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Boundedness of minimizers for spectral problems in $\mathbb{R}^N$

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Abstract – In [8] it was proved that any increasing functional of the first k eigenvalues of the Dirichlet Laplacian admits a (quasi-)open minimizer among the subsets of $\mathbb{R}^N$ of unit measure. In this paper we show that every minimizer is uniformly bounded by a constant depending only on k, N.

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1. Introduction

This paper deals with the following minimization problem:

$$\min\left\{\mathcal{F}(\lambda_1(A),\dots,\lambda_k(A))\ :\ A\subseteq\mathbb{R}^N,\ \text{quasi-open, with } |A|=1\right\}, \tag{1.1}$$

where $\lambda_i(A)$ denotes the i-th eigenvalue of the Dirichlet-Laplacian. This spectral problem is well-studied, for instance when the functional reduces to the projection on the last coordinate (see [3], [6], [7]).

Theorem A in [8] assures that, if $\mathcal{F}$ is increasing in each variable and lower semi-continuous (l.s.c.), then problem (1.1) has *at least* a bounded minimizer, where the boundedness constant depends only on k, N, but not on the functional. Moreover, in [2], with completely different techniques involving the regularity of shape sub-solutions of the torsion energy, Bucur was able to prove existence of an optimal set in the case of $F = \lambda_k$ and to show that *all* optimal sets in this case are bounded and have finite perimeter.

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The aim of this note is simply to show that *every* minimizer for problem (1.1) has diameter uniformly bounded, depending only on k, N, up to assume the functional $\mathcal{F}$ to be *weakly strictly increasing*, that is, increasing in each variable and such that for every $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{R}^k$ with $x_i < y_i$ for $i = 1, \dots, k$

$$\mathcal{F}(x_1, \dots, x_k) < \mathcal{F}(y_1, \dots, y_k).$$

The above assumption is necessary in order to avoid the trivial case of a constant functional, for which every admissible set is a minimizer. We will use the same notations as in [8] and very similar techniques. The basic idea is that, given a sequence of admissible sets that γ-converges to a minimizer, either it is uniformly bounded or it is possible to decrease all the first k eigenvalues of its sets by a uniform strictly positive constant.

The main result is the following.

THEOREM 1.1. *Let $k, N \in \mathbb{N}$ and $\mathcal{F} \colon \mathbb{R}^k \to \mathbb{R}$ be weakly strictly increasing and l.s.c.. Then every minimizer for problem (*1.1*) is contained in an N-cube Q_R with edge of length $R = R(k, N)$.*

It is important to highlight that a natural question about optimal sets for (1.1), even if unrelated with the aim of this paper, is whether they are open and not only quasi-open. The minimization for this kind of spectral optimization problems is done among quasi-open sets because they form a class with good compactness properties with respect to the γ-convergence (see [5]). This regularity issue for minimizers is a very difficult topic, due to the min-max structure of eigenvalues. A partial answer to this question was given in [4], where it is proved that for functionals F which are increasingly bi-Lipschitz in each variable, then *every* solution is an open set up to measure zero. On the other hand, for the most interesting functional $F = \lambda_k$, it is only possible to prove that every optimal set admits an eigenfunction (corresponding to the k-th eigenvalue) which is Lipschitz continuous in $\mathbb{R}^N$.

The paper is organized as follows. In Section 2. we give some useful results about capacity and quasi-open sets and we present the notations used throughout the paper. Then in Section 3. we study the "tails" of the minimizing sequence, while in Section 4. we deal with their "inner part". At last in Section 5. we put all the informations together and we prove Theorem 1.1.

2. Notations and preliminary results

First of all we recall the definitions of capacity and of quasi-open sets. For a more detailed treatment of those subjects, see [7].

DEFINITION 2.1. Let D be an open set and $A \subset\subset D$ a compactly supported subset. The *capacity of A in D* is defined as

$$\operatorname{cap}_D(A) = \inf \left\{ \int_D |Dv|^2 \ : \ v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } A \right\}. \tag{2.1}$$

Let then $A \subseteq \mathbb{R}^N$ be a bounded set and let D be an open set such that $A \subset\subset D$. The set A is called *quasi-open* if for every $\varepsilon > 0$ there exists an open set $A \subseteq A_\varepsilon \subset \subset D$ such that $\mathrm{cap}_D(A_\varepsilon \setminus A) < \varepsilon$. Clearly this definition does not depend on the choice of D. A generic subset of $\mathbb{R}^N$ is said to be *quasi-open* if its intersection with any ball is a quasi-open bounded set.

For sake of completeness we state and prove here three lemmas dealing with general properties of capacity.

Lemma 2.2. *Let $D \subseteq \mathbb{R}$ be an open set and $\Omega_1 \subseteq \Omega_2 \subset\subset D$. Then* $\mathrm{cap}_D(\Omega_2) \geq \mathrm{cap}_D(\Omega_1)$.

Proof. By definition,
(2.2)
$$\mathrm{cap}_D(\Omega_2) = \inf \left\{ \int_D |Dv|^2 \ : \ v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } \Omega_2 \right\}.$$

Since it is clear that the class of function $\{v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } \Omega_2\}$ is included in $\{v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } \Omega_1\}$, thus by definition of infimum we have the thesis. □

Lemma 2.3. *Let $D \subseteq \mathbb{R}$ be an open set and $A \subset\subset D$. Suppose that A is included in the union of two sets: $A \subseteq A_1 \cup A_2$. Then, for all $\alpha > 0$ we have*

$$\mathrm{cap}_D(A_1) + \mathrm{cap}_D(A_2) \leq \alpha \qquad \Rightarrow \qquad \mathrm{cap}_D(A) \leq 2\alpha.$$

Proof. Let $\eta > 0$. By the definition of capacity, it is possible to find two functions $v_i \in H_0^1(D)$, such that $\int_D |Dv_i|^2 \leq \mathrm{cap}_D(A_i) + \eta$ and $v_i \geq 1$ on a neighborhood of A_i, for $i = 1, 2$. The function $v_1 + v_2 \in H_0^1(D)$ and $v_1 + v_2 \geq 1$ on a neighborhood of $A_1 \cup A_2$, hence we can compute:

$$\begin{aligned}
\mathrm{cap}_D(A_1 \cup A_2) &\leq \int_D |D(v_1 + v_2)|^2 \leq \int_D |Dv_1|^2 + \int_D |Dv_2|^2 + 2\int_D Dv_1 \cdot Dv_2 \\
&\leq 2\int_D |Dv_1|^2 + 2\int_D |Dv_2|^2 \leq 2(\mathrm{cap}_D(A_1) + \mathrm{cap}_D(A_2)) + 4\eta \\
&\leq 2\alpha + 4\eta.
\end{aligned}$$

In conclusion, by arbitrariness of η, we obtain $\mathrm{cap}_D(A) \leq \mathrm{cap}_D(A_1 \cup A_2) \leq 2\alpha$. □

Remark 2.4. Throughout this paper, since we are working in a *capacitary* setting, all the sets are defined up to zero capacity. This is stronger than working up to zero Lebesgue measure: we remind that, given a set $A \subseteq \mathbb{R}^n$, $\mathrm{cap}(A) = 0$ implies $|A| = 0$, but the vice versa is not true in general.

With this last Lemma we prove that a quasi-open set with zero Lebesgue measure must also have zero capacity. This fact is well-known among experts of shape optimization, but we did not find any reference for it, so we present here a simple proof.

Lemma 2.5. *Let $A \subset \mathbb{R}^N$ a quasi-open set such that $|A| = 0$. Then A has zero capacity.*

Proof. First of all we suppose A to be bounded, and let D be an open set such that $A \subset\subset D$. We fix $\varepsilon > 0$ and we aim to prove that $\mathrm{cap}_D(A) \leq 2\varepsilon$. By definition of quasi-open set (see Definition 2.1), there exists an open set $A \subseteq A_\varepsilon \subset\subset D$ such that $\mathrm{cap}_D(A_\varepsilon \setminus A) < \varepsilon$. We can write $A = (A \cap A_\varepsilon) \cup (A \setminus A_\varepsilon)$, and clearly $\mathrm{cap}_D(A \setminus A_\varepsilon) = 0$ since $A \subseteq A_\varepsilon$. Moreover, by Lemma 2.2, $\mathrm{cap}_D(A \cap A_\varepsilon) \leq \mathrm{cap}_D(A_\varepsilon)$. Since $|A| = 0$ and $A \subseteq A_\varepsilon$, we have that the following sets are equal:

$$\begin{aligned}&\left\{v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } A_\varepsilon\right\}\\ &= \left\{v \in H_0^1(D),\ v \geq 1 \text{ in a neighborhood of } A_\varepsilon \setminus A\right\}.\end{aligned}$$

In fact for all $x \in \mathbb{R}^N$ and $\eta > 0$, $dist(x, A_\varepsilon) < \eta$ if and only if $dist(x, A_\varepsilon \setminus A) < \eta$. At last, thanks to Lemma 2.3, we have:

$$\mathrm{cap}_D(A) = \mathrm{cap}_D((A \cap A_\varepsilon) \cup (A \setminus A_\varepsilon)) \leq 2\varepsilon,$$

and by arbitrariness of ε we conclude.
If A is not bounded, we consider for all $R > 0$, $A \cap B(R)$, where $B(R)$ denotes the ball of radius R centered in the origin and we can prove, with the above argument, that $\mathrm{cap}_{B(2R)}(A \cap B(R)) = 0$ for all $R > 0$. Hence again A is a set with zero capacity. □

Throughout the paper we will not need advanced tools about γ-convergence (for more details see [7]), we remind only that, given a sequence of open sets with unit measure $(\Omega_n)_n$ such that

$$\Omega_n \xrightarrow{\gamma} \Omega, \qquad \text{as } n \to +\infty, \tag{2.3}$$

then $\lambda_i(\Omega_n) \to \lambda_i(\Omega)$ as $n \to \infty$ for all $i \in \mathbb{N}$.

It is well-known (see [1]) that there exists a constant $M = M(k, N) > 0$ such that for all $\Omega \subseteq \mathbb{R}^N$, $\frac{\lambda_k(\Omega)}{\lambda_1(\Omega)} \leq M$. Since we are interested in the minimization problem (1.1), we define $K = M\lambda_k(B_N)$ and we can consider sets with $\lambda_k(\Omega) \leq K$, otherwise $\lambda_1(\Omega) \geq \frac{\lambda_k(\Omega)}{M} \geq \lambda_k(B_N)$, hence $\mathcal{F}(\Omega) > \mathcal{F}(B_N)$, where B_N denotes the unit ball in $\mathbb{R}^N$. Note that the constant K depends only on k, N.

Now we give some definitions, following [8]. First of all we fix a small positive constant $\widehat{m} = \widehat{m}(k, N) \in (0, 1/4)$ such that

$$\frac{(4\widehat{m})^{\frac{2}{N}}}{\lambda_1(B_N)} K \leq \frac{1}{2}.$$

Let $\Omega \subseteq \mathbb{R}^N$ be an open set with unit measure and, for every $t \in \mathbb{R}$,

$$\begin{aligned}\Omega_t^l &:= \Big\{(x, y) \in \Omega : x < t\Big\}, \qquad \Omega_t := \Big\{y \in \mathbb{R}^{N-1} : (t, y) \in \Omega\Big\},\\ \Omega_t^r &:= \Big\{(x, y) \in \Omega : x > t\Big\};\end{aligned}$$

notice that Ω_t^l and Ω_t^r are subsets of $\mathbb{R}^N$, while Ω_t is a subset of $\mathbb{R}^{N-1}$. On the other hand, given $0 \leq m \leq |\Omega|$ and $0 \leq m_1 \leq m_2 \leq |\Omega|$, we define the level $\tau(\Omega, m) \in \overline{\mathbb{R}}$ and the width $W(\Omega, m_1, m_2)$ as

$$\tau(\Omega, m) := \inf \Big\{ t \in \mathbb{R} : \big|\Omega_t^l\big| \geq m \Big\} , \quad W(\Omega, m_1, m_2) := \tau(\Omega, m_2) - \tau(\Omega, m_1) .$$

Observe that one surely has $-\infty < \tau(\Omega, m) < +\infty$ whenever $0 < m < |\Omega|$, as well as $W(\Omega, m_1, m_2) < +\infty$ if $0 < m_1 \leq m_2 < |\Omega|$. At last, we remark that, even if we are working with sets defined up to sets of zero capacity, the definitions above are stable up to sets of zero Lebesgue measure.

3. Boundedness of the "tails"

Throughout this Section and the next one we consider a generic open set with unit measure $\Omega \subset \mathbb{R}^N$ such that $\lambda_k(\Omega) \leq K$. We study the "tail" of the set Ω, i.e. the set $\Omega^l_{\tau(\Omega,\widehat{m})}$. In particular we focus on the horizontal projection. We set for brevity $\bar{t} = \tau(\Omega, 2\widehat{m})$ and for every $t \leq \bar{t}$ we define

$$\Omega^+(t) = \Omega_t^r, \qquad \Omega^-(t) = \Omega_t^l, \qquad \varepsilon(t) = \mathcal{H}^{N-1}(\Omega_t). \tag{3.1}$$

It is easy to see that

$$m(t) = |\Omega^-(t)| = \int_{-\infty}^t \varepsilon(s)\, ds \leq 2\widehat{m}. \tag{3.2}$$

As usual, we call $\{u_1, u_2, \dots, u_k\}$ an orthonormal set of eigenfunctions with unit L^2 norm and corresponding to the first k eigenvalues of Ω. Then we define, for every $1 \leq i \leq k$ and every $t \leq \bar{t}$,

(3.3)

$$\delta_i(t) = \int_{\Omega_t} |Du_i(t, y)|^2 \, d\mathcal{H}^{N-1}(y), \qquad \delta(t) = \sum_{i=1}^k \delta_i(t), \qquad \phi(t) = \int_{-\infty}^t \delta(s)\, ds.$$

Moreover, for any $t \leq \bar{t}$, we define the cylinder $Q(t)$, as

$$Q(t) := \Big\{ (x, y) \in \mathbb{R}^N : t - \sigma(t) < x < t, \ (t, y) \in \Omega \Big\} = \big(t - \sigma(t), t\big) \times \Omega_t \, , \tag{3.4}$$

where for any $t \leq \bar{t}$ we set

$$\sigma(t) = \varepsilon(t)^{\frac{1}{N-1}} \, . \tag{3.5}$$

We let also $\widetilde{\Omega}(t) = \Omega^+(t) \cup Q(t)$, and we introduce $\widetilde{u}_i \in W_0^{1,2}\big(\widetilde{\Omega}(t)\big)$ as

$$\widetilde{u}_i(x, y) := \begin{cases} u_i(x, y) & \text{if } (x, y) \in \Omega^+(t) \, , \\ \dfrac{x - t + \sigma}{\sigma} \, u_i(t, y) & \text{if } (x, y) \in Q(t) \, . \end{cases} \tag{3.6}$$

We restate here without proof some useful Lemmas from [8], which will be essential for our analysis.

Lemma 3.1 ([8], Lemma 2.3). *Let Ω be an open set of unit volume, with $\lambda_k(\Omega) \le K$. Then for all $1 \le i \le k$ and $t \le \bar t$, the following inequalities hold:*

$$\int_{\Omega^-(t)} u_i^2 \le C_1 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t), \qquad \int_{\Omega^-(t)} |Du_i|^2 \le C_1 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t), \tag{3.7}$$

for some $C_1 = C_1(k, N)$.

Lemma 3.2 ([8], Lemma 2.5). *For every $t \le \bar t$ and $1 \le i \le k$, one has*

$$\mathcal{R}(\widetilde u_i, \widetilde\Omega(t)) \le \lambda_i(\Omega) + C_2 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t). \tag{3.8}$$

Moreover, for every $i \ne j \in \{1, 2 \dots, k\}$, one has

$$\left| \int_{\widetilde\Omega(t)} \widetilde u_i \widetilde u_j + D\widetilde u_i \cdot D\widetilde u_j \right| \le C_2 \left(\varepsilon(t)^{\frac{3}{N-1}} + \varepsilon(t)^{\frac{1}{N-1}} \right) \sqrt{\delta_i(t)\delta_j(t)}, \tag{3.9}$$

for some $C_2 = C_2(k, N)$.

Lemma 3.3 ([8], Lemma 2.6). *There exist a small constant $\nu = \nu(k, N) < 1$ and a constant $C_3 = C_3(k, N)$ such that, if $\varepsilon(t), \delta_i(t) \le \nu$ for every $i = 1, \dots, k$ and $t \le \bar t$, then*

$$\lambda_j(\widetilde\Omega(t)) \le \lambda_j(\Omega) + C_3 \varepsilon(t)^{\frac{1}{N-1}} \delta(t) \qquad \forall 1 \le j \le k. \tag{3.10}$$

For our purposes, a slightly different version of the above Lemma is preferrable.

Lemma 3.4. *There exist a constant $\widetilde C_3 = \widetilde C_3(k, N)$ such that, if $t \le \bar t$, then*

$$\lambda_j(\widetilde\Omega(t)) \le \lambda_j(\Omega) + \widetilde C_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}} \right) \qquad \forall 1 \le j \le k. \tag{3.11}$$

Proof. It is clear that, thanks to Lemma 3.3, whenever $\varepsilon(t), \delta_i(t) \le \nu$ for all $i = 1, \dots, k$, then the thesis is true with $\widetilde C_3 = C_3$, since $\varepsilon(t)^{\frac{1}{N-1}} \delta(t) \le \varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}$.

We can now focus on the case when either $\varepsilon(t) > \nu$ or $\delta_i(t) > \nu$ for some i. Then, we remind that, since the first eigenfunction has not orthogonality constraints, Lemma 3.2 assures:

$$\lambda_1(\widetilde\Omega(t)) \le \lambda_1(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t).$$

It is well-known (see [1] or the appendix of [8]) that there is a constant $M = M(k, N) > 0$ such that $\frac{\lambda_k(\Omega)}{\lambda_1(\Omega)} \le M$ for all $\Omega \subset \mathbb{R}^N$. Hence we can write, for all $1 \le j \le k$:

$$\lambda_j(\widetilde\Omega(t)) \le M\lambda_1(\widetilde\Omega(t)) \le M \left(\lambda_1(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}} \delta(t) \right).$$

Moreover it is possible to find a big constant $A = A(k,N)$, such that $MK \leq A\nu^{\frac{N}{N-1}}$ and then, defining $\widetilde{C}_3 = A + MC$, we can conclude the computations above:

$$\begin{aligned}\lambda_j(\widetilde{\Omega}(t)) &\leq M\left(K + C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\right) \leq A\nu^{\frac{N}{N-1}} + MC\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \\ &\leq \lambda_j(\Omega) + A\nu^{\frac{N}{N-1}} + MC\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \leq \lambda_j(\Omega) + \widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right).\end{aligned}$$

□

We are now in position to state and prove the main Lemma of this section. For sake of simplicity, we call $\widehat{\Omega}(t) = |\widetilde{\Omega}(t)|^{-1/N}\widetilde{\Omega}(t)$ the modified set rescaled till unit measure.

Lemma 3.5. *Let Ω be an open set of unit volume, with $\lambda_k(\Omega) \leq K$ and $t \leq \bar{t}$. Then there exists a constant $C_4 = C_4(k,N)$ such that exactly one of the following situations happens.*

(1) $m(t) \leq C_4\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$.

(2) *(1) does not hold and for all $1 \leq i \leq k$, $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega)$. Moreover for every $\widetilde{m} > 0$ such that $m(t) \geq \widetilde{m}$, there exists an $\eta = \eta(N, \widetilde{m})$ such that for all $1 \leq i \leq k$,*
$$\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega) - \eta.$$

Proof. From Lemma 3.4 we have

$$\lambda_i(\widetilde{\Omega}(t)) \leq \lambda_i(\Omega) + \widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \qquad \forall 1 \leq i \leq k,$$

moreover, putting in account that $|\widetilde{\Omega}(t)| = |\Omega^+(t)| + |Q(t)| = 1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}$ and the scaling of the eigenvalues, then for all $1 \leq i \leq k$

$$\begin{aligned}\lambda_i(\widehat{\Omega}(t)) &\leq \left(1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\right)^{\frac{2}{N}}\left(\lambda_i(\Omega) + \widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)\right) \\ &\leq \lambda_i(\Omega) - \frac{2}{N}\lambda_1(B_N)m(t) + \frac{2K}{N}\varepsilon(t)^{\frac{N}{N-1}} + \widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \\ &\quad - \frac{2}{N}m(t)\widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) + \frac{2}{N}\widetilde{C}_3\varepsilon(t)^{\frac{N}{N-1}}\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right).\end{aligned} \tag{3.12}$$

Then if $m(t) \leq C_4\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$, condition (1) holds true; otherwise $m(t) > C_4\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$ and we can choose $C_4 \geq 1$ so that $m(t) \geq \varepsilon(t)^{\frac{N}{N-1}}$. Thus from the two last terms of (3.12), we have

$$-\frac{2}{N}m(t)\widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) + \frac{2}{N}\widetilde{C}_3\varepsilon(t)^{\frac{N}{N-1}}\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \leq 0.$$

This allows us to conclude, choosing $C_4 \geq \frac{2K+N\widetilde{C}_3}{\lambda_1(B_N)}$ and obtaining:

$$\lambda_i(\widehat{\Omega}(t)) - \lambda_i(\Omega) \leq -\frac{\lambda_1(B_N)}{N} m(t) < 0,$$

that is condition (2). Moreover if $m(t) \geq \widetilde{m}$, then we can improve the above estimate:

$$\lambda_i(\widehat{\Omega}(t)) - \lambda_i(\Omega) \leq -\frac{\lambda_1(B_N)}{N} \widetilde{m} = -\eta(N, \widetilde{m}) < 0,$$

and the proof is concluded. □

We introduce the following notations. Given an open set Ω as in the hypotheses of Lemma 3.5, we set

$$\hat{t} = \sup \left\{ t \in (-\infty, \bar{t}) \; : \; \text{condition (2) of Lemma 3.5 holds for } t \right\}, \tag{3.13}$$

with the usual convention that $\hat{t} = -\infty$ if condition (2) is false for every $t \leq \bar{t}$. If $\hat{t} > -\infty$, then $m(\hat{t}) > 0$ and we choose some $t^\star \in [\hat{t} - 1, \hat{t}]$ for which condition (2) holds. The following Lemma concludes this Section.

Lemma 3.6. *Let $(\Omega_n)_n$ be as in the hypotheses of Lemma 3.5 and $\Omega_n \xrightarrow{\gamma} \Omega$.*

(a) *If there exists a subsequence (not relabeled) such that, for all n, $m(t^\star(n)) \geq \widetilde{m} > 0$ for some $\widetilde{m} > 0$, then Ω is not optimal for problem (1.1).*

(b) *If there exists a subsequence such that $\hat{t}(n) = -\infty$ for all n, then there exists $R_1 = R_1(k, N) > 0$ such that $W(\Omega, 0, \widehat{m}) \leq R_1$.*

(c) *If there exists a subsequence such that $m(t^\star(n)) \to 0$ as $n \to \infty$, then we have again $W(\Omega, 0, \widehat{m}) \leq R_1$.*

Proof. We introduce the following subsets of $(\hat{t}(n), \bar{t}(n))$ for all $n \in \mathbb{N}$:

$$A_1^n = \{ t \in (\hat{t}(n), \bar{t}(n)) \; : \; \varepsilon(t) \geq \delta(t) \}, \qquad A_2^n = \{ t \in (\hat{t}(n), \bar{t}(n)) \; : \; \varepsilon(t) < \delta(t) \}.$$

Then, using Lemma 3.5, it is clear that for all $t \in A_1^n$, $m(t) \leq 2C_4 \varepsilon(t)^{\frac{N}{N-1}}$, while for all $t \in A_2^n$, thanks to Lemma 3.1 and reminding (3.3), $\phi(t) \leq 2C_1 \delta(t)^{\frac{N}{N-1}}$. Hence, since $\varepsilon(t) = m'(t)$ and $\delta(t) = \phi'(t)$, we can work as in the proof of Lemma 2.2 from [8] and deduce that $|A_1^n \cup A_2^n| \leq C_5 = C_5(k, N)$.

If we are in case (b), since $\hat{t}(n) = -\infty$ for all n, then $W(\Omega_n, 0, \widehat{m}) \leq |A_1^n \cup A_2^n| \leq C_5$ and the same is true for the γ-limit Ω.

On the other hand, if case (c) happens, in principle there could be some pieces of the limit Ω outside the bounded strip, but Lemma 2.5 assures that Ω must have zero capacity and not only zero Lebesgue measure outside the bounded strip. More precisely, we can choose (up to translations) the origin such that $m(0) = \widehat{m}$. Since Ω corresponds to a capacitary measure μ, case (c) implies

$$\mu = 0 \qquad \text{in} \quad \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \; : \; x < -C_5 \right\}.$$

Hence $W(\Omega, 0, \widehat{m}) \leq C_5$.

At last we consider case (a). Thanks to Lemma 3.5, we have that for all n and for all $1 \leq i \leq k$,

$$\lambda_i(\widehat{\Omega}(t^\star(n))) < \lambda_i(\Omega) - \eta.$$

Hence, since we are supposing $\mathcal{F}$ to be weakly strictly increasing, we have a sequence $(\widehat{\Omega}(t^\star(n)))_n$ such that

$$\inf_n \mathcal{F}(\widehat{\Omega}(t^\star(n))) < \mathcal{F}(\Omega),$$

thus Ω can not be optimal for (1.1). □

Remark 3.7. Applying Lemma 3.6 to a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of Lemma 3.5 and which γ-converges to Ω^*, since *(a), (b)* and *(c)* cover all the possible situations, we deduce

$$W(\Omega^*, 0, \widehat{m}) \leq R_1(k, N).$$

4. Boundedness of the "interior"

To start with, we give the analogous of the definitions (3.1), (3.2) and (3.3) of Section 3. that we need now. More precisely, for every $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$, we set for brevity

$$t_0 := \frac{\tau(\Omega, \overline{m} + \frac{\widehat{m}}{2}) + \tau(\Omega, \overline{m} - \widehat{m})}{2}, \qquad \bar{t} := \frac{\tau(\Omega, \overline{m} + \frac{\widehat{m}}{2}) - \tau(\Omega, \overline{m} - \widehat{m})}{2};$$

keep in mind that, since $\overline{m} \in (\widehat{m}, 1-\frac{\widehat{m}}{2})$, then $-\infty < \tau(\Omega, \overline{m}-\widehat{m}) < \tau(\Omega, \overline{m}+\frac{\widehat{m}}{2}) < +\infty$. For any $0 \leq t \leq \bar{t}$, we define

$$\Omega^+(t) := \Omega^l_{t_0-t} \cup \Omega^r_{t_0+t}, \qquad \Omega^-(t) := \Omega^r_{t_0-t} \cap \Omega^l_{t_0+t} = \Omega \setminus \Omega^+(t),$$

$$\varepsilon(t) := \mathcal{H}^{N-1}(\Omega_{t_0-t}) + \mathcal{H}^{N-1}(\Omega_{t_0+t}), \qquad m(t) := \left|\Omega^-(t)\right| = \int_0^t \varepsilon(s)\, ds \leq \frac{3}{2}\,\widehat{m}.$$

Moreover, having fixed an orthonormal set $\{u_1, u_2, \dots, u_k\}$ of eigenfunctions with unit L^2 norm corresponding to the first k eigenvalues of Ω, for every $1 \leq i \leq k$ and $0 \leq t \leq \bar{t}$ we define

$$\delta_i(t) := \int_{\Omega_{t_0-t}} |Du_i|^2 + \int_{\Omega_{t_0+t}} |Du_i|^2, \qquad \mu_i(t) := \int_{\Omega_{t_0-t}} u_i^2 + \int_{\Omega_{t_0+t}} u_i^2.$$

Then we define again $\delta(t) = \sum_{i=1}^k \delta_i(t)$, and we set again

$$\phi(t) := \sum_{i=1}^k \int_{\Omega^-(t)} |Du_i|^2 = \int_0^t \delta(s)\, ds.$$

Unluckily, it is not possible to prove the analogous of Lemma 3.1 in the very same way, but a little modification is needed.

LEMMA 4.1 ([8], Lemma 2.9). *There exists a small constant $\overline{\nu} = \overline{\nu}(k, K, N) < 1$ such that, if Ω is as in Lemma 3.5, $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$ and $0 \leq t \leq \overline{t}$ is such that $\varepsilon(t), \delta(t) \leq \overline{\nu}$, then for every $1 \leq i \leq k$ one has:*

$$\int_{\Omega^-(t)} u_i^2 \leq C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,, \qquad \int_{\Omega^-(t)} |Du_i|^2 \leq C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,. \tag{4.1}$$

In analogy with Section 3., we give the following definitions. We consider the "internal cylinders"

$$Q_1 := \big(t_0 - t, t_0 - t + \sigma_1\big) \times \Omega_{t_0 - t}\,, \qquad Q_2 := \big(t_0 + t - \sigma_2, t_0 + t\big) \times \Omega_{t_0 + t}\,,$$

where

$$\sigma_1 = \mathcal{H}^{N-1}(\Omega_{t_0-t})^{\frac{1}{N-1}}\,, \qquad \sigma_2 = \mathcal{H}^{N-1}(\Omega_{t_0+t})^{\frac{1}{N-1}}\,.$$

The set $\widetilde{\Omega}(t)$ is defined as

$$\widetilde{\Omega}(t) := \Big\{(x,y) \in \mathbb{R}^N : \text{either } x \leq t_0,\ \big(x - t + \sigma_1, y\big) \in \Omega^+(t) \cup Q_1\,,$$
$$\text{or } x \geq t_0,\ \big(x + t - \sigma_2, y\big) \in \Omega^+(t) \cup Q_2\Big\}\,.$$

Notice that

$$\begin{aligned}\big|\widetilde{\Omega}(t)\big| &= \big|\Omega^+(t)\big| + \big|Q_1\big| + \big|Q_2\big| = 1 - m(t) + \mathcal{H}^{N-1}\big(\Omega_{t_0-t}\big)^{\frac{N}{N-1}} + \mathcal{H}^{N-1}\big(\Omega_{t_0+t}\big)^{\frac{N}{N-1}}\\ &\leq 1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\,.\end{aligned}$$

Moreover, we define again the rescaled set

$$\widehat{\Omega}(t) := \big|\widetilde{\Omega}(t)\big|^{-\frac{1}{N}}\,\widetilde{\Omega}(t)\,.$$

In analogy with Lemma 3.5 we can state the following. Unluckily we have to keep in account also the case in which $\varepsilon(t)$ or $\delta(t)$ are greater than $\overline{\nu}$, but clearly the proof is completely equal to Lemma 3.5.

LEMMA 4.2. *Let Ω be a set as in Lemma 3.5 and let $1 \leq t \leq \overline{t}$. There exists a constant $C_6 = C_6(k, N)$ such that exactly one of the three following conditions hold:*

(1) $\max\{\varepsilon(t), \delta(t)\} > \overline{\nu}$;
(2) *(1) does not hold and* $m(t) \leq C_6\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$;
(3) *(1) and (2) do not hold and for every $1 \leq i \leq k$, one has $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega)$. Moreover if $m(t) \geq \widetilde{m}$ for some $\widetilde{m} > 0$, then there exists $\eta = \eta(N, \widetilde{m}) > 0$ such that, for every $1 \leq i \leq k$, one has $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega) - \eta$.*

In order to prove the last Lemma, analogous to Lemma 3.6, we define $\hat{t}$ as in (3.13) by setting

$$\hat{t} := \sup\Big\{1 \leq t \leq \bar{t} : \text{ condition (3) of Lemma 4.2 holds for } t\Big\},$$

with the convention that, if condition (3) is false for every $1 \leq t \leq \bar{t}$, then $\hat{t} = 1$. Moreover if $\hat{t} > 1$, then we choose some $t^\star \in (\hat{t}-1, \hat{t}]$ for which condition (3) holds.

Lemma 4.3. *Let $(\Omega_n)_n$ be as in the hypotheses of Lemma 3.5, $\Omega_n \xrightarrow{\gamma} \Omega$ and $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$.*

(a) *If there exists a subsequence (not relabeled) such that, for all n, $m(t^\star(n)) \geq \widetilde{m} > 0$ for some $\widetilde{m}$, then Ω can not be optimal for problem (1.1).*

(b) *If there exists a subsequence such that $\hat{t}(n) = 1$ for all n, then there esists $R_2 = R_2(k, N) > 0$ such that $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \leq R_2$.*

(c) *If there exists a subsequence such that $m(t^\star(n)) \to 0$ as $n \to \infty$, then we have again $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \leq R_2$.*

Proof. First of all (see [8, Lemma 2.8]) it is admissible to assume

$$m(t) > 0 \qquad\qquad \forall\, t > 0\,. \tag{4.2}$$

We define A and B as

$$A^n := \Big\{t \in (\hat{t}(n), \bar{t}(n)) : \text{ condition (1) of Lemma 4.2 holds for } t\Big\},$$
$$B^n := \Big\{t \in (\hat{t}(n), \bar{t}(n)) : \text{ condition (2) of Lemma 4.2 holds for } t \text{ and } m(t) > 0\Big\}.$$

The same argument of the proof of Lemma 2.8 in [8] gives then

$$\big|A^n\big| + \big|B^n\big| \leq C_7 = C_7(k, K, N), \qquad \forall n\,. \tag{4.3}$$

Then it is possible to conclude as in Lemma 3.6. If we are in case (b), since $\hat{t}(n) = 1$ for all n, then $W(\Omega_n, \overline{m} - \widehat{m}, \overline{m}) \leq |A^n \cup B^n| \leq C_7 + 2$ and the same is true for the γ-limit Ω.

On the other hand, if case (c) happens, in principle there could be some pieces of the limit Ω outside the bounded strip, but Lemma 2.5 assures that Ω must have zero capacity and not only zero Lebesgue measure outside the bounded strip. More precisely, we know that Ω corresponds to a capacitary measure μ and we call

$$\widetilde{\mu} := \mu \llcorner (\tau(\Omega, \overline{m} - \widehat{m}), \tau(\Omega, \overline{m})),$$

in order to restrict ourselves to the strip we are interested in. In the hypothesis of case (c) we have that

$$\widetilde{\mu} = 0 \qquad \text{in} \quad \big\{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \;:\; \tau(\Omega, \overline{m} - \widehat{m}) < x < C_7 + 2\big\}\,.$$

Hence $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \leq C_7 + 2$.

At last we consider case (a). Analogously to Lemma 3.5, we have that for all n and for all $1 \leq i \leq k$,

$$\lambda_i(\widehat{\Omega}(t^\star(n))) < \lambda_i(\Omega) - \eta.$$

Hence, since we are supposing $\mathcal{F}$ to be weakly strictly increasing, we have a sequence $(\widehat{\Omega}(t^\star(n)))_n$ such that

$$\inf_n \mathcal{F}(\widehat{\Omega}(t^\star(n))) < \mathcal{F}(\Omega),$$

so Ω can not be optimal for (1.1). □

5. Proof of the main Theorem

We are now in position to prove the main Theorem.

Proof of Theorem 1.1. Let Ω^* be a minimizer for problem (1.1); we aim to show that it is contained in an N-cube Q_R with edge of length $R = R(k, N)$. We consider a sequence $(\Omega_n)_n$ of open sets with unit measure and such that $\lambda_k(\Omega_n) \leq K$ for all n, which γ-converges to the set Ω^*.

First of all we apply Lemma 3.6 and we have that $W(\Omega^*, 0, \widehat{m}) \leq R_1$, otherwise we contradict the optimality of Ω^*.
Then we apply Lemma 4.3 with $\overline{m} = 2\widehat{m}$ and we have that $W(\Omega^*, \widehat{m}, 2\widehat{m}) \leq R_2$. We can iterate the application of Lemma (4.3) with $\overline{m} = l\widehat{m}$ ($l \geq 3$) till $l\widehat{m} \leq 1 - \frac{\widehat{m}}{2}$, thus obtaining, with a possible last application when $\overline{m} = 1 - \widehat{m}$:

$$W(\Omega^*, 0, 1 - \widehat{m}) \leq R_1 + lR_2.$$

Now we can apply the above estimate to the symmetric of the set Ω^* with respect to the plane $\{x = 0\}$, thus obtaining:

$$W(\Omega^*, \widehat{m}, 1) \leq R_1 + lR_2.$$

In conclusion we proved that $W(\Omega^*, 0, 1) \leq 2R_1 + 2lR_2$. Now we repeat the whole construction for all the other coordinates $(e_2, \ldots, e_N)$ instead of the first one. At the end, we have proved that the set Ω^* must be contained in an N-cube Q_R with edge of length $R = 2R_1 + 2lR_2$, thus the Theorem is proved. □

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