\bar{h}\bar{m}\bar{h}^{-1})_{\mathbb{M}}\right\| \le \frac{1}{c_0} \left\|\bar{h}\bar{m}\bar{h}^{-1}\right\| \le \frac{1}{c_0} \|\bar{m}\| + \frac{C}{c_0} \left(\|\bar{m}\|^{\frac{1}{\kappa}} \left\|\bar{h}\right\|^{\frac{\kappa-1}{\kappa}} + \left\|\bar{h}\right\|^{\frac{1}{\kappa}} \|\bar{m}\|^{\frac{\kappa-1}{\kappa}} \right).$$

The other inequality is proved in the same way. Finally to prove (22) we use that $\|g\| \le \delta$ yields $\left\|\bar{h}\right\|, \|\bar{m}\| \le \delta/c_0$. ☐

From now on, we will keep the convention introduced in the proof of Corollary 2.2.13 and we will denote as $g_{\mathbb{M}}$ and $g_{\mathbb{H}}$ the components of $g \in \mathbb{G}$. More precisely our convention is as follows, when $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $\mathbb{M}$ will always be the first 'factor' and $\mathbb{H}$ the second one and $g_{\mathbb{M}} \in \mathbb{M}$ and $g_{\mathbb{H}} \in \mathbb{H}$ are the unique elements such that

$$g = g_{\mathbb{M}} g_{\mathbb{H}}. \tag{23}$$

We stress that this notation is ambiguous because each component $g_{\mathbb{M}}$ and $g_{\mathbb{H}}$ depends on both the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$ and also on the order under which they are taken. The projection maps $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_{\mathbb{H}} : \mathbb{G} \to \mathbb{H}$ are defined as

$$\mathbf{P}_{\mathbb{M}}(g) := g_{\mathbb{M}}, \qquad \mathbf{P}_{\mathbb{H}}(g) := g_{\mathbb{H}} \tag{24}$$

We will collect now a few properties of components and projection maps. In particular, in Proposition 2.2.16 we prove that projection maps $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_{\mathbb{H}} : \mathbb{G} \to \mathbb{H}$ are C^∞ (indeed polynomial) as maps from $\mathbb{G} = \mathbb{R}^n \to \mathbb{G} = \mathbb{R}^n$. Nevertheless, quite differently from Euclidean spaces, $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$, in general, are not even Lipschitz maps, from $\mathbb{G}$ to $\mathbb{M}$ or to $\mathbb{H}$, when $\mathbb{G}$, $\mathbb{M}$ and $\mathbb{H}$ are endowed with the restriction of the natural left invariant distance d of $\mathbb{G}$ (see Example 2.2.17). This fact has many unpleasant consequences. One of them is related with the difficulty of controlling in an easy way the measure, or even the Hausdorff dimension, of the projection of sets (see e.g. [9]).

First of all observe that in general, $(g_{\mathbb{M}})^{-1} \ne (g^{-1})_{\mathbb{M}}$ and $(g_{\mathbb{H}})^{-1} \ne (g^{-1})_{\mathbb{H}}$. We will frequently use the notation

$$g_{\mathbb{M}}^{-1} = (g_{\mathbb{M}})^{-1} \text{ and } g_{\mathbb{H}}^{-1} = (g_{\mathbb{H}})^{-1}.$$

The sizes of the projections $\mathbf{P}_{\mathbb{M}}(p)$ and $\mathbf{P}_{\mathbb{H}}(p)$ control the distance of $p \in \mathbb{G}$ from the complementary subspaces $\mathbb{M}$ and $\mathbb{H}$. The control is different when considering the distance of p from the first component $\mathbb{M}$ or from the second component $\mathbb{H}$ or if the second component is a normal subgroup.

Corollary 2.2.14. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of a step κ group $\mathbb{G}$ and let $\mathbb{G} = \mathbb{M}\mathbb{H}$. Then,*

$$c_0 \|\mathbf{P}_{\mathbb{H}}(p)\| \le \operatorname{dist}(p, \mathbb{M}) \le \|\mathbf{P}_{\mathbb{H}}(p)\|, \qquad \textit{for all } p \in \mathbb{G},$$

where c_0 is the constant in (19)*. Moreover, if $\mathbb{H}$ is a normal subgroup of $\mathbb{G}$ then*

$$c_0 \|\mathbf{P}_{\mathbb{M}}(p)\| \le \operatorname{dist}(p, \mathbb{H}) \le \|\mathbf{P}_{\mathbb{M}}(p)\|, \qquad \textit{for all } p \in \mathbb{G}.$$

If, on the contrary, $\mathbb{H}$ is not a normal subgroup of $\mathbb{G}$, then there is $c_1 = c_1(\mathbb{M}, \mathbb{H}) > 1$ such that

$$\frac{1}{c_1} \|\mathbf{P}_{\mathbb{M}}(p)\|^{\kappa} \le \operatorname{dist}(p, \mathbb{H}) \le c_1 \|\mathbf{P}_{\mathbb{M}}(p)\|^{1/\kappa}, \qquad \textit{if } \|p\| = 1. \tag{25}$$

Proof. Using the notation in (24),

$$\operatorname{dist}(p, \mathbb{M}) = \inf\{\|p^{-1}m\| : \ m \in \mathbb{M}\} \le \|p_{\mathbb{H}}^{-1} p_{\mathbb{M}}^{-1} p_{\mathbb{M}}\| = \|p_{\mathbb{H}}\|.$$

On the other side, for $\varepsilon > 0$, let $\bar{m} \in \mathbb{M}$ be such that $\|\bar{m}^{-1}p\| \le \operatorname{dist}(p, \mathbb{M}) + \varepsilon$ then

$$c_0 \|p_{\mathbb{H}}\| \le c_0(\|\bar{m}^{-1}p_{\mathbb{M}}\| + \|p_{\mathbb{H}}\|) \le \|\bar{m}^{-1}p\| \le \operatorname{dist}(p, \mathbb{M}) + \varepsilon.$$

To estimate the distance of p from $\mathbb{H}$ when $\mathbb{H}$ is normal in $\mathbb{G}$, observe

$$\begin{aligned}
\operatorname{dist}(p, \mathbb{H}) &= \inf\{\|h^{-1}p_{\mathbb{M}}p_{\mathbb{H}}\| : \ h \in \mathbb{H}\} \\
&= \inf\{\|p_{\mathbb{M}}p_{\mathbb{M}}^{-1}h^{-1}p_{\mathbb{M}}p_{\mathbb{H}}\| : \ h \in \mathbb{H}\} \\
&\le \|p_{\mathbb{M}}\| + \inf\{\|p_{\mathbb{M}}^{-1}h^{-1}p_{\mathbb{M}}p_{\mathbb{H}}\| : \ h \in \mathbb{H}\} \\
&= \|p_{\mathbb{M}}\|, \quad \text{having taken } h = p_{\mathbb{M}}p_{\mathbb{H}}p_{\mathbb{M}}^{-1}.
\end{aligned}$$

On the other side, given $\varepsilon > 0$ let $h \in \mathbb{H}$ be such that $\operatorname{dist}(p, \mathbb{H}) + \varepsilon > \|h^{-1}p\|$, then

$$\operatorname{dist}(p, \mathbb{H}) + \varepsilon > \|h^{-1}p_{\mathbb{M}}p_{\mathbb{H}}\| = \|p_{\mathbb{M}}p_{\mathbb{M}}^{-1}h^{-1}p_{\mathbb{M}}p_{\mathbb{H}}\| \ge c_0 \|p_{\mathbb{M}}\|,$$

by Proposition 2.2.11. This concludes the proof of the second statement. Finally, in the last case, think of $\mathbb{G}$ as $\mathbb{G} = \mathbb{H}\mathbb{M}$ and write $p = \bar{h}\bar{m}$. Then, the estimate of the distance of a point p from the first component of a splitting gives

$$\tilde{c}_0 \|\bar{m}\| \le \operatorname{dist}(p, \mathbb{H}) \le \|\bar{m}\|,$$

where $\tilde{c}_0$ is the constant in Proposition 2.2.11, but related to the splitting $\mathbb{G} = \mathbb{H}\mathbb{M}$. Now the inequalities in (22) give the thesis. □

Observe that (25) cannot be improved without additional assumptions. Indeed consider the following easy example.

Example 2.2.15. Let $\mathbb{G}$ be the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ with the group law

$$x \cdot y = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) := (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - x_2y_1)/2.$$

Let

$$\mathbb{V} = \{x = (x_1, 0, 0) : x_1 \in \mathbb{R}\} \ \text{ and } \ \mathbb{W} = \{x = (0, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}.$$

$\mathbb{V}$ and $\mathbb{W}$ are complementary subgroups in $\mathbb{H}^1$, $\mathbb{W}$ is a normal subgroup while $\mathbb{V}$ is not a normal subgroup.
For $s \in (0, 1)$ let $p_s := (1, s, -s/2) \in \mathbb{H}^1$. Then it is easy to see that as $s \to 0$,

$$\begin{aligned}
\operatorname{dist}(p_s, \mathbb{V}) &= \inf\{\|v^{-1}p_s\| : \ v \in \mathbb{V}\} \\
&= \inf\{\|(1 - x_1, s, -s(x_1 + 1)/2)\| : \ x_1 \in \mathbb{R}\} \approx \sqrt{s}; \\
\|\mathbf{P}_{\mathbb{W}}(p_s)\| &= \|(0, s, 0)\| \approx s.
\end{aligned}$$

On the other side, once more for $s \in (0, 1)$ let $q_s := (1, s, s/2) \in \mathbb{H}^1$. Now we have that, as $s \to 0$,

$$\begin{aligned}
\operatorname{dist}(q_s, \mathbb{V}) &= \inf\{\|v^{-1}q_s\| : \ v \in \mathbb{V}\} \\
&= \inf\{\|(1 - x_1, s, s(1 - x_1)/2)\| : \ x_1 \in \mathbb{R}\} \approx s \\
\|\mathbf{P}_{\mathbb{W}}(q_s)\| &= \|(0, s, s)\| \approx \sqrt{s}.
\end{aligned}$$

See also Example 2.2.17.

Proposition 2.2.16. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$, then the projection maps $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_{\mathbb{H}} : \mathbb{G} \to \mathbb{H}$ defined in* (24) *are polynomial maps. More precisely, if κ is the step of $\mathbb{G}$, there are 2κ matrices $A^1, \dots, A^\kappa, B^1, \dots, B^\kappa$, depending on $\mathbb{M}$ and $\mathbb{H}$, such that*

(i) $\qquad A^j$ *and* B^j *are (n_j, n_j)-matrices, for all $1 \le j \le \kappa$,*

and, with the notations of (2),

(ii) $\qquad \mathbf{P}_{\mathbb{M}} g = \big(A^1 g^1, A^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, A^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)\,;$

(iii) $\qquad \mathbf{P}_{\mathbb{H}} g = \big(B^1 g^1, B^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, B^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)\,;$

(iv) $\qquad A^j$ *is the identity on $\mathbb{M}^j$, and B^j is the identity on $\mathbb{H}^j$, for $1 \le j \le \kappa$.*

Proof. Recall the notation

$$g = \big(g^1, \dots, g^\kappa\big)\,, \quad \mathbf{P}_{\mathbb{M}}(g) = g_{\mathbb{M}} = \big(g^1_{\mathbb{M}}, \dots, g^\kappa_{\mathbb{M}}\big)\,, \quad \mathbf{P}_{\mathbb{H}}(g) = g_{\mathbb{H}} = \big(g^1_{\mathbb{H}}, \dots, g^\kappa_{\mathbb{H}}\big)\,.$$

Let d and $n-d$ be respectively the linear dimensions of $\mathbb{M}$ and $\mathbb{H}$. Because $\mathbb{M}$ and $\mathbb{H}$ are complementary subgroups there are a $(n-d, n)$-matrix M, a (d, n)-matrix H, such that $\mathbb{M} = \{x \in \mathbb{G} : Mx = 0\}$ and $\mathbb{H} = \{x \in \mathbb{G} : Hx = 0\}$ and such the (n, n)-matrix $\begin{bmatrix} M \\ H \end{bmatrix}$ is non singular. In particular,

$$M g_{\mathbb{M}} = 0 \text{ and } H g_{\mathbb{H}} = 0, \text{ for all } g \in \mathbb{G}.$$

Notice that both M and H have the form

$$M = \begin{bmatrix} M^1 & 0 & \cdots & 0 \\ 0 & M^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M^\kappa \end{bmatrix}, \qquad H = \begin{bmatrix} H^1 & 0 & \cdots & 0 \\ 0 & H^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & H^\kappa \end{bmatrix},$$

where, for $1 \le j \le \kappa$, M^j is an $(n_j - d_j, n_j)$ matrix, H^j is an (d_j, n_j) matrix and the (n_j, n_j) matrix $\begin{bmatrix} M^j \\ H^j \end{bmatrix}$ is non singular. Here we have denoted as d_j and $n_j - d_j$ the dimensions of the corresponding layers of $\mathbb{M}$ and $\mathbb{H}$.

By definition, $g_{\mathbb{M}}$ and $g_{\mathbb{H}}$ are the solutions of the system of $2n$ equations in the $2n$ unknowns $g_{\mathbb{M},1}, \dots, g_{\mathbb{M},n}$ and $g_{\mathbb{H},1}, \dots, g_{\mathbb{H},n}$,

$$\begin{aligned} & g^1_{\mathbb{M}} + g^1_{\mathbb{H}} = g^1, \\ & g^2_{\mathbb{M}} + g^2_{\mathbb{H}} + \mathcal{Q}^2(g^1_{\mathbb{M}}, g^1_{\mathbb{H}}) = g^2, \\ & \qquad \vdots \\ & g^\kappa_{\mathbb{M}} + g^\kappa_{\mathbb{H}} + \mathcal{Q}^\kappa(g^1_{\mathbb{M}}, \dots, g^{\kappa-1}_{\mathbb{M}}, g^1_{\mathbb{H}}, \dots, g^{\kappa-1}_{\mathbb{H}}) = g^\kappa, \\ & M g_{\mathbb{M}} = 0, \\ & H g_{\mathbb{H}} = 0. \end{aligned}$$

This system can be solved layer by layer. From what stated before, we know that the linear system of $2n_1$ equations and unknowns

$$\begin{aligned} & g^1_{\mathbb{M}} + g^1_{\mathbb{H}} = g^1, \\ & M^1 g^1_{\mathbb{M}} = 0, \\ & H^1 g^1_{\mathbb{H}} = 0, \end{aligned}$$

has a unique solution $g^1_{\mathbb{M}}$, $g^1_{\mathbb{H}}$ depending linearly on the components of g^1. We denote

$$g^1_{\mathbb{M}} = A^1 g^1 \text{ and } g^1_{\mathbb{H}} = B^1 g^1.$$

Then we find the unique solution of the linear system in the $2n_2$ unknowns $g^2_{\mathbb{M}}$ and $g^2_{\mathbb{H}}$

$$\begin{aligned} &g^2_{\mathbb{M}} + g^2_{\mathbb{H}} = g^2 - \mathcal{Q}^2(g^1_{\mathbb{M}}, g^1_{\mathbb{H}}),\\ &M^2 g^2_{\mathbb{M}} = 0,\\ &H^2 g^2_{\mathbb{H}} = 0, \end{aligned}$$

and we denote as A^2 and B^2 the matrices such that

$$\begin{aligned} g^2_{\mathbb{M}} &= A^2(g^2 - \mathcal{Q}^2(g^1_{\mathbb{M}}, g^1_{\mathbb{H}})) = A^2(g^2 - \mathcal{Q}^2(A^1g^1, B^1g^1)),\\ g^1_{\mathbb{H}} &= B^2(g^2 - \mathcal{Q}^2(g^1_{\mathbb{M}}, g^1_{\mathbb{H}})) = B^2(g^2 - \mathcal{Q}^2(A^1g^1, B^1g^1)). \end{aligned}$$

Then we iterate the procedure up to the layer κ.

In order to prove (iv), observe that if $g \in \mathbb{M}$ then $\mathbf{P}_{\mathbb{M}}(g) = g$ and $\mathbf{P}_{\mathbb{H}}(g) = 0$ hence

$$\begin{aligned} g &= \left(A^1g^1, A^2(g^2 - \mathcal{Q}^2(A^1g^1, B^1g^1)), \dots, A^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1g^1, \dots, B^{\kappa-1}g^{\kappa-1}))\right),\\ 0 &= \left(B^1g^1, B^2(g^2 - \mathcal{Q}^2(A^1g^1, B^1g^1)), \dots, B^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1g^1, \dots, B^{\kappa-1}g^{\kappa-1}))\right). \end{aligned}$$

From these we get $g^1 = A^1g^1$ and $0 = B^1g^1$, for all $g \in \mathbb{M}$. Looking at the second layer, notice that $0 = B^1g^1$ yields $\mathcal{Q}^2(A^1g^1, B^1g^1)) = 0$, hence we have $g^2 = A^2(g^2)$ and $0 = B^2g^2$, for all $g \in \mathbb{M}$. The procedure can be repeated up to the layer κ. $\square$

As we anticipated, $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ are not, in general, Lipschitz maps when $\mathbb{G}$, $\mathbb{M}$ and $\mathbb{H}$ are endowed with the restriction of the distance d of $\mathbb{G}$.

Example 2.2.17. Let $\mathbb{G}$ be the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ and let $\mathbb{V}$ and $\mathbb{W}$ be the subgroups defined in Example 2.2.15.
When we consider $\mathbb{H}^1 = \mathbb{V} \cdot \mathbb{W}$ the projections $\mathbf{P}_{\mathbb{V}}$ and $\mathbf{P}_{\mathbb{W}}$ are

$$\mathbf{P}_{\mathbb{V}}(x_1, x_2, x_3) = (x_1, 0, 0), \quad \mathbf{P}_{\mathbb{W}}(x_1, x_2, x_3) = (0, x_2, x_3 - x_1x_2/2).$$

Here $\mathbf{P}_{\mathbb{W}} : \mathbb{H}^1 \to \mathbb{W}$ is not Lipschitz. Indeed, let $q = (1, 1, 0)$ and $p_\varepsilon = (1 + \varepsilon, 1 + \varepsilon, 0)$, then $\mathbf{P}_{\mathbb{W}}q = (0, 1, -1/2)$ and $\mathbf{P}_{\mathbb{W}}p_\varepsilon = (0, 1 + \varepsilon, -(1 + \varepsilon)^2/2)$. Hence, as $\varepsilon \to 0^+$,

$$\left\|q^{-1}p\right\| = \|(\varepsilon, \varepsilon, 0)\| \approx \varepsilon, \quad \left\|(\mathbf{P}_{\mathbb{W}}q)^{-1}\mathbf{P}_{\mathbb{W}}p_\varepsilon\right\| = \left\|(0, \varepsilon, -\varepsilon - \varepsilon^2/2)\right\| \approx \sqrt{\varepsilon}$$

When we consider $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, then $\mathbf{P}_{\mathbb{V}}$ and $\mathbf{P}_{\mathbb{W}}$ take the form

$$\mathbf{P}_{\mathbb{V}}(x_1, x_2, x_3) = (x_1, 0, 0), \quad \mathbf{P}_{\mathbb{W}}(x_1, x_2, x_3) = (0, x_2, x_3 + x_1x_2/2)$$

Let $q = (1, 0, 0)$ and $p_\varepsilon = (1, \varepsilon, \varepsilon/2)$, then $\mathbf{P}_{\mathbb{W}}q = (0, 0, 0)$ and $\mathbf{P}_{\mathbb{W}}p_\varepsilon = (0, \varepsilon, \varepsilon)$. Hence

$$\left\|q^{-1}p\right\| = \|(0, \varepsilon, 0)\| \approx \varepsilon, \quad \left\|(\mathbf{P}_{\mathbb{W}}q)^{-1}\mathbf{P}_{\mathbb{W}}p_\varepsilon\right\| = \|(0, \varepsilon, \varepsilon)\| \approx \sqrt{\varepsilon}$$

as $\varepsilon \to 0^+$. In this case too, $\mathbf{P}_{\mathbb{W}}$ is not a Lipschitz map.

The example shows that both the projections either on the first factor or on the second factor can be non Lipschitz. Notice that in both cases we were considering projections on the normal factor. Indeed the projection on the complement of a normal subgroup is always metric Lipschitz continuous.

Proposition 2.2.18. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$. Then*

(i) *if $\mathbb{H}$ is a normal subgroup then $\mathbf{P}_{\mathbb{M}}$ is Lipschitz;*

(ii) *if $\mathbb{M}$ is a normal subgroup then $\mathbf{P}_{\mathbb{H}}$ is Lipschitz.*

Proof. (i) For all $g = mh$ and $\bar{g} = \bar{m}\bar{h}$, we have

$$\mathbf{P}_{\mathbb{M}}(g^{-1}\bar{g}) = \mathbf{P}_{\mathbb{M}}(h^{-1}m^{-1}\bar{m}\bar{h}) = \mathbf{P}_{\mathbb{M}}(m^{-1}\bar{m}\bar{m}^{-1}mh^{-1}m^{-1}\bar{m}\bar{h}) = m^{-1}\bar{m}.$$

Hence, for all $g, \bar{g} \in \mathbb{G}$, $\mathbf{P}_{\mathbb{M}}(g^{-1}\bar{g}) = (\mathbf{P}_{\mathbb{M}}g)^{-1}\mathbf{P}_{\mathbb{M}}\bar{g}$ and, by (19),

$$\left\|(\mathbf{P}_{\mathbb{M}}g)^{-1}\mathbf{P}_{\mathbb{M}}\bar{g}\right\| \leq \left(\left\|\mathbf{P}_{\mathbb{M}}(g^{-1}\bar{g})\right\| + \left\|\mathbf{P}_{\mathbb{H}}(g^{-1}\bar{g})\right\|\right) \leq c_0^{-1}\left\|g^{-1}\bar{g}\right\|.$$

(ii) As before, $\mathbf{P}_{\mathbb{H}}(g^{-1}\bar{g}) = (\mathbf{P}_{\mathbb{H}}g)^{-1}\mathbf{P}_{\mathbb{H}}\bar{g}$ and eventually,

$$\left\|(\mathbf{P}_{\mathbb{H}}g)^{-1}\mathbf{P}_{\mathbb{H}}\bar{g}\right\| \leq c_0^{-1}\left\|g^{-1}\bar{g}\right\|, \quad \text{for all } g, \bar{g} \in \mathbb{G}.$$

□

Even if the projections are not Lipschitz we have the following control on the measure of projected sets.

Lemma 2.2.19. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$. Denote by $d_t \leq d_m$ respectively, the topological and the metric dimensions of $\mathbb{M}$. Then there is $c = c(\mathbb{M}, \mathbb{H}) > 0$ such that,*

$$\mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(B(p,r))\right) = c\, r^{d_m},$$

for all balls $B(p,r) \subset \mathbb{G}$.

Proof. Define

$$c = c(\mathbb{M}, \mathbb{H}) := \mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(B(e,1))\right).$$

Observe that $c_1(\mathbb{M}, \mathbb{H}) > 0$. Indeed, first we observe that the Lebesgue measure $\mathcal{L}^{d_t}$ is non-zero on $\mathbb{M}$ since it is the image under the exponential map of the Lebesgue measure on the d_t-dimensional Lie algebra of $\mathbb{M}$. Moreover

$$\mathbb{M} = \bigcup_{i \in \mathbb{N}} \mathbf{P}_{\mathbb{M}}(B(e,i))$$

hence there exists $i \in \mathbb{N}$ such that $\mathcal{L}^{d_t}(\mathbf{P}_{\mathbb{M}}(B(e,i))) > 0$. By group dilations $\mathbf{P}_{\mathbb{M}}(B(e,r)) = \mathbf{P}_{\mathbb{M}}(\delta_r B(e,1)) = \delta_r \mathbf{P}_{\mathbb{M}}(B(e,1))$, for all $r > 0$. Therefore $\mathcal{L}^{d_t}(\mathbf{P}_{\mathbb{M}}(B(e,1))) > 0$ and

$$\mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(B(e,r))\right) = c\, r^{d_m}.$$

To prove that also $\mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(B(p,r))\right) = \mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(p \cdot B(e,r))\right) = c\, r^{d_m}$ we prove that, for any fixed $p \in \mathbb{G}$, the map $\Phi_p : \mathbb{M} \to \mathbb{M}$ defined as $\Phi_p(m) := \mathbf{P}_{\mathbb{M}}(p \cdot m)$, has unit Jacobian determinant. That is, we prove that, for any measurable $\mathcal{E} \subset \mathbb{G}$,

$$\mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(p \cdot \mathcal{E})\right) = \mathcal{L}^{d_t}\left(\mathbf{P}_{\mathbb{M}}(\mathcal{E})\right).$$

To complete this last step, using the notations in Lemma 2.2.16, we compute

$$\begin{aligned}\Phi_p(m) &= \mathbf{P}_{\mathbb{M}}\left(p^1 + m^1, p^2 + m^2 + \mathcal{Q}^2(p^1, m^1), \ldots, p^\kappa + m^\kappa + \mathcal{Q}^\kappa(p^1, \ldots, m^{\kappa-1})\right) \\ &= \big(A^1(p^1 + m^1), A^2(p^2 + m^2 + \mathcal{Q}^2(p^1, m^1) - \mathcal{Q}^2(A^1(p^1 + m^1), B^1(p^1 + m^1)), \ldots \\ &\qquad \ldots, A^\kappa(p^\kappa + m^\kappa + \text{function of}(m^1, \ldots, m^{\kappa-1}))\big).\end{aligned}$$

Hence the Jacobian of Φ_p has the form

$$\left[\frac{\partial \Phi_p}{\partial m}\right] = \begin{bmatrix} A^1 & 0 & \cdots & 0 \\ * & A^2 & & \vdots \\ \vdots & & \ddots & 0 \\ * & \cdots & * & A^\kappa \end{bmatrix}$$

and $\det\left[\frac{\partial \Phi_p}{\partial m}\right] = 1$ because each A^j is the identity on $\mathbb{M}^j$. □

2.2.3. *Graphs.*

Definition 2.2.20. Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. We say that a set $S \subset \mathbb{G}$ is a *(left) $\mathbb{H}$-graph* (or a *left graph in direction* $\mathbb{H}$) if S intersects each left coset of $\mathbb{H}$ in one point, at most.

If $A \subset \mathbb{G}$ parametrizes the left cosets of $\mathbb{H}$ – i.e. if A itself intersect each left coset of $\mathbb{H}$ at most one time – and if S is an $\mathbb{H}$-graph, then there is a unique function $\varphi : \mathcal{E} \subset A \to \mathbb{H}$ such that S *is the graph of* φ, that is

$$S = \operatorname{graph}(\varphi) := \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}.$$

Conversely, for any $\psi : \mathcal{D} \subset A \to \mathbb{H}$ the set $\operatorname{graph}(\psi)$ is an $\mathbb{H}$-graph.

One has an important special case when $\mathbb{H}$ admits a complementary subgroup $\mathbb{M}$. Indeed, in this case, $\mathbb{M}$ parametrizes the left cosets of $\mathbb{H}$ and we have that

$$S \text{ is a } \mathbb{H}\text{-graph if and only if } S = \operatorname{graph}(\varphi)$$

for $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. By uniqueness of the components along $\mathbb{M}$ and $\mathbb{H}$, if $S = \operatorname{graph}(\varphi)$ then φ is uniquely determined among all functions from $\mathbb{M}$ to $\mathbb{H}$.

From now on we will consider mainly graphs of functions acting between complementary subgroups. Nevertheless it is relevant to mention that examples of $\mathbb{H}$-graphs that are not, in general, graphs of functions acting between complementary subgroups have been considered inside the Heisenberg groups $\mathbb{H}^n = \mathbb{R}^{2n+1}$. These sets are given as

$$S = \{(x_1, \cdots, y_n, \varphi(x_1, \cdots, y_n))\} \subset \mathbb{H}^n.$$

In our notation such an S is a $\mathbb{T}$-graph, where $\mathbb{T}$ is the centre of $\mathbb{H}^n$. The left cosets of $\mathbb{T}$ are parametrized over the set $A = H\mathbb{H}^n_e$. We recall that the centre $\mathbb{T}$ has no complementary subgroup in $\mathbb{H}^n$ (see Example 2.2.3), and in general there isn't a couple of complementary subgroups $\mathbb{M}, \mathbb{H}$ of $\mathbb{H}^n$ and a $\psi : \mathbb{M} \to \mathbb{H}$ such that $S = \operatorname{graph}(\psi)$, even locally.

If a set $S \subset \mathbb{G}$ is an intrinsic graph then it keeps being an intrinsic graph after left translations or group dilations.

Proposition 2.2.21. *Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. If S is a $\mathbb{H}$-graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $\delta_\lambda S$ and $q \cdot S$ are $\mathbb{H}$-graphs.*

If, in particular, $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, if $S = \operatorname{graph}(\varphi)$ with $\varphi : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, then

$$\begin{aligned} &\textit{For all } \lambda > 0,\ \delta_\lambda S = \operatorname{graph}(\varphi_\lambda), \textit{ with} \\ &\varphi_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{M} \to \mathbb{H} \textit{ and} \\ &\varphi_\lambda(m) = \delta_\lambda \varphi(\delta_{1/\lambda} m), \ \textit{for } m \in \delta_\lambda \mathcal{E}. \end{aligned} \tag{26}$$

$$\begin{aligned} &\textit{For any } q \in \mathbb{G}, \quad q \cdot S = \operatorname{graph}(\varphi_q), \ \textit{where} \\ &\varphi_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}, \quad \mathcal{E}_q = \{m : (q^{-1} \cdot m)_{\mathbb{M}} \in \mathcal{E}\} \textit{ and} \\ &\varphi_q(m) = (q^{-1} \cdot m)_{\mathbb{H}}^{-1} \cdot \varphi\big((q^{-1} \cdot m)_{\mathbb{M}}\big), \ \textit{for all } m \in \mathcal{E}_q. \end{aligned} \tag{27}$$

Proof. If $x, x' \in S$ and $x \neq x'$ then, by definition of $\mathbb{H}$-graph, x, x' belong to different left cosets of $\mathbb{H}$. Then $\delta_\lambda x$, $\delta_\lambda x'$ belong to different cosets of $\mathbb{H}$, because $\mathbb{H}$ is a homogeneous subgroup, and also $q \cdot x$, $q \cdot x'$ belong to different cosets of $\mathbb{H}$, by elementary properties of cosets (see e.g. [36, chapter 2, section 4]). By definition, these facts prove that both $\delta_\lambda S$ and $q \cdot S$ are $\mathbb{H}$-graphs and that there are φ_λ and φ_q such that $\delta_\lambda S = \operatorname{graph}(\varphi_\lambda)$ and $q \cdot S = \operatorname{graph}(\varphi_q)$.

To prove (26) observe that, by uniqueness of the components, $\delta_\lambda(m \cdot \varphi(m)) = m' \cdot \varphi(m')$ implies that $\delta_\lambda m = m'$ and that $\varphi_\lambda = \delta_\lambda \circ \varphi \circ \delta_{1/\lambda}$.

To prove (27) observe that, because $p_{\mathbb{H}}^{-1} = p^{-1}\cdot p_{\mathbb{M}}$, for all $p\in\mathbb{G}$, then $(q^{-1}\cdot m)_{\mathbb{H}}^{-1} = m^{-1}\cdot q\cdot (q^{-1}\cdot m)_{\mathbb{M}}$, hence

$$\begin{aligned}
\operatorname{graph}(\varphi_q) &= \{m\cdot\varphi_q(m):\ m\in\mathcal{E}_q\}\\
&= \{m\cdot (q^{-1}\cdot m)_{\mathbb{H}}^{-1}\cdot\varphi\big((q^{-1}\cdot m)_{\mathbb{M}}\big):\ m\in\mathcal{E}_q\}\\
&= \{m\cdot m^{-1}\cdot q\cdot (q^{-1}\cdot m)_{\mathbb{M}}\cdot\varphi\big((q^{-1}\cdot m)_{\mathbb{M}}\big):\ (q^{-1}\cdot m)_{\mathbb{M}}\in\mathcal{E}\}\\
&= q\cdot\operatorname{graph}(\varphi).
\end{aligned}$$

$\square$

Remark 2.2.22. From (27) and the continuity of the projection maps $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ it follows that the continuity of a function is preserved by translations. Precisely, given $q = q_{\mathbb{M}}q_{\mathbb{H}}$ and $f:\mathbb{M}\to\mathbb{H}$, then the translated function f_q is continuous in $m\in\mathbb{M}$ if and only if the function f is continuous in the corresponding point $(q^{-1}m)_{\mathbb{M}}$.

Remark 2.2.23. The algebraic expression of φ_q in Proposition 2.2.21 can be made more explicit when $\mathbb{G}$ is a semi-direct product of $\mathbb{M},\mathbb{H}$. Precisely

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then $\varphi_q(m) = q_{\mathbb{H}}\varphi\big((q^{-1}m)_{\mathbb{M}}\big)$, for $m\in\mathcal{E}_q = q\mathcal{E}(q_{\mathbb{H}})^{-1}$.

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then $\varphi_q(m) = (q^{-1}m)_{\mathbb{H}}^{-1}\varphi(q_{\mathbb{M}}^{-1}m)$, for $m\in\mathcal{E}_q = q_{\mathbb{M}}\mathcal{E}$.

If both $\mathbb{M}$ and $\mathbb{H}$ are normal in $\mathbb{G}$ – that is if $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ – then we get the well known Euclidean formula

(iii) $$\varphi_q(m) = q_{\mathbb{H}}\varphi(q_{\mathbb{M}}^{-1}m),\quad \text{for } m\in\mathcal{E}_q = q_{\mathbb{M}}\mathcal{E}.$$

See also [8, Proposition 3.6].

3. Intrinsic Lipschitz and intrinsic Differentiable functions

3.1. **General definitions.** *Intrinsic Lipschitz functions* in $\mathbb{G}$ are functions, acting between complementary subgroups of $\mathbb{G}$, with graphs non intersecting naturally defined cones. Hence, the notion of *intrinsic Lipschitz graph* respects strictly the geometry of the ambient group $\mathbb{G}$. Precisely a $\mathbb{H}$-graph S is said to be an intrinsic Lipschitz $\mathbb{H}$-graph if S intersects *intrinsic cones* with axis $\mathbb{H}$, fixed opening and vertex on S only in the vertex. Intrinsic Lipschitz functions appeared for the first time in [26] and were studied, more diffusely, in [30, 31].

We begin with two definitions of intrinsic (closed) cones.

Definition 3.1.1. Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$, $q\in\mathbb{G}$. Then, the *cones* $X(q,\mathbb{H},\alpha)$ with *axis* $\mathbb{H}$*, vertex* q*, opening* α, $0\le\alpha\le 1$ are defined as

$$X(q,\mathbb{H},\alpha) = q\cdot X(e,\mathbb{H},\alpha),\ \text{where } X(e,\mathbb{H},\alpha) = \big\{p:\operatorname{dist}(p,\mathbb{H})\le\alpha\,\|p\|\,\big\}.$$

Notice that Definition 3.1.1 does not require that $\mathbb{H}$ is a complemented subgroup.
Frequently, while working with functions acting between complementary subgroups, it will be convenient to consider also the following family of cones.

Definition 3.1.2. If $\mathbb{M},\mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $q\in\mathbb{G}$ and $\beta\ge 0$, the cones $C_{\mathbb{M},\mathbb{H}}(q,\beta)$, with base $\mathbb{M}$, axis $\mathbb{H}$, vertex q, opening β are defined as

$$C_{\mathbb{M},\mathbb{H}}(q,\beta) = q\cdot C_{\mathbb{M},\mathbb{H}}(e,\beta),\ \text{where } C_{\mathbb{M},\mathbb{H}}(e,\beta) = \{p:\|p_{\mathbb{M}}\|\le\beta\,\|p_{\mathbb{H}}\|\}\,.$$

Observe that

$$\mathbb{H} = X(e,\mathbb{H},0) = C_{\mathbb{M},\mathbb{H}}(e,0),\qquad \mathbb{G} = X(e,\mathbb{H},1) = \overline{\cup_{\beta>0}C_{\mathbb{M},\mathbb{H}}(e,\beta)}.$$

Moreover, the cones $C_{\mathbb{M},\mathbb{H}}(q,\beta)$ are equivalent with the cones $X(q,\mathbb{H},\alpha)$ in the sense of the following Proposition.

Proposition 3.1.3. *If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ then, for any $\alpha \in (0,1)$ there is $\beta \geq 1$, depending on α, $\mathbb{M}$ and $\mathbb{H}$, such that*

$$C_{\mathbb{M},\mathbb{H}}(q, 1/\beta) \subset X(q, \mathbb{H}, \alpha) \subset C_{\mathbb{M},\mathbb{H}}(q, \beta),$$

Proof. It is enough to prove the thesis with $q = e$. Let us prove the first inequality. By definition and by dilation invariance of the cones, it is enough to prove that for each $\alpha \in (0,1)$ there is $\beta > 0$ such that

$$\|p_{\mathbb{M}}\| \leq \frac{1}{\beta} \|p_{\mathbb{H}}\| \implies \operatorname{dist}(p, \mathbb{H}) \leq \alpha, \qquad \text{for all } p \in \mathbb{G} \text{ with } \|p\| = 1. \tag{28}$$

From $\|p_{\mathbb{M}}\| \leq \frac{1}{\beta} \|p_{\mathbb{H}}\|$ and (19)

$$(1 + \frac{1}{\beta}) \|p_{\mathbb{M}}\| \leq \frac{1}{\beta}(\|p_{\mathbb{M}}\| + \|p_{\mathbb{H}}\|) \leq \frac{1}{c_0\, \beta}. \tag{29}$$

Then, from (25) and (29), there is $c_1 = c_1(\mathbb{M}, \mathbb{H}) > 0$ such that

$$\operatorname{dist}(p, \mathbb{H}) \leq c_1 \|p_{\mathbb{M}}\|^{1/\kappa} \leq \frac{c_1}{(c_0(1+\beta))^{1/\kappa}} \leq \alpha$$

if β is large enough, for all $p \in \mathbb{G}$ with $\|p\| = 1$. This completes the proof of (28).
Let us prove the second inequality in the thesis. Once more, by dilation invariance of the cones, it is enough to prove that for each $\beta > 0$ there is $\alpha \in (0,1)$ such that

$$\operatorname{dist}(p, \mathbb{H}) \leq \alpha \implies \|p_{\mathbb{M}}\| \leq \beta \|p_{\mathbb{H}}\|, \qquad \text{for all } p \in \mathbb{G} \text{ with } \|p\| = 1. \tag{30}$$

From (25) we have

$$\|p_{\mathbb{M}}\| \leq c_1^{1/\kappa} \operatorname{dist}(p, \mathbb{H})^{1/\kappa} \leq c_1^{1/\kappa} \alpha^{1/\kappa}.$$

Hence, from $\|p\| = 1$ and (19),

$$1 - c_1^{1/\kappa} \alpha^{1/\kappa} \leq 1 - \|p_{\mathbb{M}}\| \leq \|p_{\mathbb{H}}\|.$$

Finally

$$\|p_{\mathbb{M}}\| \leq \frac{c_1^{1/\kappa} \alpha^{1/\kappa}}{1 - c_1^{1/\kappa} \alpha^{1/\kappa}} \|p_{\mathbb{H}}\|,$$

and we can choose α so small that the fraction is less than β. □

Now we introduce the basic definition of this paragraph.

Definition 3.1.4. (i) Let $\mathbb{H}$ be an homogeneous subgroup, not necessarily complemented in $\mathbb{G}$. We say that an $\mathbb{H}$-graph S is an *intrinsic Lipschitz* $\mathbb{H}$*-graph* if there is $\alpha \in (0,1)$ such that,

$$S \cap X(p, \mathbb{H}, \alpha) = \{p\}, \qquad \text{for all } p \in S.$$

(ii) If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, we say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic Lipschitz* in $\mathcal{E}$ when graph (f) is an *intrinsic Lipschitz* $\mathbb{H}$*-graph*.
(iii) We say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic* L*-Lipschitz* in $\mathcal{E}$ if there is $L > 0$ such that

$$C_{\mathbb{M},\mathbb{H}}(p, 1/L) \cap \operatorname{graph}(f) = \{p\}, \quad \text{for all } p \in \operatorname{graph}(f). \tag{31}$$

The Lipschitz constant of f in $\mathcal{E}$ is the infimum of the $L > 0$ such that (31) holds.

It follows immediately from Proposition 3.1.3 that f is intrinsic Lipschitz in $\mathcal{E}$ if and only if it is intrinsic L-Lipschitz for an appropriate constant L, depending on α, f and $\mathbb{M}$.

Because of Proposition 2.2.21 and Definition 3.1.2 left translations of intrinsic Lipschitz $\mathbb{H}$-graphs, or of intrinsic L-Lipschitz functions, keeps being intrinsic Lipschitz $\mathbb{H}$-graphs, or intrinsic L-Lipschitz functions. We state these facts in the following theorem.

Theorem 3.1.5. *If $\mathbb{G}$ is a Carnot group, then for all $q \in \mathbb{G}$,*

(i) $\quad S \subset \mathbb{G}$ *is an intrinsic Lipschitz $\mathbb{H}$-graph* $\implies$ $q \cdot S$ *is an intrinsic Lipschitz $\mathbb{H}$-graph;*

(ii) $\quad f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ *is intrinsic L-Lipschitz,* $\implies$ $f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}$ *is intrinsic L-Lipschitz.*

The geometric definition of intrinsic Lipschitz graphs has equivalent algebraic forms (see also [8], [29], [31]).

Proposition 3.1.6. *Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $L > 0$. Then* (i) *to* (iv) *are equivalent.*

(i)
$$f \text{ is intrinsic } L\text{-Lipschitz in } \mathcal{E}.$$

(ii)
$$\left\|\left(\bar{q}^{-1}q\right)_{\mathbb{H}}\right\| \le L \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|, \quad \text{for all } q, \bar{q} \in \operatorname{graph}(f).$$

(iii)
$$\left\|f_{\bar{q}^{-1}}(m)\right\| \equiv \left\|\left((f(\bar{m})m)_{\mathbb{H}}\right)^{-1} f\left(\bar{m}(f(\bar{m})m)_{\mathbb{M}}\right)\right\| \le L \left\|m\right\|,$$

for all $\bar{q} = \bar{m}f(\bar{m}) \in \operatorname{graph}(f)$ and for all $m \in \mathcal{E}_{\bar{q}^{-1}}$.

(iv)
$$\left\|\left(f(m_1)^{-1}m_2\right)_{\mathbb{H}} \cdot f(m_1m_2)\right\| \le L \left\|\left(f(m_1)^{-1}m_2\right)_{\mathbb{M}}\right\|, \quad \text{for all } m_1, m_2 \in \mathcal{E}.$$

Moreover, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz if and only if the distance of two points $q, \bar{q} \in \operatorname{graph}(f)$, is bounded by the norm of their projection on the domain $\mathbb{M}$. Precisely

(v)
$$\left\|\bar{q}^{-1}q\right\| \le c_0(1+L) \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\| \implies \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{H}}\right\| \le L \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|,$$

where $c_0 < 1$ is the constant in Proposition 2.2.11, and conversely

(vi)
$$\left\|\left(\bar{q}^{-1}q\right)_{\mathbb{H}}\right\| \le L \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\| \implies \left\|\bar{q}^{-1}q\right\| \le (1+L) \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|,$$

for all $q, \bar{q} \in \operatorname{graph}(f)$.

Proof. The equivalence between (i) and (ii) follows from definition (3.1.2), observing that if $\bar{q} \in \operatorname{graph}(f)$, then $C_{\mathbb{M},\mathbb{H}}(\bar{q}, 1/L) \cap \operatorname{graph}(f) = \{\bar{q}\}$ is equivalent with $C_{\mathbb{M},\mathbb{H}}(e, 1/L) \cap \operatorname{graph}(f_{\bar{q}^{-1}}) = \{e\}$. The equivalence of (ii) and (iii) follows once more from the definition of cone and left invariance of the definition. Indeed $(\bar{q}m)_{\mathbb{H}} = (\bar{m}f(\bar{m})m)_{\mathbb{H}} = (f(\bar{m})m)_{\mathbb{H}}$ and $(\bar{q}m)_{\mathbb{M}} = (\bar{m}f(\bar{m})m)_{\mathbb{M}} = \bar{m}(f(\bar{m})m)_{\mathbb{M}}$.
(iv) follows from (iii) changing variables. Let $m_2 := (f(\bar{m})m)_{\mathbb{M}}$ that is $m = (f(\bar{m})^{-1}m_2)_{\mathbb{M}}$ then

$$\begin{aligned}
\left(f(\bar{m})m\right)_{\mathbb{H}}^{-1} \cdot f\left(\bar{m}(f(\bar{m})m)_{\mathbb{M}}\right) &= \left(f(\bar{m})(f(\bar{m})^{-1}m_2)_{\mathbb{M}}\right)_{\mathbb{H}}^{-1} \cdot f\left(\bar{m}m_2\right) \\
&= \left(f(\bar{m})f(\bar{m})^{-1}m_2\left(f(\bar{m})^{-1}m_2\right)_{\mathbb{H}}^{-1}\right)_{\mathbb{H}}^{-1} \cdot f\left(\bar{m}m_2\right) \\
&= \left(f(\bar{m})^{-1}m_2\right)_{\mathbb{H}} \cdot f\left(\bar{m}m_2\right).
\end{aligned}$$

Then conclude with $\bar{m} = m_1$.
For the last two inequalities observe that, if $\left\|\bar{q}^{-1}q\right\| \le c_0(1+L) \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|$ then

$$\left\|\left(\bar{q}^{-1}q\right)_{\mathbb{H}}\right\| \le \frac{1}{c_0} \left\|\bar{q}^{-1}q\right\| - \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\| \le L \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|.$$

Conversely

$$\left\|\bar{q}^{-1}q\right\| \le \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\| + \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{H}}\right\| \le (1+L) \left\|\left(\bar{q}^{-1}q\right)_{\mathbb{M}}\right\|.$$

$\square$

Remark 3.1.7. If $\mathbb{G}$ is the semi-direct product of $\mathbb{M}$ and $\mathbb{H}$, (ii), (iii) and (iv) of Proposition 3.1.6 take a more explicit form. Indeed, recalling Remark 2.2.23, we get

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then f is intrinsic L-Lipschitz if and only if

$$\left\|f(\bar{m})^{-1}f(m)\right\| \le L \left\|f(\bar{m})^{-1}\bar{m}^{-1}mf(\bar{m})\right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then

$$((\bar{q}m)^{-1})_{\mathbb{H}} f\big((\bar{q}m)_{\mathbb{M}}\big) = m^{-1} f(\bar{m})^{-1} m f\big(\bar{m}m\big),$$

hence (iii) of Proposition 3.1.6 becomes

$$\left\| m^{-1} f(\bar{m})^{-1} m f\big(\bar{m}m\big) \right\| \le L \left\| m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

Hence, changing variables we get that f is intrinsic L-Lipschitz if and only if

$$\left\| m'^{-1} \bar{m} f(\bar{m})^{-1} \bar{m}^{-1} m' f\big(m'\big) \right\| \le L \left\| \bar{m}^{-1} m' \right\|, \qquad \text{for all } m', \bar{m} \in \mathcal{E}.$$

(iii) If $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ we get the well known expression for Lipschitz functions

$$\left\| f(\bar{m})^{-1} f(m) \right\| \le L \left\| \bar{m}^{-1} m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

Hence in this case intrinsic Lipschitz functions are the same as the usual metric Lipschitz functions from $(\mathbb{M}, d_\infty)$ to $(\mathbb{H}, d_\infty)$.

Remark 3.1.8. It is a natural question to ask if intrinsic Lipschitz functions are metric Lipschitz functions provided that appropriate choices of the metrics in the domain or in the target spaces are made. The answer to this question is almost always negative. Nevertheless something relevant can be stated.
Given $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, we consider the function $d_{\mathbb{M},f} = d_f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ defined as

$$d_f(m_1, m_2) := \frac{1}{2}\left(\left\| (q_1^{-1} q_2)_{\mathbb{M}} \right\| + \left\| (q_2^{-1} q_1)_{\mathbb{M}} \right\| \right), \qquad \text{for all } m_1, m_2 \in \mathcal{E}$$

where $q_i := m_i \cdot f(m_i) \in \operatorname{graph}(f)$.
If f is an intrinsic L-Lipschitz function then d_f is a quasi distance in $\mathcal{E}$. By quasi distance we mean that d_f is a distance with the weaker triangular inequality (32). Moreover the parametric function Φ_f,

$$\Phi_f : \big(\mathcal{E}, d_f\big) \to (\mathbb{G}, d_\infty), \qquad \Phi_f(m) := m \cdot f(m), \qquad \text{for all } m \in \mathcal{E}.$$

is a metric Lipschitz function. That is, Φ_f is a metric Lipschitz parametrization of graph (f) provided we endow $\mathcal{E}$ with the quasi distance d_f. We stress that in general it is impossible to find a *unique* quasi distance working for *all* the intrinsic Lipschitz functions. Notice that this is possible exactly when $\mathbb{H}$ is a normal subgroup.
Let us check that d_f is a quasi distance. Clearly d_f is symmetric and $m_1 = m_2$ yields $d_f(m_1, m_2) = 0$. About triangular inequality, observe that, for all $q_1, q_2, q_3 \in \operatorname{graph}(f)$,

$$\begin{aligned} c_0 \left\| (q_1^{-1} q_2)_{\mathbb{M}} \right\| &\le \left\| q_1^{-1} q_2 \right\| \le \left\| q_1^{-1} q_3 \right\| + \left\| q_3^{-1} q_2 \right\| \\ &\le \left\| (q_1^{-1} q_3)_{\mathbb{M}} \right\| + \left\| (q_1^{-1} q_3)_{\mathbb{H}} \right\| + \left\| (q_3^{-1} q_2)_{\mathbb{M}} \right\| + \left\| (q_3^{-1} q_2)_{\mathbb{H}} \right\| \end{aligned}$$

we use here that f is intrinsic Lipschitz to bound the $\mathbb{H}$ components,

$$\le (1 + L) \left\| (q_1^{-1} q_3)_{\mathbb{M}} \right\| + (1 + L) \left\| (q_3^{-1} q_2)_{\mathbb{M}} \right\|.$$

Hence eventually we get the weaker triangular inequality

$$d_f(m_1, m_2) \le ((1 + L)/c_0) \left(d_f(m_1, m_3) + d_f(m_3, m_2) \right), \qquad \text{for all } m_1, m_2, m_3 \in \mathcal{E}. \tag{32}$$

Finally, from the preceding computations, we have also that

$$\left\| q_1^{-1} q_2 \right\| \equiv \left\| \Phi_f(m_1)^{-1} \Phi_f(m_2) \right\| \le (1 + L) d_f(m_1, m_2), \qquad \text{for all } m_1, m_2 \in \mathcal{E}. \tag{33}$$

This completes the proof that Φ_f is a Lipschitz parametrization of graph (f).

As anticipated, we have a remarkable special case *when* $\mathbb{H}$ *is a normal subgroup*. Indeed, when $\mathbb{H}$ is a normal subgroup,

$$(q_1^{-1} q_2)_{\mathbb{M}} = \left(f(m_1)^{-1} m_1^{-1} m_2 f(m_2) \right)_{\mathbb{M}} = m_1^{-1} m_2$$

hence, the distance d_f is independent from f, precisely

$$d_f(m_1,m_2) = \left\|m_1^{-1}m_2\right\| = d_\infty(m_1,m_2).$$

and $\Phi_f : (\mathbb{M}, d_\infty) \to (\mathbb{G}, d_\infty)$ is a Lipschitz parametrization of graph (f).
Inversely, if $\left\|\Phi_f(\bar m)^{-1}\Phi_f(m)\right\| \le K \left\|\bar m^{-1}m\right\|$ then

$$c_0 \left\|(\Phi_f(\bar m)^{-1}\Phi_f(m))_{\mathbb{H}}\right\| \le K \left\|\bar m^{-1}m\right\| = K \left\|(\Phi_f(\bar m)^{-1}\Phi_f(m))_{\mathbb{M}}\right\|;$$

that is f is intrinsic Lipschitz by (ii) of Proposition 3.1.6. It is worth to state this fact as an independent Proposition.

Proposition 3.1.9. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$. Assume that $\mathbb{H}$ is a normal subgroup in $\mathbb{G}$. Then $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz in $\mathcal{E}$, if and only if $\Phi_f : (\mathcal{E}, d_\infty) \to (\mathbb{G}, d_\infty)$ is metric Lipschitz, that is if and only if there is $K > 1$ such that*

$$\left\|\Phi_f(\bar m)^{-1}\Phi_f(m)\right\| \le K \left\|\bar m^{-1}m\right\|, \quad \textit{for all } \bar m, m \in \mathcal{E}. \tag{34}$$

If $\mathbb{H}$ is not a normal subgroup Proposition 3.1.9 can be false: even if f is very regular, the 'natural' parametrization of graph (f) given by Φ_f may be non metric Lipschitz. Consider the following example in $\mathbb{H}^1 \equiv \mathbb{R}^3$.
Let $\mathbb{W}$, $\mathbb{V}$ be the complementary subgroups $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$ and $f : \mathbb{W} \to \mathbb{V}$ be the constant map defined by $f(w) = (1, 0, 0) \in \mathbb{V}$. Then graph (f) is a vertical plane in $\mathbb{R}^3$, parallel to $\mathbb{W}$. The 'natural' parametrization $\Phi_f : \mathbb{W} \to$ graph $(f) \subset \mathbb{H}^1$ acts as

$$\Phi_f(w) = (1, w_2, w_3 + w_2/2).$$

Then $\Phi_f(e) = (1, 0, 0)$ and, if $\bar w = (0, \varepsilon, 0)$, $\Phi_f(\bar w) = (1, \varepsilon, \varepsilon/2)$. Hence $\left\|\Phi_f(e)^{-1} \cdot \Phi_f(\bar w)\right\|$ is comparable with $\varepsilon^{1/2}$ while $\|\bar w\|$ is comparable with ε and (34) fails.

When $\mathbb{M}$ is a normal subgroup then not only Φ_f but f itself is a metric Lipschitz function from $(\mathbb{M}, d_f)$ to $(\mathbb{H}, d_\infty)$. Indeed, in this case,

$$(q_1^{-1}q_2)_{\mathbb{M}} = f(m_1)^{-1}m_1^{-1}m_2 f(m_1) \text{ and } (q_1^{-1}q_2)_{\mathbb{H}} = f(m_1)^{-1}f(m_2).$$

Hence, using (ii) of Proposition 3.1.6, we have

$$\left\|f(m_1)^{-1}f(m_2)\right\| = \left\|(q_1^{-1}q_2)_{\mathbb{H}}\right\| \le L \left\|(q_1^{-1}q_2)_{\mathbb{M}}\right\| = L \left\|f(m_1)^{-1}m_1^{-1}m_2 f(m_1)\right\|,$$

for all $m_1, m_2 \in \mathcal{E}$. Hence, we get the metric Lipschitz continuity of f, that is

$$\left\|f(m_1)^{-1}f(m_2)\right\| \le L\, d_f(m_1,m_2), \qquad \text{for all } m_1, m_2 \in \mathcal{E}.$$

Finally we notice that it is an open problem to understand if and when metric Lipschitz parameterizations of graph (f), different from the 'natural' one Φ_f, exist. This problem was addressed in [21] where the authors proved that, if S is a codimension 1 surface in $\mathbb{H}^n$ and if it is somehow more regular than just Lipschitz, then a metric Lipschitz parametrization of S exists (with a parameter space independent of S). On the contrary, D.Vittone in [10] proves that bi-Lipschitz parameterizations may not exist.

We conclude this subsection proving that intrinsic Lipschitz functions, even if non metric Lipschitz, nevertheless are metric Holder continuos.

Proposition 3.1.10. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in a step κ group $\mathbb{G}$. Let $L > 0$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ be an intrinsic L-Lipschitz function. Then*
(i) *f is bounded on bounded subsets of $\mathcal{E}$. Precisely, for all $R > 0$ there is $C_1 = C_1(\mathbb{M}, \mathbb{H}, f, R) > 0$ such that*

$$\|f(m)\| \le C_1, \qquad \textit{for all } m \in \mathcal{E} \textit{ such that } \|m\| \le R. \tag{35}$$

(ii) *f is $\frac{1}{\kappa}$-Holder continuous on bounded subset of $\mathcal{E}$. Precisely, for all $R > 0$, there is $C_2 = C_2(\mathbb{G}, \mathbb{M}, \mathbb{H}, C_1, L, R) > 0$ such that*

$$\left\| f(\bar{m})^{-1} f(m) \right\| \le C_2 \left\| \bar{m}^{-1} m \right\|^{1/\kappa}, \qquad \text{for all } m, \bar{m} \in \mathcal{E} \text{ with } \|m\|, \|\bar{m}\| \le R. \tag{36}$$

Proof. Fix $\bar{m} \in \mathcal{E}$ with $\|\bar{m}\| \le R$. From (ii) of Proposition 3.1.6 with $q = mf(m)$ and $\bar{q} = \bar{m} f(\bar{m})$, we have

$$\left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{H}} f(m) \right\| \le L \left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{M}} \right\|, \qquad \text{for all } m \in \mathcal{E}. \tag{37}$$

Now (35) follows from (37) using (19), triangle inequality and the limitations on $\|m\|$ and on $\|\bar{m}\|$.

To obtain the Holder estimate observe that, from Corollary 2.2.13 and (35), there is $C_3 = C_3(\mathbb{M}, \mathbb{H}, C_1, R) > 0$ such that

$$\left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{M}} \right\| \le C_3 \left\| \bar{m}^{-1} m \right\|^{1/\kappa}, \quad \text{for all } \bar{m}, m \in \mathcal{E},\ \|m\|, \|\bar{m}\| \le R. \tag{38}$$

Then, from (37),

$$\left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{H}} f(m) \right\| \le L C_3 \left\| \bar{m}^{-1} m \right\|^{1/\kappa}, \quad \text{for all } \bar{m}, m \in \mathcal{E},\ \|m\|, \|\bar{m}\| \le R. \tag{39}$$

From Lemma 2.2.12 and (35) we have

$$\left\| f(\bar{m})^{-1} m^{-1} \bar{m} f(\bar{m}) \right\| \le C_4 \left\| \bar{m}^{-1} m \right\|^{1/\kappa}; \tag{40}$$

from (19), (38) and (39)

$$\begin{aligned}
\left\| f(\bar{m})^{-1} \bar{m}^{-1} m f(m) \right\| &\le \left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m f(m) \right)_{\mathbb{M}} \right\| + \left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m f(m) \right)_{\mathbb{H}} \right\| \\
&= \left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{M}} \right\| + \left\| \left(f(\bar{m})^{-1} \bar{m}^{-1} m \right)_{\mathbb{H}} f(m) \right\| \\
&\le (1 + L) C_3 \left\| \bar{m}^{-1} m \right\|^{1/\kappa},
\end{aligned}$$

for all $\bar{m}, m \in \mathcal{E}$ with $\|m\|, \|\bar{m}\| \le R$. Finally, from the last one and from (40)

$$\left\| f(\bar{m})^{-1} f(m) \right\| \le \left\| f(\bar{m})^{-1} m^{-1} \bar{m} f(\bar{m}) \right\| + \left\| f(\bar{m})^{-1} \bar{m}^{-1} m f(m) \right\| \le C_2 \left\| \bar{m}^{-1} m \right\|^{1/\kappa},$$

for all $\bar{m}, m \in \mathcal{E}$ with $\|m\|, \|\bar{m}\| \le R$. □

3.2. **Surface measure of Lipschitz graphs.** If $\mathbb{M}$, $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic Lipschitz then the metric dimension of graph (f) is the same as the metric dimension of the domain $\mathcal{E}$. That is if s is this metric dimension of $\mathcal{E}$ then

$$\mathcal{S}_d^s \left(\text{graph}(f) \cap \mathcal{U} \right) < \infty,$$

for any bounded $\mathcal{U} \subset \mathbb{G}$. A non trivial corollary of this estimate is that 1-codimensional intrinsic Lipschitz graphs are boundaries of sets of locally finite $\mathbb{G}$-perimeter.

Notice that upper and lower bounds on the Hausdorff measure of a Lipschitz graph are trivially true in Euclidean spaces. Indeed if $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is Lipschitz then the map $\Phi_f : \mathbb{R}^k \to \mathbb{R}^n$ defined as $\Phi_f(x) := (x, f(x))$ is a Lipschitz parametrization of the Euclidean graph of f; this gives the upper bound on the dimension of the graph. On the other side, the projection $\mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ is Lipschitz continuous, with Lipschitz constant 1, yielding the lower bound. Such a proof cannot work here. From one side the projection $\mathbf{P}_{\mathbb{M}}$ or $\mathbf{P}_{\mathbb{H}}$ are not Lipschitz continuous, on the other side, as observed in Remark 3.1.8, the 'natural' parametrization of graph(f), given by $\Phi_f : \mathbb{M} \to \mathbb{G}$, $\Phi_f(m) := mf(m)$, is almost never a Lipschitz continuous map between the two metric spaces $\mathbb{M}$ and $\mathbb{G}$.

Theorem 3.2.1. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Let d_m denote the metric dimension of $\mathbb{M}$. If $f : \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz in $\mathbb{M}$ then there is $c = c(\mathbb{M}, \mathbb{H}) > 0$ such that,*

$$\left(\frac{c_0}{1+L}\right)^{d_m} R^{d_m} \le \mathcal{S}_d^{d_m}\big(\mathrm{graph}\,(f) \cap B(p,R)\big) \le c(1+L)^{d_m} R^{d_m}, \tag{41}$$

for all $p \in \mathrm{graph}\,(f)$ and $R > 0$, where c_0 is the structural constant in Proposition 2.2.11. In particular, $\mathrm{graph}\,(f)$ *has metric dimension d_m.*

Proof. The lower bound for $\mathcal{S}_d^{d_m}$ (graph(f)) is a consequence of Lemma 2.2.19. Indeed, assume $\mathcal{S}_d^{d_m}\big(\mathrm{graph}\,(f) \cap B(p,R)\big) < \infty$. Fix $\varepsilon > 0$, choose $r = r(\varepsilon) > 0$ and a covering of $\mathrm{graph}\,(f) \cap B(p,R)$ with closed balls $B_i = B(p_i, r_i)$ such that $r_i \le r$ and

$$\sum_i r_i^{d_m} \le \mathcal{S}_d^{d_m}\big(\mathrm{graph}\,(f) \cap B(p,R)\big) + \varepsilon. \tag{42}$$

Now observe that if $f : \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz and $p \in \mathrm{graph}\,(f)$ then

$$\mathbf{P}_{\mathbb{M}}\left(B\left(p, \frac{c_0 R}{1+L}\right)\right) \subset \mathbf{P}_{\mathbb{M}}\left(\mathrm{graph}\,(f) \cap B(p,R)\right), \text{ for all } R > 0. \tag{43}$$

Indeed, $m \in \mathbf{P}_{\mathbb{M}}\left(B(p, \frac{c_0 R}{1+L})\right)$ if and only if there is $h \in \mathbb{H}$ such that $\|p^{-1}mh\| \le \frac{c_0 R}{1+L}$. Hence, if $m \in \mathbf{P}_{\mathbb{M}}\left(B(p, \frac{c_0 R}{1+L})\right)$ then

$$c_0 \left\|\mathbf{P}_{\mathbb{M}}\left(p^{-1}m\right)\right\| = c_0 \left\|\mathbf{P}_{\mathbb{M}}\left(p^{-1}mh\right)\right\| \le \left\|p^{-1}mh\right\| \le \frac{c_0 R}{1+L}.$$

On the other side, using also the last inequality in Proposition 3.1.6,

$$\begin{aligned} \left\|p^{-1}mf(m)\right\| &\le (1+L)\left\|\mathbf{P}_{\mathbb{M}}\left(p^{-1}mf(m)\right)\right\| \\ &= (1+L)\left\|\mathbf{P}_{\mathbb{M}}\left(p^{-1}m\right)\right\| \\ &\le R. \end{aligned}$$

Hence $mf(m) \in \mathrm{graph}\,(f) \cap B(p,R)$ and $m \in \mathbf{P}_{\mathbb{M}}(\mathrm{graph}\,(f) \cap B(p,R))$ and (43) is proved.
Denoting as $d_t \le d_m$the topological dimension of $\mathbb{M}$, from (43) and Lemma 2.2.19 we conclude that for all $\varepsilon > 0$,

$$\begin{aligned} c\left(\frac{c_0}{1+L}\right)^{d_m} R^{d_m} &= \mathcal{L}^{d_t}\big(\mathbf{P}_{\mathbb{M}}(B(p, \frac{c_0 R}{1+L}))\big) \\ &\le \mathcal{L}^{d_t}\big(\mathbf{P}_{\mathbb{M}}(\mathrm{graph}\,(f) \cap B(p,R))\big) \\ &\le \sum_i \mathcal{L}^{d_t}(\mathbf{P}_{\mathbb{M}}(B_i)) \\ &= c\sum_i r_i^{d_m} \\ &\le c\,\mathcal{S}_d^{d_m}\big(\mathrm{graph}\,(f) \cap B(p,R)\big) + \varepsilon, \end{aligned}$$

and the left hand side of (41) follows from the arbitrarety of ε.

To prove the upper bound in (41), we show that there is $c = c(\mathbb{G}, \mathbb{M}, \mathbb{H}, R, L) > 0$ such that for any $p \in \mathrm{graph}\,(f)$, $R > 0$ and $\varepsilon > 0$, it is possible to cover $\mathrm{graph}\,(f) \cap B(p,R)$ with less than

$$N = c\,\varepsilon^{-d_m}$$

metric balls of radius not exceeding 5ε. Without loss of generality we assume $p = e$ and we fix ε, $0 < \varepsilon < 1$. By a Vitali covering argument we choose a family of metric balls $B(q_i, 5\varepsilon)$, such that

$$\operatorname{graph}(f) \cap B(e,R) \subset \bigcup_{i=1}^{N} B(q_i, 5\varepsilon), \quad q_i = m_i f(m_i) \in \operatorname{graph}(f), \quad i = 1, \dots, N,$$

and the concentric smaller balls $B_i := B(q_i, \varepsilon)$ are pairwise disjointed. We have to estimate the number N. With this purpose, recall the semi metric d_f defined in Remark 3.1.8 and observe that from (33)

$$2\varepsilon \leq \left\| q_i^{-1} q_j \right\| \leq (1+L)\, d_f(m_i, m_j), \qquad \text{for all } i \neq j.$$

Hence

$$\frac{2\varepsilon}{(1+L)} \leq d_f(m_i, m_j), \qquad \text{for all } m_i \neq m_j. \tag{44}$$

Denote

$$\mathcal{E}_i := \left\{ m \in \mathbb{M} : d_f(m, m_i) < \frac{c_0\, \varepsilon}{(1+L)^2} \right\}.$$

Because of (34) and (44), if $m \in \mathcal{E}_i \cap \mathcal{E}_j$, with $i \neq j$, then

$$\frac{2\varepsilon}{1+L} \leq d_f(m_i, m_j) \leq \frac{1+L}{c_0} \left(d_f(m_i, m) + d_f(m, m_j) \right) < \frac{2\varepsilon}{1+L},$$

a contradiction. Hence the sets $\mathcal{E}_i$ are pairwise disjointed

$$\mathcal{E}_i \cap \mathcal{E}_j = \emptyset, \qquad \text{for } i \neq j. \tag{45}$$

We want to estimate from below the $\mathcal{L}^{d_m}$ measure of the sets $\mathcal{E}_i$. To this purpose observe that, from Proposition 3.1.6 and for all $q_1, q_2 \in \operatorname{graph}(f)$,

$$c_0 \left\| \mathbf{P}_{\mathbb{M}} \left(q_1^{-1} q_2 \right) \right\| \leq \left\| q_1^{-1} q_2 \right\| \leq \left\| q_2^{-1} q_1 \right\| \leq (1+L) \left\| \mathbf{P}_{\mathbb{M}} \left(q_2^{-1} q_1 \right) \right\|;$$

hence

$$\left\| \mathbf{P}_{\mathbb{M}} \left(q_1^{-1} q_2 \right) \right\| \leq ((1+L)/c_0) \left\| \mathbf{P}_{\mathbb{M}} \left(q_2^{-1} q_1 \right) \right\|,$$

and finally

$$d_f(m_1, m_2) \leq \frac{1+L}{c_0} \left\| \mathbf{P}_{\mathbb{M}} \left(q_1^{-1} q_2 \right) \right\|, \quad \text{for all } q_1, q_2 \in \operatorname{graph}(f).$$

From the last inequality we have that,

$$\mathcal{E}_i \supseteq \left\{ m \in \mathbb{M} : \left\| \mathbf{P}_{\mathbb{M}} \left(q_i^{-1} q \right) \right\| \leq \frac{\varepsilon c_0^2}{(1+L)^3},\ q = m f(m) \right\}. \tag{46}$$

Observe that, for any $\delta > 0$,

$$\begin{aligned} \left\{ m \in \mathbb{M} : \left\| \mathbf{P}_{\mathbb{M}} \left(q_i^{-1} q \right) \right\| \leq \delta,\ q = m f(m) \right\} &= \mathbf{P}_{\mathbb{M}} \left(q_i^{-1} \left\{ w \in \mathbb{M} : \|w\| \leq \delta \right\} \right) \\ &= \mathbf{P}_{\mathbb{M}} \left(q_i^{-1} (B(e,\delta) \cap \mathbb{M}) \right). \end{aligned}$$

Moreover, the map $\mathbb{M} \to \mathbb{M}$ defined as $m \mapsto \mathbf{P}_{\mathbb{M}}(qm)$ for a fixed $q \in \mathbb{G}$ has unit Jacobian. Hence, recalling also Lemma 2.2.19,

$$\mathcal{L}^{d_t}(\mathcal{E}_i) \geq \mathcal{L}^{d_t} \left(B(e, \varepsilon c_0^2 (1+L)^{-3}) \right) = c\, \varepsilon^{d_m}.$$

Because $\mathcal{E}_i \subseteq \mathbf{P}_{\mathbb{M}}\big(B(e, R+1)\big)$ there is $c = c(\mathbb{G}, \mathbb{M}, \mathbb{H}, L)$ such that

$$N \leq c R^{d_m} \varepsilon^{-d_m}$$

and the proof is concluded $\square$

3.3. **Examples of intrinsic Lipschitz functions.** Pointwise limits of intrinsic Lipschitz functions are intrinsic Lipschitz.

Proposition 3.3.1. *Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$. Let $f, f_n : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, for $n = 1, 2, \dots$ and $L > 0$.*
(i) *If the f_n's are intrinsic L-Lipschitz in $\mathcal{E}$ and if f is the pointwise limit of f_n, then f is intrinsic L-Lipschitz.*
(ii) *If the f_n's are locally equibounded and intrinsic L-Lipschitz in $\mathcal{E}$, then $\{f_n : n = 1, 2, \dots\}$ is precompact with respect to uniform convergence in bounded subsets of $\mathcal{E}$.*

Proof. The first part follows from the characterization of intrinsic Lipschitz functions given in (v) of Proposition 3.1.6, together with the continuity of projection maps stated in Proposition 2.2.16. Proposition 3.1.10 yields that the f_n's are locally equicontinuous, hence precompactness follows from Arzelà Ascoli Theorem. □

In the following proposition we identify a large class of intrinsic Lipschitz functions. In order to motivate the introduction of these functions, we notice that when $\mathbb{N}$ is 1-dimensional then graph$\,(\phi_L)$ is the boundary of the 'positive part' of an intrinsic cone. That is, with notations that will be made precise in Section 4, we have

$$\text{graph}\,(\phi_L) = \partial C^+_{\mathbb{M},\mathbb{N}}(e, 1/L),$$

where $C^+_{\mathbb{M},\mathbb{N}}(e, 1/L)$ is the part of the cone $C_{\mathbb{M},\mathbb{N}}(e, 1/L)$ contained in the 'halfspace' $S^+_{\mathbb{G}}$, i.e.

$$C^+_{\mathbb{M},\mathbb{N}}(e, 1/L) := S^+_{\mathbb{G}} \cap C_{\mathbb{M},\mathbb{N}}(e, 1/L).$$

Differently rephrased, next Proposition states that the boundary of a 'positive ' intrinsic cone is an intrinsic Lipschitz graph. In turn, this result has a technical motivation in the proof of Theorem 4.1.1.

Proposition 3.3.2. *Let $\mathbb{M}$ and $\mathbb{N}$ be complementary subgroups of $\mathbb{G}$. Choose $L > 0$, $n \in \mathbb{N}$ with $\|n\| = 1$ and define $\phi_L \equiv \phi_{L,n} : \mathbb{M} \to \mathbb{N}$ as*

$$\phi_L(m) := \delta_{L\|m\|} n.$$

If $\mathbb{N}$ is horizontal, there is $L_1 = L_1(L, \mathbb{G}) \geq L$ such that ϕ_L is intrinsic L_1-Lipschitz.

The proof relies on the following two Lemmas. The estimates in them are in the spirit of the ones in Lemma 2.2.12 but they do not follow from them. Indeed here we estimate the Euclidean norms of the vector components of $h^{-1}gh$.

Lemma 3.3.3. *If $g, h \in \mathbb{G}$ then*

$$h^{-1}gh = g + \mathcal{P}(h, g),$$

where $\mathcal{P}(h, g) = \big(\mathcal{P}^1(h, g), \dots, \mathcal{P}^\kappa(h, g)\big)$, with $\mathcal{P}^1(h, g) = 0$ and, for $2 \leq j \leq \kappa$, $\mathcal{P}^j(h, g)$ are (vector valued) polynomial functions homogeneous of degree j. Moreover, if $\mathcal{B} \subset \mathbb{G}$ is a bounded set, there exists $C_{\mathcal{B}} = C_{\mathcal{B}}(\mathbb{G}) > 0$ such that, for $j = 2, \dots, \kappa$,

$$\left\|\mathcal{P}^j(h, g)\right\|_{n_j} \leq C_{\mathcal{B}} \left(\left\|g^1\right\|_{n_1} + \dots + \left\|g^{j-1}\right\|_{n_{j-1}} \right), \quad \textit{for all } h, g \in \mathcal{B}.$$

Proof. A direct computation gives

$$h^{-1}gh = g + \mathcal{Q}(g, h) + \mathcal{Q}(-h, g + h + \mathcal{Q}(g, h))$$

and we set

$$\mathcal{P}(h, g) := \mathcal{Q}(g, h) + \mathcal{Q}(-h, g + h + \mathcal{Q}(g, h)).$$

Hence $\mathcal{P}^1(h, g) = 0$ because $\mathcal{Q}^1(h, g) = 0$, for all h, g. To estimate $\mathcal{P}^j(h, g)$ for $2 \leq j \leq \kappa$ we use (4). The main point is to observe that, as a consequence of (4), $\mathcal{P}^j(h, g)$ is the sum of monomials each one containing a positive power of some g_i, for $1 \leq i \leq m_{j-1}$. Hence the thesis follows. □

Corollary 3.3.4. *There is a vector valued polynomial function $\tilde{\mathcal{P}} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}^n$ such that*

$$h^{-1}p^{-1}qh = q - p + \tilde{\mathcal{P}}(h, p^{-1}q), \qquad \text{for all } h, p, q \in \mathbb{G}.$$

Moreover, if $\mathcal{B} \subset \mathbb{G}$ is bounded, there is $C_{\mathcal{B}} = C_{\mathcal{B}}(\mathbb{G}) > 0$ such that, for $j = 2, \dots, \kappa$,

$$\left\| \tilde{\mathcal{P}}^j(h, p^{-1}q) \right\|_{n_j} \le C_{\mathcal{B}} \left(\left\| (q-p)^1 \right\|_{n_1} + \cdots + \left\| (q-p)^{j-1} \right\|_{n_{j-1}} \right), \quad \text{for all } h, p, q \in \mathcal{B}.$$

Proof. From Lemma 3.3.3

$$h^{-1}p^{-1}qh = p^{-1}q + \mathcal{P}(h, p^{-1}q) = q - p + \mathcal{Q}(-p, q) + \mathcal{P}(h, p^{-1}q).$$

Recalling (4), for all $1 \le i \le m_j$ we have homogeneous polynomials $\mathcal{R}^i_{\ell,n}$ such that

$$\begin{aligned} \mathcal{Q}_i(-p, q) &= \sum_{\ell,n=1}^{m_{j-1}} \mathcal{R}^i_{\ell,n}(-p, q) \left(-p_\ell q_n + p_n q_\ell \right) \\ &= \sum_{\ell,n=1}^{m_{j-1}} \mathcal{R}^i_{\ell,n}(-p, q) \left(q_n(q_\ell - p_\ell) - q_\ell(q_n + p_n) \right). \end{aligned}$$

Hence

$$\begin{aligned} \left\| \mathcal{Q}^j(-p, q) \right\|_{n_j} &\le c \left(|q_1 - p_1| + \cdots + |q_{m_{j-1}} - p_{m_{j-1}}| \right) \\ &\le c \left(\left\| (q-p)^1 \right\|_{n_1} + \cdots + \left\| (q-p)^{j-1} \right\|_{n_{j-1}} \right). \end{aligned}$$

Also $\mathcal{P}(h, p^{-1}q)$ can be estimated in the same way. Indeed, from Lemma 3.3.3 and the preceding inequalities

$$\begin{aligned} \left\| \mathcal{P}^j(h, p^{-1}q) \right\|_{n_j} &\le c_{\mathcal{B}} \left(\left\| (p^{-1}q)^1 \right\|_{n_1} + \cdots + \left\| (p^{-1}q)^{j-1} \right\|_{n_{j-1}} \right) \\ &\le c_{\mathcal{B}} \sum_{\ell=1}^{j-1} \left(\left\| (q-p)^\ell \right\|_{n_\ell} + \left\| \mathcal{Q}^\ell(-p, q) \right\|_{n_\ell} \right) \\ &\le \tilde{c}_{\mathcal{B}} \sum_{\ell=1}^{j-1} \left(\left\| (q-p)^\ell \right\|_{n_\ell} \right). \end{aligned}$$

□

Proof of Proposition 3.3.2. Because $\mathbb{N}$ is horizontal then $\mathbb{M}$ is a normal subgroup and, by (i) of Remark 3.1.7 we have to prove that

$$\left\| \phi_L(\bar{m})^{-1} \phi_L(m) \right\| \le L_1 \left\| \phi_L(\bar{m})^{-1} \bar{m}^{-1} m \, \phi_L(\bar{m}) \right\|, \qquad \text{for all } m, \bar{m} \in \mathbb{M}.$$

We assume without loss of generality that $L = 1$ and that $\|m\| \le \|\bar{m}\|$. Finally, by homogeneity, it is enough to prove that there is $L_1 > 0$ such that

$$\left\| \phi_L(\bar{m})^{-1} \phi_L(m) \right\| \le L_1 \left\| \phi_L(\bar{m})^{-1} \bar{m}^{-1} m \, \phi_L(\bar{m}) \right\|, \quad \text{for all } \|m\| \le \|\bar{m}\| = 1. \tag{47}$$

(Step I) Because $\mathbb{N}$ is horizontal it is abelian and it can be identified with a linear subspace of the first layer $\mathbb{R}^{n_1}$. The group operation in $\mathbb{N}$ is the usual Euclidean sum. Hence

$$\phi_L(\bar{m})^{-1} \phi_L(m) = \phi_L(m) - \phi_L(\bar{m}) = \delta_{\|m\|} n - \delta_{\|\bar{m}\|} n.$$

In the notation of (6), $n = (n^1, 0, \dots, 0)$ and $\delta_\alpha n = (\alpha n^1, 0, \dots, 0)$, for $\alpha > 0$. Then

$$\left\| \phi_L(\bar{m})^{-1} \phi_L(m) \right\| = \big| \, \|m\| - \|\bar{m}\| \, \big| \, \|n\| = \|\bar{m}\| - \|m\|, \tag{48}$$

because $\|n\| = 1$. Moreover, there is $c_1 = c_1(\mathbb{G}) > 1$, such that

$$\|\bar{m}\| - \|m\| \le c_1 \sum_{j=1}^{\kappa} \left\|(\bar{m} - m)^j\right\|_{n_j}, \quad \text{for all } \|m\| \le \|\bar{m}\| = 1. \tag{49}$$

Here $\|\cdot\|$ is the group norm and $\|\cdot\|_n$ or $\|\cdot\|_{n_j}$ are the Euclidean norms in $\mathbb{R}^n$ or $\mathbb{R}^{n_j}$. Now remember that $|t^p - s^p| \le p|t-s|$ for $0 \le s, t \le 1$ and $p \ge 1$. Hence if $\|m\| \le \|\bar{m}\| = 1$ and if Q is the homogeneous dimension of $\mathbb{G}$,

$$\begin{aligned}
\|\bar{m}\| - \|m\| \le \|\bar{m}\|^Q - \|m\|^Q &= \max_j \{\varepsilon_j^Q \left\|\bar{m}^j\right\|_{n_j}^{\frac{Q}{j}}\} - \max_j \{\varepsilon_j^Q \left\|m^j\right\|_{n_j}^{\frac{Q}{j}}\} \\
&\le \sum_{j=1}^{\kappa} \varepsilon_j^Q \left| \left\|\bar{m}^j\right\|_{n_j}^{\frac{Q}{j}} - \left\|m^j\right\|_{n_j}^{\frac{Q}{j}} \right| \\
&\le \sum_{j=1}^{\kappa} \frac{Q\varepsilon_j^j}{j} \left| \left\|\bar{m}^j\right\|_{n_j} - \left\|m^j\right\|_{n_j} \right| \\
&\le \sum_{j=1}^{\kappa} \frac{Q\varepsilon_j^j}{j} \left\|\bar{m}^j - m^j\right\|_{n_j} \\
&\le c_1 \sum_{j=1}^{\kappa} \left\|(\bar{m} - m)^j\right\|_{n_j},
\end{aligned}$$

and this proves (49).

(Step II) From Corollary 3.3.4, with $h = \phi_L(\bar{m})$ we have

$$\begin{aligned}
&\phi_L(\bar{m})^{-1}\bar{m}^{-1}m\,\phi_L(\bar{m}) = \\
&\qquad \left((\bar{m}-m)^1, (\bar{m}-m)^2 + \tilde{\mathcal{P}}^2(\phi_L(\bar{m}), \bar{m}^{-1}m), \dots, (\bar{m}-m)^\kappa + \tilde{\mathcal{P}}^\kappa(\phi_L(\bar{m}), \bar{m}^{-1}m)\right).
\end{aligned}$$

Hence, there is $c_2 = c_2(\mathbb{G}) > 0$ such that

$$\begin{aligned}
&\left\|\phi_L(\bar{m})^{-1}\bar{m}^{-1}m\,\phi_L(\bar{m})\right\| \\
&\quad \ge c_2 \Big(\left\|(\bar{m}-m)^1\right\|_{n_1} + \left\|(\bar{m}-m)^2 + \tilde{\mathcal{P}}^2(\phi_L(\bar{m}), \bar{m}^{-1}m)\right\|_{n_2} + \dots \\
&\qquad \dots + \left\|(\bar{m}-m)^k + \tilde{\mathcal{P}}^\kappa(\phi_L(\bar{m}), \bar{m}^{-1}m)\right\|_{n_\kappa} \Big) := I
\end{aligned} \tag{50}$$

because all the terms on the right hand side are bounded.
We want to prove that $\sum_{j=1}^{\kappa} \left\|(\bar{m}-m)^j\right\|_{n_j}$ is bounded from above by a constant times the right hand side I of (50).
Once more from Corollary 3.3.4, there is $c_3 = c_3(\mathbb{G}) > 1$ such that

$$\left\|\tilde{\mathcal{P}}^j(\phi_L(\bar{m}), \bar{m}^{-1}m)\right\|_{n_j} \le c_3 \left(\left\|(m-\bar{m})^1\right\|_{n_1} + \dots + \left\|(m-\bar{m})^{j-1}\right\|_{n_{j-1}} \right) \tag{51}$$

for $j = 2, \dots, \kappa$. Here c_3 is the constant $C_{\mathcal{B}}$ of Corollary 3.3.4 where $\mathcal{B}$ is the unit ball so containing $\phi_L(\bar{m})$, m and $\bar{m}$. Then we prove

$$\begin{aligned}
&\left\|(\bar{m}-m)^1\right\|_{n_1} \le I, \\
&\left\|(\bar{m}-m)^j\right\|_{n_j} \le 2c_3(1+2c_3)^{j-2} I, \qquad \text{for } 2 \le j \le \kappa.
\end{aligned} \tag{52}$$

The first line is clear. To obtain also the second one we argument by induction on j. Fix ℓ with $2 \le \ell \le \kappa$ and assume that (52) is true for $1 \le j < \ell$. Observe that if

$$\left\|(\bar m - m)^\ell + \tilde{\mathcal{P}}^\ell(\phi_L(\bar m), \bar m^{-1} m)\right\|_{n_\ell} \ge \frac{1}{2}\left\|(\bar m - m)^\ell\right\|_{n_\ell}$$

then (52) follows with $j = \ell$. On the contrary, if

$$\left\|(\bar m - m)^\ell + \tilde{\mathcal{P}}^\ell(\phi_L(\bar m), \bar m^{-1} m)\right\|_{n_\ell} < \frac{1}{2}\left\|(\bar m - m)^\ell\right\|_{n_\ell}$$

then

$$\left\|\tilde{\mathcal{P}}^\ell(\phi_L(\bar m), \bar m^{-1} m)\right\|_{n_\ell} > \frac{1}{2}\left\|(\bar m - m)^\ell\right\|_{n_\ell}$$

hence, by (51) and by the induction assumption,

$$\begin{aligned}
\left\|(\bar m - m)^\ell\right\|_{n_\ell} &\le 2c_3\left(\left\|(\bar m - m)^1\right\|_{n_1} + \cdots + \left\|(\bar m - m)^{\ell-1}\right\|_{n_{\ell-1}}\right) \le cI \\
&\le 2c_3\left(1 + 2c_3 + \dots 2c_3(1+2c_3)^{\ell-3}\right) I \\
&= 2c_3(1+2c_3)^{\ell-2}\, I.
\end{aligned}$$

Eventually from (52) we get

$$\sum_{j=1}^{\kappa} \left\|(\bar m - m)^j\right\|_{n_j} \le c_4\, I$$

where $c_4 = (1+2c_3)^\kappa$.

Now the Proposition is proved with $L_1 = \dfrac{c_1 c_4}{c_2}$. □

3.4. **Intrinsic Differentiable Functions.** A function $f : \mathbb{M} \to \mathbb{H}$, acting between complementary subgroups of $\mathbb{G}$, is *intrinsic differentiable* in a point $m \in \mathbb{M}$ if the graph of f has a tangent homogeneous subgroup in $mf(m) \in \operatorname{graph}(f)$ (see Definition 3.4.11). This notion can be stated also in terms of the existence of an approximating intrinsic linear function. Intrinsic linear functions, acting between complementary subgroups, are those functions whose graphs are homogeneous subgroups. We begin with this second approach.

Definition 3.4.1. Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{M} \to \mathbb{H}$ is an *intrinsic linear function* if ℓ is defined on all of $\mathbb{M}$ and if $\operatorname{graph}(\ell) = \{m\,\ell(m) : m \in \mathbb{M}\}$ is an homogeneous subgroup of $\mathbb{G}$.

Before using intrinsic linear functions in the definition of intrinsic differentiability of functions we begin collecting some of their properties.

Proposition 3.4.2. *Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$.*
(i) *If $\ell : \mathbb{M} \to \mathbb{H}$ is* intrinsic linear *then the homogeneous subgroups* $\operatorname{graph}(\ell)$ *and $\mathbb{H}$ are complementary subgroups and $\mathbb{G} = \operatorname{graph}(\ell) \cdot \mathbb{H}$.*
(ii) *If $\mathbb{N}$ is an homogeneous subgroup such that $\mathbb{N}$ and $\mathbb{H}$ are complementary in $\mathbb{G}$ then there is a unique intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ such that $\mathbb{N} = \operatorname{graph}(\ell)$.*

Proof. (i): Observe that for all $g \in \mathbb{G}$ we have $g = g_{\mathbb{M}} g_{\mathbb{H}} = g_{\mathbb{M}}\ell(g_{\mathbb{M}})\ell(g_{\mathbb{M}})^{-1} g_{\mathbb{H}}$, and $g_{\mathbb{M}}\ell(g_{\mathbb{M}}) \in \operatorname{graph}(\ell)$ while $\ell(g_{\mathbb{M}})^{-1} g_{\mathbb{H}} \in \mathbb{H}$. On the other side $\operatorname{graph}(\ell) \cap \mathbb{H} = e$. Indeed, if $g = m\ell(m) \in \mathbb{H}$ then $m = 0$ and $g = \ell(0)$. By (i) of Proposition 3.4.3 it follows $\ell(0) = 0$, hence $g = 0$.
(ii): We have to prove that $\mathbb{N}$ is a graph over $\mathbb{M}$ in direction $\mathbb{H}$. That is we have to prove that each coset of $\mathbb{H}$ intersects $\mathbb{N}$ in at most one point. Indeed, if there are $m \in \mathbb{M}$ and $h_1, h_2 \in \mathbb{H}$ such that $mh_1 \in \mathbb{N}$ and $mh_2 \in \mathbb{N}$, then $h_2^{-1}h_1 = h_2^{-1}m^{-1}mh_1 \in \mathbb{N} \cap \mathbb{H}$ hence $h_1 = h_2$ and $mh_1 = mh_2$. This shows that $\mathbb{N}$ is a graph over $\mathbb{M}$. To complete the proof we have to show that each coset of $\mathbb{H}$

intersects $\mathbb{N}$. Indeed, for each $m \in \mathbb{M}$, by the assumption that $\mathbb{N}$, $\mathbb{H}$ are complementary it follows that $m = m_{\mathbb{N}} m_{\mathbb{H}}$. Hence $m m_{\mathbb{H}}^{-1} = m_{\mathbb{N}} \in m \cdot \mathbb{H} \cap \mathbb{N}$. Finally the function having $\mathbb{N}$ as graph, is intrinsic linear by definition. $\square$

Intrinsic linear functions are not necessarily group homomorphism between their domains and codomains, as the following example shows.
Let $\mathbb{V}$, $\mathbb{W}$ be the complementary subgroups of $\mathbb{H}^1$ defined as $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$. For any fixed $a \in \mathbb{R}$, the function $\ell : \mathbb{V} \to \mathbb{W}$ defined as

$$\ell(v) = (0, av_1, -av_1^2/2)$$

is intrinsic linear because $\operatorname{graph}(\ell) = \{(t, at, 0) : t \in \mathbb{R}\}$ is a 1-dimensional homogeneous subgroup of $\mathbb{H}^1$. This ℓ is not a group homomorphism from $\mathbb{V}$ to $\mathbb{W}$.

Intrinsic linear functions can be algebraically characterized as follows.

Proposition 3.4.3. *Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{M} \to \mathbb{H}$ is an* intrinsic linear function *if and only if*

(i) $$\ell(\delta_\lambda m) = \delta_\lambda(\ell(m)), \qquad \textit{for all } m \in \mathbb{M} \textit{ and } \lambda \in \mathbb{R};$$

(ii) $$\ell(m_1 m_2) = \left(\ell(m_1)^{-1} m_2\right)_{\mathbb{H}}^{-1} \ell\left(\left(\ell(m_1)^{-1} m_2\right)_{\mathbb{M}}\right), \qquad \textit{for all } m_1, m_2 \in \mathbb{M}.$$

Proof. Because $\operatorname{graph}(\ell)$ is an homogeneous subgroup, for each $m \in \mathbb{M}$ there is $\bar{m} \in \mathbb{M}$ such that $\delta_\lambda (m\ell(m)) = \bar{m}\ell(\bar{m})$; hence

$$\delta_\lambda m \, \delta_\lambda (\ell(m)) = \bar{m}\ell(\bar{m})$$

and by uniqueness of the components

$$\bar{m} = \delta_\lambda m \quad \text{and } \ell(\delta_\lambda m) = \ell(\bar{m}) = \delta_\lambda (\ell(m)) .$$

Because $\operatorname{graph}(\ell)$ is a subgroup, for all $m_1, m_2 \in \mathbb{M}$ there is $\bar{m}$ such that

$$m_1 \ell(m_1) m_2 \ell(m_2) = \bar{m}\ell(\bar{m}).$$

Hence

$$\bar{m} = (m_1 \ell(m_1) m_2 \ell(m_2))_{\mathbb{M}} = m_1 \left(\ell(m_1) m_2\right)_{\mathbb{M}},$$
$$\ell(\bar{m}) = (m_1 \ell(m_1) m_2 \ell(m_2))_{\mathbb{H}} = \left(\ell(m_1) m_2\right)_{\mathbb{H}} \ell(m_2).$$

This way we obtained

(53) $$\ell\left(m_1 \left(\ell(m_1) m_2\right)_{\mathbb{M}}\right) = \left(\ell(m_1) m_2\right)_{\mathbb{H}} \ell(m_2), \quad \text{for all } m_1, m_2 \in \mathbb{M}.$$

To get a more explicit expression we change variables. To do this, first we observe that for each couple $m \in \mathbb{M}$ and $h \in \mathbb{H}$ there is exactly one $\bar{m} \in \mathbb{M}$ such that

(54) $$m = (h\bar{m})_{\mathbb{M}}$$

and $\bar{m}$ can be explicitly defined as

(55) $$\bar{m} = \left(h^{-1} m\right)_{\mathbb{M}}.$$

Indeed

$$\left(h \left(h^{-1} m\right)_{\mathbb{M}}\right)_{\mathbb{M}} = \left(h h^{-1} m \left(h^{-1} m\right)_{\mathbb{H}}^{-1}\right)_{\mathbb{M}} = \left(m \left(h^{-1} m\right)_{\mathbb{H}}^{-1}\right)_{\mathbb{M}} = m.$$

To prove that the choice of $\bar{m}$ in (54) is unique observe that

(56) $$(h\, m_1)_{\mathbb{M}} = (h\, m_2)_{\mathbb{M}} \implies m_1 = m_2$$

for all $m_1, m_2 \in \mathbb{M}$, $h \in \mathbb{H}$. Indeed

$$h\, m_1 = (h\, m_1)_{\mathbb{M}} (h\, m_1)_{\mathbb{H}} = (h\, m_2)_{\mathbb{M}} (h\, m_1)_{\mathbb{H}} = h\, m_2 \, (h\, m_2)_{\mathbb{H}}^{-1} (h\, m_1)_{\mathbb{H}}$$

hence

$$m_1 = m_2 \, (h\, m_2)_{\mathbb{H}}^{-1} (h\, m_1)_{\mathbb{H}}$$

that gives
$$m_2^{-1}\, m_1 = (h\, m_2)_{\mathbb{H}}^{-1}\, (h\, m_1)_{\mathbb{H}} \in \mathbb{H}$$
and finally $m_2^{-1}\, m_1 = 0$ because $\mathbb{M}$ and $\mathbb{H}$ are complementary. This completes the proof of (54) and (55).
Using (54) and (55), for all $m_1, m_3 \in \mathbb{M}$ we define
$$m_2 := \left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}}.$$
so that
$$m_3 = (\ell(m_1) m_2)_{\mathbb{M}}$$
and we substitute inside (53) to get
$$\begin{aligned}
\ell(m_1 m_3) &= \left(\ell(m_1)\left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}}\right)_{\mathbb{H}} \ell(\left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}}) \\
&= \left(\ell(m_1)\ell(m_1)^{-1} m_3 \left(\ell(m_1)^{-1} m_3\right)_{\mathbb{H}}^{-1}\right)_{\mathbb{H}} \ell(\left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}}) \\
&= \left(m_3 \left(\ell(m_1)^{-1} m_3\right)_{\mathbb{H}}^{-1}\right)_{\mathbb{H}} \ell(\left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}}) \\
&= \left(\ell(m_1)^{-1} m_3\right)_{\mathbb{H}}^{-1} \ell(\left(\ell(m_1)^{-1} m_3\right)_{\mathbb{M}})
\end{aligned}$$
for all $m_1, m_3 \in \mathbb{M}$. □

Corollary 3.4.4. *Let $\mathbb{G}$ be the semidirect product of the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$. Let $\ell : \mathbb{M} \to \mathbb{H}$ be intrinsic linear. Then* (ii) *of Proposition 3.4.3 takes the following form:*
$$\begin{aligned}
&\textit{if } \mathbb{M} \textit{ is normal in } \mathbb{G}\textit{:} \quad \ell(m_1 m_2) = \ell(m_1)\, \ell\left(\ell(m_1)^{-1}\, m_2\, \ell(m_1)\right), \\
&\textit{if } \mathbb{H} \textit{ is normal in } \mathbb{G}\textit{:} \quad \ell(m_1 m_2) = m_2^{-1}\, \ell(m_1)\, m_2\, \ell(m_2), \\
&\textit{if } \mathbb{M} \textit{ and } \mathbb{H} \textit{ are normal:} \quad \ell(m_1 m_2) = \ell(m_1) \cdot \ell(m_2),
\end{aligned}$$
for all $m_1, m_2 \in \mathbb{M}$. Hence, when $\mathbb{G}$ is the direct product of $\mathbb{M}$ and $\mathbb{H}$ an intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ is an homogeneous homomorphism from $\mathbb{M}$ to $\mathbb{H}$.

Proof. If $\mathbb{M}$ is normal then $(hm)_{\mathbb{M}} = hmh^{-1}$ and $(hm)_{\mathbb{H}} = h$. Hence the first one is proved. If $\mathbb{H}$ is normal then $(hm)_{\mathbb{M}} = m$ and $(hm)_{\mathbb{H}} = m^{-1}hm$ and we get the second one. Finally the last one follows from the first two. □

Intrinsic linear functions are intrinsic Lipschitz.

Proposition 3.4.5. *Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $\ell : \mathbb{M} \to \mathbb{H}$ be intrinsic linear, then*

(i) *ℓ is a polynomial function;*

(ii) *ℓ is intrinsic Lipschitz continuous.*

Proof. (i): Let $\mathbf{P}_1 : \mathbb{M} \to \operatorname{graph}(\ell)$ be the restriction to $\mathbb{M}$ of the projection on the first component, related with the decomposition $\mathbb{G} = \operatorname{graph}(\ell) \cdot \mathbb{H}$. Then
$$\mathbf{P}_1(m) = m\ell(m), \qquad \text{for all } m \in \mathbb{M}.$$
Let $\mathbf{P}_2 : \operatorname{graph}(\ell) \to \mathbb{H}$ be the restriction to $\operatorname{graph}(\ell)$ of the projection on the second component, related with the decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. Then
$$\mathbf{P}_2(m\ell(m)) = \ell(m), \qquad \text{for all } m\ell(m) \in \operatorname{graph}(\ell).$$
Then $\ell : \mathbb{M} \to \mathbb{H}$ is $\ell = \mathbf{P}_2 \circ \mathbf{P}_2$. By Proposition 2.2.16 both $\mathbf{P}_1$ and $\mathbf{P}_2$ are polynomial maps, hence also ℓ is a polynomial map.

(ii): We use characterization (iii) of Proposition 3.1.6. We have to prove that there is $L > 0$ such that, for all m_1 and $m_2 \in \mathbb{M}$,

$$\left\| \left(\ell(m_1)^{-1} m_2\right)_{\mathbb{H}} \cdot \ell(m_1 m_2)\right\| \le L \left\| \left(\ell(m_1)^{-1} m_2\right)_{\mathbb{M}}\right\| .$$

Let $L := \max\{\|\ell(m)\| : \|m\| = 1\}$. L is finite because ℓ is continuous. From (i) of Proposition 3.4.3 we have also that $\|\ell(m)\| \le L \|m\|$, for all $m \in \mathbb{M}$. Then, because

$$\left(\ell(m_1)^{-1} m_2\right)_{\mathbb{H}} \cdot \ell(m_1 m_2) = \ell\left((\ell(m_1)^{-1} m_2)_{\mathbb{M}}\right)$$

we have

$$\left\| \left(\ell(m_1)^{-1} m_2\right)_{\mathbb{H}} \cdot \ell(m_1 m_2)\right\| = \left\|\ell\left((\ell(m_1)^{-1} m_2)_{\mathbb{M}}\right)\right\| \le L \left\|(\ell(m_1)^{-1} m_2)_{\mathbb{M}}\right\|, \quad \text{for all } m_1, m_2 \in \mathbb{M},$$

and the proof is completed. □

We use intrinsic linear functions to define intrinsic differentiability in a way that is formally completely similar to the usual definition of differentiability. First we define differentiability, at the origin, for a function whose graph contains the origin, then we extend the definition in a translation invariant way.

Definition 3.4.6. Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$. For $\bar m \in \mathcal{A}$ and $\bar p := \bar m \cdot f(\bar m) \in \operatorname{graph}(f)$ let $f_{\bar p^{-1}} : \mathcal{A}_{\bar p^{-1}} \subset \mathbb{M} \to \mathbb{H}$ be the translated function. We say that f is *intrinsic differentiable in* $\bar m \in \mathcal{A}$ if $f_{\bar p^{-1}}$ is *intrinsic differentiable in* e that is if there is an intrinsic linear map $df = df_{\bar m} : \mathbb{M} \to \mathbb{H}$ such that, for all $m \in \mathcal{A}_{\bar p^{-1}}$,

$$\text{(i)} \qquad \lim_{\|m\| \to 0} \frac{\left\|df_{\bar m}(m)^{-1} \cdot f_{\bar p^{-1}}(m)\right\|}{\|m\|} = 0.$$

The intrinsic linear map $df_{\bar m}$ is called the *intrinsic differential* of f.
Writing explicitly the expression of $f_{\bar p^{-1}}$ in (i), we can equivalently state that f is *intrinsic differentiable in* $\bar m \in \mathcal{A}$ if

$$\text{(ii)} \qquad \left\|df_{\bar m}(m)^{-1} \cdot (f(\bar m) m)_{\mathbb{H}}^{-1} \cdot f\left(\bar m (f(\bar m) m)_{\mathbb{M}}\right)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0,$$

or, changing variables, if

$$\text{(iii)} \qquad \left\|\left(df_{\bar m}((f(\bar m)^{-1} m)_{\mathbb{M}})\right)^{-1} \cdot (f(\bar m)^{-1} m)_{\mathbb{H}} \cdot f(\bar m m)\right\| = o(\left\|(f(\bar m)^{-1} m)_{\mathbb{M}}\right\|),$$

as $\left\|(f(\bar m)^{-1} m)_{\mathbb{M}}\right\| \to 0$, or finally if

$$\text{(iv)} \qquad \left\|df_{\bar m}\left((f(\bar m)^{-1} \bar m^{-1} m)_{\mathbb{M}}\right)^{-1} \cdot (f(\bar m)^{-1} \bar m^{-1} m)_{\mathbb{H}} \cdot f(m)\right\| = o(\left\|(f(\bar m)^{-1} \bar m^{-1} m)_{\mathbb{M}}\right\|),$$

as $\left\|(f(\bar m)^{-1} \bar m^{-1} m)_{\mathbb{M}}\right\| \to 0$. Here $o(t)$ is such that $\lim_{t \to 0^+} o(t)/t = 0$.

Remark 3.4.7. If a function is intrinsic differentiable it keeps being intrinsic differentiable after a left translation of the graph. Precisely, let $q_1 = m_1 f(m_1)$ and $q_2 = m_2 f(m_2) \in \operatorname{graph}(f)$, then f is intrinsic differentiable in m_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv (f_{q_1^{-1}})_{q_2}$ is intrinsic differentiable in m_2. In particular, f is intrinsic differentiable in m_1 if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e.

Proposition 3.4.8. *Let* $\mathbb{M}$, $\mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *and* $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ *with* $\mathcal{A}$ *relatively open in* $\mathbb{M}$. *If* f *is intrinsic differentiable in* $m \in \mathcal{A}$, *then* f *is continuous in* m.

Proof. As observed in Remark 2.2.22, it is enough to prove the continuity of $f_{p_1^{-1}}$ at the origin. This last fact is an immediate consequence of Definition 3.4.6 (i) and of the continuity of intrinsic linear functions. □

Remark 3.4.9. Once more we write explicitly the form taken by (i) and (ii) of Definition 3.4.6 when $\mathbb{G}$ is the semidirect product of $\mathbb{M}$ and $\mathbb{H}$.
If $\mathbb{M}$ is a normal subgroup then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar{m} \in \mathbb{M}$ if

$$\left\| df_{\bar{m}}(m)^{-1} \cdot f(\bar{m})^{-1} \cdot f\left(\bar{m} f(\bar{m}) m f(\bar{m})^{-1}\right)\right\| = o(\|m\|),$$

as $\|m\| \to 0$ or, changing variables, if

$$\left\| df_{\bar{m}}\left(f(\bar{m})^{-1}\bar{m}^{-1} m f(\bar{m})\right)^{-1} \cdot f(\bar{m})^{-1} \cdot f(m)\right\| = o(\left\|f(\bar{m})^{-1}\bar{m}^{-1} m f(\bar{m})\right\|),$$

as $\left\|f(\bar{m})^{-1}\bar{m}^{-1} m f(\bar{m})\right\| \to 0$.
If $\mathbb{H}$ is a normal subgroup then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar{m} \in \mathbb{M}$ if

$$\left\| df_{\bar{m}}(m)^{-1} \cdot m \cdot f(\bar{m})^{-1} \cdot m^{-1} \cdot f(\bar{m} m)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0,$$

or, changing variables, if

$$\left\| df_{\bar{m}}\left(\bar{m}^{-1} m\right)^{-1} \cdot (\bar{m}^{-1} m) \cdot f(\bar{m})^{-1} \cdot (\bar{m}^{-1} m)^{-1} \cdot f(m)\right\| = o(\left\|\bar{m}^{-1} m\right\|),$$

as $\left\|\bar{m}^{-1} m\right\| \to 0$.
If both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $f : \mathbb{M} \to \mathbb{H}$ is differentiable in $\bar{m} \in \mathbb{M}$ if

$$\left\| df_{\bar{m}}(m)^{-1} \cdot f(\bar{m})^{-1} \cdot f(\bar{m} m)\right\| = o(\|m\|), \quad \text{as } \|m\| \to 0.$$

Remark 3.4.10. Pierre Pansu introduced in [45] a notion of differentiability for maps between nilpotent groups, the differential being an approximating homogeneous homomorphisms. More precisely, a function f, acting between two nilpotent groups $\mathbb{G}_1$ and $\mathbb{G}_2$, is Pansu differentiable in $\bar{g} \in \mathbb{G}_1$ if there is an *homogeneous homomorphism* $h : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\left\| h(\bar{g}^{-1} g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g)\right\|_{\mathbb{G}_2} = o(\left\|\bar{g}^{-1} g\right\|_{\mathbb{G}_1}), \qquad \text{as } \left\|\bar{g}^{-1} g\right\| \to 0.$$

We remark here that Pansu differentiability and intrinsic differentiability, when both of them make sense, are in general *different notions*.
Indeed, let $\mathbb{V}$, $\mathbb{W}$ be the complementary subgroups of $\mathbb{H}^1$ defined as $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$. As observed before, an intrinsic linear function $\ell : \mathbb{V} \to \mathbb{W}$ is of the form

$$\ell(v) = (0, av_1, -av_1^2/2), \quad \text{for any fixed } a \in \mathbb{R}.$$

An homogeneous homomorhism $h : \mathbb{V} \to \mathbb{W}$ is of the form

$$h(v) = (0, av_1, 0), \quad \text{for any fixed } a \in \mathbb{R}.$$

Obviously, ℓ is intrinsic differentiable in $v = 0$ while h is Pansu differentiable in $v = 0$. On the other side, it is easy to check that neither ℓ is Pansu differentiable nor h is intrinsic differentiable in $v = 0$.

We remark also that Pansu differentiability is not conserved after graph translations. Indeed, consider once more the function $h : \mathbb{V} \to \mathbb{W}$, then $\text{graph}\,(h) = \{(t, at, at^2/2) : \ t \in \mathbb{R}\}$. Let $p := (0, p_2, 0) \in \mathbb{H}^1$, $p \neq 0$. Then $p \cdot \text{graph}\,(h) = \{(t, at + p_2, (at^2 - p_2 t)/2) : \ t \in \mathbb{R}\}$ is the graph of the function $h_p : \mathbb{V} \to \mathbb{W}$ defined as $h_p(v) = (0, av_1 + p_2, -p_2 v_1)$. It is easy to check that h_p is not Pansu differentiable in $v = 0$.

Finally, if $\mathbb{G}$ is the *direct product* of $\mathbb{M}$ and $\mathbb{H}$ it is easy to convince oneself that

$$f : \mathbb{M} \to \mathbb{H} \text{ is Pansu differentiable} \iff f \text{ is intrinsic differentiable.}$$

Finally we observe that intrinsic differentiability is equivalent to the existence of a tangent subgroup to the graph. We begin with the definition of tangent subgroup.

Definition 3.4.11. Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$ and let $\mathbb{T}$ be an homogeneous subgroup in $\mathbb{G}$. Let $m \in \mathcal{A}$ and $p = mf(m) \in \operatorname{graph}(f)$ we say that $p \cdot \mathbb{T}$ is a *tangent (affine) group* or *tangent coset* to $\operatorname{graph}(f)$ in p if for all $\varepsilon > 0$ there is $\lambda = \lambda(\varepsilon) > 0$ such that

$$\operatorname{graph}(f) \cap U(p, \lambda) \subset X(p, \mathbb{T}, \varepsilon).$$

Remark 3.4.12. When $p = e$ sometimes we speak of *tangent subgroup* to $\operatorname{graph}(f)$ in e. Notice that $p \cdot \mathbb{T}$ is the tangent coset to $\operatorname{graph}(f)$ in p if and only if $\mathbb{T}$ is the tangent subgroup to $\operatorname{graph}(f_{p^{-1}})$ at e.

Proposition 3.4.13. *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ with $\mathcal{A}$ relatively open in $\mathbb{M}$.*
(I) If f is intrinsic differentiable in $m \in \mathcal{A}$, set $\mathbb{T} := \operatorname{graph}(df_m)$. Then
(1) *$\mathbb{T}$ is an homogeneous subgroup of $\mathbb{G}$;*
(2) *$\mathbb{T}$ and $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$;*
(3) *$p \cdot \mathbb{T}$ is the tangent coset to $\operatorname{graph}(f)$ in $p := mf(m)$,*

(II) Conversely, if $p := mf(m) \in \operatorname{graph}(f)$ and if there is $\mathbb{T}$ such that (1),(2), (3) *hold, then f is intrinsic differentiable in m and the differential $df_m : \mathbb{M} \to \mathbb{H}$ is the unique intrinsic linear function such that $\mathbb{T} = \operatorname{graph}(df_m)$.*

Proof. By Remark 3.4.7 and Remark 3.4.12 we can assume without loss of generality that $m = 0$ and $f(0) = 0$.
Let us prove (I). Observe that (1) and (2) follow from the definition of intrinsic linear functions and from Proposition 3.4.2.
From (i) of Definition 3.4.6, for all $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\left\| df_0(m)^{-1} \cdot f(m) \right\| \le \varepsilon \|m\|, \qquad \text{for } \|m\| \le \delta.$$

On the other hand, because $\mathbb{T} = \operatorname{graph}(df_0)$, for all $\varepsilon > 0$ and for all $mf(m) \in \operatorname{graph}(f)$ such that $\|mf(m)\| < \delta$,

$$\begin{aligned} \operatorname{dist}(mf(m), \mathbb{T}) &= \inf\{ \left\| df_0(w)^{-1} w^{-1} mf(m) \right\| : w \in \mathbb{M} \} \\ &\le \left\| df_0(m)^{-1} f(m) \right\| \le \varepsilon \|m\| \\ &\le \varepsilon \|mf(m)\|, \end{aligned}$$

that is $mf(m) \in X(e, \mathbb{T}, \varepsilon)$.
Proof of (II).
From Lemma 3.4.2, there is an intrinsic linear function $\ell : \mathbb{M} \to \mathbb{H}$ such that $\mathbb{T} = \operatorname{graph}(\ell)$. We have to prove that $\ell = df_0$, that is, for all $\varepsilon > 0$ there is $\lambda = \lambda(\varepsilon) > 0$ such that

$$\left\| \ell(m)^{-1} f(m) \right\| \le \varepsilon \|m\|, \quad \text{if } \|m\| < \lambda(\varepsilon). \tag{57}$$

The assumption that $\mathbb{T}$ is the tangent coset to $\operatorname{graph}(f)$ in $p = 0$ tells that for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\operatorname{dist}(mf(m), \mathbb{T}) \le \varepsilon \|mf(m)\|, \quad \text{if } \|mf(m)\| < \delta(\varepsilon). \tag{58}$$

Observe that $mf(m) = m\ell(m)\ell(m)^{-1}f(m)$, hence $m\ell(m)$ and $\ell(m)^{-1}f(m)$ are, respectively, the components along $\mathbb{T}$ and $\mathbb{H}$ of $mf(m)$, in the decomposition $\mathbb{G} = \mathbb{T} \cdot \mathbb{H}$. Therefore, from Corollary 2.2.14 and (58),

$$c_0 \left\| \ell(m)^{-1} f(m) \right\| \le \operatorname{dist}(mf(m), \mathbb{T}) \le \varepsilon, \quad \text{if } \|mf(m)\| < \delta(\varepsilon). \tag{59}$$

To obtain (57) from (59) it is enough to observe that there is $c = c(\mathbb{T}) > 0$ such that $\|f(m)\| \le c\,\|m\|$. Indeed, denoting by c_ℓ the Lipschitz constant of ℓ,

$$\begin{aligned}\|f(m)\| &\le \|\ell(m)\| + \left\|\ell(m)^{-1} f(m)\right\| \\ &\le c_\ell \|m\| + \varepsilon \|m f(m)\| \le (c_\ell + \varepsilon)\|m\| + \varepsilon \|f(m)\|.\end{aligned}$$

Hence $\|f(m)\| \le \frac{c_\ell+\varepsilon}{1-\varepsilon}\|m\|$ and the proof is concluded. □

4. One Codimensional Intrinsic Graphs

Through all this section $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$, where as usual $\mathbb{M}$ and $\mathbb{N}$ are complementary homogeneous subgroups, but here we assume also that $\mathbb{N}$ is *one dimensional* and (consequently) *horizontal*. Precisely we assume the existence of $V \in \mathfrak{g}_1$ such that $\mathbb{N} = \{\exp(tV),\, t \in \mathbb{R}\}$. We notice that, under these assumptions, $\mathbb{M}$ is always a normal subgroup.
Since $\mathbb{N} = \{\exp tV\}$, $\mathbb{N}$ can be identified with $\mathbb{R}$ so that it carries an order and we can define the supremum and the infimum of families of $\mathbb{N}$-valued functions. If $f_\beta : \mathbb{M} \to \mathbb{N}$ for $\beta \in B$ with $f_\beta(m) = \exp(\varphi_\beta(m)V)$ and $\varphi_\beta : \mathbb{M} \to \mathbb{R}$, we define $\inf_{\beta\in B} f_\beta : \mathbb{M} \to \mathbb{N}$ as

$$\inf_{\beta\in B} f_\beta(m) := \exp\big(\inf_{\beta\in B} \varphi_\beta(m)V\big), \qquad \text{for all } m \in \mathbb{M} \text{ s.t. } \inf_{\beta\in B} \varphi_\beta(m) \text{ is finite.}$$

Analogously we define $\sup_{\beta\in B} f_\beta$, $\max\{f_{\beta_1}, f_{\beta_2}\}$, $\min\{f_{\beta_1}, f_{\beta_2}\}$, etc.

4.1. Extension of intrinsic Lipschitz functions.

The main result proved in this first subsection is the following one.

Theorem 4.1.1. *Let $\mathbb{M}$ and $\mathbb{N}$ be complementary subgroups with $\mathbb{N}$ one dimensional. Let $\mathcal{B} \subset \mathbb{M}$ be a Borel subset of $\mathbb{M}$ and $f : \mathcal{B} \to \mathbb{N}$, be an intrinsic L-Lipschitz function. Then there are $\tilde f : \mathbb{M} \to \mathbb{N}$ and $\tilde L = \tilde L(L, \mathbb{G}, \mathbb{M}, \mathbb{N}) \ge L$ such that*

(i) $\qquad \tilde f$ *is intrinsic $\tilde L$-Lipschitz in $\mathbb{M}$,*

(ii) $\qquad \tilde f(m) = f(m)$, *for all $m \in \mathcal{B}$.*

We need a few lemmas and notations in order to prove Theorem 4.1.1.
Denote by $S^+_{\mathbb{G}}(\mathbb{M},\mathbb{N})$ and $S^-_{\mathbb{G}}(\mathbb{M},\mathbb{N})$ two complementary 'halfspaces' where

$$S^+_{\mathbb{G}}(\mathbb{M},\mathbb{N}) = \{g : g_{\mathbb{N}} = \exp(tV), \text{with } t \ge 0\}$$

and $S^-_{\mathbb{G}}(\mathbb{M},\mathbb{N})$ is the analogous one with $t \le 0$.
If $C_{\mathbb{M},\mathbb{N}}(p,\alpha)$ is an intrinsic cone with one dimensional axis $\mathbb{N}$, we denote

$$C^\pm_{\mathbb{M},\mathbb{N}}(e,\alpha) := C_{\mathbb{M},\mathbb{N}}(e,\alpha) \cap S^\pm_{\mathbb{G}}(\mathbb{M},\mathbb{N})$$

and $C^\pm_{\mathbb{M},\mathbb{N}}(p,\alpha) := p \cdot C^\pm_{\mathbb{M},\mathbb{N}}(e,\alpha)$. Notice also that

$$\partial C^+_{\mathbb{M},\mathbb{N}}(e, 1/L) = \operatorname{graph}(\phi_L), \tag{60}$$

where the functions $\phi_L : \mathbb{M} \to \mathbb{N}$, defined as $\phi_L(m) = \delta_{L\|m\|}(\exp V) = \exp(L\|m\|V)$, are the functions considered in Propositioon 3.3.2..

Lemma 4.1.2. *For each $\alpha > 0$ there is $\alpha_1 = \alpha_1(\alpha, \mathbb{G}, \mathbb{M}, \mathbb{N})$, $0 < \alpha_1 \le \alpha$, such that*

$$C^+_{\mathbb{M},\mathbb{N}}(gn, \alpha_1) \subset C^+_{\mathbb{M},\mathbb{N}}(g,\alpha), \qquad \textit{for all } g \in \mathbb{G} \textit{ and } n = \exp(tV) \in \mathbb{N}, \textit{ with } t > 0,$$
$$C^-_{\mathbb{M},\mathbb{N}}(gn, \alpha_1) \subset C^-_{\mathbb{M},\mathbb{N}}(g,\alpha), \qquad \textit{for all } g \in \mathbb{G} \textit{ and } n = \exp(tV) \in \mathbb{N}, \textit{ with } t < 0.$$

Proof. By left translation invariance, it is enough to prove that

$$C^+_{\mathbb{M},\mathbb{N}}(n,\alpha_1) \subset C^+_{\mathbb{M},\mathbb{N}}(e,\alpha), \qquad \text{for all } n = \exp(tV) \in \mathbb{N}, \text{ with } t > 0.$$

Let $p = p_{\mathbb{M}} p_{\mathbb{N}} \in C^+_{\mathbb{M},\mathbb{N}}(e,\alpha_1)$, we have to prove that $np \in C^+_{\mathbb{M},\mathbb{N}}(e,\alpha)$. Because $\mathbb{M}$ is a normal subgroup, $(np)_{\mathbb{M}} = np_{\mathbb{M}}n^{-1}$ and $(np)_{\mathbb{N}} = np_{\mathbb{N}}$.
Because $\dim \mathbb{N} = 1$ and $\mathbb{N}$ is horizontal,

$$\|(np)_{\mathbb{N}}\| = \|p_{\mathbb{N}}\| + \|n\|.$$

Indeed, if $n = \exp \theta V$ and $\bar n = \exp \bar\theta V$, then $n\bar n = \exp\big((\theta+\bar\theta)V\big)$ and $\|n\bar n\| = |\theta+\bar\theta| = |\theta| + |\bar\theta|$ when θ and $\bar\theta$ are positive.
By Lemma 2.2.12 there is $C = C(\mathbb{G}) > 0$ such that

$$\|(np)_{\mathbb{M}}\| = \left\|np_{\mathbb{M}}n^{-1}\right\| \le \|p_{\mathbb{M}}\| + C\left(\|n\|^{1/\kappa}\|p_{\mathbb{M}}\|^{(\kappa-1)/\kappa} + \|n\|^{(\kappa-1)/\kappa}\|p_{\mathbb{M}}\|^{1/\kappa}\right).$$

For $\kappa \ge 2$ there is $c = c(\kappa) \ge 1$ such that $s^{1/\kappa}t^{(\kappa-1)/\kappa} + t^{1/\kappa}s^{(\kappa-1)/\kappa} \le \varepsilon t + c\varepsilon^{1-\kappa}s$, for all $s,t \ge 0$ and for all $\varepsilon > 0$. Hence we have

$$\|(np)_{\mathbb{M}}\| \le (1 + Cc\varepsilon^{1-\kappa})\|p_{\mathbb{M}}\| + C\varepsilon\|n\| \le \alpha_1(1 + Cc\varepsilon^{1-\kappa})\|p_{\mathbb{N}}\| + C\varepsilon\|n\|,$$

because $p \in C^+_{\mathbb{M},\mathbb{N}}(e,\alpha_1)$. Now choose $\varepsilon = \alpha/C$ and $\alpha_1 = \alpha/(1+Cc\varepsilon^{1-\kappa})$ to get

$$\|(np)_{\mathbb{M}}\| \le \alpha\|p_{\mathbb{N}}\| + \alpha\|n\| = \alpha\|(np)_{\mathbb{N}}\|$$

that shows that $np \in C^+_{\mathbb{M},\mathbb{N}}(e,\alpha)$ and completes the proof for the "positive" cones. The case of the "negative" cones is completely analogous. □

We can characterize $\mathbb{N}$ valued intrinsic Lipschitz functions using the fact that subgraphs and supergraphs contain half cones. Precisely, for $f : \mathcal{U} \subset \mathbb{M} \to \mathbb{N}$, with $f(m) = \exp(\varphi(m)V)$ and $\varphi : \mathcal{U} \to \mathbb{R}$, we define the *supergraph* E_f^+ and the *subgraph* E_f^- of f as

$$E_f^- := \{m\exp(tV) :\ m \in \mathcal{U},\, t < \varphi(m)\}, \quad E_f^+ := \{m\exp(tV) :\ m \in \mathcal{U},\, t > \varphi(m)\}.$$

Notice that, if $f : \mathbb{M} \to \mathbb{N}$ is continuous,

$$\overline{E_f^-} = \{m\exp(tV) :\ m \in \mathbb{M},\, t \le \varphi(m)\}, \quad \overline{E_f^+} = \{m\exp(tV) :\ m \in \mathbb{M},\, t \ge \varphi(m)\}.$$

Lemma 4.1.3. *$f : \mathcal{U} \subseteq \mathbb{M} \to \mathbb{N}$ is intrinsic L-Lipschitz in $\mathcal{U}$ if and only if*

$$C^+_{\mathbb{M},\mathbb{N}}(mf(m),1/L) \subset \overline{E_f^+} \quad \text{and} \quad C^-_{\mathbb{M},\mathbb{N}}(mf(m),1/L) \subset \overline{E_f^-}, \qquad \text{for all } m \in \mathbb{M}. \tag{61}$$

Proof. Let $f(m) := \exp\big(\varphi(m)V\big)$, where $\varphi : \mathbb{M} \to \mathbb{R}$. If f is intrinsic L-Lipschitz, for all $\alpha < 1/L$,

$$C_{\mathbb{M},\mathbb{N}}(mf(m),\alpha) \cap \operatorname{graph}(f) = \{mf(m)\}, \qquad \text{for all } m \in \mathbb{M}. \tag{62}$$

Assume by contradiction that there is $\bar m \in \mathbb{M}$ and $\bar t > \varphi(\bar m) \in \mathbb{R}$ such that

$$\bar m \cdot \exp(\bar t V) \in C^+_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha) \cap E_f^-;$$

but then $\bar m \cdot \exp(tV) \in C^+_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha)$, for all $t \ge \bar t$, in particular

$$\bar m \cdot \exp(\varphi(\bar m)V) \in C^+_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha)$$

contraddicting (62).
On the contrary, if (61) is true, then for all $0 < \alpha < L$,

$$\begin{aligned} C_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha) &= C^-_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha) \cup C^+_{\mathbb{M},\mathbb{N}}(\bar m f(\bar m),\alpha) \\ &\subset E_f^- \cup E_f^+ \cup \{mf(m)\}, \end{aligned}$$

proving that f is intrinsic L-Lipschitz. □

Proposition 4.1.4. *Assume that $f_\beta : \mathbb{M} \to \mathbb{N}$ for $\beta \in B$ is a family of intrinsic L-Lipschitz functions. For all $L > 0$ there is $\tilde{L} = \tilde{L}(L, \mathbb{G}, \mathbb{M}, \mathbb{N}) \geq L$ such that, if*

$$f := \inf_{\beta \in B} f_\beta,$$

then either $f \equiv -\infty$ or f is defined on all of $\mathbb{M}$ and it is intrinsic $\tilde{L}$-Lipschitz.

Proof. Clearly,

$$E_f^- = \bigcap_\beta E_{f_\beta}^-, \quad E_f^+ = \bigcup_\beta E_{f_\mathfrak{g}}^+.$$

Assume there is $\bar{m} \in \mathbb{M}$ such that $f(\bar{m}) \in \mathbb{N}$ (i.e. the infimum is not $-\infty$ in at least one point). Then, from Lemmas 4.1.2 and 4.1.3, for all $\alpha < 1/L$

$$C_{\mathbb{M},\mathbb{N}}^-(\bar{m} f(\bar{m}), \alpha_1) \subset C_{\mathbb{M},\mathbb{N}}^-(\bar{m} f_\beta(\bar{m}), \alpha) \subset E_{f_\beta}^-.$$

Hence

$$C_{\mathbb{M},\mathbb{N}}^-(\bar{m} f(\bar{m}), \alpha_1) \subset \bigcap_\beta E_{f_\beta}^- = E_f^-.$$

It follows, in particular, that $f(m) \in \mathbb{N}$ for all $m \in \mathbb{M}$ and consequently we can repeat the preceding argument for all $m \in \mathbb{M}$ obtaining

$$C_{\mathbb{M},\mathbb{N}}^-(m f(m), \alpha_1) \subset E_f^-, \qquad \text{for all } m \in \mathbb{M}.$$

On the other side

$$C_{\mathbb{M},\mathbb{N}}^+(m f(m), \alpha) \subset E_{f_\beta}^+, \qquad \text{for all } m \in \mathbb{M} \text{ and } \beta \in B,$$

so that

$$C_{\mathbb{M},\mathbb{N}}^+(m f(m), \alpha) \subset E_f^+, \qquad \text{for all } m \in \mathbb{M}.$$

The thesis then follows from Lemma 4.1.3. □

Proof of Theorem 4.1.1. For each $\bar{m} \in \mathcal{B}$ let $\phi_{L,\bar{m}} : \mathbb{M} \to \mathbb{N}$ be the translated function

$$\phi_{L,\bar{m}}(m) := f(\bar{m}) \phi_L \left(f(\bar{m})^{-1} \bar{m}^{-1} m f(\bar{m}) \right),$$

where ϕ_L was introduced in Proposition 3.3.2. By Proposition 2.2.21 and Remark 2.2.23,

$$\operatorname{graph}(\phi_{L,\bar{m}}) = \bar{m} f(\bar{m}) \operatorname{graph}(\phi_L).$$

Hence, from (60),

$$\operatorname{graph}(\phi_{L,\bar{m}}) = \partial C_{\mathbb{M},\mathbb{N}}^+(\bar{m} f(\bar{m}), 1/L). \tag{63}$$

Moreover, from Proposition 3.3.2 we know that ϕ_L is intrinsic L_1-Lipschitz, hence all the translated functions $\phi_{L,\bar{m}}$ are intrinsic L_1-Lipschitz.
Define $\theta_{\bar{m}} : \mathbb{M} \to \mathbb{R}$ and $\varphi : \mathcal{B} \to \mathbb{R}$ such that

$$\phi_{L,\bar{m}}(m) = \exp(\theta_{\bar{m}}(m) V) \text{ and } f(m) = \exp(\varphi(m) V).$$

Since $\phi_{L,\bar{m}}(\bar{m}) = f(\bar{m})$, we have

$$\theta_{\bar{m}}(\bar{m}) = \varphi(\bar{m}), \qquad \text{for all } \bar{m} \in \mathcal{B}. \tag{64}$$

We define $\tilde{f} : \mathbb{M} \to \mathbb{N}$ as

$$\tilde{f}(m) := \inf_{\bar{m} \in \mathcal{B}} \phi_{L,\bar{m}}(m), \qquad \text{for all } m \in \mathbb{M}.$$

We want to show now that $\tilde{f}(m) = f(m)$ for all $m \in \mathcal{B}$. Given (64), it is enough to show

$$\theta_{\bar{m}}(m) \geq \varphi(m), \qquad \text{for all } m, \bar{m} \in \mathcal{B}. \tag{65}$$

Now, because f is intrinsic L-Lipschitz, keeping in mind (64),

$$|\varphi(m) - \theta_{\bar m}(\bar m)| = |\varphi(m) - \varphi(\bar m)| = \left\| f(\bar m)^{-1} f(m)\right\| \le L \left\| f(\bar m)^{-1}\bar m^{-1} m f(\bar m)\right\| . \tag{66}$$

On the other hand, from (63), $\theta_{\bar m}(m) \ge \theta_{\bar m}(\bar m)$ and

$$\begin{aligned}\theta_{\bar m}(m) - \theta_{\bar m}(\bar m) &= \left\|\phi_{L,\bar m}(\bar m)^{-1}\phi_{L,\bar m}(m)\right\| = L \left\|\phi_{L,\bar m}(\bar m)^{-1}\bar m^{-1} m\, \phi_{L,\bar m}(\bar m)\right\| \\ &= L \left\| f(\bar m)^{-1}\bar m^{-1} m\, f(\bar m)\right\| .\end{aligned} \tag{67}$$

Combining (66) and (67) we get (65).
Finally we apply Proposition 4.1.4 with $L = L_1$ to get that $\tilde f$ is intrinsic $\tilde L$-Lipschitz for $\tilde L := \tilde L(L_1, \mathbb{G}, \mathbb{M}, \mathbb{N})$. $\square$

4.2. **Approximate tangent cosets and intrinsic differentiability.** We keep the assumptions that $\mathbb{G} = \mathbb{M}\cdot\mathbb{N}$, where $\mathbb{N} = \{\exp tV : t \in \mathbb{R},\ V \in \mathfrak{g}_1\}$ is a *one-dimensional*, *horizontal* homogeneous subgroup.

Here we prove that intrinsic differentiability and the weaker property of existence of an *approximate tangent coset* (see Definition 4.2.1) are equivalent notions when dealing with intrinsic Lipschitz graphs. Notice that the notion of approximate tangent coset is in the spirit of De Giorgi's blow-up results for finite perimeter sets (see [23] in Euclidean spaces, [26] for Heisenberg groups and Theorem 4.4.3 here).

Definition 4.2.1. Let $\mathcal{O}$ be relatively open in $\mathbb{M}$ and $f : \mathcal{O} \to \mathbb{N}$. We say that f is *approximately intrinsic differentiable* in $m \in \mathcal{O}$ if graph (f) has an *approximate (affine) tangent group* or *approximate coset* in $p = mf(m)$, that is if there is an homogeneous subgroup $\mathbb{T}$ such that

(i) $\mathbb{T}$ and $\mathbb{N}$ are complementary subgroups in $\mathbb{G}$,

(ii) $$\lim_{r\to 0} \mathbf{1}_{(E_f^-)_{r,p}} = \mathbf{1}_{S^-_{\mathbb{G}}(\mathbb{T},\mathbb{N})} \qquad \text{in } \mathcal{L}^1_{\mathrm{loc}}(\mathbb{G}),$$

where

$$(E_f^-)_{r,p} = \{q : p\cdot\delta_r q \in E_f^-\} = \delta_{\frac{1}{r}}\left(p^{-1}\cdot E_f^-\right), \qquad \text{for } r > 0.$$

Proposition 4.2.2. *Let $\mathbb{M}$ and $\mathbb{N}$ be complementary subgroups with $\mathbb{N}$ one dimensional and horizontal. Let $f : \mathbb{M} \to \mathbb{N}$ be an intrinsic L-Lipschitz function. Then, for all $m \in \mathbb{M}$, f is approximately intrinsic differentiable in m if and only if f is intrinsic differentiable in m.*

Proof. From one side, it is easy to see that differentiability yields approximate differentiability. On the other side, let $p := mf(m)$. As usual, replacing f by $f_{p^{-1}}$, we can assume that $m = f(m) = e$. Now the proof relies on the following two lemmata.

Lemma 4.2.3. *Let $f : \mathbb{M} \to \mathbb{N}$ be an intrinsic L-Lipschitz function with $e \in$ graph (f) and let $\mathbb{T}$ be the approximate tangent group to* graph (f) *at e. Denote as $\ell : \mathbb{M} \to \mathbb{N}$ the intrinsic linear function such that $\mathbb{T} =$ graph (ℓ) (see (ii) of Proposition 3.4.2). Define*

$$f_{1/r} := \delta_{\frac{1}{r}} \circ f \circ \delta_r, \qquad \textit{for } r > 0.$$

Then

(i) $$f_{1/r} \to \ell, \qquad \textit{as } r \to 0, \textit{ uniformly on compact sets.}$$

Clearly, eventually we have that $\ell = df_e$.

Proof. We point first that

$$\delta_{1/r}(E_f^-) = E^-_{f_{1/r}} \quad \text{and} \quad \mathrm{graph}\,(f_{1/r}) = \delta_{1/r}(\mathrm{graph}\,(f)). \tag{68}$$

For $R > 0$, the functions $\{f_{1/r} : 0 < r < 1\}$ are equibounded in $U(e,R) \cap \mathbb{M}$. Indeed

$$\|f_{1/r}(m)\| = \frac{1}{r}\|f(\delta_r m)\| \leq \frac{L}{r}\|\delta_r m\| = L\|m\| \leq LR, \quad \text{for } m \in U(e,R) \cap \mathbb{M}.$$

Moreover, the functions $\{f_{1/r} : 0 < r < 1\}$ are also intrinsic L-Lipschitz and, as proved in Proposition 3.1.10, are consequently equicontinuous in $U(e,R) \cap \mathbb{M}$. By Arzelà–Ascoli theorem and a standard diagonal argument, we obtain the existence of a function $\ell : \mathbb{M} \to \mathbb{N}$ and of a sequence $(r_k)_{k\in\mathbb{N}}$, $r_k \downarrow 0$ as $k \to \infty$, such that $f_{1/r_k} \to \ell$ as $k \to \infty$, uniformly on compact sets. We prove now that $\text{graph}\,(\ell) = \mathbb{T}$. Once this is proved, then ℓ is by definition an intrinsic linear function and it is uniquely determined by the very definition of intrinsic graph and we can get rid of the sequence as usual, achieving the proof of statement (i).

We show first that

$$\mathbb{T} \subset \text{graph}\,(\ell). \tag{69}$$

To this end, take a point $\bar{p} \notin \text{graph}\,(\ell)$, then $\bar{p} \in E_\ell^- \cup E_\ell^+$. For instance suppose $\bar{p} \in E_\ell^-$ and let us prove that $\mathbf{1}_{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})}(\bar{p}) = 1$ that is $\bar{p} \in S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})$. Indeed, take a ball $B := B(\bar{p},\rho)$ such that

$$\text{dist}\,(B, \text{graph}\,(\ell)) > 0.$$

Because of the uniform convergence of the f_{1/r_k}'s on compact sets, we can assume that $B \subset E^-_{f_{1/r_k}}$ for k large enough, so that $\mathbf{1}_{E^-_{f_{1/r_k}}}(g) = 1$ for $g \in B$ and k large. By (68), this implies that $\mathbf{1}_{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})} = 1$ a.e. in B. But this implies that $\mathbf{1}_{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})} \equiv 1$ on B and hence that $\mathbf{1}_{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})}(\bar{p}) = 1$. The case $\bar{p} \in E_\ell^+ \setminus \text{graph}\,(\ell)$ can be handled in the same way. This proves (69). To prove the reversed inclusion, notice first that a point $\bar{p} \in \text{graph}\,(\ell)$ is both the limit of a sequence p_n in E_ℓ^- and of a sequence q_n in E_ℓ^+. Indeed, if $\bar{p} = \bar{m} \cdot \ell(\bar{m}) = \bar{m} \cdot \exp(\phi_\infty(\bar{m})V)$, it is enough to choose

$$p_n = \bar{m} \cdot \exp\left((\phi(\bar{m}) - 1/n)V\right) \quad \text{and} \quad q_n = \bar{m} \cdot \exp\left((\phi(\bar{m}) + 1/n)V\right).$$

On the other hand, we have just shown that $E_\ell \subset S_{\mathbb{G}}^+(\mathbb{T},\mathbb{N})$ and $E_\ell^c \setminus \text{graph}\,(\ell) \subset S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})$, so that $\bar{p} \in \overline{S_{\mathbb{G}}^-(\mathbb{T},\mathbb{N})} \cap \overline{S_{\mathbb{G}}^+(\mathbb{T},\mathbb{N})} = \mathbb{T}$. This achieves the proof of statements (i) and (ii).

Finally, (iii) follows from Proposition 3.4.2. □

The final step consists of the following lemma.

Lemma 4.2.4. *Let $f : \mathbb{M} \to \mathbb{N}$ be an intrinsic L-Lipschitz function with $e \in \text{graph}\,(f)$ and let $\mathbb{T}$ be the approximate tangent group to $\text{graph}\,(f)$ at e. Then, for any $\alpha > 0$ there exists $\delta = \delta(\alpha) > 0$ such that*

$$\text{graph}\,(f) \cap U(e,\delta) \cap C_{\mathbb{T},\mathbb{N}}(e,\alpha) = \{e\}. \tag{70}$$

Proof. By contradiction, suppose (70) fails, then there exists a sequence of points $p_n := m_n \cdot f(m_n) \in \text{graph}\,(f) \cap C_{\mathbb{T},\mathbb{N}}(e,\alpha)$, $p_n \neq e$ but converging to e as $n \to \infty$. Set $\xi_n := \delta_{1/\|m_n\|}(m_n)$. Without loss of generality, we may assume $\xi_n \to \xi_0$ as $n \to \infty$, with $\|\xi_0\| = 1$. Then we have

$$\delta_{1/\|m_n\|}(p_n) = \xi_n \cdot \delta_{1/\|m_n\|}\big(f(\delta_{\|m_n\|}(\xi_n))\big) = \xi_n \cdot f_{1/\|m_n\|}(\xi_n).$$

Hence

$$\delta_{1/\|m_n\|}(p_n) \in C_{\mathbb{T},\mathbb{N}}(e,\alpha) \cap \text{graph}\,(f_{1/\|m_n\|}).$$

By Lemma 4.2.3, this yields that

$$\xi_0 \cdot \ell(\xi_0) \in C_{\mathbb{T},\mathbb{N}}(e,\alpha) \cap \text{graph}\,(\ell) = C_{\mathbb{T},\mathbb{N}}(e,\alpha) \cap \mathbb{T} = \{e\},$$

This implies that $\xi_0 = \{e\}$, contradicting the fact that $\|\xi_0\| = 1$, and the lemma is proved. □

Let us go back to the proof of Proposition 4.2.2. From (70) it follows

$$\operatorname{graph}(f) \cap U(e,\delta) \subset C_{\mathbb{N},\mathbb{T}}(e,\gamma), \qquad \text{for all } 0<\gamma<1/\alpha.$$

Hence, by Proposition 3.1.3, for all $\beta > 0$ there is $\delta = \delta(\beta) > 0$ such that

$$\operatorname{graph}(f) \cap U(e,\delta) \subset X(e,\mathbb{T},\beta).$$

that is equivalent to the differentiability of f in e by Proposition 3.4.13. The proof of Proposition 4.2.2 is completed. □

4.3. **Finite perimeter sets and intrinsic Lipschitz graphs.** We keep the assumptions that $\mathbb{G} = \mathbb{M} \cdot \mathbb{N}$, where $\mathbb{N} = \{\exp tV : t \in \mathbb{R},\ V \in \mathfrak{g}_1\}$ is 1-*dimensional* and *horizontal* homogeneous subgroup.

The local boundedness of the Hausdorff measure of intrinsic Lipschitz graphs, proved in Theorem 3.2.1, yields that 1-codimensional graphs of intrinsic Lipschitz functions are locally the boundary of sets with locally finite $\mathbb{G}$-perimeter (see Theorem 4.3.8). We recall a few definitions and results related to the notion of perimeter of sets in $\mathbb{G}$. For more details and proofs, see [33], [26] and [52].

If $\Omega \subset \mathbb{G}$ is open, the space of compactly supported smooth sections of $H\mathbb{G}$ is denoted by $\mathbf{C}_0^\infty(\Omega, H\mathbb{G})$. If $k \in \mathbb{N}$, $\mathbf{C}_0^k(\Omega, H\mathbb{G})$ is defined similarly.

Definition 4.3.1. We say that $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ is of *bounded variation* in Ω if $f \in L^1(\Omega)$ and if

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup \left\{ \int_\Omega f(p) \mathrm{div}_{\mathbb{G}} \phi(p)\, dp : \phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}),\ |\phi(p)|_p \le 1 \right\} < \infty. \tag{71}$$

We denote by $BV_{\mathbb{G}}(\Omega)$ the normed space of bounded variation functions and by $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ the vector space of functions in $BV_{\mathbb{G}}(\mathcal{U})$ for every open set $\mathcal{U} \subset\subset \Omega$.

Theorem 4.3.2. *If $f \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on Ω. Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(p)|_p = 1$ for $\|\nabla_{\mathbb{G}} f\|$-a.e. $p \in \Omega$, and*

$$\int_\Omega f(p) \mathrm{div}_{\mathbb{G}} \phi(p)\, dp = \int_\Omega \langle \phi, \sigma_f \rangle \, d\|\nabla_{\mathbb{G}} f\|, \qquad \text{for every } \phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}).$$

Thus, the notion of gradient $\nabla_{\mathbb{G}}$ can be extended to functions $f \in BV_{\mathbb{G}}$ defining $\nabla_{\mathbb{G}} f$ as the vector valued measure

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner \|\nabla_{\mathbb{G}} f\| = \left(-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{G}} f\|, \dots, -(\sigma_f)_{m_1} \llcorner \nabla_{\mathbb{G}} f\right),$$

where $(\sigma_f)_j$ are the components of σ_f with respect to the moving base X_j.

Definition 4.3.3. A measurable set $E \subset \mathbb{G}$ is a set with locally finite perimeter in Ω or a $\mathbb{G}$-Caccioppoli set in Ω if its characteristic function $\mathbf{1}_E \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$. In this case we call perimeter of E in Ω the measure

$$|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}} \mathbf{1}_E\| \tag{72}$$

and we call generalized horizontal inward $\mathbb{G}$-normal to ∂E in Ω the horizontal vector

$$\nu_E(p) := \sigma_{\mathbf{1}_E}(p). \tag{73}$$

Similarly as in the Euclidean setting, given $E \subset \mathbb{G}$, we define the *essential boundary* or *measure theoretic boundary* $\partial_{*,\mathbb{G}} E$ and, if E is a $\mathbb{G}$-Caccioppoli set, the *reduced boundary* $\partial_{\mathbb{G}}^* E$.

Definition 4.3.4. Let $E \subset \mathbb{G}$ be a measurable set, we say that $p \in \partial_{*,\mathbb{G}} E$ if

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(E \cap U_c(p,r))}{\mathcal{L}^n(U_c(p,r))} > 0 \qquad \text{and} \qquad \limsup_{r \to 0^+} \frac{\mathcal{L}^n(E^c \cap U_c(p,r))}{\mathcal{L}^n(U_c(p,r))} > 0.$$

Definition 4.3.5. Let E be a $\mathbb{G}$-Caccioppoli set. We say that $p \in \partial^*_{\mathbb{G}} E$ if

(i)
$$|\partial E|_{\mathbb{G}}(U_c(p,r)) > 0 \qquad \text{for any } r > 0;$$

(ii)
$$\text{there exists} \quad \lim_{r\to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}};$$

(iii)
$$\lim_{r\to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \left\| \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}} \right\|_{\mathbb{R}^{m_1}} = 1.$$

Lemma 4.3.6 (Differentiation Lemma). *Assume E is a $\mathbb{G}$-Caccioppoli set, then*

$$\lim_{r\to 0} \frac{1}{|\partial E|_{\mathbb{G}}(U_c(p,r))} \int_{U_c(p,r)} \nu_E \, d|\partial E|_{\mathbb{G}} = \nu_E(p), \qquad \textit{for } |\partial E|_{\mathbb{G}}\textit{-a.e. } p,$$

hence $|\partial E|_{\mathbb{G}}$ is concentrated on the reduced boundary $\partial^*_{\mathbb{G}} E$. Moreover we can redefine ν_E in a $|\partial E|_{\mathbb{G}}$-negligible set, by assuming that $\nu_E(p)$ is equal to the limit of the averages at all point $p \in \partial^*_{\mathbb{G}} E$.

Lemma 4.3.7. *There is $c = c(\mathbb{G}) > 0$ such that $|\partial U_c(p,R)|_{\mathbb{G}} = cR^{Q-1}$, for all $p \in \mathbb{G}$ and for a.a. $R > 0$.*

Proof. Because of the invariance of the $\mathbb{G}$-perimeter under group translations, we may assume $p = e$. Moreover, by its homogeneity with respect to group dilations, we have but to show that $|\partial U_c(e,1)|_{\mathbb{G}} < \infty$. We notice first that $\partial U_c(e,1) = \{q \in \mathbb{G}\,;\, d_c(e,q) = 1\}$, and that $|\partial U_c(e,1)| = 0$.

We put $u_k(p) := \psi_k(d_c(p,e))$, $\psi_k : [0,\infty[\to [0,1]$ is a smooth function such that $\psi_k \equiv 1$ on $[0,1]$, $\psi_k \equiv 0$ on $[1+1/k,\infty[$, $|\psi'(t)| \le 2/k$ for $t \ge 0$. Clearly, $\nabla_{\mathbb{G}} u_k$ is supported in the anulus $B_c(e,1+1/k) \setminus U_c(e,1)$. On the other hand, it is well known that $|\nabla_{\mathbb{G}} d_c(\cdot,e)| \le 1$, so that $|\nabla_{\mathbb{G}} u_k| \le 2/k$. Since $|B_c(e,1+1/k)\setminus U_c(e,1)| \sim k^{-1}$ as $k \to \infty$, it follows that its total $\mathbb{G}$-variation is bounded uniformly with respect to k. Thus, we can conclude the proof because of the lower L^1-semicontinuity of the $\mathbb{G}$-variation, since $(u_k)_{k\in\mathbb{N}}$ tends in L^1 to the characteristic function of $U_c(e,1)$. □

The following Theorem is the group version of a (special case of a) celebrated theorem of Federer. See [4.5.11] of [24] and also Proposition 3.6.2 of [2] and [53].

Theorem 4.3.8. *Let $\mathcal{O}$ be an open subset of $\mathbb{G}$. If the measure $\mathcal{S}_d^{Q-1} \llcorner \partial\mathcal{O}$ is locally finite in $\mathbb{G}$ then also $|\partial\mathcal{O}|_{\mathbb{G}}$ is locally finite in $\mathbb{G}$ and there is a geometric constant $c = c(\mathbb{G}) > 0$ such that*

$$|\partial\mathcal{O}|_{\mathbb{G}} \le c\, \mathcal{S}_d^{Q-1} \llcorner \partial\mathcal{O}.$$

In particular, if $\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty$ then $|\partial\mathcal{O}|_{\mathbb{G}}(\mathbb{G}) < \infty$.

Proof. First we assume that $\mathcal{O}$ is bounded. Then, by hypothesis, $\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty$ and, for each $\varepsilon > 0$, we can cover $\partial\mathcal{O}$ with a finite number of open metric balls $U_{\varepsilon,j}$, $j = 1,2,\dots$, with radius $r_{\varepsilon,j} < \varepsilon$, such that

$$\sum_j r_{\varepsilon,j}^{Q-1} < (1+\varepsilon)\, \mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty.$$

Denote

$$S_\varepsilon := \bigcup_j U_{\varepsilon,j} \quad \text{and} \quad \mathcal{O}_\varepsilon := \mathcal{O} \cup S_\varepsilon.$$

It follows

(74)
$$\mathcal{O}_\varepsilon \to \mathcal{O} \qquad \text{in } L^1(\mathbb{G},\mathcal{L}^n), \text{ as } \varepsilon \to 0,$$

because $\mathcal{L}^n(\mathcal{O}_\varepsilon \triangle \mathcal{O}) = \mathcal{L}^n(\mathcal{O}_\varepsilon \setminus \mathcal{O}) \le \mathcal{L}^n(S_\varepsilon) \le \omega_{\mathbb{G}}^Q \sum_j r_{\varepsilon,j}^Q < (1+\varepsilon)\omega_{\mathbb{G}}^Q\, \varepsilon \mathcal{S}_d^{Q-1}(\partial\mathcal{O})$. Here $\omega_{\mathbb{G}}^Q$ is the geometric constant such that $\mathcal{L}^n(U) = \omega_{\mathbb{G}}^Q\, r^Q$ for any metric ball U with radius r.
Now observe that

$$\partial\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}} = \emptyset, \qquad \mathrm{dist}\,(\partial\mathcal{O}_\varepsilon, \bar{\mathcal{O}}) > 0$$

and that

$$\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c = S_\varepsilon \cap \bar{\mathcal{O}}^c.$$

From these and general properties of the perimeter we get

$$\begin{aligned}|\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\mathbb{G}) = |\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\bar{\mathcal{O}}^c) &= |\partial(\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{G}}(\bar{\mathcal{O}}^c)\\ &= |\partial(S_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{G}}(\bar{\mathcal{O}}^c) = |\partial S_\varepsilon|_{\mathbb{G}}(\bar{\mathcal{O}}^c)\\ &\le |\partial S_\varepsilon|_{\mathbb{G}}(\mathbb{G})\end{aligned}$$

and

$$|\partial S_\varepsilon|_{\mathbb{G}}(\mathbb{G}) \le \sum_j |\partial U_{\varepsilon,j}|_{\mathbb{G}}(\mathbb{G}) \le c\sum_j r_{\varepsilon,j}^{Q-1} < (1+\varepsilon)c\,\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty$$

where $c = c(\mathbb{G}) > 0$ is the geometric constant such that $|\partial U_{\varepsilon,j}|_{\mathbb{G}}(\mathbb{G}) = c\, r_{\varepsilon,j}^{Q-1}$. Hence eventually we have

$$|\partial\mathcal{O}_\varepsilon|_{\mathbb{G}}(\mathbb{G}) < (1+\varepsilon)c\,\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty. \tag{75}$$

From (74), (75) and the L^1-lower semicontinuity of the perimeter it follows

$$|\partial\mathcal{O}|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial\mathcal{O}) < \infty. \tag{76}$$

Now we drop the assumption of the boundedness of $\mathcal{O}$. Let U be any fixed open ball such that $U \cap \partial\mathcal{O} \neq \emptyset$. Then, by hypothesis, $\mathcal{S}_d^{Q-1}(U \cap \partial\mathcal{O}) < \infty$. Notice that $\partial(U \cap \mathcal{O}) \subset \partial U \cup (\partial\mathcal{O} \cap U)$, hence, because $\mathcal{S}_d^{Q-1}(\partial U) < \infty$ (see Remark 2.1.5),

$$\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

Thus, applying the first part of the proof to the bounded set $U \cap \mathcal{O}$, we have

$$|\partial(U \cap \mathcal{O})|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

Once more by the locality of the $\mathbb{G}$-perimeter,

$$|\partial\mathcal{O}|_{\mathbb{G}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{G}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{G}}(\mathbb{G}) \le c\,\mathcal{S}_d^{Q-1}(\partial(U \cap \mathcal{O})) < \infty.$$

This achieves the proof of the first part of the theorem.

Finally, if $\partial\mathcal{O} \cap U$ is an intrinsic Lipschitz graph, then its measure theoretic boundary in U coincides with $\partial\mathcal{O} \cap U$ and then the assertion follows from [26], Theorem 7.1. □

Theorem 4.3.9. *If $f : \mathbb{M} \to \mathbb{N}$ is intrinsic Lipschitz then the subgraph E_f^- is a set with locally finite $\mathbb{G}$-perimeter.*

Proof. The proof is a consequence of Theorems 3.2.1 and 4.3.8. □

Lemma 4.3.10. *Let $f : \mathbb{M} \to \mathbb{N}$ be an intrinsic Lipschitz function and let $\Phi_f : \mathbb{M} \to \mathbb{G}$ be the parametrization of* graph (f) *given by $\Phi_f(m) = m \cdot f(m)$. Then there exists* $c(\mathbb{M},\mathbb{N}) > 0$ *such that*

$$(\Phi_f)_\sharp(\mathcal{L}^{n-1} \llcorner \mathbb{M}) = c(\mathbb{M},\mathbb{N})\langle \nu, V\rangle\, |\partial E_f^-|_{\mathbb{G}}$$

where $(\Phi_f)_\sharp(\mathcal{L}^{n-1} \llcorner \mathbb{M})$ denotes the image of $\mathcal{L}^{n-1} \llcorner \mathbb{M}$ under the map Φ_f and $\nu := \nu_{E_f^-}$ is the horizontal generalized inward normal to E_f^- defined in Definition 4.3.3. Notice that $\mathcal{L}^{n-1} \llcorner \mathbb{M}$ is the Haar measure of $\mathbb{M}$.

Proof. We have already fixed $V \in \mathfrak{g}^1$ such that $\mathbb{N} = \{\exp \lambda V : \lambda \in \mathbb{R}\}$. Now choose $W_2, \cdots, W_n \in \mathfrak{g}$ such that $\{V, W_2, \cdots, W_n\}$ is a base of $\mathfrak{g}$ and such that

$$\mathbb{M} = \Big\{ \exp(\sum_{i=2}^{n} \lambda_i W_i) : \lambda_i \in \mathbb{R}\Big\}.$$

Let $\Psi : \mathbb{R}^n \to \mathbb{G} = \mathbb{R}^n$ be the linear map identified by the conditions

$$\Psi(\xi_1, 0 \ldots, 0) = \exp \xi_1 V, \qquad \Psi(0, \xi_2 \ldots, \xi_n) = \exp \big(\sum_{i=2}^{n} \xi_i W_i \big),$$
$$\Psi(\xi_1, \ldots, \xi_n) = \Psi(0, \xi_2, \ldots, \xi_n) \cdot \Psi(\xi_1, 0, \ldots, 0).$$

Ψ is $1 - 1$ and we denote

$$c_1(\mathbb{M}, \mathbb{N}) := \Big| \det \frac{\partial \Psi}{\partial \xi} \Big|. \tag{77}$$

Since E_f^- has locally finite $\mathbb{G}$-perimeter then,

$$\int_{E_f^-} \mathrm{div}_{\mathbb{G}} \phi \, d\mathcal{L}^n = \int_{\mathbb{G}} \langle \nu, \phi \rangle \, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}). \tag{78}$$

Choose $\phi = \psi V$, with $\psi \in C_c^1(\mathbb{G})$. Then $\mathrm{div}_{\mathbb{G}} \phi = V(\psi)$ and from (78) we get

$$\int_{E_f^-} V(\psi) \, d\mathcal{L}^n = \int_{\mathbb{G}} \langle \nu, V \rangle \, \psi \, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \psi \in C_c^1(\mathbb{G}). \tag{79}$$

Notice also that $\dfrac{\partial}{\partial \xi_1}(\psi \circ \Psi) = V(\psi) \circ \Psi$. Denoting $f(m) = \exp(\varphi(m) V)$ with $\varphi : \mathbb{M} \to \mathbb{R}$, we have

$$\hat{E}_f^- := \Psi^{-1}(E_f^-) = \big\{ \xi \in \mathbb{R}^n : \, \xi_1 < \varphi\big(\Psi(0, \xi_2, \ldots, \xi_n)\big) \big\}, \tag{80}$$

and also

$$\Psi\left(\varphi(\Psi(0, \cdot), \cdot)\right) = \Phi_f \circ \Psi(0, \cdot). \tag{81}$$

By (80) and (77), we get

$$\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(\psi \circ \Psi) \, d\mathcal{L}^n = \frac{1}{c_1(\mathbb{M}, \mathbb{N})} \int_{E_f^-} V(\psi) \, d\mathcal{L}^n \, . \tag{82}$$

On the other hand, by Fubini theorem, (80) and (81), it follows

$$\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(\psi \circ \Psi) \, d\mathcal{L}^n = \int_{\{\xi_1 = 0\}} \psi \circ \Phi_f \circ \Psi \; d\mathcal{L}^{n-1}. \tag{83}$$

Then, by [40], Theorem 1.19, and the area formula for linear maps, there is $c_2(\mathbb{M}) > 0$ such that

$$\begin{aligned} \int_{\mathbb{G}} \psi \; d\big((\Phi_f \circ \Psi(0, \cdot))_\sharp (\mathcal{L}^{n-1} \llcorner \mathbb{M})\big) &= \int_{\{\xi_1 = 0\}} \psi \circ \Phi_f \circ \Psi(0, \cdot) \, d\mathcal{L}^{n-1} \\ &= c_2(\mathbb{M}) \int_{\mathbb{M}} \psi \circ \Phi_f \; d(\mathcal{L}^{n-1} \llcorner \mathbb{M}), \qquad \text{for all } \psi \in C_c^1(\mathbb{G}). \end{aligned}$$

By (79), (82) and (83) we get eventually

$$\int_{\mathbb{G}} \psi \; d\big((\Phi_f)_\sharp (\mathcal{L}^{n-1} \llcorner \mathbb{M})\big) = c(\mathbb{M}, \mathbb{N}) \int_{\mathbb{G}} \langle \nu, V \rangle \, \psi \, d|\partial E_f^-|_{\mathbb{G}}, \qquad \text{for all } \psi \in C_c^1(\mathbb{G}).$$

The proof is completed.

$\square$

Corollary 4.3.11. *Under the same assumptions of Lemma 4.3.10, we have*

$$(\mathcal{L}^{n-1} \llcorner \mathbb{M})\big(\mathbb{M} \setminus \mathbf{P}_{\mathbb{M}}(\partial^* E_f^-)\big) = 0.$$

Proof. By Lemma 4.3.10,

$$(\mathcal{L}^{n-1} \llcorner \mathbb{M})\big(\mathbb{M} \setminus \mathbf{P}_{\mathbb{M}}(\partial^* E_f^-)\big) = c(\mathbb{M}, \mathbb{N}) \int_{\mathbb{G} \setminus \partial^* E_f^-} \langle \nu, V \rangle \, d|\partial E_f^-|_{\mathbb{G}} = 0,$$

since $|\partial E_f^-|_{\mathbb{G}}(\mathbb{G} \setminus \partial^* E_f^-) = 0$.

□

4.4. **A Rademacher type theorem.** Let $\varphi : \mathcal{U} \subset \mathbb{M} \to \mathbb{N}$ be an intrinsic Lipschitz function, where $\mathcal{U}$ is a (relatively) open subset; we want to prove here a Rademacher's type result, that is, *if φ is intrinsic Lipschitz in $\mathcal{U}$ then φ is intrinsic differentiable almost everywhere in $\mathcal{U}$*. Such a result was known only inside Heisenberg groups. We will extend it here to a much larger class of Carnot groups, i.e. the so called groups of type $\star$ introduced in [39].

Definition 4.4.1. We say that a stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is of type $\star$ if there exists a basis $\{X_1, \dots, X_{m_1}\}$ of $\mathfrak{g}_1$ such that

$$[X_j, [X_j, X_i]] = 0, \qquad \text{for all } i, j = 1, \dots, m_1. \tag{84}$$

A Carnot group $\mathbb{G}$ is said to be of type $\star$ if its Lie algebra $\mathfrak{g}$ is of type $\star$.

Remark 4.4.2. The previous definition is clearly invariant under Lie algebra isomorphisms that respect the stratification. Step 2 Carnot groups are of type $\star$, whereas free Carnot groups of step greater than 2 are not of type $\star$. If a Carnot group of step greater than 2 is of type $\star$, then the dimension of its first layer is at least 3; hence filiform groups of step greater than 2 (and in particular Engels group) are not of type $\star$.

The Lie group $\mathbb{G}_m$ of unit upper triangular $(m+1) \times (m+1)$ matrices is a Carnot group of type $\star$, for any $m \in \mathbb{N}$ ($m > 2$ to avoid a trivial case). This group is the nilpotent group coming from Iwasawa decomposition of $GL_{m+1}(\mathbb{R})$.

The rectifiability of the reduced boundary of finite perimeter sets, proved in [26, 28] inside Heisenberg groups and more generally inside step 2 groups, can be extended to groups of type $\star$.

Theorem 4.4.3 ([39])**.** *Let $\mathbb{G}$ be a Carnot group of type $\star$. If $E \subset \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, then*

(i) $\qquad \partial^*_{\mathbb{G}} E$ *is one-codimensional $\mathbb{G}$-rectifiable,*

that is $\partial^*_{\mathbb{G}} E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a $\mathbb{G}$-regular hypersurface S_h;

(ii) $\qquad \nu_E(p)$ *is the horizontal $\mathbb{G}$-normal to S_h in p, for every $p \in K_h$,*

(iii) $$|\partial E|_{\mathbb{G}} = \theta_c \mathcal{S}_c^{Q-1} \llcorner \partial^*_{\mathbb{G}} E,$$

where $\theta_c = \theta_c(\mathbb{G}, E, x) > 0$.

The starting point in our proof of the Rademacher theorem is the fact, proved in Theorem 4.3.9, that the subgraph E_φ^- of an intrinsic Lipschitz function $\varphi : \mathbb{M} \to \mathbb{N}$ is a set with locally finite $\mathbb{G}$-perimeter. From this and Theorem 4.4.3, it follows that at $|\partial E_f^-|_{\mathbb{G}}$-almost every point of graph(φ) there is an approximate tangent coset. This in turn, together with the intrinsic Lipschitz assumption, yields the intrinsic differentiability of φ.

Theorem 4.4.4. *Let $\mathbb{M}$ and $\mathbb{N}$ be complementary subgroups of a Carnot group $\mathbb{G}$ of type $\star$, with $\mathbb{N}$ one-dimensional and horizontal. Let $\mathcal{U} \subset \mathbb{M}$ be relatively open in $\mathbb{M}$ and $\varphi : \mathcal{U} \to \mathbb{N}$ be intrinsic Lipschitz. Then φ is intrinsic differentiable $(\mathcal{L}^{n-1} \llcorner \mathbb{M})$-almost everywhere in $\mathcal{U}$. Notice that $\mathcal{L}^{n-1} \llcorner \mathbb{M}$ is the Haar measure of $\mathbb{M}$.*

Proof. By Theorem 4.1.1, we may assume that φ is intrinsic Lipschitz and defined on all of $\mathbb{M}$. Hence, by Theorem 4.3.9, we know that E_φ^- has locally finite $\mathbb{G}$-perimeter. Then, by Theorem 4.4.3 we know that there is a subset

$$\partial^*_{\mathbb{G}} E_\varphi^- \subset \partial E_\varphi^- = \operatorname{graph}(\varphi)$$

such that

$$|\partial E_f^-|_{\mathbb{G}}\big(\operatorname{graph}(\varphi) \setminus \partial^*_{\mathbb{G}} E_\varphi^-\big) \equiv |\partial E_\varphi^-|_{\mathbb{G}}\big(\partial E_\varphi^- \setminus \partial^*_{\mathbb{G}} E_\varphi^-\big) = 0,$$

and for all $p = m\varphi(m) \in \partial^*_{\mathbb{G}} E_\varphi^-$, φ is approximately intrinsic differentiable in m (remember Definition 4.2.1). By Proposition 4.2.2, φ is differentiable at any point $m \in \mathbb{M}$ such that

$$m\varphi(m) \in \partial^*_{\mathbb{G}} E_\varphi^-.$$

Finally, from Corollary 4.3.11, we have

$$(\mathcal{L}^{n-1} \llcorner \mathbb{M})\big(\mathbf{P}_{\mathbb{M}}\big(\operatorname{graph}(\varphi) \setminus \partial^*_{\mathbb{G}} E_\varphi^-\big)\big) = (\mathcal{L}^{n-1} \llcorner \mathbb{M})\big(\mathbb{M} \setminus \mathbf{P}_{\mathbb{M}} \partial^*_{\mathbb{G}} E_\varphi^-\big) = 0,$$

that completes the argument. □

4.5. **One-codimensional rectifiable sets.** The results of the previous sections can be applied to prove the equivalence of two intrinsic notions of one-codimensional rectifiable sets in $\mathbb{G}$. For a related and deeper analysis about equivalence of different notions of intrinsic rectifiable sets in $\mathbb{H}^n$ we refer to [41].

We begin by recalling the definitions of intrinsic regular hypersurfaces (or one codimensional surfaces), of their tangent groups or tangent cosets as well as the related implicit function theorem and the notion of intrinsically rectifiable set.

Definition 4.5.1. $S \subset \mathbb{G}$ is a $\mathbb{G}$-*regular hypersurface* if for every $p \in S$ there exist a neighborhood $\mathcal{U}$ of p and a function $f \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

(i) $$S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\};$$

(ii) $$df_q \neq 0 \quad \text{for all } q \in \mathcal{U}.$$

The *tangent affine group* or *tangent coset* to S at p is the coset of the kernel of df_p, i.e.

(iii) $$T_{\mathbb{G}} S(p) := \{q \in \mathbb{G} : \ df_p(p^{-1} \cdot q) = 0\}.$$

Theorem 2.35 of [28] can be restated as follows

Theorem 4.5.2. *Let S be a $\mathbb{G}$-regular hypersurface in $\mathbb{G}$. Then, for all $p \in S$ there are an open $\mathcal{U} \ni p$, complementary subspaces $\mathbb{M}$ and $\mathbb{N}$, with $\mathbb{N}$ one-dimensional and horizontal, a relatively open $\mathcal{V} \subset \mathbb{M}$ and an intrinsic Lipschitz and intrinsic differentiable function $\varphi : \mathcal{V} \to \mathbb{N}$ such that*

$$S \cap \mathcal{U} = \{m\varphi(m) : m \in \mathcal{V}\}. \tag{85}$$

Moreover, for all $q = m\varphi(m) \in S \cap \mathcal{U}$, the tangent affine group $T_{\mathbb{G}} S(q)$ coincides with the tangent coset to graph (φ) *in q as introduced in Definition 3.4.11.*

It follows, in particular, that the definition of $T_{\mathbb{G}} S(x)$ does not depend on the particular function f defining the surface S.

Finally we recall two definitions of one codimensional intrinsic rectifiable sets. Each of them mimics a natural definition used in Euclidean context.

Definition 4.5.3. $\Gamma \subset \mathbb{G}$ is said to be *one codimensional* $\mathbb{G}$-*rectifiable* if there exists a sequence of $\mathbb{G}$-regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ such that

$$\mathcal{H}_c^{Q-1}\big(\Gamma \setminus \bigcup_{j \in \mathbb{N}} S_j\big) = 0.$$

Definition 4.5.4. $\Gamma \subset \mathbb{G}$ is said to be *one codimensional $\mathbb{G}_L$-rectifiable* if there exists a sequence of one-codimensional intrinsic Lipshitz graphs G_i, $i = 1, 2, \dots$, such that,

$$\mathcal{H}_c^{Q-1}\big(\Gamma \setminus \bigcup_{j \in \mathbb{N}} G_j\big) = 0.$$

In both definitions, $\mathcal{H}_c^{Q-1}$ is the $(Q-1)$-Hausdorff measure related to the distance d_c.

Proposition 4.5.5. $\Gamma \subset \mathbb{G}$ *is one codimensional $\mathbb{G}_L$-rectifiable if and only if it is one codimensional $\mathbb{G}$-rectifiable.*

Proof. Since intrinsic regular surfaces are locally graphs of intrinsic Lipschitz functions it follows that the scope of the second definition is larger than the first one. On the other direction, by definition, each G_i is the graph of a one-dimensional valued, intrinsic Lipschitz function $\varphi_i : \mathcal{C}_i \subset \mathbb{M}_i \to \mathbb{N}_i$. By the extension Theorem 4.1.1, we can assume that $\mathcal{C}_i = \mathbb{M}_i$ for all i. Hence, by Theorem 4.3.9, the subgraph of φ_i has locally finite $\mathbb{G}$-perimeter and, eventually, it is $\mathbb{G}$-rectifiable, by the structure theorem for sets of locally finite $\mathbb{G}$-perimeter proved. This proves that all of Γ is $\mathbb{G}$-rectifiable. □

References

[1] R.Abraham, J.E.Marsden, T.Raţiu, *Manifolds, tensor analysis, and applications.* Global Analysis Pure and Applied: Series B, 2. Addison-Wesley Publishing Co., Reading, Mass. (1983).

[2] L.Ambrosio, N.Fusco, D.Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, (2000).

[3] L.Ambrosio, B.Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Annalen, **318** (2000), 527–555.

[4] L.Ambrosio, B.Kirchheim, *Currents in metric spaces,* Acta Math. **185** (2000), 1–80.

[5] L.Ambrosio, B.Kirchheim, E. LeDonne, *Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane*, The Journal of Geometric Analysis, **19**, (2009), no. 3, 509–540.

[6] L.Ambrosio, F. Serra Cassano, D. Vittone, *Intrinsic regular hypersurfaces in Heisenberg groups*, J. Geom. Anal. **16**, (2006), 187–232.

[7] G.Arena, *Intrinsic Graphs, Convexity and Calculus on Carnot Groups*, PhD Thesis, University of Trento, (2008).

[8] G.Arena, R.Serapioni, *Intrinsic regular submanifolds in Heisenberg groups are differentiable graphs*, Calc. Var. Partial Differential Equations **35** (2009), no. 4, 517–536.

[9] Z.Balogh, K. Fässler, P. Mattila, J.Tyson, *Projection and slicing theorems in Heisenberg groups*, Adv. Math. 231 (2012), no. 2, 569–604.

[10] F.Bigolin, D.Vittone, *Some remarks about parametrizations of intrinsic regular surfaces in the Heisenberg group*, Publ. Mat. **54** (2010), no. 1, 159–172.

[11] F. Bigolin, F. Serra Cassano, *Distributional solutions of Burgers' equation and intrinsic regular graphs in Heisenberg groups*, J. Math. Anal. Appl. 366 (2010), no. 2, 561568.

[12] F. Bigolin, F. Serra Cassano, *Intrinsic regular graphs in Heisenberg groups vs. weak solutions of non-linear first-order PDEs*, Adv. Calc. Var. 3 (2010), no. 1, 6997.

[13] A.Bonfiglioli, E.Lanconelli, F.Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub–Laplacians*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, New York (2007).

[14] A.Baldi, B.Franchi, N.Tchou, M.C.Tesi, *Compensated compactness for differential forms in Carnot groups and applications*, Adv. Math., **223 (5)** (2010), 1555-1607.

[15] L.Capogna, D.Danielli, S.D.Pauls, J.T.Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Birkhäuser Verlag, Basel, (2007).

[16] G. Citti, A. Sarti, *A cortical based model of perceptual completion in the roto-translation space*, J. Math. Imaging Vision, Journal of Mathematical Imaging and Vision, **24**, (2006), 3, 307–326.

[17] J.Cheeger, B. Kleiner,*Differentiating maps into L^1, and the geometry of BV functions*, Ann. of Math. (**2**) 171 (2010), no. 2, 1347–1385.

[18] J.Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geometry and Functional Analysis, **9** (1999), 428–517.

[19] G.Citti, M.Manfredini, *Implicit function theorem in Carnot-Carathodory spaces*. Commun. Contemp. Math. **8** (2006), no. 5, 657–680.

[20] G.Citti, M.Manfredini, *Blow-up in non homogeneous Lie groups and rectifiability.* Houston J. Math. **31** (2005), no. 2, 333–353.
[21] D.Cole, S.D.Pauls, *C^1 hypersurfaces of the Heisenberg group are N-rectifiable*, Houston J.Math. **32** (2006), no. 3, 307–326.
[22] G. Csató, B. Dacorogna, O. Kneuss, *The pullback equation for differential forms.* Progress in Nonlinear Differential Equations and their Applications, 83. Birkhuser/Springer, New York (2012).
[23] L.C.Evans, R.F.Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, BocaRaton, (1992).
[24] H.Federer, *Geometric Measure Theory*, Springer, (1969).
[25] G.B.Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., **13** (1975), 161–207.
[26] B.Franchi, R.Serapioni, F.Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), 479–531.
[27] B.Franchi, R.Serapioni, F.Serra Cassano, *Regular hypersurfaces, Intrinsic Perimeter and Implicit Function Theorem in Carnot Groups*, Comm. Anal. Geom., **11** (2003), no 5, 909–944.
[28] B.Franchi, R.Serapioni, F.Serra Cassano, *On the Structure of Finite Perimeter Sets in Step 2 Carnot Groups*, The Journal of Geometric Analysis. **13** (2003), no. 3, 421–466.
[29] B.Franchi, R.Serapioni, F.Serra Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups.* J. Nonlinear Convex Anal. **7** (2006), no. 3, 423–441.
[30] B.Franchi, R.Serapioni, F.Serra Cassano, *Regular submanifolds, graphs and area formula in Heisenberg Groups.* Advances in Math. **211** (2007), no.1, 152–203.
[31] B.Franchi, R.Serapioni, F.Serra Cassano, *Rademacher theorem for intrinsic Lipschitz continuous functions*, J. Geom. Anal. **21** (2011), 1044–1084.
[32] B.Franchi, M.C.Tesi, *Wave and Maxwell's Equations in Carnot Groups*, Commun. Contemp. Math., **14** (5) (2012).
[33] N.Garofalo, D.M.Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., **49** (1996), 1081–1144.
[34] M.Gromov, *Metric structures for Riemannian and Non Riemannian Spaces*, Progress in Mathematics, **152**, Birkhauser Verlag, Boston, (1999).
[35] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York (1978).
[36] I.N.Herstein, *Topics in algebra*, Blaisdell Publishing Company, London (1964).
[37] V.Magnani, *Elements of Geometric Measure Theory on Sub-Riemannian Groups*, Tesi di Perfezionamento, Scuola Normale Superiore, Pisa, (2003).
[38] V.Magnani, *Towards differential calculus in stratified groups,* to appear on Journal of Australian Mathematical Society.
[39] M.Marchi, *Rectifiability of Sets of Finite Perimeter in a Class of Carnot Groups of Arbitrary Step* Preprint (2012).
[40] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, (1995).
[41] P.Mattila, R.Serapioni, F.Serra Cassano, *Characterizations of intrinsic rectifiability in Heisenberg groups* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **9** (2010), no. 4, 687–723.
[42] J.Mitchell, *On Carnot-Caratheodory metrics*, J.Differ. Geom. **21** (1985), 35–45.
[43] R.Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, *Mathematical Surveys and Monographs* **91**, American Mathematical Society, Providence, RI (2002).
[44] S. Nicolussi Golo, *The tangent space in Sub-Riemannian Geometry*, Doctoral Thesis, Università di Trento, (2012).
[45] P.Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. **129** (1989), 1–60.
[46] S.D.Pauls, *A notion of rectifiability modelled on Carnot groups*, Indiana Univ. Math. J. **53** (2004), no. 1, 49–81.
[47] M.Rumin, *Around heat decay on forms and relations of nilpotent Lie groups*, In *Séminaire de Théorie Spectrale et Géométrie*, **19**, Univ. Grenoble I, (2001), 123–164.
[48] M.Rumin, *Differential geometry on C-C spaces and application to the Novikov-Shubin numbers of nilpotent Lie groups*, C. R. Acad. Sci. Paris Ser. I Math., **329** (1999), 985–990.
[49] S. Semmes, On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A-weights Rev. Mat. Iberoamericana **12** (1996), no. 2, 337–410.
[50] E.M.Stein, *Harmonic Analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton (1993).
[51] N.Th.Varopoulos, L.Saloff-Coste, T.Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, (1992)
[52] D.Vittone, *Submanifolds in Carnot groups*, Edizioni della Normale, Scuola Normale Superiore, Pisa, (2008).
[53] D.Vittone, *Lipschitz surfaces, Perimeter and trace Theorems for BV Functions in Carnot-Caratheéodory Spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), 11(4), (2012), 939–998.

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