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Elastic deformations on the plane and approximations

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Summary. The basis for these notes is the course given by the first author in the week June 20–24, 2011 at the SISSA of Trieste (Italy), during the intensive period "Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations".

1.1 Introduction and some history

The aim of these notes is to give a good overview on the problem of the approximation of homeomorphisms in the plane, with a special emphasis on some new results.

Let us start briefly describing what is the main problem and its history. Given a homeomorphism $u : \Omega \to \Delta$, where Ω and Δ are open subsets of $\mathbb{R}^N$, one may want to find an approximation $u_\varepsilon : \Omega \to \Delta$ of u; this means on one hand that u_ε is *"good"*, e.g. a smooth homeomorphism, and on the other hand that it is *"close"* to u, that is, the distance in a suitable sense between u and u_ε is small, say $d(u, u_\varepsilon) \leq \varepsilon \ll 1$. Probably the most important example of this situation is when u is the deformation of an elastic object, in particular a thin elastic plate for $N = 2$, or an elastic body for $N = 3$. The main reason why the existence of a smooth approximation is not trivial is that one requires u_ε to be also a homeomorphism, while the standard mollification with a smoothing kernel does not ensure u_ε to be a bijection. In particular, the mollification would work if the function u is assumed to be of class C^2, but this is a too strong assumption for most of the applications: for instance, in nonlinear elasticity an important situation is that of the piecewise affine maps, whose second (distributional) derivative is concentrated on a set of zero measure.

Of course, the whole question heavily depends on what one means by *"good"*, and on which is the distance d used to say that u_ε is *"close"* to u. Concerning the first point, a natural choice is of course to ask u_ε to be smooth on Ω; but another possibility, which is sometimes preferred in the applications, is to ask u_ε to be piecewise affine. We will give our approximation results in both contexts; in particular, it is quite clear how to recover a piecewise affine approximation starting from a smooth one, while the converse is more complicate. In fact, as we will see later, the first part of these notes is devoted to give a result which allows to build a smooth approximation from a piecewise affine one (see Theorem A in Chapter 1.2).

Before passing to observe the possible meaningful choices of the distance d, let us briefly discuss the first temptative idea that one could have, in order to build a piecewise affine approximation of u. One can start fixing an arbitrary triangulation of Ω, made by many sufficiently small triangles, and then define u_ε as the function which corresponds, in every triangle, with the affine interpolation of the values of u on the vertices. It is immediate to observe that, as soon as the triangles are small enough, the map u_ε is arbitrarily close to u, at least in the L^∞ sense; however, no matter how small are the triangles, the map u_ε could fail to be injective, as Figure 1.1 shows. An explicit example of

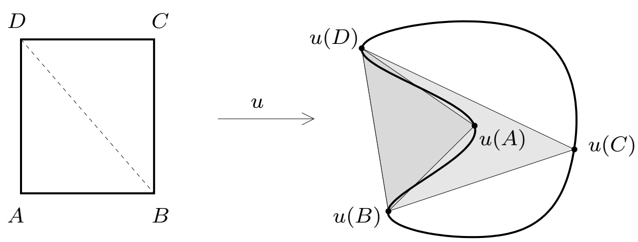

Fig. 1.1. The interpolations of the map u on the disjoint triangles ABD and BCD may overlap.

a function with such a bad behaviour on arbitrarily small scales can be found in [49]. Therefore, even the simplest case when

$$d(u,v) = \|u - v\|_{L^\infty(\Omega)} + \|u^{-1} - v^{-1}\|_{L^\infty(\Delta)} \tag{1.1}$$

is not straightforward. In this setting, the problem was heavily studied in the 1950s and the 1960s, mainly because of its importance in geometric topology: the first solution, dealing with the planar case $N = 2$, was given by Radó [47], then the problem was solved by Moise [40, 41] and Bing [9] in the spatial case $N = 3$. Other positive results in higher dimension have been found by Connell [12], Bing [10], Kirby [35] and Kirby, Siebenmann and Wall [36] (see also Rushing [48] or Luukkainen [38]), while a negative result in dimension $N = 4$ has been given by Donaldson and Sullivan [18]. It is to be mentioned that the strategy to find the approximation, in some of the above-mentioned cases, is basically the one of the affine interpolations that we described above. As we said before, the difficult part is not at all to show that the affine interpolation is close to the original map, because this is trivial: the very hard part is to show that it is always possible to select a "smart" triangulation which avoids the drawback of the figure above. Thanks to these results, basically everything is known for what concerns the L^∞ distance in (1.1).

However, this is not the end the story: indeed, for most applications the approximation with the distance d given by (1.1) is not enough. In particular, when dealing with problems in nonlinear elasticity, one wants to obtain a map u_ε corresponding to an energy close to that of u. And, even if there are several possible notions of energy of a configuration, all of them involve some derivative, and then an L^∞ bound for $u - u_\varepsilon$ is not enough. For this reason, since the 1980s Evans and Ball (see [19, 5, 6]) suggested to try to prove some approximation result for the case of $W^{1,p}$ homeomorphisms, with some distance d which involves also the L^p norm of the derivative, in place of that given by (1.1): for related functionals, see for instance [4], [7], [13], [50].

It turns out that it is not easy to give bounds to the derivative of Du_ε, if the function u_ε is obtained through the interpolation procedure described above, and for this reason the Ball–Evans question remained completely open for several years. Nevertheless, two positive results essentially using this strategy came out in recent years. In the first one, Mora-Corral [42] was able to deal with the case of a planar bi-Sobolev map, which is C^2 everywhere except than in a point (this situation is not simple at all, on the contrary it already contains most of the difficulties which arise in the general situation). In this case, it was proved the existence of a piecewise affine approximation u_ε close to u with respect to the distance

$$d(u,v) = \|u-v\|_{L^\infty(\Omega)} + \|u^{-1} - v^{-1}\|_{L^\infty(\Delta)} + \|Du - Dv\|_{L^p(\Omega)}\,; \qquad (1.2)$$

this was the first positive result for a distance involving the derivative. The other paper, by Bellido and Mora-Corral [8], proves that if the planar map u belongs to some Hölder space $\mathrm{C}^{0,\alpha}$, then there exists a piecewise affine approximation which is close under the $\mathrm{C}^{0,\beta}$ distance, where for any $\alpha \in (0,1]$ the constant $\beta = \beta(\alpha) \in (0,1)$ is explicitely determined. The results by Mora-Corral and by Bellido and Mora-Corral have been obtained defining u_ε as a piecewise affine function on a suitably constructed triangulation; as one can easily imagine, to find a proper triangulation and to define a corresponding piecewise affine function u_ε obtaining also an estimate of $Du - Du_\varepsilon$ (or a $\mathrm{C}^{0,\beta}$ estimate) is quite more complicate than it already was for the L^∞ case discussed before.

Recently, Iwaniec, Kovalev and Onninen have used a different strategy to give a positive answer, still in the two-dimensional case. More precisely, they have considered a $W^{1,p}$ map u, and they have showed the existence of a smooth map u_ε close to u again in the sense of (1.2) –more precisely, they have used the weaker distance $\|u-v\|_{W^{1,p}}$, but then it is easy to deduce the validity also for the stronger case of (1.2). Their first result was for $p = 2$ (see [30]), then they were able to extend their analysis to the case of a generic $1 < p < \infty$ (see [31]).

The methods used in these two papers are deeply different from what used in all the preceding works, so let us give a very brief and incomplete idea about the proof in [31]. A main tool in their construction is a suitable "p-harmonic replacement" technique: roughly speaking, this means the following. Take a $W^{1,p}$ map u on the open domain Ω, and let $A \subset\subset \Omega$ be a compactly supported subdomain; then, it is possible to modify u only inside A in such a way that the new map remains continuous, it becomes p-harmonic inside A, and its total energy has not increased. This result is obtained making use of a generalization due to Alessandrini and Sigalotti of the Radó–Kneser–Choquet Theorem. The construction of the approximation u_ε of u is basically done by finding a suitable partition of the domain in cells, and then applying repeatedly the p-harmonic replacement and an *ad hoc* smoothing procedure.

Summarizing, except for the case of $p = 1$, which is still open, the results by Iwaniec, Kovalev and Onninen provide the full positive answer to Ball–Evans question for the distance (1.2) and the dimension $N = 2$.

Again, also these results are still not enough to cover the relevant cases corresponding to the nonlinear elasticity, which is the main application one has in mind. The reason of this, is that the distance given by (1.2) is still not strong enough to deal with the energy. In fact, the energy related to the map u usually takes the general form $\mathcal{E}(u) = \int_\Omega W(Du)$ for some functional $W : \mathbb{R}^{N\times N} \to \mathbb{R}$, and in all the applications one has $W(M) \to +\infty$ if $\det M \to 0$ (see for instance [6, pag. 3], but also [3, 51, 20, 44]). Notice that the meaning of this assumption is simple, since it corresponds to require a high energy to compress the material (in particular, if the material is incompressible then $W(M) = +\infty$ whenever $\det M \neq 1$). As a consequence, a small distance $d(u, u_\varepsilon) \ll 1$ with the function d given by (1.2) does not ensure that $\big|\mathcal{E}(u)-\mathcal{E}(u_\varepsilon)\big| \ll 1$, and of course it is of no use an "approximation" of a map which corresponds to a significantly different energy. For this reason, one is led to a further definition of distance, even stronger than (1.2), namely,

$$\begin{aligned} d(u,v) = \|u-v\|_{L^\infty(\Omega)} &+ \|u^{-1}-v^{-1}\|_{L^\infty(\Delta)} \\ &+ \|Du-Dv\|_{L^p(\Omega)} + \|Du^{-1}-Dv^{-1}\|_{L^p(\Delta)}\,; \end{aligned} \tag{1.3}$$

this distance is strong enough to give a control on $\big|\mathcal{E}(u)-\mathcal{E}(u_\varepsilon)\big|$ in terms of $d(u, u_\varepsilon)$ in all the relevant applications. We remark that finding an approximation result with this last notion of distance was also left as an open problem in [31, Question 4.2]. The only available result with this distance was recently obtained by the first author and Daneri in [16], where it is shown that every planar bi-Lipschitz map u can be approximated in this strong sense with either smooth or piecewise affine bi-Lipschitz homeomorphisms (this is Theorem C in Chapter 1.4 of these notes). Notice that the assumption means that u, $u^{-1} \in W^{1,\infty}$, while the best possible result that one would like to have, is with the assumption that u, $u^{-1} \in W^{1,p}$. This is presently still open.

1.1.1 Plan of the notes

We will start these notes with a short overview on the theory of the mappings of finite distortion and of bi-Sobolev mappings, Section 1.1.2: even if these notions are not needed to read the rest of the notes, they are strictly connected with the problem we are dealing with. Then, in Section 1.1.3 we will list the notation that we are going to use through this work. The main part of the notes is then divided in three chapters, each devoted to present a different recent work on the subject described in the introduction.

As we said above, one may want to approximate a homeomorphism with a smooth one, or with a piecewise affine one. The two things are not easily equivalent: in fact, while it is rather simple to pass from a smooth approximation to a piecewise affine one, the converse is much more complicated. In Chapter 1.2 we will present a recent result by Mora-Corral and the first author (proven in [43]), which shows how to do so. Actually, the result that we will prove, Theorem A, is a bit less general than the one in [43], because we preferred to focus here on the bi-Lipschitz case in order to simplify the construction. We remark also that a related result was already known since the work by Munkres [45], but there was no explicit estimate of the error, which is instead essential for our purposes (see in particular (1.6) in Theorem A).

In many different proofs of an approximation theorem, a key ingredient is an extension result. The main reason is that one seeks for an approximating map which is still a homeomorphism (this is always the source of all the difficulties); therefore, once one has solved the problem in a big portion of the domain, it can be useful to get rid of the remaining part simply taking an extension of the boundary values. In particular, for our construction we will need to know the following result: given a bi-Lipschitz function defined on the boundary of the unit square, there is a bi-Lipschitz extension in the whole square. Notice that this is exactly the claim of the well-known Kirszbraun extension Theorem, except that the Lipschitz property is replaced by the bi-Lipschitz one. And again, being interested also in the inverse makes everything harder: indeed, while Kirszbraun Theorem holds in a wide generality of spaces, and the Lipschitz constant does not change with the extension, the stronger result that we need is known only in dimension 2, and the bi-Lipschitz constant increases significantly. Since this claim is of primary importance for our construction, Chapter 1.3 will be devoted to present a recent proof by Daneri and the first author (see [15]). It has to be pointed out that the same result had been proved already by Tukia [52], with a completely different strategy. But again, in Tukia's result there was no estimate on the bi-Lipschitz constant of the approximation (because the existence was obtained via a compactness argument), which is instead necessary, while in [15] it is proved that the extension is at most CL^4 bi-Lipschitz for an explicit constant C, see Theorem B. The exponent 4 is presumably not sharp, obtaining a bi-Lipschitz constant CL would be a major result.

Finally, in the last part of the notes, Chapter 1.4, we will present the proof of the main approximation result, Theorem C, which has been recently proved by Daneri and the first author in [16]: as explained in the introduction, we prove that every planar bi-Lipschitz map can be approximated by smooth or piecewise affine bi-Lipschitz homeomorphisms in the sense of the strongest distance d in (1.3).

Our strategy in proving all the three results is to give an explicit construction by means of elementary but involved geometric arguments. Therefore, there is in principle no obstruction to extend all the results even in the three-dimensional case (while, for instance, the strategy in [30, 31] cannot be extended since it needs to identify $\mathbb{R}^2$ with the complex plane). However, since many of our arguments are extremely delicate, the extension seems at the moment quite hard (in particular, for what concerns the bi-Lipschitz extension Theorem B). The main problem should be that, in the 3-dimensional situation, many more complicate topological obstructions may arise: in fact, already for the classical approximation in the L^∞ norm described before, the 3-dimensional case is much more complicate than the 2-dimensional one, see the book [41].

The three main results that we will discuss are quite technical and involved, even if they make only use of elementary geometric facts. To keep these notes easy to read, we have then tried to simplify as much as possible the presentation and the details: the interested reader can find the fully complete results in the above-cited papers, while here some results are not proved, or not in their full generality. Moreover, each chapter contains a long preliminary section, where the overall strategy of the construction is described and the main steps are depicted. Reading each introduction will be enough to give a flavour of the proof to the quick reader, and it should also considerably help the more interested reader to follow the complete proof without getting lost in the technicalities.

1.1.2 Maps of finite distortion and bi-Sobolev functions

In this section, we give a short description of the theory of the maps of finite distortion and of the bi-Sobolev mappings. Since this is a huge field, we can only present some aspects, the interested reader can find everything in the literature (just as an example, we quote here the two monographs [1, 32], as well as the very recent one [25], other references will be given later).

First of all, we recall that a *Jordan curve* is any continuous and injective map $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, or equivalently, any closed and non self-intersecting curve in $\mathbb{R}^2$, corresponding to $C = \gamma(\mathbb{S}^1)$. Any such curve divides $\mathbb{R}^2 \setminus C$ in two disjoint connected open sets: since one of them is bounded, and the other one is unbounded, the first one is often referred to as the "internal part" of C, and the second one as the "external part". For any point $x \notin C$, consider the homotopy class $h(x) \in \mathbb{Z}$ of the map

$$\mathbb{S}^1 \ni \theta \mapsto \frac{\gamma(\theta) - x}{|\gamma(\theta) - x|} \in \mathbb{S}^1 \ :$$

it can be proved that $h(x) = 0$ whenever x belongs to the external part, while $h(x) = \pm 1$ if x is in the internal one. More precisely, either $h(x) = 1$ for *all* the points in the interior part of the curve, or $h(x) = -1$ for *all* of them: in the first case, it is said that the curve γ is counterclockwise, while in the second case it is clockwise.

Definition 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an open connected set, and $u : \Omega \to \mathbb{R}^2$ be a homeomorphism onto the image. Let γ be any counterclockwise curve inside Ω; we say that u is* orientation-preserving *(resp.,* orientation-reversing*) if the curve $u(\gamma)$ is counterclockwise (resp., clockwise).*

It can be shown that the above definition does not depend on the choice of the curve γ. As a consequence, every homeomorphism of a connected planar set must be either orientation-preserving, or orientation-reversing. To determine the orientation property of a given homeomorphism, it can be of course selected any curve γ by the above definition; in particular, if Ω is a Lipschitz domain and u is continuous up to the boundary, then it is possible to select as curve the boundary of Ω itself, which is often a useful choice. Notice that, if u is smooth enough, say $u \in \mathrm{C}^2$, then being orientation-preserving is equivalent to have $\det Du > 0$ almost everywhere; in the general case of a bi-Sobolev function u the situation is much more complicate, as we will see below.

Let us now introduce the concept of distortion of a map. Through this section $\Omega \subseteq \mathbb{R}^N$ is an open, bounded and connected set, and $u : \Omega \to \mathbb{R}^N$ a map.

Definition 1.2. *Let $x \in \Omega$ be such that $Du(x)$ exists and $\det Du(x) > 0$. The* distortion of u at x *is*

$$K_u(x) := \frac{|Du(x)|^N}{N^{\frac{N}{2}} \det Du(x)}\,.$$

We remark that sometimes, in the literature, the distortion is defined also for a point x for which $\det Du(x) = 0$, and in this case one sets $K_u(x) = 1$. The presence of the constant $N^{N/2}$ in the definition has the only purpose of letting the identity have unit distortion. To understand the meaning of the distortion, it may help to concentrate for a moment on the two-dimensional case: the first-order Taylor expansion of u around x maps the unit circle into an ellipsis of axes a and $b \leq a$, while up to rotations one has

$$Du = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Therefore, the distortion of u at x is $K_u(x) = \dfrac{a^2 + b^2}{2ab} \approx \dfrac{a}{b}$. In other words, roughly speaking, the distortion is more or less the quotient between the greatest and the smallest stretching ratios of Du. A very important feature of the distortion is the following.

Lemma 1.3. *The distortion satisfies $K_u(x) \geq 1$ wherever it is defined, and equality holds if and only if $Du(x)$ is a multiple of the identity. Moreover, if both $Du(x)$ and $Du^{-1}\big(u(x)\big)$ exist, then $K_u(x) = K_{u^{-1}}\,(u(x))$.*

This fact is extremely useful; notice that, on the contrary, there is not a precise link between $|Du(x)|$ and $\big|Du^{-1}\big(u(x)\big)\big|$.

Definition 1.4. *Assume that $u \in W^{1,1}_{\rm loc}(\Omega)$, $\det Du \in L^1_{\rm loc}(\Omega)$ and $\det Du \geq 0$ almost everywhere. Then u is said to be* of finite distortion*. If, in addition, the distortion K_u belongs to $L^\infty(\Omega)$, we say that u is* of bounded distortion*, or that it is* quasiregular*. If in addition u is a homeomorphism, then it is said* quasiconformal*.*

The study of mappings of finite distortion has started with the pioneering works by Ball [2, 3], originally motivated by non-linear elasticity; the theory is nowadays very rich, a non-complete list of some other important results on this matter is [14, 17, 21, 22, 23, 27, 24, 25, 26, 28, 29, 33, 34, 37, 38, 39, 51], where one can find also all the results that we describe below.

Observe that, if the map u is bi-Lipschitz, then it has bounded distortion, since the bi-Lipschitz constraint implies both an upper and a lower bound for $|Du|$ and $\det Du$.

A first important property of the maps of finite distortion is that improved continuity results hold, with respect to usual Sobolev function. More precisely, recall that in general a Sobolev map $f \in W^{1,p}(\Omega;\mathbb{R}^N)$ is continuous if $p > N$, but not necessarily if $p \leq N$. Instead, for a function of finite distortion the following holds.

Theorem 1.5. *If u is of finite distortion and $u \in W^{1,N}_{\rm loc}(\Omega)$, then it is continuous. The same holds true if u is of finite distortion and $e^{\lambda K_u} \in L^1_{\rm loc}(\Omega)$ for some $\lambda > 0$.*

Another important question is whether or not a map satisfies the *Luzin (N) property*, which means

$$|E| = 0 \qquad \Longrightarrow \qquad |u(E)| = 0\,, \qquad \forall\, E \subseteq \Omega.$$

The meaning of this property in the context of the elastic deformations is that "mass cannot be created from nothing" in an elastic body. Moreover, the validity of this property is essential, because if it is true then the usual change of variable formula for integrals holds. In turn, it is quite easy to check that for a generic map $f \in W^{1,p}(\Omega;\mathbb{R}^N)$ the Luzin (N) property holds as soon as $p > N$, while for $p \leq N$ this is not necessarily true: a counterexample with $p = N$ was already known to Cesari [11], and for $p < N$ it is also possible to build a counterexample which is a homeomorphism, as it essentially comes from an old idea of Ponomarev [46], see [34]. And again, the assumption of finite distortion ensures the validity of the property with weaker summability assumption (see [34, 39]).

Theorem 1.6. *If u is of finite distortion and either $u \in W^{1,N}_{\rm loc}(\Omega)$, or $e^{\lambda K_u} \in L^1_{\rm loc}(\Omega)$ for some $\lambda > 0$, then u satisfies the Luzin (N) property.*

Conversely, the following result is known (see [37]) for the Luzin (N^{-1}) property, which, for an elastic deformation, means that "mass cannot disappear".

Theorem 1.7. *Assume that u is of finite distortion, $K_u \in L^{\frac{N}{N-1}}(\Omega)$, the multiplicity of u is essentially bounded and u is not constant. Then the Luzin (N^{-1}) property*

$$|u(E)| = 0 \qquad\Longrightarrow\qquad |E| = 0\,, \qquad \forall\, E \subseteq \Omega,$$

holds true, and $\det Du > 0$ *a.e. in Ω.*

We pass now to discuss a strongly linked question, namely, under which assumptions is it possible to say that $\det Du > 0$ almost everywhere (or $\det Du < 0$ almost everywhere), for a homeomorphism u: as we already pointed out before, if u is smooth enough then this is granted. In particular $\det Du > 0$ a.e. if u is orientation-preserving, and $\det Du < 0$ a.e. if u is orientation-reversing. The connection between this strict sign condition and the Luzin (N^{-1}) property is immediate to observe: indeed, assume that the change of variable formula holds for u, and that $\det Du = 0$ on a set of positive measure $E \subseteq \Omega$. Then the Luzin (N^{-1}) property does not hold, since

$$0 = \int_E \det Du = \int_{u(E)} 1 = \big|u(E)\big|\,. \tag{1.4}$$

Keeping this observation in mind, the sharpness in the claim of Theorem 1.7 becomes evident in view of the following example, presented in [32, Par. 6.5.6]: there exists a Lipschitz homeomorphism u of finite distortion whose Jacobian vanishes in a set $E \subseteq \Omega$ of positive measure, while $|u(E)| = 0$ and $K_u \in L^p(\Omega)$ for every $p < \frac{N}{N-1}$.

Given the homeomorphism u, the easiest strategy to prove that $\det Du > 0$ (or $\det Du < 0$) almost everywhere is to start showing only the weaker inequality $\det Du \geq 0$ (or $\det Du \leq 0$); then –if the summability of Du and of K_u is enough– u becomes of finite distortion and then one can apply Theorem 1.7. Let us then list the assumptions under which it is known that either the inequality $\det Du \geq 0$ or the inequality $\det Du \leq 0$ holds a.e..

Theorem 1.8. *Let $u : \Omega \to \mathbb{R}^N$ be a homeomorphism. Then one has that* $\det Du \geq 0$ *a.e., or that* $\det Du \leq 0$ *a.e., if one of the following conditions hold:*

- *$u \in W^{1,1}_{\rm loc}(\Omega)$, if $N = 2$ or $N = 3$;*
- *$u \in W^{1,p}_{\rm loc}(\Omega)$ and $p > [N/2]$, if $N > 3$.*

In particular, if $N \leq 3$ it can be also proved that if u is orientation-preserving then $\det Du \geq 0$, while if u is orientation reversing then $\det Du \leq 0$; the same is reasonable also in dimension $N \geq 4$, but the question is still open. The sharpness of the assumptions in Theorems 1.7 and 1.8 (in particular, of the essentially bounded multiplicity to get the strict sign condition) can be checked in view of the following counterexamples (which can be found in the papers [22, 27, 23], for a unified treatment see the new monograph [25]).

Example 1.9. There exists a Lipschitz homeomorphism such that $\det Du = 0$ on a set of positive measure; there also exists a Sobolev homeomorphism such that $\det Du = 0$ almost everywhere: in this last case, then, by (1.4) we observe that u maps a set of full measure in a set of zero measure, and vice versa. There exists an approximatively differentiable (but not $W^{1,1}$) homeomorphism $u : \mathcal{B} \to \mathcal{B}$, being $\mathcal{B}$ the unit ball, such that $u(x) = x$ on $\partial\mathcal{B}$ but $\det Du < 0$ on a set of positive measure. There also exists a map (but not a homeomorphism) $u \in W^{1,p}(\mathcal{B};\mathcal{B})$ with $p < N$ which is continuous and satisfies $u(x) = x$ on $\partial\mathcal{B}$, but $\det Du < 0$ almost everywhere.

The last property that we discuss is the "regularity of the inverse"; roughly speaking, if u is a homeomorphism, is it possible to deduce summability for Du^{-1} from the summability of Du? For general maps, this is never true, since there are Lipschitz homeomorphisms whose inverse is not even in $W^{1,1}_{\rm loc}$. However, again the finite distortion allows to obtain better results.

Theorem 1.10. *If $u \in W^{1,N-1}_{\rm loc}(\Omega)$ is a homeomorphism of finite distortion, then u^{-1} belongs to $W^{1,1}_{\rm loc}\big(u(\Omega)\big)$ and it is also of finite distortion. If $u \in W^{1,N-1}_{\rm loc}(\Omega)$ is a homeomorphism, then $u^{-1} \in BV_{\rm loc}\big(u(\Omega)\big)$. Moreover, if $N = 2$ and $u \in BV_{\rm loc}(\Omega)$ is a homeomorphism, then $u^{-1} \in BV_{\rm loc}\big(u(\Omega)\big)$.*

1.1.3 Notation

Let us now give a short list of the common notation that will be used from now on, in the three main parts of the notes; in addition, every part will need some further specific notation, which will be presented at the beginning of each part.

We will always work in dimension 2, and Ω and Δ will be given open planar sets. A *triangulation* of Ω is a locally finite family $\{\mathscr{T}_i\}$ of essentially disjoint closed triangles whose union is between Ω and its closure, and with the property that the intersection of two different triangles may be either empty, or a common vertex, or a common side. A triangulation is said *finite* if so is the number of triangles of the partition, while otherwise it is *countable*; every open set admits a triangulation, but only polygons admit finite triangulations.

A map $u : \Omega \to \Delta$ is said *piecewise affine* if there is a triangulation of Ω such that u is affine on every triangle of the triangulation. We say that u is *finitely piecewise affine* if there exists such a triangulation which is finite.

We will use $\mathscr{L}$ to denote the Lebesgue measure, but often we will write for brevity $|\Omega|$ instead of $\mathscr{L}(\Omega)$.

The generic points will be usually denoted by capital letters, as P, Q and so on; sometimes, we will write $P \equiv (x, y)$ if we will need to use the orthogonal coordinates in $\mathbb{R}^2$. Only in Chapter 1.2 we will use also polar coordinates, and this will require a different specific notation (which we will directly introduce there). The segment joining P and Q will be denoted by PQ, and $\ell(PQ)$ will be its length. The triangle having vertices P, Q and R will be denoted by PQR, and in general $P_1P_2\cdots P_k$ will be the polygon whose ordered vertices are the P_i's. Given three non-aligned points P, Q and R, we will call $P\widehat{Q}R \in (0, \pi)$ the corresponding angle. Sometimes, for the ease of presentation, we will write the value of angles in degrees, with the usual convention that $\pi = 180°$.

The ball centered at P and with radius ρ will be denoted by $\mathcal{B}(P, \rho)$, while the ρ-neighborhood of the set X will be $\mathcal{B}(X, \rho)$. Moreover, $\mathcal{D}(P, \rho)$ is the square centered at P, having half-side ρ, and whose sides are parallel to the coordinate axes: in particular, since we will use it extensively, we will write for brevity $\mathcal{D} = \mathcal{D}(O, 1/2)$ to denote the square of unit side centered at the origin.

Given a matrix $M \in \mathbb{R}^{2\times 2}$, we will consider as usual its norm as $|M| = \sqrt{M_{11}^2 + M_{12}^2 + M_{21}^2 + M_{22}^2}$.

The map $u : \Omega \to \mathbb{R}^2$ is said to be L *bi-Lipschitz* if for every $P, Q \in \Omega$ one has

$$\frac{1}{L} \leq \frac{\ell\big(u(P)u(Q)\big)}{\ell(PQ)} \leq L\,;$$

or course, if u is bi-Lipschitz, then in particular it is a homeomorphism, and it is always $L \geq 1$.

1.2 Part I: From piecewise affine to smooth

This first chapter is devoted to show an approximation result, which allows to pass from a piecewise affine homeomorphism to a smooth one. Our result is basically taken from [43]: in fact, we present here only the particular case of our interest, namely, the case of bi-Lipschitz maps; the result in [43] is quite more general, even though the construction is more or less the same.

In this part, we consider a given L bi-Lipschitz map $u : \Omega \to \Delta$ which is *countably piecewise affine*. As we said in Section 1.1.3, this means that Ω is the locally finite union of essentially disjoint triangles, such that two different triangles can intersect in a common vertex or in a common side, and that u is affine on every triangle. The result we are interested in is the following.

Theorem A (From piecewise affine to smooth). *Let Ω, $\Delta \subseteq \mathbb{R}^2$ be two open sets, let $u : \Omega \to \Delta$ be a countably piecewise affine L bi-Lipschitz homeomorphism, and let $1 \leq p < \infty$. For every $\varepsilon > 0$ there exists a smooth diffeomorphism $v : \Omega \to \Delta$ such that*

$$\|v - u\|_{L^\infty(\Omega)} + \|Dv - Du\|_{L^p(\Omega)} + \|v^{-1} - u^{-1}\|_{L^\infty(u(\Omega))} + \|Dv^{-1} - Du^{-1}\|_{L^p(u(\Omega))} \leq \varepsilon\,. \tag{1.5}$$

Moreover, one has that

- *v is $100L^{7/3}$ bi-Lipschitz;*
- *$u = v$ on $\partial\Omega$;*
- *if u is orientation-preserving, then so is v;*
- *for any $0 \leq q \leq 1$, it is possible to choose the function v in such a way that*

$$\begin{gathered}\big|Dv(x)\big| \leq 13L^{3-2q}\,, \qquad \big|Dv^{-1}\big(u(x)\big)\big| \leq 70L^{1+4q}\,,\\ \det Dv(x) \geq \frac{1}{30}\,L^{-4q}\,\det Du(x)\end{gathered} \tag{1.6}$$

(notice the local estimate for Dv and Dv^{-1}, but the pointwise estimate for $\det Dv$).

A first idea to show this result could be to use a standard regularization argument, such as the mollification with a smoothing kernel. Unfortunately this does not work, because it would of course produce smooth functions, but not diffeomorphisms. More precisely, a mollification can work only if the second derivatives of u are bounded, which is of course never the case for a piecewise affine function! This is why we will need the *ad hoc* construction presented here; notice that, for what just said, to prove Theorem A we could also limit ourselves to find an approximation v which is a C^2 diffeomorphism, because then the mollification procedure would complete the proof. However, we prefer to build directly a smooth diffeomorphism, since it does not require any particular care more than what the C^2 case would.

The strategy of the proof

In this introductory section we will describe the overall strategy of the proof of Theorem A.

First of all let us observe that, obviously, the map u is already smooth inside each triangle of the triangulation, since it is affine. Therefore, the whole construction has to deal with the problem that, along the sides of the triangles, the derivative of u has a discontinuity. Let us then concentrate our attention on a single triangle and its neighbours: our idea is to select a tiny region around the boundary of the triangle and to let v be a modification of u in this region, while it will be simply $v \equiv u$ in the big part of the triangle.

Actually, as one can imagine, the real difficulty is around to the vertices. In fact, along a side only two different triangles meet, and even if Du has a discontinuity, this is not "too bad", since the two matrices representing Du in the two triangles are different but rank-one connected (since u is continuous). On the other hand, around a vertex an arbitrary number of triangles can meet, and the different matrices representing Du can be also very different from each other. For this reason, we need an additional care close to the vertices; hence, the region around the boundary of the triangle that we addressed before will be further divided in different parts, either around the vertices or around the internal parts of the sides.

Let us now be more precise; as Figure 1.3 shows, it will be convenient to subdivide a triangle and the region around it in four zones, that we will call Z_1, Z_2, Z_3 and Z_4. The first two zones are concentric disks around the vertices, while the third one is done by rectangoloids along the sides, and the fourth one is the remaining part of the triangle. The images under v of these four zones will be correspondingly defined as $\widetilde{Z}_1$, $\widetilde{Z}_2$, $\widetilde{Z}_3$ and $\widetilde{Z}_4$. Let us discuss separately the role of each zone.

In the zone Z_1, we deal with the fact that many different triangles meet at a given vertex, say the point a in the figure, and then close to each vertex there are very different matrices Du. As a consequence, the value of Dv at the point a will have necessarily nothing to do with these matrices (and hence, for simplicity we will just choose $Dv(a)$ to be the identity matrix). Then, we call Z_1 a tiny small disk around a, and $\widetilde{Z}_1$ a correspondingly small disk around $u(a)$: the map $v : Z_1 \to \widetilde{Z}_1$ will only take care of the directions of the different sides. More precisely, working in polar coordinates, v will act non trivially only on the angular coordinate of the points. In this way, we will obtain the following; let us take any side of some triangle starting at a, say ab, and consider its image under v within the zone Z_1: while at the beginning this image goes in the same direction as ab, because $Dv(a) = \mathrm{Id}$, at the end of the disk Z_1 this image will go correctly in the direction of the side $u(a)u(b)$.

The zone Z_2 will then be an annulus centered at a and whose internal disk is Z_1, and it is easy to understand what is the goal of this zone. In fact, observe that the image of the (external part of the) boundary ∂Z_2, under the map u, is clearly the union of arcs of ellipses, one in the intersection of ∂Z_2

with each of the triangles which meet at a. This union is continuous, because so is u, but it is nothing better than continuous because there are corners where two different arcs meet. We will then aim the image of ∂Z_2 under v to be a smooth curve very close to this union of arcs; to do so, the map v will this time act non trivially only on the radial coordinate of the points. It will be possible to define v in such a way that it matches smoothly with the map already defined in Z_1; moreover, close to the external boundary of Z_2, v will be equal to u in the part of each triangle which is not too close to the sides (precisely, this happens close to $\partial Z_2 \cap \partial Z_4$, compare with Figure 1.3), while it will be a smooth matching between every two different affine maps close to the sides (precisely, this happens close to $\partial Z_2 \cap \partial Z_3$). Notice that the zone $\widetilde{Z}_2 = v(Z_2)$ is a sort of annulus around each vertex $u(a)$, whose internal boundary is a circle (namely, $\partial \widetilde{Z}_1$), while the external boundary is a curve which is done by different arcs of ellipses smoothly joined together.

Furthermore, the zone Z_3 is a sort of rectangoloid around the internal part of each side of some triangle, say ab, whose "short sides" are small arcs of the circles ∂Z_2, and whose "long sides" are two segments parallel to ab. Since we want to set $v \equiv u$ in the big remaining zone Z_4, the values of v at ∂Z_3 are already forced: namely, it must be $v = u$ around the long sides of each rectangoloid, while around the short sides v must coincide with the approximation defined in the zone Z_2. The definition of v on Z_3 will then be a careful interpolation of the values at the boundary, and we basically have only to make it in such a way that v remains injective. This will be technically a bit complicate, but actually this is just mainly because we will have to change our system of coordinates: indeed, while around the short sides of Z_3 the map v has been defined using a polar system centered at a (for the left short side) and at b (for the right short side), in the whole rectangoloid we will have of course to use a different system, namely, a standard orthogonal cartesian system.

Finally, as already said, we will simply set $v \equiv u$ on the big portion Z_4 of the triangle. In this way, we will have been able to build a smooth function v, which is still bi-Lipschitz (no more with constant L, but now with constant $100L^{7/3}$), and which is equal to u on every vertex of some triangle. At this point, the proof of Theorem A will be basically over, because all the properties follow immediately from the construction. In particular, the main estimate (1.5), which says that u and v are very close from the "energy" point of view, follows trivially from the fact that the zones Z_1, Z_2 and Z_3 described above can be done as small as we wish –without affecting the bi-Lipschitz constant of v! Therefore the zones Z_4, on which v and u coincide, cover an arbitrarily large percentage of the set Ω, from which the validity of (1.5) is obvious.

1.2.1 Preliminary ingredients

Before entering in the core of our proof, we briefly introduce some notation which will be used only within this part and we present some preliminary, yet fundamental, observations. In next section, we will go into the proof of Theorem A.

Notations and first geometric properties

Given a point $a \equiv (x_a, y_a) \in \mathbb{R}^2$, $\rho > 0$ and $\theta \in \mathbb{S}^1$, we will denote by $(\rho,\theta)_{P,a}$ the point in $\mathbb{R}^2$ whose polar coordinates with respect to a are ρ and θ, that is, the point $\big(x_a + \rho\cos\theta, y_a + \rho\sin\theta\big)$. Since we will use it several times, we recall the formula of the derivative of a function in polar coordinates: given two points $a \in \Omega$, $\tilde a \in \Delta$ and a C^1 function $F : \Omega \to \Delta$ locally expressed as $F\big((\rho,\theta)_{P,a}\big) = \Big(F_1(\rho,\theta),\, F_2(\rho,\theta)\Big)_{P,\tilde a}$, for any $z \in \Omega$ satisfying $z \neq a$, $F(z) \neq \tilde a$ one has (up to a rotation)

$$DF(z) = \begin{pmatrix} \dfrac{\partial F_1}{\partial\rho} & \dfrac{1}{\rho}\,\dfrac{\partial F_1}{\partial\theta} \\ \tilde\rho\,\dfrac{\partial F_2}{\partial\rho} & \dfrac{\tilde\rho}{\rho}\,\dfrac{\partial F_2}{\partial\theta} \end{pmatrix}, \tag{1.7}$$

calling $\rho = |z-a|$ and $\tilde\rho = |F(z) - \tilde a|$.

Let us now consider a vertex $a \in \Omega$ of the triangulation, and let us call T_i, $1 \leq i \leq N$, the triangles having a as one vertex. Let $\delta = \delta(a)$ be a positive constant, much smaller than the inradius of each of the triangles T_i with $1 \leq i \leq N$ (this is possible since the triangles can be countably many, but those meeting at a are only a finite number). Define $\tau_0 : \mathbb{S}^1 \to (0,\infty)$ and $\varphi_0 : \mathbb{S}^1 \to \mathbb{S}^1$ so that

$$u\big((\delta,\theta)_{P,a}\big) = \big(\delta\tau_0(\theta), \varphi_0(\theta)\big)_{P,u(a)}\,.$$

Hence, $(\delta\tau_0, \varphi_0)_{P,u(a)}$ is the image of the circle $(\delta,\theta)_{P,a}$, which by definition is the finite union of arcs of ellipses. Notice that by (1.7) we have

$$Du(x) = \begin{pmatrix} \tau_0 & \tau_0' \\ 0 & \varphi_0'\tau_0 \end{pmatrix}, \qquad Du^{-1}\big(u(x)\big) = \begin{pmatrix} \dfrac{1}{\tau_0} & -\dfrac{\tau_0'}{\varphi_0'\tau_0^2} \\ 0 & \dfrac{1}{\varphi_0'\tau_0} \end{pmatrix}, \tag{1.8}$$

up to a rotation.

Lemma 1.11. *For every $\theta \in \mathbb{S}^1$ one has*

$$\frac{1}{L} \leq \tau_0(\theta) \leq L\,, \qquad \big|\tau_0'(\theta)\big| \leq L\,, \qquad \frac{1}{L^2} \leq \varphi_0'(\theta) \leq L^2\,. \tag{1.9}$$

Proof. Take $\theta \in \mathbb{S}^1$, let $1 \leq i \leq N$ be such that $(\delta,\theta)_{P,a} \in T_i$, and call $M \in \mathbb{R}^{2\times 2}$ the matrix representing Du in the triangle i. By singular value decomposition, $M = RAQ$ with $R, Q \in SO(2)$ and

$$A = \begin{pmatrix} L_i & 0 \\ 0 & \ell_i \end{pmatrix}, \tag{1.10}$$

being $1/L \leq \ell_i \leq L_i \leq L$ the minimum and the maximum of $|M(\vartheta)|$ for $\vartheta \in \mathbb{S}^1$. Hence

$$\tau_0(\theta) = \big|AQ(\cos\theta, \sin\theta)\big| \in \Big[\min_{\nu\in\mathbb{S}^1} |A\nu|\,, \max_{\nu\in\mathbb{S}^1} |A\nu|\Big] = \big[\ell_i\,, L_i\big]\,.$$

Let now $\vartheta_0\,, \vartheta_1 \in \mathbb{S}^1$ be the angles corresponding to the rotations Q and R respectively. Then $\tau_0(\theta) = \big|\big(L_i\cos(\theta+\vartheta_0)\,, \ell_i\sin(\theta+\vartheta_0)\big)\big|$, and, hence,

$$\max_{\theta\in\mathbb{S}^1} \big|\tau_0'(\theta)\big| \leq \Big(L_i^2 - \ell_i^2\Big) \max_{\theta\in\mathbb{S}^1} \frac{\cos\theta\,\sin\theta}{\sqrt{L_i^2\cos^2\theta + \ell_i^2\sin^2\theta}} \leq \frac{L_i^2 - \ell_i^2}{\ell_i} \leq L_i\,.$$

Concerning φ_0, by definition one has

$$\varphi_0(\theta) = \arctan\left(\frac{\ell_i\sin(\theta+\vartheta_0)}{L_i\cos(\theta+\vartheta_0)}\right) + \vartheta_1\,,$$

so that a simple calculation gives

$$\varphi_0'(\theta) = \frac{L_i\ell_i}{L_i^2\cos^2(\theta+\vartheta_0) + \ell_i^2\sin^2(\theta+\vartheta_0)} \in \left(\frac{\ell_i}{L_i}\,, \frac{L_i}{\ell_i}\right) \subseteq \left(\frac{1}{L^2}\,, L^2\right).$$

Putting together all the previous estimates yields (1.9). □

Recall that we want to build a smooth approximation v of u; therefore, we are going to replace the functions τ_0 and φ_0 (which do not remain smooth across the sides of the triangles) with smooth functions τ and φ. Calling θ_i the directions corresponding to the sides of the triangles, we fix small constants $\lambda_i \ll |\theta_i - \theta_{i+1}|$, that will be better precised later. A simple mollification argument gives the following result.

Lemma 1.12. *There exist a* C^∞ *function* $\tau : \mathbb{S}^1 \to (0,\infty)$ *and a* C^∞ *diffeomorphism* $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ *so that*

(i) $\tau(\theta_i) = \tau_0(\theta_i)$, $\varphi(\theta_i) = \varphi_0(\theta_i)$, $\tau'(\theta_i) = 0$ *and* $\varphi'(\theta_i) = \frac{\max\{L_i, L_{i+1}\}}{\tau_0(\theta_i)}$ *;*
(ii) $\tau \equiv \tau_0$ *and* $\varphi \equiv \varphi_0$ *in* $\Big\{\theta \in \mathbb{S}^1 : |\theta - \theta_i| \geq \lambda_i \ \forall\, 1 \leq i \leq N\Big\}$ *;*
(iii) $\frac{1}{2}\,\varphi_0'(\theta) \leq \varphi'(\theta) \leq L^2$, $\frac{1}{2}\,\tau_0(\theta) \leq \tau(\theta) \leq 2\tau_0(\theta)$ *and* $\tau(\theta)\varphi'(\theta) \leq 2L$ *for every* $\theta \in \mathbb{S}^1$ *;*
(iv) $|\tau'(\theta)| \leq 2L_i$ *for every* $\theta \in \mathbb{S}^1$ *such that* $\big(\delta,\theta\big)_{P,a} \in T_i$ *.*

Proof. The result immediately comes by suitably modifying a regularization of τ_0 and φ_0, as soon as one checks that, for every $1 \le i \le N$ and every $\theta \in \mathbb{S}^1$,

$$\frac{\max\{L_i, L_{i+1}\}}{\tau_0(\theta_i)} \le L^2\,, \quad \varphi_0'(\theta) \le L^2\,, \quad \tau_0(\theta)\varphi_0'(\theta) \le L\,, \quad \big|\tau_0'(\theta)\big| \le L\,,$$

and all these properties are true thanks to (1.9), and by the fact that $\tau_0^2\varphi_0' = \ell_i L_i$, which in turn is immediate comparing (1.8) and (1.10). □

The auxiliary function ξ

In our construction, we will extensively use a map ξ which allows to connect in a smooth way two functions having different boundary values. More precisely, we fix six real parameters x_0, x_1, y_0, y_1, α and β, with $x_0 < x_1$, and we are seeking for a smooth function $\xi[x_0, x_1, y_0, y_1, \alpha, \beta] : [x_0, x_1] \to \mathbb{R}$ satisfying

$$\xi(x_0) = y_0\,, \qquad \xi(x_1) = y_1\,, \qquad \xi'(x_0) = \alpha\,, \qquad \xi'(x_1) = \beta\,. \tag{1.11}$$

It is obvious that it is possible to build such a function, but we want ξ to depend smoothly on the six parameters, and we also need a good estimate on the partial derivatives of ξ (which will be denoted by $\xi_{,i}$ for $1 \le i \le 6$). In fact, by means of a convolution on a suitable piecewise affine function, it is very simple to give a definition of ξ satisfying the properties below; $K \ge 6$ is an additional free parameter that do not enter in (1.11), but which effects the definition of ξ. Figure 1.2 depicts how the function ξ looks like.

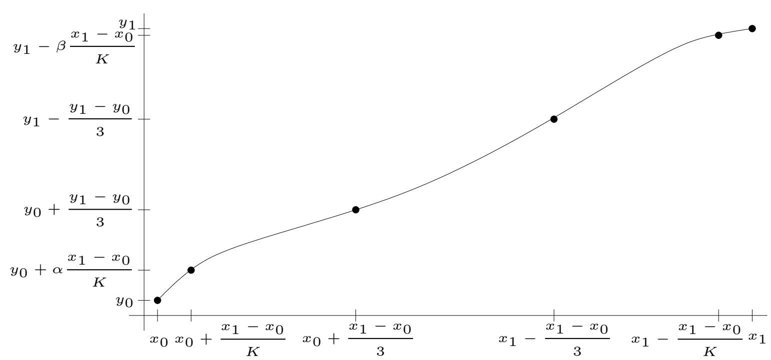

Fig. 1.2. Graph of ξ.

Lemma 1.13. *Let x_0, x_1, y_0, y_1, α, $\beta \in \mathbb{R}$ be such that $x_0 < x_1$. The following properties hold:*

(i) $\big|\xi'\big| \le \max\left\{\big|\alpha\big|,\, \big|\beta\big|,\, 2\,\dfrac{y_1 - y_0}{x_1 - x_0}\right\}$.

(ii) *If* $0 \leq \alpha,\, \beta \leq \dfrac{K(y_1 - y_0)}{6(x_1 - x_0)}$, *then* $\xi' \geq \min\left\{\alpha,\, \beta,\, \dfrac{y_1 - y_0}{2(x_1 - x_0)}\right\}$.

(iii) $0 \leq \xi_{,3} \leq 1$, $0 \leq \xi_{,4} \leq 1$, *and* $\xi_{,3} + \xi_{,4} = 1$. *Moreover,* $\xi_{,3}(t) \leq \dfrac{x_1 - t}{x_1 - x_0}$, *while* $\xi_{,4}(t) \leq \dfrac{t - x_0}{x_1 - x_0}$.

(iv) $0 \leq \xi_{,5} \leq \dfrac{x_1 - x_0}{K}$, *and* $\xi_{,5}(t) = 0$ *for* $t \geq x_0 + \dfrac{1}{3}(x_1 - x_0)$. *Similarly,* $0 \geq \xi_{,6} \geq -\dfrac{x_1 - x_0}{K}$, *and* $\xi_{,6}(t) = 0$ *for* $t \leq x_0 + \dfrac{2}{3}(x_1 - x_0)$.

(v) $|\xi_{,1}|,\, |\xi_{,2}| \leq \dfrac{|\xi'|}{x_1 - x_0}$.

1.2.2 Construction

In this section, we will describe with most details the construction of v, and we will give the proof of Theorem A. The complete proof (of a more general result) can be found in [43].

We will consider that the triangulation is not finite, and in particular that the union of the closed triangles is Ω (while in general it might arrive up to its closure): by the locally finiteness, this means that the triangles become more and more dense close to the boundary of Ω; this is clearly always true up to refine the triangulation, and it is also very useful because then all the triangles are compactly contained in Ω and this allows not to bother with $\partial\Omega$.

Taken a vertex a of the triangulation, we will work in the zone surrounding a, which will be divided in the four zones Z_1, Z_2, Z_3 and Z_4; on each zone, we will give a different definition of the map v. Thus, we will have to take care that v is smooth in each zone, but also that it remains smooth in passing from a zone to the adjacent one. We fix once for all the parameter $q \in [0, 1]$.

The zone Z_1

Recall that $\delta = \delta_a$ has been defined in Section 1.2.1 as a length much smaller than the inradius of each of the finitely many triangles meeting at a; we now fix another constant, $\eta = \eta_a$, much smaller than $1/L^2$. We call then

$$Z_1 = \mathcal{B}(a, \eta\delta)\,, \qquad\qquad \widetilde{Z}_1 = \mathcal{B}(\tilde{a}, \eta\delta L^{1-2q})\,,$$

being $\tilde{a} := u(a)$. We define then $v : Z_1 \to \widetilde{Z}_1$, using polar coordinates as in Section 1.2.1, as

$$v\big((\rho,\theta)_{P,a}\big) := \Big(L^{1-2q}\,\rho, \xi[0, \eta\delta, \theta, \varphi(\theta), 0, 0](\rho)\Big)_{P,\tilde{a}}\,.$$

The estimate that we can show for zone Z_1 is the following one.

Lemma 1.14. *The function $v : Z_1 \to \widetilde{Z}_1$ is a smooth bijection, and for every point $x \in Z_1$ one has*

$$\big|Dv\big| \leq 13L^{3-2q}\,, \quad |Dv^{-1}| \leq 26L^{1+2q}\,, \quad \det Dv(x) \geq \frac{\det Du(x)}{2L^{4q}}\,. \qquad (1.12)$$

Proof. The smoothness of v (in particular, around the pole a) follows by the definition and the properties of ξ; hence, to conclude that v is a bijection between Z_1 and $\widetilde{Z}_1$ it suffices to prove that, for each $\rho \in (0, \eta\delta)$, the smooth function $F(\theta) := \xi[0, \eta\delta, \theta, \varphi(\theta), 0, 0](\rho)$ is a bijection from $\mathbb{S}^1$ to itself. In fact, one has $F' = \xi_{,3} + \xi_{,4}\varphi'$ by definition, $\varphi' > 0$ by Lemma 1.12, $\int_0^{2\pi} \varphi' = 2\pi$ by construction, and $0 \leq \xi_{,3},\, \xi_{,4} \leq 1$, $\xi_{,3} + \xi_{,4} = 1$ by *(iii)* of Lemma 1.13. This yields that $F' > 0$ and that $\int_0^{2\pi} F' < 4\pi$, thus it must be $\int_0^{2\pi} F' = 2\pi$ and then the first part of the thesis is concluded.

Let us now pass to consider Dv and Dv^{-1}, which by formula (1.7) are given by

$$Dv = L^{1-2q} \begin{pmatrix} 1 & 0 \\ \rho\xi' & \xi_{,3} + \xi_{,4}\varphi' \end{pmatrix}\,, \quad Dv^{-1} = L^{2q-1} \begin{pmatrix} 1 & 0 \\ -\frac{\rho\xi'}{\xi_{,3}+\xi_{,4}\varphi'} & \frac{1}{\xi_{,3}+\xi_{,4}\varphi'} \end{pmatrix}\,.$$

Concerning ξ', by (i) of Lemma 1.13 and since in this case $\alpha = \beta = 0$ we know that

$$\big|\xi'\big| \leq 2\,\frac{\varphi(\theta) - \theta}{\eta\delta} \leq 4\,\frac{\pi}{\eta\delta}\,,$$

and then $\big|\rho\xi'\big| \leq 4\pi$. Moreover, by Lemma 1.12 and (1.9) we get

$$\big|\xi_{,3} + \xi_{,4}\varphi'\big| \leq |\varphi'| \leq L^2\,, \qquad \big|\xi_{,3} + \xi_{,4}\varphi'\big| \geq \frac{\ell_i}{2L_i} \geq \frac{1}{2L^2}\,,$$

where the last bound holds for every $\theta \in \mathbb{S}^1$ such that $(\delta\theta_0, \varphi_0)_{P,a} \in T_i$. By these estimates, and also recalling (1.10), we get

$$\begin{aligned} |Dv| &\leq L^{1-2q}\sqrt{1 + 16\pi^2 + L^4} \leq 13L^{3-2q}\,, \\ |Dv^{-1}| &\leq L^{2q-1}\sqrt{1 + 4L^4\big(1 + 16\pi^2\big)} \leq 26\,L^{1+2q} \\ \det Dv(x) &= L^{2-4q}\big(\xi_{,3} + \xi_{,4}\varphi'\big) \geq L^{2-4q}\,\frac{\ell_i}{2L_i} \geq \frac{\ell_i L_i}{2L^{4q}} = \frac{\det Du(x)}{2L^{4q}}\,, \end{aligned}$$

hence (1.12) is established and the proof is concluded. □

Remark 1.15. For later convenience we observe that, by the properties of ξ, for $\rho \leq \eta\delta$ close enough to $\eta\delta$ one has

$$v\big((\rho, \theta)_{P,a}\big) = \Big(L^{1-2q}\,\rho, \varphi(\theta)\Big)_{P,\tilde{a}}\,.$$

The zone Z_2

We now pass to the zone Z_2, which is the annulus $\mathcal{B}(a,\delta)\setminus\mathcal{B}(a,\eta\delta)$ surrounding Z_1, while the zone $\widetilde{Z}_2$ is defined as

$$\widetilde{Z}_2 := \Big\{(\rho,\theta)_{P,\tilde a} : \eta\delta L^{1-2q} \leq \rho \leq \delta\tau(\varphi^{-1}(\theta))\Big\}\,.$$

Also $\widetilde{Z}_2$ is surrounding $\widetilde{Z}_1$, since η has been chosen much smaller than $1/L^2$. The function $v : Z_2 \to \widetilde{Z}_2$ can be then set as

$$v\big((\rho,\theta)_{P,a}\big) := \Big(\xi[\eta\delta,\delta,\eta\delta L^{1-2q},\tau(\theta)\delta, L^{1-2q},\tau(\theta)](\rho)\ ,\ \varphi(\theta)\Big)_{P,\tilde a}\,.$$

Let us immediately check the behaviour of v close to the boundary of Z_2.

Remark 1.16. By the properties of ξ one readily observes the formulas

$$v\big((\rho,\theta)_{P,a}\big) = \Big(L^{1-2q}\,\rho,\varphi(\theta)\Big)_{P,\tilde a}\,,\qquad v\big((\rho,\theta)_{P,a}\big) = \Big(\rho\,\tau(\theta),\varphi(\theta)\Big)_{P,\tilde a}\,,$$

respectively valid for $\rho\geq\eta\delta$ close enough to $\eta\delta$, and for $\rho\leq\delta$ close enough to δ.

Let us now prove the estimates for v inside Z_2.

Lemma 1.17. *The function $v : Z_2 \to \widetilde{Z}_2$ is a smooth bijection, which matches smoothly with the function v previously defined in the zone Z_1. Moreover, for every $x\in Z_2$ it is*

$$\big|Dv\big| \leq 3L^{3-2q}\,,\quad |Dv^{-1}|\leq 33L^{1+4q}\,,\quad \det Dv(x) \geq \frac{\det Du(x)}{16L^{4q}}\,. \tag{1.13}$$

Proof. We begin observing that v is smooth because so is ξ; moreover, the two definitions of v around ∂Z_1 coincide, as one can see comparing Remarks 1.15 and 1.16. By definition, $v(\partial Z_2)=\partial\widetilde{Z}_2$, thus –since $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ is a bijection– v is a bijection of Z_2 onto $\widetilde{Z}_2$ if and only if $\xi'>0$ on the whole Z_2.

By (1.7) we evaluate

$$Dv = \begin{pmatrix} \xi' & \dfrac{1}{\rho}\,(\delta\xi_{,4}+\xi_{,6})\tau'(\theta) \\ 0 & \dfrac{\xi}{\rho}\,\varphi'(\theta)\end{pmatrix},\quad Dv^{-1} = \begin{pmatrix} \dfrac{1}{\xi'} & -\dfrac{(\delta\xi_{,4}+\xi_{,6})\tau'(\theta)}{\xi'\xi\varphi'(\theta)} \\ 0 & \dfrac{\rho}{\xi\varphi'(\theta)}\end{pmatrix},$$

therefore to obtain (1.13) we need to give bounds to ξ', ξ/ρ, $\xi_{,4}$ and $\xi_{,6}$. Concerning ξ', property (ii) of Lemma 1.13 tells us that

$$\xi' \geq \min\left\{L^{1-2q},\,\tau(\theta),\,\frac{\tau(\theta)-\eta L^{1-2q}}{2(1-\eta)}\right\} \tag{1.14}$$

as soon as

$$\frac{\max\left\{L^{1-2q},\,\tau(\theta)\right\}(1-\eta)}{K} \leq \frac{\tau(\theta)-\eta L^{1-2q}}{6}\,.$$

Since we can assume the last inequality to be true, up to choose $K \geq 24L^2$, we get the validity of (1.14). We have then

$$\frac{1}{4L^{1+2q}} \leq \frac{\tau(\theta)}{2L^{2q}} \leq \xi' \leq L\ : \tag{1.15}$$

the lower bound for ξ' is a consequence of (1.14), recalling (1.9), Lemma 1.12 and the fact that $\eta \ll 1/L^2$, while the upper bound is clear by construction. A first consequence of (1.15) is that $\xi' > 0$ and then, as pointed out before, v is a bijection between Z_2 and $\widetilde{Z}_2$.

We study now the ratio ξ/ρ: by construction of ξ, one directly obtains

$$\tau(\theta)L^{-2q} \leq \min\left\{L^{1-2q},\,\tau(\theta)\right\} \leq \frac{\xi}{\rho} \leq \max\left\{L^{1-2q},\,\tau(\theta)\right\}.$$

It is easy to see that this implies

$$\tau(\theta)L^{-2q}\varphi'(\theta) \leq \frac{\xi}{\rho}\,\varphi'(\theta) \leq 2L^{3-2q}\,. \tag{1.16}$$

Indeed, while the lower bound is an immediate consequence of the above estimate, for the upper bound we have to distinguish two cases: if $L^{1-2q} \leq \tau(\theta)$, then by (iii) of Lemma 1.12

$$\frac{\xi}{\rho}\,\varphi'(\theta) \leq \tau(\theta)\varphi'(\theta) \leq 2L \leq 2L^{3-2q}\,;$$

on the other hand, if $L^{1-2q} \geq \tau(\theta)$, then

$$\frac{\xi}{\rho}\,\varphi'(\theta) \leq L^{1-2q}\varphi'(\theta) \leq L^{3-2q}\,.$$

Let us now pass to consider $\xi_{,4}$ and $\xi_{,6}$: first, by (iii) and (iv) of Lemma 1.13, we get

$$\Big|\delta\xi_{,4}(\rho)+\xi_{,6}(\rho)\Big| \leq \rho\,; \tag{1.17}$$

second, (1.17), (1.16), (iii) of Lemma 1.12, (1.15) and (1.9) yield

$$\begin{aligned}\left|\frac{(\delta\xi_{,4}+\xi_{,6})\tau'(\theta)}{\xi'\xi\varphi'(\theta)}\right| &\leq \frac{\rho|\tau'(\theta)|}{\xi'\xi\varphi'(\theta)} \leq \frac{|\tau'(\theta)|L^{2q}}{\xi'\varphi'(\theta)\tau(\theta)} \leq \frac{2L_iL^{2q}}{\xi'\varphi'(\theta)\tau(\theta)}\\ &\leq \frac{4L_iL^{4q}}{\varphi'(\theta)\tau^2(\theta)} \leq \frac{32L_iL^{4q}}{\varphi_0'(\theta)\tau_0^2(\theta)} = \frac{32L^{4q}}{\ell_i} \leq 32L^{1+4q}\,.\end{aligned} \tag{1.18}$$

Putting together (1.15), (1.16), (1.17) and (1.18), we obtain then

$$|Dv| \leq \sqrt{L^2 + \tau'(\theta)^2 + \left(2L^{3-2q}\right)^2} \leq 3L^{3-2q}\,,$$

$$|Dv^{-1}| \leq \sqrt{16L^{2+4q} + \left(32L^{1+4q}\right)^2 + 16L^{2+4q}} \leq 33L^{1+4q}\,,$$

$$\det Dv(x) = \frac{\xi\xi'\varphi'(\theta)}{\rho} \geq \frac{\tau^2(\theta)\varphi'(\theta)}{2L^{4q}} \geq \frac{\tau_0^2(\theta)\varphi_0'(\theta)}{16L^{4q}} = \frac{L_i\ell_i}{16L^{4q}} = \frac{\det Du(x)}{16L^{4q}}\,;$$

since this is exactly the searched estimate (1.13), we have concluded. □

The zone Z_3

It is now time to pass to consider zone Z_3. While in zones Z_1 and Z_2 we have worked around the vertices of the triangulation, zone Z_3 will be around the sides. Recall that, as we pointed out in the introduction to this section, since the map u is affine –hence, smooth– in every triangle, the only “problem” are the irregularity across the sides. In fact, once we will have defined v also in Z_3, thus ruling out vertices and sides, we will simply define $v \equiv u$ in the remaining part of Ω, which will be called zone Z_4. It is also to be pointed out that Z_1, Z_2 and Z_3 will be chosen to be very small neighborhoods of the sides, thus eventually $v \equiv u$ in almost the whole set Ω. Figure 1.3 depicts how the different zones look like inside the triangle $T_i = abc$; the figure is intended only to describe the idea, but in fact the construction will be slightly different: more precisely, the common boundary of Z_3 and Z_4 will be a suitable curve instead of a segment.

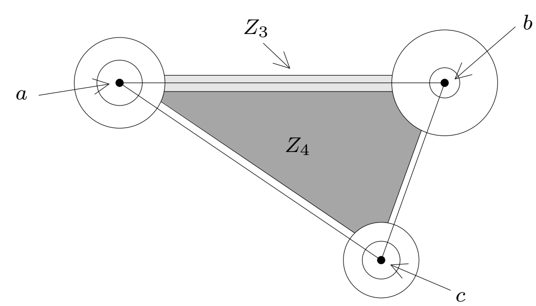

Fig. 1.3. A rough idea of the zones Z_3 and Z_4 for a triangle T_i (the small zones around the vertices are Z_1 and Z_2).

As the figure shows, the zone Z_3 is made by N disjoint narrow “stripes” around the different sides of the triangulation. Thus, we will focus on a single one, namely, we are going to work around the side ab. By the previous sections, we have already defined v in the zones Z_1 and Z_2 corresponding to a and to b; to distinguish, we will use the subscripts a and b for all the constants and the functions, for instance we will write δ_a and δ_b, $\lambda_{i,a}$ and $\lambda_{i,b}$, τ_a and τ_b and so on. To fix the ideas, we assume that both the sides ab and $\tilde a\tilde b = u(a)u(b)$ are horizontal, as well as that a and $\tilde a$ coincide with the origin of $\mathbb{R}^2$. We also call $T_{i,a} = T_{j+1,b}$ and $T_{i+1,a} = T_{j,b}$ the lower and the upper triangle having ab as a side.

We take now a small constant $h = h(a,b)$, much smaller both than δ_a and than δ_b. Since all the different constants λ are independent, we assume that

$$\begin{aligned} &\delta_a \sin\lambda_{i,a} < h\,, &\qquad &\delta_a \sin\lambda_{i+1,a} < h\,,\\ &\delta_b \sin\lambda_{j,b} < h\,, &\qquad &\delta_b \sin\lambda_{j+1,b} < h\,. \end{aligned} \tag{1.19}$$

Consider now, as in Figure 1.4 (left), P, Q, R and S the points having distance h from ab and belonging to the circles $\partial\mathcal{B}(a,\delta_a)$ and $\partial\mathcal{B}(b,\delta_b)$. Since u is affine in $T_{i,a} = T_{j+1,b}$ and ab is sent into $\tilde a\tilde b$, then also $\widetilde P = u(P)$ and $\widetilde Q = u(Q)$ have the same distance from $\tilde a\tilde b$, say h^+. Similarly, $\widetilde R = u(R)$ and $\widetilde S = u(S)$ have distance h^- from $\tilde a\tilde b$: see Figure 1.4 (right). In general, $h^+ \neq h^-$. Observe that the image under u of the circle $\partial\mathcal{B}(a,\delta_a)$ (given by $\big(\delta\tau_{0,a},\theta_{0,a}\big)_{P,a}$ in polar coordinates) is the union of two ellipses which meet on $\tilde a\tilde b$ continuously but not necessarily in a C^1 way: this curve is shown continuous in the figure. Instead, the image of this circle under v, according to the construction of Section 1.2.2 and thanks to Remark 1.16, is the smooth curve $\big(\delta\tau_a,\varphi_a\big)_{P,\tilde a}$; this curve is shown dotted. Keep in mind that, thanks to assumption (1.19), the two curves coincide where $\tau_{0,a} = \tau_a$ and $\varphi_{0,a} = \varphi_a$: thus, they differ only between P and Q.

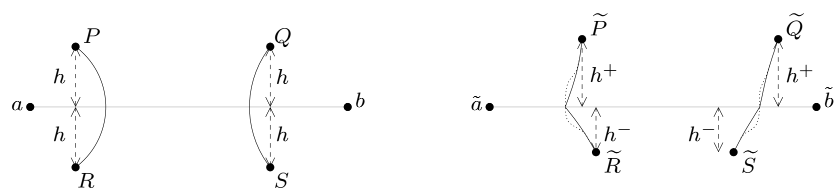

Fig. 1.4. First step in the construction of Zones Z_3 and $\widetilde Z_3$.

Observe now that, as a simple trigonometric argument shows (see Figure 1.5), the quantity $\tau_a(\theta_i)\varphi_a'(\theta_i)$ represents the speed at which the curve $\theta \mapsto v\big((\delta_a,\theta)_{P,a}\big)$ departs from the segment ab; moreover, since $\tau_a'(\theta_i) = 0$ by Lemma 1.12, this speed is vertical. Analogously, the departing speed of $\theta \mapsto v\big((\delta_b,\theta)_{P,b}\big)$ is $\tau_b(\theta_j)\varphi_b'(\theta_j)$. We can notice that the two speeds are equal, because recalling (i) of Lemma 1.12 it is

$$\begin{aligned} \tau_a(\theta_{i,a})\varphi_a'(\theta_{i,a}) &= \max\big\{L_{i,a},\, L_{i+1,a}\big\}\\ &= \max\big\{L_{j+1,b},\, L_{j,b}\big\} = \tau_b(\theta_{j,b})\varphi_b'(\theta_{j,b})\,. \end{aligned} \tag{1.20}$$

For every $-h \le t \le h$, we now denote by $P(t)$ the point on $\partial\mathcal{B}(a,\delta_a)$ having distance t from ab, so that $P = P(h)$ and $R = P(-h)$ in Figure 1.4. We define

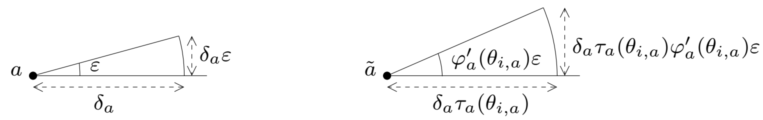

Fig. 1.5. Departing speed of the curve $\theta \mapsto v(\delta, \theta)$.

$Q(t)$ in the same way, having $Q = Q(h)$ and $S = Q(-h)$. Since the points $P(t)$ and $Q(t)$ belong to the boundary of the zones Z_2 corresponding to a and b respectively, we can also set $\widetilde{P}(t) := v\big(P(t)\big)$ and $\widetilde{Q}(t) := v\big(Q(t)\big)$. We require that the functions τ_a, τ_b, φ_a and φ_b are such that for every $t \in (-h, h)$ the distance of $\widetilde{P}(t)$ and $\widetilde{Q}(t)$ from $\tilde a \tilde b$ is the same, and we will call $\tilde t = \tilde t(t)$ this distance. Let us see why this is admissible: defining for every $t \in (-h, h)$ the angles $\theta_a(t)$ and $\theta_b(t)$ so that

$$\delta_a \sin\big(\theta_a(t)\big) = t = \delta_b \sin\big(\theta_b(t)\big)\,, \tag{1.21}$$

our request means

$$\delta_a\tau_a\big(\theta_a(t)\big)\sin\big(\varphi_a(\theta_a(t))\big) = \tilde t = \delta_b\tau_b\big(\theta_b(t)\big)\sin\big(\varphi_b(\theta_b(t))\big)\,. \tag{1.22}$$

In fact, this is admissible because (1.22) is true by definition at $t = 0$ and at $t = \pm h$ (since as we observed above one has $u = v$ on ∂Z_2 close to P, Q, R and S), and because the necessary condition which comes by differentiating (1.22) at $t = 0$, namely,

$$\delta_a\tau_a(0)\varphi'_a(0)\theta'_a(0) = \delta_b\tau_b(\pi)\varphi'_b(\pi)\theta'_b(0)\,,$$

is true. To show the validity of last equality it suffices to recall (1.20) and the fact that $\delta_a\theta'_a(0) = \delta_b\theta'_b(0) = 1$, which in turn is immediate by (1.21). Let us write down an estimate which will be useful later, and which comes by (1.22) and differentiating (1.21):

$$\begin{aligned}
\frac{\mathrm{d}\tilde t}{\mathrm{d}t} &= \delta_a\theta'_a\Big(\tau'_a(\theta_a)\sin(\varphi_a(\theta_a)) + \tau_a(\theta_a)\cos(\varphi_a(\theta_a))\varphi'_a(\theta_a)\Big)\\
&= \frac{1}{\cos\theta_a}\Big(\tau'_a(\theta_a)\sin(\varphi_a(\theta_a)) + \tau_a(\theta_a)\cos(\varphi_a(\theta_a))\varphi'_a(\theta_a)\Big)\\
&\approx \tau_a(\theta_a)\varphi'_a(\theta_a) \in \bigg(\frac{\ell_i}{4}, 3L_i\bigg)\,,
\end{aligned} \tag{1.23}$$

where we have used the facts that $\tau'_a \le 2L$ and that the angles $\varphi_a(\theta_a)$ can be assumed as small as needed up to take a small constant $h = h(a, b)$ –we will use the latter fact very often in the sequel. A consequence of (1.23) is that $t \mapsto \tilde t$ is a bijective map from $(-h, h)$ to $(-h^-, h^+)$.

Before defining Z_3 and $\widetilde{Z}_3$, we still need some more pieces of notations. First, for every $-h \le t \le h$ we call

$$\chi_0(t) := \delta_a \cos\theta_a(t)\,, \qquad \chi_1(t) := \overline{ab} - \delta_b \cos\theta_b(t)\,, \tag{1.24}$$

that is, $\chi_0(t)$ and $\chi_1(t)$ are the abscissa of $P(t)$ and $Q(t)$ respectively (by $\overline{ab}$ we mean the length of ab). As a consequence, the coordinates of the points $P(t)$ and $Q(t)$ are

$$P(t) \equiv \big(\chi_0(t), t\big)\,, \qquad Q(t) \equiv \big(\chi_1(t), t\big)\,.$$

Observe that in zones Z_1 and Z_2 it has been very useful to use polar coordinates, since the zones were centered in a particular point; on the other hand, in the present situation we are working around a segment, thus standard orthogonal coordinates are better. We define now the rectangoloid

$$\Gamma := \Big\{(\sigma, t) \in \mathbb{R}^2 :\ t \in (-h, h)\,,\, \sigma \in \big(\chi_0(t), \chi_1(t)\big)\Big\}\,,$$

whose boundary is made by two wide horizontal straight sides and two small circular lateral sides, and the map $\gamma : \Gamma \to \mathbb{R}^2$ given by

$$\gamma(\sigma, t) := \Big(\sigma, \xi[\chi_0(t), \chi_1(t), t, t, \tan\theta_a(t), -\tan\theta_b(t)](\sigma)\Big) = \Big(\sigma, \overline{\xi}(\sigma)\Big)\,.$$

Notice that, since in the construction of zone Z_3 we will use three different functions ξ, we are giving each of them a different name: we are working now with $\overline{\xi}$, then we will introduce $\tilde{\xi}$ and $\hat{\xi}$.

Figure 1.6 (left) shows the smooth curve $\sigma \mapsto \gamma(\sigma, t)$: notice that this is "almost" an horizontal curve connecting $P(t)$ and $Q(t)$; but in fact, it has a nonzero slope at the extremes in order to connect smoothly with the function defined in Z_2. The important properties of γ are shown now.

Lemma 1.18. *The map $\gamma : \Gamma \to \mathbb{R}^2$ is smooth, injective and satisfies*

$$\big|D\gamma\big| \approx 1\,, \qquad \big|D\gamma^{-1}\big| \approx 1\,, \qquad \det D\gamma(x) \approx 1\,. \tag{1.25}$$

Proof. First of all, let us calculate

$$\frac{\mathrm{d}\overline{\xi}}{\mathrm{d}t} = \overline{\xi}_{,1}\chi_0' + \overline{\xi}_{,2}\chi_1' + \overline{\xi}_{,3} + \overline{\xi}_{,4} + \overline{\xi}_{,5}\big(\tan\theta_a(t)\big)' - \overline{\xi}_{,6}\big(\tan\theta_b(t)\big)'\,. \tag{1.26}$$

By (iii) of Lemma 1.13 we know that $\overline{\xi}_{,3} + \overline{\xi}_{,4} = 1$; we are going to check that the other terms are, instead, arbitrarily small. In fact, $\overline{\xi}_{,1}$ and $\overline{\xi}_{,2}$ become as small as we wish if h is chosen small, thanks to (i) and (v) of Lemma 1.13 and since $\chi_1 - \chi_0 \approx \overline{ab}$; analogously, also χ_0' and χ_1' are small because by (1.24) and differentiating (1.21) we get

$$\chi_0'(t) = -\delta_a(t)\sin\big(\theta_a(t)\big)\theta_a'(t) = -\tan\big(\theta_a(t)\big)\,, \qquad \chi_1'(t) = \tan\big(\theta_b(t)\big)\,,$$

so $\overline{\xi}_{,1}\chi_0' + \overline{\xi}_{,2}\chi_1' \ll 1$. Concerning $\xi_{,5}$ and $\xi_{,6}$, notice that choosing a small h we get $\big(\tan\theta_a(t)\big)' \approx \theta_a' \approx 1/\delta_a$: as a consequence, and by (iv) of Lemma 1.13, if

K is big enough then also $\overline{\xi}_{,5}\big(\tan\theta_a(t)\big)' - \overline{\xi}_{,6}\big(\tan\theta_b(t)\big)' \ll 1$. And in turn, since in every different zone we can use a different K (because the value of ξ around its extremes do not depend on K), it is admissible to assume K as big as we need. These bounds and (1.26) ensure that $\mathrm{d}\overline{\xi}/\mathrm{d}t \approx 1$. Thus, by definition of $\overline{\xi}$ we get in particular that γ is injective. Since the regularity of γ is immediate from that of ξ, we are only left to show (1.25).

To do so, we observe that $\big|\overline{\xi}'\big| \leq \max\big\{|\tan\theta_a(t)|,\, |\tan\theta_b(t)|\big\} \ll 1$ by (i) of Lemma 1.13, and we calculate

$$D\gamma = \begin{pmatrix} 1 & \overline{\xi}' \\ 0 & \frac{\mathrm{d}\overline{\xi}}{\mathrm{d}t} \end{pmatrix}, \qquad D\gamma^{-1} = \begin{pmatrix} 1 & -\frac{\overline{\xi}'}{\mathrm{d}\overline{\xi}/\mathrm{d}t} \\ 0 & \frac{1}{\mathrm{d}\overline{\xi}/\mathrm{d}t} \end{pmatrix},$$

so that (1.25) follows. □

Let us now define $\widetilde{\Gamma} \subseteq \mathbb{R}^2$ and $\tilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$ in a very similar way as Γ and γ. First of all, we define $\widetilde{\chi}_0(t)$ and $\widetilde{\chi}_1(t)$ so that

$$\widetilde{P}(t) \equiv \big(\widetilde{\chi}_0(t), \tilde{t}\,\big), \qquad \widetilde{Q}(t) \equiv \big(\widetilde{\chi}_1(t), \tilde{t}\,\big),$$

that is,

$$\begin{aligned} \widetilde{\chi}_0(t) &:= \delta_a \tau_a(\theta_a(t)) \cos\big(\varphi_a\big(\theta_a(t)\big)\big), \\ \widetilde{\chi}_1(t) &:= \delta_b \tau_b(\theta_b(t)) \cos\big(\varphi_b\big(\theta_b(t)\big)\big). \end{aligned} \tag{1.27}$$

Then, we introduce the rectangoloid

$$\widetilde{\Gamma} := \Big\{(\sigma, \tilde{t}) \in \mathbb{R}^2 :\ \tilde{t} \in (-h^-, h^+),\, \sigma \in \big(\widetilde{\chi}_0(t), \widetilde{\chi}_1(t)\big)\Big\},$$

(recall that $t \mapsto \tilde{t}$ is a bijective map, so in this definition in fact $t = t(\tilde{t})$). Finally, we define the map $\tilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$ as

$$\tilde{\gamma}(\sigma, \tilde{t}) := \Big(\sigma, \tilde{\xi}(\sigma)\Big), \tag{1.28}$$

where

$$\tilde{\xi} = \xi\big[\widetilde{\chi}_0(t), \widetilde{\chi}_1(t), \tilde{t}, \tilde{t}, \tan\big(\varphi_a\big(\theta_a(t)\big)\big), -\tan\big(\varphi_b\big(\theta_b(t)\big)\big)\big].$$

Actually, we will eventually slightly modify the definition of $\tilde{\xi}$, the reason will be clear later but the modification will not effect our next proofs. Figure 1.6 (right) shows the curve $\tilde{\gamma}(\cdot, \tilde{t})$. We can extend Lemma 1.18 to the case of $\tilde{\gamma}$.

Lemma 1.19. *The map $\tilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$ is smooth, injective, and satisfies*

$$\big|D\tilde{\gamma}\big| \approx 1, \qquad \big|D\tilde{\gamma}^{-1}\big| \approx 1, \qquad \det D\tilde{\gamma}(x) \approx 1. \tag{1.29}$$

Fig. 1.6. The curves $\gamma(\cdot,t)$ and $\tilde\gamma(\cdot,\tilde t)$.

Proof. The proof is completely similar to that of Lemma 1.18, and it simply relies on showing that $\mathrm{d}\tilde\xi/\mathrm{d}\tilde t \approx 1$ and $\tilde\xi' \ll 1$. The second fact is again an immediate consequence of (i) of Lemma 1.13, so let us concentrate on the first one. Exactly as in Lemma 1.18, one observes that $\tilde\xi_{,3}+\tilde\xi_{,4}=1$, one rules out the terms with $\tilde\xi_{,5}$ and $\tilde\xi_{,6}$, and one observes that $\tilde\xi_{,1}$ and $\tilde\xi_{,2}$ are very small. The only difference with Lemma 1.18 is that in this case it is not true that $\widetilde\chi_0'$ and $\widetilde\chi_1'$ can be assumed to be arbitrarily small; however, since $\tilde\xi_{,1}$ and $\tilde\xi_{,2}$ are small, it is in fact enough to show that $\widetilde\chi_0'$ and $\widetilde\chi_1'$ are bounded. And in turn, this is true because by (1.27), and also using (iv) of Lemma 1.12,

$$\begin{aligned}\widetilde\chi_0' &= \delta_a\theta_a'\Big[\tau_a'(\theta_a)\cos\big(\varphi_a(\theta_a)\big)-\tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a)\Big]\\ &=\frac{1}{\cos(\theta_a)}\Big[\tau_a'(\theta_a)\cos\big(\varphi_a(\theta_a)\big)-\tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a)\Big]\\ &\approx\tau_a'(\theta_a)\cos\big(\varphi_a(\theta_a)\big)-\tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a)\approx\tau_a'(\theta_a)\,.\end{aligned}\tag{1.30}$$

$\square$

The last function that we need is $\Phi:\Gamma\to\widetilde\Gamma$, defined as

$$\Phi(\sigma,t):=\Big(\xi\big[\chi_0(t),\chi_1(t),\widetilde\chi_0(t),\widetilde\chi_1(t),\alpha(t),\beta(t)\big](\sigma),\tilde t\Big)=\Big(\hat\xi(\sigma),\tilde t\Big)\,,$$

where the constants α and β are given by

$$\alpha(t):=\tau_a\big(\theta_a(t)\big)\,\frac{\cos\big(\varphi_a(\theta_a(t))\big)}{\cos\big(\theta_a(t)\big)}\,,\quad \beta(t):=\tau_b\big(\theta_b(t)\big)\,\frac{\cos\big(\varphi_b(\theta_b(t))\big)}{\cos\big(\theta_b(t)\big)}\,.\tag{1.31}$$

Lemma 1.20. *The map Φ is a smooth bijection between Γ and $\widetilde\Gamma$. Moreover,*

$$\big|D\Phi\big|\leq 5L\,,\qquad \big|D\Phi^{-1}\big|\leq 35L\,,\qquad \det D\Phi(x)\geq\frac{\det Du(x)}{15}\,.\tag{1.32}$$

Proof. By construction, and observing that α and β are positive, one easily see that the image of Γ is exactly Γ, so Φ is a smooth onto function. Hence, to say that Φ is a smooth bijection we only need to check the injectivity, which by definition reduces to prove that $\hat\xi'>0$. If h is small enough, then $\alpha(t)\approx\beta(t)\approx\overline{\tilde a\tilde b}/\overline{ab}\approx\tau_a(\theta_a)$, hence by (i) and (ii) of Lemma 1.13 we deduce

$$\hat\xi'\approx\tau_a(\theta_a)\in\big(\ell_i,L_i\big)\,,\tag{1.33}$$

thus in particular the injectivity of Φ follows. To get (1.32), we write down

$$D\Phi = \begin{pmatrix} \hat\xi' & 0 \\ \dfrac{\mathrm{d}\hat\xi}{\mathrm{d}t} & \dfrac{\mathrm{d}\tilde t}{\mathrm{d}t} \end{pmatrix}, \qquad D\Phi^{-1} = \begin{pmatrix} 1/\hat\xi' & 0 \\ -\dfrac{\mathrm{d}\hat\xi/\mathrm{d}t}{\hat\xi'\cdot \mathrm{d}\tilde t/\mathrm{d}t} & \dfrac{1}{\mathrm{d}\tilde t/\mathrm{d}t} \end{pmatrix}. \tag{1.34}$$

and since we already have the estimates (1.33) for $\hat\xi'$ and (1.23) for $\mathrm{d}\tilde t/\mathrm{d}t$, we only need to take care of $\mathrm{d}\hat\xi/\mathrm{d}t = \hat\xi_{,1}\chi_0' + \hat\xi_{,2}\chi_1' + \hat\xi_{,3}\widetilde\chi_0' + \hat\xi_{,4}\widetilde\chi_1' + \hat\xi_{,5}\alpha' + \hat\xi_{,6}\beta'$.

Exactly as in the previous lemmas, all the terms are much smaller than those concerning $\hat\xi_{,3}$ and $\hat\xi_{,4}$. In fact, we already saw that χ_0' and χ_1' are arbitrarily small, and (iv) of Lemma 1.13 ensures that so are also $\hat\xi_{,5}$ and $\hat\xi_{,6}$. Thus, we only need to observe that $\hat\xi_{,1}$ and α' are bounded (since by symmetry the same will be true for $\hat\xi_{,2}$ and β'). Concerning $\hat\xi_{,1}$, we readily obtain the boundedness by (i) and (v) of Lemma 1.13 and by (1.33); on the other hand, concerning α' it is enough to recall that $\delta_a\theta_a' \approx 1$ and to derive (1.31) to get

$$\alpha' \approx \frac{\tau_a'(\theta_a)}{\delta_a},$$

thus the boundedness. Summarizing, also recalling (1.30) we get

$$\frac{\mathrm{d}\hat\xi}{\mathrm{d}t} \leq \frac{4}{3}\max\left\{\tau_a'(\theta_a),\, \tau_b'(\theta_b)\right\}. \tag{1.35}$$

Finally, assuming by symmetry that $\tau_a'(\theta_a) \geq \tau_b'(\theta_b)$, we can evaluate

$$\left|\frac{\mathrm{d}\hat\xi/\mathrm{d}t}{\hat\xi'\cdot \mathrm{d}\tilde t/\mathrm{d}t}\right| \leq 2\left|\frac{\tau_a'(\theta_a)}{\tau_a(\theta_a)^2\varphi_a'(\theta_a)}\right| \leq \frac{32L_i}{\tau_{a,0}(\theta_a)^2\varphi_{a,0}(\theta_a)} = \frac{32L_i}{\ell_i L_i} = \frac{32}{\ell_i}. \tag{1.36}$$

We conclude (1.32), thus the proof, just inserting the estimates (1.33), (1.23), (1.35) and (1.36) into (1.34). □

We are now in position to define the zones Z_3 and $\widetilde Z_3$ as

$$Z_3 := \gamma(\Gamma), \qquad\qquad \widetilde Z_3 := \tilde\gamma(\widetilde\Gamma),$$

and the function $v : Z_3 \to \widetilde Z_3$ as $v := \tilde\gamma\circ\Phi\circ\gamma^{-1}$. Putting together the results of Lemmas 1.18, 1.19 and 1.20, we then easily get the following result.

Lemma 1.21. *The function $v : Z_3 \to \widetilde Z_3$ is a smooth bijection, smoothly matching with the function v defined in the zone Z_2, and for every point $x \in Z_2$ one has*

$$\left|Dv\right| \leq 10L, \qquad |Dv^{-1}| \leq 70L, \qquad \det Dv(x) \geq \frac{\det Du(x)}{30}. \tag{1.37}$$

Proof. Thanks to Lemmas 1.18, 1.19 and 1.20, and to estimates (1.25), (1.29) and (1.32), we already know that v is a smooth bijection and that (1.37) holds. Thus, we only have to check that the two definitions of v around $\partial Z_2 \cap \partial Z_3$

match smoothly. Let us do this in the part corresponding to a, since the situation for b is clearly identical.

Recall that γ, $\tilde\gamma$ and Φ are affine for a while close to the respective boundaries; thus for $0<\varepsilon\ll 1$ an elementary calculation gives the exact formulas

$$\begin{aligned}
\gamma^{-1}\Big(\big(\delta_a+\varepsilon,\theta_a(t)\big)_{P,a}\Big) &= \Big(\chi_0(t)+\varepsilon\cos\theta_a(t),t\Big)\,,\\
\Phi\Big(\chi_0(t)+\varepsilon\cos\theta_a(t),t\Big) &= \Big(\widetilde\chi_0(t)+\tau_a\big(\theta_a(t)\big)\varepsilon\cos\big(\varphi_a\big(\theta_a(t)\big)\big),\tilde t\Big)\,,\\
v\Big(\big(\delta_a+\varepsilon,\theta_a(t)\big)_{P,a}\Big) &= \tilde\gamma\Big(\widetilde\chi_0(t)+\tau_a\big(\theta_a(t)\big)\varepsilon\cos\big(\varphi_a\big(\theta_a(t)\big)\big),\tilde t\Big)\\
&= \Big(\tau_a\big(\theta_a(t)\big)\big(\delta_a+\varepsilon\big),\varphi_a\big(\theta_a(t)\big)\Big)_{P,\tilde a}\,.
\end{aligned}$$

Since the latter, for $-1\ll\varepsilon<0$, is exactly the expression of v in Z_2 close to $\partial Z_2\cap\partial Z_3$, as pointed out in Remark 1.16, the proof is finished. $\square$

The last thing we have to do, is to determine the behaviour of v on $\partial Z_3\setminus\partial Z_2$, which will eventually coincide with $\partial Z_3\cap\partial Z_4$.

Lemma 1.22. *If $t<h$, $t\approx\pm h$, then for every $\sigma\in\big(\chi_0(t),\chi_1(t)\big)$ one has $v\big(\gamma(\sigma,t)\big)=u\big(\gamma(\sigma,t)\big)$.*

Proof. Take $t<h$, $t\approx h$, so that by (1.19) it is $\tau_a\big(\theta_a(t)\big)=\tau_{a,0}\big(\theta_a(t)\big)$ and $\varphi_a\big(\theta_a(t)\big)=\varphi_{a,0}\big(\theta_a(t)\big)$: for those t, Remark 1.16 ensures that $v=u$ around ∂Z_2, so by Lemma 1.21 we know $v\big(\gamma(\sigma,t)\big)=u\big(\gamma(\sigma,t)\big)$ as soon as σ is close enough to $\chi_0(t)$ or to $\chi_1(t)$. Take, instead, a generic $\sigma\in\big(\chi_0(t),\chi_1(t)\big)$. Calling for brevity $\begin{pmatrix}\zeta & \psi\\ 0 & \omega\end{pmatrix}$ the matrix corresponding to Du in the triangle T_i, we have

$$u\big(\gamma(\sigma,t)\big)=u\big(\sigma,\overline\xi(\sigma)\big)=\Big(\zeta\sigma+\psi\overline\xi(\sigma),\omega\overline\xi(\sigma)\Big)\,,$$

while on the other side,

$$v\big(\gamma(\sigma,t)\big)=\tilde\gamma\big(\Phi(\sigma,t)\big)=\tilde\gamma\Big(\hat\xi(\sigma),\tilde t\Big)=\Big(\hat\xi(\sigma),\tilde\xi\big(\hat\xi(\sigma)\big)\Big)\,.$$

Hence, we have to show

$$\hat\xi(\sigma)=\zeta\sigma+\psi\overline\xi(\sigma)\,,\qquad\qquad \tilde\xi\big(\hat\xi(\sigma)\big)=\omega\overline\xi(\sigma)\,. \tag{1.38}$$

Concerning the left equality, evaluating

$$\begin{aligned}
\zeta\sigma+\psi\overline\xi(\sigma) &= \zeta\sigma+\psi\xi[\chi_0,\chi_1,t,t,\tan\theta_a,-\tan\theta_b](\sigma)\\
&= \zeta\sigma+\xi[\chi_0,\chi_1,\psi t,\psi t,\psi\tan\theta_a,-\psi\tan\theta_b](\sigma)\\
&= \xi[\chi_0,\chi_1,\zeta\chi_0+\psi t,\zeta\chi_1+\psi t,\zeta+\psi\tan\theta_a,\zeta-\psi\tan\theta_b](\sigma)\,,
\end{aligned}$$

we reduce ourselves to show

$$\widetilde{\chi}_0 = \zeta\chi_0 + \psi t\,, \quad \widetilde{\chi}_1 = \zeta\chi_1 + \psi t\,, \quad \alpha = \zeta + \psi\tan\theta_a\,, \quad \beta = \zeta - \psi\tan\theta_b\,.$$

And in turn, while the first and second equality hold true because $u\big(P(t)\big) = \widetilde{P}(t)$ and $u\big(Q(t)\big) = \widetilde{Q}(t)$, the third and the fourth are immediately deduced by the fact, already observed, that the equality must be true for every σ close enough to χ_0 or χ_1. As a consequence, the left identity in (1.38) has been established.

Concerning the right identity, we start observing that

$$\tilde{t} = \omega t\,, \qquad \alpha\tan\big(\varphi_a(\theta_a)\big) = \omega\tan\theta_a\,, \qquad \beta\tan\big(\varphi_b(\theta_b)\big) = \omega\tan\theta_b\,,$$

since the first equality comes immediately by the expression of u, while the second (and, equivalently, the third) comes by (1.31), (1.21) and (1.22) as

$$\begin{aligned}\alpha\tan\big(\varphi_a(\theta_a)\big) &= \tau_a(\theta_a)\,\frac{\sin\big(\varphi_a(\theta_a)\big)}{\cos\theta_a} = \frac{\tilde{t}}{\delta_a\cos\theta_a} = \frac{\omega t}{\delta_a\cos\theta_a} \\ &= \frac{\omega\delta_a\sin\big(\theta_a\big)}{\delta_a\cos\theta_a} = \omega\tan\big(\theta_a\big)\,.\end{aligned}$$

Therefore, recalling the properties of ξ, the right identity in (1.38) reduces to

$$\tilde{\xi}\big(\hat{\xi}(\sigma)\big) = \xi\Big[\chi_0(t), \chi_1(t), \tilde{t}, \tilde{t}, \alpha(t)\tan\big(\varphi_a(\theta_a(t))\big), -\beta(t)\tan\big(\varphi_b(\theta_b(t))\big)\Big](\sigma)\,,$$

Unfortunately, the above equality is not true. Nevertheless, since $\sigma\mapsto\hat{\xi}(\sigma)$ is bijective, we can take the equality as a definition for $\tilde{\xi}$, instead of that given in (1.28). Roughly speaking, this modified definition corresponds to taking, in the definition of ξ given in Section 1.2.1, a sort of "double convolution" instead of the convolution. It is easy to check that this "new" definition of ξ does not effect neither the properties (1.11), nor the validity of Lemma 1.13. As a consequence, all our previous proofs concerning $\tilde{\xi}$ remain perfectly valid, and then (1.38) has become true, and the proof is completed. □

The zone Z_4 and the proof of Theorem A

Completing the definition of v is now an easy task: first of all, since we already have the zones Z_1, Z_2 and Z_3 in Ω, we simply define Z_4 as the remaining part. Analogously, $\widetilde{Z}_4$ is defined as the remaining part of Δ. Notice again that Z_4 and $\widetilde{Z}_4$ consist of the "interior parts" of all the triangles in Ω and Δ respectively, and also that choosing the independent constants δ and h very small one has that Z_4 and $\widetilde{Z}_4$ fill almost all Ω and Δ: in fact $\Omega\setminus Z_4$ is a small neighborhood of the 1-skeleton of the triangulation of Ω. We conclude our definition of v by setting $v\equiv u$ on Z_4; thanks to Remark 1.16 and to Lemma 1.22, v is a smooth bijection between Ω and Δ. To conclude this section, we then only need to give the proof of Theorem A.

Proof (of Theorem A). Let u be a piecewise affine L bi-Lipschitz homeomorphism and consider a compatible triangulation (which is admissible to take countable, as observed at the beginning of our construction). Let v be the map described through this section: we have that v is a smooth bijection between Ω and Δ, as well as the validity of (1.6), just putting together the results of Lemmas 1.14, 1.17, 1.21 and (1.22).

The fact that $u = v$ on $\partial\Omega$, as well as that v is orientation-preserving if so is u, are obvious by construction; instead, the fact that v is $100L^{7/3}$ bi-Lipschitz just follows by selecting the exponent $q = 1/3$ in (1.6). Therefore, we only miss (1.5).

To show (1.5), observe that for any x one has by construction that the points $u(x)$ and $v(x)$ belong to triangles having at least a point in common; hence, one has $\|u-v\|_{L^\infty} \leq \varepsilon/4$ as soon as all the triangles in Δ have diameter less than $\varepsilon/8$. And by refining the triangulation if necessary, we can obviously assume that this is true. In the same way, we can also assume $\|u^{-1}-v^{-1}\|_{L^\infty} \leq \varepsilon/4$. Let us now pass to the term

$$
\begin{aligned}
\|Dv - Du\|_{L^p(\Omega)} &= \Big(\sum\nolimits_{i\in\mathbb{N}} \|Dv - Du\|^p_{L^p(T_i)} \Big)^{1/p} \\
&\leq \big(L + 100L^{7/3}\big) \Big(\sum\nolimits_{i\in\mathbb{N}} \mathscr{L}\Big(\big\{ x \in T_i \,:\, u(x) \neq v(x) \big\} \Big) \Big)^{1/p} .
\end{aligned}
$$

Since for every triangle T_i we can let the set $\big\{x \in T_i \,:\, u(x) \neq v(x)\big\}$ be as small as we wish, up to decrease the (independent) constants δ and h, it is admissible to assume $\|Dv - Du\|_{L^p} \leq \varepsilon/4$ and, in the very same way, also $\|Dv^{-1} - Du^{-1}\|_{L^p} \leq \varepsilon/4$. As a consequence we have shown also the validity of (1.5) and then the proof is concluded. $\square$

Remark 1.23. Notice that the assumption $p < \infty$ was essential in the above proof. And in fact, the claim of Theorem A is clearly false for $p = \infty$.

1.3 Part II: Bi-Lipschitz extension Theorem

This part of the notes is devoted to show the following bi-Lipschitz extension result; through this chapter, we will write for brevity $\mathcal{D} = \mathcal{D}(O, 1/2)$ to denote the square of unit side centered at the origin.

Theorem B (bi-Lipschitz extension). *Let $u : \partial\mathcal{D} \to \mathbb{R}^2$ be a L bi-Lipschitz map. Then there exists a CL^4 bi-Lipschitz extension $v : \mathcal{D} \to \mathbb{R}^2$, and in particular if u is finitely piecewise affine there exists such a CL^4 bi-Lipschitz extension which is also finitely piecewise affine. Moreover, there exists also an extension v of u which is smooth; this smooth extension can be taken bi-Lipschitz with constant $CL^{28/3}$ if u is finitely piecewise affine, or with constant $CL^{112/3}$ in general.*

The complete proof of this result is quite involved, and beyond the purpose of these notes; we will limit ourselves to give a quite precise idea of how the construction works, but leaving many details without a formal justification. The interested reader can find the complete work in the paper [15].

The strategy of the proof

In this section we give a more or less complete overview of the strategy that we will follow in the proof. In fact, while on one hand the proof is extremely involved and technically complicate, on the other hand the ideas behind the construction are quite simple, and the tools used in the proof are elementary. We try to present all the main ideas in this introduction.

First of all, we point out that we will consider almost only the finitely piecewise affine case, because the general case will be eventually obtained through a quite simple limiting procedure. Thus, the image $u(\partial\mathcal{D})$ of the boundary of the unit square is a closed curve in $\mathbb{R}^2$ (more precisely, a polygon, since we think u to be piecewise affine), which then divides $\mathbb{R}^2$ in two parts, a bounded one and an unbounded one. We will call Δ the first one, so that actually $u : \partial\mathcal{D} \to \partial\Delta$ is a L bi-Lipschitz function, and the extension must necessarily be a bi-Lipschitz map $v : \mathcal{D} \to \Delta$.

Let us start considering for a moment a very peculiar case, namely, when Δ is strictly convex. In this case, there is an obvious way to build a piecewise affine extension v of u. Namely, take a point $\boldsymbol{O}$ in the interior of the set Δ, say, the barycenter, and for every segment AB in $\partial\mathcal{D}$ on which u is affine define v in the triangle ABO as the unique affine function coinciding with u on AB and such that $u(O) = \boldsymbol{O}$. Equivalently, we can also simply say that the image of the generic segment AO in $\mathcal{D}$ is the segment connecting $u(A)$ to $\boldsymbol{O}$ in Δ, and that v must be affine in every triangle ABO. The two points of view are clearly completely equivalent, but the latter is easier to extend when Δ is not convex.

Think now about the general case, when Δ has clearly no reason to be convex. In this case, we can still fix a point $\boldsymbol{O}$ inside Δ, and try to imagine how the image of a given segment AO in $\mathcal{D}$ should look like. Of course, it can not be a segment, if Δ is not convex; but we can guess that it must be some polygonal curve, connecting $u(A)$ to the point $\boldsymbol{O}$ remaining entirely inside Δ. In particular, this curve will have to avoid being too close to $\partial\Delta$, in order not to increase too much the bi-Lipschitz constant of v. Of course, if we consider all the "vertices" A_i of $\mathcal{D}$ (not only the four vertices of the squares, but all the extremes of segments where u is affine) then all the different piecewise affine curves between $u(A_i)$ and $\boldsymbol{O}$ will have to be disjoint, except for the final point which is always $\boldsymbol{O}$ by definition. Moreover, two "consecutive" curves will not only have not to intersect, but also they need never to become too close or too far from each other, because this would let the Lipschitz constant of v or of v^{-1} explode. This is more or less the main idea of our construction; actually, our biggest concern will be to construct suitable "good paths", that is, the piecewise affine curves just introduced. Let us now describe our proof more in detail, following the ten big steps in which the proof is divided (corresponding to ten sections of this chapter).

Our first concern is to find a suitable point $\boldsymbol{O} = u(O)$ to follow the idea described above. Actually, we will see that it is better to find, instead, a suitable "central ball" $\widehat{\boldsymbol{\mathcal{B}}}$, which will be the image of a suitable part of $\mathcal{D}$ more or less close to the center O of $\mathcal{D}$. At the end, we will see that often $\boldsymbol{O} = u(O)$ will be the center of this ball $\widehat{\boldsymbol{\mathcal{B}}}$, but in some cases it will be some other point of $\widehat{\boldsymbol{\mathcal{B}}}$. This central ball will be entirely contained in Δ, but its boundary will intersect the boundary of Δ in at least two points. For instance, Figure 1.7 shows a possible situation, where Δ is the polygon, $\widehat{\boldsymbol{\mathcal{B}}}$ is the ball, and its boundary intersects the boundary of Δ in the four points $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$. Finding a suitable ball $\widehat{\boldsymbol{\mathcal{B}}}$ and studying its properties will be the scope of Step I.

We divide then the set Δ in two parts: one is the "internal polygon", that is, the polygon whose vertices are the points of the boundary of the central ball which belong to $\partial\Delta$, and one is all the rest. In particular, this second part is of course the union of some essentially disjoint polygons, each of which having has boundary a part of $\partial\Delta$ and a side of the internal polygon. We will define "primary sector" each of these polygons: for instance, in Figure 1.7 the internal polygon is the white quadrilateral, while the primary sectors are the four coloured polygons. Notice that the internal polygon can reduce to a segment, if the points of $\partial\widehat{\boldsymbol{\mathcal{B}}} \cap \partial\Delta$ are only 2. Step II will be devoted to give all the necessary definitions about the primary sectors and to point out their first properties.

We concentrate now on a given primary sector, say, the one which contains the segment $\boldsymbol{AB}$ in its boundary: since the map v has to be piecewise affine, this sector will have to be subdivided in triangles. Hence, in Step III we will partition the sector in a finite union of essentially disjoint triangles, and we will also introduce a partial order between them. Figure 1.9 shows a sector subdivided in triangles, and the partial order is such that the triangles numbered there between 1 and 10 are an increasing sequence.

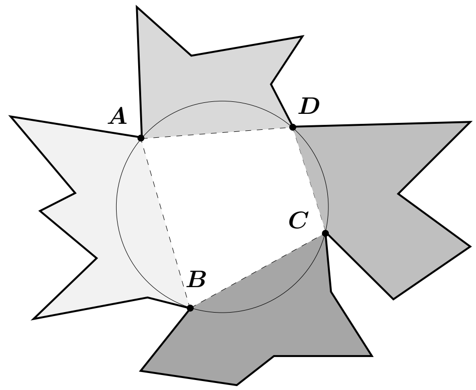

Fig. 1.7. A polygon with four (coloured) primary sectors.

We can then start building the "good paths" introduced above. More precisely, taken a vertex $\boldsymbol{P}$ on the boundary of the primary sector, in Step IV we will define a piecewise affine path which starts from $\boldsymbol{P}$ and arrives somewhere on the segment $\boldsymbol{AB}$; this path will be affine on each of the triangles defined in Step III. Since, if we call $P \in \partial\mathcal{D}$ the point such that $u(P) = \boldsymbol{P}$, the path build here will be the image under v of the first part of the segment PO, then to get the bi-Lipschitz property for v we will have to take care that the different vertices $\boldsymbol{P}$ will have to influence each other. This is probably the most complicate step in the original proof of the theorem (but in these notes, we skip many details in the proof of this step).

The next Step V will be devoted to provide a bound on the lengths of the paths constructed in Step IV: this is clearly a technical information needed to obtain our further estimates.

As we just said above, for any vertex $\boldsymbol{P} = u(P) \in \partial\Delta$ the extension v will send the first part of the segment PO onto the path defined in Step IV. However, we will notice that the best way to do so will not be to do so at constant speed, because this would eventually worsen the estimate of Theorem B. Therefore, in Step VI we will give a non constant and carefully constructed "speed function"; as before, the speed function corresponding to any vertex will influence those corresponding to the other vertices.

Putting everything together, we have then a bi-Lipschitz function defined on all the good paths inside the primary sector, and extending it in the piecewise affine way we will then get a bi-Lipschitz function defined in the whole primary sector. Step VII will be devoted to give the precise definition and to check the properties of this map.

Since we have now different bi-Lipschitz functions between suitable parts of the square $\mathcal{D}$ and each of the primary sectors, what is needed now is to send the remaining central part of $\mathcal{D}$ onto the internal polygon inscribed in the central ball. This will be fairly easy in most of the situations, such as the one depicted in Figure 1.7; however, the general case can be much more tricky, for instance when the internal polygon degenerates to a segment. In Step VIII we will show how to deal all the possible cases, and then we will have concluded the construction of a finitely piecewise affine extension $v : \mathcal{D} \to \Delta$ in the case of a finitely piecewise affine map $u : \partial\mathcal{D} \to \partial\Delta$.

To conclude the complete proof, we will now only need two very simple last steps. In Step IX, we will build an extension for a bi-Lipschitz map u which is not finitely piecewise affine. To do so, we will approximate u with a sequence of finitely piecewise affine functions u_j which uniformly converge to u; then, we will apply the construction of the Steps I–VIII to get extensions v_j of each u_j; and finally, we will only have to check that the sequence v_j uniformly converge to a function v, which is as needed.

The last Step X will provide us with a smooth extension. In fact, if u is finitely piecewise affine we will only need to apply Step VIII and Theorem A, while for a generic function u we will apply Step IX and Theorem C.

Some notation

As in the other parts, there is some specific notation which is needed only within this part, and we briefly list it here.

We will use capital letters to refer to points in the square $\mathcal{D}$, as A, B, C and so on, and bold capital letters to refer to points in Δ, as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and so on. For a quick comprehension and a shorter notation, we will use the same letter (bold or not) to denote points in $\mathcal{D}$ and in Δ which are sent to each other by u or v. This means that if $P \in \partial\mathcal{D}$ is a point on the boundary of the square, then the letter $\boldsymbol{P}$ is always used to denote the point $u(P)$. Similarly, if $Q \in \mathcal{D}$ is inside the square, then $\boldsymbol{Q} = v(Q)$.

For any two points $P, Q \in \partial\mathcal{D}$, we call as usual PQ the segment joining P and Q, which lies inside $\overline{\mathcal{D}}$; instead, by $\widehat{PQ}$ we denote the shortest path between P and Q on $\partial\mathcal{D}$; in the particular case when P and Q are opposite, by $\widehat{PQ}$ we may refer to each of the two possible paths, and we will clarify which one when needed. Moreover, $\ell(PQ)$ and $\ell(\widehat{PQ})$ are the lengths of the segment PQ and of the curve $\widehat{PQ}$ respectively: notice that $\ell(\widehat{PQ}) \in [0, 2]$. Concerning points $\boldsymbol{P} = u(P)$ and $\boldsymbol{Q} = u(Q)$ in $\partial\Delta$, to be consistent we will consider the segment $\boldsymbol{PQ}$, having length $\ell(\boldsymbol{PQ})$, only if it is entirely contained inside Δ. In addition, the curve $\widehat{\boldsymbol{PQ}}$ is defined as $u(\widehat{PQ})$, and its length is again called $\ell(\widehat{\boldsymbol{PQ}})$: notice that $\widehat{\boldsymbol{PQ}}$ might happen not be the shortest path between $\boldsymbol{P}$ and $\boldsymbol{Q}$ on $\partial\Delta$.

1.3.1 Step I: The central ball $\widehat{\boldsymbol{\mathcal{B}}}$

As anticipated above, we start considering the case when u is finitely piecewise affine; this will be the assumption of the Steps I–VIII, and only in Steps IX and X we will consider the general case.

In this first step, we select the "central ball" $\widehat{\boldsymbol{\mathcal{B}}}$, which is contained in Δ and whose boundary touches $\partial\Delta$ in at least two points. Among all the balls with this property, we are going to select one which is not too small (however, we will not simply take the biggest possible one, because this would eventually give rise to a worse estimate in Theorem B). Our result is then the following.

Lemma 1.24. *There exists an open ball $\widehat{\boldsymbol{\mathcal{B}}} \subseteq \Delta$ such that the intersection $\partial\widehat{\boldsymbol{\mathcal{B}}} \cap \partial\Delta$ consists of $N \geq 2$ points $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots \boldsymbol{A}_N$, taken in the anti-clockwise order on the circle $\partial\widehat{\boldsymbol{\mathcal{B}}}$, and with the property that for every $1 \leq i \leq N$ the path $\widehat{\boldsymbol{A}_i\boldsymbol{A}_{i+1}}$ does not intersect other points $\boldsymbol{A}_j$ than $\boldsymbol{A}_i$ and $\boldsymbol{A}_{i+1}$.*

Some preliminary remarks are now in order. First of all, since the ball $\widehat{\boldsymbol{\mathcal{B}}}$ is entirely contained in Δ, then also the points A_i on $\partial\mathcal{D}$ are ordered, in particular in the anti-clockwise sense if u is orientation-preserving and in the clockwise sense if u is orientation-reversing. Thus, the assumption that for every i the path $\widehat{\boldsymbol{A}_i\boldsymbol{A}_{i+1}}$ is the one which does not intersect other points $\boldsymbol{A}_j$ is equivalent to say that $\partial\mathcal{D}$ is the essentially disjoint union of the paths $\widehat{A_iA_{i+1}}$, with the usual convention $N+1 \equiv 1$. In addition, we can observe also what follows.

Remark 1.25. If $N = 2$ in Lemma 1.24, then it is surely $\ell(A_1A_2) = 2$, that is, the points A_1 and A_2 are opposite; otherwise, if $N \geq 3$, then the maximal distance $\ell\big(\widehat{A_iA_j}\big)$ between two vertices is at least $4/3$. Thus, whatever N is, the radius of the central ball must be at least $\frac{2}{3L}$.

Moreover, take any ball $\boldsymbol{\mathcal{B}} \subseteq \Delta$ whose boundary intersects $\partial\Delta$ in at least two points. The choice $\widehat{\boldsymbol{\mathcal{B}}} = \boldsymbol{\mathcal{B}}$ satisfies the claim of Lemma 1.24 unless there is an arc of length 2 in $\partial\mathcal{D}$ which does not contain any of the points of $\partial\Delta \cap \partial\boldsymbol{\mathcal{B}}$.

We now depict the proof of the lemma.

Proof (of Lemma 1.24). Recall that Δ is a polygon, take any point $\boldsymbol{A} \in \partial\Delta$ which is not a vertex and let $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\boldsymbol{A}) \subseteq \Delta$ be the biggest ball containing $\boldsymbol{A}$ in its boundary; thus, $\partial\boldsymbol{\mathcal{B}} \cap \partial\Delta$ must contain at least another point $\boldsymbol{B} \neq \boldsymbol{A}$. As a consequence, the set S of all the pairs $(\boldsymbol{A}, \boldsymbol{B}) \in \partial\Delta$ contained in the boundary of some ball $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \Delta$ is not empty (and, of course, symmetric). Since S is a compact subset of $(\partial\Delta)^2$, there exists a pair $(\boldsymbol{A}, \boldsymbol{B})$ maximizing $\ell(\widehat{AB})$; notice carefully that we are maximizing $\ell(\widehat{AB})$, not $\ell(\widehat{\boldsymbol{AB}})$.

Observe now that, if $\ell(\widehat{AB}) = 2$, then any ball $\widehat{\boldsymbol{\mathcal{B}}} \subseteq \Delta$ containing both $\boldsymbol{A}$ and $\boldsymbol{B}$ in its boundary satisfies our claim thanks to Remark 1.25.

We assume then that $\ell(\widehat{AB}) < 2$; among all the balls $\boldsymbol{\mathcal{B}} \subseteq \Delta$ whose boundary contains both $\boldsymbol{A}$ and $\boldsymbol{B}$, we call $\widehat{\boldsymbol{\mathcal{B}}}$ one which maximizes the radius (in most examples there is only one such ball, but it is not always the case).

Making use of the maximality of $\ell(\widehat{AB})$ among pairs $(\boldsymbol{A}, \boldsymbol{B}) \in S$, and of the maximality of the radius of $\widehat{\boldsymbol{\mathcal{B}}}$ among the admissible balls corresponding to $\boldsymbol{A}$ and $\boldsymbol{B}$, a geometric argument shows that there is some point $\boldsymbol{P} \in \partial\widehat{\boldsymbol{\mathcal{B}}} \cap \partial\Delta \setminus \widehat{\boldsymbol{AB}}$.

Consider now the three points A, B and P on $\partial\mathcal{D}$. By definition of the paths and by construction, P does not belong to the path $\widehat{AB}$. Moreover, since the length $\ell(\widehat{AB})$ is maximal, then $\widehat{AP}$ cannot contain $\widehat{AB}$, or, equivalently, B is not contained in $\widehat{AP}$; for the same reason, also A is not contained in $\widehat{BP}$. This immediately implies that every path of length 2 in $\partial\mathcal{D}$ contains at least one between A, B and P. And in turn, Remark 1.25 says that this gives our thesis. □

1.3.2 Step II: Sectors and primary sectors

In this second step we present the definition of the "sectors" in Δ, and in particular of the "primary sectors", studying their simplest main properties. The first definition is the following.

Definition 1.26. *Consider two points $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\partial\Delta$ with the property that the open segment $\boldsymbol{AB}$ entirely lies in Δ. We call* sector between $\boldsymbol{A}$ and $\boldsymbol{B}$ *the subset of Δ enclosed by $\boldsymbol{AB}$ and by the path $\widehat{\boldsymbol{AB}}$.*

Recall that, if A and B are opposite in $\mathcal{D}$, then each of the two paths connecting $\boldsymbol{A}$ and $\boldsymbol{B}$ can be referred to as $\widehat{\boldsymbol{AB}}$; as a consequence, in this case each of the two parts in which Δ is subdivided by $\boldsymbol{AB}$ is a primary sector.

Consider now the central ball $\widehat{\boldsymbol{\mathcal{B}}}$ given by Lemma 1.24, and call again $\boldsymbol{A}_i$, $1 \leq i \leq N$ the (ordered) points of $\partial\widehat{\boldsymbol{\mathcal{B}}} \cap \partial\Delta$. By construction, each open segment $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ is entirely contained in $\widehat{\boldsymbol{\mathcal{B}}}$, thus in Δ. As a consequence, we can give a name to the corresponding particular case of sectors.

Definition 1.27. *We call* primary sector *each of the sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$.*

By Lemma 1.24 we readily get that the different primary sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ are essentially disjoint. More precisely, Δ is the essentially disjoint union of the N primary sectors and the possibly degenerate polygon $\boldsymbol{A}_1\boldsymbol{A}_2 \dots \boldsymbol{A}_N$ (to visualize the situation, one can look back at Figure 1.7). Let us observe a couple of easy properties of the sectors.

Remark 1.28. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ be four points in $\partial\Delta$. If $\boldsymbol{C}, \boldsymbol{D} \in \widehat{\boldsymbol{AB}}$, then one has $\widehat{\boldsymbol{CD}} \subseteq \widehat{\boldsymbol{AB}}$. Moreover, if both the open segments $\boldsymbol{AB}$ and $\boldsymbol{CD}$ lie in the interior of Δ, then it is also $\mathcal{S}(\boldsymbol{CD}) \subseteq \mathcal{S}(\boldsymbol{AB})$.

A second very useful property, which will be used several times in our next proofs, is that the lengths of the paths in $\partial\mathcal{D}$ and of the corresponding paths in Δ can be bounded by each other. More precisely, let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two points in $\partial\Delta$ such that the closed segment $\boldsymbol{PQ}$ is entirely contained in $\overline{\Delta}$. Since $\mathcal{D}$ is a square, then $\ell(PQ) \leq \ell(\widehat{PQ}) \leq \sqrt{2}\ell(PQ)$, and since u is L bi-Lipschitz it is also $\ell(PQ)/L \leq \ell(\boldsymbol{PQ}) \leq L\ell(PQ)$, as well as $\ell(\widehat{PQ})/L \leq \ell(\widehat{\boldsymbol{PQ}}) \leq L\ell(\widehat{PQ})$. In particular, we deduce that

$$\ell\big(\widehat{PQ}\big) \leq \sqrt{2}L\,\ell(\boldsymbol{PQ})\,. \tag{1.39}$$

Observe that, of course, for the above estimate to be true it is crucial that $\widehat{PQ}$ is the shortest of the two paths in $\partial\mathcal{D}$ connecting P and Q. In fact, the reason why our definition of the central ball in Step I requires that every path $\widehat{\boldsymbol{A}_i\boldsymbol{A}_{i+1}}$ does not contain other points $\boldsymbol{A}_j$, is that then we can use the estimate (1.39) inside every primary sector $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$.

1.3.3 Step III: Subdivision of a sector in triangles

To build our extension v of u, the next ingredient that we need is to subdivide every primary sector in a finite number of triangles. Through this Step, $\mathcal{S}(\boldsymbol{AB})$ will be a fixed primary sector in Δ.

Definition 1.29. *Given three points $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ in $\mathcal{S}(\boldsymbol{AB})$, the triangle $\boldsymbol{PQR}$ is said an* admissible triangle *if each of its open sides is entirely contained either in Δ, or in $\partial\Delta$. Moreover, the segment $\boldsymbol{PR}$ is said the* exit side *of the admissible triangle $\boldsymbol{PQR}$ if $\widehat{\boldsymbol{PR}} = \widehat{\boldsymbol{PQ}} \cup \widehat{\boldsymbol{QR}}$.*

Figure 1.8 enlightens the meaning of the definition, showing five triangles in the sector $\mathcal{S}(\boldsymbol{AB})$; the triangles 1 and 3 are not admissible because they have a side which is not entirely contained neither in Δ nor in $\partial\Delta$: in particular, a side of triangle 1 (resp., triangle 3) has an open side whose intersection with Δ is half of the side (resp., a single point). On the other hand, the remaining triangles are admissible, and an arrow indicates the exit side. As the figure illustrates, and as one can readily deduce from the definition, every admissible triangle admits exactly an exit side. Notice also that every admissible triangle can have either one, or two, or three sides which are contained in Δ, and in particular so is always the exit side.

Let us briefly clarify a point in our notation. Recalling that u is assumed to be finitely piecewise affine, we will call *sides* of $\mathcal{D}$ and of Δ all the segments on which the restriction of u is affine, and *vertices* their extremes. Observe that this does not coincide with the usual sense of the words "side" and "vertex" for polygons: in particular, there can be segments which are sides of the polygon $\partial\mathcal{D}$ or $\partial\Delta$, but which are in fact a finite union of sides according to our convention. We are now in position to state the main result of this step.

Lemma 1.30. *There exists a partition of $\mathcal{S}(\boldsymbol{AB})$ in a finite number of admissible triangles such that:*

- **a)** *each vertex in $\widehat{\boldsymbol{AB}}$ is vertex of some triangle of the partition,*
- **b)** *for each triangle $\boldsymbol{PQR}$ of the partition, whose exit side is $\boldsymbol{PR}$, the orthogonal projection of $\boldsymbol{Q}$ on the straight line through $\boldsymbol{PR}$ lies in the closed segment $\boldsymbol{PR}$ (equivalently, the angles $\boldsymbol{P}\widehat{\boldsymbol{R}}\boldsymbol{Q}$ and $\boldsymbol{R}\widehat{\boldsymbol{P}}\boldsymbol{Q}$ are at most $\pi/2$).*

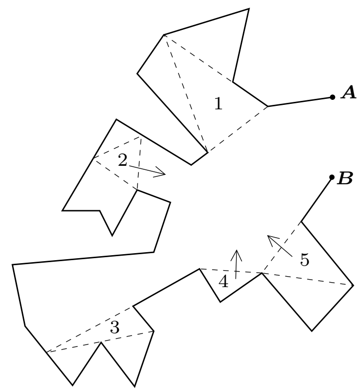

Fig. 1.8. Some (admissible or not) triangles in a sector.

For brevity, we are giving here only a brief sketch of the proof of this result (as of many of the technical results of our construction). However, it is useful to describe the main tool that one has to use, namely, the "weight" of the sectors. More precisely, let $\mathcal{S}(\boldsymbol{CD})$ be a sector, and call $m \in \mathbb{N}$ the number of its sides. For every vertex $\boldsymbol{P} \in \widehat{\boldsymbol{CD}}$, call $\boldsymbol{P}_\perp$ its orthogonal projection on the straight line passing trough $\boldsymbol{C}$ and $\boldsymbol{D}$. If there exists some vertex $\boldsymbol{P}$ such that $\boldsymbol{P}_\perp$ belongs to the closed segment $\boldsymbol{CD}$ and the open segment $\boldsymbol{PP}_\perp$ lies inside Δ, then we say that the *weight of* $\mathcal{S}(\boldsymbol{CD})$ is m. Otherwise, if no such vertex $\boldsymbol{P}$ exists, we say that the *weight of* $\mathcal{S}(\boldsymbol{CD})$ is $m + \frac{1}{2}$.

Proof (of Lemma 1.30). If the weight of $\mathcal{S}(\boldsymbol{AB})$ is 2, which is the minimal possible weight of a sector, then the sector is in fact a triangle and the claim is true simply considering the partition given by the triangle itself.

Let us then argue by induction on the (semi-integer) weight of a sector; assume then that $\mathcal{S}(\boldsymbol{AB})$ has weight $k > 2$, and that the proof has already been obtained for all the weights smaller than k. First of all, consider the case when k is integer: by definition of weight, there exists a point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ such that the segment $\boldsymbol{PP}_\perp$ is entirely contained in Δ and $\boldsymbol{P}_\perp \in \boldsymbol{AB}$. Therefore, the sector $\mathcal{S}(\boldsymbol{AB})$ is the essentially disjoint union of the two sectors $\mathcal{S}(\boldsymbol{AP})$ and $\mathcal{S}(\boldsymbol{PB})$ and of the triangle $\boldsymbol{APB}$. Moreover, each of the two sectors has weight strictly less than k, so by assumption the claim is already separately true on each of them. The searched partition of $\mathcal{S}(\boldsymbol{AB})$ is then obtained by putting together the two partitions of $\mathcal{S}(\boldsymbol{AP})$ and $\mathcal{S}(\boldsymbol{PB})$ and the triangle $\boldsymbol{APB}$.

Assume, instead, that k is not integer. Take now a point $\boldsymbol{V} \in \widehat{\boldsymbol{AB}}$ which is not a vertex, and which is then contained in some side of the sector, say $\boldsymbol{V} \in \boldsymbol{PQ} \subseteq \widehat{\boldsymbol{AB}}$. We can then split the side $\boldsymbol{PQ}$ into the two sides $\boldsymbol{PV}$ and $\boldsymbol{VQ}$; in other words, we arbitrarily decide to start considering also $\boldsymbol{V}$ as a vertex of the sector $\mathcal{S}(\boldsymbol{AB})$: observe that this is clearly admissible, though, doing so, the total number of vertices and sides increases by 1, becoming then

$k+\frac{1}{2}$ (correspondingly, the weight of the sector $\mathcal{S}(\boldsymbol{AB})$, which was previously k, becomes now $k+\frac{1}{2}$ or $k+1$). It is simple to show that, by carefully choosing the point $\boldsymbol{V}$, we can obtain that the sector $\mathcal{S}(\boldsymbol{AB})$ is the union of the two (possibly degenerate) sectors $\mathcal{S}(\boldsymbol{AV})$ and $\mathcal{S}(\boldsymbol{VB})$ with the triangle $\boldsymbol{AVB}$, and the weight of each of the sectors is at most $k-\frac{1}{2}$ (thus smaller not only of the new weight of the sector $\mathcal{S}(\boldsymbol{AB})$, but also of the old one!). Again, the inductive assumption allows then to obtain the claim. Notice that our procedure may increase the total number of sides, but only a finite number of times, so this is not a problem at all. □

From now on, we will always consider the sector $\mathcal{S}(\boldsymbol{AB})$ with a given partition as in Lemma 1.30. An explicit example of such a partition is shown in Figure 1.9. A last couple of definitions are needed to conclude this step. First of all, the triangles of the partition admit a natural partial order, induced by the relation which says that the two triangles $\boldsymbol{PRQ}$ and $\boldsymbol{PQS}$ satisfy $\boldsymbol{PRQ} \leq \boldsymbol{PQS}$ if the common side $\boldsymbol{PQ}$ is the exit side of $\boldsymbol{PQR}$. It is easy to observe that this relation is well-defined, and it is equivalent to say that $\boldsymbol{PQR} \leq \boldsymbol{STU}$ if and only if the points $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ all belong to the path $\widehat{\boldsymbol{SU}}$, being $\boldsymbol{SU}$ the exit side of $\boldsymbol{STU}$. Moreover, there is a unique greatest

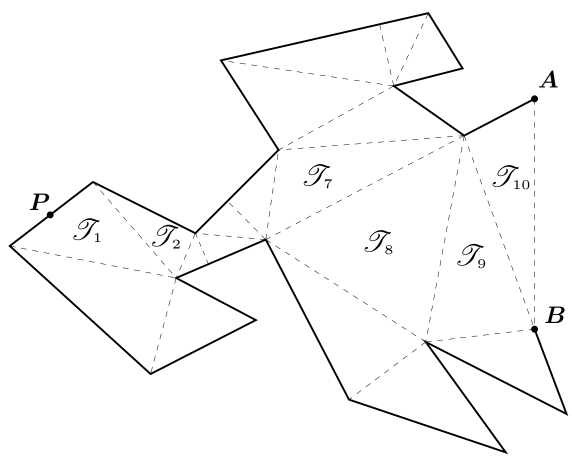

Fig. 1.9. A sector, its partition in triangles, and the natural sequence related to $\boldsymbol{P}$.

element for this order, namely, the triangle having as exit side the segment $\boldsymbol{AB}$. In addition, every triangle of the partition has exactly a unique successor, except the maximizer. Since the triangles are finitely many, in our future constructions we will argue recursively, considering a generic triangle and assuming that all the smaller triangles have been already treated. We give now the last definition.

Definition 1.31. *For every* $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$*, we call* natural sequence of triangles related to $\boldsymbol{P}$ *the sequence* $\big(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_N\big)$*, where* $\mathscr{T}_1$ *is the minimal triangle of the partition containing* $\boldsymbol{P}$*,* $\mathscr{T}_{i+1}$ *is the successor of* $\mathscr{T}_i$ *for any* $1 \leq i \leq N-1$*, and* $\mathscr{T}_N$ *is the maximal triangle. Of course,* $N = N(\boldsymbol{P})$*.*

Notice that the natural sequence of triangles related to any point $\boldsymbol{P}$ is well-defined; an example is given in Figure 1.9, with the triangles $\big(\mathscr{T}_1, \dots, \mathscr{T}_{10}\big)$.

1.3.4 Step IV: The good paths inside the sectors

The fourth is the main and most complicate step of the whole construction; what we want to do here, is to define suitable piecewise affine paths inside the sector $\mathcal{S}(\boldsymbol{AB})$, starting from each vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ and ending on the segment $\boldsymbol{AB}$. The meaning of these paths is evident: calling as usual $P = u^{-1}(\boldsymbol{P}) \in \partial\mathcal{D}$, our extension v will send the first part of the segment PO onto this piecewise affine path. Since this definition will provide the map v on some one-dimensional "skeleton" inside the square $\mathcal{D}$, the paths corresponding to different points must not become too far nor too close to each other. Let us first give the definition of what are the admissible paths for our strategy.

Definition 1.32. *Take a point* $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$*, and let* $\big(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_N\big)$ *be the corresponding natural sequence of triangles as in Definition 1.31. A* good path corresponding to $\boldsymbol{P}$ *is any piecewise affine path* $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ *such that every* $\boldsymbol{P}_i$ *belongs to the exit side of* $\mathscr{T}_i$*.*

A precise idea of what a good path is can be taken from Figure 1.10, where the sector $\mathcal{S}(\boldsymbol{AB})$ is subdivided in triangles as in Step III, and two good paths corresponding to the points $\boldsymbol{P}$ and $\boldsymbol{Q}$ are drawn. We can now

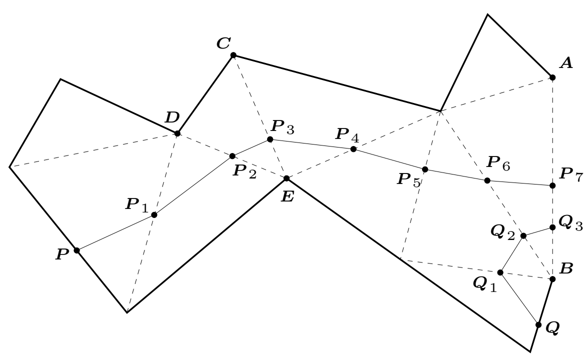

Fig. 1.10. A sector with two good paths corresponding to $\boldsymbol{P}$ and $\boldsymbol{Q}$.

present the result of this step.

Lemma 1.33. *There exist a good path* $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ *corresponding to each vertex* $\boldsymbol{P}$ *of* $\widehat{\boldsymbol{AB}}$*, being* $N = N(\boldsymbol{P})$*, so that the following holds:*

(i) for any $1 \le i \le N(\boldsymbol{P})$*, the angle between the segment* $\boldsymbol{P}_{i-1}\boldsymbol{P}_i$ *and the side of* $\mathscr{T}_i$ *to which* $\boldsymbol{P}_{i-1}$ *belongs (resp., the exit side of* $\mathscr{T}_i$*) is at least* $\arcsin\left(\frac{1}{6L^2}\right)$ *(resp., at least* $15°$*);*

(ii) $\ell\big(\widehat{\boldsymbol{PP}_N}\big) := \ell(\boldsymbol{PP}_1) + \ell(\boldsymbol{P}_1\boldsymbol{P}_2) + \cdots + \ell(\boldsymbol{P}_{N-1}\boldsymbol{P}_N) \le 4\,\ell\big(\widehat{\boldsymbol{AB}}\big)$ *;*

(iii) assume that for some $1 \le i \le N(\boldsymbol{P})$ *and* $1 \le j \le N(\boldsymbol{Q})$ *the points* $\boldsymbol{P}_i$ *and* $\boldsymbol{Q}_j$ *belong to the same exit side of some triangle; then*

$$\frac{\ell\big(\widehat{PQ}\big)}{7L} \le \ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big) \le \ell\big(\widehat{\boldsymbol{PQ}}\big)$$

and, if $i < N(\boldsymbol{P})$*, then*

$$\ell\big(\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}\big) \le \ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big)\,;$$

(iv) the piecewise affine paths $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ *are pairwise disjoint.*

Using the situation of Figure 1.10 as an example, we can quickly observe the meaning of (i)–(iv) in the lemma above. Property (i), for the point $\boldsymbol{P}$ and $i = 3$, which corresponds to the triangle $\mathscr{T} = \boldsymbol{CDE}$, means that

$$\boldsymbol{P}_3\widehat{\boldsymbol{P}_2}\boldsymbol{D} \ge \arcsin\left(\frac{1}{6L^2}\right), \qquad \boldsymbol{P}_3\widehat{\boldsymbol{P}_2}\boldsymbol{E} \ge \arcsin\left(\frac{1}{6L^2}\right),$$
$$\boldsymbol{P}_2\widehat{\boldsymbol{P}_3}\boldsymbol{C} \ge 15°\,, \qquad \boldsymbol{P}_2\widehat{\boldsymbol{P}_3}\boldsymbol{E} \ge 15°\,.$$

Property (ii) means $\ell\big(\widehat{\boldsymbol{PP}_7}\big) \le 4\,\ell\big(\widehat{\boldsymbol{AB}}\big)$, denoting $\widehat{\boldsymbol{PP}_7} := \boldsymbol{PP}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_7$. In the same way, $\ell\big(\widehat{\boldsymbol{QQ}_3}\big) \le 4\,\ell\big(\widehat{\boldsymbol{AB}}\big)$. Property (iii) means

$$\frac{\ell\big(\widehat{PQ}\big)}{7L} \le \ell\big(\boldsymbol{P}_7\boldsymbol{Q}_3\big) \le \ell\big(\boldsymbol{P}_6\boldsymbol{Q}_2\big) \le \ell\big(\widehat{\boldsymbol{PQ}}\big)\,.$$

Observe that, as in the figure, whenever the points $\boldsymbol{P}_i$ and $\boldsymbol{Q}_j$ belong to the exit side of the same triangle $\mathscr{T}$, then also $\boldsymbol{P}_{i+1}$ and $\boldsymbol{Q}_{j+1}$ belong to the exit side of the (unique) successor of $\mathscr{T}$; therefore, the same will be true for all the points $\boldsymbol{P}_{i+k}$ and $\boldsymbol{Q}_{j+k}$ until reaching the maximal triangle having $\boldsymbol{AB}$ as exit side. In particular, $N(\boldsymbol{P}) - N(\boldsymbol{Q}) = i - j$. Finally, property (iv) just means that $\widehat{\boldsymbol{PP}_7}$ and $\widehat{\boldsymbol{QQ}_3}$ do not intersect each other.

Let us now give a very short description of the different parts of the proof.

Proof (of Lemma 1.33). The thesis is obtained by an induction over the weight of the structure. In fact, assume that the lemma has been already proved for all the structures of weight less than that of $\mathcal{S}(\boldsymbol{AB})$ (which is emptily true if the weight is 2), and let $\boldsymbol{ABC}$ be the maximal triangle with respect to the order introduced in Step III.

By definition, the segment $\boldsymbol{BC}$ can lie entirely in Δ, or entirely in $\partial\Delta$; in the first case, $\mathcal{S}(\boldsymbol{BC})$ is a sector having a weight strictly less than that of $\mathcal{S}(\boldsymbol{AB})$, then by inductive assumption we already have piecewise affine paths starting from every point $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$ and arriving up to $\boldsymbol{BC}$, and satisfying (i)–(iv). In the second case, the same is emptily true with the trivial empty paths.

In any case, one has then only to connect the points $\boldsymbol{P}_{N-1}$ on $\boldsymbol{BC}$ with the segment $\boldsymbol{AB}$ (the very same has to be done, of course, also with $\boldsymbol{AC}$ in place of $\boldsymbol{BC}$). To do so, it is necessary to consider carefully all the possible positions of $\boldsymbol{C}$ with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$, and this is done in different steps.

Step A. Definition of $\boldsymbol{C}_1$.

First of all, we have to define the point $\boldsymbol{C}_1 \in \boldsymbol{AB}$; we start calling $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$ the points on the straight line passing through $\boldsymbol{A}$ and $\boldsymbol{B}$ and being at a distance $\ell\big(\boldsymbol{BC}\big)$ from $\boldsymbol{B}$, and $\ell\big(\boldsymbol{AC}\big)$ from $\boldsymbol{A}$ respectively. Then, we call $\widetilde{\boldsymbol{C}}_1 \in \boldsymbol{AB}$ the point satisfying

$$\frac{\ell\big(\widehat{AC}\big)}{\ell\big(\widehat{AB}\big)} = \frac{\ell(\boldsymbol{A}\widetilde{\boldsymbol{C}}_1)}{\ell\big(\boldsymbol{AB}\big)}\,;$$

finally, we set $\boldsymbol{C}_1$ to be the point in $\boldsymbol{C}^-\boldsymbol{C}^+$ which is closest to $\widetilde{\boldsymbol{C}}_1$: in other words, $\boldsymbol{C}_1 = \widetilde{\boldsymbol{C}}_1$ if $\widetilde{\boldsymbol{C}}_1 \in \boldsymbol{C}^+\boldsymbol{C}^-$, while $\boldsymbol{C}_1 = \boldsymbol{C}^+$ (resp. $\boldsymbol{C}_1 = \boldsymbol{C}^-$) if $\widetilde{\boldsymbol{C}}_1$ is above $\boldsymbol{C}^+$ (resp. below $\boldsymbol{C}^-$). Having defined the point $\boldsymbol{C}_1$, we have then to consider the points $\boldsymbol{P}_{N-1}$ on the segment $\boldsymbol{BC}$. This is done in a different way, depending on how the situation is.

Step B. The case when $\boldsymbol{C}_1 = \boldsymbol{C}^+$.

This is the easiest case to treat. In fact, for every vertex $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$, corresponding to a point $\boldsymbol{P}_{N-1} \in \boldsymbol{BC}$, we let $\boldsymbol{P}_N \in \boldsymbol{BC}_1$ be the point satisfying $\ell(\boldsymbol{BP}_N) = \ell(\boldsymbol{BP}_{N-1})$, so that in particular all the different segments $\boldsymbol{P}_{N-1}\boldsymbol{P}_N$ are parallel to $\boldsymbol{CC}_1$. In this case property (iv) is clearly true, and one can easily check the validity of also (i)–(iii), recalling that either $\boldsymbol{BC} \in \partial\Delta$, or the validity of (i)–(iii) is true by assumption on the sector $\mathcal{S}(\boldsymbol{BC})$.

Step C. The case when $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$ *and* $\boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C} \geq 15°$.

If $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$, it is convenient to distinguish the two cases when the angle $\boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C}$ is bigger or smaller than $15°$. In the first case, for every point $\boldsymbol{P}_{N-1} \in \boldsymbol{BC}$, corresponding to the vertex $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$, we let $\boldsymbol{P}_N \in \boldsymbol{BC}_1$ be the point such that

$$\ell\big(\boldsymbol{BP}_N\big) = \min\left\{\ell\big(\boldsymbol{BP}_{N-1}\big)\,,\,\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L}\right\}, \qquad (1.40)$$

being as usual $P \in \partial\mathcal{D}$ be given by $P = u^{-1}(\boldsymbol{P})$. It is again possible (though not so immediate), by some geometric arguments, to check that all the properties (i)–(iv) hold true.

Step D. The case when $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$ and $\boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C} \leq 15°$.

This is the hardest case to handle, since by the smallness of the angle $\boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C}$ the definition (1.40) does not work. To obtain the thesis in this situation, one has to perform a delicate geometric construction, basically shrinking the segments on $\boldsymbol{BC}$ to fit into $\boldsymbol{BC}_1$, though with a shrinking parameter which depends on the point. However, one eventually proves the validity of the searched properties (i)–(iv). $\square$

1.3.5 Step V: The lengths of the good paths

This step is devoted to find a bound for the good paths $\widehat{\boldsymbol{PP}_N}$ given by Lemma 1.33. In fact, thanks to property (ii) of that lemma, we already know that for every vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ one has $\ell(\widehat{\boldsymbol{PP}_N}) \leq 4\ell\big(\widehat{\boldsymbol{AB}}\big)$. However, it is easy to understand that this bound is not satisfactory if the point $\boldsymbol{P}$ is close to $\boldsymbol{A}$ or to $\boldsymbol{B}$. In fact, consider two points $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\widehat{\boldsymbol{AB}}$ which are close to each other, and call $N = N(\boldsymbol{P})$ and $M = N(\boldsymbol{Q})$. Since the map v that we want to define must be bi-Lipschitz, then the paths $\widehat{\boldsymbol{PP}_N}$ and $\widehat{\boldsymbol{QQ}_M}$ must remain close to each other, and their lengths must be similar. In the particular case when $\boldsymbol{Q} = \boldsymbol{A}$, it is clearly $M = 0$ and the path $\widehat{\boldsymbol{QQ}_M}$ degenerates to the single point $\boldsymbol{A}$, having then 0 length; therefore, when $\boldsymbol{P}$ approaches $\boldsymbol{A}$ (or $\boldsymbol{B}$), then also the length of $\widehat{\boldsymbol{PP}_N}$ must be very small, and so in that case the bound $\ell(\widehat{\boldsymbol{PP}_N}) \leq 4\ell\big(\widehat{\boldsymbol{AB}}\big)$ is not enough.

For this reason, we can look back more carefully at the construction of Step IV, obtaining the following estimate.

Lemma 1.34. *For any $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ it is $\ell\big(\widehat{\boldsymbol{PP}_N}\big) \leq 113 \min\Big\{\ell\big(\widehat{\boldsymbol{AP}}\big), \ell\big(\widehat{\boldsymbol{PB}}\big)\Big\}$.*

Also in this case we are not going to present the proof, but we can quickly explain the overall idea. Taken a point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$, let us consider all the different triangles of the natural sequence of triangles $\big(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_N\big)$ related to $\boldsymbol{P}$, according to Definition 1.31. Call for brevity $\boldsymbol{A}_i\boldsymbol{B}_i$ the exit side of $\mathscr{T}_i$, being $\boldsymbol{A}_i$ the point in $\widehat{\boldsymbol{AP}}$ and $\boldsymbol{B}_i$ the one in $\widehat{\boldsymbol{PB}}$: in particular, $\boldsymbol{A}_N = \boldsymbol{A}$ and $\boldsymbol{B}_N = \boldsymbol{B}$. For consistency, call also $\boldsymbol{A}_0\boldsymbol{B}_0$ the side of $\mathscr{T}_1$ which contains $\boldsymbol{P} = \boldsymbol{P}_0$. Recalling the construction of the partition of triangles of Step III, it is clear that the exit side of any triangle $\mathscr{T}_i$ coincides with a (non-exit) side of the following triangle $\mathscr{T}_{i+1}$; as a consequence, the exit sides of $\mathscr{T}_i$ and $\mathscr{T}_{i+1}$ have exactly one common point, thus either $\boldsymbol{A}_{i+1} = \boldsymbol{A}_i$ or $\boldsymbol{B}_{i+1} = \boldsymbol{B}_i$. Suppose for simplicity that $\ell(\widehat{\boldsymbol{PB}}) \leq \ell(\widehat{\boldsymbol{AP}})$: then the claim of Lemma 1.34 can be written as

$$\sum_{i=0}^{N-1} \ell(\boldsymbol{P}_i\boldsymbol{P}_{i+1}) \leq 113\bigg(\ell(\boldsymbol{P}_0\boldsymbol{B}_0) + \sum_{i=0}^{N-1} \ell(\boldsymbol{B}_i\boldsymbol{B}_{i+1})\bigg)\,, \tag{1.41}$$

and this expression of the thesis is evidently particularly useful to start a sort of comparison argument. More precisely, let $0 \leq i < N$ be a generic index: if $\boldsymbol{B}_{i+1} \neq \boldsymbol{B}_i$, then an essentially trigonometric argument based on (i) of Lemma 1.33 gives $\ell(\boldsymbol{P}_i\boldsymbol{P}_{i+1}) \leq 4\ell(\boldsymbol{B}_i\boldsymbol{B}_{i+1})$, and in turn this is clearly a good inequality in order to obtain (1.41). On the other hand, if $\boldsymbol{B}_i = \boldsymbol{B}_{i+1}$, this does not help the validity of (1.41) since $\ell(\boldsymbol{P}_i\boldsymbol{P}_{i+1}) > 0$ while $\ell(\boldsymbol{B}_i\boldsymbol{B}_{i+1}) = 0$. However, it is reasonable to guess that the property $\boldsymbol{B}_i = \boldsymbol{B}_{i+1}$ cannot hold too often; in fact, by (iii) of Lemma 1.33 we know that $\ell(\boldsymbol{P}_{i+1}\boldsymbol{B}_i) = \ell(\boldsymbol{P}_{i+1}\boldsymbol{B}_{i+1}) \leq \ell(\boldsymbol{P}_i\boldsymbol{B}_i)$, and then if $\boldsymbol{B}_i = \boldsymbol{B}_j$ then the length $\ell(\widehat{\boldsymbol{P}_i\boldsymbol{P}_j})$ cannot be excessively large. Basically, proving Lemma 1.34 means giving a quantitative estimate to this rough argument.

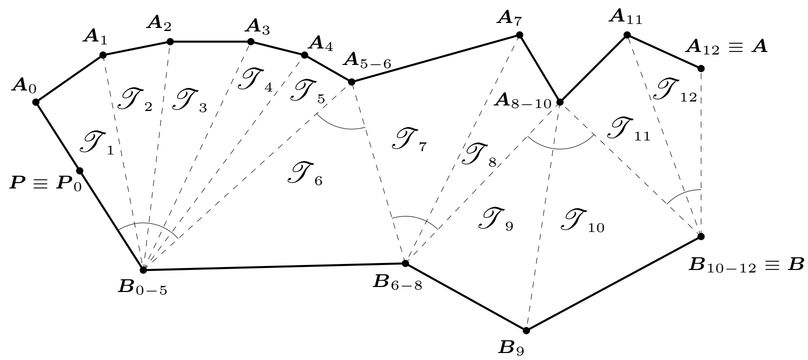

Fig. 1.11. A natural sequence of triangles $\mathscr{T}_i$ with the points $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ and the angles $\theta^{\pm}$.

Thanks to the above observations, our strategy will be to group the different triangles $\mathscr{T}_i$ of the natural sequence in different categories, more or less with the aim of studying what happens for all the triangles sharing the same point $\boldsymbol{B}_i$. We will then subdivide (see Figure 1.11) the natural sequence of triangles into sequences of consecutive triangles $\mathscr{U} = \big(\mathscr{T}_i, \mathscr{T}_{i+1}, \dots, \mathscr{T}_{i+j}\big)$ called "units", and similarly we will further subdivide the units into consecutive "systems of units" $\mathscr{S} = \big(\mathscr{U}_i, \mathscr{U}_{i+1}, \dots, \mathscr{U}_{i+j}\big)$, and finally the systems into "blocks of systems" $\mathscr{B} = \big(\mathscr{S}_i, \mathscr{S}_{i+1}, \dots, \mathscr{S}_{i+j}\big)$. This procedure will eventually allow to get (1.41), hence proving Lemma 1.34.

1.3.6 Step VI: The speed of the paths

In this step we aim to set the "speed" at which the first part of a segment PO in $\mathcal{D}$ should be mapped onto the piecewise affine path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$. To explain what this actually means, consider a given vertex $\boldsymbol{P} \in \widehat{\boldsymbol{A}\boldsymbol{B}}$ and let $\psi : [0, M] \to \mathcal{S}(\boldsymbol{A}\boldsymbol{B})$ be the arc-length parametrization of the path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$

found in Step IV, being M the length of the path. The preliminary idea that one could have, is to define the extension v on the first part of the segment PO as $v\big(tO+(1-t)P\big) := \psi(t)$. Of course, this can work only if $M<1$, which is not always true, and the first correction that comes in one's mind is to define, instead, $v\big(tO+(1-t)P\big) := \psi(\tau t)$ for a suitable constant τ so that $\tau M<1$. In other words, one would like to map the segment PO on the path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ at constant speed τ. However, some examples show that this is not the right choice: in fact, if the shape of Δ is spiraling first in one direction, and then in the other one, then there are pairs of points $\boldsymbol{P}$ and $\boldsymbol{Q}$ which are close to each other, but such that $v\big(tO+(1-t)P\big)$ and $v\big(tO+(1-t)Q\big)$ are far away for some suitable t. To avoid this problem, one can realize that it is necessary to map the segments in $\mathcal{D}$ at variable speed; since the drawback of using a constant speed is only in how the different good paths behave with respect to each other, the speeds corresponding to different vertices will have to influence each other –see (1.46) below. Let us then present how we will proceed: first of all, call $\boldsymbol{\Sigma}$ the union of all the paths $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ found in Step IV and corresponding to the different vertices $\boldsymbol{P}\in\widehat{\boldsymbol{AB}}$, which is a disjoint union by construction. We introduce then the following notation.

Definition 1.35. *The function $\tau:\boldsymbol{\Sigma}\to\mathbb{R}^+$ is said a* possible parametrization *if for every vertex $\boldsymbol{P}\in\widehat{\boldsymbol{AB}}$ it is*

- $\tau(\boldsymbol{P})=0\,,$
- *for each vertex $\boldsymbol{P}\in\widehat{\boldsymbol{AB}}$ and each $0\le i<N(\boldsymbol{P})$, the restriction of τ to the closed segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$ is affine.*

Moreover, for any $\boldsymbol{S}$ belonging to the open segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$, we shall write

$$\tau'(\boldsymbol{S}) := \frac{\tau(\boldsymbol{P}_{i+1})-\tau(\boldsymbol{P}_i)}{\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)}\,.$$

Observe that τ' corresponds to the inverse of the constant speed of the parametrization τ within the segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$. The result of this step is the following.

Lemma 1.36. *There exists a possible parametrization τ such that*

$$\frac{1}{60L}\le\tau'(\boldsymbol{S})\le 1\qquad\forall\,\boldsymbol{S}\in\boldsymbol{\Sigma}\,,\tag{1.42}$$

if $\boldsymbol{P}_i$ and $\boldsymbol{Q}_j$ belong to the same exit side of a triangle, then

$$|\tau(\boldsymbol{P}_i)-\tau(\boldsymbol{Q}_j)|\le 170L\,\ell\big(\widehat{\boldsymbol{P}\boldsymbol{Q}}\big)\,.\tag{1.43}$$

Proof. Since $\tau(\boldsymbol{P})=0$ for every vertex $\boldsymbol{P}\in\widehat{\boldsymbol{AB}}$, defining τ is equivalent to define τ'; we will build the searched function arguing again by induction over the weight of the sector.

Step A. *The case when the weight of $\mathcal{S}(\boldsymbol{AB})$ is 2.*

If the weight of the sector is 2, then the sector is in fact a triangle, and we simply set $\tau' \equiv 1$ on all $\boldsymbol{\Sigma}$. The validity of (1.42) is clearly true; concerning (1.43), we only have to consider two vertices $\boldsymbol{P}$ and $\boldsymbol{Q}$ and the points $\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ on $\boldsymbol{AB}$. But then $\tau(\boldsymbol{P}_1) = \ell\big(\boldsymbol{PP}_1\big)$ and $\tau(\boldsymbol{Q}_1) = \ell\big(\boldsymbol{QQ}_1\big)$, and then (iii) of Lemma 1.33 and the triangular inequality yield

$$|\tau(\boldsymbol{P}_1) - \tau(\boldsymbol{Q}_1)| = \big|\ell\big(\boldsymbol{PP}_1\big) - \ell\big(\boldsymbol{QQ}_1\big)\big| \leq \ell\big(\boldsymbol{PQ}\big) + \ell\big(\boldsymbol{P}_1\boldsymbol{Q}_1\big) \leq 2\ell\big(\boldsymbol{PQ}\big)\,,$$

so (1.43) is verified.

<u>Step B.</u> *The case when the weight of* $\mathcal{S}(\boldsymbol{AB})$ *is at least* 3.

Consider now the general case of a sector of weight at least 3, and let $\boldsymbol{ABC}$ be the maximal triangle in the sense of the order between triangles. By inductive assumption, it is admissible to assume that the map τ is already defined in both the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$ and satisfies (1.42) and (1.43) within the respective sectors; hence, we only have to define τ (or τ') in $\boldsymbol{\Sigma} \cap \boldsymbol{ABC}$. Notice that, given two generic vertices $\boldsymbol{P}$ and $\boldsymbol{Q}$ on $\widehat{\boldsymbol{AB}}$, and calling $N = N(\boldsymbol{P})$ and $M = N(\boldsymbol{Q})$, we already know by inductive assumption that

$$\big|\tau(\boldsymbol{P}_{N-1}) - \tau(\boldsymbol{Q}_{M-1})\big| \leq 170L\,\ell\big(\boldsymbol{PQ}\big)\,. \tag{1.44}$$

This estimate is true not only when $\boldsymbol{P}_{N-1}$ and $\boldsymbol{Q}_{M-1}$ belong both to $\boldsymbol{AC}$ or both to $\boldsymbol{BC}$, in which case (1.44) directly comes from (1.43), but also when one of the points lies in $\boldsymbol{AC}$ and the other one in $\boldsymbol{BC}$: indeed, to get (1.44) in this case just apply (1.43) once to $\boldsymbol{P}_{N-1}$ and $\boldsymbol{C}$, and once to $\boldsymbol{C}$ and $\boldsymbol{Q}_{M-1}$.

Before defining τ, we start considering the temptative map $\tilde\tau$ given by setting $\tilde\tau = \tau$ on $\mathcal{S}(\boldsymbol{AC}) \cup \mathcal{S}(\boldsymbol{BC})$, and $\tilde\tau' \equiv 1/(60L)$ on $\boldsymbol{ABC}$: in other words, for any vertex $\boldsymbol{P}$ we are defining

$$\tilde\tau(\boldsymbol{P}_N) = \tau(\boldsymbol{P}_{N-1}) + \frac{1}{60L}\,\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,. \tag{1.45}$$

It is clear that $\tilde\tau$ satisfies (1.42), but it might not verify (1.43); hence, we give the definition of τ as

$$\tau(\boldsymbol{P}_N) := \tilde\tau(\boldsymbol{P}_N) \vee \max\Big\{\tilde\tau(\boldsymbol{Q}_M) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) : \boldsymbol{Q} \in \widehat{\boldsymbol{AB}}\Big\}\,. \tag{1.46}$$

It is immediate to observe that, since $\tau(\boldsymbol{P}_N) \geq \tilde\tau(\boldsymbol{P}_N)$, then $\tau' \geq \tilde\tau' = 1/(60L)$ in $\boldsymbol{\Sigma} \cap \boldsymbol{ABC}$, so the first half of (1.42) is true. Concerning (1.43), it is a direct consequence of

$$\tau(\boldsymbol{P}_N) \geq \tau(\boldsymbol{Q}_M) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,; \tag{1.47}$$

and in turn, if $\tau(\boldsymbol{Q}_M) = \tilde\tau(\boldsymbol{Q}_M)$ then

$$\tau(\boldsymbol{P}_N) \geq \tilde\tau(\boldsymbol{Q}_M) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) = \tau(\boldsymbol{Q}_M) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,,$$

so (1.47) is true. On the other hand, if $\tau(\boldsymbol{Q}_M) = \tilde\tau(\boldsymbol{R}_K) - 170L\,\ell\big(\widehat{\boldsymbol{QR}}\big)$ for some $\boldsymbol{R} \in \widehat{\boldsymbol{AB}}$ with $K = N(\boldsymbol{R})$, then

$$\begin{aligned}\tau(\boldsymbol{P}_N) &\geq \tilde{\tau}(\boldsymbol{R}_K) - 170L\,\ell\big(\widehat{\boldsymbol{PR}}\big) \geq \tilde{\tau}(\boldsymbol{R}_K) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) - 170L\,\ell\big(\widehat{\boldsymbol{QR}}\big)\\ &= \tau(\boldsymbol{Q}_M) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,,\end{aligned}$$

hence (1.47) is again true.

Summarizing, to finish the proof we have to check the second half of (1.42), namely, that $\tau' \leq 1$, or equivalently that $\tau(\boldsymbol{P}_N) - \tau(\boldsymbol{P}_{N-1}) \leq \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)$ for every vertex $\boldsymbol{P}$. This is the hardest part of the proof of this lemma, and we skip it here; basically, assuming the existence of a vertex $\boldsymbol{P}$ such that $\tau(\boldsymbol{P}_N) - \tau(\boldsymbol{P}_{N-1}) > \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)$, and using carefully (1.44), (1.45) and (1.46), we can provide the required absurd. □

1.3.7 Step VII: The extension v onto the primary sector $\mathcal{S}(\boldsymbol{AB})$

In this Step we finally give the definition of the map v from a suitable subset $\mathcal{D}_{AB}$ of $\mathcal{D}$ onto the primary sector $\mathcal{S}(\boldsymbol{AB})$ (in fact, we will call such function u_{AB} instead of directly v, the reason will appear clear in next Step VIII). The idea is to use the good paths of Step IV and the speeds found in Step VI to send a 1−skeleton Σ inside $\mathcal{D}$ onto the set $\boldsymbol{\Sigma}$ already used in Step VI, which is simply the union of the good paths $\widehat{\boldsymbol{PP}_N}$; then, we extend this preliminary map to the whole $\mathcal{D}_{AB}$ in the piecewise affine way. For a picture of the construction, see Figure 1.12; some care is needed to do everything precisely, and to check that everything works.

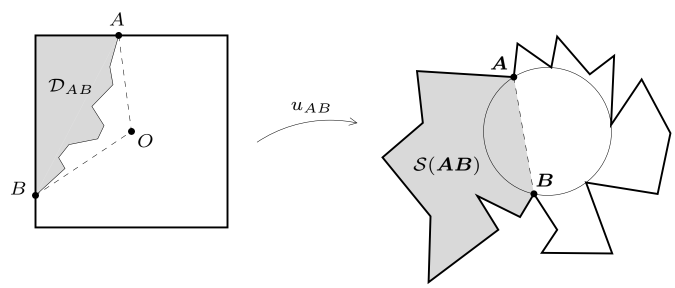

Fig. 1.12. The function $u_{AB} : \mathcal{D}_{AB} \to \mathcal{S}(\boldsymbol{AB})$.

We can directly start with the relevant definitions. First of all, taken any vertex $P \in \widehat{AB}$, we want to define the points P_i in PO for every $1 \leq i \leq N(\boldsymbol{P})$, in order to set $u_{AB}(P_i) = \boldsymbol{P}_i$: to do so, we make use of the "possible parametrization" τ constructed in Step VI, in the sense that we let

$$0 < t_{P,i} := \frac{\tau(\boldsymbol{P}_i)}{10L} \leq \frac{4}{5}\,, \qquad\qquad P_i = t_{P,i}\,O + (1 - t_{P,i})P\,. \tag{1.48}$$

The fact that $t_{P,i} \leq 4/5$ can be easily observed making use of (1.42) of Lemma 1.36, (ii) of Lemma 1.33, and the Lipschitz property of u, since

$$\tau(\boldsymbol{P}_i) \leq \tau(\boldsymbol{P}_N) \leq \sum_{j=1}^{N} \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{P}_j\big) = \ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) \leq 4\,\ell\big(\widehat{\boldsymbol{AB}}\big) \leq 4L\,\ell\big(\widehat{AB}\big) \leq 8L\,.$$

We can then define the 1-skeleton $\Sigma \subseteq \mathcal{D}$ and the set $\mathcal{D}_{AB}$. Concerning Σ, it is simply the union of the piecewise affine paths $\widehat{PP_N}$. Moreover, enumerating for a moment the vertices of $\widehat{AB}$ as $P^0 \equiv A$, P^1, ... , P^{W-1}, $P^W \equiv B$, and calling $N(i) = N(\boldsymbol{P}^i)$, the set $\mathcal{D}_{AB}$ is the polygon whose boundary is the union of $\widehat{AB}$ with the path $AP^1_{N(1)}P^2_{N(2)}\cdots P^{W-1}_{N(W-1)}B$, see again Figure 1.12.

We now define the map u_{AB} from Σ to $\boldsymbol{\Sigma}$ by setting $u_{AB}(P_i) = \boldsymbol{P}_i$ for every vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ and every $1 \leq i \leq N(\boldsymbol{P})$, and extending in the piecewise affine way: this map is injective by (iv) of Lemma 1.33. Finally, we want to extend u_{AB} from the whole $\mathcal{D}_{AB}$ to the whole $\mathcal{S}(\boldsymbol{AB})$; to do so, it is convenient to consider two consecutive vertices $\boldsymbol{P}$ and $\boldsymbol{Q}$ on $\widehat{\boldsymbol{AB}}$, and restrict our attention on the quadrilateral PQQ_MP_N and on the corresponding polygon whose boundary is $\boldsymbol{PQ}\cup\widehat{\boldsymbol{QQ}_M}\cup\boldsymbol{Q}_M\boldsymbol{P}_N\cup\widehat{\boldsymbol{P}_N\boldsymbol{P}}$, where again we call $N = N(\boldsymbol{P})$ and $M = N(\boldsymbol{Q})$. Observe that $\mathcal{D}_{AB}$ (resp., $\mathcal{S}(\boldsymbol{AB})$) is the union of

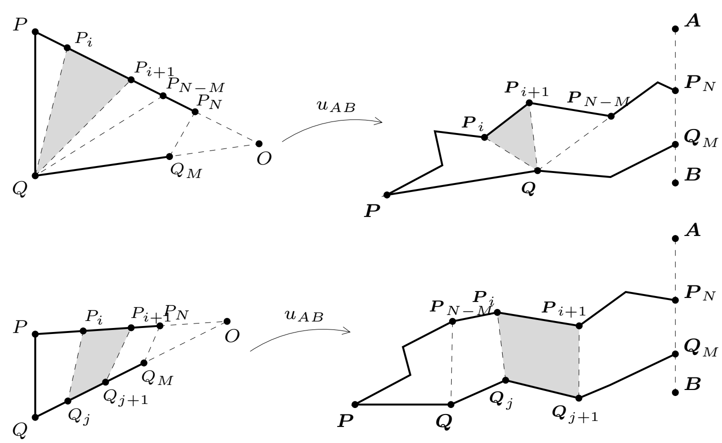

Fig. 1.13. Passing from $u_{AB} : \Sigma \to \boldsymbol{\Sigma}$ to $u_{AB} : \mathcal{D}_{AB} \to \mathcal{S}(\boldsymbol{AB})$.

these quadrilaterals (resp., polygons) over all the consecutive pairs of vertices. To extend u_{AB} to the whole $\mathcal{D}_{AB}$, let us assume by symmetry that $N > M$ (observe that $N = M$ is impossible, since $N > M$ is equivalent to say that the triangle of the partition found in Step III which contains the side $\boldsymbol{PQ}$ has $\boldsymbol{Q}$ in its exit side, and in turn this exit side must contain exactly one between $\boldsymbol{P}$ and $\boldsymbol{Q}$). The quadrilateral PQQ_MP_N is then naturally subdivided into the triangles $P_iP_{i+1}Q$ for all $0 \leq i < N-M$, and the quadrilaterals $P_iP_{i+1}Q_{j+1}Q_j$ with $j = i-(N-M)$ for $N-M \leq i < N$. Looking at Figure 1.13, it is then easy to imagine how we will extend the map u_{AB}, already defined from Σ to $\boldsymbol{\Sigma}$, to be a map from $\mathcal{D}_{AB}$ to $\mathcal{S}(\boldsymbol{AB})$. In fact, for every $0 \leq i < N-M$ we let u_{AB} to be the unique affine function sending the triangle $P_iP_{i+1}Q$ onto the triangle $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}$, while for every $N-M \leq i < N$ we let u_{AB} to be the unique affine function sending the triangle $P_iP_{i+1}Q_{j+1}$ (resp. $Q_{j+1}Q_jP_i$) onto the triangle $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}$ (resp. $\boldsymbol{Q}_{j+1}\boldsymbol{Q}_j\boldsymbol{P}_i$). It is immediate from the construction that the map $u_{AB}:\mathcal{D}_{AB}\to\mathcal{S}(\boldsymbol{AB})$ is finitely piecewise affine and bijective, as well as that $u_{AB}=u$ on $\widehat{AB}=\partial D\cap\mathcal{D}_{AB}$. The main result of this step is to give the following precise bound on the bi-Lipschitz constant of u_{AB}, which is boring to check but only needs elementary geometric estimates.

Lemma 1.37. *The map $u_{AB}:\mathcal{D}_{AB}\to\mathcal{S}(\boldsymbol{AB})$ is CL^4 bi-Lipschitz.*

1.3.8 Step VIII: The extension $\boldsymbol{v}$ onto the whole $\boldsymbol{\mathcal{D}}$

In Step VII we have been finally able to define a bi-Lipschitz function u_{AB} from $\mathcal{D}_{AB}$ to $\mathcal{S}(\boldsymbol{AB})$ for every primary sector $\mathcal{S}(\boldsymbol{AB})$. We have now to put together all these different maps and to complete the definition; recall from Step I (in particular, keep in mind Figure 1.7) that $\varDelta$ is the disjoint union of the different primary sectors, plus an internal polygon. Similarly, by the construction of Step VII (see Figure 1.12) it is immediate to realize that $\mathcal{D}$ is the disjoint union of the different sets $\mathcal{D}_{AB}$, plus an internal polygon. It is then easy to imagine a very simple way to define the whole extension v, depicted in Figure 1.14: we set $v=u_{AB}$ in each set $\mathcal{D}_{AB}$, and then we send the "internal polygon" in $\mathcal{D}$ onto the "internal polygon" in $\varDelta$ in the clear piecewise affine way (more precisely, we send each triangle $Q_jQ_{j+1}O\subseteq\mathcal{D}$ onto the corresponding triangle $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}\subseteq\varDelta$, being $\boldsymbol{O}$ the center of the central ball $\widehat{\boldsymbol{\mathcal{B}}}$). Unfortunately, the situation is not always as one can see in the figure: indeed, as we already pointed out, it may happen that there are only two primary sectors, and then the "internal polygon" in $\varDelta$ in fact degenerates to a segment; of course, this prevents our easy construction, because the internal polygon in $\varDelta$ is never degenerate, containing by (1.48) at least a region of width $1/5$ around the center. And this is not the unique bad possibility: another one is that there is indeed a non-empty internal polygon in $\varDelta$, but it does not contain the center of the central ball $\widehat{\boldsymbol{\mathcal{B}}}$, and then the definition given above makes no sense. Let us then quickly give the proof of Theorem B for the finitely piecewise affine case.

Proof (of Theorem B, finitely piecewise affine case). Let us conclude our construction for the case when u is finitely piecewise affine, that we have treated since Step I on. We apply Lemma 1.24 to get the central ball $\widehat{\boldsymbol{\mathcal{B}}}$ and the N points $\boldsymbol{A}_i$, $1\leq i\leq N$ in its boundary, so that also the N primary sectors are defined; let us also call for brevity r the radius of the central ball. For every $1\leq i\leq N$, let us call $-r<d_i<r$ the signed distance between $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ and the center of $\widehat{\boldsymbol{\mathcal{B}}}$, with the convention that the sign is positive if the center does not belong to $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, and negative otherwise: for instance, in the situation of Figure 1.14 all the four distances d_i are positive. We can then show our thesis considering three different possibilities.

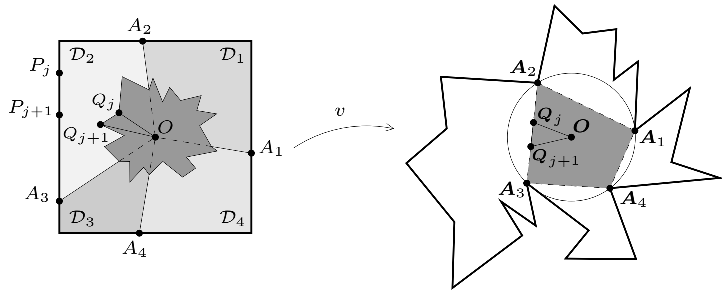

Fig. 1.14. The "easy case".

Step A. The case when $d_i \geq r/4$ for each $1 \leq i \leq N$.

This is the "easy case": enumerate as $\{P_j\}$ all the vertices in $\partial\mathcal{D}$, and call $\boldsymbol{P}_j$ the corresponding vertices in $\partial\Delta$. Moreover, as in Figure 1.14, for every j call for brevity $Q_j = (P_j)_M$ and $\boldsymbol{Q}_j = (\boldsymbol{P}_j)_M$, being $M = N(\boldsymbol{P})$. Finally, call $\boldsymbol{O}$ the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Observe now that the "internal polygon" in $\mathcal{D}$ (resp., in Δ) is the essentially disjoint union of all the triangles $Q_jQ_{j+1}O$ (resp., $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$). Let then v be the finitely piecewise affine function which corresponds with $u_{A_iA_{i+1}}$ on each $\mathcal{D}_{A_iA_{i+1}}$, and which moves in the affine way every triangle $Q_jQ_{j+1}O$ on $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$. Thanks to the assumption $d_i \geq r/4$, recalling Lemma 1.37 and by simple geometric arguments, it is easy to show that this map is CL^4 bi-Lipschitz.

Step B. The case when $-r/2 \leq d_i < r/4$ for some $1 \leq i \leq N$.

Suppose now that there is some $1 \leq i \leq N$ for which $-r/2 < d_i < r/4$: then, it is impossible that $d_j \leq -r/2$ for any other j. Also in this case we call $\boldsymbol{O}$ the center of $\widehat{\boldsymbol{\mathcal{B}}}$, but we define v in a different way than in Step A above. More precisely, take any $1 \leq i \leq N$: if $d_i \geq r/4$, then we define the map v in the whole triangle $A_iA_{i+1}O$ as before. Instead, if $d_i < r/4$, as for $i = 1$ in Figure 1.15 (left), then we give the following definition: first of all we call, as in the figure, $\boldsymbol{C} \in \partial\widehat{\boldsymbol{\mathcal{B}}}$ the point in the axis of $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$, $\boldsymbol{M}$ the middle point of $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$, and $\boldsymbol{D} \in \boldsymbol{OC}$ the point having distance $r/4$ from $\boldsymbol{O}$. Then, we call $\Phi : \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1}) \to \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ the bi-Lipschitz and piecewise affine function which moves the triangle $\boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{C}$ onto the quadrilateral $\boldsymbol{A}_i\boldsymbol{D}\boldsymbol{A}_{i+1}\boldsymbol{C}$, and leaves the rest of the sector fixed. Finally, we set $v = \Phi \circ u_{A_iA_{i+1}}$ on $\mathcal{D}_{A_iA_{i+1}}$, and for every vertex $Q_j \in \partial\mathcal{D}_{A_iA_{i+1}}$ we define again v as the affine map moving the triangle $Q_jQ_{j+1}O$ onto the triangle $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$, being this time $\boldsymbol{Q}_j = \Phi\big(u_{A_iA_{i+1}}(Q_j)\big)$. It is again possible, by means of simple but boring geometric arguments, to check that v is still CL^4 bi-Lipschitz.

Step C. The case when $d_i < -r/2$ for some $1 \leq i \leq N$.

The last possible case is when there exists some $1 \leq \bar{i} \leq N$ for which $r_{\bar{i}} < -r/2$, as it happens for $\bar{i} = 1$ in Figure 1.15 (right): notice that in this case such $\bar{i}$ is unique, and $r_i \geq r/2$ for all $i \neq \bar{i}$. In this case, $\boldsymbol{O}$ will not be

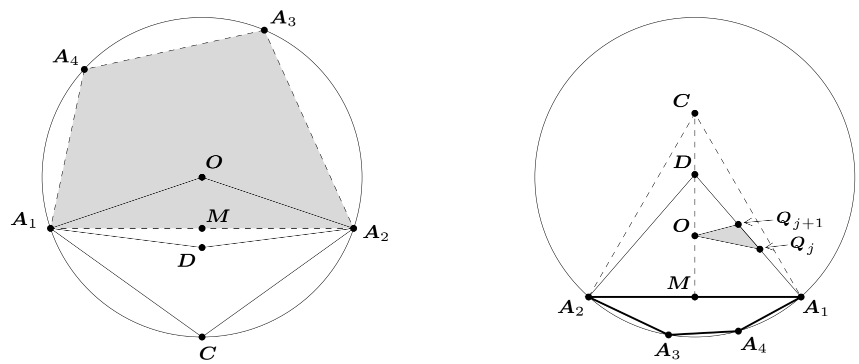

Fig. 1.15. The other two cases.

the center of $\widehat{\boldsymbol{\mathcal{B}}}$. More precisely, as in the figure, set the following definitions: let $\boldsymbol{M}$ be the midpoint of $\boldsymbol{A}_{\bar{i}}\boldsymbol{A}_{\bar{i}+1}$, $\boldsymbol{C} \in \widehat{\boldsymbol{\mathcal{B}}}$ be the point such that the triangle $\boldsymbol{A}_{\bar{i}}\boldsymbol{A}_{\bar{i}+1}\boldsymbol{C}$ is equilateral, and $\boldsymbol{D}$ and $\boldsymbol{O}$ be the two points dividing $\boldsymbol{CM}$ in three equal parts. Our map v will be such that $v(O) = \boldsymbol{O}$ as usual.

More precisely, as in Step B above we define $\Phi : \mathcal{S}(\boldsymbol{A}_{\bar{i}}\boldsymbol{A}_{\bar{i}+1}) \to \mathcal{S}(\boldsymbol{A}_{\bar{i}}\boldsymbol{A}_{\bar{i}+1})$ the piecewise affine bi-Lipschitz map which moves the triangle $\boldsymbol{A}_{\bar{i}}\boldsymbol{A}_{\bar{i}+1}\boldsymbol{C}$ onto the quadrilateral $\boldsymbol{A}_{\bar{i}}\boldsymbol{D}\boldsymbol{A}_{\bar{i}+1}\boldsymbol{C}$, and which is the identity in the rest of the sector. Then, for all $i \neq \bar{i}$, we define v on the triangle $A_iA_{i+1}O$ as in Step A (the only difference being that this time the point $\boldsymbol{O}$ has changed), while in the triangle $A_{\bar{i}}A_{\bar{i}+1}O$ we define v as in Step B. As before, checking that v is CL^4 bi-Lipschitz only requires elementary geometric calculations. □

1.3.9 Step IX: The general case of a bi-Lipschitz map $\boldsymbol{u}$

In this step we can very quickly give the proof of the general case of Theorem B, when u is just a bi-Lipschitz map and one seeks for a CL^4 bi-Lipschitz extension.

Proof (of Theorem B, general case). Let $u : \partial\mathcal{D} \to \mathbb{R}^2$ be a L bi-Lipschitz map. Then there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ of $4L$ bi-Lipschitz maps $u_j : \partial\mathcal{D} \to \mathbb{R}^2$ which are finitely piecewise affine and uniformly converge to u for $j \to \infty$: this is a very easy consequence of Lemma 1.50 below, which is a purely geometric result (not using at all the results of the present chapter, of course). We can then apply the result proved in Step VIII, finding for every j an extension $v_j : \mathcal{D} \to \mathbb{R}^2$ which is CL^4 bi-Lipschitz and finitely piecewise affine, and which coincides with u_j on $\partial\mathcal{D}$. By trivial compactness, and possibly up to a subsequence, the functions v_j uniformly converge to a CL^4 bi-Lipschitz function $v : \mathcal{D} \to \mathbb{R}^2$, and by construction $v = u$ on $\partial\mathcal{D}$. □

1.3.10 Step X: The smooth extension

We can now conclude with the last claim of Theorem B, which says that it is also possible to find a smooth extension of u.

Proof (of Theorem B, smooth extension). We are only left to show that, if $u : \partial\mathcal{D} \to \mathbb{R}^2$ is a L bi-Lipschitz map, then there exists a smooth extension, bi-Lipschitz with constant $CL^{28/3}$ if u is finitely piecewise affine, or with constant $CL^{112/3}$ in general. To prove this claim, we first take a CL^4 bi-Lipschitz extension v_0 of u: this is possible for a generic u thanks to Step IX, while if u is finitely piecewise affine this is given by Step VIII, and in this case v_0 is also finitely piecewise affine. We must then approximate v_0 with a smooth v.

If u is finitely piecewise affine, and then so is v_0, then the existence of the required smooth approximation v of v_0 is ensured by Theorem A, and the bi-Lipschitz constant of v is at most $100(CL^4)^{7/3} = CL^{28/3}$.

On the other hand, if u is generic, then the existence of the smooth approximation v of v_0 is given by Theorem C, and in this case its bi-Lipschitz constant is $C_2(CL^4)^{28/3} = CL^{112/3}$.

In both cases, v clearly fulfills all the requirement of the theorem, so the proof is finally complete. □

1.4 Part III: Approximation Theorem

This last part of the notes is devoted to present our core result, namely, the Approximation Theorem; the claim is the following.

Theorem C (Approximation for bi-Lipschitz homeomorphisms). *Let Ω and Δ be two planar bounded open sets, and let $u : \Omega \to \Delta$ be a L bi-Lipschitz homeomorphism; let also $\bar\varepsilon > 0$ and $1 \le p < \infty$. Then, there exists a piecewise affine or smooth bi-Lipschitz homeomorphism $v : \Omega \to \Delta$ such that $u = v$ on $\partial\Omega$ and*

$$\|u - v\|_{L^\infty} + \|u^{-1} - v^{-1}\|_{L^\infty} + \|Du - Dv\|_{L^p} + \|Du^{-1} - Dv^{-1}\|_{L^p} \le \bar\varepsilon\,. \tag{1.49}$$

In particular, there exists such a map v which is $C_1 L^4$ bi-Lipschitz and (countably) piecewise affine, and another such map v which is $C_2 L^{28/3}$ bi-Lipschitz and smooth, being C_1 and C_2 geometric constants not depending on u, Ω, Δ.

Notice that in the above result, since $v = u$ on $\partial\Omega$, if u is orientation-preserving then so is v. In particular, if Ω is simply connected then we already pointed out that u (then, also v) must be either orientation-preserving or orientation-reversing.

We also remark that in Theorem C we are speaking about a generic countably piecewise affine approximation, not about a *finitely* piecewise affine one, that is, associated to a finite triangulation. But in fact, it is clear that a finite piecewise affine approximation v as in the Theorem is impossible unless Ω has polygonal boundary, and u is already finitely piecewise affine on $\partial\Omega$. However, if this is the case, then there exists in fact a finitely piecewise affine approximation (thus, roughly speaking, the finitely piecewise affine interpolation exists as soon as this existence is not clearly impossible).

Theorem D (Finitely piecewise affine approximation). *Let Ω and Δ be two planar polygonal open sets, and let $u : \Omega \to \Delta$ be a L bi-Lipschitz homeomorphism which is finitely piecewise affine on $\partial\Omega$. Then, there exists a finitely piecewise affine approximation $v : \Omega \to \Delta$ as in Theorem C which is $C_1 C'(\Omega) L^4$ bi-Lipschitz.*

In these notes, we will present a complete proof of the above two theorems, but we will not keep track of the geometric constants C_1 and C_2, since we will directly call C any big geometrical constant, possibly changing from line to line. A complete proof with also explicit estimates for C_1 and C_2 can be found in [16]. Notice however that, while the constants C_1 and C_2 are purely geometric, the constant $C'(\Omega)$ of Theorem D depends on Ω. On the dependence of $C'(\Omega)$ on the domain, and on the unavoidability of this dependence, see Remark 1.53.

The strategy of the proof

Let us give a description of how the proof of Theorem C works. First of all, we observe that it is enough to find a piecewise affine approximation v of

u, since then the existence of a smooth approximation will be simply given by Theorem A. Then, let us keep in mind the following observation already done at the beginning of the introduction: if we consider a triangulation of Ω, there is an obvious way to define a piecewise affine function v similar to u, namely, the affine interpolation of u in every triangle. It is obvious that v is very close to u, together with its inverse, in L^∞, and it is reasonable to hope that the two functions are close also in $W^{1,p}$. But unfortunately, this idea can not lead to a formal proof because there is no reason why v should be still a homeomorphism –recall the situation given in Figure 1.1; and, as we said in the introduction, even taking finer and finer triangulations will not give the guarantee of the existence of some one-to-one interpolation v.

Nevertheless, the idea for proving the Theorem comes from considering a bit more the above rough strategy. The first observation is the following: let x be a "good point" in Ω, that is, a Lebesgue point for Du whose image is a Lebesgue point for Du^{-1}. The very definition of Lebesgue points suggests that, around x, the map u must be very close to an affine function; and it is easy to guess that, if an affine function is orientation preserving and not "too flat", then a map which is close enough to it will also be orientation preserving: this fact will be formally proved thanks to the "L^∞ Lemma" 1.44. Therefore, for suitably small triangles taken around a "good point" x, there is no danger of the bad behaviour described above.

The second observation is the following: taking many Lebesgue points and building arbitrary triangles around each of them is quite complicate, because the triangles would tend to overlap and/or to leave "holes" with a crazy shape, and this would be hard to work with. Instead, a much better idea is to do exactly the opposite: *first*, we select a nice and easy-to-deal triangulation in Ω, and *then*, we just divide the triangles in two categories, those which are "good triangles" (this will mean, roughly speaking, that the above bad behaviour does not occur neither in the triangle nor in triangles around it), and the other which are "bad ones". We can then define v as the standard interpolation in the union of the "good triangles", and leave the definition to be done later in the rest of Ω. In particular, the "nice triangulations" that we will take are given as follows: we fill a big portion of Ω with a regular tiling of squares all having the same side –and we imagine every square to be divided in two triangles by a diagonal. The good news here is that, since almost every point is a Lebesgue point, it can be proved that most of the squares built above are in fact made by two good triangles, and we will call them "Lebesgue squares"; more precisely, Proposition 1.43 will show that, if the side of the squares of the tiling is small enough, then an arbitrarily large portion of Ω will be filled by Lebesgue squares (see Figure 1.16, left), and then the simple affine interpolation v introduced above works. The strategy described up to now is the goal of the first part of the proof, Section 1.4.1.

Observe that then, with the first part of our construction, we have already found a suitable definition of v in an arbitrarily large portion of Ω, say Ω_ε.

The second part of the proof, contained in Section 1.4.2, will then deal with the small portion $\Omega \setminus \Omega_\varepsilon$ left out from the first part. Unfortunately, as one can easily guess, the situation in this remaining set will be quite more complicate. A good news is that, since the set on which we have to work is very small, and since u is a Lipschitz function, in extending the map v to the whole Ω one does not have to worry about the validity of the estimate (1.49), since it will automatically holds for any extension. Thus, we will be done if we can just find *any* extension of v in $\Omega \setminus \Omega_\varepsilon$.

Our idea now is to profit from Theorem B: more precisely, let us write the set $\Omega \setminus \Omega_\varepsilon$ as a countable union of squares, with sides becoming smaller and smaller to approach $\partial\Omega$ (recall that, instead, the squares built in the first part were all with the same sides): Figure 1.16, right, gives an idea of how this will be done. The utility of Theorem B at this stage is clear: instead of defining v on the whole $\Omega \setminus \Omega_\varepsilon$, small but 2-dimensional, it is enough to define it on the 1-dimensional skeleton of the triangulation of $\Omega \setminus \Omega_\varepsilon$, which is a locally finite union of curves; of course, we have to make a definition which is piecewise affine, and which matches with the function v already defined in $\partial\Omega_\varepsilon$.

The correct definition of the map v on the above-mentioned 1-skeleton will be the most complicate step for the second part of the proof, Section 1.4.2 (see in particular Proposition 1.46). In particular, we will let v be a suitable interpolation of u on points of the skeleton (actually, each side of some square will need to be subdivided in a possibly big number of segments). Our definition will be fairly easy to give while in the interior of the different sides of the squares, while it will become extremely delicate around the "crosses" between different sides; the reason of this difficulty is that the different sides joining at a generic vertex will need to have disjoint images.

Once having completed all the steps described up to now, Theorem C will easily follow in the piecewise affine case, and as we said above the smooth case will then just follow applying Theorem A. Finally, the generalization of the result to the finitely piecewise affine case, contained in Theorem D, will basically only require us another little effort, and this will be done in Section 1.4.3.

Notation and Definitions

Let us now list here some definitions which are needed only for the proof of the approximation result, and thus which will only be used within this part.

Definition 1.38 (Right polygon and r-piecewise affine function). *We say that a bounded open set $\Omega' \subseteq \mathbb{R}^2$ is a* right polygon of side-length $2r$ *(or simply an r-*polygon*) if $\partial\Omega'$ is the essentially disjoint union of finitely many segments Γ_i, each of which having length $2r$ and being parallel to one of the coordinate directions* e_1, e_2. *Moreover, a bi-Lipschitz function $u : \Omega' \to \mathbb{R}^2$ is said* r-piecewise affine on $\partial\Omega'$ *if u is affine on every segment Γ_i.*

For reasons that will be clear during the construction, it will be useful to work not directly with triangles of some triangulation, but instead with squares. Recall that we denote by $\mathcal{D}(z,r)$ the square centered at $z \in \mathbb{R}^2$, half-side r and sides parallel to e_1 and e_2; for the sake of brevity, through this part, once we will have a collection of squares $\{\mathcal{D}(z_\alpha, r_\alpha)\}_{\alpha\in\mathbb{N}}$, we will sometimes denote the generic element simply by $\mathcal{D}_\alpha$. Instead of generic triangulations, then, we will deal with "tilings", according to the next definition.

Definition 1.39 (Tiling). *If $\Omega \subseteq \mathbb{R}^2$ is a bounded open set, a* tiling of Ω *is any locally finite collection of closed squares $\{\mathcal{D}(z_\alpha, r_\alpha)\}_{\alpha\in\mathbb{N}}$ whose union is comprised between Ω and* clos Ω *and such that, $\forall \alpha \neq \beta \in \mathbb{N}$, $\mathcal{D}_\alpha \cap \mathcal{D}_\beta$ is either empty, or a common vertex of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, or a side of one of them. We call* adjacent *two squares of the tiling if their intersection is nonempty.*

Observe that some sets may admit a finite tiling (for instance, all the right polygons), but usually a set Ω admits only countable tilings.

In our construction, we will subdivide Ω in a right polygon $\Omega' \subset\subset \Omega$ and a countable tiling of $\Omega \setminus \Omega'$, locally finite in Ω. Let us give the appropriate definition.

Definition 1.40 (r-Tiling of a right polygon and tiling of (Ω, Ω')). *For any r-polygon Ω', we call* r-tiling of Ω' *the (unique) tiling $\{\mathcal{D}(z_\alpha, r)\}_{\alpha\in\mathscr{I}(r)}$ made by squares having all half-side r. If Ω is a bounded open set and $\Omega' \subset\subset \Omega$ is a r-polygon, any tiling $\{\mathcal{D}(z_j, r_j)\}_{j\in\mathbb{N}}$ of Ω such that $r \in r_j\mathbb{N}$ for every $j \in \mathbb{N}$ and whose restriction to Ω' is the r-tiling of Ω' is called a* tiling of (Ω, Ω').

Figure 1.16 shows a set Ω, a r-polygon $\Omega' \subset\subset \Omega$, the r-tiling of Ω' and a (finite subset of a) tiling of (Ω, Ω'). We give now the last two definitions: we

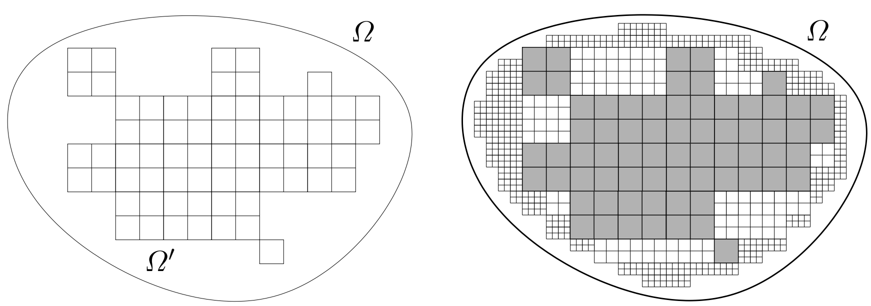

Fig. 1.16. Left: the r-tiling of an r-polygon $\Omega' \subset\subset \Omega$. Right: a tiling of (Ω, Ω'): the r-tiling of Ω' is dark.

call "grid" the skeleton of a tiling, and we call "interpolation" of a function with respect to a tiling the piecewise affine function which is affine on the grid.

Definition 1.41 (Grid). *The* grid *of the tiling* $\{\mathcal{D}_\alpha\}_{\alpha\in\mathbb{N}}$ *of some set* Ω *is the union of all the sides of the squares. We call* side *and* vertex *of the grid any side or vertex of any of the squares of the tiling.*

Definition 1.42 ((Ω', r)-interpolation of u). *Given an r-right polygon Ω' and its r-tiling* $\{\mathcal{D}_\alpha\}_{\alpha\in\mathscr{I}(r)}$*, we call* (Ω', r)-interpolation of u *the finitely piecewise affine function* $v:\Omega'\to v(\Omega')\subseteq\mathbb{R}^2$ *which equals u on the vertices of the tiling and which, for every square $\mathcal{D}_\alpha$ of the tiling, is affine on the two right triangles forming $\mathcal{D}_\alpha$ and having both as hypothenuse the north-west/south-east diagonal of $\mathcal{D}_\alpha$.*

1.4.1 Approximation on the "Lebesgue squares"

In this section we will determine an r-tiling made be some "good" squares, and we will define the approximation there. Our goal is to prove the following result. Through the section, Ω and u will always be a set and a function as in the assumptions of Theorem C.

Proposition 1.43. *For every $\varepsilon>0$ there is a right polygon $\Omega_\varepsilon\subset\subset\Omega$ of side-length $2r$ such that the (Ω_ε, r)-interpolation $v:\Omega_\varepsilon\to v(\Omega_\varepsilon)\subseteq\mathbb{R}^2$ is $L+\varepsilon$ bi-Lipschitz and satisfies*

$$\Delta_\varepsilon := v(\Omega_\varepsilon)\subset\subset\Delta\,, \tag{1.50}$$

$$\begin{aligned}\|v-u\|_{L^\infty(\Omega_\varepsilon)}+\|v^{-1}-u^{-1}\|_{L^\infty(\Delta_\varepsilon)}+\|Du-Dv\|_{L^p(\Omega_\varepsilon)}&\\ +\|Du^{-1}-Dv^{-1}\|_{L^p(\Delta_\varepsilon)}&\leq\varepsilon\,,\end{aligned} \tag{1.51}$$

$$\mathscr{L}(\Omega\setminus\Omega_\varepsilon)\leq\varepsilon\,,\quad \mathscr{L}(\Delta\setminus\Delta_\varepsilon)\leq\varepsilon\,,\quad d(\Omega_\varepsilon,\mathbb{R}^2\setminus\Omega)\geq 4r\,, \tag{1.52}$$

$$\|v-u\|_{L^\infty(\Omega_\varepsilon)}\leq\frac{\sqrt{2}r}{3L^3}\,. \tag{1.53}$$

To show this proposition, and in particular to ensure the injectivity of v, we will select only squares $\mathcal{D}_\alpha$ such that u is uniformly close to an affine function on the nine squares around $\mathcal{D}_\alpha$. More precisely, each of these affine functions will correspond to the differential of u at some Lebesgue points for Du in $\mathcal{D}_\alpha$. This is why we will denote these squares as "Lebesgue squares".

The plan of this section is the following: first we will show Lemma 1.44, which says that, if on some square Du is close in average to some bi-Lipschitz matrix M, then u is uniformly close to some affine function u_M with $Du_M=M$. Then, in Lemma 1.45 we will find the set Ω_ε, which will be obtained as union of "Lebesgue squares" of side r on which we can apply Lemma 1.44. Finally, we will prove that the (Ω_ε, r)-interpolation of u satisfies the requirements of Proposition 1.43.

The L^∞ Lemma

Here we present a fundamental L^∞ result that we will need to show Proposition 1.43. In the following, we will call for brevity $\mathbb{R}^{2\times 2}_L$ the set of those 2×2 matrices which are "L bi-Lipschitz"; more precisely, $M \in \mathbb{R}^{2\times 2}_L$ if for every vector $v \in \mathbb{R}^2$ one has

$$\frac{|v|}{L} \leq M(v) \leq L|v|\,.$$

Lemma 1.44. *For every $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ with the following property: given any $\bar z \in \Omega$, $M \in \mathbb{R}^{2\times 2}_L$ and $\rho > 0$ such that $\mathcal{D}(\bar z, \rho) \subset\subset \Omega$ and*

$$\fint_{\mathcal{D}(\bar z,\rho)} |Du(z) - M|\,dz \leq \delta, \tag{1.54}$$

there exists an affine function $u_M : \mathbb{R}^2 \to \mathbb{R}^2$ with $Du_M = M$ and such that

$$|u(z) - u_M(z)| \leq \eta\,\frac{\rho}{3} \qquad\qquad \forall\, z \in \mathcal{D}(\bar z, \rho)\,. \tag{1.55}$$

Proof. We assume for simplicity $\bar z = u(\bar z) = (0,0) \in \mathbb{R}^2$ and, fixed a constat $R \gg 1$ to be specified later, we set

$$\begin{aligned}
B^1 &:= \left\{ x \in \big[-\rho,\rho\big] :\ \int_{-\rho}^{\rho} |Du(x,t) - M|\,dt \leq \rho R\delta \right\},\\
B^2 &:= \left\{ y \in \big[-\rho,\rho\big] :\ \int_{-\rho}^{\rho} |Du(t,y) - M|\,dt \leq \rho R\delta \right\}.
\end{aligned}$$

Observe that the above integrals make sense because all the horizontal and vertical sections of u are bi-Lipschitz, since so is u. Property (1.54) and Fubini–Tonelli Theorem give

$$\big|[-\rho,\rho] \setminus B^1\big| \leq \frac{4\rho}{R}\,, \qquad\qquad \big|[-\rho,\rho] \setminus B^2\big| \leq \frac{4\rho}{R}\,. \tag{1.56}$$

Calling then $u_M(z) = Mz$ and writing for simplicity $\varphi(z) = u(z) - u_M(z)$, we can evaluate, for every $x_1,\, x_2 \in B^1$ and $y_1,\, y_2 \in B^2$,

$$\begin{aligned}
&\big|\varphi(x_1,y_1) - \varphi(x_2,y_2)\big|\\
&\qquad \leq \big|\varphi(x_1,y_1) - \varphi(x_2,y_1)\big| + \big|\varphi(x_2,y_1) - \varphi(x_2,y_2)\big|\\
&\qquad \leq \int_{x_1}^{x_2} \big|Du(t,y_1) - M\big|\,dt + \int_{y_1}^{y_2} \big|Du(x_2,t) - M\big|\,dt \leq 2\rho R\delta\,.
\end{aligned} \tag{1.57}$$

Take now any point $z \equiv (x,y) \in \mathcal{D}(\bar z, \rho)$: thanks to (1.56), we can take $x_1 \in B^1$ and $y_1 \in B^2$ so that

$$\big|x - x_1\big| \leq \frac{4\rho}{R}\,, \qquad\qquad \big|y - y_1\big| \leq \frac{4\rho}{R}\,;$$

recalling that u and u_M are L bi-Lipschitz, and thus φ is $2L$-Lipschitz, we deduce

$$\big|\varphi(x,y)-\varphi(x_1,y_1)\big| \leq \frac{8\sqrt{2}\rho L}{R}\,. \tag{1.58}$$

Applying the same argument to $(0,0)$, we find $x_2 \in B^1$ and $y_2 \in B^2$ so that

$$\big|\varphi(x_2,y_2)\big| = \big|\varphi(0,0)-\varphi(x_2,y_2)\big| \leq \frac{8\sqrt{2}\rho L}{R}\,. \tag{1.59}$$

Putting together (1.57), (1.58) and (1.59), we immediately get

$$\big|\varphi(x,y)\big| \leq \frac{16\sqrt{2}\rho L}{R} + 2\rho R\delta \leq \eta\,\frac{\rho}{3}\,,$$

where the last inequality is true as soon as we choose first R big enough and then δ small enough. This gives us (1.55), thus concluding the proof. □

A large right polygon made of Lebesgue squares

Our next objective is to find a r-right polygon $\Omega' \subset\subset \Omega$ almost filling the whole Ω and done by squares $\mathcal{D}(z_\alpha,r)$ such that assumption (1.54) of Lemma 1.44 is true in the bigger squares $\mathcal{D}(z_\alpha,3r)$. Later on, Proposition 1.43 will be shown with such a right polygon.

Lemma 1.45. *For any $\eta > 0$ there exist $r = r(\eta) > 0$ and a r-polygon $\Omega(\eta) \subset\subset \Omega$ such that $\mathscr{L}\big(\Omega \setminus \Omega(\eta)\big) \leq \eta$, and each square $\mathcal{D}(z_\alpha,r)$ of the r-tiling satisfies*

$$\mathcal{D}(z_\alpha,5r) \subset\subset \Omega\,, \qquad\qquad ⨍_{\mathcal{D}(z_\alpha,3r)} |Du(z)-M|\,dz \leq \delta\,, \tag{1.60}$$

for a suitable $M = M(\alpha) \in \mathbb{R}^{2\times 2}_L$, being $\delta = \delta(\eta)$ given by Lemma 1.44.

Proof. First of all, pick some $r_0 > 0$ and some r_0-polygon $\Omega_0 \subset\subset \Omega$ such that $\mathscr{L}\big(\Omega\setminus\Omega_0\big) \leq \eta/2$ and that every square of the r_0-tiling of Ω_0 satisfies the inclusion in (1.60). For any r satisfying $r_0 \in r\mathbb{N}$, the set Ω_0 is also a r-polygon, and we can all $\{\mathcal{D}(z_\alpha,r)\}_{\alpha\in\mathscr{I}_0(r)}$ the corresponding r-tiling. Let then

$$\mathscr{I}(r) := \Big\{\alpha \in \mathscr{I}_0(r) : ⨍_{\mathcal{D}(z_\alpha,3r)} |Du - M| \leq \delta \text{ for some } M = M(\alpha) \in \mathbb{R}^{2\times 2}_L\Big\}\,,$$
$$\Omega(\eta) := \bigcup\nolimits_{\alpha\in\mathscr{I}(r)} \mathcal{D}(z_\alpha,r)\,.$$

Since by construction $\big\{\mathcal{D}(z_\alpha,r)\big\}_{r\in\mathscr{I}(r)}$ is the r-tiling of the r-right polygon $\Omega(\eta)$ and the upper bound of (1.60) holds true, we are only left to find a suitable $r = r(\eta)$ so that $\mathscr{L}\big(\Omega_0\setminus\Omega(\eta)\big) \leq \eta/2$.

To this aim, apply the Lebesgue Differentiation Theorem to Du to find that, for $\mathscr{L}$-a.e. $z \in \Omega_0$, there exists $r(z) > 0$ such that $\mathcal{D}(z, 4r(z)) \subseteq \Omega_0$ and

$$\fint_{\mathcal{D}(z,\rho)} \big|Du(w) - Du(z)\big|\, dw \leq \frac{\delta}{2} \qquad\qquad \forall\, 0 < \rho \leq 4r(z)\,;$$

select than $r = r(\eta)$ so small that, calling

$$A(r) := \big\{ z \in \Omega_0 :\, r(z) \leq r \big\}\,,$$

one has $\mathscr{L}\left(A(r)\right) \leq \eta/2$. We claim that, for each $\alpha \in \mathscr{I}_0(r)$,

$$\mathscr{L}\Big(\mathcal{D}(z_\alpha, r) \setminus A(r)\Big) > 0 \qquad \Longrightarrow \qquad \alpha \in \mathscr{I}(r)\ : \tag{1.61}$$

this will immediately yield the thesis, since then

$$\mathscr{L}\Big(\Omega_0 \setminus \Omega(\eta)\Big) = \mathscr{L}\Big(\bigcup\nolimits_{\alpha \in \mathscr{I}_0(r)\setminus \mathscr{I}(r)} \mathcal{D}(z_\alpha, r)\Big) \leq \mathscr{L}\left(A(r)\right) \leq \frac{\eta}{2}\,.$$

Therefore, we need to show (1.61). To do so, take $\alpha \in \mathscr{I}_0(r)$ and assume that $\mathscr{L}\left(\mathcal{D}(z_\alpha, r) \setminus A(r)\right) > 0$: then, let $M = Du(z)$ for some $z \in \mathcal{D}(z_\alpha, r) \setminus A(r)$. By definition of $A(r)$ and $r(z)$ we get

$$\begin{aligned}\fint_{\mathcal{D}(z_\alpha,3r)} |Du - M| &= \frac{1}{36r^2} \int_{\mathcal{D}(z_\alpha,3r)} |Du - M| \leq \frac{1}{36r^2} \int_{\mathcal{D}(z,4r)} |Du - M| \\ &= \frac{64}{36} \fint_{\mathcal{D}(z,4r)} |Du - M| \leq \frac{8}{9}\, \delta\,,\end{aligned}$$

thus (1.61) is obtained recalling the definition of $\mathscr{I}(r)$, and then the proof is concluded. □

Proof of Proposition 1.43

We are now ready to present the proof of Proposition 1.43, which basically consists in showing that, if $\eta = \eta(\varepsilon)$ is small enough, then the set $\Omega_\varepsilon = \Omega(\eta)$ given by Lemma 1.45 satisfies the required properties.

Proof (of Proposition 1.43). Let $\varepsilon > 0$ be a given constant, and let $\eta = \eta(\varepsilon)$ be a sufficiently small constant, whose value will be precised later. Define now $\delta := \delta\big(\eta(\varepsilon)\big)$ as in Lemma 1.44, as well as $r := r\big(\eta(\varepsilon)\big)$ and $\Omega_\varepsilon := \Omega(\eta(\varepsilon))$ as in Lemma 1.45. We will prove the proposition with this choice of Ω_ε.

Let us then briefly fix some notation to be used only within this proof. First of all we call, as in the statement, $v : \Omega_\varepsilon \to \Delta_\varepsilon$ the (Ω_ε, r)-interpolation of u (see Definition 1.42) on the right polygon Ω_ε. Then, for any $\alpha \in \mathscr{I}(r)$, we pick $M_\alpha \in \mathbb{R}^{2\times 2}_L$ for which the upper bound of (1.60) holds true. Lemma 1.44, applied with $\rho = 3r$, provides then us with an affine function $u_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Du_\alpha = M_\alpha$ and

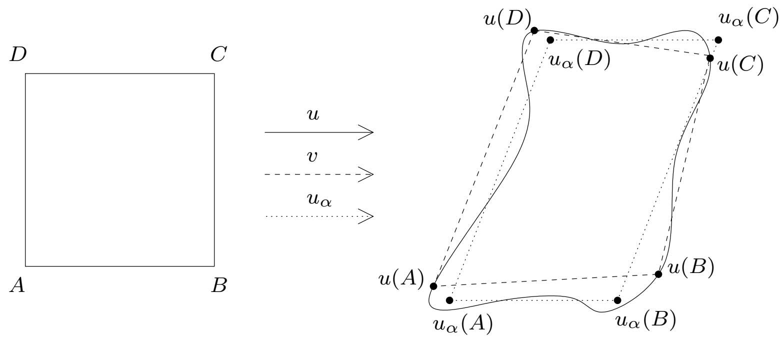

Fig. 1.17. The functions u, v and u_α on a square.

$$\big|u - u_\alpha\big| \leq \eta r \qquad \text{on } \mathcal{D}(z_\alpha, 3r) \tag{1.62}$$

(Figure 1.17 depicts the functions u, v and u_α). We can then start the proof, which will be divided in some steps for clarity.

Step I. For any $\alpha \in \mathscr{I}(r)$, $v\big(\mathcal{D}(z_\alpha, r)\big) \subseteq u\big(\mathcal{D}(z_\alpha, 3r)\big)$.

Let $\alpha \in \mathscr{I}(r)$; keeping in mind (1.62) and recalling the definition of v, we get that

$$v\big(\mathcal{D}(z_\alpha, r)\big) \subseteq \mathcal{B}\Big(u_\alpha\big(\mathcal{D}(z_\alpha, r)\big), \eta r\Big)\,, \tag{1.63}$$

where, for any set $X \subseteq \mathbb{R}^2$, $\mathcal{B}(X, r)$ is the r-neighborhood of X. Similarly, we get that

$$u\big(\mathcal{D}(z_\alpha, 3r)\big) \supseteq \Big\{x : \mathcal{B}(x, \eta r) \subseteq u_\alpha\big(\mathcal{D}(z_\alpha, 3r)\big)\Big\}\,.$$

Hence, the step is concluded if

$$\mathcal{B}\Big(u_\alpha\big(\mathcal{D}(z_\alpha, r)\big), \eta r\Big) \subseteq \Big\{x : \mathcal{B}(x, \eta r) \subseteq u_\alpha\big(\mathcal{D}(z_\alpha, 3r)\big)\Big\}\,,$$

which can be rephrased as

$$\mathcal{B}\Big(u_\alpha\big(\mathcal{D}(z_\alpha, r)\big), 2\eta r\Big) \subseteq u_\alpha\big(\mathcal{D}(z_\alpha, 3r)\big)\,.$$

And in turn, recalling that $Du_\alpha \equiv M_\alpha \in \mathbb{R}^{2\times 2}_L$, the latter inclusion is true as soon as $\eta < 1/L$.

We underline that, by this step and (1.60), we also have $\Delta_\varepsilon \subset\subset \Delta$, that is, (1.50) holds.

Step II. Injectivity of v.

For any $\alpha \in \mathscr{I}(r)$, applying (1.62) as in Step I we deduce that v is injective on $\mathcal{D}(z_\alpha, 3r) \cap \Omega_\varepsilon$ as soon as $\eta < 1/L$. In particular, it is impossible that $v(z_1) = v(z_2)$ if $z_1 \neq z_2$ belong to two adjacent squares of the r-tiling of Ω_ε. To prove the injectivity of v, then, we have to consider two non-adjacent squares $\mathcal{D}(z_\alpha, r)$ and $\mathcal{D}(z_\beta, r)$ and to show that $v\big(\mathcal{D}(z_\alpha, r)\big) \cap v\big(\mathcal{D}(z_\beta, r)\big) = \emptyset$.

And in turn, this is immediate to observe, arguing as in Step I and using (1.62) and (1.63).

Step III. *Estimate for $\|v-u\|_{L^\infty(\Omega_\varepsilon)}$ and for $\|v^{-1}-u^{-1}\|_{L^\infty(\Delta_\varepsilon)}$.*

Let us take a generic square $\mathcal{D}_\alpha$ of the r-tiling of Ω_ε. Thanks to (1.62), we know that $\|u-u_\alpha\|_{L^\infty(\mathcal{D}_\alpha)}\leq \eta r$. On the other hand, since v and u_α are both affine on the two right triangles forming $\mathcal{D}_\alpha$, and since $v=u$ on the vertices of those triangles, again (1.62) tells us that $\|v-u_\alpha\|_{L^\infty(\mathcal{D}_\alpha)}\leq \eta r$; hence $\|v-u\|_{L^\infty(\mathcal{D}_\alpha)}\leq 2\eta r$. Since this is true for every α, we get

$$\|v-u\|_{L^\infty(\Omega_\varepsilon)}\leq 2\eta r\leq \frac{\varepsilon}{4L} \tag{1.64}$$

provided that η (and hence also r) is small enough.

Since v is injective by Step II, the uniform estimate for $v^{-1}-u^{-1}$ is now easy: taken a generic point $w=v(z)\in\Delta_\varepsilon$, with $z\in\Omega_\varepsilon$, the L bi-Lipschitz property of u and (1.64) yield

$$\big|u^{-1}(w)-v^{-1}(w)\big|=\big|u^{-1}(v(z))-u^{-1}(u(z))\big|\leq L\big|v(z)-u(z)\big|\leq\frac{\varepsilon}{4}\,,$$

so that

$$\|u^{-1}-v^{-1}\|_{L^\infty(\Delta_\varepsilon)}\leq\frac{\varepsilon}{4}\,. \tag{1.65}$$

Step IV. *Estimate for $\|Dv-Du\|_{L^p(\Omega_\varepsilon)}$.*

Since by construction $|Du|\leq L$ and $|Dv|\leq\sqrt{2}L$, we start observing that

$$\begin{aligned}\|Dv-Du\|^p_{L^p(\Omega_\varepsilon)}&=\sum_{\alpha\in\mathscr{I}(r)}\|Dv-Du\|^p_{L^p(\mathcal{D}_\alpha)}\\&\leq\big(3L\big)^{p-1}\sum_{\alpha\in\mathscr{I}(r)}\|Dv-Du\|_{L^1(\mathcal{D}_\alpha)}\\&\leq\big(3L\big)^{p-1}\sum_{\alpha\in\mathscr{I}(r)}\Big(\|Dv-Du_\alpha\|_{L^1(\mathcal{D}_\alpha)}+\|Du_\alpha-Du\|_{L^1(\mathcal{D}_\alpha)}\Big)\,.\end{aligned} \tag{1.66}$$

By (1.60) we know that, for each $\alpha\in\mathscr{I}(r)$,

$$\begin{aligned}\|Du-Du_\alpha\|_{L^1(\mathcal{D}_\alpha)}&=\int_{\mathcal{D}(z_\alpha,r)}\big|Du-M_\alpha\big|\\&\leq 36r^2\fint_{\mathcal{D}(z_\alpha,3r)}\big|Du-M_\alpha\big|\leq 36\delta r^2=9\delta\big|\mathcal{D}_\alpha\big|\,.\end{aligned} \tag{1.67}$$

Let us now study $\|Dv-Du_\alpha\|_{L^1(\mathcal{D}_\alpha)}$: consider the triangle $T=z_1z_2z_3$, where

$$z_1\equiv z_\alpha+\big(-r,-r\big)\,,\qquad z_2\equiv z_\alpha+\big(r,-r\big)\,,\qquad z_3\equiv z_\alpha+\big(r,r\big)\,.$$

Since both v and u_α are affine on T, then $Dv-Du_\alpha$ is a constant linear function on T. Recalling again (1.62), let us then calculate

$$\begin{aligned}\Big|\big(Dv_{|T}-Du_\alpha\big)(2r\mathrm{e}_1)\Big| &= \Big|\big(v(z_2)-v(z_1)\big)-\big(u_\alpha(z_2)-u_\alpha(z_1)\big)\Big|\\ &= \Big|\big(u(z_2)-u(z_1)\big)-\big(u_\alpha(z_2)-u_\alpha(z_1)\big)\Big| \leq 2\eta r\,,\end{aligned}$$

and similarly

$$\Big|\big(Dv_{|T}-Du_\alpha\big)(2r\mathrm{e}_2)\Big| = \Big|\big(v(z_3)-v(z_2)\big)-\big(u_\alpha(z_3)-u_\alpha(z_2)\Big| \leq 2\eta r\,.$$

We deduce that $\|Dv-Du_\alpha\|_{L^\infty(T)} \leq \sqrt{2}\eta$, and repeating the same argument for all the triangles in which $\mathcal{D}(z_\alpha,3r)\cap\Omega_\varepsilon$ is divided we get

$$\big\|Dv-Du_\alpha\big\|_{L^\infty(\mathcal{D}(z_\alpha,3r)\cap\Omega_\varepsilon)} \leq \sqrt{2}\eta\,. \tag{1.68}$$

Inserting this estimate and (1.67) into (1.66), we get

$$\begin{aligned}\|Dv-Du\|^p_{L^p(\Omega_\varepsilon)} &\leq \big(3L\big)^{p-1}\big(9\delta+\sqrt{2}\eta\big)\sum_{\alpha\in\mathscr{I}(r)}\big|\mathcal{D}_\alpha\big|\\ &= \big(3L\big)^{p-1}\big(9\delta+\sqrt{2}\eta\big)\big|\Omega_\varepsilon\big| \leq \left(\frac{\varepsilon}{4}\right)^p,\end{aligned} \tag{1.69}$$

where as usual the last inequality holds true as soon as η, hence also δ, is small enough.

Step V. Bi-Lipschitz estimate for v.

Let z, $z'\in\Omega_\varepsilon$, and let α be such that $z\in\mathcal{D}(z_\alpha,r)$. Suppose first that $z'\in\mathcal{D}(z_\alpha,3r)$: in this case, by the definition of v, by (1.68) and by the fact that u_α is L bi-Lipschitz, we directly obtain

$$\left(\frac{1}{L}-\sqrt{2}\eta\right)\big|z-z'\big| \leq \big|v(z)-v(z')\big| \leq \big(L+\sqrt{2}\eta\big)\big|z-z'\big|\,. \tag{1.70}$$

If, instead, $z'\notin\mathcal{D}(z_\alpha,3r)$, and then $|z-z'|\geq 2r$, then the L^∞ estimate (1.64) gives on one hand

$$\begin{aligned}\big|v(z)-v(z')\big| &\leq \big|u(z)-u(z')\big|+\big|v(z)-u(z)\big|+\big|v(z')-u(z')\big|\\ &\leq L\big|z-z'\big|+4\eta r \leq \big(L+2\eta\big)\big|z-z'\big|\,,\end{aligned} \tag{1.71}$$

and on the other hand

$$\begin{aligned}\big|v(z)-v(z')\big| &\geq \big|u(z)-u(z')\big|-\big|v(z)-u(z)\big|-\big|v(z')-u(z')\big|\\ &\geq \left(\frac{1}{L}-2\eta\right)\big|z-z'\big|\,.\end{aligned} \tag{1.72}$$

Putting together (1.70), (1.71) and (1.72), we obtain that v is $L+\varepsilon$ bi-Lipschitz if η is small enough.

Step VI. Estimate for $\|Dv^{-1}-Du^{-1}\|_{L^p(\Delta_\varepsilon)}$.

Keep in mind the elementary fact that, for any two invertible matrices A and B, one has $\left|B^{-1}-A^{-1}\right| \leq \left|A^{-1}\right|\left|B^{-1}\right|\left|B-A\right|$. Fix then any $\alpha \in \mathscr{I}(r)$; recalling that u and u_α are L bi-Lipschitz, the result of Step I, $Du_\alpha \equiv M_\alpha$ on $\mathcal{D}_\alpha$ and (1.60), we get

$$\begin{aligned}
\|Du^{-1} - Du_\alpha^{-1}\|_{L^1(v(\mathcal{D}_\alpha))} &= \int_{v(\mathcal{D}(z_\alpha,r))} \left|Du^{-1}(z) - Du_\alpha^{-1}(z)\right| dz \\
&\leq L^2 \int_{u(\mathcal{D}(z_\alpha,3r))} \left|Du\big(u^{-1}(z)\big) - M_\alpha\right| dz \leq L^4 \int_{\mathcal{D}(z_\alpha,3r)} \left|Du(w) - M_\alpha\right| dw \\
&= 36r^2L^4 \fint_{\mathcal{D}(z_\alpha,3r)} \left|Du - M_\alpha\right| \leq 36r^2L^4\delta = 9L^4\delta\left|\mathcal{D}_\alpha\right|.
\end{aligned}$$

On the other hand, since u_α is L bi-Lipschitz by definition while v is $(L+\varepsilon)$ bi-Lipschitz by Step V, and by (1.68), for any $z \in \mathcal{D}_\alpha$ we obtain

$$\left|Dv^{-1}\big(v(z)\big) - Du_\alpha^{-1}\big(v(z)\big)\right| \leq L\big(L+\varepsilon\big)\sqrt{2}\,\eta \leq 2L^2\eta\,,$$

which since z is generic gives

$$\left\|Dv^{-1} - Du_\alpha^{-1}\right\|_{L^\infty(v(\mathcal{D}_\alpha))} \leq 2L^2\eta\,.$$

Arguing as in (1.66) we get then

$$\begin{aligned}
\|Dv^{-1} - Du^{-1}\|^p_{L^p(\Delta_\varepsilon)} &\leq \big(3L\big)^{p-1} \sum_{\alpha\in\mathscr{I}(r)} \|Dv^{-1} - Du^{-1}\|_{L^1(v(\mathcal{D}_\alpha))} \\
&\leq \big(3L\big)^{p-1} \sum_{\alpha\in\mathscr{I}(r)} 2L^2\eta\left|v(\mathcal{D}_\alpha)\right| + 9L^4\delta\left|\mathcal{D}_\alpha\right| \\
&= \big(3L\big)^{p-1}\Big(2L^2\eta\left|\Delta_\varepsilon\right| + 9L^4\delta\left|\Omega_\varepsilon\right|\Big) \leq \left(\frac{\varepsilon}{4}\right)^p,
\end{aligned} \tag{1.73}$$

where as usual the last estimate holds possibly decreasing η and then also δ.

Step VII. Conclusion.

We are finally ready to conclude the proof of Proposition 1.43. The fact that v is $L+\varepsilon$ bi-Lipschitz is given by Step V; the validity of (1.50) has been observed in Step I; the estimate (1.51) follows adding (1.64), (1.65), (1.69) and (1.73); the first and the third inequality in (1.52) follow by Lemma 1.45, while the second one follows by the first, by the bi-Lipschitz property of u, and by the L^∞ estimate (1.64); finally, concerning (1.53), it suffices to recall that $\|v-u\|_{L^\infty(\Omega_\varepsilon)} \leq 2\eta r$ by (1.64) and then choose $\eta \leq \sqrt{2}/(6L^3)$. □

1.4.2 Approximation out of "Lebesgue squares"

This section is devoted to the proof of Theorem C; thanks to Proposition 1.43, we are left to define a suitable countably piecewise affine approximation of u out of the r-polygon Ω_ε. Even though the construction will be quite involved, the scheme is extremely simple: first of all, we will cover $\Omega \setminus \Omega_\varepsilon$ with a suitable (locally finite) tiling; then, we will define a bi-Lipschitz piecewise affine approximation of u on the grid of this tiling; finally, we will extend the approximation to the interior of the squares of the grid by means of Theorem B. The main result of this section is the following.

Proposition 1.46. *Let $v_\varepsilon : \Omega_\varepsilon \to \Delta_\varepsilon$ be a piecewise affine bi-Lipschitz function as in Proposition 1.43. Then there exists a $C_1 L^4$ bi-Lipschitz countably piecewise affine function $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \to \Delta \setminus \Delta_\varepsilon$, satisfying $\tilde v_\varepsilon \equiv u$ on $\partial\Omega$, and $\tilde v_\varepsilon \equiv v_\varepsilon$ on $\partial\Omega_\varepsilon$.*

Let us directly see how Theorem C easily follows as a consequence of Propositions 1.43 and 1.46.

Proof (of Theorem C). Take $\bar\varepsilon > 0$, let $\varepsilon = \varepsilon(\bar\varepsilon)$ to be specified later, and apply Proposition 1.43 to get an r-polygon $\Omega_\varepsilon \subset\subset \Omega$ and a piecewise affine bi-Lipschitz function $v_\varepsilon : \Omega_\varepsilon \to \Delta_\varepsilon$; apply then Proposition 1.46 to find a $C_1 L^4$ bi-Lipschitz function $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \to \Delta \setminus \Delta_\varepsilon$. We define $v : \Omega \to \Delta$ as $v \equiv v_\varepsilon$ on Ω_ε and $v \equiv \tilde v_\varepsilon$ on $\Omega \setminus \Omega_\varepsilon$: since v_ε and $\tilde v_\varepsilon$ are bi-Lipschitz with constant $L+\varepsilon$ and $C_1 L^4$ respectively, and since $\tilde v_\varepsilon \equiv v_\varepsilon$ on $\partial\Omega_\varepsilon$, we have that v is a bi-Lipschitz homeomorphism with constant $C_1 L^4$. It remains to show that v satisfies (1.49), and by (1.51) we only have to consider what happens in $\Omega \setminus \Omega_\varepsilon$. Since $\tilde v_\varepsilon$ is bi-Lipschitz with constant $C_1 L^4$, (1.52) implies

$$\begin{aligned} \|Dv - Du\|_{L^p(\Omega\setminus\Omega_\varepsilon)} &\leq \|Dv - Du\|_{L^\infty(\Omega\setminus\Omega_\varepsilon)} \big|\Omega \setminus \Omega_\varepsilon\big|^{1/p} \\ &\leq \big(L + C_1 L^4\big)\varepsilon^{1/p}\,, \end{aligned} \tag{1.74}$$

and similarly

$$\|Dv^{-1} - Du^{-1}\|_{L^p(\Delta\setminus\Delta_\varepsilon)} \leq \big(L + C_1 L^4\big)\varepsilon^{1/p}\,. \tag{1.75}$$

Concerning the L^∞ estimates, since $\big|\Omega \setminus \Omega_\varepsilon\big| \leq \varepsilon$ then for every $z \in \Omega \setminus \Omega_\varepsilon$ there exists $z' \in \Omega_\varepsilon$ such that $|z - z'| \leq \sqrt{\varepsilon/\pi}$, hence (1.51) gives

$$\begin{aligned} |v(z) - u(z)| &\leq |v(z) - v(z')| + |v(z') - u(z')| + |u(z') - u(z)| \\ &\leq \big(L + C_1 L^4\big)\sqrt{\frac{\varepsilon}{\pi}} + \|v_\varepsilon - u\|_{L^\infty(\Omega_\varepsilon)} \leq \big(L + C_1 L^4\big)\sqrt{\frac{\varepsilon}{\pi}} + \varepsilon\,. \end{aligned}$$

Arguing in the same way to bound $\big|v^{-1}(w) - u^{-1}(w)\big|$ for a generic $w \in \Delta \setminus \Delta_\varepsilon$ yields

$$\max\Big\{\|v - u\|_{L^\infty(\Omega\setminus\Omega_\varepsilon)}, \|v^{-1} - u^{-1}\|_{L^\infty(\Delta\setminus\Delta_\varepsilon)}\Big\} \leq \big(L + C_1 L^4\big)\sqrt{\frac{\varepsilon}{\pi}} + \varepsilon\,. \tag{1.76}$$

Putting together (1.74), (1.75) and (1.76), we obtain the validity of (1.49) provided that $\varepsilon = \varepsilon(\bar\varepsilon)$ is sufficiently small. The required countably piecewise affine approximation has then been found. Concerning the smooth one, it can be obtained simply applying Theorem A to v. $\square$

Let us now work on the proof of Proposition 1.46, for which we will need some more notation. Recalling that Ω_ε is a r-polygon for some $r = r(\varepsilon)$, we select a suitable tiling $\big\{\mathcal{D}_j = \mathcal{D}(z_j, r_j)\big\}_{j\in\mathbb{N}}$ of $(\Omega, \Omega_\varepsilon)$, in the sense of Definition 1.40 (hence, $\{\mathcal{D}_j\}$ is a tiling of Ω whose restriction to Ω_ε coincides with the r-tiling of Ω_ε). We only want that the tiling $\{\mathcal{D}_j\}$ verifies

$$r_j = r \quad \forall\, j : \operatorname{clos}\mathcal{D}_j \cap \partial\Omega_\varepsilon \neq \emptyset\,, \tag{1.77}$$

$$\mathcal{D}_j \subset\subset \Omega \quad \forall\, j \in \mathbb{N}\,, \tag{1.78}$$

which is clearly possible thanks to (1.52). Notice that (1.78) forces the tiling to be countable and not finite, and the squares to become smaller and smaller when approaching $\partial\Omega$. Of course, if Ω were a r-right polygon, instead of (1.78) one could have asked the tiling to be finite, and (1.77) could have been improved asking $r_j = r$ for every j: we will discuss this possibility more in detail in Remark 1.52, since this will be the basis to show Theorem D.

From now on, then, we fix a tiling of $(\Omega, \Omega_\varepsilon)$ satisfying (1.77) and (1.78), and we denote by $\mathcal{Q}$ its associated 1-dimensional grid in the sense of Definition 1.41. We also set $\mathcal{Q}' = \mathcal{Q} \cap (\Omega \setminus \operatorname{clos}\Omega_\varepsilon)$, which is the part of the grid on which we really need to work. In words, $\mathcal{Q}'$ is the 1-dimensional set made by all the sides of the grid $\mathcal{Q}$ which lie in $\Omega \setminus \operatorname{clos}\Omega_\varepsilon$.

Let now w_α be the generic vertex of $\mathcal{Q}'$, thus the generic vertex of the grid $\mathcal{Q}$ which does not belong to Ω_ε (but it may belong to $\partial\Omega_\varepsilon$!). The point w_α is of the form $w_\alpha = z_j + (\pm r_j, \pm r_j)$ for some j, and it is an extreme of either three or four sides of $\mathcal{Q}$. To shorten the notation, we will denote the other extremes of these sides by w_α^i with $1 \leq i \leq \bar{i}(\alpha)$, and then $\bar{i}(\alpha) \in \{3, 4\}$. Finally, we will denote by ℓ_α the minimum of the lengths of the sides $w_\alpha w_\alpha^i$. In particular, if $w_\alpha \notin \partial\Omega_\varepsilon$ then w_α is one extreme of either three or four sides of $\mathcal{Q}' \subseteq \mathcal{Q}$. On the other hand, if $w_\alpha \in \partial\Omega_\varepsilon$ then by (1.77) it is extreme of four sides of $\mathcal{Q}$, either one or two of these four sides lie in $\mathcal{Q}'$, and $\ell_\alpha = r$.

Theorem B tells us that, to obtain the piecewise affine function $\tilde{v}_\varepsilon$ of Proposition 1.46, we can limit ourselves to define it, in a suitable way, on the 1-dimensional grid $\mathcal{Q}'$. This is exactly what we will do, and our main ingredients will be the following two lemmas. The first one (Lemma 1.50) states that, on any given segment inside Ω, u can be uniformly approximated as well as desired with suitable piecewise affine $4L$ bi-Lipschitz functions. While this is clearly of primary importance to define the piecewise affine approximation $\tilde{v}_\varepsilon$ of u on the sides of $\mathcal{Q}'$, it is still not enough. In fact, some additional care is needed to treat the "crosses" of $\mathcal{Q}'$ (that is, the regions around the vertices), in order to ensure that $\tilde{v}_\varepsilon$ is injective on the whole $\mathcal{Q}'$. This will be obtained thanks to the second result (Lemma 1.51).

Before stating the two lemmas, a couple of pieces of notation more are needed.

Definition 1.47 (Interpolation of u). *Let $pq \subset\subset \Omega$ be a segment, and let $\{z_i z_{i+1}\}_{0\leq i<N}$ be N essentially disjoint segments whose union is pq, with*

$z_0 = p$ and $z_N = q$. We call interpolation of u *(related to the segments $\{z_i z_{i+1}\}_{0\le i<N}$) the finitely piecewise affine function $u_{pq} : pq \to \mathbb{R}^2$ defined by*

$$u_{pq}\Big(z_i + t\big(z_{i+1} - z_i\big)\Big) = u(z_i) + t\big(u(z_{i+1}) - u(z_i)\big)$$

for every $0 \le i < N$ and every $0 \le t \le 1$.

Definition 1.48 (Adjusted function and crosses). *Let $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ be a sequence such that for any α one has $3L\xi_\alpha \le \ell_\alpha$. For any $\alpha \in \mathbb{N}$ and any $1 \le i \le \bar{i}(\alpha)$, we define ξ^i_α as the biggest number such that*

$$\begin{cases} \Big|u(w_\alpha) - u\big(w_\alpha + \xi^i_\alpha(w^i_\alpha - w_\alpha)\big)\Big| \le \xi_\alpha & \text{if } w_\alpha w^i_\alpha \subseteq \mathcal{Q}'\,, \\ \Big|u(w_\alpha) - v_\varepsilon\big(w_\alpha + \xi^i_\alpha(w^i_\alpha - w_\alpha)\big)\Big| \le \xi_\alpha & \text{if } w_\alpha w^i_\alpha \subseteq \mathcal{Q}\setminus\mathcal{Q}'\,. \end{cases}$$

We call then adjusted function *the map $u_{\rm adj} : \mathcal{Q} \to \mathbb{R}^2$ defined as follows. First, we set $u_{\rm adj} = v_\varepsilon$ on $\mathcal{Q}\setminus\mathcal{Q}'$. Then, let $w_\alpha w_\beta$ be a side of $\mathcal{Q}'$, thus being $w_\beta = w^i_\alpha$ and $w_\alpha = w^j_\beta$ for two suitable $1 \le i \le \bar{i}(\alpha)$ and $1 \le j \le \bar{i}(\beta)$. We define*

$$u_{\rm adj}\big(w_\alpha + t(w_\beta - w_\alpha)\big) := \begin{cases} u(w_\alpha) + \frac{t}{\xi^i_\alpha}\Big(u\big(w_\alpha + \xi^i_\alpha(w_\beta - w_\alpha)\big) - u(w_\alpha)\Big) & \text{in } (0, \xi^i_\alpha)\,, \\ u\big(w_\alpha + t(w_\beta - w_\alpha)\big) & \text{in } (\xi^i_\alpha, 1 - \xi^j_\beta)\,, \\ u(w_\beta) + \frac{(1-t)}{\xi^j_\beta}\Big(u\big(w_\beta + \xi^j_\beta(w_\alpha - w_\beta)\big) - u(w_\beta)\Big) & \text{in } (1 - \xi^j_\beta, 1). \end{cases}$$

In words, for any side in $\mathcal{Q}'$, $u_{\rm adj}$ coincides with u in the internal part of the side, while the two parts closest to the vertices w_α and w_β are replaced with segments. Moreover, for any vertex w_α of $\mathcal{Q}'$ we will define its associated cross *as*

$$Z_\alpha = \bigcup_{i=1}^{\bar{i}(\alpha)} \Big\{ w_\alpha + t(w^i_\alpha - w_\alpha) :\ 0 \le t \le \xi^i_\alpha \Big\}\,.$$

Remark 1.49. Some remarks are in order at this moment. First of all, since u is L bi-Lipschitz on the whole Ω, and v_ε is L bi-Lipschitz on every segment $w_\alpha w^i_\alpha \subseteq \mathcal{Q}\setminus\mathcal{Q}'$, the choice $3L\xi_\alpha \le \ell_\alpha$ directly implies $0 < \xi^i_\alpha \le 1/3$ for any α and any $1 \le i \le \bar{i}(\alpha)$. As a consequence, two different crosses have always empty intersection. For the same reason, each of the $\bar{i}(\alpha)$ extremes of the cross Z_α has a distance at least ξ_α/L from w_α. Finally, we claim that $\mathcal{B}(u(w_\alpha),\xi_\alpha) \cap \mathcal{B}(u(w_\beta),\xi_\beta) = \emptyset$ for all different α and β. Indeed, assuming without loss of generality that $\ell_\alpha \ge \ell_\beta$, we have $\big|u(w_\beta) - u(w_\alpha)\big| \ge \ell_\alpha/L$, and then

$$\xi_\alpha + \xi_\beta \le \frac{\ell_\alpha}{3L} + \frac{\ell_\beta}{3L} \le \frac{2\ell_\alpha}{3L} < \big|u(w_\beta) - u(w_\alpha)\big|\,.$$

Lemma 1.50. *Let $pq \subset\subset \Omega$ be a segment; then for every $\delta > 0$ there exists a map $u^\delta_{pq} : pq \to \Delta$ which is a $4L$ bi-Lipschitz interpolation of u with the property that $\|u^\delta_{pq} - u\|_{L^\infty(pq)} \leq \delta$.*

Lemma 1.51. *There exists a sequence $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ such that the associated adjusted function $u_{\rm adj} : \mathcal{Q} \to \mathbb{R}^2$ is $18L$ bi-Lipschitz and $u_{\rm adj}(\mathcal{Q}) \subseteq \Delta$.*

Let us immediately see how Proposition 1.46 follows from Lemmas 1.50 and 1.51; later, we will show the two lemmas.

Proof (of Proposition 1.46). The scheme of the proof is the following: first of all, we take an adjusted function $u_{\rm adj} : \mathcal{Q} \to \mathbb{R}^2$ as in Definition 1.48, where the sequence $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ is chosen as in Lemma 1.51. Since this function $u_{\rm adj}$ is piecewise affine around the crosses but not in the interior of the sides, we pass to a new function $u'_{\rm adj} : \mathcal{Q} \to \Delta$, coinciding with $u'_{\rm adj}$ near the crosses but injective and piecewise affine, by the aid of Lemma 1.50. Finally, the function $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \to \Delta \setminus \Delta_\varepsilon$ is obtained by extending $u'_{\rm adj}$ in the interior of all the squares forming $\Omega \setminus \Omega_\varepsilon$ thanks to Theorem B. The proof is divided in several steps for clarity.

<u>Step I.</u> *Definition of $u'_{\rm adj} : \mathcal{Q} \to \Delta$.*

Let $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ be a sequence as in Lemma 1.51, and let $u_{\rm adj} : \mathcal{Q} \to \mathbb{R}^2$ be the corresponding adjusted function according to Definition 1.48, which is $18L$ bi-Lipschitz and whose image is contained in Δ. We want to introduce the function $u'_{\rm adj} : \mathcal{Q} \to \Delta$. We start setting $u'_{\rm adj} \equiv u_{\rm adj} \equiv v_\varepsilon$ on $\mathcal{Q} \setminus \mathcal{Q}'$; then, let $w_\alpha w_\beta$ be a generic side in $\mathcal{Q}'$, and call pq the *internal segment* of $w_\alpha w_\beta$, that is, p and q are the extremes of the segment $w_\alpha w_\beta \setminus \big(Z_\alpha \cup Z_\beta\big)$. Let now $\delta = \delta(\alpha, \beta)$ be a small constant, to be precised later; we set then $u'_{\rm adj} = u_{\rm adj}$ on the external part of the segment, i.e., $w_\alpha w_\beta \setminus pq$, and $u'_{\rm adj} = u^\delta_{pq}$ on pq, where u^δ_{pq} is the interpolation given by Lemma 1.50.

Let us study the map $u'_{\rm adj}$: by construction, it is clearly a continuos and countably piecewise affine function on $\mathcal{Q}$; moreover, since the different constants $\delta(\alpha, \beta)$ can be chosen independently from each other and very small, and since every internal segment pq is compactly supported in Ω, we can also assume that $u'_{\rm adj}(\mathcal{Q}) \subseteq \Delta$. Notice that $u'_{\rm adj}$ has been obtained by glueing the $4L$ bi-Lipschitz functions u^δ_{pq} and the $18L$ bi-Lipschitz function $u_{\rm adj}$, thus by a trivial geometric argument $u'_{\rm adj}$ is $18\sqrt{2}L$-Lipschitz (but, *a priori*, it could be not bi-Lipschitz and not even injective!). In fact, we aim to show that $u'_{\rm adj}$ is bi-Lipschitz (so, in particular, injective); more precisely, for every two points $z, z' \in \mathcal{Q}$ we will show that

$$\big|u'_{\rm adj}(z) - u'_{\rm adj}(z')\big| \geq \frac{1}{CL}|z - z'|\,; \tag{1.79}$$

this will be the content of the Steps II–IV, where we will consider the different possible reciprocal positions of z and z'; we recall that C is a sufficiently large purely geometric constant, whose value may increase from line to line. Eventually, in Step V we will define v_ε by extending u'_{adj} to the interior of all the squares of the tiling of $\Omega \setminus \Omega_\varepsilon$, and we will check that this map satisfies the requirement of the proposition.

Step II. The case in which $z \in pq \subseteq w_\alpha w_\beta$, $z' \notin w_\alpha w_\beta$.

We start considering what happens if one of the points, say z, is in the internal segment pq of some side $w_\alpha w_\beta$, while z' does not belong to the side $w_\alpha w_\beta$. We can further subdivide the case into two subcases: if z' does not belong to any internal segment (hence, either z' belongs to some cross, or $z' \in \Omega_\varepsilon$), then $u'_{\text{adj}}(z') = u_{\text{adj}}(z')$ and then Lemma 1.51 gives

$$\begin{aligned}\big|u'_{\text{adj}}(z) - u'_{\text{adj}}(z')\big| &= \big|u^\delta_{pq}(z) - u_{\text{adj}}(z')\big|\\ &\geq \big|u_{\text{adj}}(z) - u_{\text{adj}}(z')\big| - \big|u^\delta_{pq}(z) - u_{\text{adj}}(z)\big|\\ &= \big|u_{\text{adj}}(z) - u_{\text{adj}}(z')\big| - \big|u^\delta_{pq}(z) - u(z)\big|\\ &\geq \frac{1}{18L}\,|z-z'| - \delta(\alpha,\,\beta) \geq \frac{1}{CL}\,|z-z'|\end{aligned}$$

provided that

$$\delta(\alpha,\,\beta) \leq \frac{\min\big\{\xi_\alpha,\,\xi_\beta\big\}}{20L^2} \tag{1.80}$$

(keep in mind that, as pointed out in Remark 1.49, we know that $|z-z'| \geq \xi_\alpha/L$). Hence, (1.79) is proved in this first subcase.

Consider now the other subcase, namely, when z' belongs to some other internal segment $p'q' \subseteq w_{\alpha'}w_{\beta'}$. In that case, we again know that $|z-z'| \geq \xi_\alpha/L$, as well as $|z-z'| \geq \xi_{\alpha'}/L$; hence, by (1.80) we directly have

$$\begin{aligned}\big|u'_{\text{adj}}(z) - u'_{\text{adj}}(z')\big| &= \big|u^\delta_{pq}(z) - u^{\delta'}_{p'q'}(z')\big|\\ &\geq \big|u(z) - u(z')\big| - \big|u(z) - u^\delta_{pq}(z)\big| - \big|u(z') - u^{\delta'}_{p'q'}(z')\big|\\ &\geq \frac{1}{L}\,\big|z-z'\big| - \delta(\alpha,\,\beta) - \delta(\alpha',\,\beta') \geq \frac{1}{CL}\big|z-z'\big|\,,\end{aligned}$$

and again (1.79) is established.

Step III. The case in which $z \in pq \subseteq w_\alpha w_\beta$, $z' \in w_\alpha w_\beta$.

Consider the situation when z again belongs to the internal segment $pq \subseteq w_\alpha w_\beta$, but now also z' belongs to the same side $w_\alpha w_\beta$. If also z' is in the internal part pq, then (1.79) is immediate since u'_{adj} coincides with u^δ_{pq} both in z and in z', and u^δ_{pq} is $4L$ bi-Lipschitz by Lemma 1.50. Assume then that $z' \in w_\alpha p$ (if $z' \in qw_\beta$ the situation is clearly the same).

By Definition 1.48, $u'_{\text{adj}}(z') = u_{\text{adj}}(z')$ lies in the segment $u(w_\alpha)u(p)$, which is a radius of the ball $\mathcal{B}\big(u(w_\alpha), \xi_\alpha\big)$. Hence, for every point s outside the same ball, a trivial geometric argument tells us that

$$\big|s - u_{\text{adj}}(z')\big| \geq \frac{\big|s - u(p)\big| + \big|u(p) - u_{\text{adj}}(z')\big|}{3}\,. \tag{1.81}$$

We cannot directly apply this estimate to $u'_{\rm adj}(z) = u^\delta_{pq}(z)$, because this point might lie within the ball; however, by the Definition 1.47 of interpolation, we know that $u^\delta_{pq}(z)$ belongs to a segment $u(\tilde p)u(\tilde q)$, for two points $\tilde p$, $\tilde q \in pq$; and in turn, by Definition 1.48 of ξ^i_α, we have that both $u(\tilde p)$ and $u(\tilde q)$ are outside the ball $\mathcal{B}\big(u(w_\alpha), \xi_\alpha\big)$. Summarizing, $u^\delta_{pq}(z)$ belongs to a segment whose both extremes are outside the ball. As a consequence, if

$$\big|u^\delta_{pq}(z) - u(w_\alpha)\big| \geq \xi_\alpha\,,$$

then we can apply estimate (1.81) with $s = u^\delta_{pq}(z)$; and otherwise, if

$$\big|u^\delta_{pq}(z) - u(w_\alpha)\big| < \xi_\alpha\,,$$

a simple geometric argument provides

$$\xi_\alpha - \big|u^\delta_{pq}(z) - u(w_\alpha)\big| \ll \big|u^\delta_{pq}(z) - u(p)\big|\,,$$

if $\delta(\alpha,\beta)$ is small enough with respect to ξ_α. This fact together with (1.81) readily gives

$$\begin{aligned}
\big|u^\delta_{pq}(z) - u_{\rm adj}(z')\big| &\geq \frac{\big|u^\delta_{pq}(z) - u(p)\big| + \big|u(p) - u_{\rm adj}(z')\big|}{4}\\
&= \frac{\big|u^\delta_{pq}(z) - u^\delta_{pq}(p)\big| + \big|u_{\rm adj}(p) - u_{\rm adj}(z')\big|}{4}
\end{aligned}$$

since $u(p) = u^\delta_{pq}(p) = u_{\rm adj}(p)$. Using again the fact that u^δ_{pq} is $4L$ bi-Lipschitz and $u_{\rm adj}$ is $18L$ bi-Lipschitz, as well as that z, p and z' are aligned, we obtain

$$\begin{aligned}
\big|u'_{\rm adj}(z) - u'_{\rm adj}(z')\big| &= \big|u^\delta_{pq}(z) - u_{\rm adj}(z')\big|\\
&\geq \frac{\big|u^\delta_{pq}(z) - u^\delta_{pq}(p)\big|}{4} + \frac{\big|u_{\rm adj}(p) - u_{\rm adj}(z')\big|}{4}\\
&\geq \frac{|z-p|}{16L} + \frac{|p-z'|}{72L} \geq \frac{|z-z'|}{72L}\,.
\end{aligned}$$

Hence, (1.79) is checked also in this case.

Step IV. The case in which neither z nor z' are in some internal segment.

The last case that we have to consider, to prove (1.79), is when neither z nor z' belong to some internal segment; therefore, both z and z' belong either to some cross, or to $\mathcal{Q} \setminus \mathcal{Q}'$. By definition, this means that $u'_{\rm adj}(z) = u_{\rm adj}(z)$, as well as $u'_{\rm adj}(z') = u_{\rm adj}(z')$; and then, since $u_{\rm adj}$ is $18L$ bi-Lipschitz by Lemma 1.51, (1.79) is already known. Summarizing, we have shown that (1.79) is valid, and then $u'_{\rm adj} : \mathcal{Q} \to \Delta$ is CL bi-Lipschitz.

Step V. Conclusion.

We can now finally define the searched function $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \to \Delta \setminus \Delta_\varepsilon$. To do so, consider each square $\mathcal{D}_j \subseteq \Omega \setminus \Omega_\varepsilon$: since we have a CL bi-Lipschitz function on $\partial\mathcal{D}_j \subseteq \mathcal{Q}$, namely, $u'_{\rm adj}$, we can define $\tilde v_\varepsilon$ on $\mathcal{D}_j$ as a CL^4 bi-Lipschitz extension by means of Theorem B. Notice that, since $\tilde v_\varepsilon = u'_{\rm adj}$ on $\partial\mathcal{D}_j$ for every j, then $\tilde v_\varepsilon$ is globally continuous, countably piecewise affine, and it satisfies $\tilde v_\varepsilon = u$ on $\partial\Omega$ and $\tilde v_\varepsilon = u'_{\rm adj} = v_\varepsilon$ on $\partial\Omega_\varepsilon$. Moreover, for each square $\mathcal{D}_j \subseteq \Omega\setminus\Omega_\varepsilon$ one has $\partial\big(\tilde v_\varepsilon(\mathcal{D}_j)\big) = u'_{\rm adj}(\partial\mathcal{D}_j)$, and this yields at once that the image of $\tilde v_\varepsilon$ is exactly $\Delta\setminus\Delta_\varepsilon$, and that $\tilde v_\varepsilon$ is injective. In turn, the injectivity implies that $\tilde v_\varepsilon$ is globally CL^4 bi-Lipschitz, since it is so on every square. The thesis is then concluded. □

We now make a quick remark, which we will need to show Theorem D.

Remark 1.52. Assume that Ω is a right polygon of side-length $2\bar r$ and that u is $\bar r$-piecewise affine on $\partial\Omega$, according to Definition 1.38. Consider then the r-polygon Ω_ε given by Proposition 1.43: by the construction of Section 1.4.1, it is not restrictive to assume that $\bar r \in r\mathbb{N}$ and that Ω_ε is a subset of the r-tiling of Ω. Therefore, we can repeat *verbatim* the construction of Proposition 1.46 using, as tiling, the r-tiling of Ω: hence, we are replacing assumptions (1.77) and (1.78) by asking that the tiling is finite, and that $r_j = r$ for all the squares of the tiling –see the remark right after (1.78). Notice that this makes sense because, since u is already $\bar r$-piecewise affine on the boundary of the $\bar r$-polygon Ω, then there is no need for the tiling to use smaller and smaller squares approaching the boundary. Consequently, the bi-Lipschitz approximation $\tilde v_\varepsilon$ built in Proposition 1.46 is in this case finitely piecewise affine instead of countably piecewise affine. We also remark that the assumption that u is $\bar r$-piecewise affine on $\partial\Omega$ is unavoidable, since otherwise the map $\tilde v_\varepsilon$ could not coincide with u on $\partial\Omega$.

To conclude the proof of Theorem C, we then only need to give the proofs of Lemma 1.50 and of Lemma 1.51.

Proof (of Lemma 1.50). Fix a small number $\rho>0$, to be precised later; define $t_0 = 0$ and $z_0 = p$, and then, recursively,

$$\begin{aligned}
t_{i+1} &:= \max\Big\{t_i < t \le 1:\ \big|u(z_i) - u\big(p + t(q-p)\big)\big| \le \rho\Big\}\,,\\
z_{i+1} &:= p + t_{i+1}(q-p)\,.
\end{aligned}$$

This defines a finite sequence of points $z_0 = p,\, z_1,\, \dots z_N = q$ in pq, being $N = N(p,q,\rho)$. Hence, we define u^δ_{pq} as the interpolation of u associated with the sequence $\{z_i\}$, according to Definition 1.47: by definition, u^δ_{pq} is finitely piecewise affine, and moreover it is L-Lipschitz because so is u. Since u is uniformly continuous on pq, the bound $\|u - u^\delta_{pq}\|_{L^\infty(pq)} \le \delta$ is true as soon as ρ is small enough. To conclude, we then only have to check that

$$\big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| \ge \frac{1}{4L}|z - z'| \tag{1.82}$$

for every z, z' in pq.

First of all, if both z and z' belong to the same segment z_iz_{i+1}, then (1.82) holds true because on that segment u^δ_{pq} is affine and u is L bi-Lipschitz.

Then, we can assume that $z \in z_iz_{i+1}$ and $z' \in z_jz_{j+1}$ with $j > i$. If $j = i+1$, then an immediate geometric argument ensures that

$$\begin{aligned} u^\delta_{pq}(z)\widehat{u^\delta_{pq}(z_{i+1})}u^\delta_{pq}(z') &= u^\delta_{pq}(z_i)\widehat{u^\delta_{pq}(z_{i+1})}u^\delta_{pq}(z_{i+2}) \\ &= u(z_i)\widehat{u(z_{i+1})}u(z_{i+2}) \geq \frac{\pi}{3}\,, \end{aligned}$$

and then

$$\begin{aligned} \big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| &\geq \frac{\big|u^\delta_{pq}(z) - u^\delta_{pq}(z_{i+1})\big|}{2} + \frac{\big|u^\delta_{pq}(z_{i+1}) - u^\delta_{pq}(z')\big|}{2} \\ &= |z - z_{i+1}|\,\frac{|u(z_i) - u(z_{i+1})|}{2|z_i - z_{i+1}|} + |z_{i+1} - z'|\,\frac{|u(z_{i+1}) - u(z_{i+2})|}{2|z_{i+1} - z_{i+2}|} \\ &\geq \frac{|z - z'|}{2L}\,, \end{aligned}$$

thus (1.82) is again checked.

Finally, let us assume that $j > i+1$; but then, $u^\delta_{pq}(z')$ belongs to the segment $u(z_j)u(z_{j+1})$, and both $u(z_j)$ and $u(z_{j+1})$ are outside the two balls $\mathcal{B}(u(z_i),\rho) \cup \mathcal{B}(u(z_{i+1}),\rho)$, while $u^\delta_{pq}(z) \in u(z_i)u(z_{i+1})$. A simple consequence is that

$$\big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| \geq \sqrt{3}\rho/2\,,$$

and this yields

$$\begin{aligned} \big|u(z_i) - u(z_{j+1})\big| &\leq \big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| + 2\rho \leq \left(1 + \frac{4}{3}\sqrt{3}\right)\big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| \\ &\leq 4\big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big|\,, \end{aligned}$$

which in turn implies

$$\big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\big| \geq \frac{\big|u(z_i) - u(z_{j+1})\big|}{4} \geq \frac{\big|z_i - z_{j+1}\big|}{4L} \geq \frac{\big|z - z'\big|}{4L}\,.$$

Therefore, we have shown the validity of (1.82) in every possible case, and this concludes the proof. □

Proof (of Lemma 1.51). Take a vertex w_α of the grid $\mathcal{Q}'$, and fix a constant $\xi_\alpha \leq \ell_\alpha/(3L)$, with $\xi_\alpha = \ell_\alpha/(3L)$ if $w_\alpha \in \partial\Omega_\varepsilon$. Since $\ell_\alpha \leq 2r$ for every α, in particular we have

$$\xi_\alpha \leq \frac{2r}{3L}\,. \tag{1.83}$$

Define now ξ^i_α as in Definition 1.48 and, for any $1 \leq i \leq \bar{i}(\alpha)$, let $p^i = w_\alpha + \xi^i_\alpha\big(w^i_\alpha - w_\alpha\big)$. We claim that for every α it is

$$u(w_\alpha)u(p^i) \subset\subset \Delta \qquad \forall\, 1 \le i \le \bar{i}(\alpha) \ : \tag{1.84}$$

in fact, if $w_\alpha \in \partial\Omega_\varepsilon$, then (1.84) is already ensured by (1.53) and (1.52) in Proposition 1.43; on the other hand, if $w_\alpha \in \Omega \setminus \partial\Omega_\varepsilon$ then (1.84) becomes true possibly up to decrease the value of ξ_α.

Let now $u_{\rm adj}$ be the adjusted function corresponding to the sequence $\{\xi_\alpha\}$, as in Definition 1.48; since $u_{\rm adj}(\mathcal{Q}) \subseteq \Delta$ thanks to (1.84), to obtain the thesis we only have to show that

$$\frac{|z-z'|}{18L} \le \big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \le 18L|z-z'|\,. \tag{1.85}$$

for every z, $z' \in \mathcal{Q}$. We will do it in some steps.

<u>Step I.</u> *For all α, $u_{\rm adj}^{-1}\Big(\operatorname{clos}\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)\Big) = Z_\alpha$.*

In this first step we aim to show that, for any α and for any $z \in \mathcal{Q}$, we have $\big|u_{\rm adj}(z) - u(w_\alpha)\big| \le \xi_\alpha$ if and only if $z \in Z_\alpha$. If $z \in Z_\alpha$, then $z \in w_\alpha p^i$ for some $1 \le i \le \bar{i}(\alpha)$, and since $u_{\rm adj}$ is affine in the segment $w_\alpha p^i$, while $\big|u_{\rm adj}(p^i) - u(w_\alpha)\big| = \xi_\alpha$, then of course $\big|u_{\rm adj}(z) - u(w_\alpha)\big| \le \xi_\alpha$.

Suppose, on the other hand, that $z \notin Z_\alpha$, so that we have to prove that $\big|u_{\rm adj}(z) - u(w_\alpha)\big| > \xi_\alpha$.

If $z \in w_\alpha w_\alpha^i$ for some $1 \le i \le \bar{i}(\alpha)$, then there are three possibilities. First, if $w_\alpha w_\alpha^i \subseteq \mathcal{Q} \setminus \mathcal{Q}'$, then $u_{\rm adj} = v_\varepsilon$ is affine on the whole side $w_\alpha w_\alpha^i$, so the claim is trivial. Second, if $w_\alpha w_\alpha^i \subseteq \mathcal{Q}'$ and z belongs to the cross Z_β associated to the vertex $w_\beta = w_\alpha^i$, then $u_{\rm adj}(z)$ belongs to the ball $\mathcal{B}\big(u(w_\beta),\xi_\beta\big)$, which does not intersect $\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)$ by Remark 1.49, thus the claim again follows. Third, if $w_\alpha w_\alpha^i \subseteq \mathcal{Q}'$ and $z \notin Z_\beta$, then $u_{\rm adj}(z) = u(z)$, thus the claim is again true by the definition of ξ_α^i.

To finish this step, we have then to consider a point $z \notin Z_\alpha$ which does not belong to any side of $\mathcal{Q}$ having w_α as one extreme. We distinguish again some possibilities. First, if z belongs to the cross Z_β for some β, then again the claim is true since $\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big) \cap \mathcal{B}\big(u(w_\beta),\xi_\beta\big) = \emptyset$ by Remark 1.49. Second, if z does not belong to any cross and $z \in \mathcal{Q}'$, then $u_{\rm adj}(z) = u(z)$ so the claim is true because, using the bi-Lipschitz property of u and the fact that $\xi_\alpha \le \ell_\alpha/(3L)$, we have

$$u(z) \in \mathcal{B}\big(u(w_\alpha),\xi_\alpha\big) \quad \Longrightarrow \quad \big|z - w_\alpha\big| \le \frac{\ell_\alpha}{3}\,,$$

which is impossible because $|z-w_\alpha| > \ell_\alpha$. Third and last, assume that z does not belong to any cross and $z \in \mathcal{Q}\setminus\mathcal{Q}'$. If this is the case, then $u_{\rm adj}(z) = v_\varepsilon(z)$, $|z - w_\alpha| \ge 2r$ by construction and by (1.77), and then (1.53) and (1.83) give

$$\begin{aligned}\big|u_{\rm adj}(z) - u(w_\alpha)\big| &= \big|v_\varepsilon(z) - u(w_\alpha)\big| \ge \big|u(z) - u(w_\alpha)\big| - \big|u(z) - v_\varepsilon(z)\big| \\ &\ge \frac{\big|z-w_\alpha\big|}{L} - \frac{\sqrt{2}r}{3L^3} \ge \frac{r}{L} > \xi_\alpha\,,\end{aligned}$$

thus the first step is concluded.

We fix now two points $z, z' \in \mathcal{Q}$: the proof of Lemma 1.51, thus also of Theorem C, will follow once we show the validity of (1.85). We will do this in next steps, considering all the possible mutual positions of z and z'.

Step II. Validity of (1.85) if $z, z' \in Z_\alpha$.

Let us first suppose that both z and z' belong to a same cross Z_α. By construction, $u_{\rm adj}$ is L bi-Lipschitz on each segment $w_\alpha p^i$, hence to show (1.85) we can assume without loss of generality that $z \in w_\alpha p^1$ and $z' \in w_\alpha p^2$. Therefore, we readily have

$$\begin{aligned}\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| &\leq \big|u_{\rm adj}(z) - u_{\rm adj}(w_\alpha)\big| + \big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z')\big| \\ &\leq L\big(|z - w_\alpha| + |w_\alpha - z'|\big) \leq \sqrt{2}\, L\, |z - z'| \,.\end{aligned} \tag{1.86}$$

On the other side, to estimate $\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big|$ from below, assume without loss of generality that

$$\big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z)\big| \leq \big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z')\big| \,,$$

and define $z'' \in w_\alpha z'$ so that

$$\big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z)\big| = \big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z'')\big| \,,$$

which is uniquely defined since $u_{\rm adj}$ is affine in the segment $w_\alpha z'$.

Since the triangle $u_{\rm adj}(w_\alpha)u_{\rm adj}(z)u_{\rm adj}(z'')$ is isosceles, then

$$u_{\rm adj}(z)\widehat{u_{\rm adj}(z'')}u_{\rm adj}(z') \geq \frac{\pi}{2} \,. \tag{1.87}$$

The validity of (1.85) will follow at once as soon as we prove that

$$\frac{\big|u_{\rm adj}(z) - u_{\rm adj}(z'')\big|}{\big|z - z''\big|} \geq \frac{1}{2L} \,, \tag{1.88}$$

because in this case, also recalling (1.87) and the fact that $u_{\rm adj}$ is L bi-Lipschitz on the segment $z'z'' \subseteq w_\alpha p^2$, we get

$$\begin{aligned}\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| &\geq \frac{\sqrt{2}}{2}\Big(\big|u_{\rm adj}(z) - u_{\rm adj}(z'')\big| + \big|u_{\rm adj}(z'') - u_{\rm adj}(z')\big|\Big) \\ &\geq \frac{\sqrt{2}}{2}\bigg(\frac{|z - z''|}{2L} + \frac{|z'' - z'|}{L}\bigg) \geq \frac{\sqrt{2}}{4L}|z - z'| \,,\end{aligned}$$

which recalling (1.86) allows us to write, whenever z and z' are under the assumptions of this step,

$$\frac{\sqrt{2}}{4L}|z - z'| \leq \big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \leq \sqrt{2}L|z - z'| \,, \tag{1.89}$$

which is in turn stronger than (1.85). Summarizing, to conclude this step we only need to prove the validity of (1.88).

As usual, we have to consider three possible cases. First, if both $w_\alpha p^1$ and $w_\alpha p^2$ are contained in $\mathcal{Q}'$, then by definition and recalling the definition of z''

$$\frac{\big|u_{\rm adj}(z)-u_{\rm adj}(z'')\big|}{\big|z-z''\big|}=\frac{\big|u_{\rm adj}(p^1)-u_{\rm adj}(p^2)\big|}{\big|p^1-p^2\big|}=\frac{\big|u(p^1)-u(p^2)\big|}{\big|p^1-p^2\big|}\geq\frac{1}{L}\,,$$

so (1.88) holds true. Second, if both $w_\alpha w_\alpha^1$ and $w_\alpha w_\alpha^2$ belong to $\mathcal{Q}\setminus\mathcal{Q}'$, then since v_ε is $L+\varepsilon$ bi-Lipschitz we have

$$\frac{\big|u_{\rm adj}(z)-u_{\rm adj}(z'')\big|}{\big|z-z''\big|}=\frac{\big|v_\varepsilon(z)-v_\varepsilon(z'')\big|}{\big|z-z''\big|}\geq\frac{1}{L+\varepsilon}\,,$$

so again (1.88) holds true. Finally, assume that $w_\alpha w_\alpha^1\subseteq\mathcal{Q}'$ while $w_\alpha w_\alpha^2\subseteq\mathcal{Q}\setminus\mathcal{Q}'$. In this case, it must clearly be $w_\alpha\in\partial\Omega_\varepsilon$, hence by Remark 1.49 we know that $|p^1-w_\alpha|$ and $|p^2-w_\alpha|$ are both at least $\xi_\alpha/L=2r/(3L^2)$, which implies

$$|p^1-p^2|\geq\frac{2\sqrt{2}}{r(3L^2)}\,.$$

Therefore, recalling again (1.53) we have

$$\begin{aligned}\frac{\big|u_{\rm adj}(z)-u_{\rm adj}(z'')\big|}{\big|z-z''\big|}&=\frac{\big|u_{\rm adj}(p^1)-u_{\rm adj}(p^2)\big|}{\big|p^1-p^2\big|}=\frac{\big|u(p^1)-v_\varepsilon(p^2)\big|}{\big|p^1-p^2\big|}\\&\geq\frac{\big|u(p^1)-u(p^2)\big|}{\big|p^1-p^2\big|}-\frac{\big|u(p^2)-v_\varepsilon(p^2)\big|}{\big|p^1-p^2\big|}\\&\geq\frac{1}{L}-\frac{\sqrt{2}r/(3L^3)}{2\sqrt{2}r/(3L^2)}=\frac{1}{2L}\,,\end{aligned}$$

thus (1.88) has been checked in all the three possible cases and this step is concluded.

Step III. Validity of (1.85) if for all α one has $z,\,z'\notin\operatorname{int}Z_\alpha$.

Let us assume now that neither z nor z' belong to the interior of any cross. In this case, we have $u_{\rm adj}(z)=u(z)$ if $z\in\mathcal{Q}'$, or $u_{\rm adj}(z)=v_\varepsilon(z)$ if $z\in\mathcal{Q}\setminus\mathcal{Q}'$, and the same holds for z'. Since u is L bi-Lipschitz while v_ε is $L+\varepsilon$ bi-Lipschitz, the validity of (1.85) is obvious if both $z,\,z'\in\mathcal{Q}'$, as well as if both $z,\,z'\in\mathcal{Q}\setminus\mathcal{Q}'$. Therefore, we have to concentrate only on the case in which $z\in\mathcal{Q}'$, $z'\in\mathcal{Q}\setminus\mathcal{Q}'$.

In this case, the main observation is that $|z-z'|\geq 2\sqrt{2}r/(3L^2)$, since both z and z' must be at distance at least $2r/(3L^2)$ from any vertex $w_\alpha\in\partial\Omega_\varepsilon$, because they do not belong to any cross Z_α. As a consequence, again by (1.53) we get

$$\begin{aligned}\big|u_{\rm adj}(z)-u_{\rm adj}(z')\big|&=\big|u(z)-v_\varepsilon(z')\big|\geq\big|u(z)-u(z')\big|-\big|u(z')-v_\varepsilon(z')\big|\\&\geq\frac{|z-z'|}{L}-\frac{\sqrt{2}r}{3L^3}\geq\frac{|z-z'|}{2L}\end{aligned}$$

and

$$\begin{aligned}\big|u_{\rm adj}(z)-u_{\rm adj}(z')\big| &= \big|u(z)-v_\varepsilon(z')\big| \leq \big|u(z)-u(z')\big| + \big|u(z')-v_\varepsilon(z')\big|\\ &\leq L\,|z-z'| + \frac{\sqrt{2}r}{3L^3} \leq \Big(L+\frac{1}{2L}\Big)|z-z'|\,,\end{aligned}$$

thus also in this case (1.85) is proven. In particular, under the assumptions of this step one has

$$\frac{|z-z'|}{2L} \leq \big|u_{\rm adj}(z)-u_{\rm adj}(z')\big| \leq \frac{3}{2}\,L\,|z-z'|\,. \tag{1.90}$$

<u>Step IV.</u> *Validity of (1.85) if $z \in Z_\alpha$ and for all β one has $z' \notin \operatorname{int} Z_\beta$.*

Let us now consider the case when z belongs to some cross Z_α (say, $z \in w_\alpha p^1$), while z' does not belong to the interior of any cross. To obtain the above estimate in (1.85), we start with the trivial geometric observation that there exists $1 \leq i \leq \bar{i}(\alpha)$ such that

$$|z-z'| \geq \frac{\sqrt{2}}{2}\Big(|z-p^i| + |p^i-z'|\Big)\,,$$

not necessarily being $i=1$. As a consequence, we apply estimate (1.89) of Step II for the points z and p^i –both belonging to Z_α– and the estimate (1.90) of Step III for the points p^i and z' –none of which belonging to the interior of some Z_β– to get

$$\begin{aligned}\big|u_{\rm adj}(z)-u_{\rm adj}(z')\big| &\leq \big|u_{\rm adj}(z)-u_{\rm adj}(p^i)\big| + \big|u_{\rm adj}(p^i)-u_{\rm adj}(z')\big|\\ &\leq \sqrt{2}L|z-p^i| + \frac{3}{2}\,L|p^i-z'| \leq \frac{3}{2}\,\sqrt{2}\,L|z-z'|\,.\end{aligned}$$

We now have to show the below estimate in (1.85), and we start recalling that by Step I we have

$$u_{\rm adj}(z) \in \operatorname{clos}\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)\,, \qquad\qquad u_{\rm adj}(z') \notin \mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)\,.$$

As already observed in (1.81), a trivial geometric argument ensures then that

$$\big|u_{\rm adj}(z)-u_{\rm adj}(z')\big| \geq \frac{\big|u_{\rm adj}(z)-u_{\rm adj}(p^1)\big| + \big|u_{\rm adj}(p^1)-u_{\rm adj}(z')\big|}{3}\,. \tag{1.91}$$

Thus, using the L bi-Lipschitz property of $u_{\rm adj}$ in the segment $w_\alpha p^1$, and the estimate (1.90) of Step III for p^1 and z', we get

$$\big|u_{\rm adj}(z)-u_{\rm adj}(z')\big| \geq \frac{|z-p^1|}{3L} + \frac{|p^1-z'|}{6L} \geq \frac{|z-z'|}{6L}\,.$$

Summarizing, under the assumptions of this step we have

$$\frac{|z-z'|}{6L} \leq \big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \leq \frac{3}{2}\sqrt{2}\,L|z-z'|\,, \tag{1.92}$$

hence in particular (1.85) is again checked.

<u>Step V.</u> *Validity of (1.85) if $z \in Z_\alpha$ and $z' \in Z_\beta$.*

We face now the last possible situation, that is, when z and z' belong to two different crosses. The argument will be very similar to that of Step IV; indeed, to get the above estimate in (1.85) we again start observing that for some $1 \leq i \leq \bar{i}(\alpha)$ it must be

$$|z-z'| \geq \frac{\sqrt{2}}{2}\Big(|z-p^i| + |p^i - z'|\Big)\,.$$

Therefore, applying estimate (1.89) of Step II to the points z, $p^i \in Z_\alpha$, and estimate (1.92) of Step IV to the points $z' \in Z_\beta$ and p^i –which does not belong to the interior of any cross– we find

$$\begin{aligned}\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| &\leq \big|u_{\rm adj}(z) - u_{\rm adj}(p^i)\big| + \big|u_{\rm adj}(p^i) - u_{\rm adj}(z')\big|\\ &\leq \sqrt{2}L|z-p^i| + \frac{3}{2}\sqrt{2}\,L|p^i - z'| \leq 3L|z-z'|\,.\end{aligned}$$

Finally, to get the below estimate in (1.85) we use again (1.91), which holds true exactly as in Step IV, together with the L bi-Lipschitz property of $u_{\rm adj}$ in $w_\alpha p^1$ and the estimate (1.92) of Step IV for p^1 and z', obtaining

$$\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \geq \frac{|z-p^1|}{3L} + \frac{|p^1-z'|}{18L} \geq \frac{|z-z'|}{18L}\,.$$

We have then established the validity of (1.85) in all the possible cases, hence the proof is concluded. □

1.4.3 Finitely piecewise affine approximation on polygonal domains

This last section is devoted to present a proof of Theorem D, which will be very short since only a simple modification of the arguments of Section 1.4.2 is needed.

Proof (of Theorem D). We begin by considering the particular case when Ω is a $\bar{r}$-right polygon and u is $\bar{r}$-piecewise affine on $\partial\Omega$ according to Definition 1.38. As already underlined in Remark 1.52, we can then slightly modify the proofs of Proposition 1.43 and Proposition 1.46 to get what the following results: first of all, there exist some r such that $\bar{r} \in r\mathbb{N}$, a r-right polygon $\Omega_\varepsilon \subset\subset \Omega$ which is part of the r-tiling of Ω, and a $L+\varepsilon$ bi-Lipschitz and finitely piecewise affine function $v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^2$ for which (1.50), (1.51), (1.52) and (1.53) hold. And second, there exists also a finitely piecewise affine map $\tilde{v}_\varepsilon : \Omega \setminus \Omega_\varepsilon \to \Delta \setminus \Delta_\varepsilon$ which is $C_1 L^4$ bi-Lipschitz and which coincides with

u on $\partial\Omega$ and with v_ε on $\partial\Omega_\varepsilon$. Therefore, gluing v_ε and $\tilde{v}_\varepsilon$ exactly as done in the proof of Theorem C, the very same proof provides the required C_1L^4 bi-Lipschitz and (finitely) piecewise affine approximation of u.

Consider now, instead, the general situation of a polygon Ω with a map u which is finitely piecewise affine on $\partial\Omega$. Of course, there exist a right polygon $\widehat{\Omega}$ and a (finitely) piecewise affine and bi-Lipschitz map $\Phi : \Omega \to \widehat{\Omega}$, having bi-Lipschitz constant $C = C(\Omega)$. The map $u \circ \Phi^{-1}$ is a CL bi-Lipschitz map from the right polygon $\widehat{\Omega}$ to Δ, which is piecewise affine on the boundary. Then, the first part of the proof applied in $\widehat{\Omega}$ gives an approximation $v : \widehat{\Omega} \to \Delta$ which is finitely piecewise affine and $C_1C^4L^4$ bi-Lipschitz. Finally, $v \circ \Phi : \Omega \to \Delta$ is a $C_1C^5L^4$ bi-Lipschitz approximation of u as desired. Summarizing, we have concluded the proof setting $C'(\Omega) = C^5$. □

Remark 1.53. The fact that the constant in Theorem D depends on Ω could seem at first glance unsatisfactory, since in the other results we have got purely geometric constants. On the other side, it is not possible to find a constant which does not depend on the polygon; more precisely, it is easy to observe that, for any polygon Ω, the best constant $C'(\Omega)$ (that is, the smallest one, corresponding to the "smartest" choice of $\widehat{\Omega}$ and Φ in the above proof) depends on the geometric features of Ω, such as the minimum and the maximum angles of its boundary, which in turn cannot be a priori bounded by any power of L. However, the construction above enlightens that $C'(\Omega) = 1$ whenever Ω is a right polygon.

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