VARIATIONAL ANALYSIS OF A MEAN CURVATURE FLOW ACTION FUNCTIONAL

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Abstract. We consider the reduced Allen–Cahn action functional, which arises as the sharp interface limit of the Allen–Cahn action functional and can be understood as a formal action functional for a stochastically perturbed mean curvature flow. For suitable evolutions of (generalized) hypersurfaces this functional consists of the integral over time and space of the sum of the squares of the mean curvature and of the velocity vectors. Given initial and final conditions, we investigate the associated action minimization problem, for which we propose a weak formulation, and within the latter we prove compactness and lower-semicontinuity properties of a suitably generalized action functional. Furthermore, we derive the Euler–Lagrange equation for smooth stationary trajectories and investigate some related conserved quantities. To conclude, we analyze the explicit case in which the initial and final data are concentric spheres. In this particular situation we characterize the properties of the minimizing rotationally symmetric trajectory in dependence of the given time span.

1. Introduction

Action functionals arise in the large deviation theory for stochastically perturbed ODEs and PDEs as the lowest order term in a small noise expansion. The value of the action-functional on a given deterministic path is related to the probability that the solutions of the stochastic dynamics are close to that path. For prescribed initial and final states, an action minimizer can be associated with a most likely connecting path.

As a formal approximation of a stochastic mean curvature flow evolution, we consider the Allen–Cahn equation perturbed by additive noise, i.e.

$$\varepsilon\partial_t u \,=\, \varepsilon\Delta u - \frac{1}{\varepsilon}W'(u) + \sqrt{2\gamma}\eta. \tag{1.1}$$

Here $\varepsilon, \gamma > 0$ are respectively the interface thickness and noise-intensity parameter, W is a fixed double-well potential, and η describes a time-space white noise. In general this equation admits function-valued solutions only in one spatial dimension and a regularization for the noise term is thus necessary.

The Allen–Cahn action functional has been computed in [8] and in [9],[13], in the case of one and higher spatial dimensions respectively and can be written as

$$\tilde{\mathcal{S}}_\varepsilon(u) \,:=\, \int_0^T \int_\Omega \Big(\sqrt{\varepsilon}\partial_t u + \frac{1}{\sqrt{\varepsilon}}\big(- \varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\big)\Big)^2 \,dx\,dt. \tag{1.2}$$

Expanding the square and observing that the mixed term is a time derivative, one obtains that, for fixed initial and final data, the action minimization problem is equivalent to the minimization of the functional

$$\mathcal{S}_\varepsilon(u) \,:=\, \int_0^T \int_\Omega \varepsilon(\partial_t u)^2 + \frac{1}{\varepsilon}\big(- \varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\big)^2 \,dx\,dt. \tag{1.3}$$

In a series of papers [6, 10, 13, 14, 23, 20] reduced action functionals, defined as the sharp interface limit for $\varepsilon \to 0$ of $\tilde{\mathcal{S}}_\varepsilon$ or $\mathcal{S}_\varepsilon$, have been considered. In [13] it was shown that not only

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smooth evolutions, but also evolutions that exhibit the nucleation of new components can be approximated with finite action $\tilde{\mathcal{S}}_\varepsilon$. The latter are given by families $\Sigma = (\Sigma_t)_{t\in(0,T)}$ that consist of smooth surfaces and that evolve smoothly except for finitely many 'singular times' where a new component is nucleated in the form of a double interface that subsequently generates a non-zero enclosed volume. For such evolutions a reduced action has been derived and reads (for a precise statement see [13])

$$\tilde{\mathcal{S}}_0(\Sigma) := c_0 \int_0^T \int_{\Sigma_t} \left|\vec{v}(x,t) - \vec{\mathrm{H}}(x,t)\right|^2 d\mathcal{H}^n(x)dt + 4\tilde{\mathcal{S}}_{0,nuc}(\Sigma), \tag{1.4}$$

$$\tilde{\mathcal{S}}_{0,nuc}(\Sigma) := 2c_0 \sum_i \mathcal{H}^n(\Sigma^i), \tag{1.5}$$

where $n+1$ is the space dimension, Σ^i denotes the i^{th} component of Σ at its nucleation time, $\vec{v}$ denotes the normal velocity vector for the evolution $(\Sigma_t)_{t\in(0,T)}$, $\vec{\mathrm{H}}(t,\cdot)$ denotes the mean curvature vector of Σ_t, and the constant c_0 depends only on the choice of the double-well potential W. The reduced action functional of $\mathcal{S}_\varepsilon$ is given by

$$\mathcal{S}_0(\Sigma) := c_0 \int_0^T \int_{\Sigma_t} \left(|\vec{v}(x,t)|^2 + |\vec{\mathrm{H}}(x,t)|^2\right) d\mathcal{H}^n(x)dt + 2\mathcal{S}_{0,nuc}(\Sigma), \tag{1.6}$$

$$\mathcal{S}_{0,nuc}(\Sigma) := 2c_0 \sum_i \mathcal{H}^n(\Sigma^i), \tag{1.7}$$

where the summation in the last line is now over the singular times at which nucleation or annihilation occur, and Σ^i denotes the components which are created or annihilated. In [23], in the case of one space dimension, a generalization of $\mathcal{S}_0$ has been introduced and the Γ–convergence of $\mathcal{S}_\varepsilon$ to $\mathcal{S}_0$ has been proved. In [20] a general compactness statement for the sharp interface limit of sequences with bounded action $\tilde{\mathcal{S}}_\varepsilon$ and initial or final data with uniformly controlled diffuse surface area has been shown. Moreover, a generalized reduced action functional has been proposed and a lower bound estimate has been proven.

It is well known [2, 5, 7, 12, 24] that solutions of the Allen–Cahn equation converge to the evolution by mean curvature flow of phase boundaries. Therefore, the reduced action functional can formally be considered as a mean curvature flow action functional, although at present no rigorous connection to a suitable stochastically perturbed mean curvature flow is known. The goal of this paper is to study such formal mean curvature flow action functional for evolutions of generalized hypersurfaces. We restrict here to a particular generalization of the functional $\mathcal{S}_0$ defined above, which has the property of being invariant under time-inversion. Independently of the question whether this functional represents an action functional or not, its variational analysis helps to gain a better understanding of the behavior of the Allen–Cahn action functional itself. Moreover, variational approaches to study evolutions of surfaces have some interest in its own, extending classical shape optimization techniques for surfaces to the dynamic case and providing a different characterization of solutions. Bellettini and Mugnai [3] study a functional closely related to $\tilde{\mathcal{S}}_0$ and introduce a concept of variational solutions to mean curvature flow as corresponding minimizers with respect to given initial conditions. The authors also present a weak form of the minimum problem, similar to but different from the weak formulation of the functional $\mathcal{S}_0$ that we will develop below.

The regular part of the functional $\mathcal{S}_0$ consists of a sum of a Willmore energy term and a velocity term. The Willmore functional has been studied intensively over the last decades, see for example [15, 16, 17, 22, 26, 27, 28] and is still an active field of geometric analysis. The minimization of the velocity part for given initial and final data is related to an L^2-geodesic distance between them. In [18], using minimizing sequences with highly curved structures, it has been shown that this distance degenerates and is always zero. The Willmore term in the functional $\mathcal{S}_0$ penalizes regions with high curvature and therefore represents a regularization of the velocity term. In the minimization of the action functional we therefore see an interesting interplay of a stationary and of a dynamic contribution.

In this paper we present a variational analysis of a reduced Allen–Cahn action functional. Our first goal is to prove a compactness and lower semicontinuity theorem, which ensures the applicability of the direct method of the calculus of variations, and consequently gives the existence of minimizers.
In the class of smooth evolutions a uniform bound on the action for a (minimizing) sequence does not provide sufficient control to derive a compactness statement in this class. In Section 2 we therefore provide a new generalized formulation in a specific class of evolutions of surface area measures, and in Section 3 we show compactness and lower semicontinuity properties for uniformly action bounded sequences of generalized evolutions. Lower-semicontinuity properties have not been shown in previous formulations of reduced Allen–Cahn action functionals and represent one main contribution of the present work. The analysis of generalized evolutions and the application of the direct method of the calculus of variations in the first part of this paper is complemented in the second part by the study of some properties of the smooth stationary points for the action functional. We derive the Euler–Lagrange equation for the action-minimization problem (Section 4) and in Section 5 we study some associated conserved quantities, which reveal some analogies with Lagrangian mechanics. In Section 6 we consider as a specific example the problem of finding the action-optimal connection between two concentric circles. We characterize the minimizers in the class of rotationally symmetric solutions and we describe their minimality properties in relation to the full class of smooth evolutions. The behavior of these stationary evolutions with rotational symmetry crucially depends on the time-span which is at disposal to connect the initial and the final state.

General notation. Let $n \in \mathbb{N}$ be fixed and consider for $T > 0$ the space-time domain $Q_T := \mathbb{R}^{n+1} \times (0,T)$. For a function $\eta \in C^1(Q_T)$ we respectively denote by $\nabla\eta$, $\partial_t\eta$, $\nabla'\eta$ the gradient with respect to the spatial variables, to the time derivative, and the space-time gradient. In particular we have $\nabla'\eta = (\nabla\eta, \partial_t\eta)^T$.
For a function $u \in BV(Q_T)$, we denote by $\nabla u, \partial_t u, \nabla' u$ the signed measures respectively associated with the distributional derivative of u in the x, t, and (x,t)-variables. With $|\nabla u|, |\partial_t u|, |\nabla' u|$ we denote the corresponding total variation measures. For a set $E \subset \mathbb{R}^{n+1}$ of finite perimeter we denote by $\partial^* E$ the essential boundary of E.

For a family of Radon measures $(\mu_t)_{t\in(0,T)}$ we denote by $\mu = \mu_t \otimes \mathcal{L}^1$ the product measure, i.e.

$$\mu(\eta) \;=\; \int_0^T \mu_t(\eta(\cdot,t))\,dt \qquad \text{for all } \eta \in C^0_c(Q_T).$$

Throughout the paper we identify an integral n-varifold V with its associated weight-measure $\mu = \mu_V$. For notation on geometric measure theory we refer to the book of Simon [25].

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2. Generalized action functional

A sequence of smooth evolutions with uniformly bounded action $\mathcal{S}_0$ does not necessarily converge to a smooth evolution, even up to finitely many singular times. Thus, in order to ensure lower semicontinuity and compactness properties for generalized evolutions with uniformly bounded action, we need to define a suitable class of weak evolutions and a suitably generalized expression for the action functional in this class.
We recall the definition of L^2-flows [20] and in particular a characterization of the velocity for certain evolutions of varifolds.

Definition 2.1. Let $T > 0$ be given. Consider a family $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)}$ of Radon measures on $\mathbb{R}^{n+1}$ and associate to $\boldsymbol{\mu}$ the product measure $\mu := \mu_t \otimes \mathcal{L}^1$. We call $\boldsymbol{\mu}$ an L^2*-flow* if the following properties hold:

for almost all $t \in (0,T)$

$$\mu_t \text{ is an integral } n\text{-varifold with } \sup_{0<t<T} \mu_t(\mathbb{R}^{n+1}) < \infty, \tag{2.1}$$

$$\mu_t \text{ has weak mean curvature } \vec{\mathrm{H}} \in L^2(\mu_t;\mathbb{R}^{n+1}). \tag{2.2}$$

The evolution $\boldsymbol{\mu}$ has a generalized normal velocity $\vec{v} \in L^2(\mu;\mathbb{R}^{n+1})$, i.e.

$$t \mapsto \mu_t(\psi) \quad \text{is of bounded variation in } (0,T) \text{ for all } \psi \in C^1_c(\mathbb{R}^{n+1}), \tag{2.3}$$

$$\vec{v}(x,t) \perp T_x\mu_t \quad \text{for } \mu\text{-almost all } (x,t) \in Q_T, \tag{2.4}$$

$$\sup_\eta \Big| \int_{Q_T} (\partial_t\eta + \nabla\eta\cdot\vec{v}) d\mu_t dt\Big| < \infty, \tag{2.5}$$

where the supremum is taken over all $\eta \in C^1_c(Q_T)$ with $|\eta| \le 1$.

Remark 2.2. The definition of generalized velocity for a family of varifolds is motivated by the classical formula for the time derivative of the localized surface area in the case of a smooth evolution $(\Sigma_t)_{t\in(0,T)}$ of smooth surfaces

$$\frac{d}{dt}\int_{\Sigma_t} \eta \, d\mathcal{H}^n = \int_{\Sigma_t} \big(\partial_t\eta + \nabla\eta\cdot\vec{v} - \eta\vec{v}\cdot\vec{\mathrm{H}}\big)\, d\mathcal{H}^n \,,$$

for any $\eta \in C^1_c(Q_T)$. Integrating in time and using the Cauchy–Schwarz inequality, we obtain

$$\Big|\int_0^T\int_{\Sigma_t} (\partial_t\eta + \nabla\eta\cdot\vec{v})\, d\mathcal{H}^n\, dt\Big| \le \|\eta\|_{C^0(Q_T)}\Big(\int_0^T\int_{\Sigma_t} |\vec{v}|^2\, d\mathcal{H}^n\, dt\Big)^{\frac12}\Big(\int_0^T\int_{\Sigma_t} |\vec{\mathrm{H}}|^2\, d\mathcal{H}^n\, dt\Big)^{\frac12},$$

which shows that (2.5) holds in the smooth case.

The evolution of measures $t \mapsto \mu_t(\psi)$, $\psi \in C^1_c(\mathbb{R}^{n+1})$ can be controlled only in $BV((0,T))$ and therefore the limit of a sequence of evolutions of measures with uniformly bounded action functional may exhibit jumps in the evolution of limit measures, even in the case that all the elements of the sequence are smooth. Thus, in order to formulate initial and final conditions and to be able to preserve them taking limits, we complement the evolution of measures by an evolution of phases. For the action minimization problem we will therefore consider the following class of generalized evolutions.

Definition 2.3. Let $T > 0$ be given as well as two open and bounded subsets $\Omega(0) \subset \mathbb{R}^{n+1}$ and $\Omega(T) \subset \mathbb{R}^{n+1}$ with finite perimeter. Let $\mathcal{M} = \mathcal{M}(T,\Omega(0),\Omega(T))$ be the class of tuples $\boldsymbol{\Sigma} = (\boldsymbol{\mu},\boldsymbol{u})$, $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)}$, $\boldsymbol{u} = (u(\cdot,t))_{t\in[0,T]}$, with the following properties:

the evolution $\boldsymbol{\mu}$ is an L^2-flow in the sense of Definition 2.1,
for almost all $t \in (0,T)$

$$u(\cdot,t) \in BV(\mathbb{R}^{n+1},\{0,1\}), \tag{2.6}$$

$$|\nabla u(\cdot,t)| \le \mu_t \tag{2.7}$$

holds, the evolution of phases $\boldsymbol{u}$ satisfies $u \in C^{\frac12}([0,T];L^1(\mathbb{R}^{n+1}))$, $\boldsymbol{u}$ attains the initial and final data

$$u(\cdot,0) = \mathcal{X}_{\Omega(0)}, \quad u(\cdot,T) = \mathcal{X}_{\Omega(T)}, \tag{2.8}$$

and

$$\int_{Q_T} \partial_t\eta(x,t)u(x,t)\, dx\, dt = \int_{Q_T} \eta(x,t)\vec{v}(x,t)\cdot\nu(x,t)\, d|\nabla u(\cdot,t)|\, dt \tag{2.9}$$

holds for all $\eta \in C^1_c(Q_T)$, where $\vec{v}$ is the generalized velocity of $\boldsymbol{\mu}$ and where $\nu(\cdot,t)$ denotes the generalized inner normal on $\partial^*\{u(\cdot,t)=1\}$.

The property (2.9) yields the following estimates.

Lemma 2.1. For $\boldsymbol{\Sigma} \in \mathcal{M}$ as above, we have that $u \in C^{\frac{1}{2}}([0,T];L^p(\mathbb{R}^{n+1}))$ for all $1 \le p < \infty$. For almost any $0 \le t_1 \le t_2 \le T$ it holds

$$\int_{\mathbb{R}^{n+1}} |u(x,t_2)-u(x,t_1)|\,dx \;\le\; \|\vec{v}\|_{L^2(\mu;\mathbb{R}^{n+1})}(t_2-t_1)^{\frac{1}{2}}\Big(\sup_{t_1<t<t_2}\mu_t(\mathbb{R}^{n+1})\Big)^{\frac{1}{2}} \tag{2.10}$$

Moreover, $u \in BV(Q_T)$ and

$$\left(|\nabla u| + |\partial_t u|\right)(Q_T) \;\le\; 2T\sup_{0<t<T}\mu_t(\mathbb{R}^{n+1}) + \int_{Q_T}|\vec{v}|^2\,d\mu. \tag{2.11}$$

Proof. First we deduce from (2.9) that for any $\varphi \in C^1_c((0,T))$ and any $\psi \in C^1_c(\mathbb{R}^{n+1})$

$$\Big|\int_0^T \partial_t\varphi(t)\int_{\mathbb{R}^{n+1}} u(x,t)\psi(x)\,dx\,dt\Big| \;=\; \Big|\int_0^T \varphi(t)\int_{\mathbb{R}^{n+1}} \psi(x)\vec{v}(x,t)\cdot\nu(x,t)\,d|\nabla u(\cdot,t)|\,dt\Big|.$$

Hence, the function $t \mapsto \int_{\mathbb{R}^{n+1}} u(x,t)\psi(x)\,dx$ belongs to $W^{1,2}((0,T))$ and for almost all $0<t_1<t_2<T$ we have

$$\Big|\int_{\mathbb{R}^{n+1}}\big(u(x,t_2)-u(x,t_1)\big)\psi(x)\,dx\Big| \;\le\; \|\vec{v}\|_{L^2(\mu);\mathbb{R}^{n+1}}(t_2-t_1)^{\frac{1}{2}}\left(\sup_{t_1<t<t_2}\mu_t(\mathbb{R}^{n+1})\right)^{\frac{1}{2}}\|\psi\|_{C^0_c(\mathbb{R}^{n+1})}.$$

Since $u(x,t_2)-u(x,t_1) \in BV(\mathbb{R}^{n+1},\{-1,0,1\})$, taking the supremum over $\psi \in C^0_c(\mathbb{R}^{n+1})$ with $\|\psi\| \le 1$ yields (2.10) and $u \in C^{\frac{1}{2}}([0,T];L^1(\mathbb{R}^{n+1}))$. Since $|u(x,t_2)-u(x,t_1)| \le 1$ almost everywhere, we deduce that $u \in C^{\frac{1}{2}}([0,T];L^p(\mathbb{R}^{n+1}))$ for all $1 \le p < \infty$. From (2.7) and (2.9) one gets $u \in BV(Q_T)$ and (2.11). □

In the class $\mathcal{M}$ we now define a generalized action functional.

Definition 2.4. For $\boldsymbol{\Sigma} \in \mathcal{M}$, $\boldsymbol{\Sigma} = (\boldsymbol{\mu},\boldsymbol{u})$ as above we define

$$\mathcal{S}(\boldsymbol{\Sigma}) := \mathcal{S}_+(\boldsymbol{\Sigma}) + \mathcal{S}_-(\boldsymbol{\Sigma}), \tag{2.12}$$

$$\begin{aligned}\mathcal{S}_+(\boldsymbol{\Sigma}) := \sup_\eta \Big[&2|\nabla u(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) \\ &+ \int_{Q_T} -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v}-\vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big],\end{aligned} \tag{2.13}$$

$$\begin{aligned}\mathcal{S}_-(\boldsymbol{\Sigma}) := \sup_\eta \Big[&-2|\nabla u(\cdot,T)|(\eta(\cdot,T)) + 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) \\ &+ \int_{Q_T} 2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v}+\vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big],\end{aligned} \tag{2.14}$$

where the supremum is taken over all $\eta \in C^1(\mathbb{R}^{n+1}\times[0,T])$ with $0 \le \eta \le 1$.

We remark that $\mathcal{S}$ is invariant under the time inversion $t \mapsto T-t$. Since in $\mathcal{S}_\pm$ the terms $(1-2\eta)_+\frac{1}{2}|\vec{v}\mp\vec{\mathrm{H}}|^2$ are nonnegative, we observe that a bound on the action implies the generalized velocity property (2.5) and, more precisely, the estimate

$$\Big|\int_{Q_T}\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big)\,d\mu\Big| \;\le\; \frac{1}{2}\mathcal{S}(\boldsymbol{\Sigma})\,, \tag{2.15}$$

for all $\eta \in C^1_c(Q_T)$ with $|\eta| \le 1$. By choosing $\eta = 0$ in (2.13) and in (2.14), we also have that

$$\int_{Q_T}(|\vec{v}|^2 + |\vec{\mathrm{H}}|^2)\,d\mu_t\,dt \;\le\; \mathcal{S}(\boldsymbol{\Sigma}). \tag{2.16}$$

The functional $\mathcal{S}$ takes into account the jumps in the evolution of the generalized surface measures $t \mapsto \mu_t$ (possibly also a discrepancy between $|\nabla u|(\cdot,0)$ and $\lim_{t\downarrow 0}\mu_t$, and $|\nabla u|(\cdot,T)$ and $\lim_{t\uparrow T}\mu_t$) and actually, as we now prove in the following proposition, generalizes the notion of action functional for the smooth case.

Proposition 2.5. *Let $\mathbf{\Sigma} = (\boldsymbol{\mu}, \boldsymbol{u})$ be given by an evolution $(\Omega(t))_{t\in[0,T]}$ of open sets $\Omega(t) \subset \mathbb{R}^{n+1}$, which means*

$$u(\cdot,t) \;=\; \mathcal{X}_{\Omega(t)} \qquad \text{and} \qquad \mu_t \;:=\; \mathcal{H}^n \llcorner \partial\Omega(t).$$

Assume that $(\partial\Omega(t))_{t\in[0,T]}$ represents, outside of a set (possibly empty) of singular times $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$, a smooth evolution of smooth hypersurfaces. Then

$$\mathcal{S}(\mathbf{\Sigma}) \;=\; \int_0^T \int_{\partial\Omega(t)} (|\vec{v}(\cdot,t)|^2 + |\vec{\mathrm{H}}(\cdot,t)|^2)\, d\mathcal{H}^n\, dt + 2\sum_{j=0}^{k+1} \sup_{\psi} |\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)|, \tag{2.17}$$

where the supremum is taken over all $\psi \in C^1(\mathbb{R}^{n+1})$ with $|\psi| \le 1$ and where we have set $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(0)$ for $t < 0$ and $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(T)$ for $t > T$.

Proof. We have μ-almost everywhere that

$$\begin{aligned} & -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2 && (2.18)\\ &= -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v} - \eta\vec{v}\cdot\vec{\mathrm{H}}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2 - 2\eta\vec{v}\cdot\vec{\mathrm{H}}. && (2.19)\end{aligned}$$

For $0 \le \eta \le \frac{1}{2}$, the sum of the last two terms can be estimated as

$$(1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2 - 2\eta\vec{v}\cdot\vec{\mathrm{H}} \;=\; \frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2 - \eta(|\vec{v}|^2 + |\vec{\mathrm{H}}|^2) \;\le\; \frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2\,, \tag{2.20}$$

and for $\frac{1}{2} \le \eta \le 1$ it holds

$$(1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2 - 2\eta\vec{v}\cdot\vec{\mathrm{H}} \;=\; -2\eta\vec{v}\cdot\vec{\mathrm{H}} \;\le\; 2|\vec{v}\cdot\vec{\mathrm{H}}|\mathcal{X}_{\{\vec{v}\cdot\vec{\mathrm{H}}<0\}} \;\le\; \frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2. \tag{2.21}$$

Moreover, for any $0 \le j \le k$, we have that

$$\begin{aligned}\int_{t_j}^{t_j+1}\int_{\mathbb{R}^{n+1}} 2\big(\partial_t\eta + \nabla\eta\cdot\vec{v} - \eta\vec{v}\cdot\vec{\mathrm{H}}\big)\, d\mu_t\, dt \;&=\; 2\int_{t_j}^{t_{j+1}} \frac{d}{dt}\Big(\int_{\partial\Omega(t)} \eta(\cdot,t)\, d\mathcal{H}^n\Big)\, dt\\ &=\; 2\Big(\lim_{t\nearrow t_{j+1}} \mu_t(\eta(\cdot,t)) - \lim_{t\searrow t_j} \mu_t(\eta(\cdot,t))\Big) \end{aligned} \tag{2.22}$$

and therefore

$$\begin{aligned} & 2|\nabla u(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) - \int_{Q_T} 2\big(\partial_t\eta + \nabla\eta\cdot\vec{v} - \eta\vec{v}\cdot\vec{\mathrm{H}}\big)\, d\mu_t\, dt\\ &= 2\sum_{j=0}^{k+1}\Big(\mu_{t_j+}(\eta(\cdot,t_j)) - \mu_{t_j-}(\eta(\cdot,t_j))\Big)\\ &\le 2\sum_{j=0}^{k+1}\sup_{\psi}\big(\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)\big), \end{aligned} \tag{2.23}$$

where the supremum is taken over all $\psi \in C^1(\mathbb{R}^{n+1})$ with $0 \le \psi \le 1$ and where we have used that $|\nabla u(\cdot,T)| = \mu_t$ for all $t > T$, $|\nabla u(\cdot,0)| = \mu_t$ for all $t < 0$. Thus, using (2.20) and (2.21), we deduce

$$\mathcal{S}_+(\mathbf{\Sigma}) \;\le\; \frac{1}{2}\int_0^T\int_{\partial\Omega(t)} |\vec{v}(\cdot,t) - \vec{\mathrm{H}}(\cdot,t)|^2\, d\mathcal{H}^n\, dt + 2\sum_{j=0}^{k+1}\sup_{\psi}\big(\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)\big)_+. \tag{2.24}$$

On the other hand, setting $\eta = 0$ except in an arbitrary small neighborhood of the t_j's and choosing $\eta(\cdot, t_j)$ to obtain an arbitrarily good approximation of $\sup_\psi \big(\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)\big)$, we see that (2.24) is in fact an equality. Similarly, we derive

$$\mathcal{S}_-(\mathbf{\Sigma}) \;=\; \frac{1}{2}\int_0^T \int_{\partial\Omega(t)} |\vec{v}(\cdot,t) + \vec{\mathrm{H}}(\cdot,t)|^2 \, d\mathcal{H}^n \, dt + 2\sum_{j=0}^{k+1} \sup_\psi \big(\mu_{t_j-}(\psi) - \mu_{t_j+}(\psi)\big)_+ \,, \tag{2.25}$$

where the supremum is taken over all $\psi \in C^1(\mathbb{R}^{n+1})$ with $0 \le \psi \le 1$. Summing (2.24) and (2.25), and using that we have equality in (2.24), we obtain (2.17). $\square$

Remark 2.6. The expression on the right-hand side of (2.17) corresponds to the definition $\mathcal{S}_0$ of the action functional for evolutions which are smooth outside a finite set of points in time.
The proof of Proposition 2.5 shows in particular that $\mathcal{S}_+$ penalizes upward jumps of the measure evolution $t \mapsto \mu_t$ and $\mathcal{S}_-$ penalizes downward jumps.

3. Compactness and lower-semicontinuity for uniformly action-bounded sequences

We now consider sequences of generalized evolutions with uniformly bounded action and constrained to fixed given initial and final data. The main results in this section are the following compactness and lower-semicontinuity statements.

Theorem 3.1. *Let $T > 0$ and let $\Omega(0) \subset \mathbb{R}^{n+1}$, $\Omega(T) \subset \mathbb{R}^{n+1}$ be two given open bounded sets with finite perimeter. Consider a family of evolutions $(\mathbf{\Sigma}_l)_{l\in\mathbb{N}}$ in $\mathcal{M}(T, \Omega(0), \Omega(T))$, where $\mathbf{\Sigma}_l = (\boldsymbol{\mu}^l, \boldsymbol{u}^l)$, with*

$$\mathcal{S}(\mathbf{\Sigma}_l) \;\le\; \Lambda \quad \text{for all } l \in \mathbb{N}, \tag{3.1}$$

where $\Lambda > 0$ is a fixed constant.

Then there exists a subsequence $l \to \infty$ (not relabeled) and a limit evolution $\mathbf{\Sigma} = (\boldsymbol{\mu}, \boldsymbol{u}) \in \mathcal{M}(T, \Omega(0), \Omega(T))$, $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)}$, $\boldsymbol{u} = (u(\cdot,t))_{t\in[0,T]}$, such that

$$u^l \to u \qquad \text{in } L^1(Q_T) \cap C^0([0,T]; L^1(\mathbb{R}^{n+1})), \tag{3.2}$$

$$\mu_t^l \to \mu_t \quad \text{for almost all } t \in (0,T) \text{ as integral varifolds on } \mathbb{R}^{n+1}, \tag{3.3}$$

$$\mu^l \to \mu \quad \text{as Radon measures on } Q_T, \tag{3.4}$$

and such that $u \in C^{0,1/2}([0,T]; L^1(\mathbb{R}^{n+1}))$ and $\mu \ll \mathcal{H}^{n+1}$.
Moreover it holds

$$\mathcal{S}(\mathbf{\Sigma}) \;\le\; \liminf_{l\to\infty} \mathcal{S}(\mathbf{\Sigma}_l)\,. \tag{3.5}$$

In particular, the minimum of $\mathcal{S}$ in $\mathcal{M}(T, \Omega(0), \Omega(T))$ is attained.

In the remainder of this section we prove Theorem 3.1. The line of the proof follows closely the arguments in [20], which are themselves based on [13, 14]. However, the setting is here quite different, as we do not pass to the limit with phase field approximations but with a sequence of sharp interface evolutions. Furthermore, our generalized action functional is different from the one in [20]. For these reasons, the proofs need to be adapted. Most of the statements will be proven in detail and for some others we will refer to the corresponding statements in [20].

Remark 3.2. From (2.15), (2.16), and (3.1) we first obtain the uniform bounds

$$\int_{Q_T} (|\vec{v}_l|^2 + |\vec{\mathrm{H}}_l|^2) \, d\mu_t^l \, dt \;\le\; \Lambda, \tag{3.6}$$

$$\sup_{\eta\in C_c^1(Q_T)} \Big| \int_{Q_T} \big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) \, d\mu_t^l \, dt \Big| \;\le\; \frac{1}{2}\Lambda \|\eta\|_{C_c^0(Q_T)}. \tag{3.7}$$

To the integral varifolds $(\mu_t^l)_{t\in(0,T)}$ we associate the product measures $\mu^l := \mu_t^l \otimes \mathcal{L}^1$. As a first step we show that under the assumptions above we obtain a uniform bound for the area measures, and that time differences of the area measures are controlled by means of the initial data and of Λ.

Proposition 3.3. [20, Lemma 5.1] *There exist a constant $C = C(\Omega(0), T, \Lambda)$, such that for all $l \in \mathbb{N}$ we have*

$$\sup_{t\in(0,T)} \mu_t^l(\mathbb{R}^{n+1}) \le C(\Omega(0), T, \Lambda), \tag{3.8}$$

$$\mu^l(Q_T) \le C(\Omega(0), T, \Lambda), \tag{3.9}$$

and such that for all $\psi \in C_c^1(\mathbb{R}^{n+1})$ it holds

$$\sup_{l\in\mathbb{N}} |\partial_t \mu_t^l(\psi)|((0,T)) \le C(\Omega(0), T, \Lambda)\|\psi\|_{C_c^1(\mathbb{R}^{n+1})}. \tag{3.10}$$

Proof. Choosing $\eta(x,t) = \varphi(t)$, with $\varphi \in C_c^1((0,T))$, from (3.7) we first deduce that $M_l : (0,T) \to \mathbb{R}_0^+$, $M_l(t) := \mu_t^l(\mathbb{R}^{n+1})$ is of bounded variation with

$$|M_l'|((0,T)) \le \frac{\Lambda}{2}. \tag{3.11}$$

Choosing $\eta(x,t) = \varphi(t)$, with $\varphi \in C^1([0,T])$ not necessarily compactly supported in $(0,T)$, from the very definition of $\mathcal{S}$ we obtain

$$\Big|\lim_{t\searrow 0} M_l(t) - \mathcal{H}^n(\partial^*\Omega(0))\Big| \le \frac{\Lambda}{2}, \tag{3.12}$$

$$\Big|\lim_{t\nearrow T} M_l(t) - \mathcal{H}^n(\partial^*\Omega(T))\Big| \le \frac{\Lambda}{2}, \tag{3.13}$$

where ∂^* is the reduced boundary operator. Indeed, setting

$$\varphi_k(t) := \begin{cases} 1 - kt & \text{for } 0 \le t \le \frac{1}{k} \\ 0 & \text{otherwise} \end{cases},$$

it holds

$$\Lambda \ge -2\mathcal{H}^n(\partial^*\Omega(0)) + 2k\int_0^{\frac{1}{k}} M_l(t)\,dt,$$

$$\Lambda \ge 2\mathcal{H}^n(\partial^*\Omega(0)) - 2k\int_0^{\frac{1}{k}} M_l(t)\,dt.$$

Thus, taking the limit $k \to \infty$, we have proven (3.12). Along the same lines its is possible to obtain (3.13). Using (3.11), (3.12) and (3.13), we can deduce that

$$\mu_t^l(\mathbb{R}^{n+1}) \le \mathcal{H}^n(\partial^*\Omega(0)) + \Lambda$$

holds, and (3.8) follows, as well as (3.9).
We now fix $\psi \in C_c^1(\mathbb{R}^{n+1})$ and from (3.7), with the choice $\eta(x,t) = \varphi(t)\psi(x)$, we obtain

$$\begin{aligned} |D\mu_t^l(\psi)|((0,T)) &\le \frac{1}{2}\Lambda\|\psi\|_{C_c^0(\mathbb{R}^{n+1})} + \sup_{|\varphi|\le 1}\Big|\int_0^T \varphi(t)\int_{\mathbb{R}^{n+1}} \nabla\psi\cdot\vec{v}(\cdot,t)\,d\mu_t^l\,dt\Big| \\ &\le \frac{1}{2}\Lambda\|\psi\|_{C_c^0(\mathbb{R}^{n+1})} + \|\nabla\psi\|_{L^2(\mu)}\|\vec{v}\,\|_{L^2(\mu;\mathbb{R}^{n+1})} \\ &\le \Big(\frac{1}{2}\Lambda + \big(T\sup_{0<t<T}\mu_t^l(\mathbb{R}^{n+1})\big)^{1/2}\Lambda^{1/2}\Big)\|\psi\|_{C_c^1(\mathbb{R}^{n+1})}, \end{aligned} \tag{3.14}$$

where we have used (3.6) and by means of (3.8) the estimate (3.10) now follows. □

Remark 3.4. Proposition 3.3, Lemma 2.1 and the bound (3.6) yield the uniform bounds

$$\int_{\mathbb{R}^{n+1}} |u_l(x,t_2) - u_l(x,t_1)|\,dx \;\leq\; C(\Lambda,T,\Omega(0))(t_2-t_1)^{\frac{1}{2}}, \tag{3.15}$$

$$\left(|\nabla u_l| + |\partial_t u_l|\right)(Q_T) \;\leq\; C(\Lambda,T,\Omega(0)). \tag{3.16}$$

Combining Proposition 3.3 and Lemma 2.1, we obtain a compactness statement for the characteristic functions of the sets $(\Omega_t)_{t\in[0,T]}$.

Proposition 3.5. [20, Prop 4.1]. *There exist a subsequence $l \to \infty$ (not relabeled) and a function $u \in BV(Q_T;\{0,1\})$, $u \in C^{\frac{1}{2}}([0,T];L^1(\mathbb{R}^{n+1}))$ such that (2.8) and (3.2) hold.*

Proof. By (3.16), the compactness Theorem for BV functions ensures the existence of a subsequence $l \to \infty$ and of a function $u \in BV(Q_T)$, with $u^l \to u$ in $L^1(Q_T)$. In particular, $u(x,t) \in \{0,1\}$ for almost every $(x,t) \in Q_T$.
From (3.15), we deduce that $(u^l)_{l\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}([0,T];L^1(\mathbb{R}^{n+1})$. Moreover, by (2.7) for $\mathbf{\Sigma}_l$ and by (3.8), the family $\{u_l(t) : l \in \mathbb{N}\}$ is relatively compact in $L^1(\mathbb{R}^{n+1})$ for almost any $t \in (0,T)$. Applying the Arzela-Ascoli Theorem, we deduce that, possibly after passing to another subsequence, $u^l \to u$ in $C^0([0,T];L^1(\mathbb{R}^{n+1}))$, with $u \in C^{\frac{1}{2}}([0,T];L^1(\mathbb{R}^{n+1}))$.
By (3.2), the condition (2.8) for $\mathbf{\Sigma}_l$ implies that u attains the initial and final data. □

We next show a compactness statement for the evolution of the surface area measures.

Proposition 3.6. [20, Prop 4.2] *There exists a subsequence $l \to \infty$ (not relabeled) and a family of Radon measures $(\mu_t)_{t\in(0,T)}$ on $\mathbb{R}^{n+1}$ such that (2.3), (2.7) are valid, such that*

$$\mu_t^l \to \mu_t \quad \text{for all } t \in (0,T) \text{ as Radon measures on } \mathbb{R}^{n+1}, \tag{3.17}$$

and such that (3.4) holds. Moreover,

$$\sup_{t\in[0,T]} \mu_t(\mathbb{R}^{n+1}) \;\leq\; C(\Omega(0),T,\Lambda) \tag{3.18}$$

is satisfied.

Proof. We first consider a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $C^1_c(\mathbb{R}^{n+1})$, which is dense in $C^0_0(\mathbb{R}^{n+1})$ with respect to the supremum norm. By (3.8) and (3.10), we have that for any fixed $k \in \mathbb{N}$ the sequence of functions $(t \mapsto \mu^l_t(\psi_k))_{l\in\mathbb{N}}$ is uniformly bounded in $BV(0,T)$. By means of a diagonal argument, we obtain a subsequence $l \to \infty$ and functions $m_k \in BV(0,T)$, $k \in \mathbb{N}$, such that for all $k \in \mathbb{N}$

$$\mu^l_t(\psi_k) \to m_k(t) \qquad \text{for almost-all } t \in (0,T), \tag{3.19}$$

$$D\mu^l_t(\psi_k) \to m_k' \qquad \text{as Radon measures on } (0,T). \tag{3.20}$$

Let S denote the countable set of times $t \in (0,T)$ where, for some $k \in \mathbb{N}$, the measure m_k' has an atomic part. As in [20, Prop. 4.2] one shows that (3.19) holds on $(0,T) \setminus S$ and that there exist Radon measures μ_t on $\mathbb{R}^{n+1}$, $t \in (0,T) \setminus S$ with

$$\mu^l_t \to \mu_t \quad \text{as Radon-measures on } \mathbb{R}^{n+1}$$

for the whole sequence selected in (3.19)–(3.20), and for all $t \in (0,T)$ for which (3.19) holds. This proves (3.17).
For any $\psi \in C^0_0(\mathbb{R}^{n+1})$ the map $t \mapsto \mu_t(\psi)$ has no jumps in $(0,T) \setminus S$ and, for all $\varphi \in C^1_c((0,T))$ with $|\varphi| \leq 1$, using (3.10) we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n+1}} \partial_t\varphi(t)\mu_t(\psi)\,dt\Big| &= \Big|\lim_{l\to\infty}\int_{\mathbb{R}^{n+1}} \partial_t\varphi(t)\mu^l_t(\psi)\,dt\Big| \\ &\leq \liminf_{l\to\infty} |\partial_t\mu^l_t(\psi)| \;\leq\; C(\Omega(0),T,\Lambda)\,,\end{aligned}$$

and we have proven (2.3).
Applying the dominated convergence theorem, (3.4) follows. By (3.2) we have $u_l(\cdot,t) \to u(\cdot,t)$

in $L^1(\mathbb{R}^{n+1})$ as $l \to \infty$, and (2.7) for $\mathbf{\Sigma}_l$, together with (3.3) and the lower-semicontinuity of the perimeter under L^1-convergence, allow to conclude that $|\nabla u(\cdot,t)| \leq \mu_t$, which proves (2.7). □

We next show that the measures $(\mu_t)_{t\in(0,T)}$ are integral varifolds with weak mean curvature in $L^2(\mu_t;\mathbb{R}^{n+1})$.

Proposition 3.7. [20, Thm. 4.3] *For any $t \in (0,T)$ the limit measure μ_t is an integral varifold with weak mean curvature $\vec{\mathrm{H}}(\cdot,t) \in L^2(\mu_t;\mathbb{R}^{n+1})$ and for almost all $t \in (0,T)$*

$$\mu_t^l \to \mu_t\,(l\to\infty) \qquad \text{as varifolds.} \tag{3.21}$$

The sequence $(\mu^l,\vec{\mathrm{H}}_l)_{l\in\mathbb{N}}$ converges to $(\mu,\vec{\mathrm{H}})$ as measure function pairs, i.e.

$$\int_0^T\int_{M_t^l}\eta(.,t)\cdot\vec{\mathrm{H}}_l(.,t)\,d\mathcal{H}^n\,dt \to \int_{Q_T}\eta(.,t)\cdot\vec{\mathrm{H}}(.,t)\,d\mu_t\,dt \tag{3.22}$$

holds for all $\eta \in C_c^0(\mathbb{R}^{n+2}_{0,T};\mathbb{R}^{n+1})$. Moreover, we have the estimates

$$\int_{\mathbb{R}^{n+1}}|\vec{\mathrm{H}}(\cdot,t)|^2\,d\mu_t \leq \liminf_{l\to\infty}\int_{M_t^l}|\vec{\mathrm{H}}_l(\cdot,t)|^2\,d\mathcal{H}^n \qquad \text{for almost all } t\in(0,T), \tag{3.23}$$

$$\int_{Q_T}|\vec{\mathrm{H}}|^2\,d\mu \leq \liminf_{l\to\infty}\int_0^T\int_{M_t^l}|\vec{\mathrm{H}}_l(\cdot,t)|^2\,d\mathcal{H}^n\,dt. \tag{3.24}$$

Proof. By (3.6) and Fatous Lemma we have

$$h(t) := \liminf_{l\to\infty}\int_{\mathbb{R}^{n+1}}|\vec{\mathrm{H}}_l(\cdot,t)|^2\,d\mathcal{H}^n \in L^1(0,T)$$

and in particular $h(t) < \infty$ for almost every $t \in (0,T)$. We fix such $t \in (0,T)$ and from Allards compactness Theorem (see [1]) we deduce that there exists a subsequence $l' \to \infty$ and an integral varifold $\tilde{\mu}_t$ with weak mean curvature $\vec{\mathrm{H}}(\cdot,t) \in L^2(\tilde{\mu}_t;\mathbb{R}^{n+1})$ such $\mu_t^{l'} \to \tilde{\mu}_t$ as varifolds, and such that

$$\int|\vec{\mathrm{H}}(\cdot,t)|^2\,d\tilde{\mu}_t \leq h(t) = \liminf_{l\to\infty}\int_{\mathbb{R}^{n+1}}|\vec{\mathrm{H}}_l(\cdot,t)|^2\,d\mathcal{H}^n. \tag{3.25}$$

From (3.3) we deduce that $\mu_t = \tilde{\mu}_t$. In particular, μ_t is an integral varifold with weak mean curvature in $L^2(\mu_t;\mathbb{R}^{n+1})$ which satisfies (3.23). The estimate (3.24) follows from (3.23) and Fatou's Lemma. Since an integral varifold is uniquely determined by its mass measure, we see that the whole sequence $l \to \infty$ from (3.3) converges to μ_t in the varifold topology. This shows (3.21).

The fact that the measure-function pair converges is shown as in [20, Thm. 4.3] by an identification argument of point-wise limits of

$$-\int_{\mathbb{R}^{n+1}}\vec{\mathrm{H}}_l(\cdot,t)\cdot\xi\,\mathrm{d}\mu_t^l, \quad \xi\in C_c^1(\mathbb{R}^{n+1})$$

(where at first the subsequence depends on time) and by an application of Lebesgue dominated convergence theorem. Thus (3.22) is proven. □

In the next step we show that the limit evolution has a generalized velocity.

Proposition 3.8. [20, Thm. 4.4] *There exists a subsequence $l \to \infty$ and a function $\vec{v} \in L^2(\mu;\mathbb{R}^{n+1})$ such that $(\mu^l,\vec{v}_l) \to (\mu,\vec{v})$ as measure-function pairs, i.e.*

$$\lim_{l\to\infty}\int_0^T\int_{M_t^l}\vec{v}_l(\cdot,t)\cdot\eta(\cdot,t)\,d\mathcal{H}^n\,dt = \int_{Q_T}\vec{v}(\cdot,t)\cdot\eta(\cdot,t)\,d\mu_t\,dt. \tag{3.26}$$

Moreover, we have the estimate

$$\int_{Q_T} |\vec{v}(\cdot,t)|^2 \, d\mu_t \, dt \;\le\; \liminf_{l\to\infty} \int_0^T \int_{M_t^l} |\vec{v}_l(\cdot,t)|^2 \, d\mathcal{H}^n \, dt\,, \tag{3.27}$$

and $\vec{v}$ is the generalized speed of the evolution $(\mu_t)_{t\in(0,T)}$ in the sense of Definition 2.1.

Proof. From (3.1) for $\mathbf{\Sigma}_l$, the convergence (3.4) and the compactness and lower semicontinuity property for measure-function pairs [11, Theorem 4.4.2], we conclude the existence of a subsequence $l\to\infty$ and a limit $\vec{v}\in L^2(\mu;\mathbb{R}^{n+1})$ satisfying (3.26) and (3.27).
By (3.7), (3.1) for $\mathbf{\Sigma}_l$, (3.26) and (3.27) we deduce that for any $\eta\in C^1_c(Q_T)$ with $|\eta|\le 1$

$$\begin{aligned}&\Big|\int_{Q_T} (\partial_t\eta(\cdot,x) + \nabla\eta(\cdot,x)\cdot\vec{v}(\cdot,t)) \, d\mu_t \, dt\Big|\\ \le{}& \liminf_{l\to\infty} \Big|\int_{Q_T} (\partial_t\eta(\cdot,x) + \nabla\eta(\cdot,x)\cdot\vec{v}_l(\cdot,t)) \, d\mu_t^l \, dt\Big| \;\le\; \frac{\Lambda}{2}.\end{aligned}$$

We therefore obtain (2.5). It remains to show that $\vec{v}(x,t)$ is normal to $T_x\mu_t$ for $\mu-$almost all $(x,t)\in Q_T$. The proof follows as in [20, Lemma 6.3] by an adaption of [19, Proposition 3.2]. ☐

In order to show that the limit evolution of phases satisfies (2.9) we need some preparations. First, for $r>0$ and $(x_0,t_0)\in Q_T$, we define the cylinders

$$Q_r(t_0,x_0) \;:=\; B^{n+1}(x_0,r)\times(t_0-r,t_0+r).$$

Proposition 3.9. [20, Prop 8.1] *The measure μ is absolutely continuous with respect to $\mathcal{H}^{n+1}$,*

$$\mu \;\ll\; \mathcal{H}^{n+1}. \tag{3.28}$$

Proof. The proof is an adaption of [20, Prop 8.1], in the present case we just have to substitute (8.16) in that paper by the following argument. For $t_0\in(0,T)$, $x_0\in\mathbb{R}^{n+1}$ the monotonicity formula [16, (A.6)] yields that for any $0<r<r_0<\min\{t_0,T-t_0\}$

$$\begin{aligned}&\frac{1}{r}\int_{t_0-r}^{t_0+r} r^{-n}\mu_t\big(B(x_0,r)\big)\,dt\\ \le{}&\frac{2}{r}\int_{t_0-r}^{t_0+r} r_0^{-n}\mu_t\big(B^n_{r_0}(x_0)\big)\,dt + C\frac{1}{r}\int_{t_0-r}^{t_0+r}\int_{\mathbb{R}^{n+1}} |\vec{\mathrm{H}}(\cdot,t)|^2\,d\mu_t\,dt.\end{aligned}$$

The proof then proceeds as in [20]. ☐

We now need to show that the generalized tangent plane of μ exists $\mathcal{H}^{n+1}$-almost everywhere on $\partial^*\{u=1\}$. To this aim, we first prove the following relation between the measures μ and $|\nabla' u|$.

Proposition 3.10. [20, Prop. 8.2] *For the total variation measure $|\nabla' u|$ we have*

$$|\nabla' u| \;\le\; g\mu, \tag{3.29}$$

for a function $g\in L^2(\mu)$. In particular, $|\nabla' u|$ is absolutely continuous with respect to μ. Moreover, the tangent plane to μ exists at $\mathcal{H}^{n+1}$-almost-all points of $\partial^*\{u=1\}$.

Proof. By (2.9), (2.7) and (3.2) we deduce that for any $\eta\in C^1_c(Q_T)$ with $|\eta|\le 1$

$$\Big|\int_{Q_T} -\partial_t\eta u \, d\mathcal{L}^{n+2}\Big| \tag{3.30}$$

$$= \Big|\lim_{l\to\infty}\int_{Q_T} -\partial_t\eta u_l \, d\mathcal{L}^{n+2}\Big| \;\le\; \liminf_{l\to\infty}\int_{Q_T} |\eta|(\cdot,t)|\vec{v}_l(\cdot,t)|\,d\mu_t^l\,dt. \tag{3.31}$$

By (3.6) and [11, Theorem 4.4.2], there exists a subsequence $l \to \infty$ and $\tilde{g} \in L^2(\mu)$, $\tilde{g} \geq 0$ such that $(\mu^l, |\vec{v}_l|) \to (\mu, \tilde{g})$ as $l \to \infty$ and such that

$$\int_{Q_T} \tilde{g}^2 \, d\mu \;\leq\; \int_{Q_T} |\vec{v}_l|^2 \, d\mu^l \;\leq\; \Lambda.$$

By (3.31) we therefore get

$$\Big| \int_{Q_T} -\partial_t \eta u \, d\mathcal{L}^{n+2} \Big| \;\leq\; \int_{Q_T} |\eta| \tilde{g} \, d\mu,$$

which shows that

$$|\partial_t u| \;\leq\; \tilde{g}\mu. \tag{3.32}$$

Similarly, we find

$$\begin{aligned} \Big| \int_{Q_T} -\nabla \eta u \, d\mathcal{L}^{n+2} \Big| &= \Big| \lim_{l\to\infty} \int_{Q_T} -\nabla \eta u_l \, d\mathcal{L}^{n+2} \Big| \\ &= \lim_{l\to\infty} \Big| \int_{Q_T} \eta \nu_l |\nabla u_l| \Big| \\ &\leq \liminf_{l\to\infty} \int_{Q_T} |\eta|(\cdot,t) \, d\mu_t^l \, dt \;=\; \int_{Q_T} |\eta|(\cdot,t) \, d\mu, \end{aligned}$$

which yields $|\nabla u| \;\leq\; \mu$. Putting this together with (3.32), we obtain (3.29) and deduce that $|\nabla' u|$ is absolutely continuos with respect to μ.
The final statement has been proved in Proposition [20, Proposition 8.3]. □

Proposition 3.11. *For the limit phase function u in* (3.2) *the equation* (2.9) *holds.*

Proof. We first observe that $\vec{v} \in L^1(|\nabla u|)$, since by (3.27), (3.29), and Proposition 3.10

$$\begin{aligned} \int_{Q_T} |\vec{v}| \, d|\nabla u| \;\leq\; \int_{Q_T} |\vec{v}| \, d|\nabla' u| \;&\leq\; \int_{Q_T} g|\vec{v}| \, d\mu \\ &\leq\; \|g\|_{L^2(\mu)} \|\vec{v}\,\|_{L^2(\mu)} \;<\; \infty. \end{aligned}$$

For $\vec{v} \in L^2(\mu; \mathbb{R}^{n+1})$ there exist a sequence $\varepsilon \to 0$, with $\vec{v}_\varepsilon \in C^0_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ and $\vec{v}_\varepsilon \to v$ in $L^2(\mu, \mathbb{R}^{n+1})$. By [20, Proposition 3.3] we know that for μ-almost all $(x,t) \in Q_T$ at which the tangential plane of μ exists the vector

$$\begin{pmatrix} \vec{v}(x,t) \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R} \qquad \text{is perpendicular to } T_{(x,t)}\mu. \tag{3.33}$$

By Proposition 3.10, this implies

$$\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \cdot \nu' = 0 \qquad |\nabla' u| - \text{a.e.}\,, \tag{3.34}$$

where ν' denotes the generalized inner normal of $\{u = 1\}$ on $\partial^*\{u = 1\}$. It follows that

$$\int_{Q_T} \eta \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \cdot \nu' d|\nabla' u| dt = 0\,, \tag{3.35}$$

hence

$$\begin{aligned} \Big| \int_{Q_T} \eta \begin{pmatrix} 1 \\ \vec{v}_\varepsilon \end{pmatrix} \cdot \nu' d|\nabla' u| \Big| &= \Big| \int_{Q_T} \eta \begin{pmatrix} 0 \\ \vec{v}_\varepsilon - \vec{v} \end{pmatrix} \cdot \nu' d|\nabla' u| \Big| \\ &\leq ||\eta||_{C^0(Q_T)} \int_{Q_T} \eta |\vec{v}_\varepsilon - \vec{v}| d\mu \to 0 \quad \text{for } \varepsilon \to 0\,. \end{aligned} \tag{3.36}$$

Therefore it holds

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\int_{Q_T} \eta \begin{pmatrix} 1 \\ \vec{v}_\varepsilon \end{pmatrix} \cdot \nu' d|\nabla' u| = -\lim_{\varepsilon\to 0}\int_{Q_T} (\partial_t \eta u + \nabla\cdot(\eta\vec{v}_\varepsilon)u)dxdt \\ &= \lim_{\varepsilon\to 0}\int_{Q_T} (-\partial_t\eta u + \eta\vec{v}_\varepsilon\cdot\nabla u)dxdt = -\int_{Q_T}\partial_t\eta u dxdt + \int_{Q_T}\eta\vec{v}\cdot\nu d|\nabla u|dt \end{aligned} \tag{3.37}$$

which proves (2.9). □

We are now in the position to complete the proof of Theorem 3.1, in particular we are able to show the lower-semicontinuity of the action functional.

Proof of Theorem 3.1. The compactness statements have already been proven above. The L^2-flow property has been shown in Proposition 3.6, Proposition 3.7, and Proposition 3.8. The assertions (2.6) and (2.8) have been proved in Proposition 3.5, the property (2.7) in Proposition 3.6, and (2.9) in Proposition 3.11. Moreover, $u \in C^{0,1/2}([0,T]; L^1(\mathbb{R}^{n+1}))$ follows by Lemma 2.1, and $\mu \ll \mathcal{H}^{n+1}$ by (3.28). Therefore it remains to show the lower-semicontinuity statement. By (3.22), (3.26) we deduce the weak measure-function-pair convergences

$$(\mu^l, \vec{v}_l - \vec{\mathrm{H}}_l) \to (\mu, \vec{v} - \vec{\mathrm{H}}), \qquad (\mu^l, \vec{v}_l + \vec{\mathrm{H}}_l) \to (\mu, \vec{v} + \vec{\mathrm{H}}) \qquad \text{as } l \to \infty.$$

The lower-semicontinuity property for quadratic functionals under the weak measure-function-pair convergence (see [11, Theorem 4.4.2]) implies that for any $\tilde{\eta} \in C^0(\mathbb{R}^{n+1} \times [0,T])$ with $\tilde{\eta} \geq 0$

$$\int_{Q_T} \tilde{\eta}|\vec{v} - \vec{\mathrm{H}}|^2\, d\mu \,\leq\, \liminf_{l\to\infty} \int_{Q_T} \tilde{\eta}|\vec{v}_l - \vec{\mathrm{H}}_l|^2\, d\mu^l, \tag{3.38}$$

$$\int_{Q_T} \tilde{\eta}|\vec{v} + \vec{\mathrm{H}}|^2\, d\mu \,\leq\, \liminf_{l\to\infty} \int_{Q_T} \tilde{\eta}|\vec{v}_l + \vec{\mathrm{H}}_l|^2\, d\mu^l. \tag{3.39}$$

Using (2.8) for $\mathbf{\Sigma}_l$ and (3.26), we deduce for any $\eta \in C^1(\mathbb{R}^{n+1} \times [0,T])$ with $0 \leq \eta \leq 1$

$$\begin{aligned} &2|\nabla u(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) \\ &\qquad + \int_{Q_T} -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2\, d\mu_t\, dt \\ &\leq \liminf_{l\to\infty} \Big[2|\nabla u_l(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u_l(\cdot,0)|(\eta(\cdot,0)) \\ &\qquad + \int_{Q_T} -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}_l\big) + (1-2\eta)_+\frac{1}{2}|\vec{v}_l - \vec{\mathrm{H}}_l|^2\, d\mu_t^l\, dt\Big] \\ &\leq \liminf_{l\to\infty} \mathcal{S}_+(\mathbf{\Sigma}_l). \end{aligned}$$

By taking the supremum over η we deduce

$$\mathcal{S}_+(\mathbf{\Sigma}) \,\leq \mathcal{S}_+(\mathbf{\Sigma}_l).$$

Similarly we obtain $\mathcal{S}_-(\mathbf{\Sigma}) \leq \mathcal{S}_-(\mathbf{\Sigma}_l)$ and therefore (3.5). Together with the properties proved above this implies (3.1) and thus $\mathbf{\Sigma} \in \mathcal{M}(T, \Omega(0), \Omega(T))$. □

4. Smooth stationary points of the action functional

In the following we consider smooth evolutions of smooth hypersurfaces in $\mathbb{R}^{n+1}$ and characterize stationary points, as well as some conserved quantities along stationary trajectories, for the action functional. In this part it is more convenient to describe evolutions by families of embeddings. We introduce the following setting.

Definition 4.1. Consider a smooth n-dimensional compact, orientable manifold M without boundary. Let $\boldsymbol{\phi} : M \times [0,T] \to \mathbb{R}^{n+1}$ be a smooth one parameter family of embeddings $\phi_t := \boldsymbol{\phi}(\cdot,t), t \in [0,T]$. By $\mathbf{\Sigma} := (\Sigma_t)_{t\in[0,T]}$, where $\Sigma_t := \phi_t(M)$, we denote the evolution of the hypersurfaces associated to $\boldsymbol{\phi}$. In slight abuse of notation, we use the same symbols which we used in the previous sections for the evolutions of the surface area measure and of the inner set. The family of Riemannian measures on M induced via pullback by the parametrizations ϕ_t, for $t \in [0,T]$, will be denoted with $(\bar\mu_t)_{t\in[0,T]}$. Once more in a slight abuse of notation, we denote by $\nu : M \times [0,T] \to \mathbb{R}^{n+1}$ the family of inner unit normals $\nu(\cdot,t)$ of the sets enclosed by the hypersurfaces Σ_t, and by $v(\cdot,t)$, $\mathrm{H}(\cdot,t) : M \to \mathbb{R}^{n+1}$ the scalar normal velocity and the scalar mean curvature of Σ_t given respectively by

$$v(x,t) := \partial_t\boldsymbol{\phi}(x,t)\cdot\nu(x,t), \quad \mathrm{H}(x,t) := \vec{\mathrm{H}}_{\Sigma_t}(\boldsymbol{\phi}(x,t))\cdot\nu(x,t)\,,$$

for $x \in M$ and $t \in [0,T]$. We say that $(\boldsymbol{\phi}^\varepsilon)_{-\varepsilon_0<\varepsilon<\varepsilon_0}$ is a smooth normal variation of $\boldsymbol{\phi}$ which preserves initial and final data, if the $\boldsymbol{\phi}^\varepsilon$ are given by a smooth map $\Phi : M\times[0,T]\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^{n+1}$ as $\boldsymbol{\phi}^\varepsilon = \Phi(\cdot,\cdot,\varepsilon)$ and if

$$\begin{aligned}
&\boldsymbol{\phi}^0 = \mathrm{Id}, \qquad \partial_\varepsilon|_{\varepsilon=0}\boldsymbol{\phi}^\varepsilon = f\nu,\\
&\boldsymbol{\phi}^\varepsilon(\cdot,0) = \boldsymbol{\phi}(\cdot,0), \quad \boldsymbol{\phi}^\varepsilon(\cdot,T) = \boldsymbol{\phi}(\cdot,T) \qquad \text{for all } -\varepsilon_0<\varepsilon<\varepsilon_0,
\end{aligned}$$

where $f : M\times[0,T] \to \mathbb{R}^{n+1}$, with $f(\cdot,0) = f(\cdot,T) = 0$, is smooth. We set $\phi_t^\varepsilon = \boldsymbol{\phi}^\varepsilon(\cdot,t) = \Phi(\cdot,t,\varepsilon)$ and denote by $\bar\mu_t^\varepsilon$, $\nu^\varepsilon(\cdot,t)$, $t\in[0,T]$, $-\varepsilon_0<\varepsilon<\varepsilon_0$, the pullback measures and the normal field associated with $\boldsymbol{\phi}^\varepsilon$, and by v^ε, H^ε the scalar velocity and scalar mean curvature fields on $M\times[0,T]$ associated to $\boldsymbol{\phi}^\varepsilon$. Finally, we call the vector field $X := f\nu$ the *variation field* associated to the given variation and set $X_t := X(\cdot,t)$.

Remark 4.2. Note that if $\boldsymbol{\Sigma}$ is given by a smooth evolution of smooth embeddings $\boldsymbol{\phi}$ as above, the action functional $\mathcal{S}$ reduces to

$$\mathcal{S}(\boldsymbol{\phi}) := \mathcal{S}(\boldsymbol{\Sigma}) = \int_0^T\int_M \Big(v^2(\cdot,t)+\mathrm{H}^2(\cdot,t)\Big)\,d\bar\mu_t\,dt. \tag{4.1}$$

4.1. **Variation Formulae.** In this section we make some preliminary computations which will be needed for the deduction of the smooth Euler-Lagrange equation for the functional $\mathcal{S}$. For the notation and the fundamental identities from differential geometry, we refer to the appendix. In the following ∇ and Δ refer respectively to the Levi-Civita covariant derivative and to the associated Laplace-Beltrami operator.

Lemma 4.3. *The following variation formulae hold:*

$$\partial_\varepsilon|_{\varepsilon=0}\,d\bar\mu_t^\varepsilon = -\mathrm{H}\,X\cdot\nu\,d\bar\mu_t = -\mathrm{H}f\,d\bar\mu_t\,, \tag{4.2}$$

$$\partial_\varepsilon|_{\varepsilon=0}\nu^\varepsilon = -\nabla f\,, \tag{4.3}$$

$$\partial_\varepsilon|_{\varepsilon=0}\mathrm{H}^\varepsilon = \Delta f + f|\mathrm{A}|^2\,. \tag{4.4}$$

Proof. If we denote with g and g^ε the Riemannian metrics induced respectively by the embeddings $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^\varepsilon$ in $\mathbb{R}^{n+1}$, we have

$$\begin{aligned}
\partial_\epsilon|_{\epsilon=0}g_{ij}^\varepsilon &= \partial_\varepsilon|_{\varepsilon=0}(\partial_i\phi_t^\varepsilon\cdot\partial_j\phi_t^\varepsilon) = \partial_i(X\cdot\partial_j\phi_t)+\partial_j(X\cdot\partial_i\phi_t)-2(X\cdot\partial_{ij}^2\phi_t)\\
&= \partial_i(X\cdot\partial_j\phi_t)+\partial_j(X\cdot\partial_i\phi_t)-2\Gamma_{ij}^r X\cdot\partial_r\boldsymbol{\phi}-2\mathrm{h}_{ij}X\cdot\nu\\
&= -2\mathrm{h}_{ij}X\cdot\nu = -2f\mathrm{h}_{ij}.
\end{aligned}$$

Since by definition $g_{ij}^\varepsilon(g^\varepsilon)^{jk} = \delta_i^k$, we get

$$\partial_\varepsilon|_{\varepsilon=0}(g^\varepsilon)^{ij} = 2f\mathrm{h}^{ij}\,.$$

Using the formula $\partial_\varepsilon\det(A_\varepsilon) = \det(A_\varepsilon)\operatorname{tr}[A_\varepsilon^{-1}\partial_\varepsilon A_\varepsilon]$ we obtain the following equation which describes the variation of the induced Riemannian measure

$$\begin{aligned}
\partial_\varepsilon|_{\varepsilon=0}\mathrm{d}\bar\mu_t^\varepsilon = \partial_\varepsilon|_{\varepsilon=0}\sqrt{\det(g^\varepsilon)} &= \frac{\sqrt{\det(g)}g^{ij}\partial_\varepsilon|_{\varepsilon=0}g_{ij}^\varepsilon}{2}\\
&= \frac{-2\sqrt{\det(g)}g^{ij}h_{ij}}{2}X\cdot\nu\\
&= -\mathrm{H}f\,\mathrm{d}\bar\mu_t\,.
\end{aligned}$$

For the variation of the normal vector to the hypersurface we get

$$\partial_\varepsilon(\nu^\varepsilon \cdot \partial_i \phi_t^\varepsilon)|_{\varepsilon=0} = -\nu \cdot \partial_i \partial_\varepsilon|_{\varepsilon=0}\phi_t^\varepsilon = -\nu \cdot \partial_i(f\nu) = -\partial_i f,$$

consequently it holds

$$\partial_\varepsilon|_{\varepsilon=0}\nu^\varepsilon = -\nabla f\,.$$

In order to compute the variation of the mean curvature, we start computing the variation of the second fundamental form (see (7.2) in the Appendix),

$$\partial_\varepsilon|_{\varepsilon=0}\mathrm{h}_{ij}^\varepsilon = \partial_\varepsilon|_{\varepsilon=0}(\nu^\varepsilon \cdot \partial_{ij}^2\phi_t^\varepsilon) = -\nabla f \cdot \partial_{ij}^2\phi_t + \nu \cdot \partial_{ij}^2 X\,. \tag{4.5}$$

By (7.3) and (7.2) we obtain

$$\begin{aligned}\nabla f \cdot \partial_{ij}^2\phi_t \;&=\; \nabla f \cdot (\Gamma_{ij}^r \partial_r \boldsymbol{\phi} + \mathrm{h}_{ij}\nu) \;=\; \nabla_r f \Gamma_{ij}^r, \qquad (4.6)\\ \nu \cdot \partial_{ij}^2 X \;&=\; \nu \cdot \partial_{ij}^2(f\nu) \;=\; \partial_{ij}^2 f + f\nu \cdot \partial_{ij}^2\nu\\ &=\; \partial_{ij}^2 f - f\nu \cdot \partial_i(\mathrm{h}_{jr}g^{rp}\partial_p\phi_t)\\ &=\; \partial_{ij}^2 f - f\mathrm{h}_{jr}g^{rp}\mathrm{h}_{pi}. \qquad (4.7)\end{aligned}$$

From equations (4.5) - (4.7) we finally deduce that

$$\partial_\varepsilon|_{\varepsilon=0}\mathrm{h}_{ij}^\varepsilon = \nabla_{ij}^2 f - f\mathrm{h}_{jr}g^{rp}\mathrm{h}_{pi}.$$

It then follows that for the variation of the scalar mean curvature we have

$$\partial_\varepsilon|_{\varepsilon=0}\mathrm{H}^\varepsilon = \partial_\varepsilon|_{\varepsilon=0}(g^{ij})^\varepsilon \mathrm{h}_{ij} + g^{ij}\partial_\varepsilon|_{\varepsilon=0}\mathrm{h}_{ij}^\varepsilon = \Delta f + f|\mathrm{A}|^2\,.$$

□

4.2. **The first Variation of $\mathcal{S}$.** We now compute the Euler–Lagrange equation for $\mathcal{S}$.

Theorem 4.4. *Let $\boldsymbol{\phi}$ be a smooth evolution of smooth embeddings and consider a normal variation $(\boldsymbol{\phi}^\varepsilon)_{-\varepsilon_0<\varepsilon<\varepsilon_0}$ with associated variation field $f\nu$ as in Definition 4.1. Then the first variation of $\mathcal{S}$ at $\boldsymbol{\phi}$ in the direction of f is given by*

$$\delta\mathcal{S}(\boldsymbol{\phi})(f) \;=\; \frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\boldsymbol{\phi}^\varepsilon) = \int_0^T\int_M f\,\Big[-\partial_t v + \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2 - \frac{\mathrm{H}^3}{2} + \frac{v^2\mathrm{H}}{2}\Big]\,d\bar\mu_t\,dt. \tag{4.8}$$

Consequently, the Euler-Lagrange equation for a smooth stationary point $\boldsymbol{\phi}$ of $\mathcal{S}$ is given by

$$\partial_t v = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2 - \frac{\mathrm{H}^3}{2} + \frac{v^2\mathrm{H}}{2}. \tag{4.9}$$

Proof. We start by computing the variation of the normal speed.

$$\partial_\epsilon|_{\epsilon=0}v^\varepsilon = \partial_t\partial_\epsilon|_{\epsilon=0}\phi_t^\varepsilon \cdot \nu + \partial_t\phi_t \cdot \partial_\epsilon|_{\epsilon=0}\nu^\varepsilon = \partial_t(f\nu)\cdot\nu = \partial_t f. \tag{4.10}$$

Using equations (4.2), (4.4) and (4.10), we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\boldsymbol{\phi}^\varepsilon) = \int_0^T\int_M \Big[\partial_t f v + \mathrm{H}\Delta f + \mathrm{H}f|\mathrm{A}|^2 - (v^2+\mathrm{H}^2)\frac{f\mathrm{H}}{2}\Big]\,d\bar\mu_t\,dt\,. \tag{4.11}$$

Observing that

$$\frac{d}{dt}\int_M fv\,d\bar\mu_t = \int_M \Big[\partial_t f v + f\partial_t v - fv^2\mathrm{H}\Big]\,d\bar\mu_t\,,$$

we get

$$\int_0^T\int_M \partial_t f v\,d\bar\mu_t dt = \int_0^T\int_M f\Big[-\partial_t v + \mathrm{H}v^2\Big]\,d\bar\mu_t dt\,. \tag{4.12}$$

Substituting (4.12) into (4.11), we deduce

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\boldsymbol{\phi}^\varepsilon) = \int_0^T\int_M f\,\Big[-\partial_t v + \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2 - \frac{\mathrm{H}^3}{2} + \frac{v^2\mathrm{H}}{2}\Big]\,d\bar\mu_t\,dt\,,$$

which concludes the proof. □

5. Symmetries and Conserved Quantities

In this section we will analyze some particular variations and deduce certain properties of the stationary points of the action functional which are often less obvious from the Euler–Lagrange equation. In particular, we characterize some conserved quantities along smooth evolutions which are stationary points.

5.1. **Energy Conservation.**

The functional $\mathcal{S}$ can be formally seen as the sum of a kinetic and a potential term depending on the curvature, integrated with respect to a time dependent measure. By analogy with Lagrangian mechanics, one can write the formally associated Hamiltonian and can check whether energy conservation along stationary trajectories holds. We actually obtain the following property.

Proposition 5.1. *Let $\boldsymbol{\phi} : M \times [0,T] \to \mathbb{R}^{n+1}$ be a stationary point of the Functional $\mathcal{S}$ in the class of smooth evolutions with prescribed initial and final data. Then the quantity*

$$E(\phi_t) := \int_M (v^2 - \mathrm{H}^2)\, d\bar{\mu}_t, \quad t \in [0,T] \tag{5.1}$$

does not depend on t. We thus can define $E(\boldsymbol{\phi}) := E(\phi_t)$, $t \in [0,T]$ as the energy *of the stationary point $\boldsymbol{\phi}$.*

Proof. Let us consider a time reparametrization for $\boldsymbol{\phi}$ of the form $\boldsymbol{\phi}^\varepsilon(\cdot,t) = \boldsymbol{\phi}(\cdot,t_\varepsilon)$, $t_\varepsilon := t + \varepsilon\eta$, with $\eta \in C_0^\infty(0,T)$. One easily checks that the action of $\boldsymbol{\phi}^\varepsilon$ is given by

$$\mathcal{S}(\boldsymbol{\phi}^\varepsilon) = \int_0^T \int_M ((v^\varepsilon)^2 + (\mathrm{H}^\varepsilon)^2) d\bar{\mu}_t^\varepsilon\, dt = \int_0^T \int_M \Big(\frac{v^2}{1+\varepsilon\eta'} + (1+\varepsilon\eta')\mathrm{H}^2\Big) d\bar{\mu}_t dt. \tag{5.2}$$

For the corresponding first variation of $\mathcal{S}$ we get

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{S}(\boldsymbol{\phi}^\varepsilon) = \int_0^T \eta'(t) \int_M (-v^2 + \mathrm{H}^2) d\bar{\mu}_t dt. \tag{5.3}$$

Since $\boldsymbol{\phi}$ has been supposed to be stationary, the thesis follows. □

Remark 5.2. It is also possible to deduce the energy conservation from equations (4.2), (4.4) and (4.10). Actually,

$$\begin{aligned} \frac{d}{dt}\int_M (v^2 - \mathrm{H}^2)\, d\bar{\mu}_t = \int_M \Big[& 2v(\Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 - \frac{\mathrm{H}^3}{2} + \frac{v^2\mathrm{H}}{2}) - \\ & - 2\mathrm{H}(\Delta v + v|\mathrm{A}|^2) - (v^2 - \mathrm{H}^2)v\mathrm{H}\Big] d\bar{\mu}_t \ = \ 0\,. \end{aligned} \tag{5.4}$$

The fact that the energy conservation follows just by the invariance under time reparametrization, suggests that the energy should be conserved also in settings which are more general than the smooth one.

5.2. **Conformal Variations.**

We next investigate conformal variations of the form

$$\boldsymbol{\phi}(x,t,\varepsilon) = e^{a(t,\varepsilon)}\boldsymbol{\phi}(x,t), \tag{5.5}$$

where $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a smooth function which satisfies

$$a(t,0) \ = \ 0 \qquad \text{for all } \ t \in [0,T], \tag{5.6}$$

$$a(0,\varepsilon) \ = \ a(T,\varepsilon) \ = \ 0 \qquad \text{for all } \ -\varepsilon_0 < \varepsilon < \varepsilon_0. \tag{5.7}$$

We define $\alpha(t) := \partial_\varepsilon|_{\varepsilon=0} a(t,\varepsilon)$ for $t \in [0,T]$. The following lemma describes the variation under (5.5) of the geometric quantities appearing in $\mathcal{S}$.

Lemma 5.3. *Consider an evolution $\boldsymbol{\phi}$ and a variation ϕ_t^ε as in* (5.5)-(5.7)*. For all $t \in [0,T]$ we then have:*

$$\partial_\varepsilon|_{\varepsilon=0} d\bar\mu_t^\varepsilon \;=\; n\alpha(t)\, d\bar\mu_t, \tag{5.8}$$

$$\partial_\varepsilon|_{\varepsilon=0} v^\varepsilon(\cdot,t) = \alpha'(t)\phi_t\cdot\nu(\cdot,t) + \alpha(t) v(\cdot,t), \tag{5.9}$$

$$\partial_\varepsilon|_{\varepsilon=0}\mathrm{H}^\varepsilon \;=\; -\alpha(t)\mathrm{H}. \tag{5.10}$$

Proof. For the embeddings ϕ_t^ε, the induced metric on the corresponding embedded submanifold in $\mathbb{R}^{n+1}$ reads

$$g_{ij}^\varepsilon(\cdot,t) \;=\; e^{2a(t,\varepsilon)} g_{ij}(\cdot,t)\,, \tag{5.11}$$

hence

$$\partial_\varepsilon|_{\varepsilon=0} g_{ij}^\varepsilon(\cdot,t) = 2\alpha(t) g_{ij}(\cdot,t)\,,$$

and

$$\partial_\varepsilon|_{\varepsilon=0} d\bar\mu_t^\varepsilon = \frac{1}{2}\,\mathrm{tr}(\partial_\varepsilon|_{\varepsilon=0} g_{ij}^\varepsilon(\cdot,t)) d\bar\mu_t = n\alpha(t)\mathrm{d}\bar\mu_t\,.$$

Moreover, since $\nu_\varepsilon(x,t) = \nu(x,t)$, we deduce that

$$\partial_\varepsilon|_{\varepsilon=0}\nu^\varepsilon(\cdot,t)\cdot\partial_j\phi_t = 0\,.$$

For the normal speed, we have

$$\partial_\varepsilon|_{\varepsilon=0} v^\varepsilon(\cdot,t) \;=\partial_\varepsilon|_{\varepsilon=0}(\partial_t\phi_t^\varepsilon\cdot\nu^\varepsilon(\cdot,t))|_{\varepsilon=0} =\; \partial_t(\alpha(t)\phi_t\cdot\nu = \alpha'(t)\phi_t\cdot\nu(\cdot,t) + \alpha(t)v(\cdot,t)\,. \tag{5.12}$$

For the mean curvature, since the function α does not depend on the space variables, we obtain that

$$\mathrm{H}^\varepsilon \;=\; e^{-a(\cdot,\varepsilon)}\mathrm{H} \tag{5.13}$$

and we deduce (5.10). □

By means of Lemma 5.3, we are able to compute the variation of the *kinetic* term in $\mathcal{S}$.

Proposition 5.4. *Let $\boldsymbol{\phi} : M\times[0,T]\to\mathbb{R}^{n+1}$ be a trajectory and ϕ_t^ε be a variation as in* (5.5)-(5.7)*. For all $t\in[0,T]$ we have*

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^T\int_M (v^\varepsilon)^2 d\bar\mu_t^\varepsilon\, dt \\
&\qquad =\; 2\int_0^T\alpha(t)\int_M(-\partial_t v\boldsymbol{\phi}\cdot\nu + v\boldsymbol{\phi}\cdot\nabla v + v^2\mathrm{H}\boldsymbol{\phi}\cdot\nu + \frac{n}{2}v^2)\, d\bar\mu_t\, dt,
\end{aligned} \tag{5.14}$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^T\int_M(\mathrm{H}^\varepsilon)^2 d\bar\mu_t^\varepsilon\, dt \;=\; \big(n-2\big)\int_0^T\alpha(t)\int_M \mathrm{H}^2 d\bar\mu_t\, dt, \tag{5.15}$$

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\boldsymbol{\phi}^\varepsilon) \\
&\qquad =\; 2\int_0^T\alpha(t)\int_M\Big[-\partial_t v\boldsymbol{\phi}\cdot\nu + v\boldsymbol{\phi}\cdot\nabla v + v^2\mathrm{H}\boldsymbol{\phi}\cdot\nu + \frac{n}{2}v^2 + \Big(\frac{n}{2}-1\Big)\mathrm{H}^2\Big]\, d\bar\mu_t\, dt\,.
\end{aligned} \tag{5.16}$$

Proof. From Lemma 5.3 we get

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^T\int_M(v^\varepsilon)^2\, d\bar\mu_t^\varepsilon\, dt \;=\; 2\int_0^T\int_M\big[v(\alpha'\langle\boldsymbol{\phi},\nu\rangle + \alpha v) + \frac{n}{2}\alpha v^2\big]\, d\bar\mu_t\, dt. \tag{5.17}$$

By an integration by parts in the first of the three terms in the integrand on the right-hand side, and (4.2), (4.3) we obtain

$$\begin{aligned}
\int_0^T\int_M v\alpha'\boldsymbol{\phi}\cdot\nu d\bar\mu_t dt = \int_0^T\int_M[-\partial_t v\alpha\boldsymbol{\phi}\cdot\nu - v^2\alpha + v\alpha\boldsymbol{\phi}\cdot\nabla v + v^2\mathrm{H}\alpha\boldsymbol{\phi}\cdot\nu]\, d\bar\mu_t dt \\
+ \Big[\int_M v\,\alpha\boldsymbol{\phi}\cdot\nu d\bar\mu_t\Big]_0^T.
\end{aligned}$$

Using equation (5.7) we deduce (5.14). By (5.11) and (5.13) we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^T\int_M (\mathrm{H}^\varepsilon)^2\,d\mu_t^\varepsilon\,dt \;=\; \frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^T\int_M \mathrm{H}^2 e^{(-2+n)a(\cdot,\varepsilon)}\,d\mu_t\,dt$$
$$= \int_0^T(-2+n)\alpha(t)\int_M \mathrm{H}^2\,d\mu_t\,dt,$$

which gives (5.15). Together with (5.14) we finally deduce (5.16). □

Remark 5.5. We conclude that trajectories that are stationary for the *kinetic* part of the action along conformal variations as above satisfy

$$\int_M(-\partial_t v\phi_t\cdot\nu + v\phi_t\cdot\nabla v + v^2\mathrm{H}\phi_t\cdot\nu + \frac{n}{2}v^2)d\bar{\mu}_t = 0\,. \tag{5.18}$$

On the other hand it holds

$$\frac{d}{dt}\int_M v\phi_t\cdot\nu d\bar{\mu}_t \;=\; \int_M[\partial_t v\phi_t\cdot\nu + v^2 - v\phi_t\cdot\nabla v\rangle - v^2\mathrm{H}\phi_t\cdot\nu\rangle]d\bar{\mu}_t\,. \tag{5.19}$$

Adding equations (5.18) and (5.19) we obtain that for all $t\in[0,T]$

$$\frac{d}{dt}\int_M v\phi_t\cdot\nu \mathrm{d}\bar{\mu}_t = (1+\frac{n}{2})\int_M v^2 d\bar{\mu}_t\,. \tag{5.20}$$

Integrating over time, we find

$$\Big[\int_M v\phi_t\cdot\nu d\bar{\mu}_t\Big]_0^T = (1+\frac{n}{2})\int_0^T\int_M v^2 d\bar{\mu}_t dt\,. \tag{5.21}$$

For $n=2$ the Willmore functional is invariant under dilations, and in this case the variation of the whole action functional coincides with the variation of its *kinetic* part.

We are now in the position to prove an equality which can be used to deduce the Hamilton-Jacobi equation associated to $\mathcal{S}$.

Proposition 5.6. *For $\mathcal{S}-$stationary trajectories, the following equation holds true:*

$$\Big[\int_M v\phi_t\cdot\nu d\bar{\mu}_t\Big]_0^T = \int_0^T\int_M\Big[\Big(1+\frac{n}{2}\Big)v^2+\Big(\frac{n}{2}-1\Big)\mathrm{H}^2\Big]d\bar{\mu}_t dt = 2TE(\boldsymbol{\phi}) + n\mathcal{S}(\boldsymbol{\phi})\,, \tag{5.22}$$

where $E(\boldsymbol{\phi})$ denotes the energy defined in (5.1).

Proof. Since we are considering an $\mathcal{S}-$stationary trajectory, from equation (5.16) we have that

$$\int_M\Big[(-\partial_t v\phi_t\cdot\nu + v\phi_t\cdot\nabla v + v^2\mathrm{H}\phi_t\cdot\nu + \frac{n}{2}v^2 + \Big(\frac{n}{2}-1\Big)\mathrm{H}^2\Big]d\bar{\mu}_t = 0\,. \tag{5.23}$$

adding (5.19) and (5.23) we get

$$\frac{d}{dt}\int_M v\phi_t\cdot\nu d\bar{\mu}_t = \int_M\Big[\Big(1+\frac{n}{2}\Big)v^2+\Big(\frac{n}{2}-1\Big)\mathrm{H}^2\Big]d\bar{\mu}_t = 2E(\boldsymbol{\phi}) + \frac{n}{2}\int_M(v^2+\mathrm{H}^2)\mathrm{d}\bar{\mu}_t \tag{5.24}$$

and the thesis follows integrating over time. □

5.3. **Isometric variations.** We now consider variations of the form

$$\begin{aligned} &\phi_t^\varepsilon(x) \;=\; O(t,\varepsilon)\phi_t(x) \quad \text{for } x\in M, t\in[0,T], -\varepsilon_0<\varepsilon<\varepsilon_0,\\ &O(t,\varepsilon)\;\in\; SO(n+1),\quad O(t,0)\;=\;O(0,\varepsilon)\;=\;O(T,\varepsilon)=\mathrm{Id} \quad \text{for all } -\varepsilon_0<\varepsilon<\varepsilon_0, t\in[0,T]. \end{aligned} \tag{5.25}$$

It is clear that these variations leave the area element and the mean curvature invariant. Therefore, considering the first variation of the action functional, we obtain the following property.

Proposition 5.7. *If a trajectory undergoes a variation as in* (5.25)*, the first variation of* $\mathcal{S}$ *reads as*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\boldsymbol{\phi}^{\varepsilon}) \;=\; 2\int_0^T\int_M (A'(t)\phi_t(x))\cdot\partial_t\phi_t\,d\bar{\mu}_t\,dt\,, \tag{5.26}$$

where $A(t)=\partial_\epsilon|_{\epsilon=0}O(\varepsilon,t)$.

Proof. For the normal speed of the evolution $\boldsymbol{\phi}^{\varepsilon}$ we compute

$$\begin{aligned} v^{\varepsilon} \;&=\; \partial_t\phi_t^{\varepsilon}\cdot\nu^{\varepsilon}(\cdot,t) \;=\; \partial_t\big(O(t,\varepsilon)\phi_t\big)\cdot O(t,\varepsilon)\nu(\cdot,t),\\ \partial_\varepsilon|_{\varepsilon=0}v^{\varepsilon} \;&=\; \partial_t\big(A(t)\phi_t\big)\cdot\nu(\cdot,t)+\partial_t\phi_t\cdot A(t)\nu(\cdot,t). \end{aligned}$$

Since $A(t)$ is an antisymmetric matrix for any $t\in[0,T]$ we deduce

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0}S(\boldsymbol{\phi}^{\varepsilon}) &= 2\int_0^T\int_M v(\cdot,t)\big(\partial_t\big(A(t)\phi_t\big)\cdot\nu(\cdot,t)+\partial_t\phi_t\cdot A(t)\nu(\cdot,t)\big)d\bar{\mu}_t dt\\ &= 2\int_0^T\int_M \big(\partial_t\big(A(t)\phi_t\big)\cdot\partial_t\phi_t-(A(t)\partial_t\phi_t)\cdot\partial_t\phi_t\big)d\bar{\mu}_t dt, \end{aligned} \tag{5.27}$$

and the thesis follows. $\square$

Stationarity with respect to the variations of the form (5.25) can be interpreted as conservation of the *angular momentum.*

Corollary 5.8. *Let* $\boldsymbol{\phi}$ *be a stationary point of* $\mathcal{S}$*. Then the* angular momentum

$$\int_M v(\cdot,t)\big(\nu(\cdot,t)\otimes\phi_t-\phi_t\otimes\nu(\cdot,t)\big)\,d\bar{\mu}_t\,, \tag{5.28}$$

does not depend on time.

Proof. Consider equation (5.26) and choose $A(t)=f(t)A$, with $f\in C_c^\infty(0,T)$ arbitrary and A an arbitrary antisymmetric matrix. With this choice, from the stationarity of $\boldsymbol{\phi}$ we obtain

$$A:\frac{\mathrm{d}}{\mathrm{d}t}\int_M v(\cdot,t)\nu(\cdot,t)\otimes\phi_t\,d\bar{\mu}_t=0\,, \tag{5.29}$$

which coincides with the thesis. $\square$

Remark 5.9. If $\phi_0(\cdot)$ and $\phi_T(\cdot)$ are both round spheres, being $\phi(\cdot,0)$ and $\phi(\cdot,T)$ parallel to $\nu(\cdot,0)$ and $\nu(\cdot,T)$ respectively, the integrand in the formula for the angular momentum vanishes pointwise, and thus the angular momentum itself is zero. Notice however that the vanishing of the angular momentum does not imply that the trajectory is at every time a round sphere, even if if the initial and final data are both round spheres. This point will be clarified in the following section.

6. The spherical Case

In this section, we will study the problem of finding optimal trajectories connecting concentric, round $n-$spheres in $\mathbb{R}^{n+1}$. We will characterize optimal *spherical trajectories*, which are evolutions that at each $t\in[0,T]$ consist of a round sphere. We also determine conditions under which the optimal trajectory in the class of *spherical trajectories* is an absolute minimizer of the action functional.

6.1. **Some Formulae for Graphs over Spheres.** Let $\phi : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be a smooth embedding which can be parametrized as a graph over $\mathbb{S}^n$. This means that there exists a smooth function $r : \mathbb{S}^n \to \mathbb{R}$ such that

$$\phi(x) = r(x)x \ , \ x \in \mathbb{S}^n \,. \tag{6.1}$$

The following equations follow from (6.1) by direct computations.

Proposition 6.1. *If $\phi : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$, which is parametrized as a graph over the unit $n-$sphere, we have that the naturally induced metric on $\phi(\mathbb{S}^n)$ is given by*

$$\gamma_{ij} = r^2\tau_{ij} + \hat{\nabla}_i r\hat{\nabla}_j r \,, \tag{6.2}$$

where τ_{ij} is the standard metric on the unit sphere in $\mathbb{R}^{n+1}$ with associated Levi-Civita connection $\hat{\nabla}$ and measure $d\hat{\mu}$. The inverse of the induced metric reads

$$\gamma^{ij} = \frac{1}{r^2}\left(\tau^{ij} - \frac{\hat{\nabla}^i r\hat{\nabla}^j r}{r^2 + |\hat{\nabla} r|^2}\right). \tag{6.3}$$

The inner unit normal normal vector to the embedded surface is

$$\nu(x) = -\frac{1}{\sqrt{r^2 + |\hat{\nabla} r|^2}}(rx - \tau^{ij}\hat{\nabla}_i r\hat{\nabla}_j r) \tag{6.4}$$

and the second fundamental form is given by

$$\mathrm{h}_{ij} = \nu \cdot \partial^2_{ij}\phi = \frac{1}{\sqrt{r^2 + |\hat{\nabla} r|^2}}(r^2\tau_{ij} + 2\hat{\nabla}_i r\hat{\nabla}_j r - r\hat{\nabla}_i\hat{\nabla}_j r) \,, \tag{6.5}$$

while the mean curvature can be expressed as

$$\mathrm{H} = \frac{1}{r^2(r^2 + |\hat{\nabla} r|^2)^{3/2}}\Big[(n+1)r^2|\hat{\nabla} r|^2 + nr^4 + r\hat{\nabla}_i\hat{\nabla}_j r\hat{\nabla}^i r\hat{\nabla}^j r - r\hat{\Delta} r(r^2 + |\hat{\nabla} r|^2)\Big] \,. \tag{6.6}$$

Finally, the induced area element is given by

$$d\bar{\mu} = r^{n-1}\sqrt{r^2 + |\hat{\nabla} r|^2}\, d\hat{\mu} \,. \tag{6.7}$$

6.2. **First variation around spherical trajectories.** In this section we will study the first variation of the action functional in the following family of trajectories.

Definition 6.2. Given three positive real numbers T, R_0, and R_T, we say that a smooth map $\phi_0 : \mathbb{S}^n \times [0,T] \to \mathbb{R}^{n+1}$, which for any fixed $t \in [0,T]$ is an embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, is a spherical trajectory connecting the concentric $n-$spheres of radii R_0 and R_T, if there exists a smooth map $r_0 : [0,T] \to \mathbb{R}$ such that

$$\phi_0(x,t) = r_0(t)x \ , \ x \in \mathbb{S}^n \,, \tag{6.8}$$

with $r_0(0) = R_0$ and $r_0(T) = R_T$.

We now compute the first variation of the action functional around an arbitrary spherical trajectory. Without any loss of generality, we can restrict to variations which are graphs over spheres at any time.

Lemma 6.3. *Let T, R_0, R_T be positive real numbers and let $r_0 : [0,T] \to \mathbb{R}$ be a function defining a spherical trajectory ϕ_0 as in (6.8). Let $\rho : \mathbb{S}^n \times [0,T] \to \mathbb{R}$ be a smooth function with $\rho(\cdot,0) = \rho(\cdot,T) = 0$ for any $x \in \mathbb{S}^n$ and ε a real number. Define $r_\varepsilon : \mathbb{S}^n \times [0,T] \to \mathbb{R}$ so that $r_\varepsilon(x,t) = r_0(t) + \varepsilon\rho(x,t)$ and define $\phi_\varepsilon : \mathbb{S}^n \times [0,T] \to \mathbb{R}^{n+1}$ as $\phi_\varepsilon(x,t) = (r_0(t) + \varepsilon\rho(x,t))x$. Then it holds*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{S}(\phi_\varepsilon) = -\int_0^T\int_{\mathbb{S}^n}[2\ddot{r}_0 r_0^n + n\dot{r}_0^2 r_0^{n-1} - n^2(n-2)r_0^{n-3}]\rho\, d\hat{\mu}\, dt \,, \tag{6.9}$$

where the dot denotes the partial derivative with respect to t and $\hat{\mu}$ is the surface measure of the standard unit $n-$sphere in $\mathbb{R}^{n+1}$. As a consequence, for any stationary spherical trajectory, the function r_0 satisfies the ordinary differential equation

$$2\ddot{r}_0 r_0^n + n\dot{r}_0^2 r_0^{n-1} - n^2(n-2)r_0^{n-3} = 0\,. \tag{6.10}$$

Proof. Let us define $Q_\varepsilon := r_\varepsilon^2 + |\hat{\nabla} r_\varepsilon|^2$. From the definition of normal speed and (6.4) we have

$$\begin{aligned} v_\varepsilon^2 := (\partial_t \phi_\varepsilon \cdot \nu_\varepsilon)^2 &= Q_\varepsilon^{-1}(\dot{r}_\varepsilon x \cdot (r_\varepsilon x - \tau^{ij}\hat{\nabla}_i r_\varepsilon \hat{\nabla}_j x))^2 \\ &= Q_\varepsilon^{-1}(\dot{r}_\varepsilon x, r_\varepsilon x)^2 = Q_\varepsilon^{-1}\dot{r}_\varepsilon^2 r_\varepsilon^2\,. \end{aligned} \tag{6.11}$$

Moreover, from (6.6) and since $\hat{\nabla} r_\varepsilon = \varepsilon\hat{\nabla}\rho$,we have that

$$\mathrm{H}_\varepsilon = r_\varepsilon^{-2} Q_\varepsilon^{-3/2}[(n+1)\varepsilon^2 r_\varepsilon^2 |\hat{\nabla}\rho|^2 + n r_\varepsilon^4 + \varepsilon^3 r_\varepsilon \hat{\nabla}_{ij}^2\rho\hat{\nabla}_i\rho\hat{\nabla}_j\rho - \varepsilon r_\varepsilon \hat{\Delta}\rho Q_\varepsilon]\,, \tag{6.12}$$

which we rewrite as $\mathrm{H}_\varepsilon = r_\varepsilon^{-2} Q_\varepsilon^{-3/2} W_\varepsilon$, where we have denoted with W_ε all the terms in between the square brackets.
For convenience, let us rewrite the action as

$$\mathcal{S}(\phi_\varepsilon) := \int_0^T \int_{\mathbb{S}^n} (v_\varepsilon^2 + \mathrm{H}_\varepsilon^2) d\bar{\mu}_t^\varepsilon dt = \int_0^T \int_{\mathbb{S}^n} (K_\varepsilon + P_\varepsilon) d\hat{\mu} dt\,, \tag{6.13}$$

with $K_\varepsilon := \dot{r}_\varepsilon^2 r_\varepsilon^{n+1} Q_\varepsilon^{-1/2}$ and $P_\varepsilon := r_\varepsilon^{n-5} Q_\varepsilon^{-5/2} W_\varepsilon^2$. Differentiating, we get

$$\partial_\varepsilon|_{\varepsilon=0} K_\varepsilon = 2\dot{r}_0 r_0^n \dot{\rho} + (n+1)\dot{r}_0^2 r_0^{n-1}\rho - \dot{r}_0^2 r_0^{n-1}\rho \tag{6.14}$$

and

$$\begin{aligned} \partial_\varepsilon|_{\varepsilon=0} P_\varepsilon &= n^2(n-5) r_0^{n-3}\rho - 5n^2 r_0^{n-3}\rho + 2n r_0^{n-3}(4n\rho - \hat{\Delta}\rho) \\ &= n^2(n-2) r_0^{n-3}\rho + 2n r_0^{n-3}\hat{\Delta}\rho\,. \end{aligned} \tag{6.15}$$

The thesis follows by summing the two contributions and integrating by parts. □

Remark 6.4. (1) Note that (6.10) characterizes the Euler–Lagrange equation of $\mathcal{S}$ restricted to the class of spherical symmetric evolutions. Moreover, equation (6.10) follows already from energy conservation along a spherical trajectory. Actually, in the case of spherical trajectories, (5.1) implies

$$\frac{d}{dt}(\dot{r}_0^2 r_0^n - n^2 r_0^{n-2}) = 0\,, \tag{6.16}$$

which is equivalent to (6.10). Since energy conservation is a consequence of stationarity with respect to time reparametrization, and since the class of spherical trajectories is invariant under time reparametrization, energy conservation is clearly a necessary condition for the stationarity of spherical trajectories. Here it is also sufficient.
(2) In the case of a spherical trajectory the Euler–Lagrange equation (4.9) reduces to (6.10). Thus, a spherical trajectory which is critical in the class of spherical evolutions is also critical in the larger class of smooth trajectories.

Remark 6.5. In the case $n = 1$, equation (6.16) is equivalent to $(\dot{r}_0^2 r_0 - r_0^{-1}) = E$, with $E \in \mathbb{R}$. The solutions to this equation coincide with the rotationally symmetric minima of $\mathcal{S}$ given by Okabe in [21]. We also notice, that the three classes of solution given by Okabe correspond respectively to the cases $E < 0$, $E = 0$, and $E > 0$.
If $n = 2$, from (6.16) we obtain an explicit formula for r_0 (assuming without any loss of generality that $R_0 > R_T$),

$$r_0(t) = \sqrt{\frac{R_0^2 - R_T^2}{T} t + R_0^2}. \tag{6.17}$$

In particular, the unique stationary spherical solution is a time rescaled mean curvature flow. In the following it will be convenient to compare T with the time $T_{MCF}(R_0, R_T)$ which is needed to join two concentric 2−dimensional round spheres in $\mathbb{R}^3$ having radii R_0 and R_T by (time reversed) mean curvature flow. This time is given by

$$T_{MCF} \,=\, T_{MCF}(R_0, R_T) \,=\, \frac{|R_0^2 - R_T^2|}{4}. \tag{6.18}$$

6.3. **Second variation around spherical trajectories.** We further investigate stationary spherical trajectories and in particular we determine conditions under which they are local minima of the action functional in the class of general smooth evolutions.

Lemma 6.6. *In the situation of Lemma 6.3, we have that*

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{S}(\phi_\varepsilon) = \int_0^T\int_{\mathbb{S}^n} [&2\dot\rho^2 r_0^n + ((n+1)(n-2)+2)\dot r_0^2 r_0^{n-2}\rho^2 + 2(r_0^n)^{\cdot}(\rho^2)^{\cdot} - \dot r_0^2 r_0^{n-2}|\hat\nabla\rho|^2\\ &+ n^2(n-2)(n-3)r_0^{n-4}\rho^2 + (3n^2-8n)r_0^{n-4}|\hat\nabla\rho|^2 + 2r_0^{n-4}(\hat\Delta\rho)^2]\,d\hat\mu\,dt\,.\end{aligned} \tag{6.19}$$

Proof. Adopting the same notation as in the proof of Lemma 6.3, we have that $K_\varepsilon = \dot r_\varepsilon^2 r_\varepsilon^{n+1}Q_\varepsilon^{-1/2}$. Consequently, we make the following preliminary computations

$$\partial_\epsilon|_{\epsilon=0} r_\varepsilon^{n+1} = (n+1)r_0^n\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0} r_\varepsilon^{n+1} = n(n+1)r_0^{n-1}\rho^2\,,$$

$$\partial_\epsilon|_{\epsilon=0}\dot r_\varepsilon^2 = 2\dot r_0\dot\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0}\dot r_\varepsilon^2 = 2\dot\rho^2\,,$$

$$\partial_\epsilon|_{\epsilon=0}Q_\varepsilon^{-1/2} = -r_0^{-2}\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0}Q_\varepsilon^{-1/2} = r_0^{-3}(2\rho^2 - |\hat\nabla\rho|^2)\,.$$

This way, we have that

$$\begin{aligned}\partial_\epsilon^2\Big|_{\epsilon=0}K_\varepsilon &= 2r_0^n\dot\rho^2 + n(n+1)\dot r_0^2 r_0^{n-2}\rho^2 + \dot r_0^2 r_0^{n-2}(2\rho^2 - |\hat\nabla\rho|^2)\\ &\quad + 4(n+1)\dot r_0 r_0^{n-1}\dot\rho\rho - 4\dot r_0 r_0^{n-1}\dot\rho\rho - 2(n+1)\dot r_0^2 r_0^{n-2}\rho^2\\ &= 2r_0^n\dot\rho^2 + ((n-2)(n+1)+2)\dot r_0^2 r_0^{n-2}\rho^2 + 4n\dot r_0 r_0^{n-1}\dot\rho\rho - \dot r_0^2 r_0^{n-2}|\hat\nabla\rho|^2\,.\end{aligned} \tag{6.20}$$

For $P_\varepsilon = r_\varepsilon^{n-5}Q_\varepsilon^{-5/2}W_\varepsilon^2$, we compute

$$\begin{aligned}&\partial_\epsilon|_{\epsilon=0} r_\varepsilon^{n-5} = (n-5)r_0^{n-6}\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0} r_\varepsilon^{n-5} = (n-5)(n-6)r_0^{n-7}\rho^2\,,\\ &\partial_\epsilon|_{\epsilon=0}Q_\varepsilon^{-5/2} = -5r_0^{-6}\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0}Q_\varepsilon^{-5/2} = 5r_0^{-7}(6\rho^2 - |\hat\nabla\rho|^2)\,,\\ &\partial_\epsilon|_{\epsilon=0}W_\varepsilon = 4nr_0^3\rho - r_0^3\Delta\rho, \qquad \partial_\epsilon^2\Big|_{\epsilon=0}W_\varepsilon = 2r_0^2[(n+1)|\hat\nabla\rho|^2 + 6n\rho^2 - 3\rho\hat\Delta\rho]\,,\\ &\partial_\epsilon|_{\epsilon=0}W_\varepsilon^2 = 2nr_0^7(4n\rho - \hat\Delta\rho),\\ &\partial_\epsilon^2\Big|_{\epsilon=0}W_\varepsilon^2 = 4nr_0^6[(n+1)|\hat\nabla\rho|^2 + 6n\rho^2 - 3\rho\hat\Delta\rho] + 2r_0^6(4n\rho - \hat\Delta\rho)^2\,.\end{aligned}$$

By the previous computations, we can conclude that

$$\partial_\epsilon^2\Big|_{\epsilon=0}P_\varepsilon = n^2(n-2)(n-3)r_0^{n-4}\rho^2 + (3n^2-8n)r_0^{n-4}|\hat\nabla\rho|^2 + 2r_0^{n-4}(\hat\Delta\rho)^2 + \tag{6.21}$$

$$+\, r_0^{n-4}(-4n^2+12n)\hat\nabla\cdot(\rho\hat\nabla\rho). \tag{6.22}$$

If we now sum the equations (6.20) and (6.21), and integrate over space and time, the thesis follows after imposing the boundary conditions on ρ. ☐

We now fix $n = 2$ and consider the stationary spherical evolution r_0. By (6.16), the integral over the third term in (6.19) vanishes. Evaluating (6.19) in r_0 for spatially homogeneous ρ, we observe that the second variation is positive definite. This shows that for $n = 2$ the spherical stationary point r_0 determined by (6.10) is the unique minimizer in the class of smooth spherical

evolutions. We will call r_0 the associated $\mathcal{S}$-optimal spherical trajectory.
Equation (6.19) considerably simplifies for $n = 2$. In this case, we can actually prove that the optimal spherical trajectory is not always a minimizer of the action functional.

Theorem 6.7. *Let $n = 2$. For two given positive real numbers R_0 and R_T, the $\mathcal{S}-$optimal spherical trajectory connecting the two concentric spheres of radii R_0 and R_T over the time interval $[0,T]$ is a local minimizer of $\mathcal{S}$ if $T \geq \frac{1}{3}\sqrt{3}T_{MCF}$, where T_{MCF} was defined in (6.18). Furthermore, there exists $0 < T_1 \leq \frac{1}{3}\sqrt{3}T_{MCF}$ such that for all $0 < T < T_1$ the optimal spherical trajectory connecting the given data is a not a local minimizer of $\mathcal{S}$.*

Proof. For $n = 2$, equation (6.19) reads

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{S}(\phi_\varepsilon) = \int_0^T \int_{\mathbb{S}^2} [2\dot{\rho}^2 r_0^2 + 2\dot{r}_0^2\rho^2 - \dot{r}_0^2|\hat{\nabla}\rho|^2 + 2r_0^{-2}((\hat{\Delta}\rho + \rho)^2 - \rho^2)]d\hat{\mu}dt\,, \tag{6.23}$$

where partial integration with respect to both spatial and time variables has been performed and where we have used that (6.17) implies $(r_0^2)^{\cdot\cdot} = 0$. We now choose $\rho(x,t) = \eta(t)\psi_l(x)$, where $\eta \in C_0^\infty([0,T])$ is arbitrary and where $\psi_l : \mathbb{S}^2 \to \mathbb{R}$, $l \in \mathbb{N}_0$, denotes the $l-$th spherical harmonic associated to the standard metric on $\mathbb{S}^2$, for which holds $\Delta\psi = -l(l+1)\psi$. Substituting $\dot{r}_0^2$ with its explicit expression given by equation (6.17) and recalling (6.18), we get

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{S}(\phi_\varepsilon) = 4\pi\int_0^T \Big[2\dot{\eta}^2 r_0^2 + \frac{\eta^2}{r_0^2}\Big(4(2 - l(l+1))\frac{T_{MCF}^2}{T^2} + 2((l(l+1)-1)^2 - 1)\Big)\Big]dt\,. \tag{6.24}$$

One computes that the term in the large round brackets is for all $l \in \mathbb{N}_0$ nonnegative if $T \geq \frac{1}{3}\sqrt{3}T_{MCF}$. Since we can expand any perturbation in a series of spherical harmonics with time dependent coefficients and since the expression on the right-hand side of (6.23) splits in a sum of the corresponding expressions of the spherical harmonics, we have shown that the second variation is positive definite for $T \geq \frac{1}{3}\sqrt{3}T_{MCF}$. On the other hand, for any $\eta \in C_0^\infty([0,T])$ and $l \geq 2$ fixed, we can choose $0 < T \ll 1$ such that the corresponding second variation becomes negative. □

Remark 6.8. Theorem 6.7 shows that for any pair of given concentric spherical data, the prescribed amount of time to join them determines whether the optimal spherical trajectory is a local minimum for the action functional or not. The non-minimality of some critical spherical trajectories can be understood remembering that there exist evolutions which make the velocity contribution in the action functional arbitrarily small (as shown in [18]), and that at the same time the round spheres in $\mathbb{R}^3$ are the only absolute minimizers for the Willmore functional in the class of compact surfaces without boundary. If the amount of time given to connect the initial with the final data is sufficiently small, the possibility to make the velocity contribution arbitrarily close to zero compensates a non optimal curvature term. On the other hand, if the given amount of time is sufficiently large, the Willmore minimizing property of the round spheres favors the spherically symmetric evolutions.

We complement the previous result and show a global minimizing property for the $\mathcal{S}$-optimal trajectory connecting two concentric spheres in $\mathbb{R}^3$.

Theorem 6.9. *Let $n = 2$. Let $R_0 > R_T > 0$ and $T > 0$ be positive real numbers. If $T \geq T_{MCF}(R_0, R_T)$, the $\mathcal{S}-$optimal spherical trajectory connecting the two concentric round $2-$spheres of radii R_0 and R_T over the time interval $[0,T]$ is a global minimum for $\mathcal{S}$ in the class of smooth evolutions.*

Proof. We first observe that for any $c \in \mathbb{R}$ we have

$$8\pi(R_0^2 - R_T^2) = 2\int_0^T \int_M v\mathrm{H}\, d\bar{\mu}_t dt = \int_0^T \int_M \Big(-\frac{1}{c}(v - c\mathrm{H})^2 + \frac{1}{c}v^2 + c\mathrm{H}^2\Big)d\bar{\mu}_t dt.$$

Thus, we obtain

$$\begin{aligned}\int_0^T\int_M (v^2+\mathrm{H}^2)\mathrm{d}\bar{\mu}_t\mathrm{d}t &= 8\pi c(R_0^2-R_T^2)+\int_0^T\int_M (v-c\mathrm{H})^2 d\bar{\mu}_t dt+(1-c^2)\int_0^T \mathrm{H}^2 d\bar{\mu}_t dt\\ &\geq 8\pi c(R_0^2-R_T^2)+(1-c^2)\int_0^T\int_M \mathrm{H}^2 d\bar{\mu}_t dt\,,\end{aligned}\tag{6.25}$$

where the equality holds if and only if

$$v = c\mathrm{H}\tag{6.26}$$

at each point in space and time. Moreover, if (6.26) holds, the initial and final conditions force the solution to have spherical symmetry at any time. Since $\int_M \mathrm{H}^2 d\bar{\mu}_t \geq 16\pi$, as spheres are the unique minimizer of the Willmore energy for smooth embeddings of M, we deduce from (6.25) that

$$\int_0^T\int_M (v^2+\mathrm{H}^2)\mathrm{d}\bar{\mu}_t\mathrm{d}t \;\geq\; \max_{c^2\leq 1}\Big(8\pi c(R_0^2-R_T^2)+16(1-c^2)\pi T\Big).$$

Explicit calculations show that the maximum on the right hand side is uniquely attained for $c_* = \frac{T_{MCF}}{T}$ (note that by assumption $c_* \leq 1$), and that we thereby obtain the estimate

$$\int_0^T\int_M (v^2+\mathrm{H}^2)d\bar{\mu}_t dt \;\geq\; 16\pi\Big(\frac{T_{MCF}^2}{T}+T\Big).\tag{6.27}$$

The value of the right-hand side coincides with the value of the action functional evaluated at r_0 and the optimality is proven. □

Remark 6.10. The proof of Theorem 6.9 shows that the $\mathcal{S}$-optimal spherical trajectory is optimal also in the class of non-vanishing evolutions which are piecewise smooth and have a continuous area function $t \mapsto \bar{\mu}_t(M)$. In a final remark we compare the $\mathcal{S}$-optimal spherical trajectory with a non-smooth evolution which vanishes on some positive time interval.

Remark 6.11. Let the sphere with radius R_0 evolve by mean curvature flow until it vanishes at $T_{MCF}(R_0) = \frac{R_0^2}{4}$. Consider then a point nucleation at $T - T_{MCF}(R_T) = T - \frac{R_T^2}{4}$, which evolves by time–reversed MCF up to the time T. This way, at time $t = T$ we reach the sphere of radius R_T (notice that this kind of trajectory is well defined under the further assumption $T \geq \max\{T_{MCF}(R_0), T_{MCF}(R_T)\}$). The value of the action functional for this particular trajectory is given by $8\pi(R_0^2+R_T^2)$, which is twice the sum of the area of the initial and final data. Comparing this value with the value of the action functional at the $\mathcal{S}$-optimal smooth spherical solution, which by (6.27) is given by the value $16\pi\Big(\frac{T_{MCF}^2}{T}+T\Big)$, we see that if $T > \frac{(R_0+R_T)^2}{4}$ the value of the action for the non-smooth trajectory is smaller than the value for the smooth one. Collecting all the results, we obtain the following picture. For $T < T_{MCF}(R_0, R_T)$ the minimum of the action functional is attained, but we can not say wether its minimum points are smooth or spherically symmetric trajectories. For $T_{MCF}(R_0, R_T) \leq T \leq \frac{(R_0+R_T)^2}{4}$, the optimal smooth spherically symmetric connection is the absolute action minimizer. For $T > \frac{(R_0+R_T)^2}{4}$ the optimal smooth rotationally symmetric evolution is still a local minimizer, nevertheless the absolute minimum of the action functional is attained at evolutions which are not smooth and could have nucleations.

7. Appendix. Notations and results from differential geometry

Let M be an n-dimensional smooth differentiable manifold without boundary and consider a smooth immersion $\phi : M \to \mathbb{R}^{n+1}$. Denoting with $(x^1, ..., x^n)$ a local coordinate system on M and $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) := (e_1, ..., e_n)$ the associated base for the tangent space, the Riemannian metric g induced by ϕ on M via pullback reads as follows:

$$g_{ij} := \partial_i\phi \cdot \partial_j\phi\,,\tag{7.1}$$

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. We will denote with ∇ the covariant derivative associated to the Levi-Civita connection of g.
Since $\phi(M)$ has codimension one in $\mathbb{R}^{n+1}$, it follows that M is orientable and at each point of $\phi(M)$ there is a well defined smooth inner unit normal vector field which we call ν. We define the second fundamental form of $\phi(M)$ according to

$$\mathrm{A} = (\mathrm{h}_{ij})_{ij} := (\nu \cdot \partial^2_{ij}\phi)_{ij}\,, \tag{7.2}$$

which implies that A is a symmetric 2-tensor on $\phi(M)$.
The mean curvature of the couple (M, ϕ) is defined as the trace of the second fundamental form and is denoted by $\vec{\mathrm{H}}$, while the scalar mean curvature is given by $\mathrm{H} := \vec{\mathrm{H}} \cdot \nu$.
Within this setting, the Gauss-Weingarten relations read

$$\partial^2_{ij}\phi = \Gamma^k_{ij}\partial_k\phi + \mathrm{h}_{ij}\nu \quad \text{and} \quad \partial_i\nu = -\mathrm{h}_{ik}g^{kl}\partial_l\phi\,. \tag{7.3}$$

The Bianchi identities for the curvature tensor of the immersed manifold are equivalent to

$$\nabla_i\mathrm{h}_{jk} = \nabla_j\mathrm{h}_{ik}\,. \tag{7.4}$$

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