Higher differentiability of solutions of elliptic systems with Sobolev coefficients: the case $p = n = 2$

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Abstract

We establish higher differentiability results for local solutions of elliptic systems of the type

$$\mathrm{div} A(x, Du) = 0$$

in a bounded open set in $\mathbb{R}^2$. The operator $A(x, \xi)$ is assumed to be strictly monotone and Lipschitz continuous with respect to variable ξ. The novelty of the paper is that we allow discontinuous dependence with respect to the x-variable, through a suitable Sobolev function.

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1 Introduction and statements

In this paper, we address the study of the higher differentiability of local solutions of quasilinear elliptic systems, as well as local minimizer of convex variational integrals having quadratic growth in the gradient, allowing discontinuous dependence with respect to the x-variable, through a suitable Sobolev function. In order to make our presentation precise, we shall introduce and discuss our hypotheses.
We shall consider elliptic system of the form

$$\mathrm{div} A(x, Du) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2 \tag{1.1}$$

acting on Sobolev mappings $u \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$, with $N \geq 1$. Here Ω is a bounded open set in $\mathbb{R}^2$, $Du = Du(x)$ stands for the Jacobi matrix of u, an $N \times 2$ matrix whose entries $D_j u^i$ are the first order partial derivatives of the coordinate functions, $A(x, Du)$ is an $N \times 2$ matrix with columns A_1, A_2 and

$$\mathrm{div} A(x, Du) = \sum_{i=1}^{2} D_i A_i(x, Du(x))$$

is interpreted as an R^N valued distribution. We assume that

$$A(x, \xi) : \Omega \times \mathbb{R}^{N\times 2} \to \mathbb{R}^{N\times 2}$$

is a Carathéodory mapping, satisfying for positive constants $\alpha \leq \beta$, the following set of hypotheses:

$$A(x, \lambda\xi) = \lambda A(x, \xi) \tag{H0}$$

$$\langle A(x,\xi)-A(x,\eta),\xi-\eta\rangle \geq \alpha|\eta-\xi|^2 \tag{H1}$$

$$|A(x,\xi)-A(x,\eta)| \leq \beta|\xi-\eta|, \tag{H2}$$

for all constant $\lambda > 0$, for almost every $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^{N\times 2}$.
Concerning the dependence on the x-variable, we shall assume that there exists a function k such that $k(x) \in \mathrm{L}^2(\Omega)$ and

$$|A(x,\xi)-A(y,\xi)| \leq (|k(x)|+|k(y)|)|x-y|(1+|\xi|)\,, \tag{H3}$$

for almost every $x, y \in \Omega$ and all $\xi \in \mathbb{R}^{N\times 2}$.
The function k plays the role of the derivative of the function $x \to A(x,\xi)$ and so the assumption (H3) serves to describe the continuity of the operator $A(x,\xi)$ with respect to the x-variable. Obviously, this is a weak form of continuity, since the function k may blows up at some points.
The model case we have in mind is an equation of the form

$$\operatorname{div}(a(x)Du) = 0,$$

where $a(x)$ is a function in the Sobolev class $\mathrm{W}^{1,2} \cap L^\infty$. Typical examples of operators that exhibit the above monotonicity and Lipschitz continuity properties arise from Euler-Lagrange systems of variational integrals

$$\mathfrak{F}(v,O) = \int_O F(x,Dv(x))\,\mathrm{d}x \tag{1.2}$$

with smooth integrand $F = F(x,\xi)$ satisfying, for positive constants ℓ, L and for a function $k(x) \in \mathrm{L}^2(\Omega)$ the following assumptions:

$$\xi \mapsto F(x,\xi) \text{ is a strictly convex } C^2 \text{ function for a.e. } x \in \Omega \tag{F1}$$

$$\ell|\xi|^2 \leq F(x,\xi) \leq L|\xi|^2 \tag{F2}$$

$$\langle D_{\xi\xi}F(x,\xi)\eta,\eta\rangle \geq \nu|\eta|^2 \tag{F3}$$

$$D_\xi F(x,\lambda\xi) = \lambda D_\xi F(x,\xi) \tag{F4}$$

$$|D_\xi F(x,\xi)-D_\xi F(y,\xi)| \leq L(|k(x)|+|k(y)|)|x-y|(1+|\xi|)\,, \tag{F5}$$

for all constant $\lambda > 0$, for almost every $x, y \in \Omega$ and all $\xi \in \mathbb{R}^{N\times 2}$.
There exists a wide literature concerning the regularity of solutions of the system (1.1) and of local minimizers of the integral functional (1.2), in case assumptions (H3) and (F5) are replaced by

$$|A(x,\xi)-A(y,\xi)| \leq \omega(|x-y|)(1+|\xi|) \tag{H3$'$}$$

and

$$|D_\xi F(x,\xi)-D_\xi F(y,\xi)| \leq \omega(|x-y|)(1+|\xi|)\,, \tag{F5$'$}$$

respectively. In the classical setting, the function $\omega : [0,\infty) \to [0,\infty)$ is assumed to be Hölder continuous, i.e.,

$$\omega(\rho) = \min\{\rho^\alpha, 1\} \qquad \text{for some } (\alpha, 1]. \tag{1.3}$$

For an exhaustive treatment of the regularity of solutions of elliptic systems under the assumptions (H0)–(H2) and (H3$'$), or local minimizers under the assumptions (F1)–(F4) and (F5$'$), we refer the interested reader to [17, 18] and the references therein.

In the last few years, the study of the regularity has been successfully carried out under weaker assumptions on the function $\omega(\rho)$, which, roughly speaking, measures the continuity of the operator A with respect to the x-variable. In particular, in [14] (see also [10, 11]), a partial $C^{0,\alpha}$ regularity result has been established relaxing the assumption (1.3) in a continuity assumption of the type

$$\lim_{\rho \to 0} \omega(\rho) = 0.$$

Very recently, the result of [14] has been extended in [4] to operators that have discontinuous dependence on the x-variable, through a VMO coefficient. We also recall that a continuity result for solutions of linear elliptic equations with Sobolev coefficients has been established in [27].
In [25], we established the higher differentiability of local minimizers of convex degenerate functionals, with integrand $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, having p-growth in the gradient and depending on the x-variable through a function in the Sobolev class $\mathrm{W}^{1,n}$. More precisely, we assumed (F5) for $k \in \mathrm{L}^n(\Omega)$. The higher differentiability of the gradient of the local minimizers is obtained in the case $2 \leq p < n$, by establishing higher differentiability estimates for solutions to a class of auxiliary problems. Such problems are constructed adding singular higher order perturbations to the integrand, following the techniques in [5]. We took advantage from the assumption $k \in \mathrm{L}^n$, by the use of the Sobolev imbedding Theorem, that cannot be used in the critical growth exponent case $p = n = 2$.

In this paper we fill this gap, proving the following

Theorem 1.1. *Let $A \colon \Omega \times \mathbb{R}^{N\times 2} \to \mathbb{R}^{N\times 2}$ be a Carathéodory function satisfying the assumptions (H0)–(H3). If $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$ is a local solution of the system* (1.1)*, then*

$$D^2 u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2}),$$

for any exponent p, $p < 2$. Furthermore, there exists a radius $R_0 = R_0(\alpha, \beta, p, N)$ such that, whenever $B_{2R} \subset B_{R_0} \Subset \Omega$, we have the Caccioppoli type inequality

$$\int_{B_{\frac{R}{4}}} |D^2 u|^p \,\mathrm{d}x \leq \frac{c}{R^p} \int_{B_R} |Du|^p \,\mathrm{d}x \tag{1.4}$$

for a constant $c = c(\alpha, \beta, p, N)$.

We also have

Theorem 1.2. *Let $F \colon \Omega \times \mathbb{R}^{N\times 2} \to \mathbb{R}$ satisfy the assumptions (F1)–(F5). If $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$ is a local minimizer of the functional* (1.2)*, then*

$$D^2 u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$$

where p is any exponent such that $p < 2$. Furthermore, there exists a radius $R_0 = R_0(\alpha, \beta, p, N)$ such that whenever $B_{2R} \subset B_{R_0} \Subset \Omega$ we have the Caccioppoli type inequality

$$\int_{B_{\frac{R}{4}}} |D^2 u|^p \,\mathrm{d}x \leq \frac{c}{R^p} \int_{B_R} |Du|^p \,\mathrm{d}x \tag{1.5}$$

for a constant $c = c(\alpha, \beta, p, N)$.

The proof of Theorems 1.1 and 1.2 is achieved combining a suitable a priori estimate for the second derivatives of the local solutions of the system with an approximation argument.
Our main idea in order to establish the a priori estimate is to treat the regularity of local solutions of systems with discontinuous coefficients with the tools needed to deal with functionals satisfying (p, q) growth conditions. Functionals with (p, q) growth conditions have been widely investigated both in the scalar and in the vectorial setting (see for example [1, 2, 8, 9, 12, 13, 22, 23, 24, 26]).

A classical tool in order to establish the higher differentiability of the gradient is the difference quotient method. The main difficulty here is due to the fact that we deal with the critical growth exponent $p = n = 2$. Therefore, in order to obtain suitable a priori estimates, we construct test functions obtained combining the difference quotient method with the use of the Hodge decomposition, inspired by [21]. In fact, the difference quotient method, as usual, leads to the higher differentiability result, while the Hodge decomposition allows us to establish the a priori estimates in a Sobolev space with less integrability than 2, so avoiding the difficulties due to the critical growth case.
Actually, this is not only a technical feature, since for the solutions of (1.1) under the assumptions (H0)–(H3), we can not expect second derivatives in L^2, as it is shown in Example 4.1. where it is constructed a quasilinear elliptic equation that satisfying (H3) for a function $k \in \mathrm{L}^2$, having a solution u that doesn't belong to $\mathrm{W}^{1,2}$.
Moreover, we'd like to point out that the assumption on k can not be weakened in order to have solutions with second derivatives in L^p, with p arbitrarily close to 2 (see Example 4.2).
Once Theorem 1.1 is proven, Theorem 1.2 follows by noticing that local minimizers of the functional (1.2), under the assumptions (F0)–(F5), are local solutions of the Euler Lagrange system

$$\mathrm{div} D_\xi F(x, Du) = 0$$

and the operator $D_\xi F(x, \xi)$ satisfies the assumptions (H0)–(H3).
The plan of the paper is the following. We have collected standard preliminary material in Section 2, which at the same time serves as our reference for notation. The proofs of the higher differentiability results stated in Theorems 1.1 and 1.2 are presented in Section 3. The examples are contained in Section 4

2 Preliminaries

For matrices $\xi, \eta \in \mathbb{R}^{N\times n}$ we write $\langle \xi, \eta\rangle := \mathrm{trace}(\xi^T\eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{N\times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column. Observe that $|a \otimes b| = |a||b|$, where $|a|$, $|b|$ denote the usual euclidean norms of a in $\mathbb{R}^N$, b in $\mathbb{R}^n$, respectively.
When $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is sufficiently differentiable we write

$$D_\xi F(x, \xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(x, \xi + t\eta) \quad \text{and} \quad D_{\xi\xi} F(x, \xi)[\eta, \eta] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} F(x, \xi + t\eta)$$

for $\xi, \eta \in \mathbb{R}^{N\times n}$.
Let us give the definition of local solution of the system (1.1) and of local minimizer of the integral (1.2):

Definition 2.1. *A mapping $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local solution of the system* (1.1) *if*

$$\int_{\mathrm{supp}\varphi} \langle A(x, Du), D\varphi \rangle \, \mathrm{d}x = 0$$

for any $O \Subset \Omega$ and any $\varphi \in \mathrm{C}_0^\infty(O, \mathbb{R}^N)$.

Definition 2.2. *A mapping $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local F–minimizer if*

$$\int_{\mathrm{supp}\varphi} F(x, Du) \, \mathrm{d}x \le \int_{\mathrm{supp}\varphi} F(x, Du + D\varphi) \, \mathrm{d}x$$

for any $O \Subset \Omega$ and any $\varphi \in \mathrm{C}_0^\infty(O, \mathbb{R}^N)$.

Now, we state a very well-known iteration lemma.

Lemma 2.3. *Let $\Phi\colon [\frac{R}{2}, R] \to \mathbb{R}$ be a bounded nonnegative function on the interval $[\frac{R}{2}, R]$ where $R > 0$. Assume that for all $\frac{R}{2} \le r < s \le R$ we have*

$$\Phi(r) \le \vartheta \Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\gamma}$$

where $\vartheta \in (0, 1)$, A, B, $C \ge 0$ and $0 < \gamma$ are constants. Then there exists a constant $c = c(\vartheta, \gamma)$ such that

$$\Phi\left(\frac{R}{2}\right) \le c\left(A + \frac{B}{R^2} + \frac{C}{R^\gamma}\right)$$

See for instance [18], pp. 191–192.

2.1 Difference quotient

In order to get the higher differentiability of the solutions of system (1.1) we have to use the difference quotient method. Therefore we introduce the following finite difference operator.

Definition 2.4. *For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by*

$$\tau_{s,h} F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \ldots, n\}$.

The difference quotient is defined for $h \in \mathbb{R} \setminus \{0\}$ as

$$\Delta_{s,h} F(x) = \frac{\tau_{s,h} F(x)}{h}.$$

The following proposition describes some elementary properties of the finite difference operator and can be found, for example, in [18].

Proposition 2.5. *Let f and g be two functions such that $F, G \in W^{1,p}(\Omega; \mathbb{R}^N)$, with $p \ge 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

$(d1)$ $\tau_{s,h}F \in W^{1,p}(\Omega)$ *and*

$$D_i(\tau_{s,h}F) = \tau_{s,h}(D_iF).$$

$(d2)$ *If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then*

$$\int_\Omega F\,\tau_{s,h}G\,dx = -\int_\Omega G\,\tau_{s,-h}F\,dx.$$

$(d3)$ *We have*

$$\tau_{s,h}(FG)(x) = F(x+he_s)\tau_{s,h}G(x) + G(x)\tau_{s,h}F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.6. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1,\dots,n\}$ and $F, D_sF \in L^p(B_R)$ then*

$$\int_{B_\rho} |\tau_{s,h}F(x)|^p\,dx \le |h|^p \int_{B_R} |D_sF(x)|^p\,dx.$$

Moreover

$$\int_{B_\rho} |F(x+he_s)|^p\,dx \le c(n,p)\int_{B_R} |F(x)|^p\,dx.$$

Now, we recall the fundamental Sobolev embedding property. (For the proof see, for example, [18, Lemma 8.2]).

Lemma 2.7. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1 < p < +\infty$. Suppose that there exist $\rho \in (0,R)$ and $M > 0$ such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h}F(x)|^p\,dx \le M^p|h|^p,$$

for every h with $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_\rho;\mathbb{R}^N) \cap L^{\frac{np}{n-p}}(B_\rho;\mathbb{R}^N)$. Moreover

$$||DF||_{L^p(B_\rho)} \le M$$

and

$$||F||_{L^{\frac{np}{n-p}}(B_\rho)} \le c\left(M + ||F||_{L^p(B_R)}\right),$$

with $c \equiv c(n,N,p)$.

2.2 Hodge decomposition

We recall that for a vector field $F \in L^p(\mathbb{R}^n,\mathbb{R}^n)$, with $1 < p < +\infty$, the Poisson equation

$$\Delta w = \mathrm{div}F$$

is solved by a function $w \in \mathrm{W}^{1,p}$ whose gradient can be expressed in terms of the Riesz transform as follows

$$Dw = -(\mathbf{R}\otimes\mathbf{R})(F).$$

The tensor product operator $\mathbf{R}\otimes\mathbf{R}$ is the $n\times n$ matrix whose entries are the second order Riesz transforms $R_j \circ R_k$ $(1 \le j, k \le n)$ and therefore the above identity reads as

$$D_j w = -\sum_{k=1}^{n} R_j R_k F^k,$$

where F^k denotes the $k-th$ component of the vector field F.
Setting $\mathbf{E} = -(\mathbf{R}\otimes\mathbf{R})$, since $\mathrm{div}(Dw - F) = 0$, we have that the range of the operator

$$\mathbf{B} = \mathbf{Id} - \mathbf{E}$$

consists of divergence free vector fields and the Hodge decomposition of F is given by

$$F = \mathbf{E}(F) + \mathbf{B}(F).$$

When F is a $N \times n$ matrix field, then we define $\mathbf{E}(F)$ and $\mathbf{B}(F)$ row-wise. Then $\mathbf{E}(F)$ is a $N \times n$ matrix field whose rows are rotation free and $\mathbf{B}(F)$ is a $N \times n$ matrix field whose rows are divergence free. Standard Calderon-Zygmund theory yields L^p bounds for the operators $\mathbf{E}$ and $\mathbf{B}$, whenever $1 < p < +\infty$. However, we will need a more precise estimate, which is contained in the following stability property of the Hodge decomposition.

Lemma 2.8. *Let $w \in \mathrm{W}^{1,2-\varepsilon}(\mathbb{R}^2, \mathbb{R}^N)$, for $0 < \varepsilon < 1$. Then there exist $\Phi \in \mathrm{W}^{\frac{2-\varepsilon}{1-\varepsilon}}(\mathbb{R}^2; \mathbb{R}^N)$ and $H \in \mathrm{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(\mathbb{R}^2; \mathbb{R}^N)$ with $\mathrm{div}H = 0$, such that*

$$D\Phi = Dw|Dw|^{-\varepsilon} + H. \tag{2.1}$$

Moreover

$$||D\Phi||_{\mathrm{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(\mathbb{R}^2;\mathbb{R}^N)} \le c_H ||Dw||^{1-\varepsilon}_{\mathrm{L}^{2-\varepsilon}(\mathbb{R}^2;\mathbb{R}^N)} \tag{2.2}$$

and

$$||H||_{\mathrm{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(\mathbb{R}^2;\mathbb{R}^N)} \le c_H \varepsilon ||Dw||^{1-\varepsilon}_{\mathrm{L}^{2-\varepsilon}(\mathbb{R}^2;\mathbb{R}^N)}, \tag{2.3}$$

for an absolute positive constant c_H.

The proof of previous Lemma is contained in [21, Theorem 4]. The fact that the constant is independent of the dimension and of ε can be derived as in [19, Corollary 3].

3 Proof of Theorem 1.1

This section is devoted to the proof of our main result, that will be divided in two parts. In the first one, we will establish an a priori estimate and in the second we will conclude through an approximation procedure.

Proof of Theorem 1.1.
Step 1. The a priori estimate

Suppose that the local solution $u \in W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} < r < s < t < \lambda r < R$, with $1 < \lambda < 2$. Further, let us consider a cut off function $\rho \in C_0^\infty(B_t)$ such that $\rho = 1$ on B_s , $|\nabla \rho| \leq \frac{c}{t-s}$ and for a fixed exponent p, with $1 < p < 2$, let us consider the matrix field

$$D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}.$$

The function $\rho\tau_{s,h}u$ can be considered as a $\mathrm{W}^{1,p}$ map on $\mathbb{R}^2$ that vanishes off B_t. Therefore, the Hodge decomposition of Theorem 2.8 implies that there exist $\psi \in \mathrm{W}_0^{\frac{p}{p-1}}(B_t;\mathbb{R}^N)$ and $B \in \mathrm{L}^{\frac{p}{p-1}}(B_t;\mathbb{R}^N)$ with $\mathrm{div} B = 0$ such that

$$D\psi = D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2} + B \tag{3.1}$$

and

$$||D\psi||_{\mathrm{L}^{\frac{p}{p-1}}(B_t;\mathbb{R}^N)} \leq c_H ||D(\rho\tau_{s,h}u)||^{p-1}_{\mathrm{L}^p(B_t;\mathbb{R}^N)} \tag{3.2}$$

$$||B||_{\mathrm{L}^{\frac{p}{p-1}}(B_t;\mathbb{R}^N)} \leq c_H(2-p) ||D(\rho\tau_{s,h}u)||^{p-1}_{\mathrm{L}^p(B_t;\mathbb{R}^N)}, \tag{3.3}$$

where c_H is an absolute constant. Using $\varphi = \tau_{s,-h}(\psi)$ as a test function in the system (1.1), we get

$$\int_{B_t} \langle A(x,Du), D\varphi\rangle \,\mathrm{d}x = 0,$$

which, by virtue of $(d2)$ of Proposition 2.5, is equivalent to the following

$$\int_{B_t} \langle \tau_{s,h}A(x,Du), D\psi\rangle \,\mathrm{d}x = 0. \tag{3.4}$$

We write the left hand side of (3.4) as follows

$$\begin{aligned}
&\int_{B_t} \langle \tau_{s,h}A(x,Du), D\psi\rangle \,\mathrm{d}x \\
=\ &\int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x, Du(x)), D\psi \Big\rangle \,\mathrm{d}x \\
=\ &\int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D\psi \Big\rangle \,\mathrm{d}x \\
+\ &\int_{B_t} \Big\langle A(x+sh, Du(x)) - A(x, Du(x)), D\psi \Big\rangle \,\mathrm{d}x \\
=\ &\int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2} \Big\rangle \,\mathrm{d}x \\
+\ &\int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), B \Big\rangle \,\mathrm{d}x \\
+\ &\int_{B_t} \Big\langle A(x+sh, Du(x)) - A(x, Du(x)), D\psi \Big\rangle \,\mathrm{d}x\,,
\end{aligned} \tag{3.5}$$

where the last equality is due to the Hodge decomposition (3.1). Inserting (3.5) in (3.4), we have

$$\int_{B_t} \rho\Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2} \Big\rangle \,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{B_t} (\rho-1)\Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), B\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x)) - A(x, Du(x)), D\psi\Big\rangle \,\mathrm{d}x\,. \qquad (3.6)
\end{aligned}
$$

The homogeneity of the operator $A(x,\xi)$ at (H0) yields

$$
\begin{aligned}
& \int_{B_t} \Big\langle A(x+sh, \rho Du(x+sh)) - A(x+sh, \rho Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&= -\int_{B_t} (1-\rho)\Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), B\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x)) - A(x, Du(x)), D\psi\Big\rangle \,\mathrm{d}x\,. \qquad (3.7)
\end{aligned}
$$

Adding to both sides of (3.7) the quantity

$$
\int_{B_t} \Big\langle A(x+sh, \rho Du(x+sh) + \nabla\rho\otimes\tau_{s,h}u), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x\,,
$$

we obtain

$$
\begin{aligned}
& \int_{B_t} \Big\langle A(x+sh, \rho Du(x+sh) + \nabla\rho\otimes\tau_{s,h}u) - A(x+sh, \rho Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&= \int_{B_t} \Big\langle A(x+sh, \rho Du(x+sh) + \nabla\rho\otimes\tau_{s,h}u) - A(x+sh, \rho Du(x+sh)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} (1-\rho)\Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), D(\rho\tau_{s,h}u)|D(\rho\tau_{s,h}u)|^{p-2}\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x+sh)) - A(x+sh, Du(x)), B\Big\rangle \,\mathrm{d}x \\
&- \int_{B_t} \Big\langle A(x+sh, Du(x)) - A(x, Du(x)), D\psi\Big\rangle \,\mathrm{d}x\,. \qquad (3.8)
\end{aligned}
$$

The left hand side of (3.8) can be written as

$$
\int_{B_t}\Big\langle A(x+sh, \rho Du(x+sh)+\nabla\rho\otimes\tau_{s,h}u) - A(x+sh, \rho Du(x)), \rho Du(x+sh) - \rho Du(x) + \nabla\rho\otimes\tau_{s,h}u\Big\rangle\, |D(\rho\tau_{s,h}u)|^{p-2}
$$

and so, the monotonicity assumption (H1) with $\xi = \rho Du(x+sh) + \nabla\rho\otimes\tau_{s,h}u$ and $\eta = \rho Du(x)$ yields that

$$
\begin{aligned}
& \alpha\int_{B_t} |D(\rho\tau_{s,h}u)|^p \,\mathrm{d}x \\
&\le \int_{B_t} |A(x+sh, \rho Du(x+sh) + \nabla\rho\otimes\tau_{s,h}u) - A(x+sh, \rho Du(x+sh))||D(\rho\tau_{s,h}u)|^{p-1} \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{B_t\setminus B_s} |A(x+sh, Du(x+sh)) - A(x+sh, Du(x))||D(\rho\tau_{s,h}u)|^{p-1}\,\mathrm{d}x \\
&+ \int_{B_t} |A(x+sh, Du(x+sh)) - A(x+sh, Du(x))||B|\,\mathrm{d}x \\
&+ \int_{B_t} |A(x+sh, Du(x)) - A(x, Du(x))||D\psi|\,\mathrm{d}x \\
&=: I + II + III + IV\,, \qquad (3.9)
\end{aligned}
$$

where we also used the properties of ρ. In order to estimate I, we use the assumption (H2) and Young's inequality as follows

$$
\begin{aligned}
I &\le \beta \int_{B_t\setminus B_s} |\nabla\rho||\tau_{s,h}u||D(\rho\tau_{s,h}u)|^{p-1}\,\mathrm{d}x \\
&\le \frac{\beta}{(t-s)^p}\int_{B_t\setminus B_s} |\tau_{s,h}u|^p \mathrm{d}x + \beta\int_{B_t\setminus B_s} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x \\
&\le c\frac{\beta|h|^p}{(t-s)^p}\int_{B_{\lambda r}} |Du|^p \mathrm{d}x + \beta\int_{B_t\setminus B_s} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x, \qquad (3.10)
\end{aligned}
$$

where in last line we used Lemma 2.6. The assumption (H2) and Young's inequality yield

$$
\begin{aligned}
II &\le \beta \int_{B_t\setminus B_s} |Du(x+sh) - Du(x)||D(\rho\tau_{s,h}u)|^{p-1}\,\mathrm{d}x \\
&\le \beta \int_{B_t\setminus B_s} |\tau_{s,h}Du|^p\,\mathrm{d}x + \beta\int_{B_t\setminus B_s} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

In order to estimate III, we use again the Lipschitz continuity of the operator $A(x,\xi)$ at (H2) and Hölder's inequality thus getting

$$
\begin{aligned}
III &\le \beta\int_{B_t} |Du(x+sh) - Du(x)||B|\,\mathrm{d}x \\
&\le \beta\left(\int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_t} |B|^{\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le c_H\beta(2-p)\left(\int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_t} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le 4^{p-1}c_H\frac{\beta^p}{\alpha^{p-1}}(2-p)^p\int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x + \frac{\alpha}{4}\int_{B_t} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x,
\end{aligned}
$$

where we used (3.3) and Young's inequality. Since $p < 2$, we have that $4^{p-1}c_H < 4c_H =: \tilde{c}$, for a constant $\tilde{c}$ independent of p. Therefore

$$
III \le \tilde{c}\frac{\beta^p}{\alpha^{p-1}}(2-p)^p\int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x + \frac{\alpha}{4}\int_{B_t} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x. \qquad (3.12)
$$

In order to estimate IV, we use the assumption (H3), the fact that $k(x) \in \mathrm{L}^2(\Omega)$ and Hölder's inequality thus obtaining

$$
IV \le |h|\int_{B_t} (|k(x+sh)| + |k(x)|)|Du(x)|D\psi|\,\mathrm{d}x
$$

$$
\begin{aligned}
&\leq |h| \left(\int_{B_t} |D\psi|^{\frac{p}{p-1}} \, \mathrm{d}x \right)^{\frac{p-1}{p}} \\
&\cdot \left(\int_{B_t} (|k(x+sh)| + |k(x)|)^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_{\lambda R}} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2p}} \\
&\leq c|h| \left(\int_{B_R} |k(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_t} |D\psi|^{\frac{p}{p-1}} \, \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{B_t} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2p}},
\end{aligned}
$$

where we used Lemma 2.6 and c is an absolute constant. Using (3.2) and Young's inequality in previous estimate, we obtain

$$
\begin{aligned}
IV &\leq c_H |h| \left(\int_{B_R} |k(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{B_t} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2p}} \\
&\leq \frac{\alpha}{4} \int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x + \tilde{c}\frac{|h|^p}{\alpha^{p-1}} \left(\int_{B_R} |k(x)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_t} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2}}, \quad (3.13)
\end{aligned}
$$

where $\tilde{c}$ is a constant independent of p. Inserting estimates (3.10), (3.11), (3.12) and (3.13) in (3.9), we get

$$
\begin{aligned}
&\alpha \int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \\
\leq\ & \frac{\alpha}{2} \int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x + 2\beta \int_{B_t \setminus B_s} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \\
+\ & \beta \int_{B_t \setminus B_s} |\tau_{s,h}Du|^p \, \mathrm{d}x + \tilde{c}\frac{\beta^p}{\alpha^{p-1}}(2-p)^p \int_{B_t} |\tau_{s,h}Du|^p \, \mathrm{d}x \\
+\ & \tilde{c}\frac{|h|^p}{\alpha^{p-1}} \left(\int_{B_R} |k(x)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2}} \\
+\ & c\frac{\beta|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p \, \mathrm{d}x. \qquad (3.14)
\end{aligned}
$$

Reabsorbing the first integral in the right hand side of (3.14) by the left hand side, we have

$$
\begin{aligned}
&\frac{\alpha}{2} \int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \leq 2\beta \int_{B_t \setminus B_s} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \\
+\ & \beta \int_{B_t \setminus B_s} |\tau_{s,h}Du|^p \, \mathrm{d}x + \tilde{c}\frac{\beta^p}{\alpha^{p-1}}(2-p)^p \int_{B_t} |\tau_{s,h}Du|^p \, \mathrm{d}x \\
+\ & \tilde{c}\frac{|h|^p}{\alpha^{p-1}} \left(\int_{B_R} |k(x)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{2}} \\
+\ & c\frac{\beta|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p \, \mathrm{d}x. \qquad (3.15)
\end{aligned}
$$

By virtue of the elementary inequality $(|a| + |b|)^p \leq 2^p(|a|^p + |b|^p) \leq 4(|a|^p + |b|^p)$, we get

$$
\frac{\alpha}{2} \int_{B_t} |D(\rho\tau_{s,h}u)|^p \, \mathrm{d}x \leq 9\beta \int_{B_t \setminus B_s} |\tau_{s,h}Du|^p \, \mathrm{d}x
$$

$$
\begin{aligned}
&+ \quad 8\beta \int_{B_t\setminus B_s} |D\rho|^p|\tau_{s,h}u|^p\,\mathrm{d}x + \tilde{c}\frac{\beta^p}{\alpha^{p-1}}(2-p)^p \int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x \\
&+ \quad \tilde{c}\frac{|h|^p}{\alpha^{p-1}} \left(\int_{B_R} |k(x)|^2\,\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}}\,\mathrm{d}x\right)^{\frac{2-p}{2}} \\
&+ \quad c\frac{\beta|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p\,\mathrm{d}x \\
&\leq \quad 9\beta \int_{B_t\setminus B_s} |D\tau_{s,h}u|^p\,\mathrm{d}x + \tilde{c}\frac{\beta^p}{\alpha^{p-1}}(2-p)^p \int_{B_t} |\tau_{s,h}Du|^p\,\mathrm{d}x \\
&+ \quad \tilde{c}\frac{|h|^p}{\alpha^{p-1}} \left(\int_{B_R} |k(x)|^2\,\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}}\,\mathrm{d}x\right)^{\frac{2-p}{2}} \\
&+ \quad c\frac{\beta|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p\,\mathrm{d}x + 8\beta \int_{B_t\setminus B_s} |D\rho|^p|\tau_{s,h}u|^p\,\mathrm{d}x\,, \qquad (3.16)
\end{aligned}
$$

where c and $\tilde{c}$ are absolute constants. By the properties of ρ and Lemma 2.6, from (3.16) we infer that

$$
\begin{aligned}
&\int_{B_s} |D\tau_{s,h}u|^p\,\mathrm{d}x \leq \int_{B_t} |D(\rho\tau_{s,h}u)|^p\,\mathrm{d}x \\
&\leq \quad 18\frac{\beta}{\alpha} \int_{B_t\setminus B_s} |D\tau_{s,h}u|^p\,\mathrm{d}x + \tilde{c}\frac{\beta^p}{\alpha^p}(2-p)^p \int_{B_t} |D\tau_{s,h}u|^p\,\mathrm{d}x \\
&+ \quad \tilde{c}\frac{|h|^p}{\alpha^p} \left(\int_{B_R} |k(x)|^2\,\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}}\,\mathrm{d}x\right)^{\frac{2-p}{2}} \\
&+ \quad c(\alpha,\beta)\frac{|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p\mathrm{d}x\,. \qquad (3.17)
\end{aligned}
$$

Using the hole filling trick of Widman, i.e. adding to both sides of (3.17) the quantity

$$
\frac{18\beta}{\alpha} \int_{B_t\setminus B_s} |D(\tau_{s,h}u)|^p\,\mathrm{d}x,
$$

we get

$$
\begin{aligned}
&\int_{B_s} |D(\tau_{s,h}u)|^p\,\mathrm{d}x \leq \\
&\leq \quad \frac{18\beta\alpha^{p-1} + \tilde{c}\beta^p(2-p)^p}{18\beta\alpha^{p-1} + \alpha^p} \int_{B_t} |D(\tau_{s,h}u)|^p\,\mathrm{d}x \\
&+ \quad c(\alpha,\beta)|h|^p \left(\int_{B_R} |k(x)|^2\,\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}}\,\mathrm{d}x\right)^{\frac{2-p}{2}} \\
&+ \quad c(\alpha,\beta)\frac{|h|^p}{(t-s)^p} \int_{B_{\lambda r}} |Du|^p\,\mathrm{d}x. \qquad (3.18)
\end{aligned}
$$

Setting

$$
\varphi(r) = \int_{B_r} |D(\tau_{s,h}u)|^p\,\mathrm{d}x\,,
$$

we can write inequality (3.18) as

$$\varphi(s) \leq \vartheta\varphi(t) + A + \frac{B}{(t-s)^p},$$

where

$$\vartheta = \frac{18\beta\alpha^{p-1} + \tilde{c}\beta^p(2-p)^p}{18\beta\alpha^{p-1} + \alpha^p} \qquad A = c(\alpha,\beta)|h|^p \left(\int_{B_R} |k(x)|^2 \, dx\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}} \, dx\right)^{\frac{2-p}{2}}$$

and

$$B = c(\alpha,\beta)|h|^p \int_{B_{\lambda r}\setminus B_r} |Du|^p \, dx\,.$$

Choosing $p < 2$ such that

$$\vartheta = \frac{18\beta\alpha^{p-1} + \tilde{c}\beta^p(2-p)^p}{18\beta\alpha^{p-1} + \alpha^p} < 1 \quad \text{or, equivalently,} \quad p > 2 - \frac{\alpha}{\beta}\left(\frac{1}{\tilde{c}}\right)^{\frac{1}{p}},$$

we are legitimate to apply Lemma 2.3 thus obtaining

$$\begin{aligned} &\int_{B_r} |D(\tau_{s,h}u)|^p \, dx \\ \leq\ & c(\alpha,\beta,p)|h|^p \left(\int_{B_R} |k(x)|^2 \, dx\right)^{\frac{p}{2}} \left(\int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}} \, dx\right)^{\frac{2-p}{2}} \\ +\ & c(\alpha,\beta,p)\frac{|h|^p}{(\lambda-1)^p r^p} \int_{B_{\lambda r}} |Du|^p \, dx. \end{aligned} \tag{3.19}$$

By virtue of (3.19), we can apply Lemma 2.7 in order to obtain

$$\begin{aligned} \int_{B_r} |Du|^{\frac{2p}{2-p}} \, dx \ \leq\ & c\left(\int_{B_R} |k(x)|^2 \, dx\right)^{\frac{p}{2-p}} \int_{B_{\lambda r}} |Du(x)|^{\frac{2p}{2-p}} \, dx \\ +\ & \frac{c}{(\lambda-1)^{\frac{2p}{2-p}} r^{\frac{2p}{2-p}}} \left(\int_{B_{\lambda r}} |Du|^p \, dx\right)^{\frac{2}{2-p}} \end{aligned} \tag{3.20}$$

where $c = c(\alpha,\beta,p,N)$. By the absolute continuity of the integral we can choose $R_0 = R_0(\alpha,\beta,p,N)$ such that

$$c(\alpha,\beta,p,N)\left(\int_{B_{R_0}} |k(x)|^2 \, dx\right)^{\frac{p}{2-p}} \leq \frac{1}{2} \tag{3.21}$$

so that, if $R < R_0$, estimate (3.20) becomes

$$\int_{B_r} |Du|^{\frac{2p}{2-p}} \, dx \leq \frac{1}{2}\int_{B_{\lambda r}} |Du|^{\frac{2p}{2-p}} \, dx + \frac{c(\alpha,\beta,p,N)}{(\lambda-1)^{\frac{2p}{2-p}} r^{\frac{2p}{2-p}}} \left(\int_{B_{\lambda r}} |Du|^p \, dx\right)^{\frac{2}{2-p}}. \tag{3.22}$$

Since estimate (3.22) is valid for radii $\frac{R}{2} < r < \lambda r < R < R_0$ for any $\lambda > 1$, the iteration Lemma 2.3 with $\varphi(r) = \int_{B_r} |Du|^{\frac{2p}{2-p}}\,\mathrm{d}x$ and $\vartheta = \frac{1}{2}$ implies

$$\int_{B_{\frac{R}{2}}} |Du|^{\frac{2p}{2-p}}\,\mathrm{d}x \le \frac{c(\alpha,\beta,p,N)}{R^{\frac{2p}{2-p}}} \left(\int_{B_R} |Du|^p\,\mathrm{d}x\right)^{\frac{2}{2-p}} . \tag{3.23}$$

In view of (3.23) and (3.21) and by the arbitrariness of the ball $B_R \subset B_{R_0}$, estimate (3.19) can be written as follows

$$\int_{B_{\frac{R}{4}}} |D(\tau_{s,h}u)|^p\,\mathrm{d}x \le |h|^p \frac{c(\alpha,\beta,p,N)}{R^p} \int_{B_R} |Du|^p\,\mathrm{d}x \tag{3.24}$$

and therefore, by the use of Lemma 2.7, we conclude with the estimate

$$\int_{B_{\frac{R}{4}}} |D^2u|^p\,\mathrm{d}x \le \frac{c(\alpha,\beta,p,N)}{R^p} \int_{B_R} |Du|^p\,\mathrm{d}x\,. \tag{3.25}$$

Step 2. The approximation

Fix a compact set $\Omega' \Subset \Omega$, and for a smooth kernel $\phi \in \mathrm{C}^\infty_c(B_1(0))$ with $\phi \ge 0$ and $\int_{B_1(0)} \phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put

$$k_\varepsilon = k * \phi_\varepsilon$$

and

$$A_\varepsilon(x,\xi) = \int_{B_1} \phi(\omega) A(x+\varepsilon\omega,\xi)\,\mathrm{d}\omega \tag{3.26}$$

on Ω' for each positive $\varepsilon < \text{dist}\,(\Omega',\Omega)$. The assumptions (H0)–(H2) imply that

$$A_\varepsilon(x,\lambda\xi) = \lambda A_\varepsilon(x,\xi) \tag{A0}$$

$$\langle A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta), \xi-\eta\rangle \ge \alpha|\eta-\xi|^2 \tag{A1}$$

$$|A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta)| \le \beta|\xi-\eta| \tag{A2}$$

By virtue of assumption (H3), we have that

$$|A_\varepsilon(x_1,\xi) - A_\varepsilon(x_2,\xi)| \le (|k_\varepsilon(x_1)| + |k_\varepsilon(x_2)|)|x_1-x_2|(1+|\xi|)\,. \tag{A3}$$

for almost every $x_1, x_2 \in \Omega$ and for all $\xi,\eta \in \mathbb{R}^{N\times 2}$. Let u be a local solution of the system (1.1) and let fix a ball $B_R \Subset B_{\frac{R_0}{2}} \subset \Omega'$, where R_0 is the radius determined in previous step. Let us denote by $u_\varepsilon \in \mathrm{W}^{1,2}(B_R)$ the solution of the Dirichlet problem

$$\begin{cases} \mathrm{div} A_\varepsilon(x,Dv) = 0 & \text{in } B_R \\ \\ v = u & \text{on } \partial B_R\,. \end{cases} \tag{P$_\varepsilon$}$$

It is well known that $u_\varepsilon \in \mathrm{W}^{2,2}_{\mathrm{loc}}(B_R;\mathbb{R}^N)$ and, since A_ε satisfies conditions (A0)–(A3), for ε sufficiently small, we are legitimate to apply estimate (3.25) to get

$$\int_{B_{\frac{r}{4}}} |D^2u_\varepsilon|^p\,\mathrm{d}x \le \frac{c(\alpha,\beta,p,N)}{r^p} \int_{B_r} (|Du_\varepsilon|^p)\,\mathrm{d}x\,, \tag{3.27}$$

for every ball $B_r \Subset B_R$. By the inequality (A1) and using that u_ε solves the problem (P_ε), we get that

$$\begin{aligned}\alpha \int_{B_R} |Du_\varepsilon|^2 \mathrm{d}x &\leq \int_{B_R} \langle A_\varepsilon(x, Du_\varepsilon), Du_\varepsilon\rangle \,\mathrm{d}x \\ &= \int_{B_R} \langle A_\varepsilon(x, Du_\varepsilon), Du\rangle \,\mathrm{d}x \leq \beta \int_{B_r} |Du_\varepsilon||Du| \,\mathrm{d}x \\ &\leq \frac{\alpha}{2} \int_{B_R} |Du_\varepsilon|^2 \mathrm{d}x + c(\alpha,\beta) \int_{B_R} |Du|^2 \,\mathrm{d}x\,, \end{aligned} \tag{3.28}$$

where we also used (A2) and Young's inequality. Reabsorbing the first integral in the right hand side of (3.28) by the left hand side we obtain

$$\int_{B_R} |Du_\varepsilon|^2 \mathrm{d}x \leq c(\alpha,\beta) \int_{B_R} |Du|^2 \,\mathrm{d}x\,, \tag{3.29}$$

i.e., the sequence (Du_ε) is bounded in $\mathrm{L}^2(B_R, \mathbb{R}^N)$. Therefore, there exists a not relabeled sequence u_ε such that

$$u_\varepsilon \rightharpoonup v \qquad \text{weakly} \qquad \text{in} \qquad \mathrm{W}^{1,2}(B_R; \mathbb{R}^N).$$

Our next aim is to prove that the sequence Du_ε converge to Dv in measure, i.e we shall prove that for every $\eta,\ \lambda > 0$, there exists $\nu = \nu(\eta, \lambda)$ such that

$$|\{x \in B_R :\ |Du_\varepsilon - Du_{\varepsilon'}| > \lambda\}| < \eta, \tag{3.30}$$

for every ε and ε' in $(0,\nu)$. For some $B > 1$, let us define the sets

$$\begin{aligned} E_1 &= \{x \in B_R :\ |Du_\varepsilon| > B\} \cup \{x \in B_R :\ |Du_{\varepsilon'}| > B\}, \\ E_2 &= \{x \in B_R :\ |Du_\varepsilon| \leq B,\ |Du_{\varepsilon'}| \leq B,\ |Du_\varepsilon - Du_{\varepsilon'}| > \lambda\}. \end{aligned}$$

Obviuosly, we have

$$\{x \in B_R :\ |Du_\varepsilon - Du_{\varepsilon'}| > \lambda\} \subset E_1 \cup E_2.$$

Since, by (3.29), the following estimate holds

$$||Du_\varepsilon||_{L^2(B_R)} \leq c(\alpha,\beta)||Du||_{L^2(B_R)} := C,$$

we have that $|E_1| < \frac{\eta}{2}$, for $B \geq C\sqrt{\frac{2}{\eta}}$, independently of $\varepsilon, \varepsilon'$. So from now on, we will suppose that $B = 2C\sqrt{\frac{2}{\eta}}$. The definition of A_ε and elementary calculations yield

$$\begin{aligned} &\langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \\ \geq\ & \alpha|Du_\varepsilon - Du_{\varepsilon'}|^2 - |\varepsilon - \varepsilon'|\,|k_\varepsilon + k_{\varepsilon'}||Du_\varepsilon||Du_\varepsilon - Du_{\varepsilon'}| \end{aligned}$$

and so, integarting previous estimate over the set E_2, we get

$$\alpha \int_{E_2} |Du_\varepsilon - Du_{\varepsilon'}|^2 \mathrm{d}x$$

$$
\begin{aligned}
&\leq \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \,\mathrm{d}x \\
&+ |\varepsilon - \varepsilon'| \int_{E_2} |k_\varepsilon + k_{\varepsilon'}||Du_\varepsilon||Du_\varepsilon - Du_{\varepsilon'}| \,\mathrm{d}x \\
&\leq \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \,\mathrm{d}x \\
&+ \frac{\alpha}{2}\int_{E_2} |Du_\varepsilon - Du_{\varepsilon'}|^2 \mathrm{d}x + c|\varepsilon - \varepsilon'|^2 \int_{E_2} |k_\varepsilon + k_{\varepsilon'}|^2 |Du_\varepsilon|^2 \,\mathrm{d}x\,, \qquad (3.31)
\end{aligned}
$$

where we used Young's inequality. Reabsorbing the second integral in the right hand side of (3.31) by the left hand side and using the definition of E_2, we deduce that

$$
\begin{aligned}
&\frac{\alpha}{2}\int_{E_2} |Du_\varepsilon - Du_{\varepsilon'}|^2 \mathrm{d}x \\
\leq\ & \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \mathrm{d}x + cB^2|\varepsilon - \varepsilon'|^2 \int_{B_R} (k_\varepsilon^2 + k_{\varepsilon'}^2) \,\mathrm{d}x \\
\leq\ & \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \mathrm{d}x + c\frac{|\varepsilon - \varepsilon'|^2}{\eta} \int_{B_R} (k_\varepsilon^2 + k_{\varepsilon'}^2) \mathrm{d}x, \qquad (3.32)
\end{aligned}
$$

where, in the last inequality, we used that $B = 2C\sqrt{\frac{2}{\eta}}$. We can verify, as in [3], that E_2 is a compact set. In order to estimate the first integral in the right hand side of (3.32), let us denote by $E_{2,t}$, for every $t > 0$, the set $E_{2,t} = \{x \in E_2 : \operatorname{dist}(x, \partial E_2) > t\}$. Consider the subset $L_t = E_{2,\frac{t}{2}} \setminus \overline{E_{2,t}}$ and a smooth cut-off function $\psi_t \in C_0^\infty(E_{2,\frac{t}{2}}; [0,1])$ such that $\psi_t = 1$ on $E_{2,t}$. As the thickness of the strip L_t is of order t, we have an upper bound of the form $||\nabla\psi_t||_\infty \leq \frac{c}{t}$. Using $\psi_t(u_\varepsilon - u_{\varepsilon'})$ as test function in problems (P_ε) and $(\mathrm{P}_{\varepsilon'})$, we get

$$
\begin{aligned}
&\int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
=\ & \int_{E_{2,t}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
+\ & \int_{E_2\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
=\ & \int_{E_{2,\frac{t}{2}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
+\ & \int_{E_2\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
-\ & \int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
=\ & -\int_{E_{2,\frac{t}{2}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \nabla\psi_t(u_\varepsilon - u_{\varepsilon'})\rangle dx \\
+\ & \int_{E_2\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx
\end{aligned}
$$

$$
\begin{aligned}
& - \int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x,Du_\varepsilon) - A_{\varepsilon'}(x,Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
\leq\ & \frac{c}{t}\int_{E_2} |A_\varepsilon(x,Du_\varepsilon) - A_{\varepsilon'}(x,Du_{\varepsilon'})||u_\varepsilon - u_{\varepsilon'}|dx \\
+\ & 2B\int_{E_2\setminus\overline{E_{2,t}}} |A_\varepsilon(x,Du_\varepsilon) - A_{\varepsilon'}(x,Du_{\varepsilon'})|\mathrm{d}x \\
+\ & 2B\int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} |A_\varepsilon(x,Du_\varepsilon) - A_{\varepsilon'}(x,Du_{\varepsilon'})|\mathrm{d}x \\
\leq\ & \frac{cB}{t}\int_{E_2} |u_\varepsilon - u_{\varepsilon'}|dx + cB^2\Big(|E_2\setminus\overline{E_{2,t}}| + |E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}|\Big)\,, \qquad (3.33)
\end{aligned}
$$

where we used that $|Du_\varepsilon| \leq B$ and $|Du_{\varepsilon'}| \leq B$ on the set E_2. Using Hölder's inequality and that $B = 2C\sqrt{\frac{2}{\eta}}$ in (3.33), we obtain

$$
\begin{aligned}
& \int_{E_2} \langle A_\varepsilon(x,Du_\varepsilon) - A_{\varepsilon'}(x,Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
\leq\ & \frac{c}{t\sqrt{\eta}}\left(\int_{B_R} |u_\varepsilon - u_{\varepsilon'}|^2\mathrm{d}x\right)^{\frac{1}{2}} + \frac{c}{\eta}(|E_2\setminus\overline{E_{2,t}}| + |E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}|) \\
\leq\ & \frac{c}{t\sqrt{\eta}}\left(\int_{B_R} |u_\varepsilon - u_{\varepsilon'}|^2\mathrm{d}x\right)^{\frac{1}{2}} + \frac{ct}{\eta}\,, \qquad (3.34)
\end{aligned}
$$

where in the last line we used that the thickness of the strips $E_2\setminus\overline{E_{2,t}}$ and $E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}$ is of order t. Choosing $t = \eta^2\lambda$ and inserting estimate (3.34) in (3.32), we finally obtain

$$
\begin{aligned}
\frac{\alpha}{2}\int_{E_2} |Du_\varepsilon - Du_{\varepsilon'}|^2dx \ &\leq\ \frac{c}{\eta^2\sqrt{\eta}\lambda}||u_\varepsilon - u_{\varepsilon'}||_2 + c\eta\lambda + c\frac{|\varepsilon-\varepsilon'|^2}{\eta}\int_{B_R}(k_\varepsilon^2 + k_{\varepsilon'}^2)\mathrm{d}x \\
&\leq\ \frac{c}{\eta^2\sqrt{\eta}\lambda}||u_\varepsilon - u_{\varepsilon'}||_2 + c\eta\lambda + c(||k||_2)\frac{|\varepsilon-\varepsilon'|^2}{\eta}\,, \qquad (3.35)
\end{aligned}
$$

where we used that $\lim_\varepsilon ||k_\varepsilon||_2 = ||k||_2$. The strong convergence of the sequence u_ε in L^2 implies that there exists $\nu(\eta,\lambda)$ such that for every $\varepsilon, \varepsilon' < \nu$ we have

$$
\frac{\alpha}{2}\int_{E_2} |Du_\varepsilon - Du_{\varepsilon'}|^2dx \leq c\eta\lambda, \qquad (3.36)
$$

and hence (3.30) follows. The convergence in measure of (Du_ε) together with the weak convergence of (Du_ε) to Dv in $\mathrm{L}^2(B_R,\mathbb{R}^N)$ implies that

$$
Du_\varepsilon \to Dv \qquad \text{strongly in } \mathrm{L}^p(B_R,\mathbb{R}^N), \qquad (3.37)
$$

for every $1 < p < 2$. Next we show that v solves the Dirichlet problem

$$
\begin{cases}
\mathrm{div}A(x,Dv) = 0 & \text{in } B_R \\
\\
v = u & \text{on } \partial B_R\,.
\end{cases} \qquad (P)
$$

In fact, for every test function $\varphi \in C_0^\infty(B_R;\mathbb{R}^N)$, we have

$$
\begin{aligned}
& \left|\int_{B_R} \langle A(x,Dv), D\varphi\rangle \,\mathrm{d}x\right| \\
\leq\ & \left|\int_{B_R} \langle A(x,Dv), D\varphi\rangle \,\mathrm{d}x - \int_{B_R} \langle A_\varepsilon(x,Dv), D\varphi\rangle \,\mathrm{d}x\right| \\
+\ & \left|\int_{B_R} \langle A_\varepsilon(x,Dv), D\varphi\rangle \,\mathrm{d}x - \int_{B_R} \langle A_\varepsilon(x,Du_\varepsilon), D\varphi\rangle \,\mathrm{d}x\right| \\
\leq\ & ||D\varphi||_\infty \left\{\varepsilon\Big(\int_{B_R} (|k|^2+|k|_\varepsilon^2)\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{B_R} |Dv|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} + \beta\int_{B_R} |Dv-Du_\varepsilon|\,\mathrm{d}x\right\} \quad (3.38)
\end{aligned}
$$

where we used that

$$\int_{B_R} \langle A_\varepsilon(x,Du_\varepsilon), D\varphi\rangle \,\mathrm{d}x = 0.$$

Since $Du_\varepsilon \to Dv$ strongly in $\mathrm{L}^p(B_R,\mathbb{R}^N)$, taking the limit as $\varepsilon \to 0$ in (3.38) we obtain

$$\int_{B_R} \langle A(x,Dv), D\varphi\rangle \,\mathrm{d}x = 0.$$

Since $v = u$ on ∂B_R in the sense of traces, we can use $u - v$ as a test function in the system thus obtaining

$$\alpha\int_{B_R} |Du-Dv|^2\,\mathrm{d}x \leq \int_{B_R} \langle A(x,Dv)-A(x,Du), Du-Dv\rangle \,\mathrm{d}x = 0$$

and therefore

$$u \equiv v \qquad\qquad \text{in } \ B_R.$$

By passing to the limit as $\varepsilon \searrow 0$ in (3.27), thanks to Fatou's Lemma and (3.37), we finally get

$$\int_{B_{\frac{r}{2}}} |D^2u|^p\,\mathrm{d}x \leq \frac{c(\alpha,\beta,p,N)}{r^p}\int_{B_r} |Du|^p\,\mathrm{d}x \tag{3.39}$$

which is the conclusion. □

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let u be a local minimizer of the functional (1.2). It is well known that, thanks to the assumptions (F1)-(F2), u is a local solution of the Euler Lagrange system

$$\mathrm{div} D_\xi F(x,Du) = 0.$$

One can easily check that the operator $D_\xi F(x,\xi)$ satisfies the assumptions (H0)–(H3) and we conclude by virtue of Theorem 1.1. □

4 Examples

Our first example shows that the conclusion of Theorem 1.1 is sharp. In fact, for the second derivatives of the solution of the system (1.1) one can not expects better integrability than L^p, for p arbitrarily close to 2. In fact we have the following

Example 4.1. *There exists a linear elliptic equations whose coefficients matrix satisfies assumptions (H0)–(H3) for a function $k \in L^2(B_{\frac{1}{e}})$, that admits a solution $u \in \mathrm{W}^{2,p}(B_{\frac{1}{e}})$, for every $p < 2$ but*

$$u \notin \mathrm{W}^{2,2}(B_{\frac{1}{e}}).$$

Proof. Let us consider the function $u:\ B\left(0, \frac{1}{e}\right) \to \mathbb{R}$ defined as

$$u(x_1, x_2) = x_1(1 - \log|x|)$$

Define the matrix

$$A(x) = \begin{pmatrix} \frac{\log|x|-1}{\log|x|} + \frac{2\log|x|-1}{\log|x|(\log|x|-1)} \frac{x_2^2}{|x|^2} & -\frac{1}{\log|x|-1} \frac{x_1 x_2}{|x|^2} \\ & \\ -\frac{1}{\log|x|-1} \frac{x_1 x_2}{|x|^2} & \frac{\log|x|-1}{\log|x|} + \frac{2\log|x|-1}{\log|x|(\log|x|-1)} \frac{x_1^2}{|x|^2} \end{pmatrix}$$

One can easily check that the operator $A\colon\ (x, \eta) \in B_{\frac{1}{e}} \times \mathbb{R}^2 \to A(x) \cdot \eta \in \mathbb{R}^2$ satisfies the following :

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \frac{1}{2}|\xi - \eta|^2 \tag{H1}$$

$$|A(x, \xi) - A(x, \eta)| \leq 2|\xi - \eta| \tag{H2}$$

and

$$|A(x_1, \xi) - A(x_2, \xi)| \leq (|k(x_1)| + |k(x_2)|)|x_1 - x_2|(1 + |\xi|)\,. \tag{H3}$$

where k is defined as

$$k(x) = \frac{1}{|x|} \frac{1}{1 - \log|x|}.$$

Notice that $k \in \mathrm{L}^2(B_{\frac{1}{e}})$, and, since

$$|D^2 u| \simeq \frac{1}{|x|},$$

we have that $u \in \mathrm{W}^{2,p}(B_{\frac{1}{e}})$ for every $p < 2$, but

$$u \notin \mathrm{W}^{2,2}(B_{\frac{1}{e}}).$$

□

Our second example shows that the assumption $k(x) \in \mathrm{L}^2$ in Theorem 1.1 is sharp in order to obtain that the second derivatives of the solution of the system (1.1) have degree of integrability arbitrarily close to 2. In fact we have the following

Example 4.2. *There exists a linear elliptic equations whose coefficients matrix satisfies assumptions*

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle \geq \alpha |\eta - \xi|^2 \tag{H1}$$

$$|A(x,\xi) - A(x,\eta)| \leq \frac{1}{\alpha} |\xi - \eta| \tag{H2}$$

for a fixed $\alpha \in (0,1)$*. Moreover*

$$|A(x_1,\xi) - A(x_2,\xi)| \leq (|k(x_1)| + |k(x_2)|)|x_1 - x_2|(1 + |\xi|) \tag{H3}$$

for a function $k \in L^p(B_{\frac{1}{e}})$*, for every* $p < 2$*. Furthermore the equation*

$$\mathrm{div} A(x) \nabla u = 0$$

admits a solution $u \in \mathrm{W}^{2,p}(B_{\frac{1}{e}})$*, for every*

$$p < \frac{2}{2-\alpha}.$$

Proof. Let us consider the function $u : B(0,1) \to \mathbb{R}$ defined as

$$u(x_1, x_2) = x_1 |x|^{\alpha - 1}$$

for a fixed $\alpha \in (0,1)$. Let us define the matrix $A(x)$ as

$$A(x) = \begin{pmatrix} \frac{1}{\alpha} + \frac{\alpha^2-1}{\alpha} \frac{x_2^2}{|x|^2} & -\frac{\alpha^2-1}{\alpha} \frac{x_1 x_2}{|x|^2} \\ \\ -\frac{\alpha^2-1}{\alpha} \frac{x_1 x_2}{|x|^2} & \frac{1}{\alpha} + \frac{\alpha^2-1}{\alpha} \frac{x_1^2}{|x|^2} \end{pmatrix}$$

One can easily check that

$$\mathrm{div}(A(x) \nabla u) = 0$$

and

$$\alpha |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \frac{1}{\alpha} |\xi|^2 .$$

Moreover

$$|A(x_1,\xi) - A(x_2,\xi)| \leq (|k(x_1)| + |k(x_2)|)|x_1 - x_2|(1 + |\xi|) \tag{H3}$$

for the function

$$k(x) \simeq \frac{1}{|x|}$$

and therefore $k \in L^p(B_{\frac{1}{e}})$, for every $p < 2$. For the function u, we have that

$$|D^2 u| \simeq |x|^{\alpha - 2},$$

hence for the second derivatives of u, there is no better integrability than L^q, for $q < \frac{2}{2-\alpha}$. □

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