INITIAL-BOUNDARY VALUE PROBLEMS FOR CONTINUITY EQUATIONS WITH BV COEFFICIENTS

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Abstract. We establish well-posedness of initial-boundary value problems for continuity equations with BV (bounded total variation) coefficients. We do not prescribe any condition on the orientation of the coefficients at the boundary of the domain. We also discuss some examples showing that, regardless the orientation of the coefficients at the boundary, uniqueness may be violated as soon as the BV regularity deteriorates at the boundary.

Keywords: Continuity equation, transport equation, initial-boundary value problem, low regularity coefficients, uniqueness.

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1. Introduction

This work is devoted to the study of the initial-boundary value problem for the continuity equation

$$\partial_t u + \operatorname{div}(bu) = c\,u + f. \tag{1.1}$$

where $b :]0,T[\times\Omega \to \mathbb{R}^d$ is a given vector field, $c :]0,T[\times\Omega \to \mathbb{R}$ and $f :]0,T[\times\Omega \to \mathbb{R}$ are given functions and the unknown is $u :]0,T[\times\Omega \to \mathbb{R}$. Finally, $\Omega \subseteq \mathbb{R}^d$ is an open set and div denotes the divergence computed with respect to the space variable only. Note that, in the particular case when $c = \operatorname{div} b$ and $f \equiv 0$, equation (1.1) reduces to the transport equation

$$\partial_t u + b \cdot \nabla u = 0.$$

The analysis of (1.1) in the case when b has low regularity has recently drawn considerable attention: for an overview of some of the main contributions, we refer to the lecture notes by Ambrosio and Crippa [3]. Here, we only quote the two main breakthroughs due to DiPerna and Lions [17] and to Ambrosio [2], which deal with the case when $\operatorname{div} b$ is bounded and b enjoys Sobolev and BV (bounded total variation) regularity, respectively. More precisely, in [17] and [2] the authors establish existence and uniqueness results for the Cauchy problem posed by coupling (1.1) with an initial datum in the case when $\Omega = \mathbb{R}^d$.

In the classical framework where all functions are smooth up to the boundary, the initial-boundary value problem is posed by prescribing

$$\begin{cases} \partial_t u + \operatorname{div}(bu) = c\,u + f & \text{in }]0,T[\times\Omega \\ u = \bar g & \text{on } \Gamma^- \\ u = \bar u & \text{at } t = 0, \end{cases} \tag{1.2}$$

where $\bar u$ and $\bar g$ are bounded smooth functions and Γ^- is the portion of $]0,T[\times\partial\Omega$ where the characteristics are entering the domain $]0,T[\times\Omega$. Note, however, that if b and u are not sufficiently regular (if, for example, u is only an L^∞ function), then their values on negligible sets are not, a priori, well defined. In § 3.2 we provide the distributional formulation of (1.2) by relying on the theory of normal traces for weakly differentiable vector fields, see the works by Anzellotti [6] and, more recently, by Chen and Frid [9], Chen, Torres and Ziemer [10] and by Ambrosio, Crippa and Maniglia [4].

Our main positive result reads as follows:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Assume that the vector field b satisfies the following hypotheses:*

1. $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$;
2. $\operatorname{div} b \in L^\infty(]0,T[\times\Omega)$;
3. *for every open and bounded set* $\Omega_* \subseteq \Omega$, $b \in L^1_{\mathrm{loc}}([0,T[;BV(\Omega_*;\mathbb{R}^d))$.

Assume moreover that $c \in L^\infty(]0,T[\times\Omega)$ and $f \in L^\infty(]0,T[\times\Omega)$. Then, given $\bar u \in L^\infty(\Omega)$ and $\bar g \in L^\infty(\Gamma^-)$, problem (1.2) admits a unique distributional solution $u \in L^\infty(]0,T[\times\Omega)$.

Some remarks are here in order. First, we recall that Γ^- is a subset of $]0,T[\times\partial\Omega$ and we point out that $L^\infty(\Gamma^-)$ denotes the space $L^\infty(\Gamma^-, \mathscr{L}^1 \otimes \mathscr{H}^{d-1})$.

Second, we refer to the book by Leoni [19, Definition 12.10] for the definition of open set with uniformly Lipschitz boundary. In the case when Ω is bounded, the definition reduces to the classical condition that Ω has Lipschitz boundary. This regularity assumption guarantees that classical results on the traces of Sobolev and BV functions apply to the set Ω, see again Leoni [19] for an extended discussion.

Third, several works are devoted to the analysis of the initial-boundary value problem (1.2). In particular, we refer to Bardos [7] for an extended discussion on the case when b enjoys Lipschitz regularity, and to Mischler [20] for the case when the continuity equation in (1.2) is the Vlasov equation. Also, we quote reference [8], where Boyer establishes uniqueness and existence results for (1.2) and investigates space continuity properties of the trace of the solution on suitable surfaces. The main assumption in [8] is that b has Sobolev regularity, and besides this there are the technical assumptions that $\operatorname{div} b \equiv 0$, $c \equiv 0$ and that Ω is bounded. See also the analysis by Girault and Ridgway Scott [18] for the case when b enjoys Sobolev regularity and is tangent to the boundary. Note that the extension of Boyer's proof to the case when b has BV regularity is not straightforward.

Our approach is quite different from Boyer's: indeed, the analysis in [8] is based on careful estimates on the behavior of b and u close to the boundary and involves the introduction of a system of normal and tangent coordinates at $\partial\Omega$, and the use of a local regularization of the equation. Conversely, as mentioned above, in the present work we rely on the theory of normal traces for weakly differentiable vector fields. From the point of view of the results we obtain, the main novelties of the present work can be summarized as follows.

- We establish well-posedness of (1.2) (see Theorem 1.1) under the assumptions that b enjoys BV regularity, while in [8] Sobolev regularity is required. Note, however, that the main novelty in Theorem 1.1 is the uniqueness part, since existence can be established under the solely hypotheses that $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$ and $\operatorname{div} b$, c, $f \in L^\infty(]0,T[\times\Omega)$ by closely following the same argument as in [8], see [11] for the technical details. We point out in passing that, for the Cauchy problem, the extension of the uniqueness result from Sobolev to BV regularity is one of the main achievement in Ambrosio's paper [2]. Also, this extension is crucial in view of the applications to some classes of nonlinear PDEs like systems of conservation laws in several space dimensions, see the lecture notes by De Lellis [14], the overview by Crippa and Spinolo [12] and the references therein.
- We exhibit some counterexamples (see Proposition 1.2, Theorem 1.3 and Corollary 1.4 below) showing that, regardless the orientation of b at the boundary, uniqueness may be violated as soon as b enjoys BV regularity in every open set Ω_* compactly contained in Ω, but the regularity deteriorates at the boundary of Ω. Also, as the proof of Theorem 1.3 shows, if BV regularity deteriorates at the domain boundary, it may happen that the normal trace of b at $\partial\Omega$ is identically zero, while the normal trace of bu is identically 1, see § 3.2 for the definition of normal trace of b and bu. Note, moreover, that, while in the proof of Proposition 1.2 we heavily rely on a previous example due to Depauw [16], the construction of the most surprising of our counter-examples (the one that we exhibit in the proof of Theorem 1.3) is new and does not rely on [1, 16].
- In [8, § 7.1], Boyer establishes a space continuity property for the solution of (1.2) in directions trasversal to the vector field b under the assumption that b enjoys Sobolev regularity. Proposition 3.5 in the present work ensures that an analogous property holds under BV regularity assumptions. The property we establish is loosely speaking the following: assume Σ_r is a family of surfaces which continuously depend on the parameter r and assume moreover that the surfaces are all transversal to a given direction. Then the normal trace of the vector field ub on Σ_r strongly converges to the normal trace of ub on Σ_{r_0} as $r \to r_0$.

Here is our first counterexample. In the statement of Proposition 1.2, $\operatorname{Tr} b$ denotes the normal trace of b along the outward pointing, unit normal vector to $\partial\Omega$, as defined in § 3.2.

Proposition 1.2. *Let Ω be the set $\Omega :=]0,+\infty[\times\mathbb{R}^2$. Then there is a vector field $b :]0,1[\times\Omega \to \mathbb{R}^3$ such that*

i) $b \in L^\infty(]0,1[\times\Omega;\mathbb{R}^3)$*;*
ii) $\operatorname{div} b \equiv 0$*;*

iii) *for every open and bounded set* Ω_* *such that its closure* $\bar{\Omega}_* \subseteq \Omega$, *we have* $b \in L^1([0,1[; BV(\Omega_*; \mathbb{R}^3))$;

iv) $\mathrm{Tr}\, b \equiv -1$ *on* $]0,1[\times \partial\Omega$;

v) *the initial-boundary value problem*

$$\begin{cases} \partial_t u + \mathrm{div}\,(bu) = 0 & \text{in }]0,1[\times\Omega \\ u = 0 & \text{on }]0,1[\times\partial\Omega \\ u = 0 & \text{at } t=0 \end{cases} \tag{1.3}$$

admits infinitely many different solutions.

Some remarks are here in order. First, since the vector field b is divergence-free, then any solution of (1.3) is a solution of the transport equation

$$\partial_t u + b \cdot \nabla u = 0$$

satisfying zero boundary and initial conditions. Second, the proof of Proposition 1.2 is, basically, a reformulation of an intriguing construction due to Depauw [16] which was inspired by a previous example by Aizenman [1].

Finally, note that property iv) in the statement of Proposition 1.2 states that the vector field b is inward pointing at the boundary $\partial\Omega$. This fact is actually crucial for our argument because it allows us to build on Depauw's construction.

When the vector field is outward pointing, one could heuristically expect that the solution would not be affected by the loss of regularity of b at the domain boundary. Indeed, in the smooth case the solution is simply "carried out" of the domain along the characteristics and, consequently, the behavior of the solution inside the domain is not substantially affected by what happens close to the boundary. Hence, one would be tempted to guess that, even in the non smooth case, when $\mathrm{Tr}\, b > 0$ on the boundary the solution inside the domain is not affected by boundary behaviors and uniqueness should hold even when the BV regularity of b deteriorates at the boundary. The example discussed in the statement of Theorem 1.3 shows that this is actually not the case and that, even if b is outward pointing at $\partial\Omega$, then uniqueness may be violated as soon as the BV regularity deteriorates at the boundary.

Theorem 1.3. *Let* Ω *be the set* $\Omega :=]0,+\infty[\times\mathbb{R}^2$. *Then there is a vector field* $b :]0,1[\times\Omega \to \mathbb{R}^3$ *such that*

i) $b \in L^\infty(]0,1[\times\Omega; \mathbb{R}^3)$;

ii) $\mathrm{div}\, b \equiv 0$;

iii) *for every open and bounded set* Ω_* *such that its closure* $\bar{\Omega}_* \subseteq \Omega$, *we have* $b \in L^1([0,1[; BV(\Omega_*; \mathbb{R}^3))$;

iv) $\mathrm{Tr}\, b \equiv 1$ *on* $]0,1[\times\partial\Omega$;

v) *the initial-boundary value problem*

$$\begin{cases} \partial_t u + \mathrm{div}\,(bu) = 0 & \text{in }]0,1[\times\Omega \\ u = 0 & \text{at } t=0 \end{cases} \tag{1.4}$$

admits infinitely many different solutions.

We make some observations. First, the proof of Theorem 1.3 does not use Depauw's example [16] and it relies on a new construction. The basic idea of the proof is the following. In the case when b is smooth, if $\mathrm{Tr}\, b > 0$ then b is outward pointing at the domain boundary and hence the solution of the continuity equation is "carried out" the domain along the characteristic lines. Conversely, in the proof of Theorem 1.3 we construct a nontrivial solution u which roughly speaking "enters" the domain $]0,1[\times\Omega$, although $\mathrm{Tr}\, b > 0$. This nontrivial solution satisfies $u(t,x) \geq 0$ for a.e. $(t,x) \in]0,1[\times\Omega$, but $\mathrm{Tr}\ (bu) < 0$. This apparently self-contradictory behavior is possible because the vector field b is constructed in such a way that there are infinitely many regions where b is inward pointing and infinitely many regions where b is outward pointing. These regions mix at finer and finer scales as one approaches the domain boundary and this accounts for the breakdown of the BV regularity. The key point in the construction is that, although the total "averaged" effect is that b is outward pointing at the domain boundary, the presence of infinitely many regions where b is inward pointing allows to construct a nontrivial solution u "entering" the domain.

Second, by a trivial modification of the proof one can exhibit a vector field b satisfying properties i), ii), iii) and v) above and, instead of property iv), $\mathrm{Tr}\, b \equiv 0$ on $]0,1[\times\partial\Omega$. Hence, even in the case when b is tangent at the domain boundary, uniqueness may be violated as soon as the BV regularity deteriorates at the domain boundary.

Third, since $\mathrm{Tr}\, b \equiv 1$ on $]0,1[\times\partial\Omega$, then in (1.4) we do not prescribe the value of the solution u at the boundary. Note that by a slight abuse of notation we still term (1.4) "initial-boundary value problem" because the equation is defined in a domain with a non trivial boundary. Also, this is consistent with Definition 3.4.

Finally, in the proof of Theorem 1.3 we exhibit infinitely many different solutions of (1.4) and in general different solutions attain different values on $]0,1[\times\partial\Omega$. However, by refining the proof of Theorem 1.3 we obtain the following result.

Corollary 1.4. *Let Ω be the set $\Omega :=]0,+\infty[\times\mathbb{R}^2$, then there is a vector field $b :]0,1[\times\Omega \to \mathbb{R}^3$ satisfying requirements* i), . . . , iv) *in the statement of Theorem 1.3 and such that* (1.4) *admits infinitely many solutions that satisfy* $\mathrm{Tr}\,(bu) \equiv 0$ *on* $]0,1[\times\partial\Omega$.

The additional condition $\mathrm{Tr}\,(bu) \equiv 0$ in the corollary can be heuristically interpreted as (a weak version of) $u \equiv 0$ on $]0,1[\times\partial\Omega$.

We also point out that, again by a trivial modification of the proof, one can exhibit a vector field b satisfying properties i), ii), iii) and v) in the statement of Corollary 1.4 and, instead of property iv), $\mathrm{Tr}\, b \equiv 0$ on $]0,1[\times\partial\Omega$. Also, for any given real constant k, one can actually construct infinitely many solutions of (1.4) that satisfy $\mathrm{Tr}\,(bu) = k$ on $]0,1[\times\partial\Omega$.

Outline. The paper is organized as follows. In § 2 we recall some results on normal traces of vector fields established in [4]. In § 3 we establish the uniqueness part of the proof of Theorem 1.1 and the space continuity property. In § 4 we construct the counter-examples that prove Proposition 1.2, Theorem 1.3 and Corollary 1.4.

Notation.

- $\mathscr{L}^n$: the n-dimensional Lebesgue measure.
- $\mathscr{H}^m$: the m-dimensional Hausdorff measure.
- $\mu \llcorner E$: the restriction of the measure μ to the measurable set E.
- $\mathbf{1}_E$: the characteristic function of the set E.
- Ω: an open set in $\mathbb{R}^d$ having uniformly Lipschitz continuous boundary.
- $L^\infty(]0,T[\times\partial\Omega) := L^\infty(]0,T[\times\partial\Omega, \mathscr{L}^1 \otimes \mathscr{H}^{d-1})$, where we denote with $\otimes$ the (tensor) product of two measures.
- $\operatorname{div} b$: the distributional divergence of the vector field $b :]0,T[\times\Omega \to \mathbb{R}^d$, computed with respect to the $x \in \Omega$ variable only.
- $\operatorname{Div} B$: the standard "full" distributional divergence of the vector field B. In particular, when $B :]0,T[\times\Omega \to \mathbb{R}^{d+1}$, then $\operatorname{Div} B$ is the divergence computed with respect to the $(t,x) \in]0,T[\times\Omega$ variable .
- $\nabla\varphi$: the gradient of the smooth function $\varphi :]0,T[\times\Omega \to \mathbb{R}^d$, computed with respect to the $x \in \Omega$ variable only.
- $\operatorname{Tr}(b,\Sigma)$: the normal trace of the vector field b on the surface $\Sigma \subseteq \Omega$, as defined in [4] (see also § 2 in here).
- $\operatorname{Tr} b$: the normal trace of the vector field b on $]0,T[\times\partial\Omega$, defined as in § 3.2.
- $|x|$: the Euclidian norm of the vector $x \in \mathbb{R}^d$.
- $\operatorname{supp} \rho$: the support of the smooth function $\rho : \mathbb{R}^N \to \mathbb{R}$.
- $B_R(0)$: the ball of radius $R > 0$ and center at 0.
- $\mathcal{M}_\infty(\Lambda)$: the class of bounded, measure-divergence vector fields, namely the functions $B \in L^\infty(\Lambda;\mathbb{R}^N)$ such that, for every $R > 0$, the distributional divergence $\operatorname{Div} B$ is a bounded Radon measure on the bounded open set $B_R(0) \cap \Lambda \subseteq \mathbb{R}^N$.

2. Normal traces of bounded, measure-divergence vector fields

We collect in this section some definitions and properties concerning weak traces of measure-divergence vector fields. Our presentation follows [4, §3].

Given an open set $\Lambda \subseteq \mathbb{R}^N$, we denote by $\mathcal{M}_\infty(\Lambda)$ the family of bounded, measure-divergence vector fields, namely the functions $B \in L^\infty(\Lambda;\mathbb{R}^N)$ such that the distributional divergence $\operatorname{Div} B$ is a bounded Radon measure on $\Lambda \cap B_R(0)$, for every $R > 0$.

We first define the normal trace of B on the boundary $\partial\Lambda$.

Definition 2.1. Assume that $\Lambda \subseteq \mathbb{R}^N$ is a domain with uniformly Lipschitz continuous boundary. Let $B \in \mathcal{M}_\infty(\Lambda)$, then the normal trace of B on $\partial\Lambda$ can be defined as a distribution by the identity

$$\langle \mathrm{Tr}(B, \partial\Lambda), \varphi \rangle = \int_\Lambda \nabla\varphi \cdot B\, dx + \int_\Lambda \varphi\, d(\mathrm{Div}\, B) \qquad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N). \tag{2.1}$$

This definition is consistent with the Gauss-Green formula if the vector field B is sufficiently smooth. In this case the distribution is induced by the integration of $B \cdot \vec{n}$ on $\partial\Lambda$, where $\vec{n}$ is the outward pointing, unit normal vector to $\partial\Lambda$.

We now quote [4, Proposition 3.2].

Lemma 2.2. *The above defined distribution is induced by an L^∞ function on $\partial\Lambda$, which we can still call $\mathrm{Tr}(B, \partial\Lambda)$, with*

$$\|\mathrm{Tr}(B, \partial\Lambda)\|_{L^\infty(\partial\Lambda)} \le \|B\|_{L^\infty(\Lambda)}. \tag{2.2}$$

Moreover, if Σ is a Borel set contained in $\partial\Lambda_1 \cap \partial\Lambda_2$ and if $\vec{n}_1 = \vec{n}_2$ on Σ, then

$$\mathrm{Tr}(B, \partial\Lambda_1) = \mathrm{Tr}(B, \partial\Lambda_2) \quad \mathscr{H}^{N-1} - a.e. \text{ on } \Sigma. \tag{2.3}$$

Starting from the identity (2.3), it is possible to introduce the notion of normal trace on general bounded, oriented, Lipschitz continuous hypersurfaces $\Sigma \subseteq \mathbb{R}^N$. Indeed, once the orientation of $\vec{n}_\Sigma$ is fixed, we can find $\Lambda_1 \subseteq \mathbb{R}^N$ such that $\Sigma \subseteq \partial\Lambda_1$ and the normal vectors n_Σ and n_1 coincide. Then we can define

$$\mathrm{Tr}^-(B, \Sigma) := \mathrm{Tr}(B, \partial\Lambda_1). \tag{2.4}$$

Analogously, if $\Lambda_2 \subseteq \mathbb{R}^N$ is an open subset such that $\Sigma \subseteq \partial\Lambda_2$, and $\vec{n}_2 = -\vec{n}_\Sigma$, we can define

$$\mathrm{Tr}^+(B, \Sigma) := -\mathrm{Tr}(B, \partial\Lambda_2). \tag{2.5}$$

Note that we have the formula

$$(\mathrm{Div}\, B) \llcorner \Sigma = \Big(\mathrm{Tr}^+(B, \Sigma) - \mathrm{Tr}^-(B, \Sigma) \Big) \mathscr{H}^{N-1} \llcorner \Sigma. \tag{2.6}$$

In particular, Tr^+ and Tr^- coincide $\mathscr{H}^{N-1}$-a.e. on Σ if and only if Σ is a $(\mathrm{Div}\, B)$-negligible set. Note, moreover, that the measure $(\mathrm{Div}\, B) \llcorner \Sigma$ does not depend on the orientation of Σ. By recalling definition (2.5), one can verify that this is consistent with property (2.6).

We now go over some space continuity results established in [4, §3]. We first recall the definition of a family of graphs.

Definition 2.3. Let $I \subseteq \mathbb{R}$ be an open interval. A family of oriented surfaces $\{\Sigma_r\}_{r \in I} \subseteq \mathbb{R}^N$ is a family of graphs if there are

- a bounded open set $D \subseteq \mathbb{R}^{N-1}$;
- a Lipschitz function $f : D \to \mathbb{R}$;
- a system of coordinates $(x_1, \cdots, x_N)$

such that the following holds. For every $r \in I$,

$$\Sigma_r = \Big\{ (x_1, \ldots, x_N) : \ f(x_1, \ldots, x_{N-1}) - x_N = r \Big\}$$

and Σ_r is oriented by the normal $(-\nabla f, 1)/\sqrt{1 + |\nabla f|^2}$.

We now quote [4, Theorem 3.7].

Theorem 2.4. *Let $B \in \mathcal{M}_\infty(\mathbb{R}^N)$ and let $\{\Sigma_r\}_{r\in I}$ be a family of graphs as in Definition 2.3. Given $r_0 \in I$, we define the functions $\alpha_0, \alpha_r : D \to \mathbb{R}$ by setting*

$$\alpha_0(x_1, \dots, x_{N-1}) := \mathrm{Tr}^-(B, \Sigma_{r_0})\big(x_1, \dots, x_{N-1}, f(x_1, \dots, x_{N-1}) - r_0\big)$$

and

$$\alpha_r(x_1, \dots, x_{N-1}) := \mathrm{Tr}^+(B, \Sigma_r)\big(x_1, \dots, x_{N-1}, f(x_1, \dots, x_{N-1}) - r\big).$$

Then we have

$$\alpha_r \overset{*}{\rightharpoonup} \alpha_0 \quad \text{weakly}^* \text{ in } L^\infty(D, \mathscr{L}^{N-1} \llcorner D) \text{ as } r \to r_0^+.$$

3. Proof of Theorem 1.1

3.1. **Preliminary results.** In this section we establish some results that are preliminary to the distributional formulation of problem (1.2).

Lemma 3.1. *Let B be a locally bounded vector field on $\mathbb{R}^N$ and let $\{\rho_\varepsilon\}_{0<\varepsilon<1}$ be a standard family of mollifiers satisfying $\operatorname{supp}\rho_\varepsilon \subseteq B_\varepsilon(0)$ for every $\varepsilon \in]0,1[$.*

The divergence of B is a locally finite measure if and only if for any K compact in $\mathbb{R}^N$ there exists a positive constant C such that the inequality

$$\|\mathrm{Div}\, B * \rho_\varepsilon\|_{L^1(K)} \le C \tag{3.1}$$

holds uniformly in $\varepsilon \in]0,1[$.

Proof. If $\mathrm{Div}\, B$ is a locally finite measure the inequality (3.1) is satisfied on any compact K for some constant C independent from ε.

On the other hand, the sequence $(\mathrm{Div}\, B) * \rho_\varepsilon = \mathrm{Div}\,(B * \rho_\varepsilon)$ converges to $\mathrm{Div}\, B$ in the sense of distributions and the uniform bound (3.1) implies that we can extract a subsequence which converges weakly in the sense of measures. □

Lemma 3.2. *Let $\Lambda \subseteq \mathbb{R}^N$ be an open subset with uniformly Lipschitz continuous boundary and let B belong to $\mathcal{M}_\infty(\Lambda)$. Then the vector field*

$$\tilde{B}(z) := \begin{cases} B(z) & z \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\mathcal{M}_\infty(\mathbb{R}^N)$.

Proof. We only need to check that the distributional divergence of $\tilde{B}$ is a locally bounded Radon measure. Given $\varepsilon \in]0,1[$ we define the ε-neighborhood of $\partial\Lambda$ as

$$\partial\Lambda_\varepsilon = \{z \in \mathbb{R}^N \ : \ \mathrm{dist}(z, \partial\Lambda) < \varepsilon\}.$$

Any compact subset K of $\mathbb{R}^N$ can be decomposed as follows:

$$K = \big(K \cap (\Lambda \setminus \partial\Lambda_\varepsilon)\big) \cup \big(K \cap \partial\Lambda_\varepsilon\big) \cup \big(K \setminus (\Lambda \cup \partial\Lambda_\varepsilon)\big). \tag{3.2}$$

Also, note that $\mathrm{Div}\,(\tilde{B} * \rho_\varepsilon)$ is zero on $K \setminus (\Lambda \cup \partial\Lambda_\varepsilon)$ and that its L^1 norm is uniformly bounded on $K \cap (\Lambda \setminus \partial\Lambda_\varepsilon)$. Moreover,

$$\begin{aligned}\int_{K\cap\partial\Lambda_\varepsilon} |\mathrm{div}\,(\tilde{B} * \rho_\varepsilon)|\, dz &\le \int_{K\cap\partial\Lambda_\varepsilon} |\tilde{B}| * |\nabla\rho_\varepsilon|\, dz \\ &\le \|\tilde{B}\|_{L^\infty(\mathbb{R}^N)} \|\nabla\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \mathscr{L}^N(K \cap \partial\Lambda_\varepsilon).\end{aligned} \tag{3.3}$$

We observe that $\|\tilde{B}\|_{L^\infty(\mathbb{R}^N)} = \|B\|_{L^\infty(\Lambda)}$, that $\mathscr{L}^N(K \cap \partial\Lambda_\varepsilon) \le C_* \varepsilon$ and that $\|\nabla \rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \le C_{**}/\varepsilon$ for suitable constants $C_* > 0$ and $C_{**} > 0$. Hence,

$$\int_{K \cap \partial\Lambda_\varepsilon} |\mathrm{div}\,(\tilde{B} * \rho_\varepsilon)|\, dz \le \|B\|_{L^\infty(\Lambda)} C_* C_{**}$$

and by relying on Lemma 3.1 we conclude. □

3.2. **Distributional formulation of problem (1.2).** We can now discuss the distributional formulation of (1.2). The following result provides a distributional formulation of the normal trace of b and bu on $]0, T[\times\partial\Omega$.

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary and let $T > 0$. Assume that $b \in L^\infty(]0, T[\times\Omega; \mathbb{R}^d)$ is a vector field such that $\mathrm{div}\, b$ is a finite Radon measure on $]0, T[\times(\Omega \cap B_R(0))$ for every $R > 0$. Then there is a unique function, which in the following we denote by $\mathrm{Tr}\, b$, that belongs to $L^\infty(]0, T[\times\partial\Omega)$ and satisfies*

$$\int_0^T \int_{\partial\Omega} \mathrm{Tr}\, b\, \varphi\, d\mathscr{H}^{d-1} dt = \int_0^T \int_\Omega b \cdot \nabla\varphi\, dx dt + \int_0^T \int_\Omega \varphi\, d(\mathrm{div}\, b) \qquad \forall \varphi \in \mathcal{C}^\infty_c([0, T[\times\mathbb{R}^d). \tag{3.4}$$

Also, if $w \in L^\infty(]0, T[\times\Omega)$, $c \in L^\infty(]0, T[\times\Omega)$ and $f \in L^\infty(]0, T[\times\Omega)$ satisfy

$$\int_0^T \int_\Omega w\big(\partial_t \eta + b \cdot \nabla\eta\big) dx dt + \int_0^T \int_\Omega f \eta\, dx dt + \int_0^T \int_\Omega c w \eta\, dx dt = 0 \qquad \forall \eta \in \mathcal{C}^\infty_c(]0, T[\times\Omega), \tag{3.5}$$

then there are two uniquely determined functions, which in the following we denote by $\mathrm{Tr}\,(bw) \in L^\infty(]0, T[\times\partial\Omega)$ and $w_0 \in L^\infty(\Omega)$, that satisfy

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \mathrm{Tr}\,(bw) \varphi\, d\mathscr{H}^{d-1} dt - \int_\Omega \varphi(0, \cdot) w_0\, dx \\ &\qquad = \int_0^T \int_\Omega w\big(\partial_t \varphi + b \cdot \nabla\varphi\big) dx dt + \int_0^T \int_\Omega f \varphi\, dx dt + \int_0^T \int_\Omega c w \varphi\, dx dt \\ &\qquad \forall\, \varphi \in \mathcal{C}^\infty_c([0, T[\times\mathbb{R}^d). \end{aligned} \tag{3.6}$$

Three remarks are here in order. First, note that requirement (3.5) is nothing but the distributional formulation of the equation

$$\partial_t w + \mathrm{div}\,(bw) = cu + f \quad \text{in }]0, T[\times\Omega. \tag{3.7}$$

Second, a possible heuristic interpretation of Lemma 3.3 is the following. If equation (3.7) is satisfied *inside* the domain $]0, T[\times\Omega$, then the initial datum of w and the normal trace of bw at the domain boundary are uniquely defined, at least in a distributional sense. Finally, note that the existence of the function w_0 follows from Lemma 1.3.3 in [13], the novelty of Lemma 3.3 is establishing the existence of the function $\mathrm{Tr}\,(bw)$.

Proof. We first establish the existence of a function $\mathrm{Tr}\, b$ satisfying (3.4). Note that the uniqueness of such a function follows from the arbitrariness of the test function φ. We define the vector field $B : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by setting

$$B(t,x) := \begin{cases} (1,b) & (t,x) \in]0,T[\times\Omega \\ 0 & \text{elsewhere in } \mathbb{R}^{d+1} \end{cases} \tag{3.8}$$

and we note that $\mathrm{Div}\, B\big|_{]0,T[\times\Omega} = \mathrm{div}\, b$, therefore B satisfies the hypotheses of Lemma 3.2 provided that $\Lambda :=]0,T[\times\Omega$. Hence, $B \in \mathcal{M}_\infty(\mathbb{R}^{d+1})$. We apply Lemma 2.2 and we observe that $\mathrm{Tr}\,(B,\partial\Lambda)\big|_{\{0\}\times\Omega} \equiv -1$. We can then conclude by setting

$$\mathrm{Tr}\, b := \mathrm{Tr}\,(B,\partial\Lambda)\big|_{]0,T[\times\partial\Omega}$$

and by observing that, since $\varphi \in \mathcal{C}^\infty_c([0,T[\times\mathbb{R}^d)$, then

$$-\int_\Omega \varphi(0,x)dx = \int_0^T \int_\Omega \partial_t\varphi\, dxdt.$$

The existence of the function $\mathrm{Tr}\,(bw)$ satisfying (3.6) can be established by setting

$$C(t,x) := \begin{cases} (w,bw) & (t,x) \in]0,T[\times\Omega \\ 0 & \text{elsewhere in } \mathbb{R}^{d+1} \end{cases} \tag{3.9}$$

and observing that condition (3.5) implies that $\mathrm{Div}\, C\big|_{]0,T[\times\Omega} = cu + f$. We can then conclude by using the same argument as before, by setting

$$w_0 := -\mathrm{Tr}\,(C,\partial\Lambda)\big|_{\{0\}\times\Omega} \quad \text{and} \quad \mathrm{Tr}\,(bw) := \mathrm{Tr}\,(C,\partial\Lambda)\big|_{]0,T[\times\partial\Omega}. \tag{3.10}$$

□

We now provide the precise formulation of problem (1.2). In the following definition the functions u_0, $\mathrm{Tr}\,(bu)$ and $\mathrm{Tr}\, b$ are defined as in Lemma 3.3.

Definition 3.4. Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Assume that $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$ is a vector field such that $\mathrm{div}\, b$ is a finite Radon measure on $]0,T[\times(\Omega\cap B_R(0))$ for every $R > 0$. Assume furthermore that $c,\ f \in L^\infty(]0,T[\times\Omega)$. A distributional solution of (1.2) is a function $u \in L^\infty(]0,T[\times\Omega)$ such that

i) u satisfies equation (3.5);
ii) $u_0 = \bar u$;
iii) $\mathrm{Tr}\,(bu) = \bar g \mathrm{Tr}\, b$ on the set Γ^- which is defined as follows:

$$\Gamma^- := \big\{(t,x) \in]0,T[\times\partial\Omega :\ (\mathrm{Tr}\ b)(t,x) < 0\big\}.$$

3.3. **Proof of Theorem 1.1.** First, we observe that the existence of a solution of (1.2) is established in [11] by closely following an argument due to Boyer [8]. More precisely, in [11] we introduce a second order approximation of (1.2) and we establish an existence and uniqueness result for the approximate problem by relying on classical parabolic techniques. Next, we establish suitable uniform a-priori estimates and pass to the limit.

Second, we point out that, by linearity, establishing uniqueness amounts to show that, if $f \equiv 0$, $\bar{g} \equiv 0$ and $\bar{u} \equiv 0$, then the solution of problem (1.2) satisfies $u \equiv 0$. The argument is organized in two main steps: in § 3.3.1 we show that, under the hypotheses of Theorem 1.1, distributional solutions of (1.2) enjoy renormalization properties. Next, in § 3.3.2 we conclude by relying on a by now standard argument based on the Gronwall Lemma.

3.3.1. *Renormalization properties.* We assume $\bar{u} \equiv 0$ and $f \equiv 0$ and we proceed according to the following steps.

STEP 1: we use the same argument as in Ambrosio [2] to establish renormalization properties "inside" the domain. More precisely, the Renormalization Theorem [2, Theorem 3.5] implies that the function u^2 satisfies

$$\int_0^T \int_\Omega u^2 \big(\partial_t \psi + b \cdot \nabla \psi\big) dxdt + \int_0^T \int_\Omega u^2 (2c - \operatorname{div} b)\, \psi \, dxdt = 0 \qquad \forall \psi \in \mathcal{C}^\infty_c([0, T[\times \Omega). \tag{3.11}$$

STEP 2: we establish a trace renormalization property.

First, we observe that by combining hypothesis 3 in the statement of Theorem 1.1 with Theorem 3.84 in the book by Ambrosio, Fusco and Pallara [5] we obtain that the vector field B defined as in (3.8) satisfies $B(t, \cdot) \in BV(\Omega_*)$ for every open and bounded set $\Omega_* \subseteq \mathbb{R}^d$ and for $\mathscr{L}^1$-a.e. $t \in]0, T[$.

Next, we recall that the proof of Lemma 3.3 ensures that the vector field uB belongs to $\mathcal{M}_\infty(\mathbb{R}^{d+1})$. We can then apply [4, Theorem 4.2], which implies the following trace renormalization property:

$$\mathrm{Tr}\,(u^2 b)(t, x) = \begin{cases} \left(\dfrac{\mathrm{Tr}\,(ub)}{\mathrm{Tr}\; b} \right)^2 \mathrm{Tr}\; b & \mathrm{Tr}\; b(t, x) \neq 0 \\[2ex] 0 & \mathrm{Tr}\; b(t, x) = 0. \end{cases} \tag{3.12}$$

Some remarks are here in order. First, to define $\mathrm{Tr}\,(u^2 b)$ we recall (3.11), use Lemma 3.3 and set

$$\mathrm{Tr}\,(u^2 b) := \mathrm{Tr}\,(u^2 B, \partial \Lambda)\big|_{]0,T[\times \partial\Omega}, \tag{3.13}$$

where $\Lambda =]0, T[\times \Omega$.

Second, note that, strictly speaking, the statement of [4, Theorem 4.2] requires that the vector field B has BV regularity with respect to the (t, x)-variables, which in our case would imply some control on the time derivative of b. However, by examining the proof of [4, Theorem 4.2] and using the particular structure of the vector field B one can see that only space regularity is needed to establish (3.12).

STEP 3: by combining (3.11) with (3.13) and recalling Lemma 3.3 we infer that

$$\begin{aligned}\int_0^T \int_{\partial\Omega} \mathrm{Tr}\,(u^2 b)\varphi \, d\mathscr{H}^{d-1} dt &= \int_0^T \int_\Omega u^2 \big(\partial_t \varphi + b \cdot \nabla \varphi\big) dx dt \\ &+ \int_0^T \int_\Omega u^2 (2c - \mathrm{div}\, b)\, \varphi \, dx dt \qquad \forall\, \varphi \in \mathcal{C}^\infty_c([0,T[\times \mathbb{R}^d).\end{aligned} \tag{3.14}$$

3.3.2. *Conclusion of the proof of Theorem 1.1.* We conclude by following a by now standard argument, see for example the expository work by De Lellis [15, Proposition 1.6]. Also, note that in the remaining part of the proof we use [13, Lemma 1.3.3] and we identify u^2 with its representative satisfying that the map $t \mapsto u^2(t, \cdot)$ is continuous in $L^\infty_{\mathrm{loc}}(\Omega)$ endowed with the weak* topology. We proceed according to the following steps.

STEP A: we fix $\bar t \in]0, T[$ and we construct a sequence of test functions φ_n as follows. First, we choose a function $h : [0, +\infty[\to \mathbb{R}$ such that

$$h \in \mathcal{C}^\infty_c([0, +\infty[), \quad h \geq 0 \text{ and } h' \leq 0 \text{ everywhere in } [0, +\infty[. \tag{3.15}$$

Next, we set

$$\nu(t, x) := h\big(|x| - \|b\|_{L^\infty} |t - \bar t|\big) \tag{3.16}$$

and we observe that ν satisfies

$$\partial_t \nu = \|b\|_{L^\infty} h', \quad \nabla \nu = \frac{x}{|x|} h' \quad \text{for } \mathscr{L}^{d+1}\text{-a.e. } (t,x) \in]0, \bar t\,[\times \mathbb{R}^d.$$

We recall that $h' \leq 0$ and we conclude that

$$\partial_t \nu + b \cdot \nabla \nu \leq (\|b\|_{L^\infty} - \|b\|_{L^\infty}) h' = 0 \quad \text{for } \mathscr{L}^{d+1}\text{-a.e. } (t,x) \in]0, \bar t\,[\times \mathbb{R}^d. \tag{3.17}$$

We then choose a sequence of cut-off functions $\chi_n \in \mathcal{C}^\infty_c([0, +\infty[)$ satisfying

$$\chi_n \equiv 1 \text{ on } [0, \bar t\,], \quad \chi_n \equiv 0 \text{ on } [\,\bar t + 1/n, +\infty[, \quad \chi_n' \leq 0 \text{ everywhere on } [0, +\infty[. \tag{3.18}$$

Finally, we set

$$\varphi_n(t, x) := \chi_n(t) \nu(t, x) \quad (t, x) \in [0, T[\times \mathbb{R}^d$$

and we observe that $\varphi_n \geq 0$ everywhere on $[0, +\infty[\times \mathbb{R}^d$ and that φ_n is compactly supported in $[0, T[\times \mathbb{R}^d$ provided that n is sufficiently large.

STEP B: we use φ_n as a test function in (3.14). First, we observe that by recalling that $\bar g \equiv 0$ and by using the renormalization property (3.12) we obtain that the left hand side of (3.14) is nonnegative, namely

$$\begin{aligned}0 \leq \int_0^T \int_\Omega u^2 \nu \chi_n' dx dt &+ \int_0^T \int_\Omega \chi_n u^2 \big(\partial_t \nu + b \cdot \nabla \nu\big) dx dt \\ &+ \int_0^T \int_\Omega (2c - \mathrm{div}\, b)\, u^2\, \nu \chi_n \, dx dt\,.\end{aligned}$$

Next, we let $n \to +\infty$ and by recalling properties (3.17) and (3.18) we obtain

$$\int_\Omega \nu(\bar t, \cdot)\, u^2(\bar t, \cdot) dx \leq (2\|c\|_{L^\infty} + \|\mathrm{div}\, b\|_{L^\infty}) \int_0^{\bar t} \int_\Omega \nu\, u^2 dx dt.$$

We can finally conclude by using the Gronwall Lemma and the arbitrariness of the function h in (3.16). This concludes the proof of Theorem 1.1. □

3.4. **Space continuity property.** We now state the analogue of the space continuity property established in the Sobolev case by Boyer in [8, § 7.1].

Proposition 3.5. *Let b, c and f be as in the statement of Theorem 1.1, $u \in L^\infty(]0,T[\times\Omega)$ be the distributional solution of (1.2) and $B \in \mathcal{M}_\infty(\mathbb{R}^{d+1})$ be the same vector field as in (3.8). Given a family of graphs $\{\Sigma_r\}_{r\in I} \subseteq \mathbb{R}^d$ as in Definition 2.3, we fix $r_0 \in I$ and we define the functions $\gamma_0, \gamma_r :]0,T[\times D \to \mathbb{R}$ by setting*

$$\gamma_0(t, x_1, \dots, x_{d-1}) := \mathrm{Tr}^-(uB,]0,T[\times\Sigma_{r_0})\big(t, x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}) - r_0\big)$$

and

$$\gamma_r(t, x_1, \dots, x_{d-1}) := \mathrm{Tr}^+(uB,]0,T[\times\Sigma_r)\big(t, x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}) - r\big).$$

Then

$$\gamma_r \to \gamma_0 \quad \text{strongly in } L^1(]0,T[\times D) \text{ as } r \to r_0^+. \tag{3.19}$$

Proof. The argument is organized in three steps.

Step 1: we make some preliminary considerations and introduce some notation. With a slight abuse of notation, we consider b as a vector field defined on $\mathbb{R}^{d+1}$, set equal to zero out of $]0,T[\times\Omega$.

By combining hypothesis 3 in the statement of Theorem 1.1 with [5, Theorem 3.84] we obtain that $b(t,\cdot) \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. Hence, the classical theory of BV functions (see for instance [5, Section 3.7]) ensures that the outer and inner traces $b(t,\cdot)^+_{\Sigma_r}$ and $b(t,\cdot)^-_{\Sigma_r}$ are well-defined, vector valued functions for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and for every r.

Step 2: given B as in (3.8), we define the functions $\beta_0, \beta_r :]0,T[\times D \to \mathbb{R}$ by setting

$$\beta_0(t, x_1, \dots, x_{d-1}) := \mathrm{Tr}^-(B,]0,T[\times\Sigma_{r_0})\big(t, x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}) - r_0\big)$$

and

$$\beta_r(t, x_1, \dots, x_{d-1}) := \mathrm{Tr}^+(B,]0,T[\times\Sigma_r)\big(t, x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}) - r\big).$$

We claim that

$$\beta_r \to \beta_0 \quad \text{strongly in } L^1(]0,T[\times D) \text{ as } r \to r_0^+. \tag{3.20}$$

To establish (3.20), we first observe that by using [5, Theorem 3.88] and an approximation argument one can show that for every $r \in I$ we have

$$\beta_r = b^+_{\Sigma_r} \cdot \vec{m}, \quad \text{and} \quad \beta_0 = b^-_{\Sigma_0} \cdot \vec{m} \quad \text{for } \mathscr{L}^d\text{-a.e. } (t,x) \in]0,T[\times D.$$

In the previous expression, $\vec{m} = (-\nabla f, 1)/\sqrt{1 + |\nabla f|^2}$ is the unit normal vector defining the orientation of Σ_r. Also, by again combining [5, Theorem 3.88] with an approximation argument we get that

$$\int_0^T \int_D |\beta_r - \beta_0| dx_1 \dots d_{x_{d-1}} dt \le \int_0^T |Db(t,\cdot)|(S) dt,$$

which implies (3.20). In the previous expression, $|Db(t,\cdot)|$ denotes the total variation of the distributional derivative of $b(t,\cdot)$, and S is the set

$$S := \Big\{ (x_1, \dots, x_{d-1}, x_d) \ : (x_1, \dots, x_{d-1}) \in D \ \text{ and } \\ f(x_1, \dots, x_{d-1}) - r < x_d < f(x_1, \dots, x_{d-1}) - r_0 \Big\}.$$

STEP 3: we conclude the proof of Proposition 3.5. First, we observe that due to Theorem 2.4 we have that

$$\gamma_r \rightharpoonup \gamma_0 \text{ weakly in } L^2(]0,T[\times D) \text{ as } r \to r_0^+. \tag{3.21}$$

Next, we recall that γ_r is the normal trace of uB and that β_r is the trace of B, so that by applying [4, Theorem 4.2] we get

$$\gamma_r^2 = \beta_r \mathrm{Tr}^+(u^2 B,]0,T[\times \Sigma_r) \quad \text{and} \quad \gamma_0^2 = \beta_0 \mathrm{Tr}^-(u^2 B,]0,T[\times \Sigma_{r_0}). \tag{3.22}$$

By combining (3.20) with the uniform bound $\|\beta_r\|_{L^\infty} \leq \|b\|_{L^\infty}$ we infer that $\beta_r \to \beta_0$ strongly in $L^2(]0,T[\times D)$. Then we apply Theorem 2.4 to $\mathrm{Tr}^+(u^2 B,]0,T[\times \Sigma_r)$ and hence by recalling (3.22) we conclude that

$$\gamma_r^2 \rightharpoonup \gamma_0^2 \text{ weakly in } L^2(]0,T[\times D) \text{ as } r \to r_0^+. \tag{3.23}$$

By using (3.21), we get that (3.23) implies that $\gamma_r \to \gamma_0$ strongly in $L^2(]0,T[\times D)$ and from this we eventually get (3.19). □

4. COUNTER-EXAMPLES

4.1. Some notation and a preliminary result. For the reader's convenience, we collect here some notation we use in this section.

- Throughout all § 4, Ω denotes the set $]0,+\infty[\times\mathbb{R}^2$.
- We use the notation $(r,y) \in]0,+\infty[\times\mathbb{R}^2$ or, if needed, the notation $(r,y_1,y_2) \in]0,+\infty[\times \mathbb{R} \times \mathbb{R}$ to denote points in Ω.
- div denotes the divergence computed with respect to the (r,y)-variable.
- Div denotes the divergence computed with respect to the (t,r,y)-variable.
- div_y denotes the divergence computed with respect to the y variable only.
- We decompose $]0,1[\times\Omega$ as $]0,1[\times\Omega = \Lambda^+ \cup \Lambda^- \cup \mathcal{S}$, where

$$\Lambda^+ := \{(t,r,y) \in]0,1[\times\Omega : \ r > t\} \tag{4.1}$$

and

$$\Lambda^- := \{(t,r,y) \in]0,1[\times\Omega : \ r < t\}, \tag{4.2}$$

while $\mathcal{S}$ is the surface

$$\mathcal{S} := \{(t,r,y) \in]0,1[\times\Omega : \ r = t\}. \tag{4.3}$$

We also observe that, owing to [13, Lemma 1.3.3], up to a redefinition of $u(t,x)$ in a negligible set of times, we can assume that the map $t \mapsto u(t,\cdot)$ is continuous from $]0,1[$ in $L^\infty(\Omega)$ endowed with the weak-$*$ topology, and in particular

$$u(t,\cdot) \overset{*}{\rightharpoonup} u_0 \quad \text{in } L^\infty(\Omega) \text{ as } t \to 0^+,$$

where u_0 the value attained by u at $t = 0$, as in Lemma 3.3.

4.2. **Proof of Proposition 1.2.** The proof is organized in three steps.

STEP 1: we recall an intriguing example due to Depauw [16] which is pivotal to our construction. Note that that, in this step, r should be regarded as a *time* variable, while $y \in \mathbb{R}^2$ is the only space variable.

In [16], Depauw explicitly exhibits a time dependent vector field $d :]0,1[\times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following properties:

a) $d \in L^\infty(]0,1[\times \mathbb{R}^2; \mathbb{R}^2)$.
b) For every $r > 0$, $d(r, \cdot)$ is piecewise smooth and, for almost every $y \in \mathbb{R}^2$, the characteristic curve trough y is well defined.
c) $\mathrm{div}_y d \equiv 0$.
d) $d \in L^1_{\mathrm{loc}}\big(]0,1[; BV_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)\big)$, but $d \notin L^1\big([0,1[; BV_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)\big)$. Namely, the BV regularity deteriorates as $r \to 0^+$.
e) The Cauchy problem

$$\begin{cases} \partial_r w + \mathrm{div}_y(dw) = 0 & \text{on }]0,1[\times \mathbb{R}^2 \\ w = 0 & \text{at } r = 0 \end{cases} \tag{4.4}$$

admits a nontrivial bounded solution, which in the following we denote by $v(r, y)$.

STEP 2: we exhibit a vector field b satisfying properties i), ..., v) in the statement of Theorem 1.3. We recall that the sets Λ^+, Λ^- and $\mathcal{S}$ are defined by (4.1), (4.2) and (4.3), respectively. We define the vector field $b :]0,1[\times \Omega \to \mathbb{R}^3$ by setting

$$b(t, r, y) := \begin{cases} \big(1, d(r, y)\big) & \text{if } r \le 1 \\ \big(1, 0\big) & \text{if } r > 1 \end{cases} \tag{4.5}$$

In the previous expression, d is Depauw's vector field as in STEP 1, extended to the constant vector field $(1, 0)$ for $r > 1$. By relying on properties a), c) and d) in STEP 1 one can show that b satisfies properties i), ii), iii) in the statement of Proposition 1.2.

Next, we recall that the initial-boundary value problem (1.3) admits the trivial solution $u \equiv 0$ and that the linear combination of solutions is again a solution. Hence, establishing property v) in the statement of Proposition 1.2 amounts to exhibit a nontrivial solution of (1.3). We define the function u by setting

$$u(t, r, y) := \begin{cases} v(r, y) & \text{in } \Lambda^- \\ 0 & \text{in } \Lambda^+, \end{cases} \tag{4.6}$$

where v is the same function as in STEP 1.

STEP 3: we show that the function u is a distributional solution of (1.3). We set $C := (u, bu)$ and we observe that by construction $\mathrm{Div}\, C \equiv 0$ on Λ^+. Also, property e) in STEP 1 implies that $\mathrm{Div}\, C \equiv 0$ on Λ^-. Finally, by recalling (2.6) we infer that $\mathrm{Div}\, C \llcorner \mathcal{S} = 0$ since the normal trace is 0 on both sides.

We are left to show that the initial and boundary data are attained. First, we observe that $u(t, \cdot) \stackrel{*}{\rightharpoonup} 0$ as $t \to 0^+$ and hence $u_0 \equiv 0$ by the weak continuity of u with respect to time. Next, we fix an open and bounded set $D \subseteq \mathbb{R}^2$ and we define the family of graphs $\{\Sigma_r\}_{r\in]0,1[} \subseteq]0,1[\times\Omega$ by setting

$$\Sigma_r := \Big\{(t, r, y_1, y_2) : \ t \in]0, 1[\text{ and } (y_1, y_2) \in D\Big\}. \tag{4.7}$$

The orientation is given by the vector $(0, -1, 0, 0)$. We point out that requirement e) in Step 1 implies that $v(r, \cdot) \overset{*}{\rightharpoonup} 0$ as $r \to 0^+$. Hence, by recalling that b is given by (4.5), we obtain that $\mathrm{Tr}^+(C, \Sigma_r) \overset{*}{\rightharpoonup} 0$ as $r \to 0^+$. By recalling Theorem 2.4, we infer that $\mathrm{Tr}^-(C, \Sigma_0) \equiv 0$. On the other hand, by looking at the proof of Lemma 3.3 we realize that $\mathrm{Tr}^-(C, \Sigma_0) = \mathrm{Tr}\,(bu)$ on $]0, 1[\times\partial\Omega$. This concludes the proof of Proposition 1.2. □

4.3. **Proof of Theorem 1.3.** The proof is divided in three main steps:

(1) in § 4.3.1 we construct the auxiliary vector field β_k, which will serve as a "building block" for the construction of the vector field b;
(2) in § 4.3.2 we define the vector field b;
(3) finally, in § 4.3.3 we exhibit a non trivial solution of (1.4). Since the problem is linear, any linear combination of solutions is also a solution and hence the existence of a nontrivial solution implies the existence of infinitely many different solutions.

4.3.1. *Construction of the vector field β_k.* We fix $k \in \mathbb{N}$ and we construct the vector field β_k, which is defined on the cell

$$(r, y_1, y_2) \in]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[.$$

We split the r-interval $]0, 4 \cdot 2^{-k}[$ into four equal sub-intervals and we proceed according to the following steps.
Step 1: if $r \in]0, 2^{-k}[$, we consider a "three-colors chessboard" in the (y_1, y_2)-variables at scale 2^{-k} as in Figure 1, left part. The vector field β_k attains the values $(1, 0, 0)$, $(-5, 0, 0)$ and $(0, 0, 0)$ on dashed, black and white squares, respectively. Note that β_k satisfies

$$\mathrm{div}\,\beta_k \equiv 0 \quad \text{on }]0, 2^{-k}[\times]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[\tag{4.8}$$

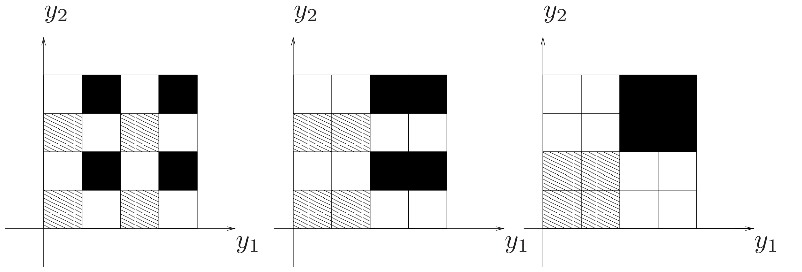
since β_k is piecewise constant and tangent at its discontinuity surfaces.
Here is the rigorous definition of β_k: we set

$$D_k := \bigcup_{n,m=0,1}](2n)2^{-k}, (2n + 1)2^{-k}[\times](2m)2^{-k}, (2m + 1)2^{-k}[$$

and

$$B_k := \bigcup_{n,m=0,1}](2n + 1)2^{-k}, (2n + 2)2^{-k}[\times](2m + 1)2^{-k}, (2m + 2)2^{-k}[. \tag{4.9}$$

FIGURE 1. The vector field $\beta_k(r, y_1, y_2)$ for different values of r: the dashed, black and white squares are the region where β_k attains the values $(1,0,0)$, $(-5,0,0)$ and $(0,0,0)$, respectively.



Note that D_k and B_k are represented in the left part of Figure 1 by dashed and black regions, respectively. Next, we define

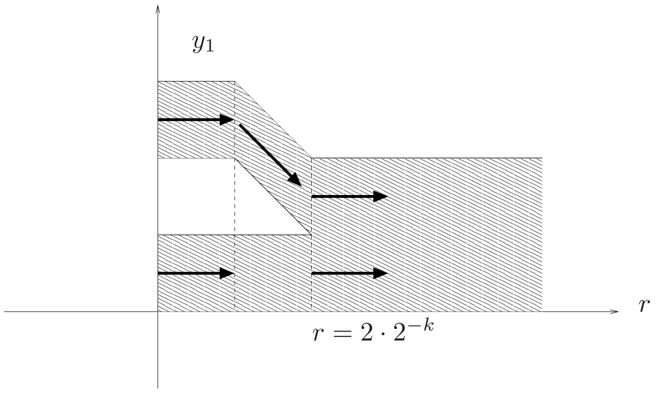
$$\beta_k(r, y_1, y_2) := \begin{cases} (1,0,0) & \text{if } (y_1, y_2) \in D_k \\ (-5,0,0) & \text{if } (y_1, y_2) \in B_k \\ (0,0,0) & \text{elsewhere on } \left]0, 4 \cdot 2^{-k}\right[\times \left]0, 4 \cdot 2^{-k}\right[\end{cases} \tag{4.10}$$

STEP 2: if $r \in]2^{-k}, 2 \cdot 2^{-k}[$, then the heuristic idea to define β_k is that we want to (i) horizontally leftward slide the rightmost dashed squares and (ii) horizontally rightward slide the leftmost black squares. The final goal is that at $r = 2 \cdot 2^{-k}$ we have reached the configuration of the vector field described in Figure 1, center part. The nontrivial issue is that we also require that

$$\operatorname{div} \beta_k \equiv 0 \quad \text{on }]0, 2 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[. \tag{4.11}$$

To achieve (4.11), we employ the construction illustrated in Figure 2: the vector field β_k attains the value $(1,0,0)$ on the horizontal part of the dashed region, the value $(1,-1,0)$ on the inclined part of the dashed region and the value $(0,0,0)$ elsewhere. Note that (4.11) is satisfied because β_k is piecewise constant and it is tangent at its discontinuity surfaces on the interval $r \in]2^{-k}, 2 \cdot 2^{-k}[$. We conclude by recalling (4.8) and by observing that the normal trace is continuous at the discontinuity surface $r = 2^{-k}$ and hence no divergence is created there.

FIGURE 2. The vector field $\beta_k(r, y_1, y_2)$ for $y_2 \in]0, 2^{-k}[$ and $y_2 \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[$. The field β_k attains the value $(1, -1, 0)$ in the inclined dashed region.



Here is the rigorous definition of β_k for $r \in]2^{-k}, 2 \cdot 2^{-k}[$:

$$\beta_k(r, y_1, y_2) := \begin{cases} (1,0,0) & \text{if } (y_1, y_2) \in]0, 2^{-k}[\times]0, 2^{-k}[\\ & \text{or } (y_1, y_2) \in]0, 2^{-k}[\times]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\\ (1,-1,0) & \text{if } -r + 3 \cdot 2^{-k} < y_1 < -r + 4 \cdot 2^{-k} \\ & \text{and } y_2 \in]0, 2^{-k}[\text{ or } y_2 \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\\ (-5,0,0) & \text{if } (y_1, y_2) \in]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[\times]2^{-k}, 2 \cdot 2^{-k}[\\ & \text{or } (y_1, y_2) \in]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[\times]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[\\ (-5,-5,0) & \text{if } r < y_1 < r + 2^{-k} \\ & \text{and } y_2 \in]2^{-k}, 2 \cdot 2^{-k}[\text{ or } y_2 \in]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[\\ (0,0,0) & \text{elsewhere on } \big]0, 4 \cdot 2^{-k}\big[\times \big]0, 4 \cdot 2^{-k}\big[\end{cases} \tag{4.12}$$

STEP 3: if $r \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[$, the heuristic idea is defining β_k in such a way that (i) we push up the lower black region in Figure 1, central part, (ii) we pull down the upper dashed region in Figure 1, central part and (iii) we satisfy the requirement that β_k is divergence-free. This is done by basically using the same construction as in STEP 2. Note that at $r = 3 \cdot 2^{-k}$ we have reached the configuration described in Figure 1, right part.

Here is the rigorous definition of β_k for $r \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[$:

$$\beta_k(r, y_1, y_2) := \begin{cases} (1,0,0) & \text{if } (y_1, y_2) \in]0, 2 \cdot 2^{-k}[\times]0, 2^{-k}[\\ (1,0,-1) & \text{if } y_1 \in]0, 2 \cdot 2^{-k}[\\ & \text{and } -r + 4 \cdot 2^{-k} < y_2 < -r + 5 \cdot 2^{-k} \\ (-5,0,0) & \text{if } (y_1, y_2) \in]2 \cdot 2^{-k}, 4 \cdot 2^{-k}[\times]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[\\ (-5,0,-5) & \text{if } y_1 \in]2 \cdot 2^{-k}, 4 \cdot 2^{-k}[\\ & \text{and } r - 2^{-k} < y_2 < r \\ (0,0,0) & \text{elsewhere on } \big]0, 4 \cdot 2^{-k}\big[\times \big]0, 4 \cdot 2^{-k}\big[\end{cases}$$

STEP 4: if $r \in]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[$, then we consider the "three colors chessboard" in the (y_1, y_2)-variables at scale $2 \cdot 2^{-k}$ illustrated in Figure 1, right part. The vector field β_k attains the value $(1,0,0)$, $(-5,0,0)$ and $(0,0,0)$ on dashed, black and white regions, respectively.

Here is the rigorous definition of β_k for $r \in]3 \cdot 2^{-k}, 4 \cdot 2^{-k}[$:

$$\beta_k(r, y_1, y_2) := \begin{cases} (1,0,0) & \text{if } (y_1, y_2) \in]0, 2 \cdot 2^{-k}[\times]0, 2 \cdot 2^{-k}[\\ (-5,0,0) & \text{if } (y_1, y_2) \in]2 \cdot 2^{-k}, 4 \cdot 2^{-k}[\times]2 \cdot 2^{-k}, 4 \cdot 2^{-k}[\\ (0,0,0) & \text{elsewhere on } \big]0, 4 \cdot 2^{-k}\big[\times \big]0, 4 \cdot 2^{-k}\big[\end{cases}$$

Note that by construction

$$\operatorname{div} \beta_k \equiv 0 \quad \text{on }]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[\times]0, 4 \cdot 2^{-k}[. \tag{4.13}$$

4.3.2. *Construction of the vector field b.* We now define the vector field b by using as a "building block" the vector field β_k defined in § 4.3.1. We proceed in two steps.

STEP A: we extend β_k to $]0, 2^{2-k}[\times\mathbb{R}^2$ by imposing that it is 2^{2-k}-periodic in both y_1 and y_2, namely we set

$$\beta_k(r, y_1 + m2^{2-k}, y_2 + n2^{2-k}) := \beta_k(r, y_1, y_2) \tag{4.14}$$

for every $m, n \in \mathbb{Z}$ and $(y_1, y_2) \in \big]0, 2^{2-k}\big[\times \big]0, 2^{2-k}\big[$. We recall (4.13) and we observe that β_k is tangent at the surfaces $y_1 = m2^{2-k}$ and $y_2 = n2^{2-k}$, $m, n \in \mathbb{Z}$. We therefore get

$$\operatorname{div} \beta_k \equiv 0 \quad \text{on }]0, 2^{2-k}[\times\mathbb{R}^2. \tag{4.15}$$

STEP B: we define the vector field b. To this end, we introduce the decomposition

$$]0, 1[:= \mathcal{N} \cup \bigcup_{k=3}^{\infty} I_k, \tag{4.16}$$

where $\mathcal{N}$ is an $\mathscr{L}^1$-negligible set and

$$I_k :=]2^{2-k}, 2^{3-k}[, \quad k \geq 3.$$

We then set

$$b(t,r,y_1,y_2) := \begin{cases} \beta_k(r - 2^{2-k}, y_1, y_2) & \text{if } r \in I_k \\ \\ (1,0,0)\mathbf{1}_D + (-5,0,0)\mathbf{1}_B & \text{if } r \geq 1. \end{cases} \tag{4.17}$$

In the previous expression, $\mathbf{1}_D$ denotes the characteristic function of the set

$$D := \Big\{(t,r,y_1,y_2) : \beta_3(1/2, y_1, y_2) \cdot (1,0,0) = 1\Big\}$$

and $\mathbf{1}_B$ is the characteristic function of the set

$$B := \Big\{(t,r,y_1,y_2) : \beta_3(1/2, y_1, y_2) \cdot (1,0,0) = -5\Big\}.$$

Some remarks are here in order. First, note that actually the vector field b is constant with respect to t. Second, the most interesting behavior occurs for $r \leq 1$. Indeed, the vector field b behaves like β_3 on the interval $r \in]1/2, 1[$, like β_4 on the interval $r \in]1/4, 1/2[$, like β_5 on the interval $r \in]1/8, 1/4[$, and so on. Loosely speaking, as $r \to 0^+$ the r-component of vector field b oscillates between the values 1, -5 and 0 on a finer and finer "three-colors chessboard". The vector field b is constant in r for $r > 1$ and it is defined in such a way that no divergence is created at the surface $r = 1$.

Finally, we recall (4.15) and we observe that the vector field is continuous at the surfaces $r = 2^{3-k}$, $k \geq 3$. Also, for $r > 1$ the vector field b is tangent at the discontinuity surfaces. Hence,

$$\operatorname{div} b \equiv 0 \quad \text{on }]0,1[\times\Omega. \tag{4.18}$$

4.3.3. *Construction of a nontrivial solution of* (1.4). To exhibit a nontrivial solution of (1.4) we proceed as follows: first, we give the rigorous definition, next we make some heuristic remark and finally we show that u is actually a distributional solution of (1.4).

We recall the decomposition $]0,1[\times\Omega = \Lambda^+ \cup \Lambda^- \cup \mathcal{S}$, where Λ^+, Λ^- and $\mathcal{S}$ are defined by (4.1), (4.2) and (4.3), respectively. We define the function $u :]0,1[\times\Omega \to \mathbb{R}$ by setting

$$u(t,r,y_1,y_2) := \begin{cases} 1 & (t,r,y_1,y_2) \in \Lambda^- \text{ and } b(t,r,y_1,y_2)\cdot(1,0,0) = 1 \\ 0 & \text{elsewhere in }]0,1[\times\Omega. \end{cases} \tag{4.19}$$

The heuristic idea behind this definition is as follows. We have defined the vector field b in such a way that, although b is overall outward pointing (namely, $\operatorname{Tr} b > 0$), there are actually countably many regions where b is inward pointing (namely its r-component is strictly positive) which accumulate and mix at the domain boundary: these regions are represented by the dashed square in Figure 1. The function u is defined in such a way that u is transported along the characteristics (which are well-defined for a.e. (r, y) in the domain interior) and it is nonzero only on the regions where b is inward pointing. As a result, although b is overall outward pointing, it actually carries into the domain the nontrivial function u. This behavior is made possible by the breakdown of the BV regularity of b at the domain boundary.

We now show that u is a distributional solution of (1.4). First, we observe that $u(t,\cdot) \stackrel{*}{\rightharpoonup} 0$ as $t \to 0^+$ and hence the weak continuity of u with respect to the time implies that the initial datum is satisfied.

We then set $C := (u, bu)$ and we observe that $\mathrm{Div}\, C = 0$ on Λ^+ because C is identically 0 there. Next, we recall that the vector field b is constant with respect to t and, by using (4.19), we infer that u is also constant with respect to t in Λ^-. Hence, showing that $\mathrm{Div}\, C \equiv 0$ in Λ^- amounts to show that $\mathrm{div}\,(bu) \equiv 0$ in Λ^-. This follows by the same argument that we have used to infer that $\mathrm{div}\, b = 0$ for $r < 1$.

Finally, we observe that the normal vector to the surface $\mathcal{S}$ is (up to an arbitrary choice of the orientation) $\vec{n} := (1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$. Hence, by construction the normal trace of C is zero on both sides of the surface $\mathcal{S}$ and hence $\mathrm{Div}\, C \llcorner \mathcal{S} = 0$. This concludes the proof of Theorem 1.3.

As a side remark, we point out that by relying on Theorem 2.4 one can show that $\mathrm{Tr}\,(bu) = -1/4$ on $]0,1[\times\partial\Omega$. □

4.4. **Proof of Corollary 1.4.** We first describe the heuristic idea underlying the construction of the vector field b. Loosely speaking, we proceed as in the proof of Theorem 1.3, but we modify the values of the "building block" β_k on the subinterval $r \in]0, 2^{-k}[$. Indeed, instead of defining β_k as in Step 1 of § 4.3.1, we introduce nontrivial components in the (y_1, y_2)-directions. These non-trivial components are reminiscent of the construction in Depauw [16] and the resulting vector field can be actually regarded as a localized version of Depauw's vector field. In particular, they enable us to construct a solution that oscillates between 1, -1 and 0 and undergoes a finer and finer mixing as $r \to 0^+$.

The technical argument is organized in two steps: in § 4.4.1 we introduce the "localized version" of Depauw vector field, while in § 4.4.2 we conclude the proof of Corollary 1.4. Before proceeding, we introduce the following notation:

- Q_k is the square $(y_1, y_2) \in]0, 2^{-k}[\times]0, 2^{-k}[$;
- S_k is the square $(y_1, y_2) \in]0, 2^{2-k}[\times]0, 2^{2-k}[$.

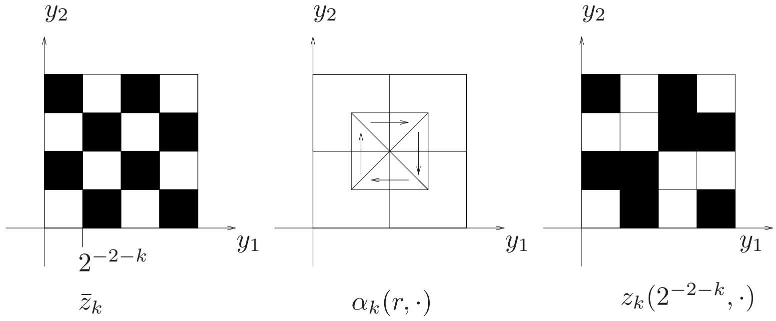
4.4.1. *A localized version of Depauw* [16] *vector field.* We construct the vector field α_k, which is defined on the cell $(r, y_1, y_2) \in]0, 2^{-k}[\times Q_k$. Also, for this construction we regard r as a time-like variable and we describe how a given initial datum evolves under the action of α_k. The argument is divided into steps.

Step 1: we construct the "building block" a_k, which is defined on the square $(y_1, y_2) \in]-2^{-2-k}, 2^{-2-k}[\times]-2^{-2-k}, 2^{-2-k}[$ by setting

$$a_k(y_1, y_2) = \begin{cases} (0, -2y_1) & |y_1| > |y_2| \\ (2y_2, 0) & |y_1| < |y_2|. \end{cases} \tag{4.20}$$

Note that a_k takes values in $\mathbb{R}^2$, it is divergence free and it is tangent at the boundary of the square.

FIGURE 3. The action of the vector field $\alpha_k(r,\cdot)$ on the solution z_k on the interval $r\in]0,2^{-2-k}[$. The solution z_k attains the values 1 and -1 on black and white regions, respectively.



STEP 2: we define the function $\bar z_k : Q_k \to \mathbb{R}$ by considering the chessboard illustrated in Figure 3, left part. The function $\bar z_k$ attains the value -1 and 1 on white and black squares, respectively.

STEP 3: we begin the construction of the vector field $\alpha_k :]0,2^{-k}[\times Q_k \to \mathbb{R}^2$. If $r\in]0,2^{-2-k}[$, then $\alpha_k(r,\cdot)$ is defined by setting

$$\alpha_k(r,y_1,y_2)=\begin{cases} a_k(y_1-2^{-1-k},y_2-2^{-1-k}) \\ \qquad \text{if } (y_1,y_2)\in]2^{-2-k},3\cdot 2^{-2-k}[\times]2^{-2-k},3\cdot 2^{-2-k}[\\ \\ (0,0) \text{ elsewhere on } Q_k. \end{cases} \tag{4.21}$$

See Figure 3, central part, for a representation of the values attained by α_k on the interval $r\in]0,2^{-2-k}[$.

We term z_k the solution of the problem

$$\begin{cases} \partial_r z_k + \mathrm{div}_y(\alpha_k z_k) = 0 & \text{in }]0,2^{-k}[\times Q_k \\ z_k = \bar z_k & \text{at } r=0, \end{cases} \tag{4.22}$$

where $\bar z_k$ is defined as in STEP 2. Note that by construction $\mathrm{div}_y \alpha_k = 0$ and therefore the first line of (4.22) is actually a transport equation. Hence, the value attained by the function z_k can be determined by the classical method of characteristics. In particular, the function $z_k(2^{-2-k},\cdot)$ is represented in Figure 3, right part, and it attains the values 1 and -1 on black and white squares, respectively.

STEP 4: if $r\in]2^{-2-k},3\cdot 2^{-2-k}[$, then $\alpha_k(r,\cdot)$ is defined by setting

$$\alpha_k(r,y_1,y_2)=a_k(y_1-i2^{-2-k},y_2-j2^{-2-k})$$

if

$$(y_1,y_2)\in](i-1)2^{-2-k},(i+1)\cdot 2^{-2-k}[\times](j-1)2^{-2-k},(j+1)\cdot 2^{-2-k}[,$$

where i,j can be either 1 or 3. See Figure 4, central part, for a representation of the values attained by α_k on the interval $r\in]2^{-2-k},3\cdot 2^{-2-k}[$. Note that by construction $\mathrm{div}_y\alpha_k\equiv 0$ on $]0,3\cdot 2^{-2-k}[\times Q_k$ and hence the solution z_k of (4.22)

FIGURE 4. The action of the vector field $\alpha_k(r,\cdot)$ on the solution z_k on the interval $r \in]2^{-2-k}, 3\cdot 2^{-2-k}[$. The solution z_k attains the values 1 and -1 on black and white regions, respectively.

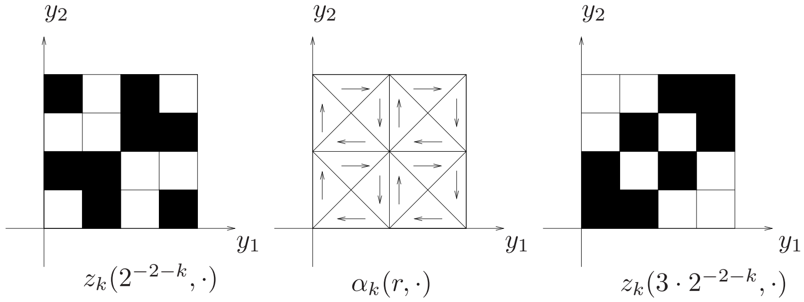
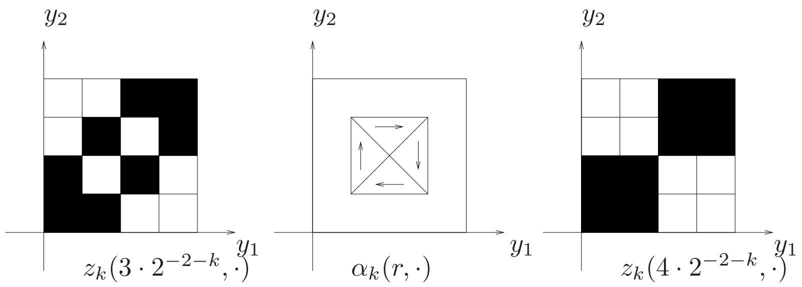


FIGURE 5. The action of the vector field $\alpha_k(r,\cdot)$ on the solution z_k on the interval $r \in]3\cdot 2^{-2-k}, 4\cdot 2^{-2-k}[$. The solution z_k attains the values 1 and -1 on black and white regions, respectively.



evaluated at $r = 3\cdot 2^{-2-k}$ is as in Figure 4, right part: as usual, the black and white squares represent the regions where $z_k(3\cdot 2^{-2-k},\cdot)$ attain the values 1 and -1, respectively.

STEP 5: if $r \in]3\cdot 2^{-2-k}, 4\cdot 2^{-2-k}[$, then $\alpha_k(r,\cdot)$ is again defined by (4.21). Hence, the values attained by $z_k(2^{-k},\cdot)$ are those represented in Figure 5, right part.

4.4.2. *Conclusion of the proof.* Loosely speaking, the proof of Corollary 1.4 is concluded by combining the construction described in § 4.4.1 with the proof of Theorem 1.3. The argument is divided in four steps.

STEP A: we define the vector field $\tilde\beta_k$ and the solution u_k on $(r, y_1, y_2) \in]0, 2^{-k}[\times S_k$, where S_k is the square $]0, 2^{2-k}[\times]0, 2^{2-k}[$.

We recall the definition of B_k provided by (4.9) and we set

$$\tilde{\beta}_k(r, y_1, y_2) := \begin{cases} \big(1, \alpha_k(r, y_1, y_2)\big) & (y_1, y_2) \in]0, 2^{-k}[\times]0, 2^{-k}[\\ \big(1, \alpha_k(r, y_1 - 2 \cdot 2^{-k}, y_2)\big) & (y_1, y_2) \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\times]0, 2^{-k}[\\ \big(1, \alpha_k(r, y_1, y_2 - 2 \cdot 2^{-k})\big) & (y_1, y_2) \in]0, 2^{-k}[\times]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\\ \big(1, \alpha_k(r, y_1 - 2 \cdot 2^{-k}, y_2 - 2 \cdot 2^{-k})\big) & (y_1, y_2) \in]2^{-k}, 3 \cdot 2^{-k}[\times]2^{-k}, 3 \cdot 2^{-k}[\\ (-5, 0, 0) & (y_1, y_2) \in B_k \\ (0, 0, 0) & \text{elsewhere in } S_k \end{cases}$$

Note that, basically, the definition of $\tilde{\beta}_k$ is obtained from (4.10) by changing the value of the vector field on D_k and inserting as a component in the (y_1, y_2)-directions the vector field α_k constructed in § 4.4.1.

Also, we define the function u_k by setting

$$u_k(r, y_1, y_2) := \begin{cases} z_k(r, y_1, y_2) & (y_1, y_2) \in]0, 2^{-k}[\times]0, 2^{-k}[\\ z_k(r, y_1 - 2 \cdot 2^{-k}, y_2) & (y_1, y_2) \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\times]0, 2^{-k}[\\ z_k(r, y_1, y_2 - 2 \cdot 2^{-k}) & (y_1, y_2) \in]0, 2^{-k}[\times]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\\ z_k(r, y_1 - 2 \cdot 2^{-k}, y_2 - 2 \cdot 2^{-k}) & (y_1, y_2) \in]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\times]2 \cdot 2^{-k}, 3 \cdot 2^{-k}[\\ 0 & \text{elsewhere in } S_k, \end{cases}$$

where z_k is the same function as in § 4.4.1.

STEP B: we define the vector field $\tilde{\beta}_k$ and the solution u_k for $(r, y_1, y_2) \in]2^{-k}, 2^{2-k}[\times S_k$.

We set $\tilde{\beta}_k(r, y_1, y_2) := \beta_k(r, y_1, y_2)$, where β_k denotes the same vector field as in § 4.3.1. The function u_k satisfies

$$\partial_r u_k + \mathrm{div}_y(\tilde{\beta}_k u_k) = 0.$$

Since $\mathrm{div}_y \tilde{\beta}_k = 0$, the values attained by u_k for $(r, y_1, y_2) \in]2^{-k}, 2^{2-k}[\times S_k$ can be computed by the classical method of characteristics. To provide an heuristic intuition of the behavior of u_k, we refer to Figure 1, center and right part, and we point out that u_k attains the value 0 on white and black areas, while on dashed areas it attains the same values as in Figure 5, right part.

STEP C: we extend $\tilde{\beta}_k$ and u_k to $]0, 2^{2-k}[\times\mathbb{R}^2$ by periodicity by proceeding as in (4.14).

STEP D: we finally define a vector field b and the function u. We recall the decomposition (4.16) and we define b as in (4.17), replacing β_k with $\tilde{\beta}_k$. Also, we define u by setting

$$u(t, r, y_1, y_2) = \begin{cases} u_k(r, y_1, y_2) & \text{in } \Lambda^-, \text{ when } r \in I_k \\ 0 & \text{in } \Lambda^+. \end{cases}$$

By arguing as in the proof of Theorem 1.3, one can show that u and b satisfy requirements i), ..., v) in the statement of Theorem 1.3 and that moreover $\mathrm{Tr}\,(bu) \equiv 0$. This concludes the proof of Corollary 1.4. □

Acknowledgments

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