Multiphase shape optimization problems

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Abstract

This paper is devoted to the analysis of multiphase shape optimization problems, which can formally be written as

$$\min\Big\{g\big((F_1(\Omega_1),\dots,F_h(\Omega_h)\big)+m\big|\bigcup_{i=1}^h\Omega_i\big|:\ \Omega_i\subset D,\ \Omega_i\cap\Omega_j=\emptyset\Big\},$$

where $D\subseteq\mathbb{R}^d$ is a given bounded open set, $|\Omega_i|$ is the Lebesgue measure of Ω_i and m is a positive constant. For a large class of such functionals, we analyse qualitative properties of the cells and the interaction between them. Each cell is itself subsolution for a (single-phase) shape optimization problem, from which we deduce properties like finite perimeter, inner density, separation by open sets, absence of triple junction points, etc. As main examples we consider functionals involving the eigenvalues of the Dirichlet Laplacian of each cell, i.e. $F_i=\lambda_{k_i}$.

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1 Introduction

Let $D\subseteq\mathbb{R}^d$ be a bounded open set and $m\geq 0$. We study multiphase shape optimization problems of the form

$$\min\Big\{g\big(F_1(\Omega_1),\dots,F_h(\Omega_h)\big)+m\big|\bigcup_{i=1}^h\Omega_i\big|:\ \Omega_i\subset D,\ \Omega_i\cap\Omega_j=\emptyset\Big\},\tag{1.1}$$

where $|\Omega|$ is the Lebesgue measure of Ω. To each cell Ω_i, we associate a shape functional F_i, the interaction between cells being described by the function $g:\mathbb{R}^h\to\mathbb{R}$. If one fixes $h-1$ cells of an optimal configuration, and let formally free only one, this cell is a shape subsolution. In a neighborhood of the junction points, it can be compared only with its inner perturbations. One of the main questions raised by such a shape optimization problem concerns precisely the interaction between the cells. The functionals F_i we consider here, involve quantities related to the Dirichlet Laplacian operator on each cell as for example the eigevalues $(\lambda_k(\Omega_i))_{k\in\mathbb{N}}$ of the Laplace operator with Dirichlet boundary conditions on a quasi-open set Ω_i.

For a very particular choice of g and F_i, this topic was intensively studied in the last years, essentially for functionals involving the first eigenvalue

$$g\big(F_1(\Omega_1),\dots,F_h(\Omega_h)\big)=\sum_{i=1}^h\lambda_1(\Omega_i)\qquad\text{and}\qquad g\big(F_1(\Omega_1),\dots,F_h(\Omega_h)\big)=\max_{i=1,\dots,h}\lambda_1(\Omega_i).\tag{1.2}$$

For $m=0$, we refer the reader to the papers [18, 19, 20, 25, 15] and the references therein, while for $m>0$, only the case $h=1$ was studied in [4, 5].

Many interesting qualitative results were obtained for (1.2), among which regularity properties of the boundaries and information on the junction points.

In this paper we intend to discuss general functionals F_i, precisely functionals which have a variation controlled by the Dirichlet energy (see Definition 5.1 below) e.g. the k-th eigenvalue of the

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Dirichlet Laplacian, in a context where the measure constraint is relevant ($m > 0$). For example, problems of the form

$$\min\Big\{\sum_{i=1}^{n}\lambda_{k_i}(\Omega_i)+m|\Omega_i|:\ \Omega_i\subset D,\ \Omega_i \text{ quasi-open},\ \Omega_i\cap\Omega_j=\emptyset\Big\}. \tag{1.3}$$

fit in our framework. If $m > 0$, the sets Ω_i will not in general cover D and a void region will appear, so the solution will be a sort of lacunary partition of D. As we consider general functionals F_i, the same tools used for the regularity of the free boundaries in [18, 15] can not be adapted. Even if F_i is simply the k-th eigenvalue of the Dirichlet Laplacian, obtaining a regularity result is a complicated task, since the k-th eigenvalue is itself a critical point and not a minimizer as the first eigenvalue is.

We refer the reader to the survey papers [11, 26] and the books [7, 27, 28] for a detailed introduction to the topic of shape optimization problems. Existence of a solution for (1.1) in the class of quasi-open sets was proved in [8] and is a consequence of a general result due to Buttazzo and Dal Maso (see [12, 13]).

We focus in this paper on the analysis of the geometric interaction between cells. Our main tool involves the analysis of the *shape subsolutions for the torsional energy*, i.e. quasi-open sets $\Omega\subset\mathbb{R}^d$ which satisfy for some $m > 0$

$$E(\Omega)+m|\Omega|\le E(\widetilde\Omega)+m|\widetilde\Omega|,\ \forall\widetilde\Omega\subset\Omega, \tag{1.4}$$

where the torsional energy $E(\Omega)$ is defined as

$$E(\Omega):=\min\Big\{\frac12\int_\Omega|\nabla u|^2\,dx-\int_\Omega u\,dx:u\in H^1_0(\Omega)\Big\}.$$

Under mild assumptions on g and for a quite large class of functionals F_i, every cell of the optimal solution of (1.1) is a shape subsolution of the torsional energy.

Analyzing the properties of the subsolutions we prove that (Sections 4 and 5)

- each cell satisfies inner density estimates and has finite perimeter;
- there are no triple junction points, i.e. $\partial\Omega_i\cap\partial\Omega_j\cap\partial\Omega_k=\emptyset$, for different i,j,k;
- each (quasi-open) cell Ω_i can be isolated by an open set D_i from the other cells, and solves the problem
$$\min\Big\{F_i(\Omega):\ \Omega\subseteq D_i,\ \Omega \text{ quasi-open},\ |\Omega|=|\Omega_i|\Big\};$$
- if F_i depends in (1.1) only on the first and the second eigenvalues, there exists a solution consisting of open cells;
- in $\mathbb{R}^2$, for $m = 0$, every solution of (1.3) is equivalent to a solution consisting of open sets.

We emphasize that a subsolution is not, in general, an open set, as Remark 3.17 shows. Even for the solutions of some simple one-phase shape optimization problems, as

$$\min\Big\{\lambda_k(\Omega):\ \Omega\subseteq D,\ \Omega \text{ quasi-open},\ |\Omega|=m\Big\}, \tag{1.5}$$

with $k\ge 3$, the optimal set Ω is, a priori, no more than a quasi-open set. Until recently, the only functionals which were known to have (smooth) open sets as solutions were the first eigenvalue (see [5]) and the Dirichlet Energy (see [4]).

The study of triple junction points goes through a multiphase monotonicity formula (Lemma 2.14) in the spirit of [14] and [18, Lemmas 4.2 and 4.3], which is proved in the Appendix. Precisely, if $u_i\in H^1(B_1)$, $i=1,2,3$, are three non-negative functions with disjoint supports and such that $\Delta u_i\ge -1$, for each $i=1,2,3$, then there are dimensional constants $\varepsilon>0$ and $C_d>0$ such that for each $r\in(0,\frac12)$

$$\prod_{i=1}^{3}\left(\frac{1}{r^{2+\varepsilon}}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right)\le C_d\left(1+\sum_{i=1}^{3}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right)^3. \tag{1.6}$$

The main gain of this multiphase monotonicity formula is that for junction points of three cells (or more), at least one gradient decays faster than $r^{\varepsilon/2}$, which contradicts the super linear decay which is expected for subsolutions, cf. Lemma 3.8.

2 Preliminaries

In this section we recall some of the notions and results that we need in this paper.

2.1 Measure theoretic tools

We shall use throughout the paper the notions of a measure theoretic closure $\overline{\Omega}^M$ and a measure theoretic boundary $\partial^M\Omega$ of a Lebesgue measurable set $\Omega\subset\mathbb{R}^d$, which are defined as:

$$\overline{\Omega}^M=\Big\{x\in\mathbb{R}^d:\ |B_r(x)\cap\Omega|>0,\ \forall r>0\Big\},$$

$$\partial^M\Omega=\Big\{x\in\mathbb{R}^d:\ |B_r(x)\cap\Omega|>0,\ |B_r(x)\cap\Omega^c|>0,\ \forall r>0\Big\}.$$

Moreover, for every $0\le\alpha\le 1$, we define the set of points of density α as

$$\Omega_{(\alpha)}=\Big\{x\in\mathbb{R}^d:\ \lim_{r\to 0}\frac{|B_r(x)\cap\Omega|}{|B_r|}=\alpha\Big\}.$$

If Ω has finite perimeter in sense of De Giorgi, i.e. the distributional gradient $\nabla\mathbb{1}_\Omega$ is a measure of finite total variation $|\nabla\mathbb{1}_\Omega|(\mathbb{R}^d)<+\infty$, the generalized perimeter of Ω is given by

$$P(\Omega)=|\nabla\mathbb{1}_\Omega|(\mathbb{R}^d)=\mathcal{H}^{d-1}(\partial^*\Omega),$$

where $\partial^*\Omega$ is the reduced boundary of Ω.

The s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. To simplify notations and when no ambiguity occurs, we shall use the notation $|\partial B_r(x)|$ for the $(d-1)$ Hausdorff measure of the boundary of the ball centered in x of radius r.

2.2 Capacity and quasi-open sets

As we mentioned in the introduction, for our purposes it is convenient to extend the notion of a Sobolev space and Laplace operator to measurable sets. One has to use the notion of capacity of a set $E\subset\mathbb{R}^d$, which is defined as

$$\mathrm{cap}(E)=\inf\Big\{\|u\|^2_{H^1}:\ u\in H^1(\mathbb{R}^d),\ u\ge 1\text{ in a neighbourhood of }E\Big\},\tag{2.1}$$

where $\|u\|^2_{H^1}=\|u\|^2_{L^2}+\|\nabla u\|^2_{L^2}$ (see, for example, [28] for more details).

- We say that a property $\mathcal{P}$ holds quasi-everywhere (shortly q.e.) in $\mathbb{R}^d$, if the set of points E, where $\mathcal{P}$ does not hold, is of zero capacity ($\mathrm{cap}(E)=0$).
- We say that a set $\Omega\subset\mathbb{R}^d$ is quasi-open, if for each $\varepsilon>0$ there is an open set ω_ε of capacity $\mathrm{cap}(\omega_\varepsilon)\le\varepsilon$ such that $\Omega\cup\omega_\varepsilon$ is an open set.
- A function $u:\mathbb{R}^d\to\mathbb{R}$ is quasi-continuous, if for each $\varepsilon>0$ there is an open set ω_ε of capacity $\mathrm{cap}(\omega_\varepsilon)\le\varepsilon$ such that the restriction of u on the closed set $\mathbb{R}^d\setminus\omega_\varepsilon$ is a continuous function.

We note that any function $u\in H^1(\mathbb{R}^d)$ has a quasi-continuous representative $\widetilde{u}:\mathbb{R}^d\to\mathbb{R}$, which is unique up to sets of zero capacity (see [28]). Moreover, if the sequence $u_n\in H^1(\mathbb{R}^d)$ converges strongly in $H^1(\mathbb{R}^d)$ to the function $u\in H^1(\mathbb{R}^d)$, then there is a subsequence converging quasi-everywhere.

In $\mathbb{R}^d$ the canonical quasi-continuous representative $\widetilde{u}$ of $u\in H^1(\mathbb{R}^d)$ has a pointwise definition, i.e. for quasi-every $x\in\mathbb{R}^d$ the following limit exists

$$\widetilde{u}(x)=\lim_{r\to 0}\fint_{B_r(x)}u(y)\,dy.\tag{2.2}$$

We define the Sobolev space $H^1_0(\Omega)$, for every measurable set $\Omega\subset\mathbb{R}^d$,

$$H^1_0(\Omega)=\Big\{u\in H^1(\mathbb{R}^d):\ \widetilde{u}=0\ \text{ q.e. on }\Omega^c\Big\}.\tag{2.3}$$

In the case when Ω is an open set, $H^1_0(\Omega)$ coincides with the classical Sobolev space defined as the closure of the smooth functions with compact support $C^\infty_c(\Omega)$, with respect to the norm $\|\cdot\|_{H^1}$ (see [28]).

2.3 The torsion function of a set of finite measure

Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure $|\Omega| < +\infty$ and let $f \in L^2(\Omega)$ be a given function. We say that $u \in H^1_0(\Omega)$ is a solution of the equation

$$-\Delta u = f, \qquad u \in H^1_0(\Omega), \tag{2.4}$$

if u minimizes the functional $J_f : H^1_0(\Omega) \to \mathbb{R}$, where for every $v \in H^1_0(\Omega)$

$$J_f(v) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \int_{\mathbb{R}^d} uf\,dx.$$

We note that, for every $f \in L^2(\Omega)$, a solution u of (2.4) exists and is unique. Moreover, for every $v \in H^1_0(\Omega)$ we have

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla v\,dx = \int_{\mathbb{R}^d} vf\,dx,$$

and, taking $v = u$, we get

$$\min_{v \in H^1_0(\Omega)} J_f(v) = J_f(u) = -\frac{1}{2}\int_{\mathbb{R}^d} uf\,dx =: E_f(\Omega). \tag{2.5}$$

Above, $E_f(\Omega)$ is called Dirichlet energy of Ω. In the case when $f \equiv 1$, we denote with w_Ω the solution of (2.4) and with $E(\Omega)$ the quantity $E_1(\Omega)$. Then $E(\Omega)$ is the *torsional energy* and w_Ω the *torsion function* of Ω. In the Proposition below, we list a few properties of w_Ω.

Proposition 2.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is a set of finite measure and that $w_\Omega \in H^1_0(\Omega)$ is the energy function of Ω. Then we have*

(a) w_Ω is bounded and

$$\|w_\Omega\|_{L^\infty} \le \frac{|\Omega|^{2/d}}{d|B_1|^{2/d}},$$

where B_1 is the unit ball in $\mathbb{R}^d$.

(b) $\Delta w_\Omega + \mathbb{1}_{\{w_\Omega > 0\}} \ge 0$ in sense of distributions on $\mathbb{R}^d$.

(c) Every point of $\mathbb{R}^d$ is a Lebesgue point for w_Ω.

(d) For every $x_0 \in \mathbb{R}^d$ and every $r > 0$, we have the inequalities

$$w_\Omega(x_0) \le \frac{r^2}{2d} + \fint_{\partial B_r(x_0)} w_\Omega\,d\mathcal{H}^{d-1} \qquad \text{and} \qquad w_\Omega(x_0) \le \frac{r^2}{2d} + \fint_{B_r(x_0)} w_\Omega\,dx. \tag{2.6}$$

(e) w_Ω is upper semi-continuous on $\mathbb{R}^d$.

(f) $H^1_0(\Omega) = H^1_0\big(\{w_\Omega > 0\}\big)$.

Proof. We set for simplicity $w := w_\Omega$. Point *(a)* follows by the classical result of Talenti (see [31]) or by a direct argument, as in [10]. For $t \ge 0$ and $\varepsilon > 0$, we consider the test function $w_\varepsilon := (w \wedge t) + (w - t - \varepsilon)^+ \in H^1_0(\Omega)$. Thus the inequality $J_1(w) \le J_1(w_\varepsilon)$ provides us with the estimate

$$\varepsilon\big|\{w > t\}\big| \ge \int_\Omega (w - w_\varepsilon)\,dx \ge \frac{1}{2}\int_{\{t \le w \le t+\varepsilon\}} |\nabla w|^2\,dx = \frac{1}{2}\int_t^{t+\varepsilon}\left(\int_{\{w=s\}} |\nabla w|\,d\mathcal{H}^{d-1}\right)ds, \tag{2.7}$$

where in the last equality we used the co-area formula. Passing to the limit as $\varepsilon \to 0$ and applying the Cauchy-Scwatz inequality, we get

$$\begin{aligned}\big|\{w > t\}\big| &\ge \frac{1}{2}\int_{\{w=t\}} |\nabla w|\,d\mathcal{H}^{d-1} \ge \frac{1}{2}\Big(\mathcal{H}^{d-1}(\{w = t\})\Big)^2\left(\int_{\{w=t\}} |\nabla w|^{-1}\,d\mathcal{H}^{d-1}\right)^{-1} \\ &\ge \frac{d^2|B_1|^{2/d}}{2}\big|\{w > t\}\big|^{\frac{2(d-1)}{d}}\left(\int_{\{w=t\}} |\nabla w|^{-1}\,d\mathcal{H}^{d-1}\right)^{-1},\end{aligned} \tag{2.8}$$

where the last estimate is due to the isoperimetric inequality in $\mathbb{R}^d$. Setting $\phi(t) := \big|\{w > t\}\big|$, again by the co-area formula we note that $\phi'(t) = -\int_{\{w=t\}} |\nabla w|^{-1}\, d\mathcal{H}^{d-1}$. By (2.8) ϕ satisfies the differential inequality

$$\big[\phi(t)^{2/d}\big]' \leq -d|B_1|^{2/d}, \qquad \phi(0) \leq |\Omega|, \tag{2.9}$$

and so, the claim.

Claim *(b)* can be found in [28, Lemma 7.2.5] and we report it here for sake of completeness. Consider a non-negative function $\varphi \in C_c^\infty(\mathbb{R}^d)$. For each $n \geq 1$, define the function $p_n : \mathbb{R} \to \mathbb{R}$ as

$$p_n(t) = 0, \quad \text{if } t \leq 0; \qquad p_n(t) = nt, \quad \text{if } t \in [0, 1/n]; \qquad p_n(t) = 1, \quad \text{if } t \geq 1/n. \tag{2.10}$$

Since p_n is Lipschitz and vanishes in zero, we have that $p_n(w) \in H_0^1(\Omega)$ and $p_n(w)\varphi \in H_0^1(\Omega)$. Now we compute

$$\begin{aligned} \int_\Omega \varphi p_n(w)\, dx &= \int_{\mathbb{R}^d} \nabla w \cdot \nabla\big(\varphi p_n(w)\big)\, dx = \int_{\mathbb{R}^d} \varphi p_n'(w)|\nabla w|^2\, dx + \int_{\mathbb{R}^d} p_n(w)\nabla w \cdot \nabla\varphi\, dx \\ &\geq \int_\Omega p_n(w)\nabla w \cdot \nabla\varphi\, dx. \end{aligned} \tag{2.11}$$

Since $p_n(w) \uparrow \mathbb{1}_{\{w>0\}}$, as $n \to \infty$, we obtain *(b)*.

In order to prove *(c)* we consider $x_0 \in \mathbb{R}^d$ and the ball $B_1(x_0)$. By *(b)* we have that the function

$$u(x) := w(x) + \frac{|x - x_0|^2}{2d},$$

is sub-harmonic and positive on $B_1(x_0)$. Then the function $r \mapsto \fint_{B_r(x_0)} u\, dx$ is decreasing and so the limit $\lim_{r\to 0} \fint_{B_r(x_0)} u\, dx$ exists. Since $u - w$ is smooth, the limit $\lim_{r\to 0} \fint_{B_r(x_0)} w\, dx$ also exists. Defining $u(x_0)$ and $w(x_0)$ as the above limits, we have

$$\begin{aligned} \fint_{B_r(x_0)} |w(x) - w(x_0)|\, dx &\leq \fint_{B_r(x_0)} |u(x) - u(x_0)|\, dx + \fint_{B_r(x_0)} |x - x_0|^2\, dx \\ &= \fint_{B_r(x_0)} u\, dx - u(x_0) + \fint_{B_r(x_0)} |x - x_0|^2\, dx \to 0, \end{aligned}$$

which proves *(c)*.

For claim *(d)* consider the auxiliary function $\phi_r(x) := \frac{1}{2d}\big(r^2 - |x - x_0|^2\big)$. By *(b)* we have that $\Delta(w - \phi_r) \geq 0$ on $\mathbb{R}^d$. Moreover, $0 \leq \phi_r \leq \frac{r^2}{2d}$ on $B_r(x_0)$ and so,

$$w(x_0) - \frac{r^2}{2d} = w(x_0) - \phi_r(x_0) \leq \fint_{B_r} (w - \phi_r)\, dx \leq \fint_{B_r} w\, dx,$$

which gives the second inequality in (2.6), the other one being analogous.

In order to prove *(e)*, we consider a sequence $x_n \to x_\infty$ in $\mathbb{R}^d$. By (2.6), we have that

$$\begin{aligned} \limsup_{n\to\infty} w(x_n) &\leq \frac{r^2}{2d} + \limsup_{n\to\infty} \fint_{B_r(x_n)} w\, dx \\ &= \frac{r^2}{2d} + \fint_{B_r(x_\infty)} w\, dx, \end{aligned}$$

and the claim follows by sending $r \to 0$.

We now sketch the proof of *(f)* (for the detailed proof we refer, for example, to [33, Chapter 3]). Since $\{w > 0\} \subset \Omega$, it is sufficient to prove that every function $u \in H_0^1(\Omega)$ is also in $H_0^1(\{w > 0\})$, i.e. that $\mathrm{cap}(\{u > 0\} \setminus \{w > 0\}) = 0$. For this we may suppose that $0 \leq u \leq 1$. Let u_ε be the minimizer of the functional $G : H_0^1(\Omega) \to \mathbb{R}$

$$G(v) = \int_\Omega |\nabla v|^2\, dx + \frac{1}{\varepsilon}\int_\Omega |v - u|^2\, dx.$$

Then u_ε is the solution of

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon}u_\varepsilon = \frac{1}{\varepsilon}u, \qquad u_\varepsilon \in H^1_0(\Omega).$$

Since $u \le 1$, by the comparison principle, we have that $u_\varepsilon \le \frac{1}{\varepsilon}w$ and so $\mathrm{cap}(\{u_\varepsilon > 0\}\setminus\{w > 0\}) = 0$. On the other hand, since $G(u_\varepsilon) \le G(u)$, we have that $u_\varepsilon \to u$ in L^2 and weakly in $H^1_0(\Omega)$. By using the equation for u_ε, we get that

$$\int_\Omega |\nabla(u-u_\varepsilon)|^2\,dx = \int_\Omega \nabla u\cdot\nabla(u-u_\varepsilon)\,dx - \frac{1}{\varepsilon}\int_\Omega |u-u_\varepsilon|^2\,dx \le \int_\Omega \nabla u\cdot\nabla(u-u_\varepsilon)\,dx \to 0,$$

which shows that the $u_\varepsilon \to u$ strongly in $H^1(\mathbb{R}^d)$ and so quasi-everywhere on $\mathbb{R}^d$, which concludes the proof that $\mathrm{cap}(\{u > 0\} \setminus \{w > 0\}) = 0$. $\square$

Remark 2.2. Claim *(d)* of Proposition 2.1 in particular shows that the quasi-open sets are the natural domains for the Sobolev spaces. Indeed, for any measurable set Ω, the set $\{w_\Omega > 0\} \subset \Omega$ is quasi-open and such that $H^1_0(\Omega) = H^1_0\big(\{w_\Omega > 0\}\big)$. On the other hand, if Ω is quasi-open, then there is a function $u \in H^1_0(\Omega)$ such that $\Omega = \{u > 0\}$ up to a set of zero capacity. Since $u \in H^1_0(\{w_\Omega > 0\})$, we have that $\mathrm{cap}(\{u > 0\} \setminus \{w_\Omega > 0\}) = 0$ and so the sets Ω and $\{w_\Omega > 0\}$ coincide quasi-everywhere.

Remark 2.3. From now on we identify w_Ω with its representative defined through the equality

$$w_\Omega(x_0) = \lim_{r\to 0} \fint_{B_r(x_0)} w_\Omega\,dx, \qquad \forall x_0 \in \mathbb{R}^d.$$

Thus, we identify every quasi-open set $\Omega \subset \mathbb{R}^d$ with its representative $\{w_\Omega > 0\}$. With this identification, we have the following simple observations:

- Let Ω be a quasi-open set, Then the measure theoretical and the topological closure of Ω coincide $\overline{\Omega} = \overline{\Omega}^M$. Indeed, we have $\overline{\Omega}^M \subset \overline{\Omega}$. On the other hand, if $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}^M$, then there is a ball $B_r(x_0)$ such that $w_\Omega = 0$ on $B_r(x_0)$ and so, $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$. Thus we have also $\mathbb{R}^d \setminus \overline{\Omega}^M \subset \mathbb{R}^d \setminus \overline{\Omega}$, which proves the claim.

- Let Ω_1 and Ω_2 be two quasi-open sets. If $|\Omega_1\cap\Omega_2| = 0$, then $\Omega_1\cap\Omega_2 = \emptyset$. Indeed, we note that $\Omega_1\cap\Omega_2 = \big\{x \in \mathbb{R}^d:\ w_{\Omega_1}(x)w_{\Omega_2}(x) > 0\big\}$. Since $|\Omega_1\cap\Omega_2| = 0$, we have that $\int_{\mathbb{R}^d} w_1 w_2\,dx = 0$. Note that every point of $x \in \mathbb{R}^d$ is a Lebesgue point for the product w_1w_2, we have that $w_1w_2 = 0$ everywhere on $\mathbb{R}^d$.

- Let Ω_1 and Ω_2 be two disjoint quasi-open sets. Then the measure theoretical and the topological common boundaries coincide

$$\partial\Omega_1 \cap \partial\Omega_2 = \overline{\Omega}_1 \cap \overline{\Omega}_2 = \overline{\Omega}_1^M \cap \overline{\Omega}_2^M = \partial^M\Omega_1 \cap \partial^M\Omega_2.$$

2.4 The γ and weak γ-convergence

The identification of the quasi-open sets Ω and their torsional function w_Ω leads naturally to the following (more functional than geometrical) distance. We define the so called γ-distance between two quasi-open sets of finite measure Ω_1 and Ω_2 by

$$d_\gamma(\Omega_1, \Omega_2) = \int_{\mathbb{R}^d} \big|w_{\Omega_1} - w_{\Omega_2}\big|\,dx.$$

Notice that if $\Omega_1 \subseteq \Omega_2$ then $d_\gamma(\Omega_1, \Omega_2) = \frac{1}{2}\big[E(\Omega_1) - E(\Omega_2)\big]$.

Definition 2.4. *In the family of quasi-open sets of finite measure, it is said that the sequence Ω_n γ-converges to Ω if $d_\gamma(\Omega_n, \Omega) \to 0$, as $n \to \infty$.*

Remark 2.5. Sometimes, the γ-distance is defined using the L^2-norm of $w_{\Omega_1} - w_{\Omega_2}$. In a family of sets with uniformly bounded measure, the two distances provide us with the same convergence. Indeed, we have

$$\int_{\mathbb{R}^d} \big|w_{\Omega_1} - w_{\Omega_2}\big|\,dx \le \big(|\Omega_1| + |\Omega_2|\big)^{1/2}\left(\int_{\mathbb{R}^d} \big|w_{\Omega_1} - w_{\Omega_2}\big|^2\,dx\right)^{1/2},$$

$$\int_{\mathbb{R}^d} \left|w_{\Omega_1} - w_{\Omega_2}\right|^2 dx \le \frac{|\Omega_1|^{2/d} + |\Omega_2|^{2/d}}{d|B_1|^{2/d}} \int_{\mathbb{R}^d} \left|w_{\Omega_1} - w_{\Omega_2}\right| dx.$$

For the purposes of our paper, it is more convenient to use the L^1-norm.

Definition 2.6. *In the family of quasi-open sets of finite measure, it is said that the sequence Ω_n weak γ-converges to Ω if the sequence of the corresponding torsional functions w_{Ω_n} converges in $L^1(\mathbb{R}^d)$ to some function $w \in H^1(\mathbb{R}^d)$ and $\Omega = \{w > 0\}$.*

Remark 2.7. The γ and the weak-γ-convergences are not equivalent. An example of weak-γ-convergent sequence which is not γ-convergent, is given by the classical example of Cioranescu-Murat [17]. Moreover, from the density result in [21], for every $\Omega \subset \mathbb{R}^d$ of bounded measure and every potential $V : \Omega \to [0, +\infty]$ there is a sequence Ω_n such that w_{Ω_n} converges in $L^1(\mathbb{R}^d)$ to the weak solution of the equation

$$-\Delta w + V w = 1, \qquad w \in H^1_0(\Omega).$$

Remark 2.8. For every (quasi-) open set D of finite Lebesgue measure the set

$$\mathcal{A}_{cap}(D) = \Big\{\Omega :\ \Omega \text{ quasi-open},\ \Omega \subset D\Big\},$$

is sequentially compact for the weak γ-convergence. Indeed, let $\Omega_n \in \mathcal{A}_{cap}(D)$ be a sequence of quasi-open sets and let w_n be the sequence of corresponding torsional functions. Using the equation for w_n and the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\int_D |\nabla w_n|^2 \, dx = \int_D w_n \, dx \le |\Omega_n|^{\frac{d+2}{2d}} \|w_n\|_{L^{\frac{2d}{d-2}}} \le C_d |D|^{\frac{d+2}{2d}} \|\nabla w_n\|_{L^2},$$

and so, w_n is bounded in $H^1_0(D)$. The compactness of $\mathcal{A}_{cap}(D)$ now follows by the compactness of the inclusion $H^1_0(D) \subset L^2(D)$.

Remark 2.9. As a consequence of the Fatou Lemma, the Lebesgue measure is lower semi-continuous with respect to the weak γ-convergence in $\mathcal{A}_{cap}(D)$. Moreover, if the sequence $\Omega_n \in \mathcal{A}_{\text{cap}}(D)$ weak γ-converges to Ω, then, for a suitable subsequence, there is a sequence of quasi-open sets ω_k such that $\omega_k \supset \Omega_{n_k}$ and ω_k γ-converges to Ω (see for example [7]).

The weak γ-convergences is used to establish existence results for shape optimization problems where the shape functional is γ-continuous and decreasing for inclusions. We recall here a general existence result, proved in [8], which is a multiphase version of the classical Butazzo-Dal Maso Theorem (see [13]).

Theorem 2.10. *Let $D \subset \mathbb{R}^d$ be a quasi-open set of finite Lebesgue measure and let $\mathcal{F} : [\mathcal{A}_{\text{cap}}(D)]^h \to \mathbb{R}$ satisfy*

(i) $\mathcal{F}$ is decreasing with respect to the inclusion, i.e. if $\omega_i \subset \Omega_i$, for all $i = 1, \dots, h$, then

$$\mathcal{F}(\Omega_1, \dots, \Omega_h) \le \mathcal{F}(\omega_1, \dots, \omega_h);$$

(ii) $\mathcal{F}$ is lower semi-continuous with respect to the γ-convergence, i.e. if Ω_i^n γ-converges to Ω_i, for every $i = 1, \dots, h$, then

$$\mathcal{F}(\Omega_1, \dots, \Omega_h) \le \liminf_{n\to\infty} \mathcal{F}(\Omega_1^n, \dots, \Omega_h^n).$$

Then the multiphase shape optimization problem

$$\min\Big\{\mathcal{F}(\Omega_1, \dots, \Omega_h) + m\sum_{i=1}^h |\Omega_i| :\ \Omega_i \in \mathcal{A}_{\text{cap}}(D),\ \forall i;\ \Omega_i \cap \Omega_j = \emptyset,\ \forall i \ne j\Big\}, \tag{2.12}$$

has a solution for every $m \ge 0$.

The proof is a consequence of Remarks 2.8 and 2.9, the essential point being the fact that a decreasing shape functional which is γ-lower semicontinuous, is also weak γ-lower semicontinuous.

Remark 2.11. There is a large class of functionals which are known to be decreasing and lower semi-continuous with respect to the γ-convergence (see [7, 11], for more details). Typical examples are

- the Dirichlet Energy $E_f(\Omega)$, with respect to a function $f \in L^2(D)$, defined in (2.5);
- the eigenvalues of the Dirichlet Laplacian, i.e.

$$\lambda_k(\Omega) = \min_{S_k \subset H_0^1(\Omega)} \max \left\{ \int_\Omega |\nabla u|^2 \, dx : \ u \in S_k, \ \int_\Omega u^2 \, dx = 1 \right\}, \tag{2.13}$$

  where the minimum is over all k-dimensional subspaces $S_k \subset H_0^1(\Omega)$.

2.5 Monotonicity theorems

We recall the following two-phase monotonicity formula due to Caffarelli, Jerison and Kenig [14].

Theorem 2.12. (Caffarelli, Jerison, Kenig) *Let $u_1, u_2 \in H^1(B_1)$ be two non-negative functions such that $\Delta u_i \geq -1$, for $i = 1, 2$, and $\int_{\mathbb{R}^d} u_i u_j \, dx = 0$. Then there is a dimensional constant C_d such that for each $r \in (0, 1)$ we have*

$$\prod_{i=1}^{2} \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}} \, dx \right) \leq C_d \left(1 + \sum_{i=1}^{2} \int_{B_1} \frac{|\nabla u_i|^2}{|x|^{d-2}} \, dx \right)^2. \tag{2.14}$$

In paper [14], Theorem 2.12 was stated with the additional assumption that the functions u_1 and u_2 are continuous. An inspection of the original proof shows that this assumption is not necessary, as it will be seen in the proof of Lemma 2.14, in the Appendix.

The following monotonicity lemma is due to Conti, Terracini and Verzini and holds in two dimensions.

Theorem 2.13. (Conti, Terracini, Verzini) *In $\mathbb{R}^2$, let $u_1, u_2, u_3 \in H^1(B_1)$ be three non-negative subharmonic functions such that $\int_{\mathbb{R}^2} u_i u_j \, dx = 0$. Then the function*

$$r \mapsto \prod_{i=1}^{3} \left(\frac{1}{r^3} \int_{B_r} |\nabla u_i|^2 \, dx \right) \tag{2.15}$$

is nondecreasing on $[0, 1]$.

As in our problem the functions are not subharmonic, the argument we search is closer to Theorem 2.12 than to Theorem 2.13. We give a multiphase monotonicity formula in the spirit of Theorem 2.12. We are not able to obtain optimal decreasing rates as in Theorem 2.13, but the estimate below will be sufficient for our purposes and holds in any dimension of the space.

Lemma 2.14 (Three-phase monotonicity lemma)**.** *Let $u_i \in H^1(B_1)$, $i = 1, 2, 3$, be three non-negative Sobolev functions such that $\Delta u_i \geq -1$, for each $i = 1, 2, 3$, and $\int_{\mathbb{R}^d} u_i u_j \, dx = 0$, for each $i \neq j$. Then there are dimensional constants $\varepsilon > 0$ and $C_d > 0$ such that, for every $r \in (0, \frac{1}{2})$, we have*

$$\prod_{i=1}^{3} \left(\frac{1}{r^{2+\varepsilon}} \int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}} \, dx \right) \leq C_d \left(1 + \sum_{i=1}^{3} \int_{B_1} \frac{|\nabla u_i|^2}{|x|^{d-2}} \, dx \right)^3. \tag{2.16}$$

The proof of this result follows the main arguments and steps of Theorem 2.12. For the convenience of the reader, we report it in the Appendix, with an emphasis on the technical differences brought by the lack of continuity and the presence of the third phase.

3 Shape subsolutions for the torsional energy

In this section we study the quasi-open sets of finite measure which are minimal for the functional $E(\cdot) + m|\cdot|$, with respect to internal variations of the domain. Sets satisfying this property will be called energy subsolutions. More precisely, we give the following:

Definition 3.1. *Let F be a functional on the family of quasi-open sets $\mathcal{A}_{\rm cap}(\mathbb{R}^d)$.*

- *We say that $\Omega \in \mathcal{A}_{\rm cap}(\mathbb{R}^d)$ is a* **shape subsolution for** *F, if*

$$F(\Omega) \le F(\omega), \quad \forall \textit{quasi-open} \quad \omega \subset \Omega. \tag{3.1}$$

- *We say that $\Omega \in \mathcal{A}_{\rm cap}(\mathbb{R}^d)$ is a* **local shape subsolution for** *F, if there is $\varepsilon > 0$ such that*

$$F(\Omega) \le F(\omega), \quad \forall \textit{quasi-open} \quad \omega \subset \Omega \quad \textit{such that} \quad d_\gamma(\Omega, \omega) < \varepsilon. \tag{3.2}$$

- *We say that $\Omega \in \mathcal{A}_{\rm cap}(\mathbb{R}^d)$ is an* **energy subsolution** *(with constant m) if Ω is a local subsolution for the functional $F(\Omega) = E(\Omega) + m|\Omega|$, where $m > 0$ is a given constant.*

Remark 3.2. We note that by the maximum principle we have $w_\Omega \ge w_\omega$, whenever $\omega \subset \Omega$ are quasi-open sets of finite measure. Thus, we have that

$$d_\gamma(\omega, \Omega) = \int_{\mathbb{R}^d} (w_\Omega - w_\omega)\, dx = 2\big(E(\omega) - E(\Omega)\big).$$

In particular, a set $\Omega \in \mathcal{A}_{\rm cap}(\mathbb{R}^d)$ is an energy subsolution, if and only if,

$$2m|\Omega \setminus \omega| \le d_\gamma(\omega, \Omega), \quad \forall \text{ quasi-open} \quad \omega \subset \Omega \quad \text{such that} \quad d_\gamma(\omega, \Omega) < \varepsilon. \tag{3.3}$$

Remark 3.3. If Ω is an energy subsolution with constant m and $m' \le m$, then Ω is also an energy subsolution with constant m'.

Remark 3.4. We recall that if $\Omega \subset \mathbb{R}^d$ is a quasi-open set of finite measure and $t > 0$ is a given real number, then we have

$$w_{t\Omega}(x) = t^2 w_\Omega(x/t) \qquad \text{and} \qquad E(t\Omega) = t^{d+2} E(\Omega).$$

Thus, if Ω is an energy subsolution with constants m and ε, then $\Omega' = t\Omega$ is an energy subsolution with constants $m' = 1$ and $\varepsilon' = \varepsilon t^{d+2}$, where $t = m^{-1/2}$.

Remark 3.5. If the energy subsolution $\Omega \subset \mathbb{R}^d$ is smooth, then writing the optimality condition for local perturbations of the domain Ω with smooth vector fields (see, for example, [28, Chapter 5] for the shape derivative tool) we obtain

$$|\nabla w_\Omega|^2 \ge 2m \quad \text{on} \quad \partial\Omega.$$

The energy subsolutions play an important role in the study of the optimal domains even for very general spectral optimization problems. In fact, in [6] the following Theorem was proved:

Theorem 3.6. *Suppose that the quasi-open set of finite measure Ω is a local shape subsolution for the functional $F = \lambda_k + m|\cdot|$, where $k \in \mathbb{N}$ and $m > 0$. Then Ω is an energy subsolution (with possibly different constant m').*

In particular, using this result, in [6] and [9], was proved boundedness of the optimal sets of some spectral optimization problems. In this section, we exploit the notion of a subsolution differently, obtaining an inner density estimate, which we use later in Section 5 to study the solutions of general multiphase problems.

Lemmas 3.7 and 3.8 below are implicitly contained in the paper of Alt and Caffarelli [1, Lemma 3.4]. We adapt them in the context of shape subsolutions of the torsional energy and rephrase them in two separate statements. For the sake of completeness we report here the proofs.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^d$ be an energy subsolution with constant m and let $w = w_\Omega$. Then there exist constants C_d, depending only on the dimension d, and r_0, depending on ε, such that for each $x_0 \in \mathbb{R}^d$ and each $0 < r < r_0$ we have the following estimate:*

$$\begin{aligned} &\frac{1}{2}\int_{B_r(x_0)} |\nabla w|^2\, dx + m\big|B_r(x_0) \cap \{w > 0\}\big| \\ &\qquad\le \int_{B_r(x_0)} w\, dx + C_d \left(r + \frac{\|w\|_{L^\infty(B_{2r}(x_0))}}{2r}\right) \int_{\partial B_r(x_0)} w\, d\mathcal{H}^{d-1}, \end{aligned} \tag{3.4}$$

Proof. Without loss of generality, we can suppose that $x_0 = 0$. We denote with B_r the ball of radius r centered in 0 and with A_r the annulus $B_{2r} \setminus \overline{B_r}$.
Let $\psi : A_1 \to \mathbb{R}^+$ be the solution of the equation:

$$\Delta\psi = 0 \quad \text{on} \quad A_1, \qquad \psi = 0 \quad \text{on} \quad \partial B_1, \qquad \psi = 1 \quad \text{on} \quad \partial B_2.$$

We can also give the explicit form of ψ, but for our purposes, it is enough to know that ψ is bounded and positive. With $\phi : A_1 \to \mathbb{R}^+$ we denote the solution of the equation:

$$-\Delta\phi = 1 \quad \text{on} \quad A_1, \qquad \phi = 0 \quad \text{on} \quad \partial B_1, \qquad \phi = 0 \quad \text{on} \quad \partial B_2.$$

For an arbitrary $r > 0$, $\alpha > 0$ and $k > 0$, we have that the solution v of the equation

$$-\Delta v = 1 \quad \text{on} \quad A_r, \qquad v = 0 \quad \text{on} \quad \partial B_r, \qquad v = \alpha \quad \text{on} \quad \partial B_{2r},$$

is given by

$$v(x) = r^2\phi(x/r) + \alpha\psi(x/r), \tag{3.5}$$

and its gradient is of the form

$$\nabla v(x) = r(\nabla\phi)(x/r) + \frac{\alpha}{r}\nabla\psi(x/r). \tag{3.6}$$

Let v be as in 3.5 with $\alpha \geq \|w\|_{L^\infty(B_{2r})}$. Consider the function $w_r = w\,\mathbb{1}_{B_{2r}^c} + (w \wedge v)\,\mathbb{1}_{B_{2r}}$ and note that, by the choice of α, we have that $w_r \in H_0^1(\Omega)$ and denote with Ω_r the quasi-open set $\Omega_r := \{w_r > 0\} = \Omega \setminus \overline{B_r}$. Since Ω is an energy subsolution, choosing r small enough, we have the inequality

$$\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega w\,dx + m\big|\{w > 0\}\big| \leq \frac{1}{2}\int_\Omega |\nabla w_r|^2\,dx - \int_\Omega w_r\,dx + m\big|\{w_r > 0\}\big|.$$

Since $w_r = 0$ in B_r and $w_r = w$ in $(B_{2r})^c$, we have that

$$\begin{aligned}
\frac{1}{2}\int_{B_r} |\nabla w|^2\,dx + m\big|B_r \cap \{w > 0\}\big| &\leq \frac{1}{2}\int_{A_r} |\nabla w_r|^2 - |\nabla w|^2\,dx + \int_{B_{2r}} (w - w_r)\,dx \\
&\leq \int_{A_r} \nabla w_r \cdot \nabla(w_r - w)\,dx + \int_{B_{2r}} (w - w_r)\,dx \\
&= -\int_{A_r} \nabla \cdot v\nabla((w - v)^+)\,dx + \int_{B_{2r}} (w - v)^+\,dx \\
&= \int_{\partial B_r} w\frac{\partial v}{\partial n}\,d\mathcal{H}^{d-1} + \int_{B_r} w\,dx \\
&\leq \Big(r\|\nabla\phi\|_\infty + \frac{\alpha}{r}\|\nabla\psi\|_\infty\Big)\int_{\partial B_r} w\,d\mathcal{H}^{d-1} + \int_{B_r} w\,dx,
\end{aligned} \tag{3.7}$$

where the last inequality is due to (3.6). Taking $\alpha = \|w\|_{L^\infty(B_{2r})}$, we have the claim. □

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^d$ be an energy subsolution with constant 1 and let $w = w_\Omega$. Then there exist constants $C_d > 0$ (depending only on the dimension) and $r_0 > 0$ (depending on the dimension d and on the constant ε from Definition 3.1) such that for every $x_0 \in \mathbb{R}^d$ and every $0 < r < r_0$ the following implication holds:*

$$\Big(\|w\|_{L^\infty(B_r(x_0))} \leq C_d r\Big) \Rightarrow \Big(w = 0 \text{ on } B_{r/2}(x_0)\Big). \tag{3.8}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$. By the trace theorem for $W^{1,1}$ functions (see [2, Theorems 3.87 and 3.88]), we have that

$$\begin{aligned}\int_{\partial B_{r/2}} w\, d\mathcal{H}^{d-1} &\le C_d\Big(\frac{2}{r}\int_{B_{r/2}} w\,dx + \int_{B_{r/2}} |\nabla w|\,dx\Big)\\ &\le C_d\Big(\frac{2}{r}\int_{B_{r/2}} w\,dx + \frac{1}{2}\int_{B_{r/2}} |\nabla w|^2\,dx + \frac{1}{2}\big|\{w>0\}\cap B_{r/2}\big|\Big)\\ &\le 2C_d\Big(\frac{2}{r}\|w\|_{L^\infty(B_{r/2})} + \frac{1}{2}\Big)\Big(\frac{1}{2}\int_{B_{r/2}} |\nabla w|^2\,dx + \big|\{u>0\}\cap B_{r/2}\big|\Big),\end{aligned} \tag{3.9}$$

where the constant $C_d > 0$ depends only on the dimension d.
We define the Dirichlet energy of w on the ball B_r as

$$E(w, B_r) := \frac{1}{2}\int_{B_r} |\nabla w|^2\,dx + \big|B_r \cap \{w>0\}\big|. \tag{3.10}$$

Combining (3.9) with the estimate from Lemma 3.7, we have

$$\begin{aligned}E(w, B_{r/2}) &\le \int_{B_{r/2}} w\,dx + C_d\Big(r + \frac{2}{r}\|w\|_{L^\infty(B_r)}\Big)\int_{\partial B_{r/2}} w\,d\mathcal{H}^{d-1}\\ &\le \bigg(\|w\|_{L^\infty(B_{r/2})} + C_d\Big(\frac{2}{r}\|w\|_{L^\infty(B_{r/2})} + \frac{1}{2}\Big)\Big(r + \frac{1}{r}\|w\|_{L^\infty(B_r)}\Big)\bigg)\,E(w, B_{r/2}),\end{aligned} \tag{3.11}$$

where the constants C_d depend only on the dimension d. The claim follows by observing that if

$$\|w\|_{L^\infty(B_r)} \le cr,$$

for some small c and r, then by (3.11) we obtain $E(w, B_{r/2}) = 0$. □

In other words, Lemma 3.8 says that in a point of $\overline{\Omega}^M$ (the measure theoretic closure of the energy subsolution Ω) the function w_Ω has at least linear growth. In particular, the maximum of w_Ω on $B_r(x)$ and the average on $\partial B_r(x)$ are comparable for $r > 0$ small enough.

Corollary 3.9. *Suppose that $\Omega \subset \mathbb{R}^d$ is an energy subsolution with $m = 1$ and let $w = w_\Omega$. Then there exists $r_0 > 0$, depending on the dimension and the constant ε from Definition 3.1, such that for every $x_0 \in \overline{\Omega}^M$ and every $0 < r < r_0$, we have*

$$2^{-d-2}\|w\|_{L^\infty(B_r(x_0))} \le \fint_{\partial B_{2r}(x_0)} w\,d\mathcal{H}^{d-1} \le \|w\|_{L^\infty(B_{2r}(x_0))}. \tag{3.12}$$

Proof. Suppose that $x_0 = 0$ and consider the function $\varphi_{2r}(x) := \dfrac{(2r)^2 - |x|^2}{2d}$. By Proposition 2.1 we have that $\Delta(w - \varphi_{2r}) \ge 0$ on $\mathbb{R}^d$ and $0 \le \varphi_{2r} \le 2r^2/d$ on B_{2r}. Comparing $w - \varphi_{2r}$ with the harmonic function on B_{2r} with boundary values w, we obtain that for every $x \in B_r$, we have

$$w(x) - \varphi_{2r}(x) \quad \le \frac{4r^2 - |x|^2}{d\omega_d 2r}\int_{\partial B_{2r}} \frac{w(y)}{|y-x|^d}\,d\mathcal{H}^{d-1}(y) \le 2^d \fint_{\partial B_{2r}} w\,d\mathcal{H}^{d-1}. \tag{3.13}$$

For $0 < r < \min\Big\{r_0, \dfrac{d\,C_d}{8}, 1\Big\}$, where r_0 and C_d are the constants from Lemma 3.8, we choose $x_r \in B_r$ such that

$$w(x_r) > \frac{1}{2}\|w\|_{L^\infty(B_r)} > \frac{rC_d}{2}.$$

Then we have

$$\frac{\|w\|_{L^\infty(B_r)}}{2} \le w(x_r) \le 2^d \fint_{\partial B_{2r}} w\,d\mathcal{H}^{d-1} + \frac{2r^2}{d} \le 2^d \fint_{\partial B_{2r}} w\,d\mathcal{H}^{d-1} + \frac{\|w\|_{L^\infty(B_r)}}{4}, \tag{3.14}$$

which proves the claim. □

Remark 3.10. In particular, for every energy subsolution $\Omega \subset \mathbb{R}^d$, there are constants c and r_0 such that if $x_0 \in \overline{\Omega}^M$, then for every $0 < r \le r_0$, we have that

$$cr \le \fint_{\partial B_r(x_0)} w_\Omega \, d\mathcal{H}^{d-1}.$$

Moreover, since $\displaystyle\int_{B_r} w_\Omega \, dx = \int_0^r \int_{\partial B_s} w_\Omega \, d\mathcal{H}^{d-1} \, ds$, we have $\displaystyle cr \le \fint_{B_r(x_0)} w_\Omega \, dx.$

As a consequence of Corollary 3.9, we can simplify (3.4). Precisely, we have the following result.

Corollary 3.11. *Suppose that $\Omega \subset \mathbb{R}^d$ is an energy subsolution with $m = 1$ and let $w := w_\Omega$. Then there are constants $C_d > 0$, depending only on the dimension d, and r_0, depending on d and ε from Definition* 3.1, *such that for every $x_0 \in \overline{\Omega}^M$ and every $0 < r < r_0$, we have*

$$\frac{1}{2} \int_{B_r(x_0)} |\nabla w|^2 \, dx + \big|\{w > 0\} \cap B_r(x_0)\big| \le C_d \frac{\|w\|_{L^\infty(B_{2r}(x_0))}}{2r} \int_{\partial B_r(x_0)} w \, d\mathcal{H}^{d-1}. \tag{3.15}$$

Proof. By Lemma 3.8 and Corollary 3.9, for $r > 0$ small enough, we have

$$\frac{1}{r}\|w\|_{L^\infty(B_r(x_0))} \ge C_d, \qquad \frac{1}{r} \fint_{\partial B_r(x_0)} w \, d\mathcal{H}^{d-1} \ge 2^{-d-3} C_d. \tag{3.16}$$

Thus, for r small enough, we have

$$\int_{B_r(x_0)} w(x) \, dx \le |B_r| \frac{d 2^{-d-3} C_d}{r} \|w\|_{L^\infty(B_r(x_0))} \le \frac{1}{r} \|w\|_{L^\infty(B_r(x_0))} \int_{\partial B_r(x_0)} w \, d\mathcal{H}^{d-1}, \tag{3.17}$$

and so, it remains to apply the above estimate to (3.4). $\square$

Relying on inequality (3.15) and Lemma 3.8 we get the following inner density estimate, which is much weaker than the density estimates from [1]. The main reason is that we work only with subsolutions and not with minimizers of a free boundary problem.

Proposition 3.12. *Suppose that $\Omega \subset \mathbb{R}^d$ is an energy subsolution and let $w = w_\Omega$. Then there exists a constant $c > 0$, depending only on the dimension, such that for every $x_0 \in \overline{\Omega}^M$, we have*

$$\limsup_{r \to 0} \frac{\big|\{w > 0\} \cap B_r(x_0)\big|}{|B_r|} \ge c. \tag{3.18}$$

Proof. Without loss of generality, we can suppose that $x_0 = 0$ and by rescaling we can assume that $m = 1$. Let r_0 and C_d be as in Lemma 3.8 and let $0 < r < r_0$. By the Trace Theorem in $W^{1,1}(B_r)$, we have

$$\begin{aligned}
\int_{\partial B_r} w \, d\mathcal{H}^{d-1} &\le C_d \Big(\int_{B_r} |\nabla w| \, dx + \frac{1}{r} \int_{B_r} w \, dx \Big) \\
&\le C_d \left(\Big(\int_{B_r} |\nabla w|^2 \, dx \Big)^{1/2} \big|\{w > 0\} \cap B_r\big|^{1/2} + \frac{\|w\|_{L^\infty(B_r)}}{r} \big|\{w > 0\} \cap B_r\big| \right) \\
&\le C_d \left(\frac{\|w\|_{L^\infty(B_{2r})}}{2r} \int_{\partial B_r} w \, d\mathcal{H}^{d-1} \right)^{1/2} \big|\{w > 0\} \cap B_r\big|^{1/2} \\
&\qquad + C_d \frac{\|w\|_{L^\infty(B_r)}}{r} \big|\{w > 0\} \cap B_r\big|,
\end{aligned} \tag{3.19}$$

where the last inequality is due to Corollary 3.11 and C_d denotes a constant which depends only on the dimension d. Let

$$\begin{aligned} X &= \left(\int_{\partial B_r} w\,d\mathcal{H}^{d-1}\right)^{1/2}, \\ \alpha &= C_d\left(\frac{\|w\|_{L^\infty(B_{2r})}}{2r}\right)^{1/2}\big|\{w>0\}\cap B_r\big|^{1/2}, \\ \beta &= C_d\frac{\|w\|_{L^\infty(B_r)}}{r}\big|\{w>0\}\cap B_r\big|. \end{aligned} \tag{3.20}$$

Then, we can rewrite (3.19) as

$$X^2 \le \alpha X + \beta.$$

But then, since $\alpha, \beta > 0$, we have the estimate $X \le \alpha + \sqrt{\beta}$. Taking the square of both sides, we obtain

$$\begin{aligned} \int_{\partial B_r} w\,d\mathcal{H}^{d-1} &\le C_d\big|\{w>0\}\cap B_r\big|\left(\frac{\|w\|_{L^\infty(B_{2r})}}{2r} + \frac{\|w\|_{L^\infty(B_r)}}{r}\right) \\ &\le 3C_d\big|\{w>0\}\cap B_r\big|\frac{\|w\|_{L^\infty(B_{2r})}}{2r}. \end{aligned} \tag{3.21}$$

By Corollary 3.9, we have that

$$\frac{\|w\|_{L^\infty(B_{r/2})}}{r/2} \le \frac{C_d\big|\{w>0\}\cap B_r\big|}{|B_r|}\frac{\|w\|_{L^\infty(B_{2r})}}{2r}, \tag{3.22}$$

for some dimensional constant $C_d > 0$. We choose the constant c from (3.18) as $c = (2C_d)^{-1}$ and we argue by contradiction. Suppose, by absurd, that we have

$$\limsup_{r\to 0}\ C_d\frac{\big|\{w>0\}\cap B_r\big|}{|B_r|} < \frac{1}{2}. \tag{3.23}$$

Setting, for $r > 0$ small enough,

$$f(r) := \frac{\|w\|_{L^\infty(B_r)}}{r},$$

and using (3.22), we have that for each $n \in \mathbb{N}$ the following inequality holds

$$f(r4^{-(n+1)}) \le \frac{C_d\big|\{w>0\}\cap B_{2r4^{-(n+1)}}\big|}{|B_{2r4^{-(n+1)}}|}f(r4^{-n}), \tag{3.24}$$

and so

$$f(r4^{-(n+1)}) \le f(r)\prod_{k=0}^{n}\frac{C_d\big|\{w>0\}\cap B_{2r4^{-(k+1)}}\big|}{|B_{2r4^{-(k+1)}}|}. \tag{3.25}$$

By equation (3.23), we have that $f(r4^{-n}) \to 0$, which is a contradiction with Lemma 3.8. □

Theorem 3.13. *Suppose that the quasi-open set $\Omega \subset \mathbb{R}^d$ is an energy subsolution with constant $m > 0$. Then, we have that:*

(i) Ω is a bounded set. Moreover, there is a constant $C > 0$ such that, for every $r > 0$ small enough, Ω can be covered with less than Cr^{-d-1} balls of radius r;

(ii) Ω is of finite perimeter and

$$\sqrt{\frac{m}{2}}\mathcal{H}^{d-1}(\partial^*\Omega) \le |\Omega|; \tag{3.26}$$

(iii) Ω is equivalent a.e. to a closed set. More precisely, $\Omega = \overline{\Omega}^M$ a.e., $\overline{\Omega}^M = \mathbb{R}^d \setminus \Omega_{(0)}$ and $\Omega_{(0)}$ is an open set. Moreover, if Ω is given through its canonical representative from Remark 2.3, then $\overline{\Omega} = \overline{\Omega}^M$.

Proof. Suppose that *(i)* does not hold. Then we construct the sequence $x_1, x_2, \dots, x_N$, for some $N > C_d r^{-d-1}$, as follows:

$$x_1 \in \Omega, \qquad x_{j+1} \in \Omega \setminus \left(\bigcup_{i=1}^{j} \overline{B_r(x_j)} \right).$$

Then the balls $B_{r/2}(x_j)$ are disjoint and $x_j \in \Omega$. By Remark 3.10, there is a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} w_\Omega \, dx \geq \sum_{j=1}^{N} \int_{B_{r/2}(x_j)} w_\Omega \, dx \geq N c r^{d+1},$$

which is a contradiction for $C > 0$ large enough.

The bound on the perimeter of Ω was implicitly proved in [6, Theorem 2.2].

For *(iii)*, it is sufficient to prove that $\Omega_{(0)}$ satisfies

$$\Omega_{(0)} = \mathbb{R}^d \setminus \overline{\Omega}^M = \Big\{ x \in \mathbb{R}^d : \text{ exists } r > 0 \text{ such that } |B_r(x) \cap \Omega| = 0 \Big\}, \tag{3.27}$$

where the second equality is just the definition of $\overline{\Omega}^M$. We note that $\Omega_{(0)} \subset \mathbb{R}^d \setminus \overline{\Omega}^M$ trivially holds for every measurable Ω. On the other hand, if $x \in \overline{\Omega}^M$, then, by Proposition 3.12, there is a sequence $r_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|B_{r_n}(x) \cap \Omega|}{|B_{r_n}|} \geq c > 0,$$

and so $x \notin \Omega_{(0)}$, which proves the opposite inclusion and the equality in (3.27). □

Remark 3.14. The second statement of Theorem 3.13 implies, in particular, that the energy subsolutions cannot be too small. Indeed, by the isoperimetric inequality, we have

$$c_d \sqrt{\frac{m}{2}} |\Omega|^{\frac{d-1}{d}} \leq \sqrt{\frac{m}{2}} \mathcal{H}^{d-1}(\partial^* \Omega) \leq |\Omega| \leq C_d [\mathcal{H}^{d-1}(\partial^* \Omega)]^{\frac{d}{d-1}},$$

and so

$$c_d m^{\frac{d}{2}} \leq |\Omega|, \qquad c_d m^{\frac{d-1}{2}} \leq \mathcal{H}^{d-1}(\partial^* \Omega),$$

for some dimensional constant c_d.

The results of this section can be adapted to the local subsolutions for the functional $F(\Omega) = \lambda_1(\Omega) + m|\cdot|$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $\Omega \subset \mathbb{R}^d$. We note that by Theorem 3.6, we have that the local subsolutions for $\lambda_1 + m|\cdot|$ are also energy subsolutions. Moreover, we have the following new, or more precise, statements.

Theorem 3.15. *Suppose that the quasi-open set $\Omega \subset \mathbb{R}^d$ is a subsolution for the first eigenvalue of the Dirichlet Laplacian. Then, we have that:*

(i)
$$\sqrt{m} \mathcal{H}^{d-1}(\partial^* \Omega) \leq \lambda_1(\Omega) |\Omega|^{1/2}; \tag{3.28}$$

(ii) Ω is quasi-connected, i.e. if $A, B \subset \Omega$ are two quasi-open sets such that $A \cup B = \Omega$ and $\mathrm{cap}(A \cap B) = 0$, then $\mathrm{cap}(A) = 0$ or $\mathrm{cap}(B) = 0$;

(iii) $\Omega = \{u > 0\}$, up to a set of zero capacity, where u is the first Dirichlet eigenfunction on Ω.

Proof. In order to prove the bound (3.28), we follow the idea from [6]. Let u be the first, normalized in $L^2(\Omega)$, eigenfunction on Ω. Since $\lambda_1(\{u > 0\}) = \lambda_1(\Omega)$, we have that $|\{u > 0\} \Delta \Omega| = 0$. Consider the set $\Omega_\varepsilon = \{u > \varepsilon\}$. In order to use Ω_ε to test the (local) subminimality of Ω, we first note that Ω_ε γ-converges to Ω. Indeed, the family of torsion functions w_ε of Ω_ε is decreasing in ε and converges in L^2 to the torsion function w of $\{u > 0\}$, as $\varepsilon \to 0$, since

$$\lambda_1(\Omega) \int_\Omega (w - w_\varepsilon) u \, dx = \int_\Omega \nabla w \cdot \nabla u \, dx - \int_{\Omega_\varepsilon} \nabla w_\varepsilon \cdot \nabla (u - \varepsilon)^+ \, dx = \int_\Omega u - (u - \varepsilon)^+ \, dx \to 0.$$

Now, using $(u-\varepsilon)^+ \in H^1_0(\Omega_\varepsilon)$ as a test function for $\lambda_1(\Omega_\varepsilon)$, we have

$$
\begin{aligned}
\lambda_1(\Omega)+m|\Omega| &\le \lambda_1(\Omega_\varepsilon)+m|\Omega_\varepsilon| \\
&\le \frac{\int_\Omega |\nabla(u-\varepsilon)^+|^2\,dx}{\int_\Omega |(u-\varepsilon)^+|^2\,dx}+m|\Omega_\varepsilon| \\
&\le \int_\Omega |\nabla(u-\varepsilon)^+|^2\,dx+\lambda_1(\Omega)\frac{\int_\Omega \big(u^2-|(u-\varepsilon)^+|^2\big)\,dx}{\int_\Omega |(u-\varepsilon)^+|^2\,dx}+m|\Omega_\varepsilon| \\
&\le \int_\Omega |\nabla(u-\varepsilon)^+|^2\,dx+\lambda_1(\Omega)\frac{2\varepsilon\int_\Omega u\,dx}{1-2\varepsilon\int_\Omega u\,dx}+m|\Omega_\varepsilon| \\
&\le \int_\Omega |\nabla(u-\varepsilon)^+|^2\,dx+\frac{2\varepsilon\lambda_1(\Omega)|\Omega|^{1/2}}{1-2\varepsilon\int_\Omega u\,dx}+m|\Omega_\varepsilon|.
\end{aligned}
\tag{3.29}
$$

Thus, we obtain

$$
\int_{\{0<u\le\varepsilon\}} |\nabla u|^2\,dx+m\big|\{0<u\le\varepsilon\}\big| \le 2\varepsilon\lambda_1(\Omega)|\Omega|^{1/2}\Big(1-2\varepsilon\int_\Omega u\,dx\Big)^{-1}. \tag{3.30}
$$

The mean quadratic-mean geometric and the Hölder inequalities give

$$
\begin{aligned}
2m^{1/2}\int_{\{0<u\le\varepsilon\}} |\nabla u|\,dx &\le 2m^{1/2}\bigg(\int_{\{0<u\le\varepsilon\}} |\nabla u|^2\,dx\bigg)^{1/2}\big|\{0<u\le\varepsilon\}\big|^{1/2} \\
&\le 2\varepsilon\lambda_1(\Omega)|\Omega|^{1/2}\Big(1-2\varepsilon\int_\Omega u\,dx\Big)^{-1}.
\end{aligned}
\tag{3.31}
$$

Using the co-area formula, we obtain

$$
\frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^{d-1}\big(\partial^*\{u>t\}\big)\,dt \le m^{-1/2}\lambda_1(\Omega)|\Omega|^{1/2}\Big(1-2\varepsilon\int_\Omega u\,dx\Big)^{-1}, \tag{3.32}
$$

and so, passing to the limit as $\varepsilon\to 0$, we obtain (3.28).

Let us now prove (ii). Suppose, by absurd that $\mathrm{cap}(A)>0$ and $\mathrm{cap}(B)>0$ and, in particular, $|A|>0$ and $|B|>0$. Since $\mathrm{cap}(A\cap B)=0$, we have that $H^1_0(\Omega)=H^1_0(A)\oplus H^1_0(B)$ and so, $\lambda_1(\Omega)=\min\{\lambda_1(A),\lambda_1(B)\}$. Without loss of generality, we may suppose that $\lambda_1(\Omega)=\lambda_1(A)$. Then, we have

$$
\lambda_1(A)+m|A|<\lambda_1(A)+m(|A|+|B|)=\lambda_1(\Omega)+m|\Omega|,
$$

which is a contradiction with the subminimality of Ω.

In order to see (iii), it is sufficient to prove that for every quasi-connected Ω, we have $\Omega=\{u>0\}$. Indeed, let $\omega=\{u>0\}$ and consider the torsion functions w_ω and w_Ω. We note that, by the weak maximum principle, we have $w_\omega\le w_\Omega$. Setting $\lambda=\lambda_1(\Omega)$, we have

$$
\int_\Omega \lambda u w_\omega\,dx=\int_\Omega \nabla u\cdot\nabla w_\omega\,dx=\int_\Omega u\,dx,
$$

$$
\int_\Omega \lambda u w_\Omega\,dx=\int_\Omega \nabla u\cdot\nabla w_\Omega\,dx=\int_\Omega u\,dx.
$$

Subtracting, we have

$$
\int_\Omega u(w_\Omega-w_\omega)\,dx=0, \tag{3.33}
$$

and so, $w_\Omega=w_\omega$ on ω. Consider the sets $A=\Omega\cap\{w_\Omega=w_\omega\}$ and $B=\Omega\cap\{w_\Omega>w_\omega\}$. By construction, we have that $A\cup B=\Omega$ and $A\cap B=\emptyset$. Moreover, we observe that $A=\omega\ne\emptyset$. Indeed, one inclusion $\omega\subset A$, follows by (3.33), while the other inclusion follows, since by strong maximum principle for w_ω and w_Ω we have the equality

$$
\Omega\cap\{w_\Omega=w_\omega\}=\{w_\Omega>0\}\cap\{w_\Omega=w_\omega\}\subset\{w_\omega>0\}=\omega.
$$

By the quasi-connectedness of Ω, we have that $\mathrm{cap}(B) = 0$, which gives that $\omega = \Omega$ up to a set of zero capacity. $\square$

Remark 3.16. If Ω is a subsolution for the first Dirichlet eigenvalue, then we have the following bound on $\lambda_1(\Omega)$:

$$\lambda_1(\Omega) \geq c_d m^{\frac{2}{d+2}}, \tag{3.34}$$

where c_d is a dimensional constant. In fact, by (3.28) and the isoperimetric inequality, we have

$$\lambda_1(\Omega)|\Omega|^{1/2} \geq \sqrt{m}P(\Omega) \geq c_d\sqrt{m}|\Omega|^{\frac{d-1}{d}},$$

and so

$$\lambda_1(\Omega) \geq c_d\sqrt{m}|\Omega|^{\frac{d-2}{2d}}.$$

By the Faber-Krahn inequality $\lambda_1(\Omega)|\Omega|^{2/d} \geq \lambda_1(B)|B|^{2/d}$, we obtain

$$\lambda_1(\Omega) \geq c_d\sqrt{m}\left(|\Omega|^{\frac{2}{d}}\right)^{\frac{d-2}{4}} \geq c_d\sqrt{m}\left(\lambda_1(\Omega)^{-1}\lambda_1(B)|B|^{2/d}\right)^{\frac{d-2}{4}} \geq c_d\sqrt{m}\lambda_1(\Omega)^{-\frac{d-2}{4}}.$$

Remark 3.17. Even if the subsolutions have some nice qualitative properties, their local geometry might be very irregular. In fact, one may construct subsolutions for the first Dirichlet eigenvalue (and thus, energy subsolutions) with empty interior in sense of the Lebesgue measure, i.e. the set $\Omega_{(1)}$ of points of density 1 has empty interior. Consider a bounded quasi-open set D with empty interior as, for example,

$$D = (0,1)\times(0,1)\setminus\left(\bigcup_{i=1}^{\infty}\overline{B}_{r_i}(x_i)\right) \subset \mathbb{R}^2,$$

where $\{x_i\}_{i\in\mathbb{N}} = \mathbb{Q}$ and r_i is such that

$$\sum_{i\in\mathbb{N}}\mathrm{cap}(\overline{B}_{r_i}(x_i)) < +\infty \qquad \text{and} \qquad \sum_{i\in\mathbb{N}}\pi r_i^2 < \frac{1}{2}.$$

Let $\Omega \subset D$ be the solution of the problem

$$\min\Big\{\lambda_1(\Omega) + |\Omega| : \Omega \subset D,\ \Omega \text{ quasi-open}\Big\}.$$

Since, Ω is a global minimizer among all sets in D, it is also a subsolution. On the other hand, D has empty interior and so does Ω.

4 Interaction between energy subsolutions

In this section we consider configurations of disjoint quasi-open sets $\Omega_1, \dots, \Omega_h$ in $\mathbb{R}^d$, each one being an energy subsolution. In particular, we will study the behaviour of the energy functions w_{Ω_i}, $i = 1, \dots, h$, around the points belonging to more than one of the measure theoretical boundaries $\partial^M\Omega_i$.

We start our discussion with a result which is useful in multiphase shape optimization problems, since it allows to separate by an open set each quasi-open cell from the others.

Lemma 4.1. *Suppose that the disjoint quasi-open sets Ω_1 and Ω_2 are energy subsolutions. Then the corresponding energy function w_1 and w_2 vanish on the common boundary $\partial^M\Omega_1 \cap \partial^M\Omega_2$.*

Proof. Recall that, by Remark 2.3, we may suppose that $\Omega_i = \{w_i > 0\}$ and that, by Proposition 2.1, every point $\mathbb{R}^d$ is a Lebesgue point for both w_1 and w_2.

Let $x_0 \in \partial^M\Omega_1 \cap \partial^M\Omega_2$. Then, for each $r > 0$ we have $\big|\{w_1 > 0\} \cap B_r(x_0)\big| > 0$ and so, by Proposition 3.12, there is a sequence $r_n \to 0$ such that

$$\lim_{n\to\infty}\frac{\big|\{w_1 > 0\} \cap B_{r_n}(x_0)\big|}{|B_{r_n}|} \geq c > 0. \tag{4.1}$$

Since $\big|\{w_1>0\}\cap\{w_2>0\}\big|=0$, we have that

$$\limsup_{n\to\infty}\frac{\big|\{w_2>0\}\cap B_{r_n}(x_0)\big|}{|B_{r_n}|}\le 1-c<1. \tag{4.2}$$

Since x_0 is a Lebesgue point for w_2, we have

$$\begin{aligned} w_2(x_0) &= \lim_{n\to\infty}\fint_{B_{r_n}(x_0)} w_2\,dx \\ &\le \limsup_{n\to\infty}\|w_2\|_{L^\infty(B_{r_n}(x_0))}\limsup_{n\to\infty}\frac{\big|\{w_2>0\}\cap B_{r_n}(x_0)\big|}{|B_{r_n}|} \\ &\le (1-c)\limsup_{n\to\infty}\|w_2\|_{L^\infty(B_{r_n}(x_0))}\le (1-c)w_2(x_0), \end{aligned}$$

where the last inequality is due to the upper semi-continuity of w_2 (see Proposition 2.1). Thus, we conclude that $w_2(x_0)=0$ and, analogously $w_1(x_0)=0$. □

Proposition 4.2. *Suppose that the disjoint quasi-open sets Ω_1 and Ω_2 are energy subsolutions. Then there are open sets $D_1, D_2\subset\mathbb{R}^d$ such that $\Omega_1\subset D_1$, $\Omega_2\subset D_2$ and $\Omega_1\cap D_2=\Omega_2\cap D_1=\emptyset$, up to sets of zero capacity.*

Proof. Define $D_1=\mathbb{R}^d\setminus\overline{\Omega}_2^M$ and $D_2=\mathbb{R}^d\setminus\overline{\Omega}_1^M$, which by the definition of a measure theoretic closure are open sets. As in Lemma 4.1, we recall that $\Omega_i=\{w_i>0\}$ and that every point of Ω_i is a Lebesgue point for the energy function $w_i\in H_0^1(\Omega_i)$. Since $\Omega_i\subset\overline{\Omega}_i^M$, we have to show only that $\Omega_1\subset D_1$ and $\Omega_2\subset D_2$ or, equivalently, that $\Omega_1\cap\overline{\Omega}_2^M=\Omega_2\cap\overline{\Omega}_1^M=\emptyset$. Indeed, if this is not the case there is a point $x_0\in\overline{\Omega}_2^M$ such that $w_1(x_0)>0$, which is a contradiction with Lemma 4.1. □

The rest of this section is dedicated to the proof of the fact that no three energy subsolutions can meet in a single point. Our main tool will be the three-phase monotonicity formula from Lemma 2.14. We note that the monotonicity formula involves terms, which are basically of the form $\fint_{B_r}|\nabla w|^2\,dx$, while the condition that the subsolution property provides concerns the mean of the function, i.e. $\fint_{\partial B_r} w\,d\mathcal{H}^{d-1}\ge cr$. These two terms express in different ways the non-degeneracy of w on the boundary, but the connection between them raises some technical issues, which esentially concern the regularity of the free boundary.

Remark 4.3 (Application of the monotonicity formula). Let Ω_1, Ω_2 and Ω_3 be three disjoint quasi-open sets of finite measure in $\mathbb{R}^d$. Let $w_i\in H_0^1(\Omega_i)$, for $i=1,2,3$, be the corresponding energy function and suppose that there is a constant $c>0$ such that

$$\fint_{B_r(x_0)}|\nabla w_i|^2\,dx\ge c,\qquad \forall r\in(0,1),\ \forall x_0\in\mathbb{R}^d,\ \forall i=1,2,3. \tag{4.3}$$

Then, by Theorem 2.12 Lemma 6.1 in the Appendix, we have that for every $x_0\in\partial^M\Omega_1\cap\partial^M\Omega_2$, we have

$$\fint_{B_r(x_0)}|\nabla w_i|^2\,dx\le\frac{C_d}{c}\left(1+\int_{\mathbb{R}^d}w_1^2\,dx+\int_{\mathbb{R}^d}w_2^2\,dx\right)^2,\qquad\forall r\in(0,1)\quad\text{and}\quad i=1,2. \tag{4.4}$$

As a consequence, using the three-phase monotonicity formula, the set of triple points $\partial^M\Omega_1\cap\partial^M\Omega_2\cap\partial^M\Omega_3$ is empty. Indeed, if $x_0\in\partial^M\Omega_1\cap\partial^M\Omega_2\cap\partial^M\Omega_3$, by Lemma 2.14 and the assumption (4.3), we would have

$$r^{-3\varepsilon}c^3\le\prod_{i=1}^3\left(\frac{1}{r^{d+\varepsilon}}\int_{B_r(x_0)}|\nabla w_i|^2\,dx\right)\le C_d\left(1+\sum_{i=1}^3\int_{\mathbb{R}^d}w_i^2\,dx\right)^2,$$

which is false for $r>0$ small enough.

Remark 4.4 (The two dimensional case). In dimension two, the energy subsolutions satisfy condition (4.3). Indeed, let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two disjoint energy subsolution with $m = 1$ and let $x_0 \in \partial^M \Omega_1 \cap \partial^M \Omega_2$. Setting $x_0 = 0$, by Corollary 3.9, we get that for each $0 < r \leq r_0$ the following estimates hold:

$$cr \leq \fint_{\partial B_r} w_1 \, d\mathcal{H}^1 \qquad \text{and} \qquad cr \leq \fint_{\partial B_r} w_2 \, d\mathcal{H}^1. \tag{4.5}$$

In particular, we get that $\partial B_r \cap \{w_1 = 0\} \neq \emptyset$ and $\partial B_r \cap \{w_2 = 0\} \neq \emptyset$. We now notice that for almost every $r \in (0, r_0)$ the restriction of w_1 and w_2 to ∂B_r are Sobolev functions. Thus, we have

$$2\pi c^2 r^3 \leq \frac{1}{|\partial B_r|} \left(\int_{\partial B_r} w_i \, d\mathcal{H}^1 \right)^2 \leq \int_{\partial B_r} w_i^2 \, d\mathcal{H}^1 \leq \frac{r^2}{\pi^2} \int_{\partial B_r} |\nabla w_i|^2 \, d\mathcal{H}^1,$$

where $\lambda < +\infty$ a constant. Dividing by r^2 and integrating for $r \in [0, R]$, where $R < r_0$, we obtain that (4.3) for some constant $c > 0$.

In particular, we obtain that if $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^2$ are three disjoint energy subsolutions then there are no triple points, i.e. the set $\partial^M \Omega_1 \cap \partial^M \Omega_2 \cap \partial^M \Omega_3$ is empty.

In higher dimension the inequality (4.3) on the common boundary points will be deduced by the following Lemma, which is implicitly contained in the proof of [1, Lemma 3.2].

Lemma 4.5. *For every $r > 0$ and every function $u \in H^1(B_r)$ we have the following estimate:*

$$\frac{1}{r^2} \big|\{u = 0\} \cap B_r\big| \left(\fint_{\partial B_r} u \, d\mathcal{H}^{d-1} \right)^2 \leq C_d \int_{B_r} |\nabla u|^2 \, dx, \tag{4.6}$$

where C_d is a constant that depends only on the dimension d.

Proof. We note that it is sufficient to prove the result in the case $u \geq 0$. Let $v \in H^1(B_r)$ be the solution of the obstacle problem

$$\min \left\{ \int_{B_r} |\nabla v|^2 \, dx : \; u - v \in H_0^1(B_r), \; v \geq u \right\}.$$

Then v is super-harmonic on B_r and harmonic on the quasi-open set $\{v > u\}$. Reasoning as in [1, Lemma 2.3], we have

$$\frac{1}{r^2} \big|\{u = 0\} \cap B_r\big| \left(\fint_{\partial B_r} u \, d\mathcal{H}^{d-1} \right)^2 \leq C_d \int_{B_r} |\nabla (u - v)|^2 \, dx. \tag{4.7}$$

Now the claim follows by the harmonicity of v on $\{v > u\}$ and the calculation

$$\int_{B_r} |\nabla (u - v)|^2 \, dx = \int_{B_r} |\nabla u|^2 - |\nabla v|^2 \, dx + 2 \int_{B_r} \nabla v \cdot \nabla (v - u) \, dx \leq \int_{B_r} |\nabla u|^2 \, dx.$$

□

Theorem 4.6. *Suppose that $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^d$ are three mutually disjoint energy subsolutions. Then the set $\partial^M \Omega_1 \cap \partial^M \Omega_2 \cap \partial^M \Omega_3$ is empty.*

Proof. Suppose for contradiction that there is a point $x_0 \in \partial^M \Omega_1 \cap \partial^M \Omega_2 \cap \partial^M \Omega_3$. Without loss of generality $x_0 = 0$. Using the inequality (3.22), we have

$$\prod_{i=1}^{3} \frac{\|w_i\|_{L^\infty(B_{r/2})}}{r/2} \leq C_d \left(\prod_{i=1}^{3} \frac{\big|\{w_i > 0\} \cap B_r\big|}{|B_r|} \right) \left(\prod_{i=1}^{3} \frac{\|w_i\|_{L^\infty(B_{2r})}}{2r} \right),$$

and reasoning as in Proposition 3.12, we obtain that there is a constant $c > 0$ and a decreasing sequence of positive real numbers $r_n \to 0$ such that

$$c \leq \prod_{i=1}^{3} \frac{\big|\{w_i > 0\} \cap B_{r_n}\big|}{|B_{r_n}|}, \qquad \forall n \in \mathbb{N},$$

Since $\big|\{w_i > 0\} \cap B_{r_n}\big| \leq |B_{r_n}|$, for each $i = 1, 2, 3$, we have

$$c \leq \frac{\big|\{w_i > 0\} \cap B_{r_n}\big|}{|B_{r_n}|}, \qquad \forall n \in \mathbb{N},$$

and since $\{w_1 > 0\}$, $\{w_2 > 0\}$ and $\{w_3 > 0\}$ are disjoint, we get

$$2c \leq \frac{\big|\{w_i = 0\} \cap B_{r_n}\big|}{|B_{r_n}|}, \qquad \forall n \in \mathbb{N}, \quad \forall i = 1, 2, 3.$$

Thus, we may apply Lemma 4.5 and then Lemma 3.8 and Corollary 3.9 , to obtain that there is a constant $\tilde{c} > 0$ such that for every $n \in \mathbb{N}$

$$\tilde{c} \leq \frac{\big|\{w_i = 0\} \cap B_{r_n}\big|}{|B_{r_n}|} \left(\frac{1}{r_n} \fint_{\partial B_{r_n}} u \, d\mathcal{H}^{d-1} \right)^2 \leq C_d \fint_{B_{r_n}} |\nabla w_i|^2 \, dx, \tag{4.8}$$

which proves that (4.3) holds for a sequence $r_n \to 0$. The conclusion follows as in Remark 4.3. $\square$

5 Multiphase shape optimization problems

Let $D \subset \mathbb{R}^d$ be a bounded open set. In this section we consider shape optimization problems of the form

$$\min \Big\{ g\big(F_1(\Omega_1), \dots, F_h(\Omega_h)\big) + m \sum_{i=1}^{h} |\Omega_i| : \ \Omega_i \in \mathcal{A}_{\mathrm{cap}}(D), \ \forall i; \ \Omega_i \cap \Omega_j = \emptyset, \ \forall i \neq j \Big\}, \tag{5.1}$$

where $g : \mathbb{R}^h \to \mathbb{R}$ is increasing in each variable and l.s.c., $F_1, \dots, F_h : \mathcal{A}_{\mathrm{cap}}(D) \to \mathbb{R}$ are decreasing with respect to inclusions and continous for the γ-convergence, and $m \geq 0$ is a given constant. Problem (5.1) admits a solution following Theorem 2.10.

Definition 5.1. *We say that $F : \mathcal{A}_{\mathrm{cap}}(D) \to \mathbb{R}$ is locally γ-Lipschitz for sub domains (or simply γ-Lip), if for each $\Omega \in \mathcal{A}_{\mathrm{cap}}(D)$, there are constants $C > 0$ and $\varepsilon > 0$ such that*

$$\big|F(\omega) - F(\Omega)\big| \leq C d_\gamma(\omega, \Omega),$$

for every quasi-open set $\omega \subset \Omega$, such that $d_\gamma(\omega, \Omega) \leq \varepsilon$.

Remark 5.2. Following Theorem 3.6, we have that the functional associated to the k-th eigenvalue of the Dirichlet Laplacian $\Omega \mapsto \lambda_k(\Omega)$ is γ-Lip, for every $k \in \mathbb{N}$.

Theorem 5.3. *Assume that g is locally Lipschitz continuous, each of the functionals F_i, $i = 1, \dots, h$ is γ-Lip and $m > 0$ and $(\Omega_1, \dots, \Omega_h)$ is a solution of* (5.1)*. Then every quasi-open set Ω_i, $i = 1, \dots, h$, is an energy subsolution.*

Proof. Let $\omega_1 \subset \Omega_1$ be a quasi-open set such that $d_\gamma(\omega_1, \Omega_1) < \varepsilon$. By the Lipschitz character of g and $F_1, \dots, F_h$, and the minimality of $(\Omega_1, \dots, \Omega_h)$, we have

$$\begin{aligned} m\big(|\Omega_1| - |\omega_1|\big) &\leq g\big(F_1(\omega_1), F_2(\Omega_2), \dots, F_h(\Omega_h)\big) - g\big(F_1(\Omega_1), F_2(\Omega_2), \dots, F_h(\Omega_h)\big) \\ &\leq L\big(F_1(\omega_1) - F_1(\Omega_1)\big) \leq CL d_\gamma(\omega_1, \Omega_1), \end{aligned}$$

where L is the Lipschitz constant of g and C the constant from Definition 5.1. Repeating the argument for Ω_i, we obtain that it is an energy subsolution with Lagrange multiplier $(CL)^{-1}m$. $\square$

Remark 5.4. As a consequence, Theorem 3.13, Proposition 4.2 and Theorem 4.6 apply so we have all information about the perimeter of the cells and their interaction. In particular, there exists a family of open sets $\{D_1, \dots, D_h\} \subset D$ such that

$$\Omega_i \subset D_i, \ \forall i \in \{1, \dots, h\} \qquad \text{and} \qquad \mathrm{cap}(\Omega_i \cap D_j) = 0, \ \forall i \neq j \in \{1, \dots, h\}.$$

Moreover, Ω_i is a solution of the problem

$$\min \Big\{ F_i(\Omega) : \ \Omega \subset D_i, \ \Omega \text{ quasi-open}, \ |\Omega| = |\Omega_i| \Big\}. \tag{5.2}$$

Remark 5.5. We note that Theorem 5.3 also holds in the case of *subsolutions* of (5.1).

In the next two subsections we will consider various examples of multiphase shape optimization problems. We aim to apply Theorem 5.3 and the results from Section 3.1 to deduce some qualitative properties of the optimal partitions.

5.1 Multiphase optimization problems for energy functionals

In this subsection we consider shape optimization problems of the form (5.1) involving the energy functionals E_{f_i}, defined in (2.5), for a function $f_i \in L^\infty(\mathbb{R}^d)$. We recall that for a quasi-open set Ω of finite measure and $f_i \in L^\infty(\Omega)$, we have

$$E_{f_i}(\Omega) = -\frac{1}{2}\int_\Omega f_i w_{\Omega,f_i}\,dx,$$

where $w_{\Omega,f_i} \in H^1_0(\Omega)$ is the solution of the equation

$$-\Delta w_{\Omega,f_i} = f_i, \qquad w_{\Omega,f_i} \in H^1_0(\Omega),$$

and the minimizer in $H^1_0(\Omega)$ of the functional

$$J_{f_i}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f_i u\,dx.$$

We concentrate our attention on the model problem

$$\min\Big\{\sum_{i=1}^h E_{f_i}(\Omega_i) + |\Omega_i| :\ \Omega_i \in \mathcal{A}_{\mathrm{cap}}(D),\ \forall i;\ \Omega_i \cap \Omega_j = \emptyset,\ \forall i \neq j\Big\}, \tag{5.3}$$

where $D \subset \mathbb{R}^d$ is a bounded open set. We note that (5.3) can be written in the form

$$\min\Big\{\sum_{i=1}^h \int_D \Big(\frac{1}{2}|\nabla u_i|^2 - f_i u_i + \mathbb{1}_{\{u_i\neq 0\}}\Big)\,dx :\ u_i \in H^1_0(D),\ u_i u_j = 0\ \text{ for }\ i \neq j\Big\}. \tag{5.4}$$

Remark 5.6 (The energy is γ-Lipschitz). We note that the functionals $E_{f_i} : \mathcal{A}_{\mathrm{cap}}(D) \to \mathbb{R}$ are γ-Lip in the sense of Definition 5.1. Indeed, if $\omega \subset \Omega$ and $M_i := \|f_i\|_{L^\infty}$, then by the maximum principle, we have

$$\begin{aligned} 2\big(E_{f_i}(\omega) - E_{f_i}(\Omega)\big) &= \int_\Omega f_i(w_{\Omega,f_i} - w_{\omega,f_i})\,dx \le M_i \int_\Omega (w_{\Omega,f_i} - w_{\omega,f_i})\,dx \\ &\le M_i \int_\Omega (w_{\Omega,M_i} - w_{\omega,M_i})\,dx = M_i^2 \int_\Omega (w_\Omega - w_\omega)\,dx, \end{aligned}$$

which gives the γ-Lip condition

$$\big|E_{f_i}(\Omega) - E_{f_i}(\omega)\big| \le M_i^2 d_\gamma(\omega,\Omega).$$

Remark 5.7 (Gradient estimate). Suppose that $w_i \in H^1(B_r)$ is such that $-\Delta w_i = f_i$ in the ball B_r for some function $f_i \in L^\infty(B_r)$. Then we have the gradient estimate

$$\|\nabla w_i\|_{L^\infty(B_{r/2})} \le C_d\left(\|f_i\|_{L^\infty(B_r)} + \frac{\|w_i\|_{L^\infty(B_r)}}{r}\right), \tag{5.5}$$

where $C_d > 0$ is a dimensional constant.

In order to obtain the Lipschitz regularity of the state functions $w_i := w_{\Omega_i,f_i}$ of the optimal sets Ω_i, we will need some regularity assumption on the box D.

Remark 5.8 (The external ball condition). We recall that a bounded open set $D \subset \mathbb{R}^d$ is said to satisfy *an external ball condition*, if there is some $R > 0$ such that for every $x \in \partial D$, there is a ball $B_R(y) \subset \mathbb{R}^d$ such that $B_R(y) \cap D = \emptyset$ and $x \in \partial B_R(y)$. We note that if D satisfies the external ball condition, then the energy function w_D is Lipschitz and $\|\nabla w_D\|_{L^\infty} \leq C_{R,d,\mathrm{diam}(D)}$, where $C_{r,d,\mathrm{diam}(D)}$ depends on R, d and the diameter of D. Indeed, let $x_0 \in D$ and let $r = \mathrm{dist}(x_0, \partial D)$. Applying the gradient estimate (5.5) for w_D, we get

$$\begin{aligned}\|\nabla w_D\|_{L^\infty(B_{r/2}(x_0))} &\leq C_d \left(1 + \frac{\|w_D\|_{L^\infty(B_r(x_0))}}{r}\right)\\ &\leq C_d \left(1 + \frac{\|w_A\|_{L^\infty(B_r(x_0))}}{r}\right) \leq C_d \left(1 + C_{r,d,\mathrm{diam}(D)}\right),\end{aligned}$$

where we used the weak maximum principle $w_D \leq w_A$ for the annulus $A = B_{R+\mathrm{diam}(D)}(y) \setminus B_R(y)$, where $y \in D^c$ is the center of the external ball that touches ∂D in the same point as $B_r(x_0)$

Theorem 5.9. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $f_1, \dots, f_h \in L^\infty(D, \mathbb{R}^+)$ be given. Then every solution of* (5.3) *consists of open sets, which are also energy subsolutions. If, moreover the box D satisfies an external ball condition, then the state functions w_{Ω_i,f_i} are Lipschitz continuous on $\mathbb{R}^d$.*

Proof. The existence of a quasi-open solution $(\Omega_1, \dots, \Omega_h)$ follows by Theorem 2.10. By Theorem 5.3, we have that each of the optimal cells Ω_i is an energy subsolution. Thus, by Remark 5.4, there is an open set $D_i \subset D$ such that Ω_i is a solution of the problem

$$\min\Big\{E_{f_i}(\Omega) :\ \Omega \subset D_i,\ \Omega \text{ quasi-open},\ |\Omega| = |\Omega_i|\Big\}. \tag{5.6}$$

Thus, by [4, Theorem 1.1] we have that Ω_i is open for every $i = 1, \dots, h$.

Suppose that $f_i \geq 0$, for all $i = 1, \dots, h$, and that D satisfies the external ball condition. We will prove that the functions $w_i := w_{\Omega_i,f_i}$ are Lipschitz using the gradient estimate (5.5). In order to do that we will study the growth of the function w_i close to the boundary $\partial\Omega_i$.

Step 1 (Boundary points on ∂D). We first prove that there are no two-phase points on the boundary of the box ∂D. Indeed, suppose by contradiction that there is a point $x_0 \in \partial\Omega_i \cap \partial\Omega_j \cap \partial D$. Since D has the external ball condition, there is a ball $B \subset \mathbb{R}^d$ such that $B \cap \Omega_i = \emptyset$, $B \cap \Omega_j = \emptyset$ and $x_0 \in \partial B$. On the other hand, every ball is an energy subsolution for some constant $m > 0$ depending on the radius. Thus, by Theorem 4.6 we have a contradiction. Thus, for every $i = 1, \dots, h$, such that $\partial\Omega_i \cap \partial D \neq \emptyset$, there is an open set $U_i \subset \mathbb{R}^d$ such that:

$$\partial\Omega_i \cap \partial D \subset U_i \qquad \text{and} \qquad 2\,\mathrm{dist}(\partial U_i, \partial\Omega_i \cap \partial D) < \mathrm{dist}(\partial U_i, \partial\Omega_k), \qquad \forall k \neq i.$$

Moreover, for $d_i > 0$ small enough, we can choose U_i of the form

$$U_i = \Big\{x \in \mathbb{R}^d :\ \mathrm{dist}(x, \partial\Omega_i \cap \partial D) < d_i\Big\}. \tag{5.7}$$

Step 2 (Growth of w_i around the two-phase boundary points). Let Ω_i, Ω_j be two sets from the optimal configuration. Without loss of generality, we set $i = 1$, $j = 2$. Consider the common boundary $\partial\Omega_1 \cap \partial\Omega_2 \subset D$. Since the optimal cells are energy subsolutions, we have that there is an open set $U \subset D$ such that

$$\partial\Omega_1 \cap \partial\Omega_2 \subset U \qquad \text{and} \qquad U \cap \Omega_k = \emptyset, \quad \text{for} \quad k \neq 1, 2.$$

Moreover, we can suppose that the ∂U is smooth. We note that the state functions $w_1 := w_{\Omega_1,f_1}$ and $w_2 := w_{\Omega_2,f_2}$ solve the following multiphase problem in U:

$$\min\Big\{\sum_{i=1,2} \int_U \left(\frac{1}{2}|\nabla u_i|^2 - f_i u_i + \mathbb{1}_{\{u_i>0\}}\right) dx :\ u_i - w_i \in H_0^1(U),\ u_i \geq 0,\ u_1 u_2 = 0\Big\}. \tag{5.8}$$

We now set $f := f_1 \mathbb{1}_{\{w_1>0\}} - f_2 \mathbb{1}_{\{w_2>0\}}$. Then for any $u \in H^1(U)$ such that $u - w_1 + w_2 \in H^1_0(U)$, we have that $u = u^+ - u^-$, $u^+ - w_1 \in H^1_0(U)$ and $u^- - w_2 \in H^1_0(U)$.

$$\begin{aligned}
&\int_U \left(\frac{1}{2}|\nabla u|^2 - fu + \mathbb{1}_{\{u\neq 0\}}\right) dx \\
&\qquad = \int_U \left(\frac{1}{2}|\nabla u^+|^2 - fu^+ + \mathbb{1}_{\{u>0\}}\right) dx + \int_U \left(\frac{1}{2}|\nabla u^-|^2 + fu^- + \mathbb{1}_{\{u<0\}}\right) dx \\
&\qquad \geq \int_U \left(\frac{1}{2}|\nabla u^+|^2 - f_1u^+ + \mathbb{1}_{\{u>0\}}\right) dx + \int_U \left(\frac{1}{2}|\nabla u^-|^2 - f_2u^- + \mathbb{1}_{\{u<0\}}\right) dx \\
&\qquad \geq \int_U \left(\frac{1}{2}|\nabla w_1|^2 - f_1w_1 + \mathbb{1}_{\{w_1>0\}}\right) dx + \int_U \left(\frac{1}{2}|\nabla w_2|^2 - f_2w_2 + \mathbb{1}_{\{w_2>0\}}\right) dx \\
&\qquad = \int_U \left(\frac{1}{2}|\nabla w|^2 - fw + \mathbb{1}_{\{w\neq 0\}}\right) dx,
\end{aligned}$$

where we set $w := w_1 - w_2$. Thus, w is a solution of the problem

$$\min\Big\{ \int_U \left(\frac{1}{2}|\nabla u|^2 - fu + \mathbb{1}_{\{u\neq 0\}}\right) dx : \ u - w \in H^1_0(U)\Big\}. \tag{5.9}$$

Applying again [4, Theorem 1.1], we get that w is locally Lipschitz in U and so w_1 and w_2 are also locally Lipschitz in U. We deduce that for every $i \neq j$, there is an open set U_{ij} such that

$$\partial\Omega_i \cap \partial\Omega_j \subset D_{ij}, \qquad U_{ij} \cap \Omega_k = \emptyset, \quad \text{for} \quad k \neq 1, 2,$$

and constants $C_{ij} > 0$ and $r_{ij} > 0$ such that

$$\|w_i\|_{L^\infty(B_r(x))} \leq C_{ij} r, \qquad \forall x \in \partial\Omega_i \cap U_{ij},\ \forall r \leq r_{ij}. \tag{5.10}$$

Moreover, by choosing r_{ij} small enough, we can suppose that U_{ij} is of the form

$$U_{ij} = \Big\{x \in D : \ \operatorname{dist}(x, \partial\Omega_i \cap \partial\Omega_j) < r_{ij}\Big\}. \tag{5.11}$$

Step 3 (Growth of w_i around the one-phase boundary points). Suppose that $x_i \in \partial\Omega_i$ is such that there is a ball $B_{r_i}(x_i)$, which does not intersect the cells Ω_j for $j \neq i$. Then w_i solves the problem

$$\min\Big\{ \int_{B_{r_i}(x_i)} \left(\frac{1}{2}|\nabla u_i|^2 - f_iu_i + \mathbb{1}_{\{u_i>0\}}\right) dx : \ u_i - w_i \in H^1_0(B_{r_i}(x_i)),\ u_i \geq 0\Big\}. \tag{5.12}$$

For any $\tilde{x} \in B_{r_i/2}(x_i) \cap \partial\Omega_i$ and any $r \leq r_i/2$, we consider the function $u = w_i \mathbb{1}_{B^c} + v\mathbb{1}_B$, where $B = B_r(\tilde{x})$ and $v \in H^1(B)$ is the solution of the problem

$$\min\Big\{ \int_B |\nabla v|^2\, dx : \ v \in H^1(B),\ v - w_i \in H^1_0(B),\ v \geq w_i\Big\}.$$

We note that v is super-harmonic on B and harmonic on $\{v > w_i\}$. Thus $v > 0$ on B and by the optimality of w_i, we have

$$\begin{aligned}
|\{w_i = 0\} \cap B| &= |\{v > 0\} \cap B| - |\{w_i > 0\} \cap B| \\
&\geq \frac{1}{2}\int_B |\nabla w_i|^2 - |\nabla v|^2\, dx + \int_B f(v - w_i)\, dx \\
&\geq \frac{1}{2}\int_B |\nabla w_i|^2 - |\nabla v|^2\, dx = \frac{1}{2}\int_B |\nabla(w_i - v)|^2\, dx \\
&\geq C_d \big|\{w_i = 0\} \cap B\big| \left(\frac{1}{r} \fint_{\partial B_r(\tilde{x})} w_i\, d\mathcal{H}^{d-1}\right)^2,
\end{aligned}$$

where the last inequality is due to Lemma 4.7. By the non-degeneracy of w_i (Lemma 3.8) and the estimate from Corollary 3.9, we get that for every single-phase point $x_i \in \partial\Omega_i$ there are constants $r_i > 0$ and $C_i > 0$ (we note that r_i depends on x_i, but C_i depends only on M_i and the dimension) such that

$$\|w_i\|_{L^\infty(B_r(x))} \le C_i r, \qquad \forall x \in \partial\Omega_i \cap B_{r_i}(x_i),\ \forall r \le r_i. \tag{5.13}$$

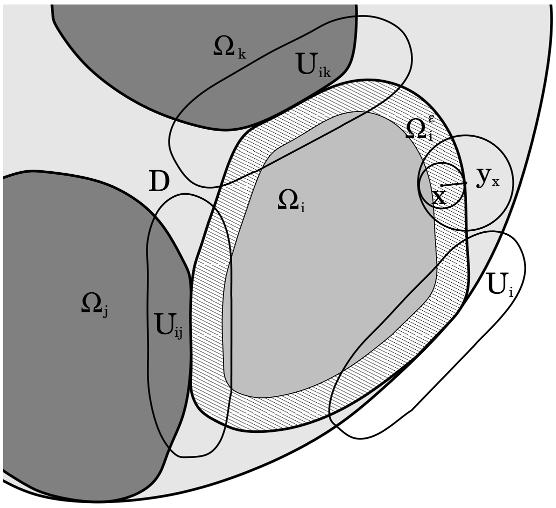

Figure 1: The three types of boundary points of an optimal cell Ω_i.

Step 4 (Lipschitz continuity of w_i). We denote with V_i the closed set

$$V_i := \partial\Omega_i \setminus \Big(U_i \cup \Big(\bigcup_{i\neq j} U_{ij}\Big)\Big),$$

where U_i and U_{ij} are as in *Step 1* and *Step 2*. Then there is some $R_i > 0$ such that

$$\mathrm{dist}(V_i, \partial\Omega_k) \ge R_i, \qquad \forall k \neq i.$$

Let $\varepsilon := \dfrac{1}{10}\min\big\{d_i, r_{ij}, R_i\big\}$ and

$$\Omega_i^\varepsilon := \Big\{x \in \Omega_i :\ \mathrm{dist}(x, \partial\Omega_i) < \varepsilon\Big\}.$$

For every $x \in \Omega_i^\varepsilon$, let $y_x \in \partial\Omega_i$ be such that $r_x := \mathrm{dist}(x, y_x) = \mathrm{dist}(x, \partial\Omega_i)$.

- If $x \in \Omega_i^\varepsilon$ is such that $y_x \in \partial\Omega_i \cap U_{ij}$, then by the gradient estimate (5.5) and (5.10), we have

$$\begin{aligned}\|\nabla w_i\|_{L^\infty(B_{r_x/2}(x))} &\le C_d\left(M_i + \frac{\|w_i\|_{L^\infty(B_{r_x}(x))}}{r_x}\right)\\ &\le C_d\left(M_i + \frac{\|w_i\|_{L^\infty(B_{2r_x}(y_x))}}{r_x}\right) \le C_d\left(M_i + 2C_{ij}\right).\end{aligned}$$

- If $x \in \Omega_i^\varepsilon$ is such that $\mathrm{dist}(x, \partial D) \le 3r_x$, then we have

$$\begin{aligned}\|\nabla w_i\|_{L^\infty(B_{r_x/2}(x))} &\le C_d\left(M_i + \frac{\|w_i\|_{L^\infty(B_{r_x}(x))}}{r_x}\right)\\ &\le C_d M_i\left(1 + \frac{\|w_D\|_{L^\infty(B_{4r_x}(z_x))}}{r_x}\right) \le C_d M_i\Big(1 + 4\|\nabla w_D\|_{L^\infty}\Big),\end{aligned}$$

  where $z_x \in \partial D$ is such that $|x - z_x| \le 3r_x$.

- If $x \in \Omega_i^\varepsilon$ is such that $\mathrm{dist}(x, \partial D) \geq 3r_x$ and $y_x \notin U_{ij}$, then we have that

$$B_{2r_x}(y_x) \subset D \qquad \text{and} \qquad B_{2r_x}(y_x) \cap \Omega_k = \emptyset, \qquad \forall k \neq i.$$

  Thus, by *Step 3*, we have

$$\begin{aligned}\|\nabla w_i\|_{L^\infty(B_{r_x/2}(x))} &\leq C_d \left(M_i + \frac{\|w_i\|_{L^\infty(B_{r_x}(x))}}{r_x} \right) \\ &\leq C_d \left(M_i + \frac{\|w_i\|_{L^\infty(B_{2r_x}(y_x))}}{r_x} \right) \leq C_d\big(M_i + 2C_i\big),\end{aligned}$$

  which concludes the proof.

□

5.2 Multiphase optimization problems for eigenvalues

In this subsection, we consider multiphase shape optimization problems involving the eigenvalues of the Dirichlet Laplacian $F_i = \lambda_{k_i}$, defined in (2.13). In particular, we prove that the problem

$$\min\Big\{ \sum_{i=1}^h \lambda_{k_i}(\Omega_i) + m|\Omega_i| : \ \Omega_i \subset D \ \textbf{open}, \ \forall i; \ \Omega_i \cap \Omega_j = \emptyset, \forall i \neq j \Big\}, \tag{5.14}$$

admits a solution in the case $k_i \in \{1, 2\}$. Or main result is the following:

Theorem 5.10. *Let $D \subset \mathbb{R}^d$ be a bounded open set and $m > 0$. Let $k_i \in \mathbb{N}$, $i = 1, \dots, h$ and $(\Omega_1, \dots, \Omega_h)$ be a solution of*

$$\min\Big\{ \sum_{i=1}^h \lambda_{k_i}(\Omega_i) + m|\Omega_i| : \ \Omega_i \subset D \ \textit{quasi-open}, \ \forall i; \ \Omega_i \cap \Omega_j = \emptyset, \forall i \neq j \Big\}. \tag{5.15}$$

Then, for every $i = 1, \dots, h$ the quasi-open set Ω_i is an energy subsolution. If, moreover, $k_i \in \{1, 2\}$, then there exist open sets $\omega_i \subseteq \Omega_i$ such that $(\omega_1, \dots, \omega_h)$ is also a solution of (5.15). *In particular,* (5.14) *has a solution.*

Proof. The fact that each Ω_i is a subsolution relies on the γ-Lip property of the k-th eigenvalue. If $k_i \in \{1, 2\}$, we use the existence of an open set D_i such that Ω_i is solution of

$$\min\Big\{ \lambda_{k_i}(\Omega) + m|\Omega| : \ \Omega \subset D_i, \ \Omega \text{ quasi-open} \Big\}. \tag{5.16}$$

If $k_i = 1$, following [5], the set Ω_i is open. If $k_i = 2$, we note that the functional λ_2 can be alternatively defined as

$$\lambda_2(\Omega) = \min\Big\{ \max\big\{\lambda_1(\Omega_a), \lambda_1(\Omega_b)\big\} : \ \Omega_a, \Omega_b \subset \Omega \text{ quasi-open}, \ \Omega_a \cap \Omega_b = \emptyset \Big\}.$$

Thus, if $(\Omega_a, \Omega_b) \in [\mathcal{A}_{\mathrm{cap}}(D_i)]^2$ is a solution of (5.1) with $g(x_1, x_2) = \max\{x_1, x_2\}$ and $F_a = F_b = \lambda_1$, then the set $\Omega = \Omega_a \cup \Omega_b$ is a solution of (5.16). Now, the quasi-open sets Ω_a and Ω_b can be isolated by open sets D_a and D_b. Thus, Ω_a and Ω_b minimize the first Dirichlet eigenvalue with a fixed measure constraint in D_a and D_b, respectively. Relying again on the regularity result from [5], we obtain that Ω_a and Ω_b are open sets. □

In particular the following holds.

Corollary 5.11. *Let $D \subset \mathbb{R}^d$ be a bounded open set and $m > 0$. For every solution Ω of the problem*

$$\min\Big\{ \lambda_2(\Omega) + m|\Omega| : \ \Omega \ \textit{quasi-open}, \ \Omega \subset D \Big\}, \tag{5.17}$$

there exists an open set $\omega \subseteq \Omega$ which is also solution and has the same measure as Ω.

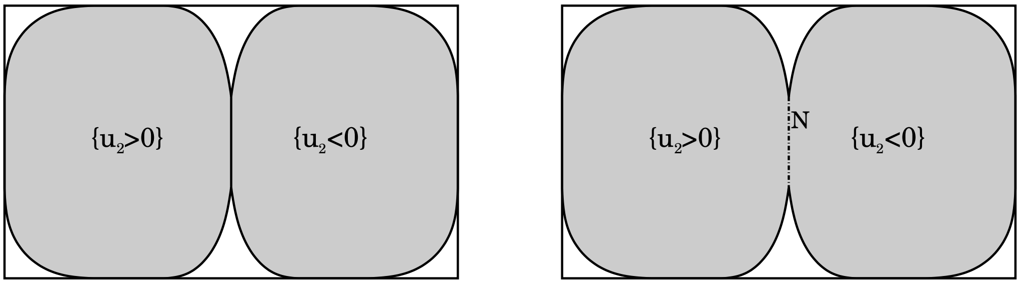

Figure 2: The optimal disconnected set for λ_2 (on the left) and a non-open optimal set (on the right).

We expect that in some cases, problem (5.17) may have solutions which are quasi-open but not open.

In fact, for some suitably chosen box D and constant $m > 0$ small enough we expect that the nodal domains, $\{u_2 > 0\}$ and $\{u_2 < 0\}$ of the eigenfunction $u_2 \in H_0^1(\Omega)$ on the optimal set $\Omega \subset D$ solving (5.17), are touching in a (sufficiently smooth) nodal set of dimension $d-1$ (see Figure 2). If this is the case then both the set $\{u_2 \neq 0\}$ and the set $\{u_2 \neq 0\} \cup N$ are solutions of (5.17), where N is a subset of the nodal set $\partial\{u_2 > 0\} \cap \partial\{u_2 < 0\}$ of measure zero. Choosing N appropriately, one may construct an optimal set, which is not open (but is equivalent to an open set in sense of the Lebesgue measure).

A somehow similar result for functionals involving higher eigenvalues holds for $m = 0$ in dimension 2. In this case, the existence of an optimal open partition was already proved in [3]; below we prove that every optimal partition is equivalent to an open one.

Theorem 5.12. *Let $D \subseteq \mathbb{R}^2$ be a bounded, open and smooth set, let $m = 0$ and $k_i \in \mathbb{N}$, $i = 1, \dots, h$. Let $(\Omega_1, \dots, \Omega_h)$ be a solution of* (5.15)*. There exists a solution $(\widetilde{\Omega}_1, \dots, \widetilde{\Omega}_h)$ consisting of open sets such that, for every $i = 1, \dots, h$,*

$$\lambda_{k_i}(\widetilde{\Omega}_i) = \lambda_{k_i}(\Omega_i) = \lambda_{k_i}(\Omega_i \cap \widetilde{\Omega}_i).$$

Moreover, every eigenfunction $u_{k_i}(\widetilde{\Omega}_i)$ is Hölder continuous on $\overline{D}$.

Proof. By [3, Theorem 2.1], we have that for each $\varepsilon > 0$ there are open sets $(A_1^\varepsilon, \dots, A_h^\varepsilon)$ such that $A_i^\varepsilon \cap A_j^\varepsilon = \emptyset$, for every $i \neq j \in \{1, \dots, h\}$ and A_i^ε γ-converges to Ω_i, for every $i = 1, \dots, h$. By choosing appropriate subsets of each A_i^ε, we may suppose that the connected components of the open sets $A_1^\varepsilon, \dots A_h^\varepsilon$ are polygons. For each A_i^ε let $E_i^\varepsilon \subset A_i^\varepsilon$ be a union of at most k_i connected components of A_i^ε and such that $\lambda_{k_i}(E_i^\varepsilon) = \lambda_{k_i}(A_i^\varepsilon)$. By the compactness of the weak γ-convergence, we may suppose that E_i^ε weak γ-converges to some quasi-open set $\omega_i \subset \Omega_i$. Moreover, we have that $\omega_i \cap \widetilde{\Omega}_j = \emptyset$, for $i \neq j$, and

$$\lambda_{k_i}(\omega_i) \leq \liminf_{\varepsilon \to 0} \lambda_{k_i}(E_i^\varepsilon) = \lim_{\varepsilon \to 0} \lambda_{k_i}(A_i^\varepsilon) = \lambda_{k_i}(\Omega_i),$$

and, by the optimality of $\Omega_1, \dots, \Omega_h$, we have $\lambda_{k_i}(\omega_i) = \lambda_{k_i}(\Omega_i)$.

We now enlarge each E_i^ε in order to obtain a partition which covers D. We claim that for each ε there are disjoint open sets $F_1^\varepsilon, \dots, F_h^\varepsilon$ such that $E_i^\varepsilon \subset F_i^\varepsilon$, F_i^ε has at most k_i connected components, $D \cap \partial F_i^\varepsilon$ is piecewise linear and $\overline{D} = \bigcup_{i=1}^h \overline{F_i^\varepsilon}$. One can obtain the family $F_1^\varepsilon, \dots, F_h^\varepsilon$ from $E_1^\varepsilon, \dots, E_h^\varepsilon$, considering all the connected components of $D \setminus \left(\bigcup_{i=1}^h \overline{E_i^\varepsilon}\right)$ and adding them, one by one, to one of the sets $E_1^\varepsilon, \dots, E_h^\varepsilon$, with which they have common boundary. We note that for every $i = 1, \dots, h$, the number of connected components of $\mathbb{R}^2 \setminus F_i^\varepsilon$ is bounded uniformly in ε. Thus, by Sverak's Theorem (see, for example, [7, Theorem 4.7.1]), there are disjoint open sets $\widetilde{\Omega}_1, \dots, \widetilde{\Omega}_h$ such that F_i^ε γ-converges to $\widetilde{\Omega}_i$. Moreover, we have $\omega_i \subset \widetilde{\Omega}_i$ and since,

$$\lambda_{k_i}(\widetilde{\Omega}_i) \leq \liminf_{\varepsilon \to 0} \lambda_{k_i}(F_i^\varepsilon) \leq \liminf_{\varepsilon \to 0} \lambda_{k_i}(E_i^\varepsilon) = \lambda_{k_i}(\Omega_i),$$

by the optimality of $\Omega_1, \dots, \Omega_h$, we have that $\lambda_{k_i}(\widetilde{\Omega}_i) = \lambda_{k_i}(\Omega_i) = \lambda_{k_i}(\omega_i)$.

Each eigenfunction belongs to $C^{0,\alpha}(\overline{D})$ as a consequence of the fact that the sets $\mathbb{R}^2 \setminus \widetilde{\Omega}_i$ have a finite number of connected components, hence they satisfy a uniform capacity density condition (see for instance [7, Theorem 4.6.7]). □

6 Appendix: Proof of the Monotonicity Lemma

The proof of Lemma 2.14 follows the main steps and arguments of Theorem 2.12, for which we refer the reader to [14]. Nevertheless, the proof of Lemma 2.14 is simplified by the use of the conclusion of Theorem 2.12. For the convenience of the reader, we use similar notations as in [14]. We report here only the technical difficulties brought by the absence of continuity of the functions u_i and presence of the third phase.

We start with recalling some preliminary results from [14]. The first lemma was proved in [14, Remark 1.5].

Lemma 6.1. *Suppose that $u \in H^1(B_2)$ is a non-negative Sobolev function such that $\Delta u + 1 \geq 0$ on $B_2 \subset \mathbb{R}^d$. Then, there is a dimensional constant C_d such that*

$$\int_{B_1} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx \leq C_d \left(1 + \int_{B_2\setminus B_1} u^2\,dx\right). \tag{6.1}$$

Proof. Let $u_\varepsilon = \phi_\varepsilon * u$, where $\phi_\varepsilon \in C_c^\infty(B_\varepsilon)$ is a standard molifier. Then $u_\varepsilon \to u$ strongly in $H^1(B_2)$, $u_\varepsilon \in C^\infty(B_2)$ and $\Delta u_\varepsilon + 1 \geq 0$ on $B_{2-\varepsilon}$. By [14, Remark 1.5] the estimate (6.1) holds for u_ε. The claim follows by passing to the limit as $\varepsilon \to 0$. □

The next Lemma is implicitly contained in [14, Lemma 2.8] and is precisely the estimate in which the continuity of u_i was used.

Lemma 6.2. *Let $u \in H^1(B_2)$ be a non-negative function such that $\Delta u + 1 \geq 0$ on B_2. Then for Lebesgue almost every $r \in (0,1)$ we have the estimate*

$$\frac{1}{r^4}\int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx \leq C_d \left(1 + \frac{r^{-2}}{\sqrt{\lambda(u,r)}}\Big(\fint_{\partial B_r} |\nabla u|^2\,d\mathcal{H}^{d-1}\Big)^{\frac{1}{2}}\right) + \frac{d\omega_d r^{-3}}{2\alpha(u,r)}\fint_{\partial B_r} |\nabla u|^2\,d\mathcal{H}^{d-1}, \tag{6.2}$$

where

$$\lambda(u,r) := \min\left\{\frac{\int_{\partial B_r} |\nabla v|^2\,d\mathcal{H}^{d-1}}{\int_{\partial B_r} v^2\,d\mathcal{H}^{d-1}} \;:\; v \in H^1(\partial B_r),\ \mathcal{H}^{d-1}\big(\{v \neq 0\} \cap \{u = 0\}\big) = 0\right\}, \tag{6.3}$$

and $\alpha(u,r) \in \mathbb{R}^+$ is the characteristic constant of $\{u > 0\} \cap \partial B_r$, i.e. the non-negative solution of the equation

$$\alpha(u,r)\left(\alpha(u,r) + \frac{d-2}{r}\right) = \lambda(u,r). \tag{6.4}$$

Proof. We start by determining the subset of the interval $(0,1)$ for which we will prove that (6.2) holds. Let $u_\varepsilon := u * \phi_\varepsilon$, where ϕ_ε is a standard molifier. Then we have that:

(i) for almost every $r \in (0,1)$ the restriction of u to ∂B_r is Sobolev. i.e. $u_{|\partial B_r} \in H^1(\partial B_r)$;

(ii) for almost every $r \in (0,1)$ the sequence of restrictions $(\nabla u_\varepsilon)_{|\partial B_r}$ converges strongly in $L^2(\partial B_r; \mathbb{R}^d)$ to $(\nabla u)_{|\partial B_r}$.

We now consider $r \in (0,1)$ such that both *(i)* and *(ii)* hold. By using the scaling $u_r(x) := r^{-2}u(rx)$, we can suppose that $r = 1$. Reasoning as in [14, Lemma 23], we have that

$$\begin{aligned}2\int_{B_1} \frac{|\nabla u_\varepsilon|^2}{|x|^{d-2}}\,dx \quad &\leq C_d + C_d\left(\int_{\partial B_1} u_\varepsilon^2\,d\mathcal{H}^{d-1}\right)^{1/2} + 2\left(\int_{\partial B_1} u_\varepsilon^2\,d\mathcal{H}^{d-1}\right)^{\frac{1}{2}}\left(\int_{\partial B_1} \left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 d\mathcal{H}^{d-1}\right)^{\frac{1}{2}}\\ &+(d-2)\int_{\partial B_1} u_\varepsilon^2\,d\mathcal{H}^{d-1},\end{aligned}$$

and passing to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned} 2\int_{B_1} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx \quad &\le C_d + C_d\left(\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1}\right)^{1/2} + 2\left(\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1}\right)^{\frac12}\left(\int_{\partial B_1}\left|\frac{\partial u}{\partial n}\right|^2 d\mathcal{H}^{d-1}\right)^{\frac12} \\ &\quad +(d-2)\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1}. \end{aligned} \tag{6.5}$$

Now we have two cases:

Case 1. If $\mathcal{H}^{d-1}\big(\{u=0\}\cap\partial B_1\big)=0$, then $\lambda=0$. Now if $\int_{\partial B_1}|\nabla u|^2\,d\mathcal{H}^{d-1}>0$, then the inequality (6.2) is trivial. If on the other hand, $\int_{\partial B_1}|\nabla u|^2\,d\mathcal{H}^{d-1}=0$, then u is a constant on ∂B_1 and so, we may suppose that $u=0$ on $\mathbb{R}^d\setminus B_1$, which again gives (6.2), by choosing C_d large enough.

Case 2. Suppose now that $\mathcal{H}^{d-1}\big(\{u=0\}\cap\partial B_1\big)>0$, the constant λ defined in (6.3) is strictly positive. Then, by (6.5) we have

$$\begin{aligned} 2\int_{B_1}\frac{|\nabla u|^2}{|x|^{d-2}}\,dx \quad &\le C_d + \frac{C_d}{\sqrt{\lambda}}\left(\int_{\partial B_1}|\nabla u|^2\,d\mathcal{H}^{d-1}\right)^{1/2} + \frac{d-2}{\lambda}\int_{\partial B_1}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^{d-1} \\ &\quad + \frac{2}{\sqrt{\lambda}}\left(\int_{\partial B_1}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^{d-1}\right)^{1/2}\left(\int_{\partial B_1}\left|\frac{\partial u}{\partial n}\right|^2 d\mathcal{H}^{d-1}\right)^{1/2} \\ &\le C_d + \frac{C_d}{\sqrt{\lambda}}\left(\int_{\partial B_1}|\nabla u|^2\,d\mathcal{H}^{d-1}\right)^{1/2} \\ &\quad + \frac{1}{\alpha}\int_{\partial B_1}\left|\frac{\partial u}{\partial n}\right|^2 d\mathcal{H}^{d-1} + \frac{\alpha+(d-2)}{\lambda}\int_{\partial B_1}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^{d-1}, \end{aligned}$$

which concludes the proof since by the definition of α, we have $\alpha^{-1}=\dfrac{\alpha+(d-2)}{\lambda}$. □

For u_1, u_2 and u_3 as in Theorem 2.14, we use the notation

$$A_{u_i}(r) = \int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx, \qquad \text{for} \quad i=1,2,3. \tag{6.6}$$

Remark 6.3. The function $A_u(r)$ is bounded and increasing for $r\in(0,1)$. Moreover, $A_u(r)$ is invariant with respect to the rescaling $u_r(x):=u(rx)$.

Below, A_{u_i} will be simply denoted A_i

Lemma 6.4. *Let u_1, u_2 and u_3 be as in Lemma 2.14. Then there are dimensional constants $C_d>0$ and $\varepsilon>0$ such that if $A_i(1/4)\ge C_d$, for every $i=1,2,3$, then*

$$\frac{d}{dr}\left[\frac{A_1(r)A_2(r)A_3(r)}{r^{6+3\varepsilon}}\right] \ge -C_d\left(\frac{1}{\sqrt{A_1(r)}}+\frac{1}{\sqrt{A_2(r)}}+\frac{1}{\sqrt{A_3(r)}}\right)\frac{A_1(r)A_2(r)A_3(r)}{r^{6+3\varepsilon}}, \tag{6.7}$$

for Lebesgue almost every $r\in[1/4,1]$.

Proof. We set, for $i=1,2,3$ and $r>0$,

$$B_i(r) = \int_{\partial B_r}|\nabla u_i|^2\,d\mathcal{H}^{d-1}.$$

Since A_i, for $i=1,2,3$, are increasing functions they are differentiable almost everywhere on $\mathbb{R}$. Thus for almost every $r\in(0,1)$, we can compute the derivative in the l.h.s. of (6.7).

$$\frac{d}{dr}\left[\frac{A_1(r)A_2(r)A_3(r)}{r^{6+3\varepsilon}}\right] = \left(-\frac{6+3\varepsilon}{r}+\frac{r^{2-d}B_1(r)}{A_1(r)}+\frac{r^{2-d}B_2(r)}{A_2(r)}+\frac{r^{2-d}B_3(r)}{A_3(r)}\right)\frac{A_1(r)A_2(r)A_3(r)}{r^{6+3\varepsilon}}.$$

Thus, it is sufficient to prove that for almost every $r\in[1/4,1]$ we have

$$-\frac{6+3\varepsilon}{r}+r^{2-d}\left(\frac{B_1(r)}{A_1(r)}+\frac{B_2(r)}{A_2(r)}+\frac{B_3(r)}{A_3(r)}\right)\geq -C_d\left(\frac{1}{\sqrt{A_1(r)}}+\frac{1}{\sqrt{A_2(r)}}+\frac{1}{\sqrt{A_3(r)}}\right). \tag{6.8}$$

Using the rescaling $u_{i,r}(x)=r^{-2}u_i(rx)$ we have

$$\int_{\partial B_1}|\nabla u_{i,r}|^2\,d\mathcal{H}^{d-1}=\frac{1}{r^{d+1}}\int_{\partial B_r}|\nabla u_i|^2\,d\mathcal{H}^{d-1}\qquad\text{and}\qquad\int_{B_1}\frac{|\nabla u_{i,r}|^2}{|x|^{d-2}}\,dx=\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx,$$

and so, in (6.8) we may assume that $r=1$. We consider two cases.

Case 1. Suppose that there is some $i=1,2,3$, say $i=1$, such that $(6+3\varepsilon)A_1(1)\leq B_1(1)$. Then we have

$$-(6+3\varepsilon)+\frac{B_1(1)}{A_1(1)}+\frac{B_2(1)}{A_2(1)}+\frac{B_3(1)}{A_3(1)}\geq -(6+3\varepsilon)+\frac{B_1(1)}{A_1(1)}\geq 0,$$

which proves (6.8) and the lemma.

Case 2. Suppose that for each $i=1,2,3$ we have $(6+3\varepsilon)A_i(1)\geq B_i(1)$. Since, for every $i=1,2,3$ we have $A_i(1)\geq C_d$, we can apply Lemma 6.2 with the additional notation $\alpha_i:=\alpha(u_i,1)$ and $\lambda_i:=\lambda(u_i,1)$ and, by choosing C_d large enough, we get

$$(2-\varepsilon)A_i(1)\leq C_d\sqrt{B_i(1)/\lambda_i}+B_i(1)/\alpha_i\leq C_d\sqrt{A_i(1)/\lambda_i}+B_i(1)/\alpha_i\leq C_d\sqrt{A_i(1)}/\alpha_i+B_i(1)/\alpha_i.$$

Multiplying both sides by $\alpha_i/A_i(1)$ and summing for $i=1,2,3$, we obtain

$$(2-\varepsilon)(\alpha_1+\alpha_2+\alpha_3)\leq C_d\sum_{i=1}^3\frac{1}{\sqrt{A_i(1)}}+\sum_{i=1}^3\frac{B_i(1)}{A_i(1)},$$

and so, in order to prove (6.8), it is sufficient to prove that

$$\alpha_1+\alpha_2+\alpha_3\geq\frac{6+3\varepsilon}{2-\varepsilon}. \tag{6.9}$$

Let $\Omega_1^*,\Omega_2^*,\Omega_3^*\subset\partial B_1$ be the optimal partition of the sphere ∂B_1 for the characteristic constant α, i.e. the triple $\{\Omega_1^*,\Omega_2^*,\Omega_3^*\}$ is a solution of the problem

$$\min\Big\{\alpha(\Omega_1)+\alpha(\Omega_2)+\alpha(\Omega_3):\ \Omega_i\subset\partial B_1\ ,\forall i;\ \mathcal{H}^{d-1}(\Omega_i\cap\Omega_j)=0,\forall i\neq j\Big\}. \tag{6.10}$$

We recall that for a set $\Omega\subset\partial B_1$, the characteristic constant $\alpha(\Omega)$ is the unique positive real number such that $\lambda(\Omega)=\alpha(\Omega)(\alpha(\Omega)+d-2)$, where

$$\lambda(\Omega)=\min\left\{\frac{\int_{\partial B_1}|\nabla v|^2\mathcal{H}^{d-1}}{\int_{\partial B_1}v^2\mathcal{H}^{d-1}}:\ v\in H^1(\partial B_1),\ \mathcal{H}^{d-1}\big(\{u\neq 0\}\setminus\Omega\big)=0\right\}.$$

We note that, by [24], $\alpha(\Omega_i^*)+\alpha(\Omega_j^*)\geq 2$, for $i\neq j$ and so summing on i and j, we have

$$6\leq\alpha(\Omega_1^*)+\alpha(\Omega_2^*)+\alpha(\Omega_3^*)\leq\alpha_1+\alpha_2+\alpha_3.$$

Moreover, the first inequality is strict. Indeed, if this is not the case, then $\alpha(\Omega_1^*)+\alpha(\Omega_2^*)=2$, which in turn gives that Ω_1^* and Ω_2^* are two opposite hemispheres (see for example [16]). Thus $\Omega_3^*=\emptyset$, which is impossible[1] Choosing ε to be such that $\frac{6+3\varepsilon}{2-\varepsilon}$ is smaller than the minimum in (6.10), the proof is concluded. □

Lemma 6.5. *Let u_1, u_2 and u_3 be as in Lemma* 2.14. *Then, there are dimensional constants $C_d>0$ and $\varepsilon>0$ such that the following implication holds: if for some $r>0$*

$$\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\geq C_d,\qquad\textit{for all}\quad i=1,2,3,$$

[1] For example, it is in contradiction with the equality $\alpha(\Omega_1^*)+\alpha(\Omega_3^*)=2$, which is also implied by the contradiction assumption.

then we have the estimate

$$4^{(6+3\varepsilon)}A_1\left(\frac{r}{4}\right)A_2\left(\frac{r}{4}\right)A_3\left(\frac{r}{4}\right)\le \big(1+\delta_{123}(r)\big)A_1(r)A_2(r)A_3(r), \tag{6.11}$$

where

$$\delta_{123}(r):=C_d\sum_{i=1}^{3}\left(\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right)^{-1/2}. \tag{6.12}$$

Proof. We first note that the (6.11) is invariant under the rescaling $u_r(x)=r^{-2}u(xr)$. Thus, we may suppose that $r=1$. We first note that if for some $i=1,2,3$, say $i=1$, we have $4^{6+3\varepsilon}A_1(1/4)\le A_1(1)$, then (6.11) holds for any positive δ_{123}.

Suppose now that for every $i=1,2,3$, we have $4^{6+3\varepsilon}A_i(1/4)\ge A_i(1)\ge C_d$. Then, we can apply Lemma 6.4, obtaining that

$$\begin{aligned}A_1(1)A_2(1)A_3(1)\ -\ & 4^{6+3\varepsilon}A_1(1/4)A_2(1/4)A_3(1/4)\\ &\ge -C_d\int_{1/4}^{1}\left(\sum_{i=1}^{3}\frac{1}{\sqrt{A_i(r)}}\right)A_1(r)A_2(r)A_3(r)\,dr\\ &\ge -3C_d4^{2+\frac{3}{2}\varepsilon}\left(\sum_{i=1}^{3}\frac{1}{\sqrt{A_i(1)}}\right)A_1(1)A_2(1)A_3(1),\end{aligned}$$

which gives the claim. □

Proof of Lemma 2.14. For $i=1,2,3$, we adopt the notation

$$A_i^k:=A_i(4^{-k}),\qquad b_i^k:=4^{4k}A_i(4^{-k})\qquad\text{and}\qquad \delta_k:=\delta_{123}(4^{-k}), \tag{6.13}$$

where A_i was defined in (6.6) and δ_{123} in (6.12).

Let $M>0$ and let

$$S(M)=\Big\{k\in\mathbb{N}:\ 4^{(6+3\varepsilon)k}A_1^kA_2^kA_3^k\le M\big(1+A_1^0+A_2^0+A_3^0\big)^3\Big\}.$$

We will prove that there is M large enough such that $k\in S(M)$, for every $k\in\mathbb{N}$. We first note that if $k\notin S(M)$, then we have

$$\begin{aligned}M\big(1+A_1^0+A_2^0+A_3^0\big)^3 &\le 4^{(6+3\varepsilon)k}A_1^kA_2^kA_3^k\\ &\le 4^{-(2-3\varepsilon)k}b_1^k4^{4k}A_2^kA_3^k\\ &\le 4^{-(2-3\varepsilon)k}b_1^kC_d\big(1+A_1^0+A_2^0+A_3^0\big)^2,\end{aligned}$$

and so $b_1^k\ge C_d^{-1}M4^{(2-3\varepsilon)k}$, where C_d is the constant from Theorem 2.12. Thus, choosing $\varepsilon<2/3$ and $M>0$ large enough, we can suppose that, for every $i=1,2,3$, $b_i^k>C_d$, where C_d is the constant from Lemma 6.5.

Suppose now that $L\in\mathbb{N}$ is such that $L\notin S(M)$ and let

$$l=\max\Big\{k\in\mathbb{N}:\ k\in S(M)\cap[0,L]\Big\}<L,$$

where we note that the set $S(M)\cap[0,L]$ is non-empty for large M, since for $k=0,1$, we can apply Theorem 2.12. Applying Lemma 6.5, for $k=l+1,\dots,L-1$ we obtain

$$\begin{aligned}4^{(6+3\varepsilon)L}A_1^LA_2^LA_3^L &\le \Big(\textstyle\prod_{k=l+1}^{L-1}(1+\delta_k)\Big)\,4^{(6+3\varepsilon)(l+1)}A_1^{l+1}A_2^{l+1}A_3^{l+1}\\ &\le \Big(\textstyle\prod_{k=l+1}^{L-1}(1+\delta_k)\Big)\,4^{(6+3\varepsilon)(l+1)}A_1^lA_2^lA_3^l\\ &\le \Big(\textstyle\prod_{k=l+1}^{L-1}(1+\delta_k)\Big)\,4^{6+3\varepsilon}M\,\big(1+A_1^0+A_2^0+A_3^0\big)^2,\end{aligned} \tag{6.14}$$

where δ^k is the variable from Lemma 6.5.

Now it is sufficient to notice that for $k = l+1, \dots, L-1$, the sequence δ_k is bounded by a geometric progression. Indeed, setting $\sigma = 4^{-1+3\varepsilon/2} < 1$, we have that, for $k \notin S(M)$, $\delta_k \le C\sigma^k$, which gives

$$\begin{aligned}\prod_{k=l+1}^{L-1}(1+\delta_k) &\le \prod_{k=l+1}^{L-1}(1+C\sigma^k)\\ &= \exp\Big(\sum_{k=l+1}^{L-1}\log(1+C\sigma^k)\Big)\\ &\le \exp\Big(C\sum_{k=l-1}^{L+1}\sigma^k\Big) \le \exp\Big(\frac{C}{1-\sigma}\Big),\end{aligned} \tag{6.15}$$

which concludes the proof. □

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