Some symmetry results for entire solutions of an elliptic system arising in phase separation

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Abstract

We study the one dimensional symmetry of entire solutions to an elliptic system arising in phase separation for Bose-Einstein condensates with multiple states. We prove that any monotone solution, with *arbitrary algebraic growth* at infinity, must be one dimensional in the case of two spatial variables. We also prove the one dimensional symmetry for *half-monotone solutions*, i.e., for solutions having only one monotone component.

1 Introduction and main results

We study smooth solutions of the elliptic system

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N \\ \Delta v = vu^2 & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N \end{cases} \tag{1.1}$$

where the pair (u, v) has at most algebraic growth at infinity and $N \geq 2$.

Problems and solutions of this type naturally arise in the study of phase separation phenomena for Bose-Einstein condensates with multiple states (cfr. [1], [2] and the references therein).

In order to motivate our study and to understand the difficulties that one has to face when dealing with system (1.1), we review the known results about the considered problem. The one dimensional case was studied in [1]. The Authors of [1] proved the existence, symmetry, monotonicity and the growth estimates for the solutions to (1.1). In particular they proved that

$$\exists\, x_0 \in \mathbb{R} \quad : \quad u(x-x_0) = v(x_0 - x) \qquad \forall\, x \in \mathbb{R},$$

$$\text{either} \quad u' > 0,\, v' < 0 \quad \text{or} \quad u' < 0,\, v' > 0$$

and

$$u(x) + v(x) \leq C(1 + |x|) \qquad \forall\, x \in \mathbb{R}.$$

Uniqueness (up to translations, scaling and reflection) of the solution to the one dimensional system (1.1) has been recently settled in [2]. Thus, the one dimensional case is well-understood. On the other hand, the higher dimensional case is more involved and much less is known in that situation.

In [5] it is proved that, in any dimension $N \geq 1$, there are no solutions to (1.1) with sublinear growth, i.e., such that

$$\exists \quad \alpha \in (0,1) \quad : \quad u(x) + v(x) \leq C(1 + |x|)^{\alpha} \qquad \forall\, x \in \mathbb{R}^N.$$

On the other hand, when $N = 2$, solutions with arbitrary integer algebraic growth at infinity has been recently constructed in [2]. In particular, these solutions are not one dimensional when the growth at infinity is superlinear, showing thus the great difference between the one dimensional case (where all the solutions have linear growth) and the higher dimensional case.

Inspired by a celebrated conjecture of E. De Giorgi [3] about monotone solutions to the Allen-Cahn equation (see also [4] for a recent review on the conjecture of De Giorgi and related topics) and motivated by the results in the one dimensional case, Berestycki, Lin, Wei and Zhao [1] raised the following

Question ([1]) *Let $N > 1$. Under what conditions is it true that all monotone solutions to (1.1), i.e., such that*

$$\frac{\partial u}{\partial x_N} > 0, \quad \frac{\partial v}{\partial x_N} < 0 \quad \textit{in } \mathbb{R}^N, \tag{1.2}$$

are one dimensional ? (That is, there exist $U, V : \mathbb{R} \to \mathbb{R}$ and a unit vector ν such that $(u(x), v(x)) = (U(\nu \cdot x), V(\nu \cdot x))$ for every $x \in \mathbb{R}^N$?)

They gave a positive answer to the above question if

$$N = 2 \qquad \textit{and} \qquad u(x) + v(x) \leq C(1 + |x|) \qquad \forall\, x \in \mathbb{R}^2. \tag{1.3}$$

Both the assumptions are crucial in their proof. In particular, in their approach it is not possible to replace the growth condition in (1.3) by the more general condition

$$u(x) + v(x) \leq C(1 + |x|)^{1+\epsilon} \qquad \forall\, x \in \mathbb{R}^2,$$

for some $\epsilon > 0$.

Our first result states that, for $N = 2$, monotone solutions to (1.1) with at most *arbitrary algebraic growth* at infinity, must be one dimensional.

Theorem 1.1. *Let $N = 2$. Then any* monotone *solution (u, v) to (1.1) with at most* arbitrary *algebraic growth at infinity, must be one dimensional.*

Our proof uses a different strategy based on the *Almgren frequency function*

$$\mathcal{N}(r) := \frac{r \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2}, \qquad r > 0. \tag{1.4}$$

We shall describe it in Section 3.

To state our second result we need the following

Definition (Half-monotone solution) *A solution (u, v) to (1.1) is said to be* half-monotone *if it has* one monotone component *(that is, if either $\frac{\partial u}{\partial x_N} > 0$ or $\frac{\partial v}{\partial x_N} < 0$ in $\mathbb{R}^N$).*

Our second result states that, for $N = 2$, half-monotone solutions to (1.1) with at most *arbitrary algebraic growth* at infinity, must be one dimensional.

Theorem 1.2. *Let $N = 2$. Then any* half-monotone *solution (u, v) to (1.1) with at most* arbitrary algebraic growth *at infinity, must be one dimensional.*

Theorem 1.1 is proved in Section 3, while Section 4 will be devoted to Theorem 1.2.

2 Some auxiliary results

In this section we prove some preliminary results which will be used in the course of the main theorems.

We first recall that the *Almgren frequency function* defined by (1.4) is *non-decreasing* in r (cfr. Proposition 5.2 of [2]) and then we prove the following result.

Lemma 2.1. *Assume $N \geq 1$ and let (u, v) be a solution to (1.1) with algebraic growth at infinity, i.e., satisfying*

$$\exists \quad \alpha \geq 1 \quad : \quad u(x) + v(x) \leq C(1 + |x|)^\alpha \qquad \forall\, x \in \mathbb{R}^N. \tag{2.1}$$

Then

$$\mathcal{N}(\infty) := \lim_{r \to +\infty} \mathcal{N}(r) \leq \alpha. \tag{2.2}$$

Proof. For $r > 0$ we set $H(r) := r^{1-N} \int_{\partial B_r(0)} u^2 + v^2$ and $q(r) = \frac{H(r)}{r^{2\mathcal{N}(r_0)}}$. A direct computation gives that

$$\forall\, r_0 > 0 \qquad r \to q(r) \qquad \textit{is nondecreasing for} \quad r > r_0. \tag{2.3}$$

Indeed, a direct calculation yields $H'(r) = 2r^{1-N} \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2$ (cfr. Section 5 of [2]) and so $q'(r) = 2r^{-2\mathcal{N}(r_0)}[r^{-1}\mathcal{N}(r)H(r) - r^{-1}\mathcal{N}(r_0)H(r) + \int_{B_r(0)} u^2v^2] \geq 2r^{-2\mathcal{N}(r_0)}[r^{-1}\mathcal{N}(r)H(r) - r^{-1}\mathcal{N}(r_0)H(r)] \geq 0$ for any $r > r_0$, where in the latter we have used the monotonicity of the Almgren frequency function $\mathcal{N}$.

From (2.3) we infer that

$$\forall\, r_0 > 0 \quad \exists\, c(r_0) > 0 \quad : \quad c(r_0)r^{2\mathcal{N}(r_0)} \leq H(r) \qquad \forall\, r > r_0 \tag{2.4}$$

and thus, by (2.1),

$$\forall\, r_0 > 0 \quad \exists\, c(r_0) > 0 \quad : \quad c(r_0)r^{2\mathcal{N}(r_0)} \leq c_1 r^{2\alpha} \qquad \forall\, r > r_0 \tag{2.5}$$

where c_1 is a positive constant depending only on the dimension N and on the constant C appearing in (2.1). From (2.5) we immediately get

$$\forall r_0 > 0 \qquad \mathcal{N}(r_0) \leq \alpha \tag{2.6}$$

and the desired conclusion (2.2) follows from the monotonicity of the Almgren frequency function $\mathcal{N}$. □

Now we prove a Liouville-type theorem which will be useful in Section 4.

Proposition 2.2. *Assume $N \geq 1$. Let $v, \sigma \in C^2(\mathbb{R}^N)$ be functions satisfying $v > 0$ on $\mathbb{R}^N$,*

$$-\,div(v^2 \nabla \sigma) \leq 0 \quad \textit{in } \mathbb{R}^N \tag{2.7}$$

and

$$\int_{B_R(0)} (v\sigma^+)^2 \leq CR^2 \qquad \forall\, R >> 1, \tag{2.8}$$

for some positive constant C independent of R.
Then $\sigma^+ = const.$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \phi \leq 1$, and

$$\phi(x) := \begin{cases} 1 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 2. \end{cases} \tag{2.9}$$

For $R > 1$ and $x \in \mathbb{R}^N$, let $\phi_R(x) = \phi(\frac{x}{R})$. Multiplying (2.7) by $\sigma^+\phi_R^2$ and integrating by parts, we find

$$\int \phi_R^2 v^2 |\nabla \sigma^+|^2 \leq -2 \int \phi_R v^2 \sigma^+ (\nabla \phi_R \cdot \nabla \sigma) \leq$$

$$\le C_1 \Big[\int_{\{R\le|x|\le 2R\}} \phi_R^2 v^2 |\nabla \sigma^+|^2 \Big]^{\frac12} \Big[\frac{1}{R^2} \int_{\{|x|\le R\}} (v\sigma^+)^2 \Big]^{\frac12},$$

for some positive constant C_1 independent of R. Now, the assumption (2.8) yields :

$$\int \phi_R^2 v^2 |\nabla \sigma^+|^2 \le C_1 C^{\frac12} \Big[\int_{\{R\le|x|\le 2R\}} \phi_R^2 v^2 |\nabla \sigma^+|^2 \Big]^{\frac12} \qquad \text{for} \quad R >> 1, \tag{2.10}$$

which implies $v^2|\nabla\sigma^+|^2 \in L^1(\mathbb{R}^N)$. Using the latter information in (2.10) and letting $R \to +\infty$, we obtain $v^2|\nabla\sigma^+|^2 \equiv 0$, which implies $\sigma^+ = const$. $\square$

We close the present section by recalling a result proved in [2] (cfr. Theorem 1.4. therein).

Theorem 2.3 ([2]). *Assume $N \ge 2$. Let (u, v) be a solution to (1.1) such that $\mathcal{N}(\infty)$ is finite. Then*

$$\mathcal{N}(\infty) = d \in \mathbb{N}^\star \tag{2.11}$$

and there is a homogeneous harmonic polynomial of degree d, denoted by Ψ, such that the blow-down sequence defined by :

$$(u_R(x), v_R(x)) := \Big(\frac{1}{L(R)} u(Rx), \frac{1}{L(R)} v(Rx) \Big), \qquad R > 0, \tag{2.12}$$

$$\textit{where} \qquad L(R) > 0 \quad : \quad \int_{\partial B_1} u_R^2 + v_R^2 = 1, \tag{2.13}$$

converges (up to a subsequence) to (Ψ^+, Ψ^-) uniformly on compact sets of $\mathbb{R}^N$. In addition, if $\mathcal{N}(\infty) = 1$ then (u, v) has linear growth at infinity.

Here, and in the sequel, we denote by w^+ the positive part of the function w and by w^- the negative part of w.

3 Monotone solutions

Proof of Theorem 1.1. By Lemma 2.1 we get that $\mathcal{N}(\infty)$ is finite. This enables us to use Theorem 2.3 with $N = 2$. The monotonicity assumption implies that $\frac{\partial u_R}{\partial x_2} > 0$ and $\frac{\partial v_R}{\partial x_2} < 0$ in $\mathbb{R}^2$, for every $R > 0$. Therefore, for every $x = (x_1, x_2) \in \mathbb{R}^2$, every $t > 0$ and every $R > 0$ we have

$$u_R(x_1, x_2 + t) \ge u_R(x_1, x_2), \qquad v_R(x_1, x_2 + t) \le v_R(x_1, x_2), \tag{3.1}$$

and an application of Theorem 2.3 immediately yields that Ψ^+ is nondecreasing with respect to x_2, while Ψ^- is nonincreasing with respect to x_2. In particular we obtain that $\frac{\partial \Psi}{\partial x_2} \ge 0$ in $\mathbb{R}^2$.

To conclude the proof we invoke the subsequent Proposition 3.1, which tells us that Ψ must be a linear function. Hence, $d = 1$ in (2.11), which means that (u, v) has at most linear growth at infinity, that is (1.3) is satisfied. The desired result then follows from [1], as discussed in the Introduction. $\square$

Now we turn to Proposition 3.1, which deals with entire monotone harmonic functions.

Proposition 3.1. *Assume $N \geq 2$ and let H be a harmonic function on $\mathbb{R}^N$ such that*

$$\frac{\partial H}{\partial x_N} \geq 0 \quad \textit{in } \mathbb{R}^N.$$

Then

$$H(x) = \gamma x_N + h(x_1, ..., x_{N-1}) \qquad \forall\, x \in \mathbb{R}^N, \tag{3.2}$$

where h is a harmonic function on $\mathbb{R}^{N-1}$ and $\gamma \in \mathbb{R}$.
In particular, H must be an affine function when $N = 2$.

Proof of Proposition 3.1. By assumption we have that $\frac{\partial H}{\partial x_N}$ is a *nonnegative* entire harmonic function. So, it must be constant by the classical Liouville Theorem, say $\frac{\partial H}{\partial x_N} \equiv \gamma \in \mathbb{R}$. In particular, the function $h := H - \gamma x_N$ satisfies $\frac{\partial h}{\partial x_N} \equiv 0$ and thus, it must be an entire harmonic function depending only on the variables $x_1, ..., x_{N-1}$. This gives (3.2). When $N = 2$, h must be affine (since in this case h depends only on one variable). This concludes the proof. $\square$

Corollary 3.2. *Assume $N \geq 3$ and let H be a homogeneous harmonic polynomial of degree $d \geq 1$ such that*

$$\frac{\partial H}{\partial x_N} \geq 0 \quad \textit{in } \mathbb{R}^N.$$

Then we have the following alternative :

(i) either H is a linear function

(ii) or H is a homogeneous harmonic polynomial of degree $d \geq 2$ in the first $N - 1$ variables.

Proof. If $d = 1$, H is cleary linear. If $d \geq 2$, then $\gamma = 0$ in (3.2), since H is also a homogeneous function of degree d. This proves the corollary. $\square$

To conclude the section we prove a proposition which will be crucial in the proof of Theorem 1.2.

Proposition 3.3. *Assume $N = 2$ and let P be a homogeneous harmonic polynomial of degree $d \geq 1$ such that P^+ is nondecreasing with respect to x_2.*
Then P must be a linear function.

Remark 1. *In the above proposition the homogeneous harmonic polynomial P cannot be replaced by an arbitrary harmonic function as shown by $H(x, y) = e^y \sin(x)$ in $\mathbb{R}^2$.*

Proof of Proposition 3.3 . If P vanishes at a point $z_0 = (x_0, y_0) \in \mathbb{R}^2 \setminus \{0\}$, it must vanishes on the entire straight line passing through z_0 and the origin. To see this, we first observe that P vanishes on the half-line $\{tz_0 : t \geq 0\}$ by homogeneity. On the other hand, the restriction of P to the entire straight line passing through z_0 and the origin is a polynomial of one variable, which is identically zero on the half-line $\{tz_0 : t \geq 0\}$. This clearly implies that P must vanish identically on the entire straight line containing $\{tz_0 : t \geq 0\}$.

Since $d \geq 1$, the polynomial P must vanish somewhere outside the origin.

Suppose that $P(z_0) = 0$ for some $z_0 = (x_0, y_0)$ with $x_0 \neq 0$ and denote by $\mathcal{S}_{z_0}$ the straight line passing through z_0 and the origin. By the monotonicity assumption on P^+ we get that $P \leq 0$ on the open half-plane lying below $\mathcal{S}_{z_0}$. Indeed, if $P(x, y) > 0$, then $P(x, s) > 0$ for every $s \geq y$, since P^+ is nondecreasing on $\mathbb{R}^2$.

Now, since P is harmonic and nonconstant, the strong maximum principle implies that $P < 0$ everywhere on the open half-plane lying below $\mathcal{S}_{z_0}$. Thus, an application of Hopf's Lemma gives that $|\nabla P| > 0$ on the straight line $\mathcal{S}_{z_0}$. In particular $|\nabla P(0)| > 0$, which clearly implies $d = 1$ (by the homogeneity of P) and $P(x, y) = \alpha x + \beta y$, with $\beta > 0$.

Next we suppose that $P(z_0) = 0$ for some $z_0 = (0, y_0)$ and $y_0 \neq 0$. In this case P vanishes on the straight line $\{(0, y) : y \in \mathbb{R}\}$ and the above argument tell us that P cannot vanish on $\mathbb{R}^2 \setminus \{(0, y) : y \in \mathbb{R}\}$. Hence, either $P > 0$ or $P < 0$ on the open half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. Applying once again Hopf's Lemma we get $d = 1$ and then $P(x, y) = \alpha x$. This concludes the proof. □

4 Half-monotone solutions

Proof of Theorem 1.2. Without loss generality we can suppose that $\frac{\partial u}{\partial x_2} > 0$ in $\mathbb{R}^2$. If we prove that $\frac{\partial v}{\partial x_2} < 0$ in $\mathbb{R}^2$ we are done, since in this case the desired conclusion will follow from Theorem 1.1. To this end, we first prove that (u, v) has at most linear growth at infinity.

By Lemma 2.1 we get that $\mathcal{N}(\infty)$ is finite and so we can use Theorem 2.3 with $N = 2$. Since (u, v) is half-monotone we see that $\frac{\partial u_R}{\partial x_2} > 0$ in $\mathbb{R}^2$, for every $R > 0$. Therefore, for every $x = (x_1, x_2) \in \mathbb{R}^2$, every $t > 0$ and every $R > 0$ we have

$$u_R(x_1, x_2 + t) \geq u_R(x_1, x_2) \tag{4.1}$$

and an application of Theorem 2.3 immediately yields that Ψ^+ is nondecreasing with respect to x_2. The latter enables us to invoke Proposition 3.3 from which we infer that Ψ is linear. Hence $d = 1$ and (u, v) has at most linear growth at infinity. With this information in our hands we are ready to prove that $\frac{\partial v}{\partial x_2} < 0$

in $\mathbb{R}^2$. The latter claim follows from the next result. This completes the proof of Theorem 1.2. $\square$

Theorem 4.1. *Let (u, v) be a solution of*

$$\begin{cases} \Delta u = uv^2 & in\ \mathbb{R}^2 \\ \Delta v = vu^2 & in\ \mathbb{R}^2 \\ u, v > 0 & in\ \mathbb{R}^2 \end{cases} \tag{4.2}$$

such that

$$u(x) + v(x) \le C(1 + |x|) \qquad \forall\, x \in \mathbb{R}^2, \tag{4.3}$$

$$\frac{\partial u}{\partial x_2} > 0 \quad in\ \mathbb{R}^2. \tag{4.4}$$

Then

$$\frac{\partial v}{\partial x_2} < 0 \quad in\ \mathbb{R}^2.$$

Proof. Set $u_2 = \frac{\partial u}{\partial x_2}$ and $v_2 = \frac{\partial v}{\partial x_2}$, then differentiating the second equation in (1.1) we get

$$\Delta v_2 = v_2 u^2 + 2vuu_2 > v_2 u^2 \quad in\ \mathbb{R}^2. \tag{4.5}$$

Here we have used (4.4) and $u, v > 0$ in $\mathbb{R}^2$.

On the other hand, if we set $\sigma = \frac{v_2}{v}$ a direct calculation gives

$$\Delta v_2 = \frac{div(v^2 \nabla \sigma)}{v} + v_2 u^2 \quad in\ \mathbb{R}^2.$$

Thus,

$$- div(v^2 \nabla \sigma) \le 0 \quad in\ \mathbb{R}^2. \tag{4.6}$$

Testing the second equation in (1.1) with $v\phi_r^2$, where ϕ_r is the standard cut-off function defined in the proof of Proposition 2.2, we have :

$$\int_{B_r} |\nabla v|^2 \le Cr^2 \qquad \forall\, r > 0 \tag{4.7}$$

where C is a positive constant independent of r. Note that in the latter estimate we have used in a crucial way the linear growth of v, i.e., the assumption (4.3).

We observe that

$$0 \le v\sigma^+ = v\Big(\frac{v_2}{v}\Big)^+ = v_2^+ \le |\nabla v| \qquad \text{on} \quad \mathbb{R}^2$$

together with (4.6) and (4.7), enables us to apply Proposition 2.2 to infer that $\sigma^+ = const. = \lambda \ge 0$. We claim that $\lambda = 0$. Indeed, $\lambda > 0$ implies

$$v_2 = \lambda v > 0 \qquad \text{and} \qquad \Delta v_2 = \lambda \Delta v = \lambda v u^2 = v_2 u^2, \tag{4.8}$$

which is in contradiction with (4.5). Hence, $\sigma^+ \equiv \lambda = 0$ and so $v_2 \le 0$ on $\mathbb{R}^2$. The strong maximum principle applied to (4.5) then gives $v_2 < 0$ on $\mathbb{R}^2$. $\square$

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