HIGHER INTEGRABILITY FOR MINIMIZERS OF THE MUMFORD-SHAH FUNCTIONAL

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Abstract. We prove higher integrability for the gradient of local minimizers of the Mumford-Shah energy functional, providing a positive answer to a conjecture of De Giorgi [5].

1. introduction

Free discontinuity problems are a class of variational problems which involve pairs (u, K) where K is some closed set and u is a function which minimizes some energy outside K. One of the most famous examples is given by the Mumford-Shah energy functional, which arises in image segmentation [10]: given a open set $\Omega \subset \mathbb{R}^n$, for any $K \subset \Omega$ relatively closed and $u \in W^{1,2}(\Omega \setminus K)$, one defines the *Mumford-Shah energy* of (u, K) in Ω to be

$$MS(u,K)[\Omega] := \int_{\Omega\setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K \cap \Omega).$$

We say that the pair (u, K) is a *local minimizer* for the Mumford Shah energy in Ω if, for every ball $B = B_\varrho(x) \Subset \Omega$,

$$MS(u,K)[B] \le MS(v,H)[B]$$

for all pairs (v, H) such that $H \subset \Omega$ is relatively closed, $v \in W^{1,2}(\Omega \setminus H)$, $K \cap (\Omega \setminus B) = H \cap (\Omega \setminus B)$, and $u = v$ almost everywhere in $(\Omega \setminus B) \setminus K$. We denote the set of local minimizers in Ω by $\mathcal{M}(\Omega)$.

The existence of local minimizers is by now well-known [6, 3, 2, 4]. In [5], De Giorgi formulated a series of conjectures on the properties of local minimizers. One of them states as follows [5, Conjecture 1]:

Conjecture (De Giorgi): *If (u, K) is a (local) minimizer of the Mumford-Shah energy inside Ω, then there exists $\gamma \in (1, 2)$ such that $|\nabla u|^2 \in L^\gamma(\Omega' \setminus K)$ for all $\Omega' \subset\subset \Omega$.*

A positive answer to the above conjecture was given in [7] when $n = 2$. The proof there strongly relies on the two-dimensional assumption, since it uses the description of minimal Caccioppoli partitions. The aim of this note is to provide a positive answer in arbitrary dimension. Since our proof avoids any compactness argument, our constants are potentially computable[1]. This is our main result:

Theorem 1.1. *There exist dimensional constants $\bar{C} > 0$ and $\bar{\gamma} = \bar{\gamma}(n) > 1$ such that, for all $(u, K) \in \mathcal{M}(B_2)$,*

$$\int_{B_{1/2}\setminus K} |\nabla u|^{2\bar{\gamma}} \le \bar{C}. \tag{1.1}$$

[1] To be precise, the constants $\bar{C}$ and $\bar{\gamma}$ can be explicitely expressed in terms of the dimension and the constants C_0 and C_ε appearing in Proposition 2.1. While C_0 is computable, the constant $\varepsilon(n)$ appearing in proposition 2.1 (iv), from which C_ε depends (see [8, 11] and Remark 2.3), is obtained in [1] using a compactness argument. However it seems likely that the compactness step could be avoided arguing as in [12], but since this would not give any new insight to the problem, we do not investigate further this point.

By a simple covering/rescaling argument, one deduces the validity of the conjecture with $\gamma = \bar\gamma$. We also remark that our result applies with trivial modifications to the "full" Mumford-Shah energy

$$MS_g(u,K)[\Omega] := \int_{\Omega\setminus K} |\nabla u|^2 + \alpha \int_\Omega |u-g|^2 + \beta \mathcal{H}^{n-1}(K\cap\Omega), \tag{1.2}$$

where $\alpha, \beta > 0$, and $g \in L^2(\Omega)\cap L^\infty(\Omega)$.

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2. Preliminaries

In the next proposition we collect the main known properties of local minimizers that will be used in the sequel.

Proposition 2.1. *There exists a dimensional constant C_0 such that for all $(u,K)\in\mathcal{M}(B_2)$, the following properties hold true.*

(i) *u is harmonic in $B_2\setminus K$.*

(ii) *For all $x\in B_1$ and all $\varrho<1$*

$$\int_{B_\varrho(x)\setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K\cap B_\varrho(x)) \le C_0 \varrho^{n-1}.$$

(iii) *For all $x\in K\cap B_1$ and all $\varrho<1$,*

$$\mathcal{H}^{n-1}(K\cap B_\varrho(x)) \ge \varrho^{n-1}/C_0.$$

(iv) *There is a dimensional constant $\varepsilon(n)>0$ such that, for every $\varepsilon\in(0,\varepsilon(n))$, there exists $C_\varepsilon>0$ for which the following statement holds true:*
For all $x\in K\cap B_1$ and all $\varrho<1$ there exists a $y\in B_{\varrho/2}(x)\cap K$, a unit vector $\bar\nu$ and a $C^{1,1/4}$ function $f:\mathbb{R}^{n-1}\to\mathbb{R}$ such that

$$K\cap B_{2\varrho/C_\varepsilon}(y) = \big[y+\mathrm{graph}_{\bar\nu}(f)\big]\cap B_{2\varrho/C_\varepsilon}(y), \tag{2.1}$$

where

$$\mathrm{graph}_{\bar\nu}(f) := \Big\{z\in\mathbb{R}^n:\ z\cdot\bar\nu = f\big(z-(\bar\nu\cdot z)z\big)\Big\}. \tag{2.2}$$

Moreover

$$f(0)=0, \qquad \|\nabla f\|_\infty + \varrho^{1/4}\|\nabla f\|_{C^{1/4}} \le C_0\varepsilon, \tag{2.3}$$

and

$$\sup_{B_{2\varrho/C_\varepsilon}(y)} \varrho|\nabla u|^2 \le C_0\varepsilon. \tag{2.4}$$

Proof. Point (i) is easy. Point (ii) is well known and it can be proved by comparison, see [2, Lemma 7.19]. Point (iii) has been proved by Carriero, De Giorgi and Leaci in [6], see also [2, Theorem 7.21]. Point (iv) expresses the *porosity* of the set where K is not a smooth graph. This has been proved in [1, 8, 11], see also [9]. More precisely, in these papers it has been proved that for any fixed positive ε there exists a constant C_ε such that, for all $x\in K\cap B_1$ and $\varrho<1$, there exists a point $y\in B_\varrho(x)\cap K$ and a ball $B_r(y)\subset B_\varrho(x)$, with $r\ge 2\varrho/C_\varepsilon$, such that

$$\frac{1}{r^{n-1}}\int_{B_r(y)} |\nabla u(z)|^2\,dz + \frac{1}{r^{n+1}}\inf_{\nu\in S^{n-1}}\int_{K\cap B_r(y)} |(z-y)\cdot\nu|^2\,d\mathcal{H}^{n-1}(z) \le \varepsilon, \tag{2.5}$$

see [8, Theorem 1.1]. From this one applies the ε-regularity theorem, [2, Theorems 8.2 and 8.3] to deduce (2.1) and (2.3). Finally, (2.4) follows from (2.3), (2.5) and classical estimates for the Neumann problem, see for instance [2, Theorem 7.53]. $\square$

The following simple geometric lemma will be useful:

Lemma 2.2 (A geometric lemma)**.** *Let G be closed set such that*

$$G \cap B_2 = \mathrm{graph}_{e_n}(f)$$

for some Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$f(0) = 0 \qquad \text{and} \qquad \|\nabla f\|_\infty \le \varepsilon. \tag{2.6}$$

Then, provided $\varepsilon \le 1/15$,

$$\mathrm{dist}\left(x, (\overline{B}_{1+2\delta} \setminus B_{1+\delta}) \cap G\right) \le \frac{3}{2}\delta \qquad \forall\, \delta \in (0, 1/2),\, x \in (\overline{B}_{1+\delta} \setminus B_1) \cap G.$$

Proof. First notice that, by (2.6),

$$\|f\|_{L^\infty(B_2)} + \|\nabla f\|_{L^\infty(B_2)} \le 3\varepsilon.$$

Given a point $x = (x', f(x')) \in (\overline{B}_{1+\delta} \setminus B_1) \cap G$, set $\alpha := \frac{1+5\delta/4}{|x|}$ and let us consider the point $\bar{x} := \big(\alpha x', f(\alpha x')\big)$. Since $|x| \ge 1$ we have $0 < (\alpha - 1)|x| \le 5\delta/4$, hence

$$\begin{aligned}
|f(\alpha x') - \alpha f(x')| &\le |f(\alpha x') - f(x')| + (\alpha - 1)|f(x')| \\
&\le (\alpha - 1)\Big(\|\nabla f\|_\infty |x'| + \|f\|_\infty\Big) \\
&\le 2(\alpha - 1)\varepsilon |x| = \frac{15}{4}\varepsilon\delta,
\end{aligned}$$

and

$$|\alpha x| = 1 + \frac{5}{4}\delta.$$

Thus, provided $\varepsilon \le 1/15$ we get

$$\left| |\bar{x}| - \left(1 + \frac{5}{4}\delta\right) \right| \le |\bar{x} - \alpha x| = |f(\alpha x') - \alpha f(x')| \le \frac{15}{4}\varepsilon\delta \le \frac{\delta}{4}$$

and

$$|\bar{x} - x| \le |\bar{x} - \alpha x| + (\alpha - 1)|x| = |f(\alpha x') - \alpha f(x')| + (\alpha - 1)\,|x| \le \frac{15}{4}\varepsilon\delta + \frac{5}{4}\delta \le \frac{3}{2}\delta,$$

which imply that $\bar{x} \in (\overline{B}_{1+2\delta} \setminus B_{1+\delta}) \cap G$, concluding the proof. $\square$

Remark 2.3. In the sequel we will apply Proposition 2.1 only with $\varepsilon := \min\{\varepsilon(n)/C_0, 1/(15C_0)\}$, where $\varepsilon(n)$ and C_0 are as in Proposition 2.1, and the factor $1/15$ comes from Lemma 2.2. Hence, with this choice, also the constant C_ε will be dimensional.

3. Proof of Theorem 1.1

Let $M \gg 1$ to be fixed, and for $h \in \mathbb{N}$ define the following set

$$A_h := \big\{x \in B_2 \setminus K \text{ such that } |\nabla u|^2(x) \geq M^{h+1}\big\}. \tag{3.1}$$

Notice that the sets A_h depend on M. However, later M will be fixed to be a large dimensional constant, so for notational simplicity we drop the dependence on M. We will use the notation $\mathcal{N}_\varrho(E)$ to denote the ϱ-neighbourhood of a set E, i.e., the set of points at distance less than ϱ from E.

The idea of the proof is the following: since u is harmonic outside K and the integral of $|\nabla u|^2$ over a ball of radius r is controlled by r^{n-1} (see Proposition 2.1(ii)), it follows by elliptic regularity that A_h is contained in a M^{-h}-neighborhood of K (Lemma 3.1). However, for the set K we have a porosity estimate which tells us that inside every ball of radius ϱ there is a ball of comparable radius where $|\nabla u|^2 \leq C/\varrho$ (see Proposition 2.1(iv)). Hence, this implies that the size of A_h is smaller than what one would get by just using that $A_h \subset \mathcal{N}_{M^{-h}}(K)$. Indeed, by induction over h we can show that A_h is contained in the M^{-h}-neighborhood of a set K_h obtained from K_{h-1} by removing the "good balls" where (2.4) hold (Lemma 3.3). Since the $\mathcal{H}^{n-1}$ measure of K_h decays geometrically (see (3.7)), this allows us to obtain a stronger estimate on the size of A_h which immediately implies the higher integrability. To make this argument rigorous we actually have to suitably localize our estimates, and for this we need to introduce some suitable sequences of radii (Lemma 3.2).

Lemma 3.1. *There exists a dimensional constant M_0 such that for $M \geq M_0$ and all $(u, K) \in \mathcal{M}(B_2)$ and $R \leq 1$*

$$A_h \cap B_{R-2M^{-h}} \subset \mathcal{N}_{M^{-h}}(K \cap \overline{B}_R) \quad \text{for all } h \in \mathbb{N}.$$

Proof. Let $x \in A_h \cap B_{R-2M^{-h}}$, $d := \operatorname{dist}(x, K)$, and $z \in K$ a point such that $|x - z| = d$. If $d > M^{-h}$ then

$$B_{M^{-h}}(x) \cap K = \emptyset \quad \text{and} \quad u \text{ is harmonic on } B_{M^{-h}}(x).$$

Hence, by the definition of A_h, the mean value property for subharmonic function[2], and Proposition 2.1(ii), we get

$$M^{h+1} \leq |\nabla u(x)|^2 \leq \fint_{B_{M^{-h}}(x)} |\nabla u|^2 \leq \frac{C_0}{|B_1|} M^h,$$

which is impossible if M is large enough. Moreover, since $x \in B_{R-2M^{-h}}$ and $d \leq M^{-h}$ we see that $z \in B_R$, proving the claim. □

Lemma 3.2 (Good radii)**.** *There are dimensional positive constants M_1 and C_1 such that for $M \geq M_1$ we can find three sequences of radii $\{R_h\}_{h\in\mathbb{N}}$, $\{S_h\}_{h\in\mathbb{N}}$ and $\{T_h\}_{h\in\mathbb{N}}$ for which the following properties hold true for every $(u, K) \in \mathcal{M}(B_2)$.*

(i) $1 \geq R_h \geq S_h \geq T_h \geq R_{h+1}$,

[2] Notice that, because u is harmonic, $|\nabla u|^2$ is subharmonic. Instead, when one deals with the full functional (1.2) (or if one wants to consider more general energy functionals than $\int |\nabla u|^2$), the mean value estimate has to be replaced by the one-sided Harnack inequality for subsolutions to uniformly elliptic equations, which in the case of minimizers of (1.2) reads as:

$$|\nabla u(x)|^2 \leq C(n)\Big(\fint_{B_{M^{-h}}(x)} |\nabla u|^2 + \alpha^2 M^{-2h} \|g\|_\infty^2\Big).$$

(ii) $R_h - R_{h+1} \leq M^{-\frac{(h+1)}{2}}$ *and* $S_h - T_h = T_h - R_{h+1} = 4M^{-(h+1)}$,

(iii) $\mathcal{H}^{n-1}\big(K \cap (\overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}})\big) \leq C_1 M^{-\frac{(h+1)}{2}}$,

(iv) $R_\infty = S_\infty = T_\infty \geq 1/2$.

Proof. We set $R_1 = 1$. Given R_h we show how to construct S_h, T_h and R_{h+1}. For every $a \in \mathbb{R}$ let us denote with $\lfloor a \rfloor$ the biggest integer less or equal than a. If we set $N_h = \lfloor M^{\frac{h+1}{2}}/8 \rfloor$, then

$$\overline{B}_{R_h} \setminus \overline{B}_{R_h - M^{-\frac{(h+1)}{2}}} \supset \bigcup_{i=1}^{N_h} \overline{B}_{R_h-(i-1)8M^{-(h+1)}} \setminus \overline{B}_{R_h - i8M^{-(h+1)}}. \tag{3.2}$$

Note that we can choose M_1 sufficiently big to ensure that $N_h \geq M^{\frac{h+1}{2}}/16$. Hence, being the annulii in the right hand side of (3.2) disjoint, there is at least an index $\bar{i}$ such that

$$\begin{aligned}\mathcal{H}^{n-1}\Big(K \cap \overline{B}_{R_h-(\bar{i}-1)8M^{-(h+1)}} \setminus \overline{B}_{R_h-\bar{i}8M^{-(h+1)}}\Big) &\leq 16M^{-\frac{(h+1)}{2}}\mathcal{H}^{n-1}\big(K \cap \overline{B}_{R_h} \setminus \overline{B}_{R_h - M^{-\frac{(h+1)}{2}}}\big)\\ &\leq 16M^{-\frac{(h+1)}{2}}\mathcal{H}^{n-1}(K \cap \overline{B}_1) \leq C_1 M^{-\frac{(h+1)}{2}},\end{aligned}$$

where in the last inequality we have taken into account Proposition 2.1(ii). If we set

$$S_h := R_h - (\bar{i}-1)8M^{-(h+1)}, \quad R_{h+1} := R_h - \bar{i}8M^{-(h+1)}, \quad T_h := (S_h + R_{h+1})/2,$$

then properties (i), (ii) and (iii) trivially hold, while (iv) follow from (ii) by choosing M_1 large enough. □

Lemma 3.3. *Let $C_0, \varepsilon, C_\varepsilon, C_1, M_1$ be as in Proposition 2.1 and Lemma 3.2, with ε as in Remark 2.3. There exist dimensional constants $C_2, M_2, \eta > 0$, with $M_2 \geq M_1$, such that, for every $M \geq M_2$, $(u, K) \in \mathcal{M}(B_2)$, and $h \in \mathbb{N}$, we can find h families of disjoint balls*

$$\mathcal{F}_j = \Big\{B_{M^{-j}/C_\varepsilon}(y_i), \quad y_i \in K, \quad i = 1, \ldots, N_j\Big\}, \qquad j = 1, \ldots, h,$$

such that

(i) *If $B^1, B^2 \in \bigcup_{j=1}^h \mathcal{F}_j$ are distinct balls, then $\mathcal{N}_{4M^{-(h+1)}}(B^1) \cap \mathcal{N}_{4M^{-(h+1)}}(B^2) = \emptyset$.*

(ii) *If $B_{M^{-j}/C_\varepsilon}(y_i) \in \mathcal{F}_j$ then there is a unit vector ν and a C^1 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, with*

$$f(0) = 0 \quad \textit{and} \quad \|\nabla f\|_\infty \leq \varepsilon,$$

such that

$$K \cap B_{2M^{-j}/C_\varepsilon}(y_i) = \big[y_i + \mathrm{graph}_\nu(f)\big] \cap B_{2M^{-j}/C_\varepsilon}(y_i) \quad \textit{and} \quad \sup_{B_{2M^{-j}/C_\varepsilon}(y_i)} |\nabla u|^2 < M^{j+1}.$$

(iii) *Let $\{R_h\}_{h\in\mathbb{N}}$, $\{S_h\}_{h\in\mathbb{N}}$ and $\{T_h\}_{h\in\mathbb{N}}$ be the sequences of radii constructed in Lemma 3.2 and define*

$$K_h := \big(K \cap \overline{B}_{S_h}\big) \setminus \Big(\bigcup_{j=1}^h \bigcup_{B\in\mathcal{F}_j} B\Big), \tag{3.3}$$

and

$$\widetilde{K}_h := \big(K \cap \overline{B}_{T_h}\big) \setminus \Big(\bigcup_{j=1}^h \bigcup_{B\in\mathcal{F}_j} \mathcal{N}_{2M^{-(h+1)}}(B)\Big) \subset K_h. \tag{3.4}$$

Then there exists a finite set of points $\mathcal{C}_h := \{x_i\}_{i\in I_h} \subset \widetilde{K}_h$ *such that*

$$|x_j - x_k| \geq 3M^{-(h+1)} \qquad \forall\, j,k \in I_h,\, j \neq k, \tag{3.5}$$

$$\mathcal{N}_{M^{-(h+1)}}(K_h \cap \overline{B}_{R_{h+1}}) \subset \bigcup_{x_i\in\mathcal{C}_h} B_{8M^{-(h+1)}}(x_i). \tag{3.6}$$

Moreover

$$\mathcal{H}^{n-1}(K_{h+1}) \leq (1-\eta)\mathcal{H}^{n-1}(K_h) + C_1 M^{-\frac{h+1}{2}}, \tag{3.7}$$

$$\left|\mathcal{N}_{M^{-(h+1)}}(K_h \cap \overline{B}_{R_{h+1}})\right| \leq C_2 M^{-(h+1)}\mathcal{H}^{n-1}(K_h). \tag{3.8}$$

(iv) *Let* A_h *be as in* (3.1)*. Then*

$$A_{h+2} \cap B_{R_{h+2}} \subset \mathcal{N}_{M^{-(h+1)}}(K_h \cap B_{R_{h+1}}). \tag{3.9}$$

Proof. We proceed by induction. For $h=1$ we set $\mathcal{F}_1 = \emptyset$, so that $K_1 = K\cap \overline{B}_{S_1}$ and $\widetilde{K}_1 = K \cap \overline{B}_{T_1}$. We also choose $\mathcal{C}_1$ to be a maximal family of points at distance $3M^{-2}$ from each other. Clearly (i), (ii), and (3.7) are true. The other properties can be easily obtained as in the steps below and the proof is left to the reader.

Assuming we have constructed h families of balls $\{\mathcal{F}_j\}_{j=1}^h$ as in the statement of the Lemma, we show how to construct the family $\mathcal{F}_{h+1}$. For this, let $\mathcal{C}_h = \{x_i\}_{i\in I_h} \subset \widetilde{K}_h$ be a family of points satisfying (3.5), and let us consider the family of disjoint balls

$$\mathcal{G}_{h+1} := \left\{B_{M^{-(h+1)}}(x_i)\right\}_{i\in I_h}.$$

Step 1. We show that

$$B_{M^{-(h+1)}}(x_i) \cap K_h = B_{M^{-(h+1)}}(x_i) \cap K \qquad \forall\, x_i \in \mathcal{C}_h.$$

Indeed, assume by contradiction there is a point $x \in B_{M^{-(h+1)}}(x_i) \cap (K\setminus K_h)$. First of all notice that, by Lemma 3.2(ii), since $x_i \in B_{T_h}$ we get that $x \in B_{S_h}$. Hence, by the definition of K_h, there is a ball $\widetilde{B} \in \mathcal{F}_j$, $j\leq h$, such that $x \in \widetilde{B}$. But then

$$\mathrm{dist}(x_i, \widetilde{B}) \leq |x - x_i| \leq M^{-(h+1)} < 2M^{-(h+1)},$$

a contradiction to the fact that $x_i \in \widetilde{K}_h$.

Step 2. We claim that there exists a positive dimensional constant η_0 such that, if N_h is the cardinality of I_h, then

$$N_h M^{-(h+1)(n-1)} \geq \eta_0 \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}})$$

Indeed, by (3.6) and Proposition 2.1(ii),

$$\begin{aligned}\mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}}) &= \mathcal{H}^{n-1}\Big(K_h \cap \overline{B}_{R_{h+1}} \cap \bigcup_{x_i\in\mathcal{C}_h} B_{8M^{-(h+1)}}(x_i)\Big)\\ &\leq C_0 N_h \big(8M^{-(h+1)}\big)^{n-1} = \frac{1}{\eta_0} N_h M^{-(h+1)(n-1)},\end{aligned}$$

where $\eta_0 := 1/(C_0 8^{n-1})$.

Step 3. By Proposition 2.1(iv) and Remark 2.3, for every ball $B_{M^{-(h+1)}}(x_i) \in \mathcal{G}_{h+1}$ there exists a ball

$$B_{M^{-(h+1)}/C_\varepsilon}(y_i) \subset B_{M^{-(h+1)}}(x_i) \tag{3.10}$$

such that
$$\sup_{B_{2M^{-(h+1)}/C_\varepsilon}(y_i)} |\nabla u|^2 \le \varepsilon M^{h+1} < M^{h+2}.$$
and
$$K \cap B_{2M^{-(h+1)}/C_\varepsilon}(y_i) = \big[y_i + \mathrm{graph}_\nu(f)\big] \cap B_{2M^{-(h+1)}/C_\varepsilon}(y_i),$$
for some unit vector ν and some C^1 function f such that
$$f(0) = 0 \quad \text{and} \quad \|\nabla f\|_\infty \le \varepsilon.$$
We define
$$\mathcal{F}_{h+1} := \big\{B_{M^{-(h+1)}/C_\varepsilon}(y_i)\big\}_{i\in I_h}.$$
In this way property (ii) in the statement of the lemma is satisfied. Moreover, since the balls $\{B_{3M^{-(h+1)}/2}(x_i)\}_{i\in I_h}$ are disjoint (because $|x_j - x_k| \ge 3M^{-(h+1)}$) and do not intersect
$$\bigcup_{j=1}^{h} \bigcup_{B\in\mathcal{F}_j} \mathcal{N}_{\frac{M^{-(h+1)}}{2}}(B)$$
it follows from (3.10) that also property (i) is satisfied provided we choose M sufficiently large.
We define K_{h+1} and $\widetilde{K}_{h+1}$ as in the statement of the lemma and we take $\mathcal{C}_{h+1} = \{x_i\}_{i\in I_{h+1}}$ a maximal sets of points in $\widetilde{K}_{h+1}$ satisfying
$$|x_j - x_k| \ge 3M^{-(h+2)} \qquad \forall\, j \ne k.$$

Step 4. The set of points $\mathcal{C}_{h+1}$ defined in the previous step satisfies by construction (3.5). We now prove it also satisfies (3.6). For this, let $x \in \mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}})$ and let $\bar{x} \in K_{h+1} \cap \overline{B}_{R_{h+2}}$ be such that
$$|x - \bar{x}| = \mathrm{dist}\big(x, K_{h+1} \cap \overline{B}_{R_{h+2}}\big) \le M^{-(h+2)}.$$
In case $\bar{x} \in \widetilde{K}_{h+1}$, by maximality there exists a point $x_i \in \mathcal{C}_{h+1}$ such that $|\bar{x} - x_i| \le 3M^{-(h+2)}$, hence $x \in B_{5M^{-(h+2)}}(x_i)$ and we are done. So, let us assume that
$$\bar{x} \in \big(K_{h+1} \cap \overline{B}_{R_{h+2}}\big) \setminus \widetilde{K}_{h+1}.$$
In this case, by the definition of K_{h+1} and $\widetilde{K}_{h+1}$, there exists a ball $\widetilde{B} \in \bigcup_{j=1}^{h+1} \mathcal{F}_j$ such that
$$\bar{x} \in K \cap \mathcal{N}_{2M^{-(h+2)}}(\widetilde{B}) \setminus \widetilde{B}.$$
Thanks to property (ii) we can apply (a scaled version of) Lemma 2.2 to find a point
$$y \in K \cap \mathcal{N}_{4M^{-(h+2)}}(\widetilde{B}) \setminus \mathcal{N}_{2M^{-(h+2)}}(\widetilde{B})$$
such that
$$|\bar{x} - y| \le 3M^{-(h+2)}.$$
Since $\bar{x} \in \overline{B}_{R_{h+2}}$ and $T_{h+1} = R_{h+2} + 4M^{-(h+2)}$
$$y \in \Big(K \cap B_{T_{h+1}} \cap \mathcal{N}_{4M^{-(h+2)}}(\widetilde{B})\Big) \setminus \mathcal{N}_{2M^{-(h+2)}}(\widetilde{B}) \subset \widetilde{K}_{h+1},$$
where the last inclusion follows by property (i) and the definition of $\widetilde{K}_h$. Again by maximality, there exists a point $x_i \in \mathcal{C}_{h+1}$ such that $|y - x_i| \le 3M^{-(h+2)}$, hence
$$|x_i - x| \le |x_i - y| + |y - \bar{x}| + |\bar{x} - x| \le 7M^{-(h+2)},$$
which completes the proof of (3.6).

Step 5. We prove (3.7). Notice that, being the balls in $\mathcal{F}_{h+1}$ disjoint, thanks to Step 1, the density estimates in Proposition 2.1(iii), Step 2, and choosing $\eta := \eta_0/C_0C_\varepsilon^{(n-1)}$ we get

$$\begin{aligned}\mathcal{H}^{n-1}(K_{h+1}) &\le \mathcal{H}^{n-1}\Big(K_h \setminus \bigcup_{i\in I_h} B_{M^{-(h+1)}/C_\varepsilon}(y_i)\Big)\\ &= \mathcal{H}^{n-1}(K_h) - \sum_{i\in I_h} \mathcal{H}^{n-1}\big(K_h \cap B_{M^{-(h+1)}/C_\varepsilon}(y_i)\big)\\ &\le \mathcal{H}^{n-1}(K_h) - \frac{N_h}{C_0C_\varepsilon^{(n-1)}} M^{-(h+1)(n-1)}\\ &\le \mathcal{H}^{n-1}(K_h) - \frac{\eta_0}{C_0C_\varepsilon^{(n-1)}} \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}})\\ &= (1-\eta)\mathcal{H}^{n-1}(K_h) + \eta\big[\mathcal{H}^{n-1}(K_h) - \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}})\big]\\ &\le (1-\eta)\mathcal{H}^{n-1}(K_h) + \mathcal{H}^{n-1}\big(K \cap \overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}}\big)\\ &\le (1-\eta)\mathcal{H}^{n-1}(K_h) + C_1 M^{-\frac{(h+1)}{2}},\end{aligned}$$

where in the last step we used Lemma 3.2(iii).

Step 6. We prove (3.8). By (3.6),

$$\mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}}) \subset \bigcup_{x_i \in C_{h+1}} B_{8M^{-(h+2)}}(x_i),$$

hence, denoting with N_{h+1} the cardinality of I_{h+1},

$$\big|\mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}})\big| \le 8^n M^{-(h+2)} N_{h+1} M^{-(h+2)(n-1)}. \tag{3.11}$$

Also, by Step 1 with h replaced by $h+1$,

$$B_{M^{-(h+2)}}(x_i) \cap K_{h+1} = B_{M^{-(h+2)}}(x_i) \cap K \qquad \forall\, x_i \in \mathcal{C}_{h+1},$$

hence by the density estimates in Proposition 2.1(iii),

$$M^{-(h+2)(n-1)} \le C_0 \mathcal{H}^{n-1}\big(K_{h+1} \cap B_{M^{-(h+2)}}(x_i)\big).$$

The above equation and (3.11), together with the disjointness of the balls $\big\{B_{M^{-(h+2)}}(x_i)\big\}_{i\in I_{h+1}}$ imply

$$\big|\mathcal{N}_{M^{-(h+2)}}(K_{h+1}\cap\overline{B}_{R_{h+2}})\big| \le C_0 8^n M^{-(h+2)} \sum_{i\in I_{h+1}} \mathcal{H}^{n-1}\big(K_{h+1}\cap B_{M^{-(h+2)}}(x_i)\big) \le C_2 M^{-(h+2)} \mathcal{H}^{n-1}(K_{h+1}).$$

Step 7. We are left to show point (iv). Let $x \in A_{h+3} \cap B_{R_{h+3}}$. By Lemma 3.2 $R_{h+2} - R_{h+3} \ge 8M^{-(h+3)}$, hence, by Lemma 3.1,

$$A_{h+3} \cap B_{R_{h+3}} \subset \mathcal{N}_{M^{-(h+3)}}(K \cap \overline{B}_{R_{h+2}}) \subset \mathcal{N}_{M^{-(h+2)}}(K \cap \overline{B}_{R_{h+2}}).$$

Let $\bar{x} \in K \cap \overline{B}_{R_{h+2}}$ a point realizing the distance and assume by contradiction that $\bar{x} \in K \setminus K_{h+1}$. By the definition of K_{h+1} and since $R_{h+2} \le S_{h+1}$, this means that there is a ball $\widetilde{B} \in \bigcup_{j=1}^{h+1} \mathcal{F}_j$ such that $\bar{x} \in \widetilde{B}$. Since $|\bar{x} - x| \le M^{-(h+2)}$ and the radius of $\widetilde{B}$ is at least $M^{-(h+1)}/C_\varepsilon$, we can choose M large enough so that $x \in 2\widetilde{B}$. But then, by property (ii) of the statement,

$$|\nabla u(x)|^2 < M^{h+2},$$

a contradiction to the fact that $x \in A_{h+3}$. □

We are now in the position to prove Theorem 1.1:

Proof of Theorem 1.1. Iterating (3.7) we obtain

$$\mathcal{H}^{n-1}(K_h) \leq C_1 \sum_{i=0}^{h} (1-\eta)^{h-i} M^{-\frac{i}{2}} \leq C_1 h \max\big\{(1-\eta)^h, M^{-h/2}\big\}. \tag{3.12}$$

We now fix $M := M_2$ where M_2 is the constant appearing in Lemma 3.3, and choose $\alpha \in (0, 1/4)$ such that $(1-\eta) \leq M^{-2\alpha}$. In this way, since $2\alpha < 1/2$ it follows from (3.12) that

$$\mathcal{H}^{n-1}(K_h) \leq C_1 h M^{-2\alpha h}.$$

Hence, by (3.9), (3.8), and the above equation, we obtain

$$|A_{h+2} \cap B_{R_{h+2}}| \leq \big|\mathcal{N}_{M^{-(h+1)}}(K_h \cap \overline{B}_{R_{h+1}})\big| \leq C_1 C_2 h M^{-h(1+2\alpha)} \qquad \forall\, h \geq 1,$$

so Lemma 3.2(iv) and the definition of A_h (see (3.1)) finally give

$$\big|\{x \in B_{1/2} \setminus K : |\nabla u|^2(x) \geq M^h\}\big| \leq C_1 C_2 M^{2+4\alpha} h M^{-h(1+2\alpha)} \qquad \forall\, h \geq 3. \tag{3.13}$$

Since

$$\begin{aligned}\int_{B_{1/2}\setminus K} |\nabla u|^{2\gamma} &= \gamma \int_0^\infty t^{\gamma-1} |(B_{1/2} \setminus K) \cap \{|\nabla u|^2 \geq t\}|\, dt \\ &\leq M^\gamma \sum_{h=0}^{\infty} M^{h\gamma} \big|(B_{1/2} \setminus K) \cap \{|\nabla u|^2 \geq M^h\}\big|,\end{aligned}$$

(3.13) implies the validity of (1.1) with, for instance, $\bar{\gamma} = 1 + \alpha$. □

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