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Existence, Uniqueness, Optimization and Stability for low Eigenvalues of some Nonlinear Operators

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Existence, uniqueness, optimization and stability for low eigenvalues of some nonlinear operators

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Introduction

This thesis is devoted to introduce and discuss the most noteworthy features of what it will be referred to as a *nonlinear eigenvalue*. A ultimate definition of such a mathematical object lacks in the literature, and may be missing also in the future. In fact, despite looking as a *contradictio in terminis* ("eigentheories" are all-linear theories), the vague idiomatic expression of *nonlinear eigenvalue* does however arise naturally from the variational viewpoint. During the last decades, non-linear eigenvalue problems captured the interest of several researchers from different areas of mathematical analysis. The model problem is driven by the so-called p-Laplacian, defined by

$$\Delta_p u \coloneqq \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

for all smooth functions $u : \mathbb{R}^N \to \mathbb{R}$. Note that taking $p = 2$ one is back to the familiar Laplace operator. Almost all features of a nonlinear eigenvalue problem are encoded in this nonlinear operator, which is singular or degenerate depending on whether $p < 2$ or $p > 2$. Several existence, uniqueness and stability results about the p-Laplacian are easily extended to rather general nonliner partial differential equations.

In the thesis, an account is given also of the eigenvalues of some non-local operators. After being studied for a long time in potential theory and harmonic analysis, fractional operators defined via singular integrals are riveting attention since equations involving the fractional Laplacian or similar nonlocal operators naturally surface in several applications.

A great attention in the thesis is devoted to carefully define a suitable notion of eigenvalues when the exponent itself p is replaced by a function $p(x)$. The definition of $p(x)$-eigenvalues seems to be new. The viscosity theory for second order differential equations allows one to study the asymptotic behaviour of $p(x)$-eigenvalues as $p(x)$ approaches a *"variable infinity"* $\infty(x)$. This passage to infinity is accomplished replacing the variable exponent by $jp(x)$ and sending $j \to \infty$. The limit problem is identified, and it has a nice geometric interpretation.

Owing to the importance of superposition principles in nature, the eigenvalues of linear second order elliptic operators appear in areas of mathematical physics ranging from classical to quantum mechanics. For example, the normal modes in the small oscillations near stable equilibria are determined by eigenvalues. Furthermore, in a quantum system the eigenvalues of Schrödinger operator represent the possible energy levels. Linear eigenvalues also play a crucial role for a better understanding of qualitative properties and long time behaviour of solutions to several partial differential equations governing many physical phaenomena.

A model case of elliptic linear eigenvalue problem is given by the celebrated Helmholtz equation

$$-\Delta u = \lambda u, \qquad \text{in } \Omega,$$

with the Dirichlet conditions $u = 0$ on the boundary $\partial\Omega$ of the open set $\Omega \subset \mathbb{R}^N$. The eigenvalues, i.e. the numbers λ such that the above problem is solvable, are the critical values of the Dirichlet integral

$$\int_\Omega |\nabla u|^2 \, dx$$

subject to the constraint

$$\int_\Omega u^2 \, dx = 1.$$

Reading the other way round, by computing the first variation of the Rayleigh quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}$$

one ends up with an eigenvalue problem for the linear Laplace operator $-\Delta$. Note that eigenfunctions can be multiplied by constants. In addition to that, the equation is also additive. The linearity is due to the quadratic growth in the integrals. If the square is replaced by a different power the linearity is destroyed. Nonetheless the homogeneity is preserved.

The fact that the eigenfunctions may be multiplied by constants is an expedient feature of the eigenvalue problems. Despite being non-linear, the problems considered in this thesis do however satisfy this property. Let $H(x, z)$ be convex, even and positively homogeneous of degree $p > 1$ in the variable $z \in \mathbb{R}^N$. Then, the critical values λ of the variational integrals

$$\int_\Omega H(x, \nabla u) \, dx$$

subject to the constraint

$$\int_\Omega |u|^p \, dx = 1$$

are the numbers λ such that the Euler-Lagrange equation

$$-\frac{1}{p} \operatorname{div} \big(\nabla_z H(x\nabla u)\big) = \lambda |u|^{p-2} u, \qquad \text{in } \Omega,$$

admits a non-trivial weak solution attaining zero Dirichlet conditions on the boundary of Ω. The equation fails to be linear unless $p = 2$. Nevertheless, if u is solves the equation, then so does cu. Thus the critical values λ of the quotient

$$\frac{\displaystyle\int_\Omega H(x, \nabla u(x))\, dx}{\displaystyle\int_\Omega |u(x)|^p\, dx}$$

will be called *eigenvalues*. The corresponding critical points solve the Euler-Lagrange equation and they are said to be *eigenfunctions*. The same names are used for critical values and critical points of the quotient if the Dirichlet integral is replaced by the double integral

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |u(y) - u(x)|^p\, \mathsf{K}(x, y)\, dx\, dy$$

where $\mathsf{K}(x, y)$ is some convolution kernel that makes the integral meaningful. In that case the (weak) Euler-Lagrange equation reads

$$\iint\limits_{\mathbb{R}^N\ \mathbb{R}^N} |u(y) - u(x)|^{p-2}(u(y) - u(x))(\varphi(y) - \varphi(x))\mathsf{K}(x, y)\, dx\, dy = \lambda \int_\Omega |u(x)|^{p-2}u(x)\varphi(x)\, dx.$$

When $p = 2$, a suitable choice of the kernel leads to the eigenvalue problem for the linear operator formally definded by

$$-(-\Delta)^s u(x) = C_{s,N} \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(y)}{|y|^{N+2s}}\, dy,$$

where $C_{N,s}$ is some normalization constant. This is called the fractional s-Laplacian. For some accounts about the integro-differential equation involving this non-local operator, the interested reader is referred to the survey [**F3**] written with Enrico Valdinoci.

Basic preliminaries on nonlinear eigenvalues. It is well known that the Dirichlet Laplace operator admits infinitely many eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \to \infty.$$

This basically relies on the Rellich compactness Theorem for the embedding of the Sobolev space $H^1_0(\Omega)$ into $L^2(\Omega)$, which makes compact the "resolvent", i.e. the self-adjoint operator mapping each right hand side $f \in L^2(\Omega)$ to the solution $u \in H^1_0(\Omega)$ of equation

$$-\Delta u = f, \qquad \text{in } \Omega,$$

with Dirichlet boundary conditions on $\partial\Omega$. Hence by spectral theorem the resolvent admits a sequence of eigenvalues μ_n converging to zero, and the corresponding u_n's are in fact eigenfunctions of $-\Delta$ with eigenvalues $\lambda_n = \mu_n^{-1}$. Moreover, these eigenfunctions give an Hilbert basis of $L^2(\Omega)$.

To prove the existence of (nonlinear) eigenvalues obtained by minimizing non-quadratic quotients, the lack of linearity makes a bit inefficient the standard methods used in the linear case, and tools of nonlinear analysis may be of help. There is no reliable spectral theory for producing a "basis" of eigenfunctions. However, there is a plenty of classical procedures to produce eigenvalues. The first chapter is focused on a well established formula for defining a non-decreasing unbounded sequence of critical values of a convex p-homogeneous functional $\mathcal{F}$, defined on some Banach space X, along the one-codimensional manifold $M = \mathcal{G}^{-1}(\{1\})$, where $\mathcal{G}$ is another convex and p-homogeneous functional on X. Namely, one sets

$$\lambda_n = \inf_f \max_\omega \mathcal{F}(f_\omega)$$

for all $n \in \mathbb{N}$. Here ω ranges among all unit vectors in $\mathbb{R}^n$ and the infimum is performed on the class of all odd continuous mappings $\omega \mapsto f_\omega$ from the unit sphere of $\mathbb{R}^n$ to M.

According to the main existence result of Chapter 1, the λ_n's are critical values of $\mathcal{F}$ along M provided that the Palais-Smale condition holds, see Theorem 1.3.3. Basically, that compactness condition reads as follows:

$$\mathcal{F}(u_n) \to \lambda \qquad \Longrightarrow \qquad u_n \to u \text{ strongly}$$

for all sequences such that the differential of $\mathcal{F}$ along M goes to zero in the cotangent norm. This restriction induces the functional to be strongly coercive along sequences of *almost critical points*. Very likely, the requirement should be fullfield if some strong monotonocity of the differential holds. Namely, condition

$$\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), u_n - u \rangle \to 0 \qquad \Longrightarrow \qquad u_n \to u \text{ strongly}$$

will do. The convexity of $\mathcal{F}$ gives the pairing a sign, but does not ensure that the above holds. Nevertheless, in the applications the functionals have a nice modulus of strict convexity. That allows to apply the full existence machinery. The results are applied in particular to the case when

$$\mathcal{F}(u, \Omega) = \int_\Omega H(x, \nabla u)\, dx, \qquad \mathcal{G}(u, \Omega) = \int_\Omega |u(x)|^p d\mu$$

where μ stands either for the Lebesgue measure of for the $(N-1)$-dimensional Hausdorff measure of the boundary. In the second case Ω is assumed to be smooth enough and the second integral is understood in the sence of traces.

Classical Elliptic Regularity for eigenfunctions. This chapter is devoted to surveying the main achievements of regularity theory that are needed in the thesis. The strong minimum principle for non-negative eigenfunctions u (i.e., either $u > 0$ or $u \equiv 0$) is often helpful. That is provided by Harnack inequality, which holds for the eigenfunctions if the integral $\int_\Omega H(x, \nabla u)$ satisfies natural growth conditions. Moreover, the eigenfunctions are Hölder continuous. The first section of the chapter summarizes these classical results. Actually, most eigenvalue problems are solvable in $C^{1,\alpha}$, the eigenfunctions being analytic functions out of their critical set and higher differentiability holds with some distinctions between the singular ($p < 2$) and the degenerate case ($p > 2$), but those results are not used anywhere in the thesis.

Then explicit bounds for the eigenfunctions are provided. This discussion is restricted to the case $H(z) = \|z\|^p$ where $\|\cdot\|$ denotes the norm associated with a (symmetric) convex body in $\mathbb{R}^N$. Similar bounds hold valid if the Dirichlet integral is replaced by a Gagliardo-type (semi)norm

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}}\,dxdy$$

where $s \in (0,1)$.

Hidden convexity for eigenfunctions and applications. Chapter 3 is based on the paper [**F2**] written with Lorenzo Brasco. The purpose is that of relating some well-known facts about the positive eigenfunctions of the p-Laplacian to the convexity of the energy functional

$$t \longmapsto \int_\Omega H(x, \gamma_t(x))\,dx$$

along suitable curves $\gamma : [0,1] \to M$ laying on the level set M of $\mathcal{G}$. Incidentally, such curves are constant speed geodesics for a suitable distance between positive functions belonging to M (different from the Finsler metric induced by the Sobolev space). In Theorem 3.2.1, this geodesic convexity is used to trivialize the global analysis, proving that the energy functional *can not have any critical point, other than its global minimizer on* M.

As a byproduct, the only possible eigenfunctions having constant sign are the ones associated with $\lambda_1(\Omega)$. This is a well known result which had been derived in various places for the p-Laplacian

$$H(z) = |z|^p, \qquad \mu = \mathcal{L}^N$$

under different assumptions on the regularity of Ω (see [**4, 61, 72**] and [**79**] for example). The most simple and direct proof of this fact was given by Kawohl and Lindqvist ([**61**]), in turn inspired by [**79**]. The proof in [**61**] is based on a clever use of the equation, but it does not clearly display the reason behind such a remarkable result.

The advantage of the viewpoint introduced in the paper [**F2**] is to reduce those well-known uniqueness results to a convexity-based device which applies to rather general nonlinear eigenvalue problems.

Spectral gap. This chapter focuses on the second variational eigenvalue $\lambda_2(\Omega)$ of the p-Laplacian (and similar nonlinear operators). Theorem 4.3.2 gives a new proof of the existence of a spectral gap: *there is no eigenvalue between* $\lambda_1(\Omega)$ *and* $\lambda_2(\Omega)$. This fact had been originally proved in [**73**].

Another very classical result in this topic is the so-called Mountain-Pass. Theorem 4.1.3 contains a simple (new) proof of this characterization of the second variational eigenvalue. Namely

$$\lambda_2(\Omega) = \inf_{\gamma} \max_{u\in\gamma} \int_\Omega H(x, \nabla u)\,dx$$

where γ ranges among all continuous paths on M connecting the first eigenfuction u_1 to its opposite function $-u_1$. For the p-Laplacian, this formula is due to [**32**].

Then the attention is turned to $\lambda_2(\Omega)$ in the case when Ω is a disconnected set. In this case, the eigenvalues on the domain are obtained by gathering the eigenvalues on the single connected components. Note that the first eigenvalue may be multiple (for example, that is the case if Ω consists of two equal balls) or simple (think of two disjoint balls with different radii). In the second case, it turns out that

$$\lambda_2(\Omega) = \min\Big\{\lambda > \lambda_1(\Omega)\colon \lambda \text{ is an eigenvalue}\Big\}.$$

This is proved in Theorem 4.1.3. Some care is taken about the consistency of the well-posedness of the minimum. On the contrary, according to Theorem 4.3.3, if the first eigenvalue is multiple then the second variational eigenvalue "collapses" on the first one.

Optimization of low Dirichlet p-eigenvalues. This chapter concerns the stability of optimal shapes for the second variational eigenvalue of the p-Laplacian. The results reported were obtained in collaboration with Lorenzo Brasco in the paper [**F5**]. A quantitative version of the so-called Hong-Krahn-Szego inequality for $\lambda_2(\Omega)$ is derived in Theorem 5.3.1. As a consequence, the disjoint union of two equal balls is proved to be a *stable* minimizer for the second variational eigenvalue.

For $n \geq 3$, very little is known about the *spectral optimization problem* of minimizing

$$\lambda_n(\Omega) \tag{0.0.1}$$

among all open sets Ω having a prescribed volume. Here $\lambda_n(\Omega)$ is the $n-$th variational Dirichlet eigenvalue of the $p-$Laplace operator. Even in the linear case $p = 2$, existence, regularity and characterization of optimal shapes for a problem like (0.0.1) are still open issues. Concerning the existence, a general (positive) answer has been given only very recently, independently by Bucur [**21**] and Mazzoleni and Pratelli [**76**].

On the contrary, the solutions to the problems

$$\min\{\lambda_1(\Omega) : |\Omega| = c\} \tag{0.0.2}$$

and

$$\min\{\lambda_2(\Omega) \,:\, |\Omega| = c\} \tag{0.0.3}$$

are well-known. Under the volume constraint, the first eigenvalue is uniquely minimized by the ball of volume c. This is the *Faber-Krahn inequality*

$$|\Omega|^{p/N}\lambda_1(\Omega) \geq |B|^{p/N}\lambda_1(B). \tag{0.0.4}$$

The second problem is uniquely solved by the union of two disjoint balls of the same volume[1]. That amounts to say that inequality

$$|\Omega|^{p/N}\lambda_2(\Omega) \geq 2^{p/N}|B|^{p/N}\lambda_1(B) \tag{0.0.5}$$

holds for all open set Ω of finite measure. In the linear case $p = 2$, This "isoperimetric" property of balls has been discovered (at least) three times: first by Edgar Krahn ([**64**]) in the '20s, but then the result has been probably neglected, since in 1955 George Pólya attributes this observation to Peter Szego (see the final remark of [**83**]). However, almost in the same years as Pólya's paper, there appeared the paper [**56**] by Imsik Hong, giving once again a proof of this result. It has to be noticed that Hong's paper appeared in 1954, just one year before Pólya's one. For this reason, (0.0.5) is referred to as the *Hong-Krahn-Szego inequality.*

The chapter then addresses some stability issues. Roughly speaking, a positive answer to the question

$$\lambda_n(\Omega) \cong \text{ optimal} \qquad \overset{?}{\Longrightarrow} \qquad \Omega \cong \text{ optimal}$$

is given for $n = 1, 2$. Once the optimal shape Ω_n^* is known, that can be accomplished by proving estimates of the type

$$|\Omega|^{p/N}\lambda_n(\Omega) - |\Omega_n^*|^{p/N}\,\lambda_n(\Omega_n^*) \geq \Phi(\mathsf{d}(\Omega, \mathcal{O}_n)),$$

where $\mathsf{d}(\cdot, \mathcal{O}_n)$ is a suitable "distance" from the "manifold" $\mathcal{O}_n$ of optimizers (open sets having the same shape as Ω_n^*) and Φ is some continuous strictly increasing function, with $\Phi(0) = 0$.

Given an open set $\Omega \subset \mathbb{R}^N$ having $|\Omega| < \infty$, its *Fraenkel asymmetry* is defined by

$$\mathcal{A}(\Omega) = \inf\left\{\frac{\|1_\Omega - 1_B\|_{L^1}}{|\Omega|} \,:\, B \text{ is a ball such that } |B| = |\Omega|\right\}.$$

This is a scaling invariant quantity such that $0 \leq \mathcal{A}(\Omega) < 2$, with $\mathcal{A}(\Omega) = 0$ if and only if Ω coincides with a ball, up to a set of measure zero. Note that the Fraenkel asymmetry may be regarded to as an L^1 distance from the set of balls. A quantitative version of the Faber-Krahn inequality (0.0.4) in terms of $\mathcal{A}$ is provided in the paper [**49**] by Fusco, Maggi and Pratelli and reads as follows

$$|\Omega|^{p/N}\lambda_1(\Omega) \geq |B|^{p/N}\left(1 + C_{N,p}\mathcal{A}(\Omega)^{2+p}\right).$$

In the planar case, the quantitative Faber-Krahn inequality was proved previously by Bhattacharya in his paper [**14**] with the better exponent 3. Moreover, for convex sets Melas [**75**]

[1]On assuming an additional convexity constraint, the problem has been conjectured by Troesch [**88**] to be solved by the convex envelope of the two balls, called *stadium.* That in fact is false, which was first proved by Henrot and Oudet [**55**]. For a proof of this fact based on over-determined problems, see [**43**]. In fact in the planar case the minimizer contains no arc of circle: according to [**68**], the sharp regularity of the minimizer is $C^{1,1/2}$

had also provided a similar result. His estimate was given in terms of the Hausdorff asymmetry, a sort of L^∞ distance, which is natural under the convexity constraint. For the linear Laplace operator, another proof was given by Hansen and Nadirashvili [**52**]. Eventually, the quantitative estimate for the second eigenvalue of the Laplacian was also proved by some probabilists (see Sznitman [**86**] for the planar case and Povel [**84**] in higher dimensional spaces).

In the case of the Hong-Krahn-Szego inequality, the relevant notion of asymmetry is the *Fraenkel* $2-$*asymmetry*, introduced in [**19**]. It is defined for all open sets Ω of finite measure by setting

$$\mathcal{A}_2(\Omega) = \inf\left\{\frac{\|1_\Omega - 1_{B_1\cup B_2}\|_{L^1}}{|\Omega|}\ :\ B_1, B_2 \text{ balls such that } |B_1\cap B_2| = 0,\ |B_i| = \frac{|\Omega|}{2},\ i=1,2\right\}.$$

Then in Theorem 5.3.1 the following quantitative estimate

$$|\Omega|^{p/N}\lambda_2(\Omega) \geq 2^{p/N}\,|B|^{p/N}\,\lambda_1(B)\,\left[1 + C_{N,p}\,\mathcal{A}_2(\Omega)^{\kappa_2}\right],$$

is proved. The exponent κ_2 depends on the dimension and on the sharp exponent κ_1 for the quantitative Faber-Krahn inequality. The analysis covers the whole range of p. Indeed, the same proof can be adapted to cover the cases $p=1$ and $p=\infty$ as well, when λ_2 becomes the *second Cheeger constant* and the *second eigenvalue of the* $\infty-$*Laplacian*, respectively.

Optimization of a nonlinear p-Stekloff eigenvalue. This chapter reports some results obtained in collaboration with Lorenzo Brasco in the paper [**F4**] about the optimization of the first nontrivial eigenvalue $\sigma_{2,p}(\Omega)$ of the so-called *pseudo* $p-$*Laplacian operator*

$$\widetilde{\Delta}_p u := \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

In the linear case $p=2$ this operator coincides with the usual Laplacian and $\sigma_2(\Omega)$ has the value of the best constant in the following Poincaré-Wirtinger trace inequality

$$c_\Omega\int_{\partial\Omega}|u(x)-\overline{u}_{\partial\Omega}|^2\,d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla u(x)|^2\,dx, \qquad u\in W^{1,2}(\Omega),$$

where $\mathcal{H}^{N-1}$ stands for the $(N-1)-$dimensional Hausdorff measure and $\overline{u}_{\partial\Omega}$ denoted the average of the trace of the function u on the boundary.

In analogy with the well-known Dirichlet and Neumann cases (see [**54**, Chapters 3 and 7]), one may be interested in the spectral optimization problem of maximizing[2] σ_2 under volume constraint. A well-known result asserts that the (unique) solutions to this problem are given by balls. This is the so-called *Brock-Weinstock inequality* (see [**20, 91**]). For ease of completeness, it is worth mentioning that Weinstock's result (valid only in dimension $N=2$) is even stronger, since it asserts that disks are still maximizers among simply connected set of given perimeter. By observing that σ_2 scales like a length to the power -1 and that

[2]On the contrary, it is not difficult to see that the problem of minimizing σ_2 is always trivial.

$\sigma_2(B_R) = R^{-1}$ for a ball of radius R, the Brock-Weinstock inequality can be written in scaling invariant form as follows

$$\sigma_2(\Omega) \leq \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{1}{N}}, \tag{0.0.6}$$

where ω_N is the measure of the $N-$dimensional ball of radius 1.

In the non-linear case $p \neq 2$, the pseudo p-Laplacian is an anisotropic operator, which considerably differs from the more familiar $p-$Laplacian. Its first non-trivial eigenvalue $\sigma_{2,p}(\Omega)$ coincides with the best constant in

$$c_\Omega \left[\min_{t\in\mathbb{R}} \int_{\partial\Omega} |u+t|^p \,\varrho\, d\mathcal{H}^{N-1}\right] \leq \sum_{i=1}^N \int_\Omega |u_{x_i}|^p \, dx, \qquad u \in W^{1,p}(\Omega).$$

By adapting Brock's method of proof (Theorems 6.5.2 and 6.5.3) it follows that

$$\sigma_{2,p}(\Omega) \leq \left(\frac{|B_p|}{|\Omega|}\right)^{\frac{p-1}{N}}, \tag{0.0.7}$$

where B_p is the $N-$dimensional ℓ^p unit ball, i.e. $B_p = \{x \in \mathbb{R}^N \,:\, |x_1|^p + \cdots + |x_N|^p < 1\}$. The previous inequality can be seen as a nonlinear counterpart of (6.5.3).

Anisotropic weighted Wulff inequalites. This chapter concerns the weighted anisotropic perimeter discussed in the paper [**F4**] written with Lorenzo Brasco. Besides recalling some basics about convex geometries in $\mathbb{R}^N$, the main result discussed here is the following *weighted Wulff inequality*

$$\int_{\partial\Omega} V(\|x\|)\,\|\nu_\Omega\|_*\, d\mathcal{H}^{N-1} \geq N\,|K|^{1/N}\,|\Omega|^{\frac{N-1}{N}} V\left((|\Omega|/|K|)^{\frac{1}{N}}\right) \tag{0.0.8}$$

which is proved in Theorem 7.3.4, generalizing the results of [**18**]. Here $\|\cdot\|$ and $\|\cdot\|_*$ denote two dual norms, respectively defined as the Minkowski gauge and the support function of a convex body K, whereas ν_Ω stands for the outward pointing unit normal to the boundary of the Lipschitz set Ω. The function $V : \mathbb{R}_+ \to \mathbb{R}_+$ is called the weight. Equality can hold if and only if $\Omega = K$, up to a scaling factor. The proof of (0.0.8) is an adaptation of the calibration technique used in [**18**]. Under suitable additional assumptions on the regularity of the weight, Theorem 7.4.1 provides a quantitative version of the above anisotropic weighted Wulff inequality, which reads as follows:

$$\int_{\partial\Omega} V(\|x\|)\|\nu_\Omega\|_* d\mathcal{H}^{N-1} \geq N\omega_{K,N}^{\frac{1}{N}}|\Omega|^{1-\frac{1}{N}}\left[V\left(\left(\frac{|\Omega|}{\omega_{K,N}}\right)\right)^{\frac{1}{N}} + C_{N,V,|\Omega|}\left(\frac{|\Omega\Delta(T_\Omega K)|}{|\Omega|}\right)^2\right],$$

where $\omega_{K,N} \coloneqq |K|$ and $T_\Omega K$ is the dilation of K having the same volume as Ω.

An eigenvalue problem with variable exponents. The last chapter concerns the eigenvalue problem introduced in collaboration with Peter Lindqvist in the recent paper [**F6**] about the minimization of the "Rayleigh quotient"

$$\frac{\|\nabla u\|_{p(x),\Omega}}{\|u\|_{p(x),\Omega}} \tag{0.0.9}$$

among all functions belonging to the Sobolev space $W_0^{1,p(x)}(\Omega)$ with variable exponent $p(x)$. The norm is the so-called *Luxemburg norm*.

If $p(x) = p$, a constant in the range $1 < p < \infty$, one reduces to the eigenvalue problem for the Dirichlet p-Laplacian. It is decisive that homogeneity holds: if u is a minimizer, so is cu for any non-zero constant c. On the contrary, the quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,dx}{\displaystyle\int_\Omega |u|^{p(x)}\,dx} \tag{0.0.10}$$

with variable exponent does not possess this expedient property, in general. Therefore its infimum over all $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$, is often zero and no mimizer appears in the space $W_0^{1,p(x)}(\Omega)$, except the trivial $\varphi \equiv 0$, which is forbidden. For an example, see [**42**, pp. 444–445]. A way to avoid this collapse is to impose the constraint

$$\int_\Omega |u|^{p(x)}\,dx = \text{constant}.$$

Unfortunately, in this setting the minimizers obtained for different normalization constants are difficult to compare in any reasonable way, except, of course, when $p(x)$ is constant. For a suitable $p(x)$, it can even happen that any positive λ is an eigenvalue for some choice of the normalizing constant. Thus (0.0.10) is not the proper generalization of the eigenvalue problem for the p-Laplacian to the case of a variable exponent.

A way to avoid this situation is to use the Rayleigh quotient (0.0.9), where the notation

$$\|f\|_{p(x),\Omega} = \inf\left\{\gamma > 0 : \int_\Omega \left|\frac{f(x)}{\gamma}\right|^{p(x)} \frac{dx}{p(x)} \le 1\right\} \tag{0.0.11}$$

was used for the Luxemburg norm. This restores the homogeneity. In the integrand, the use of $p(x)^{-1}\,dx$ (rather than $p(x)$) has no bearing, but it simplifies the equations a little. The existence of minimizers follows easily by the direct method in the Calculus of Variations. The Euler-Lagrange equation is obtained by computing the first variation of the Luxembourg norms and reads

$$\operatorname{div}\left(\left|\frac{\nabla u}{K}\right|^{p(x)-2}\frac{\nabla u}{K}\right) + \frac{K}{k}S\left|\frac{u}{k}\right|^{p-2}\frac{u}{k} = 0, \tag{0.0.12}$$

where the K, k, S are constants depending on u.

Then the passage to infinity is accomplished so that $p(x)$ is replaced by $jp(x)$, $j = 1, 2, 3\dots$ The viscosity theory for second order equations allows one to identify the limit equation which is

$$\max\left\{\Lambda_\infty - \frac{|\nabla u|}{u},\ \Delta_{\infty(x)}\left(\frac{u}{K}\right)\right\} = 0, \tag{0.0.13}$$

where

$$K = \|\nabla u\|_{\infty,\Omega}, \qquad \Lambda_\infty = \frac{1}{\max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)} \tag{0.0.14}$$

and

$$\Delta_{\infty(x)} v = \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} + |\nabla v|^2 \ \ln\big(|\nabla v|\big)\big\langle \nabla v, \nabla \ln p\big\rangle. \tag{0.0.15}$$

For a constant exponent, this has been treated first in [**59**] (see also [**60, 58, 27**]). An interesting interpretation in terms of optimal mass transportation is given in [**28**]. According to a recent manuscript by Hynd, Smart and Yu, there are domains such that there can exist several linearly independent positive eigenfunctions, see [**57**]. Thus the eigenvalue Λ_∞ is not always simple.

If Λ_∞ is given the value (0.0.14), the same as for a constant exponent, then the existence of a non-trivial solution is guaranteed. A *local uniqueness* result also holds, cf. Theorem 8.4.4. Namely, in a sufficiently interior domain the solution cannot be perturbed continuously.

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CHAPTER 1

Basic preliminaries on nonlinear eigenvalues

If X is a normed vector space, X^* will denote the strong dual space, consisting of all the linear functionals on X that are continuous with respect to the topology induced on X by the norm and $\langle \cdot\,, \cdot \rangle$ will denote the duality pairing between X and X^*. If Y is another normed space, then $\mathcal{L}(X, Y)$ will stand for the space of all continuous linear mappings from X to Y.

1. Differentiable functions

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces, $A \subset X$, u an interior point of A and $v \in X$. The *directional derivative* of a function $\mathcal{J} : A \to Y$ at u along the direction v is defined by

$$\partial_v \mathcal{J}(u) = \lim_{t \to 0} \frac{\mathcal{J}(u + tv) - \mathcal{J}(u)}{t},$$

provided the limit exists in Y.

The function $\mathcal{J}$ is said to be *(Fréchet) differentiable* at u if there exists a continuous and linear mapping $L \in \mathcal{L}(X, Y)$ such that the limit

$$\lim_{h \to 0} \frac{\mathcal{J}(u + h) - \mathcal{J}(u) - L(h)}{\|h\|_X} = 0 \tag{1.1.1}$$

holds in Y. If there exists such a linear function L, then it is uniquely determined, is denoted

$$L = \mathcal{J}'(u),$$

and is said to be the *(Fréchet) differential* of $\mathcal{J}$ at u. Moreover,

$$\|\mathcal{J}(u + h) - \mathcal{J}(u)\|_Y \leq \|L(h)\|_Y + o(\|h\|_X),$$

as $h \to 0$ in Y and $\mathcal{J}$ is continuous at u.

The function $\mathcal{J}$ is called *Gâteaux differentiable* at u if there exists a linear mapping $T \in \mathcal{L}(X, Y)$ such that the limit

$$\lim_{\varepsilon \to 0} \frac{\mathcal{J}(u + \varepsilon v) - \mathcal{J}(u)}{\varepsilon} = T(v) \tag{1.1.2}$$

holds in Y. If there exists such a linear mapping T, then it is unique, we denote it by

$$T = D\mathcal{J}(u),$$

and we call it the *Gâteaux differential* of $\mathcal{J}$ at u. Moreover,

$$\|\mathcal{J}(u+\varepsilon v)-\mathcal{J}(u)\|_Y \leq \varepsilon\|T(v)\|_Y + o(\varepsilon),$$

as $\varepsilon \to 0$ and $\mathcal{J}$ is continuous along all the straight lines passing through u.

Clearly, if $\mathcal{J}$ is differentiable then it is Gâteaux differentiable and the two differentials coincide, that is

$$D\mathcal{J}(u)[v] = \mathcal{J}'(u)(v),$$

for all $v \in X$, but the converse does not hold. For example, let $\mathcal{J} \colon X \to \mathbb{R}$ be defined by $\mathcal{J}(u) = \langle v^*, u\rangle$ if $\|u - \langle v^*, u, \rangle v\| = |\langle v^*, u\rangle|^2$, where v^* is the projection onto the closed vector space spanned by $v \in X \setminus \{0\}$, and $\mathcal{J}(u) = 0$ otherwise. Then, allthough $\mathcal{J}$ is not differentiable at the origin, all its directional derivatives at the origin do however exists and and depend linearly on the direction (in fact, they are all equal to zero).

Of course, if $\mathcal{J}$ is Gâteaux differentiable at u then there exists the derivative of $\mathcal{J}$ along all directions, and one has

$$\partial_v\mathcal{J}(u) = D\mathcal{J}(u)[v],$$

for all $v \in X$. Again, the converse does not hold, since the map $v \mapsto \partial_v\mathcal{J}(u)$, which is always homogeneous, may fail to be additive, even if there exists the directional derivative of $\mathcal{J}$ along all directions. For example, the directional derivative $\partial_u\mathcal{J}(0)$ of an odd and positively 1-homogeneous functional $\mathcal{J} : X \to \mathbb{R}$ equals the value $\mathcal{J}(u)$ that the functional takes at u, hence its dependance on u must not be linear as soon as the functional itself is nonlinear in u.

It is worth recalling that any linear mapping from X to Y such that (1.1.1) holds is automatically continuous. Indeed,

$$L(v) = \mathcal{J}(u+v) - \mathcal{J}(u) + o(\|v\|_X),$$

as $v \to 0$ in X. Thus, by the continuity at u of $\mathcal{J}$, L is continuous at the origin. Being L linear, the continuity of L follows. On the contrary, if X is infinite-dimensional, a linear mapping T from X to Y may well fail to depend continuously on v even if (1.1.2) holds.

We recall a sort of mean value property holds for the functions admitting directional derivatives. Namely, if there exists the directional derivative of $\mathcal{J}$ at the point u along the direction v then the inequality

$$\|\mathcal{J}(u+tv)-\mathcal{J}(u)\|_Y \leq t \sup_{s\in[0,t]} \|\partial_v\mathcal{J}(u+sv)\|_Y, \tag{1.1.3}$$

holds for all $t \geq 0$. One can employ (1.1.3) to prove the following criterion for the Fréchet differentiability of functions, which is of remarkable use.

Proposition 1.1.1. *Let X, Y be normed spaces, $A \subset X$, u an interior point of A, and let $\mathcal{J} : A \to Y$ be Gâteaux differentiable in a neighborhood of u in X. If the Gâteaux differential $D\mathcal{J}$ is continuous at u, then $\mathcal{J}$ is Fréchet differentiable at u.*

Proof. The proof is standard. For every $s \in [0,1]$, the function

$$R_s(v) = \mathcal{J}(u+sv) - \mathcal{J}(u) - s\partial_v \mathcal{J}(u),$$

is Gâteaux differentiable in a neighborhood of the origin sufficiently small, and

$$\partial_w R_s(v) = s\big(\partial_w \mathcal{J}(u+sv) - \partial_w \mathcal{J}(u)\big),$$

for all $w \in X$, provided $\|v\|_X$ is small enough. Note that

$$R_t(sv) = R_{ts}(v),$$

for all $t \in [0,1]$. Thus, since $R_1(0) = 0$, by (1.1.3) it follows that

$$\begin{aligned}
\|R_1(v)\|_Y \le \sup_{s\in[0,1]} \|\partial_v R_1(sv)\|_Y &= \sup_{s\in[0,1]} \|\partial_v R_s(v)\|_Y \\
&\le \sup_{s\in[0,1]} s\cdot \big\|\partial_v \mathcal{J}(u+sv) - \partial_v \mathcal{J}(u)\big\|_Y \\
&\le \sup_{s\in[0,1]} s\cdot \|D\mathcal{J}(u+sv) - D\mathcal{J}(u)\|_{\mathcal{L}(X,Y)} \|v\|_X \\
&\le \sup_{s\in[0,1]} \|D\mathcal{J}(u+sv) - D\mathcal{J}(u)\|_{\mathcal{L}(X,Y)} \|v\|_X,
\end{aligned}$$

provided $\|v\|_X$ is sufficiently small. Since $D\mathcal{J}$ is continuous at u,

$$\|D\mathcal{J}(u+sv) - D\mathcal{J}(u)\|_{\mathcal{L}(X,Y)} \le 2 \max_{\|w\|_X \le \varepsilon} \|D\mathcal{J}(u+w)\|_{\mathcal{L}(X,Y)} < +\infty,$$

provided ε is small enough. Thus, the above implies

$$\lim_{\|v\|_X \to 0} \frac{\|R_1(v)\|_Y}{\|v\|_X} = 0,$$

which is precisely the Fréchet differentiability of $\mathcal{J}$ at u. □

Let X, Y be normed spaces. Recall that

$$\|P\|_{\mathcal{L}_2(X\times X,Y)} = \sup\{\|P(u,v)\|_X : \|u\|_X, \|v\|_X \le 1\},$$

defines a norm on the vector space of all continuous mappings from $X \times X$ to Y that are bilinear, that is linear in each variable. By setting

$$\big(\phi(P)(u)\big)(v) = P(u,v), \qquad u, v \in X,$$

for all $P \in \mathcal{L}(X\times X, Y)$, one defines an isometry $\phi\colon \mathcal{L}_2(X\times X,Y) \to \mathcal{L}(X,\mathcal{L}(X,Y))$.

Let A be an open set in X. A differentiable function $\mathcal{J}$ from A to Y be is said to be *twice (Fréchet) differentiable* at $u \in A$ if the function $\mathcal{J}' : A \to \mathcal{L}(X,Y)$ is itself differentiable at u. If this is the case, we denote by

$$\mathcal{J}''(u)$$

the continuous and bilinear map which is uniquely associated with $(\mathcal{J}')'(u)$ via the isometry $\mathcal{L}(X,\mathcal{L}(X,Y)) \cong \mathcal{L}_2(X\times X,Y)$ described above. It can be proved that $\mathcal{J}''(u)$, that we call the *second (Fréchet) differential* of $\mathcal{J}$ at u, is in fact a symmetric bilinear form. The

n-th (Fréchet) differential of a mapping is defined inductively and is a symmetric continuous n-linear mapping.

Let X, Y be normed space and $A \subset X$ be an open set. A function $\mathcal{J} : A \to Y$ is said to be of class C^k on A if its k-th differential is continuous on A. We say that φ is a C^k diffeomorphism with its image if it is one-to-one it is of class C^k with its inverse function.

1.1. Local inversion of differentiable functions. If X, Y, Z are normed spaces $u_0 \in X, v_0 \in Y$ and $\mathcal{J} : X \times Y \to Z$ is a function, we denote by $\mathcal{J}'_X(u_0, v_0)$ the differential at u_0 of the function $u \mapsto \mathcal{J}(u, v_0)$.

Theorem 1.1.2 (Implicit function theorem)**.** *Let X, Y, Z be Banach spaces, A an open subset of $X \times Y$, $(u_0, v_0) \in A$ and $\mathcal{J} : X \times Y \to Z$ be a continuous function. Assume that $\mathcal{J}'_Y$ exists and is continuous in A. If the mapping $\mathcal{J}'_Y(u_0, v_0)$ if an isomorphism from Y to Z then there exist a neighborhood U of u_0 in X, a neighborhood V of v_0 in Y and a continuous function $\phi : U \to V$ such that*

$$\mathcal{J}^{-1}(\{0\}) = \mathrm{graph}(\phi).$$

If, in addition, $\mathcal{J}$ is of class C^k then so is ϕ. If this is the case, then

$$\phi'(u_0) = -\big[\mathcal{J}'_Y(u_0, v_0)\big]^{-1} \circ \mathcal{J}'_X(u_0, v_0).$$

Theorem 1.1.3 (Local inversion Theorem)**.** *Let X, Y be Banach spaces, A be an open subset of X, $u_0 \in A$ and $\mathcal{J}$ a C^1 function from A to Y. If $\mathcal{J}'(u_0)$ is an isomorphism from X to Y, then there exists an open neighborhood U of u_0 such that the restriction of $\mathcal{J}$ to U is an C^1 diffeomorphism with its image V and*

$$(\mathcal{J}^{-1})'(v_0) = \big(\mathcal{J}'(u_0)\big)^{-1}.$$

We refer to [**9**] for the proof of the above classical theorems.

2. Constrained critical levels and eigenvalues

A topological space M is said to be a C^k Banach manifold modelled on the Banach space X if there exist a set I, an open covering $\{U_\imath\}_{\imath \in I}$ of M, a family of closed vector subspaces $X_\imath$ of X and a collection of mappings $\varphi_\imath \colon U_\imath \to X_\imath$ which are homeomorphisms with their images, such that $\varphi_\imath(U_\imath \cap U_\jmath)$ is open in X and $\varphi_\jmath \circ \varphi_\imath^{-1}$ induces a C^k diffeomorphism of $\varphi_\imath(U_\imath \cap U_\jmath)$ onto $\varphi_\jmath(U_\imath \cap U_\jmath)$. The pairs $(U_\imath, \varphi_\imath)$ are called *charts*.

When it happens that all the $X_\imath$'s are one-codimensional subspaces of X, M is said to be a one-codimensional Banach manifold. Since in this Thesis we aim to adress some issues regarding real eigenvalues, which are nothing but critical levels of functionals along one-codimensional manifolds, we restrict ourselves to this case.

Let $\mathcal{G} : X \to \mathbb{R}$ be a C^1 functional, such that the topological subspace M of X defined by

$$M = \big\{u \in X \,:\, \mathcal{G}(u) = 1\big\}, \tag{1.2.1}$$

consists of regular points for $\mathcal{G}$, that is $X_u = \ker \mathcal{G}'(u) \neq X$, for all $u \in M$. Then M is a one-codimensional C^1 Banach manifold modelled on X. We call the tangent space to M at its point u the vector space

$$T_u M = \Big\{ \varphi \in X : \langle \mathcal{G}'(u), \varphi \rangle = 0 \Big\},$$

consisting of all tangent vectors to M at u. That recovers the abstract definition via derivations. Obviously, the norm of X makes the tangent space at u a Banach space. The strong dual of $T_u M$ is the cotangent space to M at u and is denoted by $T_u^* M$. By Hahn-Banach Theorem, it is isomorphic to a closed vector subspace of X^* with the norm defined by

$$\|\Lambda\|_* = \max \Big\{ \big\langle \Lambda \, , \, \varphi \big\rangle : \varphi \in T_u M, \ \|\varphi\|_X = 1 \Big\},$$

for all $\Lambda \in T_u^* M$, where $\langle \cdot, \cdot \rangle$ stands for the natural duality pairing.

If $\mathcal{F} : X \to \mathbb{R}$ is a C^1 functional, then its restriction to M is also C^1, its differential at a point $u \in M$ being nothing but the restriction $\mathcal{F}'(u)_{|T_u M}$ of the differential $\mathcal{F}'(u)$. Thus, a number c is a *critical value of* $\mathcal{F}$ *along* M if $\mathcal{F}(u) = c$ and there exist a point $u \in M$ such that

$$\langle \mathcal{F}'(u), \varphi \rangle = 0, \qquad \text{for all } \varphi \in T_u M, \tag{1.2.2}$$

and if this happens u is called a *critical point of* $\mathcal{F}$ *along* M corresponding to the critical value λ.

By Lagrange multipliers' rule, a point $u \in M$ is a critical point of $\mathcal{F}$ along M is such that

$$\mathcal{F}'(u) = \lambda \mathcal{G}'(u), \tag{1.2.3}$$

in X^* for some real number λ. Indeed, by (1.2.2) the kernel of $\mathcal{F}'(u)$ contains the kernel of $\mathcal{G}'(u)$. These differentials are linear mappings. Hence there has to be a number λ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathcal{F}'(u)"] \arrow[d, "\mathcal{G}'(u)"'] & \mathbb{R} \\ \mathbb{R} \arrow[ur, dashed, "\lambda"'] & \end{tikzcd}$$

is commutative, and (1.2.3) follows.

Let $\mathcal{F}$, $\mathcal{G}$ be C^1 functionals on a Banach space X. In addition, assume that both the functionals are even and positive homogenous of degree $p \geq 1$. Then $M = \mathcal{G}^{-1}(\{1\})$ is a regular one-codimensional manifold in X. Indeed, by the homogeneity it follows that

$$\big\langle \mathcal{G}'(u), u \big\rangle = p \, \mathcal{G}(u) = p,$$

whence $\ker \mathcal{G}'(u) \neq X$, for all $u \in M$.

Definition 1.2.1 (Nonlinear eigenvalues)**.** Let $\mathcal{F}, \mathcal{G}$ be C^1 even and positively homogeneous functionals of degree $p \geq 1$ on the Banach space X, and $M = \mathcal{G}^{-1}(\{1\})$. A real number λ is said to be an *eigenvalue* of the pair $(\mathcal{F}, \mathcal{G})$ if there exists $u \in X \setminus \{0\}$ such that

$$\langle \mathcal{F}'(u), v\rangle = \lambda \langle \mathcal{G}'(u), v\rangle, \tag{1.2.4}$$

holds for all $v \in X$. If this is the case, then u is called an *eigenvector* corresponding to λ.

Note that eigenvectors and eigenvalues of the pair $(\mathcal{F}, \mathcal{G})$ are precisely given by the critical points and critical values $\mathcal{F}$ along M. To see that, note that (1.2.2) holds for all eigenvectors $u \in M$ corresponding to the eigenvalue λ. Conversely, if $u \in M$ is a constrained critical point associated with the critical value c, then equation (1.2.4) holds with $\lambda = c$. Indeed, there has to be λ such that (1.2.3) holds, and by plugging $u = v$ in, one gets

$$\lambda = \frac{1}{p}\,\lambda\,\langle \mathcal{G}'(u), u\rangle = \frac{1}{p}\,\langle \mathcal{F}'(u), u\rangle = \mathcal{F}(u) = c.$$

3. Existence of eigenvalues: minimization and global analysis

We discuss the existence of eigenvalues for a pair $(\mathcal{F}, \mathcal{G})$ of C^1 functionals which are even and positively homogeneous of degree $p > 1$. Min-max formulae of the type

$$\lambda_n = \inf_f \max_\omega \frac{\mathcal{F}(f_\omega)}{\mathcal{G}(f_\omega)}$$

play a role. The maximum is taken among all unit vectors ω in $\mathbb{R}^n$, whereas f ranges over all odd and continuous mappings $\omega \mapsto f_\omega$ from the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ into $M = \mathcal{G}^{-1}(\{1\})$. A mapping f from $\mathbb{S}^{n-1}$ to M is said to be odd if $f_{-\omega} = -f_\omega$, for all $\omega \in \mathbb{S}^{n-1}$.

This is a well established method for producing eigenvalues of the pair $(\mathcal{F}, \mathcal{G})$. The procedure hardly would deserve a comment. Yet, for sake of completeness we discuss a proof of this existence result in next section 3.1, nonetheless. The λ_n's are *almost critical levels*, see sections 3.2 and 3.3. The conclusion that they actually are *eigenvalues* holds provided that a suitable compactness on the *almost critical* sequences is valid. This is discussed in next section, see Theorem 1.3.3.

3.1. Palais-Smale condition and existence of eigenvalues. We introduce the following condition, which dates back to the work of Palais and Smale [**81**] on the generalized Morse theory.

Definition 1.3.1. Let X be a Banach space, $\mathcal{M}$ a one-codimensional C^1 regular manifold in X, and Φ be a C^1 functional on X.

(*i*) We call $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ a $(PS)_\lambda$ sequence for Φ if

$$\lim_{k\to\infty} \Phi(u_k) = \lambda, \qquad \lim_{k\to\infty} \|\Phi'(u)_{|T_u\mathcal{M}}\|_* = 0,$$

(*ii*) The functional Φ is said to satisfy the Palais-Smale condition at level λ on $\mathcal{M}$ if any $(PS)_\lambda$ sequence has a strongly converging subsequence.

Moreover, we say that Φ satisfy the Palais-Smale condition on $\mathcal{M}$ if it satisfies the Palais-Smale condition on $\mathcal{M}$ at any level λ.

Remark 1.3.2. If $X \cong \mathbb{R}^m$ and $\mathcal{M}$ is, say, a smooth compact hypersurface, then by Bolzano–Weierstrass Theorem any $(PS)_\lambda$ sequence admits a subsequence converging to a critical level. The same conclusion can be drawn if $X \cong \mathbb{R}^m$, $\mathcal{M}$ is any compact hypersurface, and Φ is bounded by below and coercive. As a matter of fact, if X is infinite dimensional then there may well exist functionals Φ which are coercive, bounded from below and do not satisfy the (PS) condition on some manifold $\mathcal{M}$. For example, the coercive functional

$$\int_\Omega \left(|\nabla u| - 1\right)_+^p \, dx$$

does not satisfy the Palais-Smale condition on the $L^p(\Omega)$ sphere in $W_0^{1,p}(\Omega)$.

The following theorem contains the existence result. This will be applied in Section 4 for producing a sequence of variational eigenvalues of some nonlinear operators.

Theorem 1.3.3. *Let X be a uniformly convex Banach space, $\mathcal{F}, \mathcal{G}$ be two even and positively homogeneous C^1 functionals of degreee $p > 1$ on X. For all $n \in \mathbb{N}$, denote by V_n the set of all odd and continuous mappings $\omega \mapsto f_\omega$ from $\mathbb{S}^{n-1}$ to $M = \mathcal{G}^{-1}(\{1\})$ and set*

$$\lambda_n = \inf_{f \in V_n} \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_\omega).$$

Then, if $\mathcal{F}$ satisfies the Palais-Smale condition on M, the λ_n's are an increasing divergent sequence of eigenvalues of the pair $(\mathcal{F}, \mathcal{G})$.

The proof of the theorem requires some technical results, that are discussed in next section.

3.2. Deformations and pseudo-gradient vector fields. In order to prove Theorem 1.3.3, a standard strategy is that of deforming the sublevels of the functional $\mathcal{F}$ in such a way that the values around a noncritical level are suitably lowered down. This would yield a contradiction if the λ_n's were regular values. Namely, by deformation we mean the following.

Definition 1.3.4. A continuous mapping $\eta : M \to M$ is said to be a deformation if it is homotopic to the identity map, namely if there exists a continuous function $\mathfrak{H} : [0,1] \times M \to M$ such that

$$\mathfrak{H}(0, u) = u, \qquad \mathfrak{H}(1, u) = \eta(u),$$

for all $u \in M$.

The deformation η lowering the non-critical values down can be manifactured by pushing the points of M forward via a gradient flow, provided that the functional is $C^{1,1}$. Indeed, if this is the case the first variation of the functional $\mathcal{F}$ defines a locally Lipschitz vector field. Then, the associated initial value problem, accompanied by the initial condition given by a point $u \in M$, admits a unique solution $\Phi_t(u)$ by the classical Cauchy-Lipschitz theory for ordinary differential equations. For small $t > 0$, this flow yields the desired deformation.

Since here the functional $\mathcal{F}$ may well be not sufficiently regular, the technique described above can not be applied. Thus one needs the notion of pseudo-gradient vector field, which seems to be due to Palais [**80**]. Recall that, in general, by vector field on M it is meant any right inverse of the natural projection

$$\pi : \bigcup_{u \in M} \{u\} \times T_u M \to M.$$

The disjoint union of the tangent spaces to M at its points is called the tangent bundle, is denoted by TM and naturally inherits a Finsler metric structure from M. In fact, a vector field V from M to TM is locally Lipschitz continuous if there exists, for every compact subset K of M, a positive constant L_K such that

$$\|V(u) - V(w)\|_X \leq L_K \|u - w\|_X,$$

for all $u, w \in K$.

Definition 1.3.5. Let Σ consist of all critical points of $\mathcal{F}$ on M. A locally Lipschitz vector field $V : M \to TM$ is said to be a pseudo gradient vector field on M for $\mathcal{F}$ if

$$\|V(u)\| \leq 2\|\mathcal{F}'(u)_{|_{T_u M}}\|_*, \qquad \Big\langle \mathcal{F}'(u), V(u) \Big\rangle \geq \|\mathcal{F}'(u)_{|_{T_u M}}\|_*^2, \tag{1.3.1}$$

for all $u \in M \setminus \Sigma$.

A deformation can be constructed by considering the flow associated with a locally Lipschitz pseudo-gradient vector field on M, even if the functional $\mathcal{F}$ is merely of class C^1, see Proposition 1.3.9 below. The existence of pseudo-gradient vector fields being locally Lipschitz continuous is a little demanding even in the unconstrained case, for which we refer to [**8**]. However, for sake of completeness, we prove the following Lemma. The idea of the proof is to patch all the steep directions tangent to M "pushed" by $\mathcal{F}'$ by a suitable partition of unity, consisting of locally Lipschitz continuous functions.

Lemma 1.3.6. *There exists an odd locally Lipschitz pseudo–gradient vector field on M for $\mathcal{F}$.*

Proof. Let u be a regular point of $\mathcal{F}$ on M. Owing to the definition of

$$\|\mathcal{F}'(u)_{|_{T_u M}}\|_* = \sup \Big\{ \langle \mathcal{F}'(u), v \rangle : v \in T_u M, \|v\|_X = 1 \Big\},$$

there exists $v \in T_u M$ such that $\|v\|_X = 1$ and

$$\frac{2}{3}\|\mathcal{F}'(u)_{|T_u M}\|_* < \langle \mathcal{F}'(u), v\rangle. \tag{1.3.2}$$

Note that the right hand side in (1.3.2) changes sign if u is replaced by $-u$. Indeed, the differential $\mathcal{F}'$ is odd, since the functional $\mathcal{F}$ is even. Therefore, there exists an odd vector field $v : M \to TM$ such that (1.3.2) holds with $v = v(u)$, for all $u \in M$.

Let now $W : M \to TM$ be the odd vector field defined by

$$W(u) = \frac{3}{2}\|\mathcal{F}'(u)_{|T_u M}\|_* v(u),$$

for all $u \in M$. Using (1.3.2),

$$\langle \mathcal{F}'(u), W(u)\rangle > \|\mathcal{F}'(u)_{|T_u M}\|_*^2, \qquad \|W(u)\| < 2\|\mathcal{F}'(u)_{|T_u M}\|_*.$$

Let us denote by Σ the set of all critical points of $\mathcal{F}$ on M. Since $u \in M \setminus \Sigma$ was arbitrary and $\mathcal{F}'$ is continuous, for every $u \in M \setminus \Sigma$ there exists a radius $\varrho > 0$ and a ball

$$B_\varrho(u) = \{w \in M : \|u - w\|_X \le \varrho\} = \{w \in X(\Omega) : \|u - w\|_X \le \varrho\} \cap M$$

such that

$$\langle \mathcal{F}'(w), W(u)\rangle > \|\mathcal{F}'(u)_{|T_u M}\|_*^2, \qquad \|W(u)\| < 2\|\mathcal{F}'(w)_{|T_w M}\|_*, \tag{1.3.3}$$

for all $w \in B_\varrho(u)$. This defines an open covering $M \setminus \Sigma \subset \bigcup_{u\in M} B_{\varrho_u}(u)$, which can be refined by taking a locally finite one, which we denote by

$$\mathcal{O} = \{B_{\varrho_\imath}(u_\imath) \,:\, \imath \in I\}.$$

We now consider the collection $\mathcal{O}^{sym}$ of the balls

$$B_\imath := B_{\varrho_\imath}(u_\imath), \qquad \text{and} \qquad B_{-\imath} := B_{\varrho_\imath}(-u_\imath), \qquad \imath \in I.$$

This is still a locally finite open convering of $M \setminus \Sigma$. Note that by construction one has

$$u \in B_\imath \quad \Longleftrightarrow \quad -u \in B_{-\imath}, \tag{1.3.4}$$

for all $\imath \in I$.

We now construct a partition of the unit associated with $\mathcal{O}^{sym}$, consisting of locally Lipschitz continuous functions. Let $d_\imath$ (respectively, $d_{-\imath}$) denote the distance (induced by the norm) to the complementary of the ball $B_\imath$ (resp., $B_{-\imath}$). Namely, for every $\imath \in I$,

$$d_\imath(u) = \inf_{\phi\in E\setminus B_{\varrho_\imath}(u_\imath)} \|u - \phi\|,$$

for all $u \in M \setminus \Sigma$, and a similar formula holds for $d_{-\imath}$.

The distance functions to a subset of a metric space are always Lipschitz continuous. Hence the functions defined, for every $\imath \in I$, by setting

$$\kappa_\imath(u) = \frac{d_\imath(u)}{\sum_{\jmath\in I} d_\jmath(u)}, \qquad \text{and} \qquad \kappa_{-\imath}(u) = \frac{d_{-\imath}(u)}{\sum_{\jmath\in I} d_\jmath(u)},$$

for all $u \in M \setminus \Sigma$, are locally Lipschitz continuous, by composition. Indeed, the denominator is always strictly positive, as the open covering is locally finite. Moreover, $\sum_{\imath \in I} \kappa_\imath(u) = 1$. Furthermore, by (1.3.4), for all points $u \in M$ and all indexes $\imath \in I$ one has

$$\kappa_\imath(-u) = \kappa_{-\imath}(u). \tag{1.3.5}$$

We claim that

$$V(u) = \sum_{\imath \in I} \kappa_\imath(u) W(u_\imath), \qquad u \in M \setminus \Sigma,$$

defines an odd locally Lipschitz continuous function.

Indeed, all the sums defining V are finite sums, as $\kappa_{\pm\imath}(u) \neq 0 \Leftrightarrow u \in B_{\pm\imath}$ and u belongs at most to a finite number of the balls. To prove the claim, it is then sufficient to prove that V is odd. To this end, note that

$$V(-u) = \sum_{\imath \in I} \kappa_\imath(-u) W(u_\imath) = \sum_{-u \in B_\imath} \kappa_\imath(-u) W(u_\imath) = \sum_{u \in B_{-\imath}} \kappa_{-\imath}(u) W(u_\imath).$$

The last equality follows by (1.3.4) and (1.3.5). On the other hand W is odd. Thus

$$\sum_{u \in B_{-\imath}} \kappa_{-\imath}(u) W(u_\imath) = -\sum_{u \in B_{-\imath}} \kappa_{-\imath}(u) W(-u_\imath) = -\sum_{\imath \in I} \kappa_{-\imath}(u) W(-u_\imath) = -V(u).$$

and the claim is proved.

Eventually, (1.3.3) entails (1.3.1) by a straightforward computation. □

By means of an odd pseudo–gradient vector field, an odd deformation may be manifactured by taking the flow associated with the corresponding initial value problem. To this aim, we need the following two elementary lemmas.

Lemma 1.3.7. *Let $\Psi : M \to TM$ be a locally Lipschitz continuous vector field such that*

$$\sup_{w \in M} \|\Psi(w)\|_{X(\Omega)} < +\infty. \tag{1.3.6}$$

For every $u \in M$ let $\alpha(u,t)$ denote the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\alpha\big(u,t\big) = \Psi\big(\alpha\big(u,t\big)\big), \\[2ex] \alpha\big(u,0\big) = u. \end{cases} \tag{1.3.7}$$

Then the maximal time

$$\sup\Big\{T > 0 : \alpha(u,t) \text{ is defined for all } t \leq T\Big\} \tag{1.3.8}$$

is equal to $+\infty$ for all $u \in M$, and the function $x \mapsto \alpha(u(x),t)$ belongs to M, for all $t > 0$.

Proof. Let $u \in M$. Arguing by contradiction, assume that the maximal time T_u defined by (1.3.8) is finite. Then, one has

$$\alpha(u,r) - \alpha(u,s) = \int_r^s \frac{d}{dt}\alpha(u,t)\,dt = \int_r^s \Psi(\alpha(u,t))\,dt,$$

for all $0 < r, s < T_u$. Hence, by (1.3.6), there exists a positive constant $C > 0$ such that

$$\big\|\alpha(u,t_j) - \alpha(u,t_k)\big\|_X \leq \int_{t_j}^{t_k} \big\|\Psi\big(\alpha(u,t)\big)\big\|_X \;dt \leq C|t_j - t_k|,$$

for all sequences $(t_m)_{m\in\mathbb{N}} \subset (0, T_u)$. Since all the Cauchy sequences converge in X, it follows that the limit

$$\lim_{t \nearrow T_u} \alpha(u,t)$$

exists in X, let us denote it by v_u. Note that the Cauchy problem

$$\begin{cases} \frac{d}{dt}\beta(v_u,t) = \Psi(v_u,t), \\ \\ \beta(v_u,T_u) = v_u, \end{cases}$$

admits a solution β defined in a neighborhood $(T_u - \varepsilon, T_u + \varepsilon)$ of the initial time T_u. Thus, setting

$$\gamma(u,t) = \begin{cases} \alpha(u,t), & 0 < t \leq T_u, \\ \\ \beta(v_u,t), & T_u < t < T_u + \varepsilon, \end{cases}$$

it turns out that γ is a solution of the initial value problem (1.3.7), contradicting the definition of T_u. □

Lemma 1.3.8. *Let $\Psi : M \to TM$ be an odd locally Lipschitz continuous vector field such that (1.3.6) holds. For all $u \in M$, let $\alpha(u,\cdot) : [0,\infty) \to M$ be the unique solution of the differential equation $d\alpha/dt = \Psi(\alpha)$, with the initial data $\alpha(0) = u$. Then,*

$$\alpha(-u,t) = -\alpha(u,t),$$

for all $t \geq 0$ and all $u \in M$.

Proof. Let us denote $\beta(u,t) = -\alpha(u,t)$. Then the conclusion readily follows by observing that

$$\frac{d\beta}{dt} = -\frac{d\alpha}{dt} = -\Psi(\alpha) = \Psi(-\alpha) = \Psi(\beta),$$

and $\beta(0) = -\alpha(0) = -u$. □

Now we can prove the existence of a deformation.

Proposition 1.3.9. *Let $\lambda, \delta > 0$ be such that*

$$\Big|\mathcal{F}(u) - \lambda\Big| \le 2\delta \quad \Longrightarrow \quad \|\mathcal{F}'(u)_{|T_uM}\|_* \ge \delta, \tag{1.3.9}$$

for all $u \in M$. Then, there exists an odd deformation $\eta \in C(M, M)$ such that

$$\mathcal{F}(u) \le \lambda + \delta \quad \Longrightarrow \quad \mathcal{F}(\eta(u)) \le \lambda - \delta \tag{1.3.10}$$

$$\mathcal{F}(u) \le \lambda - 2\delta \quad \Longrightarrow \quad \eta(u) = u, \tag{1.3.11}$$

for all $u \in M$.

Proof. We adapt the proof of [**8**, Lemma 8.4] from the "flat" to the constrained case. All the details remain the same, but we report the proof for sake of completeness. Set

$$A = \{u \in M : \lambda - \delta \le \mathcal{F}(u) \le \lambda + \delta\},$$

and

$$B = \{u \in M : \mathcal{F}(u) \le \lambda - 2\delta\} \cup \{u \in M : \mathcal{F}(u) \ge \lambda + 2\delta\},$$

and define

$$d_A(u) = \inf_{w \in A} \|u - w\|_X, \quad d_B(u) = \inf_{z \in B} \|u - z\|_X,$$

for all $u \in M$. Note that d_A, d_B are Lipschitz continuous on M, being distance functions from a subset of X. Moreover,

$$d_A(u) + d_B(u) \ge \inf_{\substack{w \in A \\ z \in B}} \|w - z\| > 0, \tag{1.3.12}$$

where the second inequality holds because $\mathcal{F}$ is continuous. Thus, the real-valued function g defined on M by

$$g(u) = \frac{d_B(u)}{d_A(u) + d_B(u)},$$

for all $u \in M$, is also Lipschitz continuous by composition. Indeed, by (1.3.12) the denominator is always greater than a positive constant. Note also that

$$0 \le g(u) \le 1, \quad g \equiv 0 \text{ on } B, \quad \text{and } g \equiv 1 \text{ on } A. \tag{1.3.13}$$

Moreover, since the functional $\mathcal{F}$ is even, the function g is also even, i.e.

$$g(-u) = g(u), \qquad \text{for all } u \in M. \tag{1.3.14}$$

Let Σ denote the set of all critical points of $\mathcal{F}$ along M and let $V : M \setminus \Sigma \to TM$ be a locally Lipschitz pseudo-gradient vector field on M for $\mathcal{F}$, whose existence follows by Lemma 1.3.6. Fix a function $\xi : \mathbb{R} \to \mathbb{R}$ such that $\xi(t) = 1$ for $t \in [0, 1]$, and $\xi(t) = 1/t$ for all $t \ge 1$, and define

$$\Psi(u) = \begin{cases} -g(u)\xi\Big(\|\mathcal{F}'(u)_{|T_uM}\|_*\Big)V(u), & u \in M \setminus \Sigma, \\ \\ 0 & u \in \Sigma, \end{cases} \tag{1.3.15}$$

for all $u \in M \setminus \Sigma$.

By (1.3.9), Σ is contained in B, where $g \equiv 0$ by (1.3.13). Hence, Ψ is a locally Lipschitz continuous. Moreover, note that

$$\sup_{u \in M} \|\Psi(u)\|_X < +\infty. \tag{1.3.16}$$

Indeed, since V is a pseudogradient vector field for $\mathcal{F}$ on M, (1.3.1) holds. Thus

$$\|\Psi(u)\|_X = g(u)\xi(\|\mathcal{F}'(u)_{|T_uM}\|_*)\|V(u)\|_X \leq 2,$$

for all $u \in M$. Furthermore, by (1.3.14) and (1.3.15), it follows that Ψ is odd.

Thus, by Lemma 1.3.7 and Lemma 1.3.8, for every $u \in M$ the initial value problem

$$\begin{cases} \frac{d}{dt}\alpha(u,t) = \Psi(\alpha(u,t)) \\ \\ \alpha(u,0) = u, \end{cases} \tag{1.3.17}$$

admits a unique solution, which we denote by $\alpha(u,t)$, belonging to M and globally defined for all $t \geq 0$. Moreover,

$$\alpha(-u,t) = -\alpha(u,t), \tag{1.3.18}$$

for all $t \geq 0$ and all $u \in M$.

Let $u \in M$. We claim that the function

$$t \mapsto \mathcal{F}\big(\alpha(u,t)\big),$$

is non-increasing. Indeed,

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(\alpha(u,t)) &= \langle \mathcal{F}'(u), \frac{d}{dt}\alpha(u,t)\rangle \\ &= \langle \mathcal{F}'(u), \Psi(\alpha(u,t))\rangle \\ &= -g(\alpha(u,t))\xi\big(\|\mathcal{F}'(\alpha(u,t))\|_*\big)\big\langle \mathcal{F}'(\alpha(u,t)), V(u)\big\rangle, \end{aligned} \tag{1.3.19}$$

for all $t \geq 0$. Recall that V is a pseudo-gradient vector field. Thus,

$$\xi\big(\|\mathcal{F}'(\alpha(u,t))_{|T_{\alpha(u,t)}M}\|_*\big)\big\langle \mathcal{F}'(\alpha(u,t)), V(u)\big\rangle \geq \|\mathcal{F}'(\alpha(u,t))_{|T_{\alpha(u,t)}M}\|_*, \tag{1.3.20}$$

for all $t \geq 0$. Indeed, (1.3.20) follows by the definition of the auxiliary function ξ and the second inequality of (1.3.1). Since $g \geq 0$, (1.3.20) and (1.3.19) imply that

$$\frac{d}{dt}\mathcal{F}\big(\alpha(t,u)\big) \leq 0,$$

for all $t \geq 0$, and the claim is proved.

We now prove that the desidered odd deformation can be obtained by setting

$$\eta(u) = \alpha(u, 2/\delta), \qquad u \in M.$$

The fact that η is odd is a consequence of (1.3.18), and one is left to prove that both (1.3.10) and (1.3.11) hold. To do so, let $u \in M$ be fixed.

First, let us prove that (1.3.10) holds. Arguing by contradiction, assume that there exists $u \in M$ such that $\mathcal{F}(u) \leq \lambda + \delta$ and $\mathcal{F}(\eta(u)) > \lambda - \delta$. By the above claim, it follows that

$$\alpha(u,t) \in A, \qquad \text{for all } t \in [0, 2/\delta]. \tag{1.3.21}$$

Therefore, by (1.3.13), we have that $g(\alpha(u,t)) = 1$, for all $t \in [0, 2/\delta]$. Thus, by (1.3.9),

$$\begin{aligned}
\mathcal{F}(\eta(u)) - \mathcal{F}(u) &= \int_0^{2/\delta} \frac{d}{dt}\mathcal{F}\big(\alpha(u,t)\big)\,dt \\
&= \int_0^{2/\delta} \big\langle \mathcal{F}'\big(\alpha(u,t),\big), \frac{d}{dt}\alpha(u,t)\big\rangle\,dt = \int_0^{2/\delta} \big\langle \mathcal{F}'\big(\alpha(u,t)\big), \Psi\big(\alpha(u,t)\big)\big\rangle\,dt \\
&= -\int_0^{2/\delta} g(\alpha(u,t))\xi\big(\|\mathcal{F}'(\alpha(u,t))_{|T_{\alpha(u,t)}M}\|_*\big)\big\langle \mathcal{F}'\big(\alpha(u,t)\big), V(\alpha(u,t))\big\rangle\,dt \\
&= -\int_0^{2/\delta} \xi\big(\|\mathcal{F}'(\alpha(u,t))_{|T_{\alpha(u,t)}M}\|_*\big)\big\langle \mathcal{F}'\big(\alpha(u,t)\big), V(\alpha(u,t))\big\rangle\,dt \\
&\leq -\int_0^{2/\delta} \|\mathcal{F}'(\alpha(u,t))_{|T_{\alpha(u,t)}M}\|_*\,dt \leq -\frac{2}{\delta}\,\delta^2 = -2\delta,
\end{aligned}$$

whence

$$\mathcal{F}\big(\eta(u)\big) \leq \mathcal{F}\big(u\big) - 2\delta \leq \lambda + \delta - 2\delta = \lambda - \delta,$$

that is a contradiction. Since $u \in M$ was arbitrary, (1.3.10) is proved.

In order to prove (1.3.11), assume that $u \in M$ is such that $\mathcal{F}(u) \leq \lambda - 2\delta$. Recall that the function $t \mapsto \mathcal{F}(\alpha(u,t))$ is non-increasing. Thus,

$$\alpha(u,t) \in B, \qquad \text{for all } t \in [0, 2/\delta].$$

Then, by (1.3.13) one has that $g(\alpha(u,t)) \equiv 0$. Hence,

$$\mathcal{F}(\eta(u)) = \mathcal{F}(u) + \int_0^{2/\delta} \big\langle \mathcal{F}'\big(\alpha(u,t)\big), \Psi\big(\alpha(u,t)\big)\big\rangle\,dt = \mathcal{F}(u).$$

Since $u \in M$ was arbitrary, (1.3.11) follows. □

3.3. Proof of Theorem 1.3.3. Using the odd and continuous deformation provided by Proposition 1.3.9, the following theorem can be proved arguing by contradiction. Then Theorem 1.3.3 plainly follows by the definition of Palais-Smale condition.

Theorem 1.3.10. *Let X be a uniformly convex Banach space, $\mathcal{F}, \mathcal{G}$ be two even and positively homogeneous C^1 functionals of degreee $p > 1$ on X. For all $n \in \mathbb{N}$, denote by V_n the set of all odd and continuous mappings $\omega \mapsto f_\omega$ from $\mathbb{S}^{n-1}$ to $M = \mathcal{G}^{-1}(\{1\})$ and set*

$$\lambda_n = \inf_{f \in V_n} \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_\omega).$$

Then, there exist a sequence $\{u_k\}_{k\in\mathbb{N}} \subset M$ such that

$$\mathcal{F}(u_k) \to \lambda_n, \qquad \|\mathcal{F}'(u_k)\big|_{T_u M}\|_* \to 0, \tag{1.3.22}$$

as $k \to \infty$.

Proof. The antithesis is the existence of a positive number δ bounding from below the cotangent norm

$$\left\| \mathcal{F}'(u)\big|_{T_u M} \right\|_* \geq \delta,$$

for all $u \in M$ such that

$$|\mathcal{F}(u) - \lambda_n| \leq 2\delta.$$

Hence, by Proposition 1.3.9, there exists an odd deformation $\eta \in C(M, M)$ such that

$$\mathcal{F}(u) \leq \lambda_n + \delta \quad \Longrightarrow \quad \mathcal{F}(\eta(u)) \leq \lambda_n - \delta, \tag{1.3.23}$$

for all $u \in M$.

The number λ_n is defined as an infimum among connected and symmetric n-paths on M. Hence there exists a sequence $\{f_k\}_{k \in \mathbb{N}}$ of odd and continuous mappings from the unit sphere $\mathbb{S}^{n-1}$ to M such that

$$0 \leq \mathsf{F}_n(f_k) - \lambda_n \leq 2^{-k}\delta, \tag{1.3.24}$$

for all $k \in \mathbb{N}$, where

$$\mathsf{F}_n(f_k) = \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_k(\omega)), \qquad k \in \mathbb{N}. \tag{1.3.25}$$

Let $k \in \mathbb{N}$, and ω_k be a unit vector in $\mathbb{R}^n$ realizing the maximum in (1.3.25). Now, on the one hand, if $\nu \in \mathbb{S}^{n-1}$, then

$$\mathcal{F}(f_k(\nu)) \leq \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_k(\omega)) = \mathsf{F}_n(f_k) \leq \lambda_n + 2^{-k}\delta \leq \lambda_n + \delta.$$

Thus by (1.3.23)

$$\mathcal{F}(\eta(f_k(\nu))) \leq \lambda_n - \delta.$$

Therefore by taking the maximum among all unit vectors $\nu \in \mathbb{S}^{n-1}$

$$\mathsf{F}_n(\eta \circ f_k) = \max_{\nu \in \mathbb{S}^{n-1}} \mathcal{F}(\eta(f_k(\nu))) \leq \lambda_n - \delta.$$

On the other hand, since $\eta : M \to M$ is odd and continuous, the composite function $g_k = \eta \circ f_k$ is an odd and continuous mapping from $\mathbb{S}^{n-1}$ to M, hence an admissible competitor for the infimum defining λ_n and one has

$$\mathsf{F}_n(\eta \circ f_k) = \mathsf{F}_n(g_k) \geq \inf_{g \in V_n} \mathsf{F}_n(g) = \lambda_n,$$

a contradiction. $\square$

Theorem 1.3.3 plainly follows by the last theorem due to the definition of the Palais-Smale condition.

3.4. PS sequences and convex energies. In the following, we need the next two technical lemmas. The first is a generalized Hölder inequality for convex homogeneous functionals. The second is a sufficient condition for the convergence of convex and weakly lower semicontinuous energies.

Lemma 1.3.11. *Let $u, v \in X$. Let $\mathcal{G}$ be a Gâteaux differentiable, even convex and positively p-homogeneous functional on a normed space X. Then*

$$|\langle \mathcal{G}'(u), v\rangle| \leq p\mathcal{G}(u)^{\frac{p-1}{p}} \mathcal{G}(v)^{\frac{1}{p}},$$

for all $v \in X$.

Proof. Since $\mathcal{G}$ is positively homogeneous of degree p, one has

$$\langle \mathcal{G}'(u), u\rangle = p\mathcal{G}(u).$$

The functional $\mathcal{G}$ is Gâteaux differentiable at $u \in X$. Thus

$$\langle D\mathcal{G}(u), v\rangle = \frac{\mathcal{G}(u + t(v-u)) - \mathcal{G}(u)}{t} + o(1) + p\mathcal{J}(u)$$

as $t \to 0^+$. But the convexity implies that

$$\mathcal{G}(u + t(v-u)) - \mathcal{G}(u) \leq t\Big(\mathcal{G}(v) - \mathcal{G}(u)\Big),$$

for all $t \in [0, 1]$. Thus sending $t \to 0^+$ gives

$$\langle D\mathcal{G}(u), v\rangle - \mathcal{G}(v) \leq (p-1)\mathcal{G}(u).$$

Note that the first summand in the left hand side is positively homogeneous of degree 1, with respect to the variable v, whereas the second one is homogeneous of degree p. Thus, one has

$$\langle D\mathcal{G}(u), v\rangle\, s - \mathcal{G}(v)\, s^p \leq (p-1)\mathcal{G}(u),$$

for all $s > 0$. Hence by elementary optimization

$$\langle \mathcal{G}'(u), v\rangle \leq p\mathcal{G}(u)^{\frac{p-1}{p}} \mathcal{G}(v)^{\frac{1}{p}}$$

for all $u, v \in X$. Since the functional is even, by possibly replacing v by $-v$, the thesis follows. $\square$

Lemma 1.3.12. *Let $\mathcal{F}$ be a convex and weakly lower semicontinuous functional on X. If $u_n \rightharpoonup u$ weakly in X and*

$$\lim_{n\to\infty} \langle \mathcal{F}'(u_n), u - u_n\rangle = 0$$

then $\mathcal{F}(u_n) \to \mathcal{F}(u)$ as $n \to \infty$.

Proof. The proof is one line. By convexity one has

$$\mathcal{F}(u) \geq \limsup_{n\to\infty} \left(\mathcal{F}(u_n) + \langle \mathcal{F}'(u_n), u - u_n\rangle\right) = \limsup_{n\to\infty} \mathcal{F}(u_n).$$

On the other hand by the weak lower semicontinuity

$$\mathcal{F}(u) \leq \liminf_{n\to\infty} \mathcal{F}(u_n).$$

Thus $\mathcal{F}(u_n) \to \mathcal{F}(u)$. $\square$

Lemma 1.3.13. *Let $\mathcal{F}, \mathcal{G}$ be even, convex C^1 functionals on a uniformly convex Banach space X which are positively homogeneous of degree $p > 1$, and $M = \mathcal{G}^{-1}(\{1\})$. Assume that $\mathcal{G}$ is compact and $\mathcal{F}$ is coercive. Let $\{u_n\}_{n\in\mathbb{N}} \subset M$ be a sequence such that*

$$\lim_{n\to\infty} \mathcal{F}(u_n) = \lambda, \tag{1.3.26}$$

$$\lim_{n\to\infty} \|\mathcal{F}'(u_n)_{|T_{u_n}M}\|_* = 0. \tag{1.3.27}$$

Then by possibly passing to a subsequence

$$\lim_{n\to\infty} \big\langle \mathcal{F}'(u_n), u - u_n \big\rangle = 0. \tag{1.3.28}$$

Proof. Since $\mathcal{F}$ is coercive, the sequence u_n is bounded in X. By reflexivity, up to relabelling there exists a weak limit $u \in X$. Then, the sequence of numbers

$$\delta_n = \frac{1}{p} \langle \mathcal{G}'(u_n), u_n - u\rangle$$

tends to zero as $n \to \infty$. Indeed, $\mathcal{G}(u - u_n) \to 0$, since $u_n - u \rightharpoonup 0$, and Lemma 1.3.11 implies

$$\big|\langle \mathcal{G}'(u_n), u_n - u\rangle\big| \leq p\, \mathcal{G}(u - u_n)^{\frac{1}{p}}. \tag{1.3.29}$$

Note that

$$P_{u_n}(v) = v - \frac{\big\langle \mathcal{G}'(u_n), v\big\rangle}{p}\, u_n,$$

defines an element of the tangent space $T_{u_n}M$ to M at its point u_n. Moreover,

$$P_{u_n}(u - u_n) = u - (1 - \delta_n)u_n.$$

Up to subsequences, inequality

$$\big|\big\langle \mathcal{F}'(u_n), \varphi\big\rangle\big| \leq 2^{-n}\|\varphi\|_X, \qquad \text{for all } \varphi \in T_{u_n}M,$$

follows by (1.3.27). Plugging $\varphi = P_{u_n}(u - u_n)$ in yields

$$\big|\big\langle \mathcal{F}'(u_n), u - (1 - \delta_n)u_n\big\rangle\big| \leq C\, 2^{-n}.$$

The constant $C > 0$ is independent of $n \in \mathbb{N}$. By sending $n \to \infty$ one gets (1.3.28). $\square$

Remark 1.3.14. If M is a C^1 level set of some convex homogeneous compact functional $\mathcal{G}$, then convex homogeneous coercive energies $\mathcal{F}$ are weakly continuous along the Palais-Smale sequences $\{u_n\}_{n\in\mathbb{N}}$ on M. Indeed, by Lemma 1.3.12 condition (1.3.28) implies the convergence $\mathcal{F}(u_n) \to \mathcal{F}(u)$ as $n \to \infty$. Moreover, along Palais-Smale sequences the differentials $\mathcal{F}'$ are strongly monotone in the following sense.

Theorem 1.3.15. *Let $\mathcal{F}, \mathcal{G}$ be even, convex C^1 functionals on a uniformly convex Banach space X which are positively homogeneous of degree $p > 1$, and $M = \mathcal{G}^{-1}(\{1\})$. Assume that $\mathcal{G}$ is compact and $\mathcal{F}$ is coercive. Then*

$$\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), u_n - u \rangle \to 0$$

for all a Palais-Smale sequences $\{u_n\}_{n\in\mathbb{N}}$ on M.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Palais-Smale sequence in M. Since $\mathcal{F}$ is coercive and it is bounded on $\{u_n\}_{n\in\mathbb{N}}$ by definition, up to relabelling we may assume that the sequence converges weakly to some limit $u \in M$. Then

$$\langle \mathcal{F}'(u), u - u_n \rangle \to 0.$$

By Lemma 1.3.13, one also has $\langle \mathcal{F}'(u_n), u - u_n \rangle \to 0$. Subtracting concludes the proof. □

3.5. Comments on the variational eigenvalues. The min-max formula using odd and continuous mappings defined on unit sphere seems to have been introduced in the paper [**37**] (see also [**30**]). There exists another one, that relies on sophisticated topological index theories involving the notion of Krasnoselskii genus (see Remark 1.3.16 below). In that case, the infimum is taken among the objects having a prescribed genus, cf. equation (1.3.30). At variance with that, the admissible competitors for the infimum defining the λ_n's are "parametric objects", i.e. odd and continuous images of $\mathbb{S}^{n-1}$. They can be seen as symmetric, connected and compact "n-paths" along the "symmetric landscape" given by the graph of the even functional $\mathcal{F}$ on M.

Remark 1.3.16. For reader's convenience, we recall that the Krasnoselskii genus of a compact, nonempty and symmetric subset $A \subset X$ of a Banach space is defined by

$$\gamma(A) = \inf \left\{ n \in \mathbb{N} \,:\, \exists \text{ a continuous odd mapp } f : A \to \mathbb{S}^{n-1} \right\},$$

with the convention that $\gamma(A) = +\infty$, if no such an integer n exists. Using the Krasnoselskii genus, an infinite sequence of critical values of $\mathcal{F}$ is usually produced as follows (see [**50, 87**])

$$\widetilde{\lambda}_n = \inf_{\gamma(A)\geq n} \max_{u\in A} \frac{\mathcal{F}(u)}{\mathcal{G}(u)}, \qquad k \in \mathbb{N}. \tag{1.3.30}$$

It seems to be an interesting open problem to establish whether or not the two minimax procedures actually give the same sets of values. It is known (see [**37**] and the reference therein) that $\widetilde{\lambda}_n \leq \lambda_n$, for all $n \in \mathbb{N}$. So far, equality is known to hold only for $n \in \{1, 2\}$.

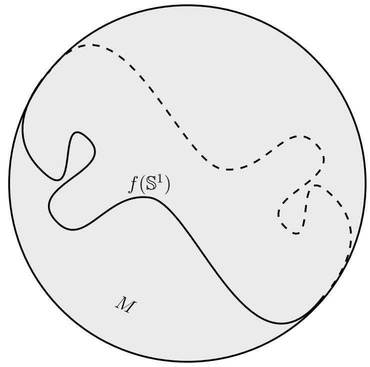

FIGURE 1. How a path would look like if M was 2-dimensional

Remark 1.3.17. For example, consider (6.2.3) in the case $n = 1$. Any continuous odd mapping f from $\mathbb{S}^0 \cong \{\pm 1\}$ to M can be identified with the choice of an antipodal pair $u_f, -u_f$ on the symmetric manifold M and the functional $\mathcal{F}$ is even, thus the infimum of

$$\mathsf{F}_1(f) = \max\{\mathcal{F}(u_f), \mathcal{F}(-u_f)\} = \mathcal{F}(u_f),$$

among all the admissible pairs $f = \{u_f, -u_f\} \subset M$ is in fact the minimum of the Rayleigh quotients.

In second place, in order to compute (6.2.3) when $n = 2$, one should minimize the quantity

$$\mathsf{F}_2(f) = \max_{\omega \in \mathbb{S}^1} \mathcal{F}(f_\omega),$$

among all odd and continuous mappings from the unit circle to M, compare with Figure 1.

In general, λ_n is obtained via minimization of the quantity

$$\mathsf{F}_n(f) = \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_\omega), \tag{1.3.31}$$

upon the class V_n of admissible n-paths.

4. Existence of eigenvalues for variational integrals

Let Ω be an open set having finite N-dimensional Lebesgue measure. We apply Theorem 1.3.3 to the case of some functionals defined on $W^{1,p}(\Omega)$ by

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u(x), \nabla u(x))\, dx$$

and

$$\mathcal{G}(u, \Omega) = \int_\Omega G(x, u(x))\, dx.$$

In order to make sure that the min-max formula of Theorem 1.3.3 applies, some assumptions on the the Lagrangians F, G are needed so that $\mathcal{F}$ satisfies the Palais-Smale condition on $M = \mathcal{G}^{-1}(\{1\})$. Namely,

$$F(x,u,z) = H(x,\nabla u) + b(x)|u|^p, \qquad G(x,u) = \rho(x)|u|^p,$$

where $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that

$$z \mapsto H(x,z) \quad \text{is } C^1, \text{ convex, even and positively homogeneous of degree } p > 1, \tag{1.4.1}$$

and $0 < c_1 < b(x), \rho(x) < c_2 < \infty$ are measurable functions. Assume also that the *growth conditions*

$$c_1(H)|z|^p \le H(x,z) \le c_2(H)|z|^p, \tag{1.4.2}$$

hold for all $(x,z) \in \Omega \times \mathbb{R}^N$. Moreover, suppose that such that

(1.4.3)
$$\lim_{n\to\infty} \int_\Omega \big\langle \nabla_z H(x,\nabla u_n) - \nabla_z H(x,\nabla u), \nabla u_n - \nabla u \big\rangle \, dx = 0 \Rightarrow \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p \, dx = 0.$$

for all $x \in \Omega$ and all sequences $\{u_n\}_{n\in\mathbb{N}}$ in M.

Theorem 1.4.1. *Let $X(\Omega)$ be either $W_0^{1,p}(\Omega)$ or*[1] *$W^{1,p}(\Omega)$. Assume that the structure conditions* (1.4.1), (1.4.2) *and* (1.4.3) *hold. For every $n \in \mathbb{N}$ define*

$$\lambda_n(\Omega) = \inf_{f \in \mathscr{C}_n} \max_{\omega \in \mathbb{S}^{n-1}} \mathcal{F}(f_\omega, \Omega) \tag{1.4.4}$$

where $\mathscr{C}_n$ denotes the class of all odd and continuous mappings from $\mathbb{S}^{n-1}$ to the C^1 one-codimensional manifold $M = \mathcal{G}^{-1}(\{1\})$ of $X(\Omega)$. Then each $\lambda_n(\Omega)$ is an eigenvalue of the pair $(\mathcal{F}, \mathcal{G})$. Moreover,

$$0 \le \lambda_1(\Omega) \le \lambda_2(\Omega) \le \ldots \le \lambda_n(\Omega) \le \ldots$$

and $\lambda_n(\Omega) \to +\infty$ as $n \to \infty$.

Proof. The functionals $\mathcal{F}, \mathcal{G}$ are convex, even and positively homogeneous of degree $p > 1$. By Theorem 1.3.3, it is enough to prove that $\mathcal{F}$ satisfies the Palais-Smale condition on the manifold

$$M = \left\{ u \in X(\Omega) : \int_\Omega \rho(x)|u(x)|^p \, dx = 1 \right\}.$$

To this aim, we use the structure assumptions. The growth conditions (1.4.2) in particular imply that $\mathcal{F}$ is coercive. Hence every Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $X(\Omega)$ and admits a weakly converging subsequence $\{u_{n_\nu}\}_{\nu\in\mathbb{N}}$. Since Ω has finite N-dimensional

[1]In the second case, assume also Ω has a Lipschitz boundary.

Lebesgue measure the embedding of $X(\Omega)$ into $L^p(\Omega)$ is compact[2]. Then $\mathcal{G}$ is compact. Thus by Theorem 1.3.15 the quantity

$$\int_\Omega \big\langle \nabla_z H(\nabla u_{n_\nu}) - \nabla_z H(\nabla u), \nabla u_{n_\nu} - \nabla u \big\rangle \, dx + \int_\Omega b(x)\big(|u_{n_\nu}|^{p-2}u_{n_\nu} - |u|^{p-2}u\big)(u_n - u)\, dx$$

goes to zero as $n \to \infty$. That implies the strong convergence of the sequence u_{n_ν}. Indeed, by (1.4.3)

$$\lim_{\nu\to\infty} \int_\Omega |\nabla u_{n_\nu} - \nabla u|^p \, dx = 0,$$

and

$$\lim_{\nu\to\infty} \int_\Omega |u_{n_\nu} - u|^p \, dx = 0$$

by Proposition A.3.2. We divide the rest of the proof in steps.

The sequence is non-decreasing. Let $n \in \mathbb{N}$ and $f : \mathbb{S}^{n-1} \to M$ be a an odd continuous mapping. Then, let E be an n-dimensional vector subspace of $\mathbb{R}^{n+1}$ and consider the restriction g_E of f to the intersection $\mathbb{S}^n \cap E \cong \mathbb{S}^{n-1}$. One has

$$\begin{aligned}
\max_{u\in f(\mathbb{S}^n)} \int_\Omega F(x,u,\nabla u)\, dx &\geq \max_{u\in f(\mathbb{S}^n\cap E)} \int_\Omega F(x,u,\nabla u)\, dx \\
&= \max_{u\in g_E(\mathbb{S}^{n-1})} \int_\Omega F(x,u,\nabla u)\, dx \\
&\geq \inf_{g\in C_o(\mathbb{S}^{n-1};M)} \max_{u\in g(\mathbb{S}^{n-1})} \int_\Omega F(x,u,\nabla u)\, dx = \lambda_n(\Omega).
\end{aligned}$$

Since f was arbitrary in $\mathscr{C}_{n+1}$, passing to the infimum among all $f \in \mathscr{C}_{n+1}$ yields

$$\lambda_{n+1}(\Omega) \geq \lambda_n(\Omega).$$

The sequence is unbounded. To prove of this fact given below uses the argument of [**50**, Proposition 5.4] (for a different proof, avoiding the use of Schauder bases, one could adapt the argument of [**F1**, Theorem 5.2]).

Recall that the $X(\Omega)$ is denoting either $W^{1,p}(\Omega)$ or its closed vector subspace $W_0^{1,p}(\Omega)$. The Banach space $X(\Omega)$ admits a Schauder basis (see [**47, 74**]). Namely, there exists an ordered countable set of elements $\{e_n\}_{n\in\mathbb{N}} \subset X(\Omega)$ with the property that for all $u \in X(\Omega)$, we have

$$u = \sum_{j=1}^\infty \alpha_j \, e_j$$

for a (uniquely determined) sequence of scalars $\{\alpha_j\}_{j\in\mathbb{N}}$. Here the converge of the series above has to be understood in the sense of the norm topology. Denote by

$$E_n = \mathrm{Vect}(\{e_1, \dots, e_n\}),$$

[2]Here it is where the smoothness assumption on the boundary is necessary, if $X(\Omega)$ is denoting $W^{1,p}(\Omega)$.

the linear envelope of the first n elements of the basis. Then it is clear that the union $\bigcup_{n\in\mathbb{N}} E_n$ is dense in $X(\Omega)$. Set also

$$F_n = \overline{\mathrm{Vect}(\{e_k\}_{k>n})},$$

which is the topological supplement of the finite-dimensional vector space E_n, and define the new sequence

$$\mu_n(\Omega) = \inf_{f\in\mathscr{C}_n} \max_{u\in f(\mathbb{S}^{n-1})\cap F_{n-1}} \int_\Omega F(x,u,\nabla u)\,dx, \qquad n\in\mathbb{N}.$$

At first, we verify that such a sequence is actually well defined. Indeed, let f be an odd and continuous map from the unit sphere $\mathbb{S}^{n-1}$ to M and assume that the intersection $f(\mathbb{S}^{n-1})\cap F_{n-1}$ is empty: this implies that for every $\omega\in\mathbb{S}^{n-1}$, the element $f(\omega)$ always has at least a nontrivial component on E_{n-1}. By composing f with the continuous odd operator

$$P_{n-1} : X(\Omega) \to E_{n-1},$$

given by the natural projection on the linear space E_{n-1}, the map $P_{n-1}\circ f$ is odd, continuous and $P_{n-1}\circ f(\omega)\neq 0$, for every $\omega\in\mathbb{S}^{n-1}$. That is, we constructed an odd continuous map from $\mathbb{S}^{n-1}$ to $E_{n-1}\setminus\{0\}\simeq\mathbb{R}^{n-1}\setminus\{0\}$. That is in contradiction with Borsuk-Ulam theorem[3]. Hence the image of any $f\in\mathscr{C}_n$ has to intersect F_{n-1}, for every $n\in\mathbb{N}$.

Obviously $\mu_n(\Omega)\le\lambda_n(\Omega)$. Therefore it sufficies now to show that

$$\lim_{k\to\infty}\mu_n(\Omega) = +\infty.$$

At this aim, assume by contradiction that $\mu_n(\Omega)<\mu$, for all $n\in\mathbb{N}$. Then, for every $n\in\mathbb{N}$, we can take a mapping $f\in C_n$ and $u_n\in f\big(\mathbb{S}^{n-1}\big)\cap F_{n-1}$ such that

$$\int_\Omega F(x,u_n,\nabla u_n)\,dx < \mu. \tag{1.4.5}$$

Since $u_n\in M$ for all $n\in\mathbb{N}$, equation (1.4.5) implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $X(\Omega)$ and weakly converges (up to a subsequence) to some limit function $u\in M$.

For every $k\in\mathbb{N}$, consider the functional ϕ_k defined on $X(\Omega)$ by

$$\phi_k(u)=\alpha_k, \qquad \text{if} \quad u=\sum_{j=1}^{+\infty}\alpha_j e_j\in X(\Omega).$$

By definition of Schauder basis such functionals are linear and they also turn out to be continuous, cf. [**10**, page 83]. Thus the weak convergence of the sequence $\{u_n\}_{n\in\mathbb{N}}$ to u implies that $\lim_{n\to\infty}\langle\phi_k,u_n\rangle=\langle\phi_k,u\rangle$, for all $k\in\mathbb{N}$. Since $u_n\in F_{n-1}$, we have that

$$\phi_k(u_n)=0, \qquad \text{for every } k\le n-1.$$

[3]We recall that the Borsuk-Ulam states the following:

"*for every continuous map $f:\mathbb{S}^n\to\mathbb{R}^n$, there exists $x_0\in\mathbb{S}^n$ such that $f(x_0)=f(-x_0)$*".

Since our function $P_{n-1}\circ f$ is odd, this would give that $0\in\mathrm{Im}(P_{n-1}\circ f)$, that is a contradiction.

Thus $\phi_k(u) = 0$ for all $k \in \mathbb{N}$. This means that $u = 0$, contradicting the fact that $u \in M$. $\square$

Remark 1.4.2. Let Ω be a bounded Lipschitz open set. A close inspection shows that the above proof can be repeated verbatim in the case when $X(\Omega) = W^{1,p}(\Omega)$ and

$$\mathcal{G}(u, \Omega) = \int_{\partial\Omega} \rho(x)|u(x)|^p d\mathcal{H}^{N-1}$$

for all $u \in W^{1,p}(\Omega)$ and $0 < c_1 \leq \rho(x) \leq c_2 < +\infty$ is a measurable function. The integral has to be understood in the sense of traces.

Note that the Lagrangian function H has in particular to satisfy the strong convexity condition (1.4.3). Owing to the elementary inequalites of the Appendix, one can to apply the formula to produce eigenvalues of the two model operators: the p-Laplacian and the pseudo p-Laplacian.

Corollary 1.4.3. *Let $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$ and $1 < p < \infty$. Let $\|\cdot\|$ denote either the euclidean norm or the ℓ^p norm in $\mathbb{R}^N$. Then there exists a non-decreasing unbounded sequence of eigenvalues for the Rayleigh quotient*

$$\frac{\displaystyle\int_\Omega \|\nabla u(x)\|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}, \qquad u \in W_0^{1,p}(\Omega).$$

CHAPTER 2

Classical elliptic regularity for eigenfunctions

Throughout this chapter there is no claim of originality. The results are extremely classical. Yet, for sake of completeness it is worth to perform some explicit computations nonetheless. Some well known tools of the classical elliptic regularity are used to give an explicit bound for the eigenfunctions.

1. L^∞ bounds

In this first section some explicit L^∞ bounds are provided for the eigenfunctions related to the variational integrals

$$\mathcal{F}(u,\Omega) := \int_\Omega H(x,\nabla u)\,dx$$

subject to the constraint

$$\mathcal{G}(u,\Omega) := \int_\Omega |u|^q\,dx = 1.$$

Here Ω is an open set of finite Lebesgue measure in $\mathbb{R}^N$, $1 < p < N$, $1 < q < p^* := Np/(N-p)$ and H is some convex and p-homogeneous C^1 function. The p-growth conditions

$$C_1(H)|z|^p \le H(x,z) \le C_2(H)|z|^p, \qquad z \in \mathbb{R}^N, \tag{2.1.6}$$

are assumed to be valid for all $(x,z) \in \Omega \times \mathbb{R}^N$ with two suitable constants $C_1, C_2 > 0$.

Unless $q = p$, the problem is sligthly different from the ones adressed in the thesis. On the other hand, even if $q \neq p$ the scaling invariance of the Rayleigh quotient

$$\frac{\displaystyle\int_\Omega H(x,\nabla u)\,dx}{\left(\displaystyle\int_\Omega |u(x)|^q\,dx\right)^{\frac{p}{q}}} \tag{2.1.7}$$

holds true and its critical levels have many features in common with the eigenvalues of the corresponding problem with $q = p$. Minimizers and other stationary points of the quotient satisfy the Euler-Lagrange equation

$$\frac{1}{p}\int_\Omega \big\langle \nabla_z H(\nabla u)\,,\,\nabla\varphi\big\rangle\,dx = \lambda\|u\|^{p-q}_{L^q(\Omega)}\int_\Omega |u|^{q-2}u\,\varphi\,dx \tag{2.1.8}$$

for all $\varphi \in W^{1,p}_0(\Omega)$. Note that the problem is non-local if $q \neq p$.

As a model case is obtained by the choice

$$H(z) = \|z\|, \qquad z \in \mathbb{R}^N, \tag{2.1.9}$$

where the symbol $\|\cdot\|$ is denoting a general norm associated with some convex body K in $\mathbb{R}^N$ (see Chapter 7). For further details the reader is referred to [**F1**], where the corresponding eigenvalue problem was carefully discussed. Namely, for any critical point u of the Rayleigh quotient the equation[1]

(2.1.10)
$$\int_\Omega \|\nabla u(x)\|^{p-1} \left\langle \frac{\nu_K\left(\frac{\nabla u(x)}{\|\nabla u(x)\|}\right)}{\left\|\nu_K\left(\frac{\nabla u(x)}{\|\nabla u(x)\|}\right)\right\|_*}, \nabla\varphi(x)\right\rangle dx = \lambda \|u\|_{L^q(\Omega)}^{p-q} \int_\Omega |u(x)|^{q-2} u(x)\varphi(x)\, dx,$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$. Here ν_K denotes the outward pointing unit normal to ∂K and $\|\cdot\|_*$ stands for the support function of the convex body K, cf. Chapter 7.

Theorem 2.1.1. *Let Ω be an open set of finite measure in $\mathbb{R}^N$, $1 < p < \infty$, $1 < q < p^*$ and $H \colon \mathbb{R}^N \to \mathbb{R}_+$ be a convex p-homogeneous C^1 function. Let $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)$ be a solution of the Euler-Lagrange equation* (2.1.8)*. Then, there exists a positive constant M, independent of u, λ, Ω, such that*

$$\|u\|_{L^\infty(\Omega)} \le M \lambda^{1/\delta p} \|u\|_{L^1(\Omega)}, \tag{2.1.11}$$

where $\delta = 1/N$ if $q \le p$, and $\delta = 1/q - 1/p + 1/N$ otherwise.

Remark 2.1.2. There exists a constant $c > 0$, such that

$$\|w\|_{L^q(\Omega)} \le c |A_w|^\gamma \|\nabla w\|_{L^p(\Omega)}, \tag{2.1.12}$$

for all $w \in W_0^{1,p}(\Omega)$, where $A_w = \{x \in \Omega : w(x) \neq 0\}$. Here

$$\gamma = \begin{cases} 1/N, & \text{if } \ q \le p, \\ 1/q - 1/p + 1/N, & \text{if } \ q > p. \end{cases} \tag{2.1.13}$$

For istance, one may take

$$c = \begin{cases} (q - q/p + 1)^{-1/q}, & \text{if } \ N = 1, \\ p(N-1)/(N-p), & \text{if } \ 1 \le p < N, \\ q(N-1)/N, & \text{if } \ p > N. \end{cases} \tag{2.1.14}$$

This is a consequence of the well-known Sobolev embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ and the Hölder inequality, that makes the volume term appear. Allthough in the case $1 < p < N$ the constant c is not the sharp constant of the Sobolev inequality, the explicit value of c has however no influence in the proof.

[1] To compute the Euler-Lagrange equation, formula (7.3.4) in Chapter 7 is helpful.

PROOF OF THEOREM 2.1.1. There is no restriction assuming (2.1.9), since the argument is the same as for a more general H. Then, let u be a solution of equation (2.1.10). We first prove the quantitative bound (2.1.11).

To this aim, we assume without any loss of generality that $u \geq 0$. Since the the purpose is to prove the validity of the homogeneous estimate (2.1.11), one can also assume that

$$\int_\Omega |u(x)|^q \, dx = 1. \tag{2.1.15}$$

Indeed, the general case follows by a simple scaling argument.

Since the first variation of the Rayleigh quotients has to vanish at the critical point u, it follows that equation (2.1.10) holds for all $\varphi \in W_0^{1,p}(\Omega)$, where ν_K denotes the outward pointing normal at the boundary of the convex body K and $\|\cdot\|_*$ stands for the support function associated with K. Here we used that $u(x) \geq 0$ almost everywhere in Ω. Note that also the term $\|u\|_{L^q(\Omega)}^{p-q}$ was ruled out via the normalization condition (2.1.15). Let $k > 1$ and plug $\varphi = (u-k)_+$ in as a test function. Then

$$\int_{A_k} \|\nabla u(x)\|^p \, dx = \lambda \int_{A_k} u(x)^{q-1}\Big(u(x) - k\Big)\, dx, \tag{2.1.16}$$

where we set

$$A_k = \{x \in \Omega : u(x) > k\}.$$

Note that

$$k|A_k| \leq \|u\|_{L^1(\Omega)}, \tag{2.1.17}$$

for all $k > 1$. Let us consider the nonnegative function defined by

$$f(k) = \int_{A_k} (u-k)\, dx = \int_k^{+\infty} |A_t|\, dt,$$

for all $k > 1$, and set

$$\varepsilon = \begin{cases} p\gamma/(p-1), & \text{if } q \leq p, \\ \\ p\gamma/(q-1), & \text{if } q > p. \end{cases} \tag{2.1.18}$$

We claim that there exists a constant $\varkappa = C\lambda^{\varepsilon/\gamma p}$, with C independent of λ, such that

$$f(k) \leq \varkappa^\varepsilon k(-f'(k))^{1+\varepsilon} \tag{2.1.19}$$

holds for all numbers k larger than or equal to

$$k_0 = \varkappa \|u\|_{L^1(\Omega)}. \tag{2.1.20}$$

Indeed, by separating variables and integrating, by (2.1.19) one gets

$$k^{\frac{\varepsilon}{1+\varepsilon}} - k_0^{\frac{\varepsilon}{1+\varepsilon}} \leq \varkappa^{\frac{\varepsilon}{1+\varepsilon}}\Big(f(k_0)^{\frac{\varepsilon}{1+\varepsilon}} - f(k)^{\frac{\varepsilon}{1+\varepsilon}}\Big),$$

for all $k > k_0$ such that $f(k) > 0$. By using (2.1.19) again, this implies that

$$k^{\frac{\varepsilon}{1+\varepsilon}} \leq k_0^{\frac{\varepsilon}{1+\varepsilon}}(1 + \varkappa^\varepsilon |A_{k_0}|^\varepsilon),$$

for all $k > k_0$ such that $f(k) > 0$. The rough estimate (2.1.17) combined with (2.1.20) gives $|A_{k_0}|^\varepsilon < 1/\varkappa$, so that

$$\varkappa |A_{k_0}| \leq 1,$$

whence

$$k \leq 2^{\frac{1+\varepsilon}{\varepsilon}} k_0, \quad \text{whenever} \quad f(k) > 0, \tag{2.1.21}$$

This gives the desired estimate with the constant

$$M = 2^{\frac{1+\varepsilon}{\varepsilon}} C^{1/\varepsilon}, \tag{2.1.22}$$

and the claim is proved.

To sake of completeness, in order to get the decay estimate (2.1.19), it is worth to distinguish whether q is above or below the threshold given by p, even though the proof is quite the same. Let us first assume that $q \leq p$. Then, by the equivalence of all norms on $\mathbb{R}^N$, there exists a positive constant $c_1 = c_1(K)$ only depending on K such that

$$\int_{A_k} \|\nabla u(x)\|^p \, dx \geq \left(c_1 \, c_2 |A_k|^{1/N}\right)^{-p} \int_{A_k} (u-k)^p \, dx,$$

for all $k > 1$, where $c_2 = c_2(N, p, q)$ is given by (2.1.14). On the other hand, since $q \leq p$, we have that $u^{q-1} \leq u^{p-1}$ on the set A_k, provided $k \geq 1$. Thus,

$$\int_{A_k} u(x)^{q-1}(u(x) - k) \, dx \leq c_3^p \left(\int_{A_k} (u-k)^p \, dx + k^{p-1} \int_{A_k} (u-k) \, dx \right),$$

for all $k > 1$, where $c_3 = c_3(p) = 2^{1-1/p}$. Now, we interpolate between the latter and the former, by using the identity (2.1.16). Notice that by (2.1.17) we have that

$$1 - \lambda (c_1 c_2 c_3 |A_k|^{1/N})^p \geq 1/2$$

as soon as $k \geq k_0$, where k_0 is defined according to (2.1.20) setting

$$k_0 = 2^{N/p} (c_1 c_2 c_3)^N \lambda^{N/p} \, \|u\|_{L^1(\Omega)} = \varkappa \|u\|_{L^1(\Omega)},$$

where

$$\varkappa := 2^{\frac{N}{p}} (c_1 c_2 c_3)^N \lambda^{\frac{N}{p}}.$$

Thus, after a simple absorption, we get

$$\int_{A_k} (u-k)^p \, dx \leq 2\lambda (c_1 c_2 c_3)^p |A_k|^{p/N} k^{p-1} \int_{A_k} (u-k) \, dx,$$

for all $k \geq k_0$. By using Hölder inequality on the left and dividing out, one obtains

$$\left(\int_{A_k} (u-k) \, dx \right)^{p-1} \leq 2\lambda (c_1 c_2 c_3)^p |A_k|^{p/N+p-1} k^{p-1}$$

whence, taking the $(p-1)$-th root, it follows that

$$\int_{A_k} (u-k)\,dx \le 2^{\frac{1}{p-1}} \lambda^{\frac{1}{p-1}} (c_1c_2c_3)^{\frac{p}{p-1}} |A_k|^{\frac{p}{(p-1)N}+1} k = H^\varepsilon |A_k|^{\varepsilon+1} k,$$

for all $k \ge k_0$, and the conclusion follows by the claim. Precisely, by (2.1.21) and (2.1.22) we have that the function u only takes values less than $M\lambda^{N/p}\|u\|_{L^1(\Omega)}$, where

$$M = 2^{N+1}(c_1c_2c_3)^N. \tag{2.1.23}$$

Let us pass to the case when $q > p$. On the one hand, there exists $c_1 = c_1(K)$ such that

$$\int_{A_k} \|\nabla u\|^p\,dx \ge (c_1c_2|A_k|^\gamma)^{-p} \left(\int_{A_k} (u-k)^q\,dx\right)^{\frac{p}{q}},$$

where $c_2 = c_2(N,p,q)$ is defined by (2.1.14) and $\gamma = 1/q - 1/p + 1/N$. On the other hand,

$$\int_{A_k} u^{q-1}(u-k)\,dx \le c_3^p \left(\int_{A_k} (u-k)^q\,dx + k^{q-1}\int_{A_k} (u-k)\,dx\right),$$

where $c_3 = c_3(p,q) = 2^{(q-1)/p}$. Similarly as above, we would like now to use identity (2.1.16) and absorb an integral term in the left. To do so, we use that $q > p$ to estimate from below

$$\left(\int_{A_k} (u-k)^q\,dx\right)^{\frac{p}{q}} = \left(\int_{A_k} (u-k)^q\,dx\right)^{\frac{p-q}{q}} \int_{A_k} (u-k)^q\,dx \ge \int_{A_k} (u-k)^q\,dx,$$

where we also used that $\|u\|^{p-q}_{L^q(\Omega)} = 1$. Arguing now as above it follows that

$$\int_{A_k} (u-k)^q\,dx \le 2^{q-1}\lambda(c_1c_2c_3)^p |A_k|^{\gamma p} \int_{A_k} (u-k)\,dx\ \ k^{q-1},$$

for all k larger than or equal to

$$k_0 = 2^{\frac{1}{\gamma p}}(c_1c_2c_3)^{\frac{1}{\gamma}}\lambda^{\frac{1}{\gamma p}}\|u\|_{L^1(\Omega)} =: \varkappa\|u\|_{L^1(\Omega)}.$$

Again, by using Hölder inequality on the left, after dividing out and taking the $(q-1)$-th root we end up with the decay estimate

$$\int_{A_k} (u-k)\,dx \le 2^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}(c_1c_2c_3)^{\frac{p}{q-1}}|A_k|^{\frac{\gamma p}{q-1}+1}\,k = \varkappa^\varepsilon|A_k|^\varepsilon\,k,$$

for all $k \ge k_0$. Once again, the conclusion follows by the claim, and, in the estimate (2.1.11), we may take as a constant

$$M = 2^{\frac{q+\gamma p}{\gamma p}}(c_1c_2c_3)^{1/\gamma}.$$

For the second part of the statement, the reader is referred to discussion in next section. $\square$

Remark 2.1.3. An interested reader may have noticed that nothing has been said about the dependance on p, q, N of the constant M appearing in (2.1.11). A close direct inspection in the proof above shows that setting

$$M = 2^{N(2-1/p)+1}\Big(c_K \cdot c(N,p,q)\Big)^N,$$

if $q \leq p$, and

$$M = 2^{\frac{2q-1+\gamma p}{\gamma p}}\Big(c_K \cdot c(N,p,q)\Big)^{1/\gamma},$$

otherwise, will do. Here $C(N,p,q)$ is defined in both cases according to (2.1.14) and

$$c_K = \max_{z\in K} |z|, \tag{2.1.24}$$

for istance $c_K = 1$ in the case of the euclidean ball $K = B(0,1)$. If the p-th power of the norm $\|\cdot\|$ is replaced by a more general Lagrangian satisfying the p-growth conditions (2.1.6), then c_K is replaced by $C_1(H)$.

A similar bound can be obtained for the fractional eigenfunctions discussed by Lindgren and Lindqvist [**70**]. Recall that they are the stationary points of the non-local Rayleigh quotient

$$\frac{\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}}dxdy}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p\,dx}.$$

Theorem 2.1.4. *Let Ω be a bounded Lipschitz set in $\mathbb{R}^N$, $s \in (0,1)$ and $p \in (1,\infty)$. Let u be a stationary point of the non-local Rayileigh quotient with critical value λ. Then*

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where the constant only depends on N, p, s, λ, Ω.

The decay estimate on the level sets

$$\int_k^{+\infty} |\{u > t\}|\,dt \leq c\,k|\{u > k\}|^{1+\varepsilon}$$

holds for all $k > 0$ with the exponent $\varepsilon = sp/N(p-1)$ and a constant $c = c(N,p,s,\lambda,\Omega)$. Then the proof runs as in the local case.

2. Hölder continuity of eigenfunctions

The proof of the Harnack inequality below can be found in the Trudinder's work [**89**].

Theorem 2.2.1. *Let Ω be a domain of finite measure in $\mathbb{R}^N$, $1 < p < N$, $1 < q < p^*$ and $H\colon \mathbb{R}^N \to \mathbb{R}_+$ be a convex p-homogeneous C^1 function satisfying the p-growth conditions* (2.1.6)*. Let B_ϱ be a ball such that the concentric ball $B_{3\varrho}$ is contained in Ω and let u be a non-negative solution of the Euler-Lagrange equation* (2.1.8)*. Then*

$$\sup_{B_\varrho} u \le C \inf_{B_\varrho} u,$$

where the constant $C > 0$ only depends on p, q, N, $C_1(H)$, $C_2(H)$ and the supremum of u on $B_{3\varrho}$.

The following strong minimum principle is a plain consequence of the Harnack inequality.

Theorem 2.2.2. *Let u be a non-negative eigenfunction on a connected set Ω. Then*

$$\textit{either} \quad u > 0 \qquad \textit{or} \quad u \equiv 0.$$

Another important consequence of the Harnack inequality is the Hölder continuity of the eigenfunctions[2].

Theorem 2.2.3. *Let Ω be an open set of finite measure in $\mathbb{R}^N$, $1 < p < N$ and $1 < q < p^*$. Then the eigenfunctions are Hölder continuous.*

Remark 2.2.4. A stronger regularity often holds. Namely, eigenfunctions also have Hölder continuous derivatives. Thus, in fact, they are analytic out of the set where their gradient vanishes, due to the classical uniformly elliptic regularity, see [**51**]. That is the case of the standard p-Laplace operator Δ_p. For the $C^{1,\alpha}$-solvability of the corresponding eigenvalue problem, see [**34, 69**]. Then one could expect such a result to hold for similar nonlinear operators, such as the pseudo p-Laplacian $\widetilde{\Delta}_p$ considered in Chapter 6.

However, we point out that the by now classical results of [**34, 69**] do not apply directly to the case of $\widetilde{\Delta}_p$, since the type of degeneracy is quite different. Low regularity (like L^∞ or $C^{0,\alpha}$) is assured by Theorem 2.2.3 above. On the contrary, higher regularity is not clear.

For example even the Lipschitz continuity of solutions seems not to be fully understood. We mention [**12**] where that is proved for the case $p \ge 2$ (see also [**90**] for some previous results valid for $p > 3$).

Moreover, Dirichlet eigenfunctions attain their boundary values in the classical sense out of a set small in p-capacity, and they are uniformly Hölder continuous with their derivatives up to the boundary if Ω is smooth, cf. [**89**, Corollary 4.2]. Different boundary conditions require another discussion.

It is worth pointing out that the eigenfunctions are quasiminimizers of some variational integrals (possibly different from $\mathcal{F}$ and $\mathcal{G}$) satisfying natural growth conditions, cf. [**51**,

[2]Actually, the Hölder regularity can be proved both for the solutions of the Euler equation and for the quasiminimizers of the Rayileigh quotient by a different method, see the discussion after Theorem 2.2.3.

Theorem 6.1]. Thus they belong to a De Giorgi class. Recall that this amounts to say that a Caccioppoli inequality

$$\int_{A(k,\varrho)} |\nabla u|^p \, dx \le \frac{M}{(R-\varrho)^p} \int_{A(k,R)} (u-k)^p \, dx + M|A(k,R)|^{1-\frac{1}{\gamma}} \tag{2.2.1}$$

holds for the (non-negative) eigenfunctions u, for all $k, \varrho > 0$. Here $M > 0$, $\gamma > N/p$ are suitable constants and $A(k,r)$ denotes $B_r \cap \{u > k\}$, where B_r is a small ball centered at some point $x_0 \in \Omega$. Once that is proved, the Hölder continuity follows. For istance, see [**51**, Theorem 7.6].

To get (2.2.1), one considers a cut off function $\zeta \in C^\infty(\Omega)$, compactly supported at a ball B_R, such that $\zeta \equiv 1$ in the concentric ball of radius $\varrho < R$ and $|\nabla\zeta| \le 2/(R-\varrho)$. Denote $H(z) = \|z\|$. Then the Euler equation (2.1.10) with $\varphi = (u-k)_+\zeta^p$ gives

$$\int_{A(k,R)} H(\nabla u)\zeta^p \, dx \le \int_{A(k,R)\setminus A(k,\varrho)} \big|\langle \nabla_z H(\nabla u), \nabla\zeta\rangle\big| \zeta^p (u-k) \, dx + c \int_{A(k,R)} (u-k)\zeta^p \, dx$$

where for istance $c = \lambda|\Omega|^{\frac{p}{q}-1}\|u\|^{p-1}_{L^\infty(\Omega)}$. This choice is possible according to the previous section. By Lemma 1.3.11 in Chapter 1, it follows that

$$\langle \nabla_z H(z), w\rangle \le H(z)^{\frac{p-1}{p}} H(w)^{\frac{1}{p}}$$

for all $z, w \in \mathbb{R}^N$. Applying Young inequality

$$H(z)^{\frac{p-1}{p}} H(w)^{\frac{1}{p}} \le \frac{p-1}{p}\delta H(z) + \frac{1}{\delta^{p-1}p} H(w)$$

with $z = \nabla u$ and $w = \nabla\zeta$, one of the integrals is absorbed in the left hand-side and inequality

$$\begin{aligned} \int_{A(k,R)} |\nabla u|^p \zeta^p \, dx \le C_1 \int_{A(k,R)} H(\nabla u)\zeta^p \, dx \le & C_2 \Bigg(\int_{A(k,R)\setminus A(k,\varrho)} |\nabla\zeta|^p (u-k)^p \, dx \\ & + \frac{1}{p} \int_{A(k,R)} (u-k)^p \zeta^p \, dx + \frac{p-1}{p} \int_{A(k,R)} \zeta^p \, dx \Bigg) \end{aligned} \tag{2.2.2}$$

holds for suitable constants $C_1, C_2 > 0$. Let $\gamma > N/p$. Since

$$\frac{1}{\gamma} + \frac{\gamma p - N}{\gamma N} = \frac{p}{N},$$

one has

$$|A(k,R)|^{p/N} \le |\Omega|^{1/\gamma} |B_R|^{\frac{\gamma p - N}{\gamma N}}.$$

In the right hand-side of (2.2.2), a term can be estimated further by Sobolev inequality

$$\int_{A(k,R)} (u-k)^p \zeta^p \le |A(k,R)|^{\frac{p}{N}} \left(\int (u-k)_+^{p^*} \zeta^{p^*} \right)^{\frac{p}{p^*}} \le |\Omega|^{1/\gamma} |B_R|^{\frac{\gamma p - N}{\gamma N}} \int |\nabla\left((u-k)_+\zeta\right)|^p.$$

Observe that the exponent $(\gamma p - N)/\gamma N$ is positive. Therefore by taking R small enough the latter can be absorbed in the left hand-side of (2.2.2) after adding the term $\int |\nabla\zeta|^p (u-k)_+^p$ to both sides, which yields

$$\int_{A(k,\varrho)} |\nabla((u-k)_+\zeta)|^p \le C\left(\int |\nabla\zeta|^p (u-k)^p + \int_{A(k,R)} \zeta^p \, dx\right).$$

Since $\nabla\zeta = 0$ everywhere with the exception of the annulus $A(k,R) \setminus A(k,\varrho)$, where the estimate $|\nabla\zeta| \le 2/(R-\varrho)$ is valid, the Caccioppoli inequality (2.2.1) follows from the latter.

CHAPTER 3

Hidden convexity for eigenfunctions and applications

Many variational eigenvalue problems have the following properties: *all the positive eigenfunctions minimize the Rayleigh quotient* and *all the minimizer are proportional.*

For istance, that is the case for the eigenvalue problem coming from the minimization of the Rayleigh quotient ($p > 1$)

$$\frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}$$

among all functions belonging to $W_0^{1,p}(\Omega)$, i.e. the eigenvalue problem for the p-Laplace operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

with zero Dirichlet condition on the boundary. Besides, the same conclusion can be drawn for different conditions, such as Neumann, on the boundary: in this case the Rayleigh quotient is considered on the whole of $W^{1,p}(\Omega)$ and its minimization is trivally accomplished by constant functions. Again, all the minimizers are proportional (of course all constants are) and positive eigenfunctions can only correspond to the least eigenvalue. In fact, these appear as expedient features of eigenvalue problems for other boundary conditions. They hold true if the p-Laplacian is replaced by a slightly different operator (correspondingly, if the Lagrangian density in the Rayleigh quotient is replaced by the p-th power of a different norm of the gradient).

According to the small note [**F2**], the two uniqueness properties are derived by a general principle based on the convexity of the energy $\int_\Omega |\nabla u|^p$ along particular curves.

1. Hidden convexity Lemma

Let Ω be an open set in $\mathbb{R}^N$, $p \geq 1$ and $H : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ be a nonnegative measurable function such that

$$z \longmapsto H(x, z) \text{ is convex and positively homogeneous of degree } p \tag{3.1.3}$$

for almost all $x \in \Omega$ and $b(x), \rho(x)$ are two non-negative bounded measurable functions bounded away from zero on Ω.

The curves of functions

$$\sigma_t(x) = \Big((1-t)\, u_0(x)^p + t\, u_1(x)^p\Big)^{\frac{1}{p}}, \qquad t \in [0,1], \tag{3.1.4}$$

play an important role. Remarkably, they are constant speed geodesics on the manifold of normalized positive Sobolev functions

$$M_+ = \left\{ u \in W_0^{1,p}(\Omega) : u > 0, \int_\Omega u(x)^p\, \rho(x) dx = 1 \right\}$$

equipped with the distance

$$d(u_0, u_1) = \left(\int_\Omega \big| u_0(x)^p - u_1(x)^p \big|\, \rho(x) dx \right)^{\frac{1}{p}}.$$

Indeed, we have

$$\begin{aligned} d(\sigma_t, \sigma_s) &= \left(\int_\Omega |\sigma_t(x)^p - \sigma_s(x)^p| \rho(x)\, dx \right)^{\frac{1}{p}} \\ &= |t-s| \left(\int_\Omega |u_0(x)^p - u_1(x)^p| \rho(x)\, dx \right)^{\frac{1}{p}} = |t-s|\, d(u_0, u_1), \end{aligned}$$

for all $s, t \in [0,1]$.

The convexity of a functional $\mathcal{K}$ defined on M_+ along the curves $t \mapsto \sigma_t$ is equivalent to a simple inequality.

Lemma 3.1.1. *Let $\mathcal{K} \colon M_+ \to [0, +\infty]$, $u_0, u_1 \in M_+$ and σ_t be defined as in* (3.1.4)*. Assume that*

$$\mathcal{K}(u_i) < +\infty, \qquad i = 1, 2.$$

Then

$$\textit{the function } t \longmapsto \mathcal{K}(\sigma_t) \textit{ is convex}$$

if and only if

$$\mathcal{K}(\sigma_t) \le (1-t)\mathcal{K}(u_1) + t\mathcal{K}(u_0)$$

for all $t \in [0,1]$.

Proof. The proof is elementary. For all pair of functions $u, v \in M_+$ and $t \in (0,1)$, denote

$$\sigma_t[u,v] = \Big((1-t)v^p + tu^p\Big)^{\frac{1}{p}}.$$

A direct computation shows that

$$\sigma_{(1-\lambda)\, t_0 + \lambda\, t_1}[u_0, u_1] = \sigma_\lambda\Big[\sigma_{t_0}[u_0,u_1], \sigma_{t_1}[u_0,u_1]\Big],$$

for all $\lambda \in [0,1]$. Therefore if one sets

$$t = \lambda, \quad u = \sigma_{t_0}[u_0, u_1], \quad \text{and } v = \sigma_{t_1}[u_0, u_1]$$

inequality

$$\mathcal{K}(\sigma_t[u,v]) \leq (1-t)\mathcal{K}(v) + t\mathcal{K}(u)$$

reads

$$\mathcal{K}(\sigma_{(1-\lambda)\,t_0+\lambda\,t_1}[u_0,u_1]) \leq (1-\lambda)\mathcal{K}(\sigma_{t_1}[u_0,u_1]) + \lambda\mathcal{K}(\sigma_{t_0}[u_0,u_1])$$

and this concludes the proof. □

The following simple lemma shall found an interesting application in next sections.

Lemma 3.1.2 (Hidden Convexity)**.** *Let Ω be an open set in $\mathbb{R}^N$, $p>1$ and $H:\Omega\times\mathbb{R}^N \to \mathbb{R}_+$ be a nonnegative measurable function such that* (3.1.3) *holds, and set*

$$\mathcal{K}(u,\Omega) = \int_\Omega H(x,\nabla u(x))\,dx.$$

Let $u, v \in W^{1,p}(\Omega)$ be nonnegative functions and set

$$\sigma_t[u,v](x) = \Big(tu(x)^p + (1-t)v(x)^p\Big)^{\frac{1}{p}},$$

for all $x \in \Omega$ and all $t \in [0,1]$.

$$\mathcal{K}(\sigma_t,\Omega) \leq (1-t)\mathcal{K}(v,\Omega) + t\mathcal{K}(u,\Omega), \qquad t\in[0,1]. \tag{3.1.5}$$

Proof. Let $\varepsilon > 0$. To abbreviate the notation, set $u_\varepsilon = u+\varepsilon$, $v_\varepsilon = v+\varepsilon$. Then the formula

$$\Big((1-t)v_\varepsilon^p + tu_\varepsilon^p\Big)^{\frac{1}{p}}$$

defines an element σ_t^ε of $W^{1,p}(\Omega)$, that is given by the composition of the vector-valued Sobolev map

$$\Phi_t^\varepsilon = \Big((1-t)^{\frac{1}{p}}\,v_\varepsilon, t^{\frac{1}{p}}\,u_\varepsilon\Big) \in W^{1,p}(\Omega;\mathbb{R}^2),$$

with the ℓ_p norm, i.e. $\|(x,y)\|_{\ell^p} = (|x|^p+|y|^p)^{1/p}$. Indeed, recall that the latter is a C^1 function outside the origin and

$$\|\Phi_t^\varepsilon\|_{\ell^p} \geq \varepsilon,$$

for all $t\in[0,1]$, by construction. Thus the usual chain rule formula holds, and we obtain

$$\begin{aligned}
\nabla\sigma_t^\varepsilon(x) &= \sigma_t^\varepsilon(x)^{1-p}\Big[(1-t)\,\nabla v_\varepsilon(x)\,v_\varepsilon(x)^{p-1} + t\,\nabla u_\varepsilon(x)\,u_\varepsilon(x)^{p-1}\Big] \\
&= \sigma_t^\varepsilon(x)\,\left[\frac{(1-t)\,v_\varepsilon(x)^p}{\sigma_t^\varepsilon(x)^p}\,\nabla\log v_\varepsilon(x) + \frac{t\,u_\varepsilon(x)^p}{\sigma_t^\varepsilon(x)^p}\,\nabla\log u_\varepsilon(x)\right]
\end{aligned}$$

almost everywhere in Ω, for all $t\in[0,1]$. Observe that the latter is a convex combination of $\nabla\log u_\varepsilon$ and $\nabla\log v_\varepsilon$. By the convexity and the homogeneity of the function H in the z

variable it follows that

$$\begin{aligned} H(x, \nabla\sigma_t^\varepsilon) &\le (1-t)\, v_\varepsilon(x)^p\, H\Big(x, \nabla \log v_\varepsilon(x)\Big) + t\, u_\varepsilon(x)^p\, H\Big(x, \nabla \log u_\varepsilon(x)\Big) \\ &= (1-t)\, H\Big(x, \nabla v_\varepsilon(x)\Big) + t\, H\Big(x, \nabla u_\varepsilon(x)\Big) \\ &= (1-t)\, H\Big(x, \nabla v(x)\Big) + t\, H\Big(x, \nabla u(x)\Big), \end{aligned} \tag{3.1.6}$$

for almost all $x \in \Omega$, for all $t \in [0,1]$. In the last passage we simply used the fact that $\nabla u_\varepsilon = \nabla u$ and $\nabla v_\varepsilon = \nabla v$. Sending $\varepsilon \to 0^+$ gives

$$H\Big(x, \nabla\sigma_t[u,v]\Big) \le (1-t)\, H\Big(x, \nabla v(x)\Big) + t\, H\Big(x, \nabla u(x)\Big)$$

almost everywhere in Ω, for all $t \in [0,1]$ and now (3.1.5) follows just by integrating over Ω this inequality. $\square$

The geodesic convexity property discussed above holds true if the variational energy integral is replaced by a nonlocal functional, such as a fractional Gagliardo (semi)norm

$$\mathcal{K}(u,\Omega) = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(y)-u(x)\big|^p}{|y-x|^{N+sp}}\, dxdy \tag{3.1.7}$$

on $W^{s,p}(\Omega)$. Here $0 < s < 1$ and $W^{s,p}(\Omega)$ stands for the fractional Sobolev space.

Lemma 3.1.3. *Let Ω be an open set in $\mathbb{R}^N$, $s \in (0,1)$ and $p \ge 1$. Let $\mathcal{K}$ be as in* (3.1.7) *and*

$$\sigma_t(x) = \Big((1-t)v(x)^p + tu(x)^p\Big)^{\frac{1}{p}},$$

for all $x \in \Omega$ and $t \in [0,1]$. Then

$$\mathcal{K}(\sigma_t,\Omega) \le (1-t)\mathcal{K}(v,\Omega) + t\mathcal{K}(u,\Omega)$$

for all $t \in [0,1]$.

Proof. Since

$$\sigma_t = \Big\|\Big(t^{\frac{1}{p}}u, (1-t)^{\frac{1}{p}}v\Big)\Big\|_{\ell^p},$$

the conclusion follows by the triangle inequality

$$\big|\|\xi\|_{\ell^p} - \|\eta\|_{\ell^p}\big| \le \|\xi-\eta\|_{\ell^p}$$

by taking $\xi = (t^{1/p}u(y), (1-t)^{1/p}v(y))$ and $\eta = (t^{1/p}u(x), (1-t)^{1/p}v(x))$ for $x, y \in \Omega$ and integrating the resulting inequality against the kernel on $\mathbb{R}^N \times \mathbb{R}^N$. $\square$

2. Uniqueness of positive eigenfunctions

In the present section, we apply the hidden convexity of eigenvalue problems to address uniqueness issues. The core of the uniqueness proof is that any positive function may be connected to the global minimizer of the Rayleigh quotient by a curve σ_t. By the convexity of the energy along such curve, it will follow that all the positive eigenfunctions are minimizers of the quotient, i.e. they correspond to the least eigenvalue. The assumptions on the structure of the eigenvalue problem are rather mild. After the proof of the theorem and some comments on its application, an analogous result for a nonlocal energy is discussed.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, having finite measure and $p > 1$. Let $H : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ be a function such that*

$$\begin{aligned} z \mapsto H(x,z) \ \ & \textit{is } C^1 \textit{ convex and homogeneous of degree } p, \textit{ i.e.} \\ & H(x,tz) = |t|^p H(x,z) \textit{ for every } t \in \mathbb{R}, (x,z) \in \Omega \times \mathbb{R}^N. \end{aligned} \tag{3.2.1}$$

Assume that the variational problem

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega H\big(x, \nabla u(x)\big) \ + b(x)|u(x)|^p dx \ : \ \int_\Omega |u(x)|^p \rho(x)\, dx = 1 \right\}, \tag{3.2.2}$$

admits at least one solution. If there exist λ and a strictly positive $v \in W_0^{1,p}(\Omega)$ such that

$$\frac{1}{p} \int_\Omega \langle \nabla H(x, \nabla v), \nabla \varphi \rangle \, dx + \int_\Omega b \, |v|^{p-2} \, v \, \varphi \, dx = \lambda \int_\Omega |v|^{p-2} \, v \, \varphi \rho(x) \, dx, \tag{3.2.3}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, then

$$\lambda = \lambda_1(\Omega). \tag{3.2.4}$$

Proof. First of all, we can assume that v is normalized so as to be an admissible competitor for the minimum problem defining $\lambda_1(\Omega)$, since equation (3.2.3) is $(p-1)$-homogeneous and $v \not\equiv 0$. Moreover, by testing the equation with $\varphi = v$ and by homogeneity of H, we get

$$\int_\Omega H(x, \nabla v) + b v^p \, dx = \frac{1}{p} \int_\Omega \langle \nabla H(x, \nabla v), \nabla v \rangle \ + b \, v^p \, dx = \lambda. \tag{3.2.5}$$

Then we take a minimizer $u \in W_0^{1,p}(\Omega)$ for (3.2.4). Thanks to the homogeneity of H, we can suppose that $u \geq 0$ without loss of generality. Indeed, the function $\widetilde{u} = |u|$ is nonnegative and still satisfies the constraint. Since $H(x,z) = H(x,-z)$, we get $H(x, \nabla \widetilde{u}) = H(x, \nabla u)$ almost everywhere and

$$\int_\Omega H(x, \nabla \widetilde{u}(x)) + b(x) \, \widetilde{u}(x)^p \, dx = \int_\Omega H(x, \nabla u(x)) + b(x)|u(x)|^p \, dx = \lambda_1(\Omega). \tag{3.2.6}$$

For every $\varepsilon \ll 1$, we set for simplicity

$$u_\varepsilon = u + \varepsilon \qquad \text{and} \qquad v_\varepsilon = v + \varepsilon.$$

To simplify the notation, let $d\mu(x) = \rho(x)dx$. We claim that

$$\begin{aligned}\int_\Omega \Big[H(x,\nabla u(x)) - H(x,\nabla v(x))\Big]\,dx + \int_\Omega b(x)\big(u_\varepsilon(x)^p - v_\varepsilon(x)^p\big)\,dx \\ \geq p\lambda \int_\Omega v(x)^{p-1}\left(\frac{\sigma_t(x)-v_\varepsilon(x)}{t}\right)d\mu(x) \\ + p\int_\Omega b(x)\big(v_\varepsilon(x)^{p-1} - v(x)^{p-1}\big)\left(\frac{\sigma_t(x)-v_\varepsilon(x)}{t}\right)dx,\end{aligned} \tag{3.2.7}$$

for all $t \in [0,1]$.

In order to prove the claim, one defines the usual curve of functions

$$\sigma_t(x) = \Big((1-t)\,v_\varepsilon(x)^p + t\,u_\varepsilon(x)^p\Big)^{\frac{1}{p}}, \qquad x\in\Omega,\ t\in[0,1],$$

connecting the non-negative functions v_ε and u_ε (in fact they are strictly positive, which will be used later in this proof). By applying Lemma 3.1.2 to the functional

$$\begin{aligned}\int_\Omega H(x,\nabla\sigma_t(x))\,dx &\leq (1-t)\int_\Omega H(x,\nabla v_\varepsilon(x))\,dx + t\int_\Omega H(x,\nabla u_\varepsilon(x))\,dx \\ &= t\left[\int_\Omega H(x,\nabla u(x))\,dx - \int_\Omega H(x,\nabla v(x))\,dx\right] \\ &\quad + \int_\Omega H(x,\nabla v(x))\,dx, \qquad t\in[0,1],\end{aligned}$$

where the fact that $\nabla u_\varepsilon = \nabla u$ and $\nabla v_\varepsilon = \nabla v$ was also used. Moreover,

$$\int_\Omega b(x)\sigma_t(x)^p\,dx = (1-t)\int_\Omega b(x)v_\varepsilon(x)^p\,dx + t\int_\Omega b(x)u_\varepsilon(x)^p\,dx, \quad t\in[0,1].$$

Thus, one has

$$\begin{aligned}\int_\Omega \Big[H(x,\nabla u(x)) - H(x,\nabla v(x))\Big]\,dx + \int_\Omega b(x)\big(u_\varepsilon(x)^p - v_\varepsilon(x)^p\big)\,dx \\ \geq \int_\Omega \left(\frac{H(x,\nabla\sigma_t(x)) - H(x,\nabla v(x))}{t} + b(x)\frac{\sigma_t(x)^p - v_\varepsilon(x)^p}{t}\right)dx.\end{aligned} \tag{3.2.8}$$

By standard convexity, the inequalities

$$H(x,\nabla\sigma_t(x)) - H(x,\nabla v(x)) \geq \Big\langle \nabla_z H(\nabla v(x)), \nabla\big(\sigma_t(x) - v_\varepsilon(x)\big)\Big\rangle,$$

and

$$\sigma_t(x)^p - v_\varepsilon(x)^p \geq p v_\varepsilon(x)^{p-1}\left(\sigma_t(x) - v_\varepsilon(x)\right),$$

hold pointwise almost everywhere for all $t \in [0,1]$. Hence, the right-hand side of (3.2.8) is estimated from below by

$$\int_\Omega \left\langle \nabla_z H(\nabla v(x)), \nabla\left(\frac{\sigma_t(x) - v_\varepsilon(x)}{t}\right)\right\rangle dx + p\int_\Omega b(x) v_\varepsilon(x)^{p-1}\left(\frac{\sigma_t(x) - v_\varepsilon(x)}{t}\right),$$

for all $t \in [0,1]$. In turn, by testing equation (3.2.3) with $\varphi = \sigma_t(x) - v_\varepsilon(x)$, the latter can be estimated from below by

$$p\lambda \int_\Omega v(x)^{p-1}\left(\frac{\sigma_t(x) - v_\varepsilon(x)}{t}\right) d\mu(x)$$
$$+ p\int_\Omega b(x)\big(v_\varepsilon(x)^{p-1} - v(x)^{p-1}\big)\left(\frac{\sigma_t(x) - v_\varepsilon(x)}{t}\right) dx,$$

and the claim follows. Note that the left hand side in (3.2.7) is independent of t. Moreover, by the concavity of the p-th root, one has

$$\frac{\sigma_t(x) - v_\varepsilon(x)}{t} \geq u(x) - v(x), \qquad \text{a.e. in } \Omega,$$

hence, the first (*resp.,* the second) integrand appearing in the right hand side of (3.2.7) can be estimated from below by a function, independent of $t \in [0,1]$, which does belong to $L^1(\Omega, d\mu)$ (*resp.,* to $L^1(\Omega)$), by Hölder inequality. Then, by Fatou's Lemma, passing to the inferior limit of both sides of (3.2.7) as $t \to 0^+$ gives

$$\int_\Omega \Big[H(x,\nabla u(x)) - H(x,\nabla v(x))\Big] dx + \int_\Omega b(x)\big(u_\varepsilon(x)^p - v_\varepsilon(x)^p\big) dx$$
$$\geq \lambda \int_\Omega \left(\frac{v(x)}{v_\varepsilon(x)}\right)^{p-1} \left(u_\varepsilon(x)^p - v_\varepsilon(x)^p\right) d\mu(x)$$
$$+ \int_\Omega b(x)\left(1 - \left(\frac{v(x)}{v_\varepsilon(x)}\right)^{p-1}\right)\left(u_\varepsilon(x)^p - v_\varepsilon(x)^p\right) dx. \tag{3.2.9}$$

By (3.2.5) and (3.2.6),

$$\lambda_1(\Omega) - \lambda \geq \int_\Omega \big(H(x,\nabla u) - H(x,\nabla v)\big) dx + \int_\Omega b(u_\varepsilon^p - v_\varepsilon^p)\, dx$$
$$+ \int_\Omega b(u^p - u_\varepsilon^p)\, dx + \int_\Omega b(v_\varepsilon^p - v^p)\, dx.$$

In the latter, the last term is positive hence it can be dropped and the inequality holds true. Thus, by (3.2.9) it follows that

$$\begin{aligned}\lambda_1(\Omega) - \lambda \geq \int_\Omega \left(\frac{v}{v_\varepsilon}\right)^{p-1} (u_\varepsilon^p - v_\varepsilon^p)\, d\mu + \int_\Omega b \left(1 - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right) (u_\varepsilon^p - v_\varepsilon^p)\, dx \\ + \int_\Omega b(u^p - u_\varepsilon^p)\, dx.\end{aligned} \tag{3.2.10}$$

Here, by Dominated Convergence Theorem the right-hand side goes to zero as $\varepsilon \to 0^+$. Indeed, in the second and third summand the integrands themselves go to zero. Since $v > 0$ in Ω, the first summand converges to

$$\int_\Omega (u^p - v^p)\, d\mu,$$

which is zero because of the normalization condition imposed on the functions u, v. Thus

$$\lambda_1(\Omega) - \lambda \geq 0.$$

Note that the reverse inequality holds by minimality of $\lambda_1(\Omega)$, so that the theorem is proved. □

Remark 3.2.2. Note the requirement: the solution v to (3.2.3) has to be *strictly* positive on Ω. That is not a big deal, since in many situations of interest Harnack's inequality is at disposal for the nonnegative minimizers of problem (3.2.2). For example the inequality is valid for all the functionals of Calculus of Variations whose Lagrangian density $H : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ is homogeneous and satisfies the growth conditions

$$c_1\, |z|^p \leq H(x, z) \leq c_2\, |z|^p, \qquad (x, z) \in \Omega \times \mathbb{R}^N,$$

with two positive constants $c_1 \geq c_2 > 0$, see Theorem 2.2.1. According to Theorem 2.2.2, Harnack inequality prevents nontrivial nonnegative solutions of (3.2.3) from vanishing at interior points of Ω.

Corollary 3.2.3. *Let Ω be an open set of finite measure in $\mathbb{R}^N$, λ be a real number and $u \in W_0^{1,p}(\Omega)$ be a non-trivial weak solution of the eigenvalue problem for the p-Laplacian*

$$-\Delta_p u = \lambda |u|^{p-2} u, \qquad \text{in } \Omega,$$

with Dirichlet conditions $u = 0$ on the boundary. Assume that $u \geq 0$. Then $\lambda = \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the minimum of the Rayleigh quotient on $W_0^{1,p}(\Omega)$.

Proof. Since $u \geq 0$, then in fact $u > 0$ by the strong minimum principle, cf. Theorem 2.2.2. Thus the Corollary follows by applying Theorem 3.2.1 with $H(z) = |z|^p$. □

Several comments are appropriate.

Remark 3.2.4. Assume that Ω is a Lipschitz open set. In the proof of Theorem 3.2.1, $\rho(x)dx$ may be replaced by $\rho(x)d\mathcal{H}^{N-1}(x)$ where $\rho : \partial\Omega \to \mathbb{R}$ is a bounded $\mathcal{H}^{N-1}$-measurable function away from zero, provided that all the integrals of Sobolev functions with respect to μ are understood in the sense of traces. This allows one to apply the uniqueness result of Theorem 3.2.1 to the eigenvalue problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u, & \text{in } \Omega, \\ |\nabla u|^{p-2}\langle \nabla u, \nu_\Omega\rangle = \lambda |u|^{p-2}u, & \text{on } \partial\Omega \end{cases}$$

considered in [**77**]. The minimization of the corresponding Rayleigh quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \, dx}{\displaystyle\int_{\partial\Omega} |u|^p \, d\mathcal{H}^{N-1}}$$

yields the best constant in a Sobolev trace inequality.

Remark 3.2.5. Let Ω be a Lipschitz open set. By the proof of Theorem 3.2.1, it is clear that the claim of the theorem holds true if we replace at each occurence $W_0^{1,p}(\Omega)$ by the entire Sobolev space $W^{1,p}(\Omega)$. This allows one to recover, for example, the case of the eigenvalue problem for the p-Laplacian with either Neumann or Stekloff boundary conditions on the boundary. But in those cases the conclusion is quite trivial: any eigenfunction having constant sign (*resp.,* constant sign on the boundary) of the Neumann (*resp.,* Stekloff) p-Laplacian is in fact a constant function.

Remark 3.2.6. The same conclusions as in Corollary 3.2.3 and Remark 3.2.4 can be drawn for the positive eigenfunctions of the so called *pseudo $p-$Laplacian* $\widetilde{\Delta}_p$, defined by

$$\widetilde{\Delta}_p u := \sum_{i=1}^N \partial_{x_i} \left(|\partial_{x_i} u|^{p-2} \, \partial_{x_i} u \right).$$

Here the eigenvalue problem (introduced in [**12**]) consists in finding the positive numbers $\lambda > 0$, such that the equation

$$-\widetilde{\Delta}_p u = \lambda \, |u|^{p-2} \, u,$$

has nontrivial weak solutions in $W_0^{1,p}(\Omega)$. In this case, Theorem 3.2.1 should be applied with $H(z) = \|z\|^p$ where $\|\cdot\|$ is the ℓ^p-norm in $\mathbb{R}^N$. Other anisotropic operators can be treated by taking a different norm.

3. Uniqueness of ground states

We apply the convexity of the energy

$$\mathcal{K}(u, \Omega) = \int_\Omega H(x, \nabla u) \, dx$$

along the curves σ_t to prove a further well-known uniqueness result: that all the minimizers of the Rayleigh quotient are proportional. In next theorem we agree the following: depending on whether the minimum (3.3.1) is taken over $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, μ will denote either the N-dimensional Lebesgue measure or a weighted $(N-1)$-dimensional Hausdorff measure of the boundary[1]. That allows one to apply the theorem to both to Dirichlet eigenvalues and to different boundary conditions.

Theorem 3.3.1. *Let the assumptions of Lemma 3.1.2 be valid. Assume, in addition, that $p > 1$, $z \mapsto H(x,z)$ is strictly convex, and Ω is a connected open set where the variational problem*

$$\lambda_1(\Omega) = \min\left\{ \int_\Omega H\big(x, \nabla u(x)\big) \; + b(x)|u(x)|^p dx \; : \; \int_\Omega |u(x)|^p \, d\mu(x) = 1 \right\}, \tag{3.3.1}$$

admits at least one positive solution. Then all the positive minimizers are proportional.

Proof. Let u, v be two normalized positive minimizers. Since $u, v > 0$ one may repeat verbatim the proof of Lemma 3.1.2 with $\varepsilon = 0$ to get

$$H(x, \nabla \log \sigma_t) \le \frac{(1-t)v^p}{\sigma_t^p} H(x, \nabla \log v) + \frac{tu^p}{\sigma_t^p} H(x, \nabla \log u). \tag{3.3.2}$$

where $\sigma_t = ((1-t)v^p + tu^p)^{\frac{1}{p}}$. Note that for all $t \in [0,1]$ the function σ_t is an admissible competitor for the minimum problem defining $\lambda_1(\Omega)$. Thus

$$\int_\Omega H(x, \nabla \sigma_t)\, dx \ge \lambda_1(\Omega).$$

On the other hand, by Lemma 3.1.2 one gets

$$\begin{aligned} &\int_\Omega H(x, \nabla \sigma_t) + b(x)|\sigma_t(x)|^p \, dx \\ &\le (1-t) \int_\Omega H(x, \nabla v) + b(x) v(x)^p \, dx + t \int_\Omega H(x, \nabla u(x)) + b(x) u(x)^p \, dx = \lambda_1(\Omega). \end{aligned}$$

Hence all the inequalities are in fact equalities. Therefore, (3.3.2) holds as an equality, pointwise almost everywhere. Note that the right-hand side is a convex combination of two values taken by H and

$$\nabla \log \sigma_t(x) = \frac{(1-t)v(x)^p}{\sigma_t(x)^p} \nabla \log v(x) + \frac{tu(x)^p}{\sigma_t(x)^p} \nabla \log u(x).$$

By the strict convexity of H it follows that

$$\nabla \log u = \nabla \log v, \qquad \text{a.e. in } \Omega$$

[1] In the second case, the boundary integrals are understood in the sense of the traces, and we also agree that Ω should be at least Lipschitz.

which is equivalent to say that $\nabla(u/v) = 0$ a.e. in Ω. Since the domain is connected, it follows that u and v are proportional. $\square$

By the strong minimum principle of Theorem 2.2.2, the requirement that $v > 0$ is fullfield by non-negative eigenfuctions provided that the Lagrangian function H satisfies natural p-growth conditions. That is the case if $H(z) = |z|^p$. Thus last theorem implies the corollary.

Corollary 3.3.2. *Let Ω be an open connected set of finite measure in $\mathbb{R}^N$. Then the first Dirichlet eigenvalue of the p-Laplacian*

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}$$

is simple, i.e. all the minimizers are proportional.

Remark 3.3.3. The same conclusion can be drawn if one replaces Dirichlet boundary value problem with the one briefly discussed in Remark 3.2.4. Similarly, the simplicity of the first eigenvalue holds true also if the p-Laplacian is replaced by the pseudo p-Laplacian.

4. Uniqueness of positive fractional eigenfunctions

Let $s \in (0, 1)$ and Ω be an open set of finite measure. We use Lemma 3.1.3 to study uniqueness issues for positive eigenfunctions of problem coming from the minimization of the nonlocal Rayleigh quotient

$$\frac{\displaystyle\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|y - x|^{N+sp}} \, dxdy}{\displaystyle\int_{\mathbb{R}^N} |u|^p \, dx}$$

on $W_0^{s,p}(\Omega)$. For a recent overview on the fractional Sobolev space, the reader is referred to [**36**].

The strategy just amounts to adapt the proof of Theorem 3.2.1 to the variational double integral

$$\mathcal{K}(u, \Omega) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|y - x|^{N+sp}} \, dxdy.$$

By computing the first variation, the Euler equation satisfied by a critical point $u \in W_0^{s,p}(\Omega)$ of the fractional Rayleigh quotient, corresponding to the critical value λ is

(3.4.1)
$$\iint\limits_{\mathbb{R}^N\,\mathbb{R}^N} \frac{\Big|u(y)-u(y)\Big|^{p-2}\Big(u(y)-u(x)\Big)\Big(\phi(y)-\phi(x)\Big)}{|y-x|^{N+sp}}\,dxdy = \lambda \int_{\mathbb{R}^N} |u(x)|^{p-2}u(x)\,\phi(x)\,dx,$$

for all $\phi \in C_0^\infty(\Omega)$.

The nonlocal Rayleigh quotient $\mathcal{R}$ attains its minimum $\lambda_{1,p}^s(\Omega)$ which gives the least eigenvalue. An introduction to fractional eigenvalues is provided in the paper [**70**], where the following result is proved for large values of p. The proof given here considerabily lowers the assumptions on that exponent.

Theorem 3.4.1. *Let $1 < p < \infty$, $s \in (0,1)$ and Ω be a bounded Lipschitz set in $\mathbb{R}^N$. Let $v \in W_0^{s,p}(\Omega)$ be a solution of (3.4.1). If $v > 0$, then*

$$\lambda = \lambda_{1,p}^s(\Omega)$$

where $\lambda_{1,p}^s(\Omega)$ denotes the minimum of the fractional Rayleigh quotients $\mathcal{R}$ on $W_0^{s,p}(\Omega)$.

Proof. Assume that $v \in W_0^{s,p}(\Omega)$ is a strictly positive solution of (3.4.1). There is no loss of generality if we assume that the function v is normalized in $L^p(\Omega)$. Let $u \in W_0^{s,p}(\Omega)$ be a solution of the minimum problem

$$\lambda_{1,p}^s(\Omega) = \min\Big\{\mathcal{K}(u,\Omega) :\ u \in W_0^{s,p}(\Omega), \int_\Omega |u(x)|^p\,dx = 1\Big\}$$

For the well-posedness of the minimization of the fractional Rayleigh quotient, we refer to [**70**].

To simplify the notation a little, let u_ε and v_ε denote the functions $u+\varepsilon$ and $v+\varepsilon$, respectively. Set

$$\sigma_t^\varepsilon(x) = \Big(tu_\varepsilon(x)^p + (1-t)v_\varepsilon(x)^p\Big)^{\frac{1}{p}}, \qquad x \in \Omega, t \in [0,1],$$

By Lemma 3.1.3, $t \mapsto \sigma_t^\varepsilon$ is a curve of functions belonging to $W_0^{s,p}(\Omega)$ along which the the energy is convex. Hence

$$\iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|\sigma_t^\varepsilon(y)-\sigma_t^\varepsilon(x)|^p}{|y-x|^{N+sp}}\,dxdy - \iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|v(y)-v(x)|^p}{|y-x|^{N+sp}}\,dxdy,$$
$$\leq t\left(\iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}}\,dxdy - \iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|v(y)-v(x)|^p}{|y-x|^{N+sp}}\,dxdy\right) = t\Big(\lambda_{1,p}^s(\Omega)-\lambda\Big),$$

for all $t \in [0,1]$ and all $\varepsilon \ll 1$. By the (standard) convexity of the map $\tau \mapsto |\tau|^p$, it follows that the left-hand side in the latter can be estimated from below as it follows

$$\iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|\sigma_t^\varepsilon(y)-\sigma_t^\varepsilon(x)|^p}{|y-x|^{N+sp}}\,dxdy - \iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|v(y)-v(x)|^p}{|y-x|^{N+sp}}\,dxdy,$$

$$\geq \iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}(v(y)-v(x))}{|y-x|^{N+sp}}\Big(\sigma_t^\varepsilon(y)-\sigma_t^\varepsilon(x)-(v(y)-v(x))\Big)\,dxdy,$$

for all $t \in [0,1]$ and $\varepsilon \ll 1$. Moreover, since $u, v \in W_0^{s,p}(\Omega)$, the function σ_t^ε also belong to $W_0^{s,p}(\Omega)$. Thus, it does take sense to plug $\phi = \sigma_t^\varepsilon - v_\varepsilon$ as a test function into the Euler-Lagrange equation which holds for the eigenfunction v, whence the identity

$$\iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}(v(y)-v(x))}{|y-x|^{N+sp}}\Big(\sigma_t^\varepsilon(y)-\sigma_t^\varepsilon(x)-\big(v_\varepsilon(y)-v_\varepsilon(x)\big)\Big)\,dxdy,$$

$$= \lambda \int\limits_\Omega v(z)^{p-1}\,\sigma_t^\varepsilon(z) - v(z)\,dz,$$

follows for all $\varepsilon \ll 1$. Here the fact that $v(y)-v(x) = v_\varepsilon(y)-v_\varepsilon(x)$ was used. Thus,

$$\lambda \int\limits_\Omega v(z)^{p-1}\frac{\sigma_t^\varepsilon(z)-v_\varepsilon(z)}{t}\,dz \leq \lambda_{1,p}^s(\Omega) - \lambda,$$

for all $t \in [0,1]$, and all $\varepsilon \ll 1$. Note that by the concavity of the p-th root, the integrand in the latter is estimated pointwise almost everywhere in Ω from below by the function

$$v(z)^{p-1}\big(u_\varepsilon(z)-v_\varepsilon(z)\big),$$

which does belong to $L^1(\Omega)$. Hence, by Fatou's Lemma,

$$\lambda \int\limits_\Omega \left(\frac{v(z)}{v(z)+\varepsilon}\right)^{p-1}\Big((u(z)+\varepsilon)^p - (v(z)-\varepsilon)^p\Big)\,dz$$

$$\leq \lambda \liminf_{t\to 0^+} \int\limits_\Omega v(z)^{p-1}\frac{\sigma_t^\varepsilon(z)-v_\varepsilon(z)}{t}\,dz \leq \lambda_{1,p}^s(\Omega)-\lambda,$$

for all ε small enough. By dominated convergence Theorem and the normalization in $L^p(\Omega)$ of both the functions u, v, sending $\varepsilon \to 0^+$ yields

$$0 \leq \lambda_{1,p}^s(\Omega) - \lambda.$$

The desired conclusion now follows, since $\lambda_{1,p}^s(\Omega)$ is the least possible s-eigenvalue and the converse inequality is obvious. $\square$

For sake of completeness, we prove the following result.

Theorem 3.4.2. *Let Ω be a connected open set, $1 < p < \infty$ and $s \in (0,1)$. Then all the positive eigenfunctions corresponding to $\lambda^s_{1,p}(\Omega)$ are proportional.*

Proof. Let u, v be two positive normalized functions $W^{s,p}_0(\Omega)$ and σ_t denote the usual constant speed geodesic connecting u to v. Recall the convexity inequality of Lemma 3.1.3

$$\mathcal{K}(\sigma_t, \Omega) \le (1-t)\mathcal{K}(v,\Omega) + t\mathcal{K}(u,\Omega).$$

If the equality holds, then for almost all $x, y \in \mathbb{R}^N$ the triangle inequality

$$\big| \|\xi\|_{\ell^p} - \|\eta\|_{\ell^p} \big| \le \|\xi - \eta\|_{\ell^p},$$

holds as an inequality with the choice

$$\xi = \Big(t^{\frac{1}{p}} u(y), (1-t)^{\frac{1}{p}} v(y)\Big), \quad \eta = \Big(t^{\frac{1}{p}} u(x), (1-t)^{\frac{1}{p}} v(x)\Big).$$

Since $p > 1$ there exists $\alpha(x,y) \in \mathbb{R}$ such that

$$u(y) = \alpha(x,y) u(x), \qquad v(y) = \alpha(x,y) v(x),$$

for almost all $x, y \in \mathbb{R}^N$. Therefore

$$\frac{u(y)}{v(y)} = \frac{u(x)}{v(x)}$$

and there is a constant β such that $u = \beta v$ almost everywhere. $\square$

Remark 3.4.3. It is worth pointing out that a stronger version of Theorem 3.4.2, claiming that $\lambda^s_{1,p}(\Omega)$ is simple, was proved in [**70**, Theorem 14] under the restriction that the exponent p is large. Indeed this additional assumption assures the equivalence of weak and viscosity solutions of the Euler-Lagrange equation associated with the fractional Rayleigh quotient. In fact, a strong minimum principle is easily obtained for viscosity supersolutions, cf. [**70**, Lemma 12].

CHAPTER 4

Spectral gap

One of the significant consequences of the uniqueness properties discussed in Chapter 3 is the existence of a gap between the least eigenvalue and the infimum of all the higher eigenvalues. We will consider the Rayleigh quotient

$$\frac{\mathcal{F}(u,\Omega)}{\mathcal{G}(u,\Omega)}$$

where the functionals $\mathcal{F},\mathcal{G}$ are defined by

$$\mathcal{F}(u,\Omega)=\int_\Omega \|\nabla u(x)\|^p\,dx,\quad \mathcal{G}(u,\Omega)=\int_\Omega |u(x)|^p\,d\mu(x),$$

for all functions u belonging to the Sobolev space $W_0^{1,p}(\Omega)$. For the exponent, we assume $1<p<\infty$ and Ω is any open set having finite Lebesgue measure in $\mathbb{R}^N$. Here μ is denoting a measure with a density $\rho(x)\in L^\infty(\Omega)$ bounded away from zero. In the Dirichlet energy, $\|\cdot\|$ denotes a norm in $\mathbb{R}^N$.

Remark 4.0.4. We restrict our discussion to the case of Dirichlet boundary conditions. As a matter of fact, the definition of eigenvalues on disconnected domains given here for Dirichlet boundary condition may be rephrased to embrace different problems. Namely, the proofs of this chapter can be repeated verbatim to deal with the case when the Lagrangian of $\mathcal{F}(u,\Omega)$ has an extra term $b(x)|u|^p$, the functionals are acting on the whole of $W^{1,p}(\Omega)$ and $d\mu=\rho(x)d\mathcal{H}^{N-1}$, $\mathcal{H}^{N-1}$ being the restriction of the $(N-1)$-dimensional Hausdorff measure to the boundary $\partial\Omega$. In that case, we agree that Ω is a bounded Lipschitz set and the integrals are understood in the sense of traces.

1. A Mountain Pass lemma

The second variational eigenvalue

$$\lambda_2(\Omega):=\inf_{\substack{f\colon \mathbb{S}^1\longrightarrow M\\ \text{odd, continuous}}}\ \max_{u\in f(\mathbb{S}^1)}\int_\Omega \|\nabla u(x)\|^p\,dx$$

admits a mountain-pass characterization, which holds no matter if Ω is connected or not. Note that by the Poincaré inequality

$$\left\{\int_\Omega \|\nabla u\|^p\,dx\right\}^{\frac{1}{p}}$$

defines an equivalent norm on the Sobolev space $W_0^{1,p}(\Omega)$. First, one needs the following preliminary result.

Lemma 4.1.1. *Let $u, v \in M$ be non-negative functions. Then the curve $\gamma\colon [0,1] \to M$*

$$\gamma_t(x) = \big((1-t)v(x)^p + tu(x)^p\big)^{\frac{1}{p}}$$

is continuous with respect to the $W^{1,p}$-topology.

Proof. First note that γ is Lipschitz continuous in the L^p topology: using the Hölder continuity of the p-th root

$$\int_\Omega |\gamma_t - \gamma_s|^p \, dx \le \left|(1-s)^{\frac{1}{p}} - (1-t)^{\frac{1}{p}}\right|^p \int_\Omega |v|^p \, dx + \left|t^{\frac{1}{p}} - s^{\frac{1}{p}}\right|^p \int_\Omega |u|^p \, dx \le |t-s| \left(\int_\Omega |u|^p \, dx + |v|^p \, dx\right)$$

for all $t, s \in [0,1]$. Let $s \in [0,1]$ and $0 \le t_\nu \le 1$ be a sequence converging to s. The claim is that

$$\lim_{\nu\to\infty} \int_\Omega |\nabla\gamma_{t_\nu} - \nabla\gamma_s|^p \, dx = 0.$$

Recall that γ_t was used in Chapter 3 to prove some uniqueness results. In particular it was shown that inequality

$$\int_\Omega \|\nabla\gamma_t(x)\|^p \, dx \le (1-t) \int_\Omega \|\nabla v(x)\|^p \, dx + t \int_\Omega \|\nabla u(x)\|^p \, dx \tag{4.1.2}$$

holds for all $t \in [0,1]$, so that

$$t \longmapsto \int_\Omega \|\nabla\gamma_t(x)\|^p \, dx$$

is a convex function, see Lemma 3.1.2. Since convex functions of one real variable are continuous,

$$\lim_{\nu\to\infty} \int_\Omega \|\nabla\gamma_{t_\nu}\|^p \, dx = \int_\Omega \|\nabla\gamma_s\|^p \, dx.$$

Therefore, to conclude it is sufficient to prove that $\partial_{x_i}\gamma_{t_\nu}$ converges to $\partial_{x_i}\gamma_s$ weakly in $L^p(\Omega)$, for all $i = 1, \dots, N$. But (4.1.2) implies that $\{\gamma_{t_\nu}\}_\nu$ is bounded in $W_0^{1,p}(\Omega)$. Thus by possibly passing to a subsequence $\partial_{x_i}\gamma_{t_\nu}$ converges weakly in $L^p(\Omega)$ to some limit w_i. Then the (weak) convergence in $L^p(\Omega)$ of γ_{t_ν} to γ_s implies that in fact $w_i = \partial_{x_i}\gamma_s$, see Lemma A.3.1. □

The following lemma is helpful.

Lemma 4.1.2. *Let $u, v \in M$, with $v \ge 0$ on Ω and u satisfying one of the following assumptions:*

(i) $u \ge 0$ on Ω;

(iii) the positive and negative parts of u are both not identically zero and

$$\frac{\displaystyle\int_\Omega \|\nabla u_+\|_{\ell^p}^p\,dx}{\displaystyle\int_\Omega u_+^p\,d\mu} \le \frac{\displaystyle\int_\Omega \|\nabla u_-\|_{\ell^p}^p\,dx}{\displaystyle\int_\Omega u_-^p\,d\mu}, \tag{4.1.3}$$

Then there exists a continuous curve $\gamma:[0,1]\to M$, such that

$$\int_\Omega \|\nabla\gamma_t(x)\|_{\ell^p}^p\,dx \le \max\left\{\int_\Omega \|\nabla u(x)\|_{\ell^p}^p\,dx, \int_\Omega \|\nabla v(x)\|_{\ell^p}^p\,dx\right\}, \qquad t\in[0,1].$$

Proof. The proof is different in the two cases.
(*i*) *Constant sign case.* If u is positive on Ω, one considers the curve $\gamma:[0,1]\to M$ defined by

$$\gamma_t(x) = \Big((1-t)\,u(x)^p + t\,v(x)^p\Big)^{\frac{1}{p}}, \qquad x\in\Omega, t\in[0,1]. \tag{4.1.4}$$

(*ii*) *"Nodal" case.* Suppose that u_+ and u_- are both non identically zero on Ω and that (4.1.3) holds. Set

$$\sigma_t(x) = \frac{u_+(x)-\cos(\pi\,t)\,u_-(x)}{\|u_+-\cos(\pi\,t)\,u_-\|_{L^p(\Omega,d\mu)}}, \qquad t\in\left[0,\frac{1}{2}\right].$$

Then σ_t is a continuous curve on M, connecting u to its (renormalized) positive part. Since u_+ and u_- have disjoint supports

$$\int_\Omega \|\nabla\sigma_t\|^p\,dx = \frac{\displaystyle\int_\Omega \|\nabla u_+\|^p\,dx + |\cos(\pi\,t)|^p \int_\Omega \|\nabla u_-\|^p\,dx}{\displaystyle\int_\Omega u_+^p\,d\mu + |\cos(\pi\,t)|^p\int_\Omega u_-^p\,d\mu}.$$

Note that

$$h(s) = \frac{a^2+s\,b^2}{c^2+s\,d^2}, \qquad s\in\mathbb{R},$$

is increasing if $b^2/d^2 \ge a^2/c^2$, so that $h(0)\le h(1)$. Thus (4.1.3) implies

$$\int_\Omega \|\nabla\sigma_t\|^p\,dx \le \int_\Omega \|\nabla\sigma_0\|^p\,dx = \int_\Omega \|\nabla u\|^p\,dx,$$

for all $t\in[0,1/2]$. In order to conclude, it is now sufficient to connect the (renormalized) positive part of u to v. To do so set

$$\widetilde{\sigma}_t(x) = \left((2-2\,t)\,\frac{u_+(x)^p}{\|u_+\|_{L^p(\Omega,d\mu)}^p} + (2\,t-1)\,v(x)^p\right)^{\frac{1}{p}}, \quad t\in\left[\frac{1}{2},1\right].$$

This is a continuous curve. To see that, the convexity of the functional along this curve may be exploited similarly as in step (i). Finally, one glues together the two curves defining

$$\gamma_t(x) = \sigma_t(x), \quad t \in \left[0, \frac{1}{2}\right] \qquad \text{and} \qquad \gamma_t(x) = \widetilde{\sigma}_t(x), \quad t \in \left[\frac{1}{2}, 1\right],$$

and the desired conclusion follows. □

Some further notations are needed. Given a pair of functions $u, v \in M \cap W_0^{1,p}(\Omega)$, denote by $\Gamma_\Omega(u, v)$ the set of continuous (in the $W^{1,p}$ topology) paths in $M \cap W_0^{1,p}(\Omega)$ connecting u to v, i.e.

$$\Gamma_\Omega(u, v) = \left\{\gamma : [0, 1] \to M \cap W_0^{1,p}(\Omega) \,:\, \gamma \text{ is continuous and } \gamma(0) = u,\, \gamma(1) = v\right\}.$$

Then $\lambda_2(\Omega)$ has the following further characterization. This is a well-known result, see [**32**, Theorem 3.1] for istance. The following proof contains an easier argument.

Theorem 4.1.3. *Let Ω be an open set of finite measure in $\mathbb{R}^N$, not necessarily connected. Then $\lambda_2(\Omega)$ has the following mountain pass characterization*

$$\lambda_2(\Omega) = \inf_{\gamma \in \Gamma_\Omega(u_1, -u_1)} \max_{u \in \gamma([0,1])} \int_\Omega \|\nabla u(x)\|^p \, dx. \tag{4.1.5}$$

If $\lambda_1(\Omega)$ is not simple, this characterization is independent of the particular u_1 we choose.

Proof. We distinguish two cases: either $\lambda_1(\Omega)$ is simple or not.

Case $\lambda_1(\Omega)$ simple. For every $\gamma \in \Gamma_\Omega(u_1, -u_1)$, the closed path on M obtained by gluing γ and $-\gamma$ is in fact the image of some odd continuous mapping f_γ from $\mathbb{S}^1$ to M. By the definition of $\lambda_2(\Omega)$,

$$\lambda_2(\Omega) \le \max_{u \in f_\gamma(\mathbb{S}^1)} \int_\Omega \|\nabla u(x)\|^p \, dx = \max_{u \in \gamma} \int_\Omega \|\nabla u(x)\|^p \, dx.$$

We agree that $u \in \gamma$ means $u \in \gamma([0, 1])$. Taking the infimum among all possible γ yields

$$\lambda_2(\Omega) \le \inf_{\gamma \in \Gamma_\Omega(-u_1, u_1)} \max_{u \in \gamma} \int_\Omega \|\nabla u(x)\|^p \, dx.$$

To prove the reverse inequality take a minimizing sequence $\{f_n\}_{n \in \mathbb{N}}$ of odd continuous mappings from the unit circle to M, say

$$\max_{u \in f_n(\mathbb{S}^1)} \int_\Omega \|\nabla u(x)\|^p \, dx \le \lambda_2(\Omega) + \frac{1}{n}.$$

Now for every $n \in \mathbb{N}$ pick $u^n \in f_n(\mathbb{S}^1)$ such that one of the hypotheses in Lemma 4.1.2 is satisfied[1]. Then there exists a continuous M-valued curve γ^n which connects the first

[1] Observe that it is always possible to make such a choice, since $f_n(\mathbb{S}^{k-1})$ is symmetric, i.e. if $u \in f_n(\mathbb{S}^1)$, then $-u \in f_n(\mathbb{S}^1)$ as well.

eigenfunction u_1 to u^n accomplishing the bound

$$\int_\Omega \|\nabla \gamma_t^n(x)\|^p\,dx \le \int_\Omega \|\nabla u^n(x)\|^p\,dx \le \lambda_2(\Omega) + \frac{1}{n}.$$

Symmetrically, the path $-\gamma^n$ connects $-u_1$ to $-u^n \in f_n(\mathbb{S}^1)$. The estimate above holds true on the new symmetrized path, since the functional is even. Therefore gluing the three paths γ^n, $-\gamma^n$ and f_n one gets a continuous curve $\Sigma^n \in \Gamma_\Omega(u_1, -u_1)$ such that

$$\max_{u\in\Sigma^n} \int_\Omega \|\nabla u(x)\|^p\,dx \le \lambda_2(\Omega) + \frac{1}{n}.$$

Passing to the infimum over $\Gamma_\Omega(u_1, -u_1)$ one obtains

$$\inf_{\gamma\in\Gamma_\Omega(u_1,-u_1)} \max_{u\in\gamma} \int_\Omega \|\nabla u\|^p\,dx \le \lambda_2(\Omega) + \frac{1}{n}.$$

To conclude the proof in this case, send $n \to \infty$.

Case $\lambda_1(\Omega)$ is multiple. Let Ω be not connected and let the first eigenvalue be not simple. Let $u_1, u_2 \in M$ be two linearly independent non-negative eigenfunctions corresponding to $\lambda_1(\Omega)$. Then the supports of u_1 and u_2 are included in different connected components of Ω. For all $t \in [0,1]$ set

$$\gamma_t(x) = \frac{\cos(\pi\,t)u_1(x) + t(1-t)u_2(x)}{\left(|\cos(\pi\,t)|^p + t^p(1-t)^p\right)^{1/p}}.$$

It is easy to see that γ has the following properties

$$\gamma_t \in M, \quad \text{for every } t\in[0,1], \qquad \text{and} \qquad \gamma_0 = u_1,\ \gamma_1 = -u_1,$$

i.e. the curve γ is admissible for the variational problem (4.1.5). In addition $\gamma_{1/2} = u_2$. Hence

$$\int_\Omega \|\nabla\gamma_t(x)\|^p\,dx = \frac{|\cos(\pi\,t)|^p\,\lambda_1(\Omega) + t^p(1-t)^p\lambda_1(\Omega)}{|\cos(\pi\,t)|^p + t^p(1-t)^p} = \lambda_1(\Omega_1)$$

for all $t \in [0,1]$. The fact that the functions u_1 and u_2 have disjoint supports was used. It follows that

$$\inf_{\gamma\in\Gamma_\Omega(u_1,-u_1)} \max_{t\in[0,1]} \int_\Omega \|\nabla\gamma_t(x)\|^p\,dx \le \lambda_1(\Omega).$$

In fact the equality holds, since $\lambda_1(\Omega)$ is the minimum of the Dirichlet energy on M. Moreover $\lambda_1(\Omega) = \lambda_2(\Omega)$, because of the multiplicity of the first eigenvalue. This shows that the Mountain-Pass formula gives $\lambda_2(\Omega)$. Observe that exchanging the role of u_1 and u_2 has no bearing in the above. Thus in this case formula (4.1.5) is independent of the choice of the particular first eigenfunction. $\square$

2. Low variational eigenvalues on disconnected domains

It is interesting to define and discuss the eigenvalues of the variational integrals

$$\mathcal{F}(u,\Omega)=\int_\Omega \|\nabla u(x)\|^p dx, \quad \mathcal{G}(u,\Omega)=\int_\Omega \rho(x)|u(x)|^p\, dx$$

in the case of a disconnected open set Ω in $\mathbb{R}^N$. For istance, in some spectral optimization problems with volume constraint the optimal shape is not connected.

Remark 4.2.1. The set of the eigenvalues on Ω is made of the collection of the eigenvalues on each single connected component of Ω. The eigenvalues are obtained by gathering and ordering increasingly the eigenvalues on the single pieces; correspondingly, each Dirichlet eigenfunction on Ω is an eigenfunction on one of the connected components Ω_i with zero Dirichlet conditions on $\partial\Omega_i$ and vanishes on the others.

For all domains Ω having finite N-dimensional Lebesgue measure, the minimum of the Rayleigh quotient

$$\lambda_1(\Omega) := \min_{u\in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \|\nabla u(x)\|^p\, dx}{\displaystyle\int_\Omega |u(x)|^p\, dx}$$

is attained. This follows by a standard compactness argument. Note that it is still denoted by the same symbol as the one used in the case of a connected domain. Besides, the minimum is a positive number, since the inequality

$$\int_\Omega \rho(x)|u(x)|^p\, dx \le C\int_\Omega \|\nabla u(x))\|^p\, dx$$

holds for all functions $u\in W_0^{1,p}(\Omega)$ with the same constant $C>0$. By the equivalence of all norms in $\mathbb{R}^N$, this can be seen via a straighforward contradiction argument based on the compactness of the Rellich-Kondrachev embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. As a matter of fact, the constant $C>0$ might depend on the set Ω. Nevertheless, a closer inspection shows that in fact

$$\int_\Omega \rho(x)|u(x)|^p\, dx \le C(N,p,q,a,\rho)|\Omega|^{\frac{p}{N}}\int_\Omega \|\nabla u(x)\|^p\, dx$$

where the constant now is independent of Ω. This can be rephrased by saying that in the scaling invariant shape optimization problem

$$\min_\Omega |\Omega|^{\frac{p}{N}}\lambda_1(\Omega)$$

the value of the minimum stays above some universal positive constant. As a consequence,

$$|\Omega_i|\to 0 \qquad \Longrightarrow \qquad \lambda_1(\Omega_i)\to +\infty. \tag{4.2.1}$$

Let now Ω be a disconnected set having finite measure and Ω_i be its connected components. Then the *first eigenvalue* on Ω is the least value among the first eigenvalues on the single connected components, i.e.

$$\lambda_1(\Omega) = \min_{i\in\mathbb{N}} \lambda_1(\Omega_i).$$

The well-posedness of the minimum and the equality follow by (4.2.1). Next proposition summarizes some apparent features of the first eigenvalue on (possibly disconnected) domains.

Proposition 4.2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set, having finite measure. Let $u \in W_0^{1,p}(\Omega)$ be a Dirichlet eigenfunction relative to some eigenvalue λ. If u has constant sign in Ω, then $\lambda = \lambda_1(\Omega_0)$ for some connected component Ω_0 of Ω, i.e. u is a first eigenfunction of Ω_0. In particular $\lambda = \lambda_1(\Omega)$ if Ω is connected.*

PROOF. Let first Ω be connected and u have constant sign on Ω, say $u \geq 0$. Then in fact $u > 0$ by the strong minimum principle of Theorem 2.2.2. Thus Theorem 3.2.1 implies that u is a first eigenfunction.

On the other hand, if Ω is disconnected, then λ has to be a Dirichlet eigenvalue of a certain connected component Ω_0; correspondigly u is an eigenfunction of Ω_0, having constant sign. Then it sufficies to apply the first part to conclude. $\square$

3. The Spectral gap theorem

The *second eigenvalue on* Ω *counted with multiplicity* is defined as

$$\lambda^*(\Omega) := \begin{cases} \lambda_1(\Omega), & \text{if } \lambda_1(\Omega) \text{ is not simple,} \\ \\ \min\Big\{\lambda > \lambda_1(\Omega)\colon \lambda \text{ is an eigenvalue}\Big\}, & \text{if } \lambda_1(\Omega) \text{ is simple.} \end{cases} \tag{4.3.1}$$

This well posed, see Remark 4.3.1. In fact $\lambda^*(\Omega)$ is has precisely the value given by the formula

$$\lambda_2(\Omega) := \inf_{f\in\mathscr{C}} \max_{u\in f(\mathbb{S}^1)} \int_\Omega \|\nabla u(x)\|^p \, dx,$$

where $\mathscr{C} = \{f\colon \mathbb{S}^1 \to M \mid f \text{ odd, continuous}\}$ and $M = \{u \in W_0^{1,p}(\Omega) : \int_\Omega \rho(x)|u|^p \, dx = 1\}$. Incidentally, by Theorem 4.1.3 that implies that $\lambda^*(\Omega)$ has the same value as the mountain pass-type formula.

Remark 4.3.1. Let Ω_i $(i = 1, 2, \ldots)$ be a labelling of the connected components of Ω, with the convention that $\lambda_1(\Omega) = \lambda_1(\Omega_1)$. Then

$$\lambda_*(\Omega) = \begin{cases} \lambda_1(\Omega), & \text{if } \exists\, i > 1\colon \lambda_1(\Omega_i) = \lambda_1(\Omega), \\ \\ \min\Big\{ \min\{\lambda_1(\Omega_i)\colon i > 1\}, \lambda_2(\Omega_1)\Big\}, & \text{otherwise.} \end{cases}$$

Note that the minimum inside is achieved by (4.2.1). Indeed, since $|\Omega| < \infty$, one has $|\Omega_i| \to 0$ as $i \to \infty$. Hence $\lambda_1(\Omega_i) \to +\infty$ as $i \to \infty$. This prevents the first eigenvalues $\lambda_1(\Omega_i)$ strictly larger than the least one from accumulating to the value taken by $\lambda_1(\Omega) = \lambda_1(\Omega_1)$.

In the following, a proof of the existence of a spectral gap is given.

Theorem 4.3.2. *Let Ω be an open set of finite measure, not necessarily connected. If $\lambda_1(\Omega)$ is simple, then* there is a spectral gap*, i.e.*

$$\lambda_1(\Omega) < \lambda_2(\Omega) \tag{4.3.2}$$

and

$$\lambda_2(\Omega) = \min\Big\{\lambda > 0\colon \lambda \textit{ is an eigenvalue larger than } \lambda_1(\Omega)\Big\}. \tag{4.3.3}$$

PROOF. Assume that the first eigenvalue $\lambda_1(\Omega)$ is simple. We divide the proof in steps. *Gap inequality.* Its "eigenspace" intersects the manifold M in two opposite points, say $u_1, -u_1$. If $0 < \varepsilon < 1/2$ the sets[2]

$$B_\varepsilon^+ = \{u \in M : \|u - u_1\|_{L^p(\Omega,d\mu)} < \varepsilon\}, \quad B_\varepsilon^- = \{u \in M : \|u + u_1\|_{L^p(\Omega,d\mu)} < \varepsilon\}$$

[2]For brevity we denote $\|\phi\|^p_{L^p(\Omega,d\mu)} = \int_\Omega |\phi|^p \, d\mu$.

are disjoint. To prove inequality (4.3.2) one argues by contradiction assuming that the limit

$$\lim_{j\to\infty} \max_{u\in f_j(\mathbb{S}^1)} \int_\Omega \|\nabla u(x)\|^p \, dx = \lambda_1(\Omega) \tag{4.3.4}$$

holds through a sequence $f_j \in \mathscr{C}$ of odd and continuous mappings from $\mathbb{S}^1$ to M. Since the functions f_j are odd and continuous, their images $f_j(\mathbb{S}^1)$ are symmetric and connected subsets of M. Then, for all $j \in \mathbb{N}$ one can pick an element u_j of $f_j(\mathbb{S}^1)$ which belongs neither to B_ε^+ nor to B_ε^-. The growth assumptions on the Lagrangian density H imply that the sequence is bounded in $W_0^{1,p}(\Omega)$. Indeed by (4.3.4)

$$\sup_{j\in\mathbb{N}} \int_\Omega \|\nabla u(x)\|^p dx < +\infty.$$

Owing to Rellich Kondrachev theorem, up to relabelling the sequence there exists $u \in M$ such that u_j converges to u weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. By the Dirichlet integral is lower semicontinuous with respect to the weak convergence

$$\int_\Omega \|\nabla u(x)\|^p \, dx \le \liminf_{j\to\infty} \int_\Omega \|\nabla u_j(x)\|^p \, dx.$$

Using (4.3.4) again, it follows that u is a first eigenfunction, say $u = u_1$. On the other hand

$$u_j \in f_j(\mathbb{S}^1) \setminus \left(B_\varepsilon^+ \cup B_\varepsilon^-\right), \quad j \in \mathbb{N},$$

hence $\|u_j - u_1\|_{L^p(\Omega, d\mu)} \ge \varepsilon$ for all $j \in \mathbb{N}$. That gives the desired contradiction, since the weighted L^p norm is continuous with respect to the L^p convergence.

Any nodal eigenvalue is larger or equal than $\lambda_2(\Omega)$. The idea of the proof is from [**58**]. Let λ be an eigenvalue and $u \in W_0^{1,p}(\Omega)$ be an eigenfunction corresponding to λ. Assume that $u > 0$ on A, $u < 0$ on B, where $A, B \subset \Omega$ are non-empty open sets and denote by $u_A = u\chi_A$ (resp., $u_B = u\chi_B$) the restriction of u to A (resp., to B). Testing the Euler-Lagrange equation with $\varphi = u_A$ yields

$$\int_A \|\nabla u(x)\|^p \, dx = \lambda \int_A |u|^p \, d\mu.$$

A similar identity is obtained using u_B rather than u_A. The function $f \colon \mathbb{S}^1 \to M$ defined by

$$f(\alpha, \beta) = \alpha u_A + \beta u_B,$$

for all $(\alpha, \beta) \in \mathbb{S}^1$, is odd and continuous. Thus the definition of $\lambda_2(\Omega)$ implies that

$$\max_{(\alpha,\beta)\in\mathbb{S}^1} \int_\Omega \|\nabla f(\alpha, \beta)\|^p \, dx \ge \lambda_2(\Omega).$$

On the other hand,

$$\int_\Omega \|\nabla f(\alpha,\beta)\|^p\,dx = |\alpha|^p \int_A \|\nabla u(x)\|^p + |\beta|^p \int_B \|\nabla u(x)\|^p$$
$$= \lambda\left\{|\alpha|^p \int_A |u|^p\,d\mu + |\beta|^p \int_B |u|^p\,d\mu\right\} = \lambda$$

for all $(\alpha,\beta)\in\mathbb{S}^1$. Hence

$$\max_{(\alpha,\beta)\in\mathbb{S}^1} \int_\Omega \|\nabla f(\alpha,\beta)\|^p\,dx = \lambda.$$

Thus $\lambda \geq \lambda_2(\Omega)$.

$\lambda_2(\Omega)$ *is the least higher eigenvalue.* The purpose of the final step is to prove characterization (4.3.3). By Theorem 1.4.1, if Ω is connected there is nothing to prove. Indeed, in that case any eigenfunction corresponding to $\lambda_2(\Omega)$ has to change sign. Thus, assume that Ω has a countable family of connected components Ω_i. By definition of eigenvalues on a disconnected set, we have $\lambda_1(\Omega)=\lambda_1(\Omega_1)$, up to relabelling. Now

$$\lambda^*(\Omega) = \min\Big\{\lambda>0\colon \lambda \text{ is an eigenvalue larger than } \lambda_1(\Omega)\Big\}.$$

It remains to show that $\lambda^*(\Omega)=\lambda_2(\Omega)$. Pick an eigenfunction u_* corresponding to $\lambda^*(\Omega)$, normalized so that it has unit norm in $L^p(\Omega,d\mu)$. The following alternative holds:

$$\text{either} \qquad \lambda^*(\Omega)=\lambda_2(\Omega_1) \qquad \text{or} \qquad \lambda^*(\Omega)=\min_{i>1}\lambda_1(\Omega_i).$$

Assume that $\lambda^*(\Omega)=\lambda_2(\Omega_1)$. By the first step $\lambda_2(\Omega_1)>\lambda_1(\Omega_1)$, since Ω_1 is connected. Therefore, according to Proposition 4.2.2 u_* has to change sign. Hence $\lambda^*(\Omega)\geq\lambda_2(\Omega)$ by the previous step. To prove the reverse inequality, note that the support of u_* is contained in Ω_1 and

$$\lambda^*(\Omega)=\lambda_2(\Omega_1)=\inf_{f\in\mathscr{C}_1}\max_{u\in f(\mathbb{S}^1)}\int_\Omega \|\nabla u(x)\|^p\,dx,$$

where $\mathscr{C}_1$ is the subfamily of $\mathscr{C}$ consisting of all odd and continuous M-valued mappings f defined on $\mathbb{S}^1$ such that the function f_ω is only supported on the connected component Ω_1, rather than in the whole of Ω. Since the infimum is performed on a smaller family than the one involved in the definition of $\lambda_2(\Omega)$, one has

$$\lambda^*(\Omega)\geq\lambda_2(\Omega).$$

This concludes the proof for the first alternative.

Now, assume that $\lambda_1(\Omega)=\lambda_1(\Omega_1)$ and $\lambda^*(\Omega)=\lambda_1(\Omega_2)$. Pick an eigenfunction u_* corresponding to $\lambda^*(\Omega)$. Since u_* is a first eigenfunction on Ω_2, which is connected, up to a sign $u_*>0$ due to the strong minimum principle. Let u_1 be a (normalized) first eigenfunction on Ω (i.e., on Ω_1). Consider the continuous path $\gamma:[0,1]\to M$

$$\gamma_t(x)=\frac{\cos(\pi\,t)u_1(x)+t(1-t)u_*(x)}{\left(|\cos(\pi\,t)|^p+t^p(1-t)^p\right)^{1/p}},\ x\in\Omega,$$

for all $t \in [0,1]$. It is easy to see that γ has the following properties

$$\gamma_t \in M \cap W_0^{1,p}(\Omega), \quad \text{for every } t \in [0,1], \qquad \text{and} \qquad \gamma_0 = u_1, \ \gamma_1 = -u_1.$$

Denote by f its odd extension to $[-1,1]$. Up to a re-parametrization, f is an admissible competitor for the infimum in the min-max formula, hence

$$\lambda_2(\Omega) \leq \max_{0 \leq t \leq 1} \int_\Omega \|\nabla \gamma_t(x)\|^p \, dx.$$

Note that u_1 and u_* have disjoint supports. Then

$$\|\nabla \gamma_t(x)\|^p = \frac{|\cos(\pi t)|^p \|\nabla u_1(x)\|^p + t^p(1-t)^p \|\nabla u_*(x)\|^p}{|\cos(\pi t)|^p + t^p(1-t)^p}, \qquad \text{a.e. in } \Omega,$$

for all $t \in [0,1]$. Integrating on Ω one gets

$$\max_{t \in [0,1]} \int_\Omega H(x, \nabla \gamma_t) = \max_{t \in [0,1]} \frac{|\cos(\pi t)|^p \lambda_1(\Omega_1) + t^p(1-t)^p \lambda_1(\Omega_2)}{|\cos(\pi t)|^p + t^p(1-t)^p} = \lambda_1(\Omega_2).$$

The fact that

$$\max_{t \in [0,1]} \frac{|\cos(\pi t)|^p A_1 + t^p(1-t)^p A_2}{|\cos(\pi t)|^p + t^p(1-t)^p} = \max\{A_1, A_2\}, \qquad A_1, A_2 > 0,$$

was used. Therefore $\lambda_2(\Omega) \leq \lambda_1(\Omega_2) = \lambda^*(\Omega)$. By the first step, $\lambda_2(\Omega) > \lambda_1(\Omega)$. Since by definition $\lambda^*(\Omega)$ is the least eigenvalue on Ω strictly larger than $\lambda_1(\Omega)$, it follows that $\lambda_2(\Omega) = \lambda^*(\Omega)$. $\square$

To conclude the chapter, it is proved that the existence of several linearly independent first eigenfunctions makes $\lambda_2(\Omega)$ end in collapsing to the same value as $\lambda_1(\Omega)$.

Theorem 4.3.3. *Let Ω be a disconnected domain of finite Lebesgue measure. If $\lambda_1(\Omega)$ is not simple then $\lambda_2(\Omega) = \lambda_1(\Omega)$.*

Proof. The proof is from [**F5**]. Given two linearly independent first eigenfuctions $u, v > 0$, the continuous curve $\gamma\colon [0,1] \to M$ defined by

$$\gamma_t(x) = \frac{\cos(\pi t) u(x) + t(1-t) v(x)}{\big(|\cos(\pi t)|^p + t^p(1-t)^p\big)^{\frac{1}{p}}}$$

joins u to $-u$. According to Theorem 4.1.3, the class of admissible competitors in the mountain pass formula (4.1.5) is independent of the choice of the first eigenfuction. Thus γ is an admissible competitor in that infimum problem so that

$$\lambda_2(\Omega) \leq \int_\Omega \|\nabla \gamma_t(x)\|^p \, dx = \lambda_1(\Omega).$$

Conversely, $\lambda_1(\Omega)$ is the least eigenvalue, hence opposite inequality obviously holds. That concludes the proof. $\square$

Summarizing, the second term $\lambda_2(\Omega)$ in the sequence of variational min-max eigenvalues recovers the value of the *second eigenvalue counted with multiplicity* defined by (4.3.1).

Corollary 4.3.4. *Let Ω be an open set in $\mathbb{R}^N$, not necessarily connected. Then*

$$\lambda_2(\Omega) = \begin{cases} \min\Big\{\lambda > \lambda_1(\Omega)\colon \lambda \text{ is an eigenvalue}\Big\}, & \text{if } \lambda_1(\Omega) \text{ is simple,} \\ \\ \lambda_1(\Omega), & \text{otherwise.} \end{cases}$$

PROOF. The statement plainly follows by Theorems 4.3.2 and 4.3.3. $\square$

CHAPTER 5

Optimization of low Dirichlet p-eigenvalues

In this chapter we shall focus on the nonlinear operator acting on scalar functions u defined on some open set Ω in $\mathbb{R}^N$, namely the so-called p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

The operator relies on the variational integral

$$\int_\Omega |\nabla u|^p \, dx$$

subject to the contraint

$$\int_\Omega |u|^p \, dx = \text{const}\,.$$

The optimal shape Ω for the low eigenvalues of $-\Delta_p$, with volume constraint, are exhibited and the stability of optimizers is discussed.

Given an open set $\Omega \subset \mathbb{R}^N$ having finite measure and $p \in (1,\infty)$, we define the L^p unitary sphere

$$M_p(\Omega) = \{u \in L^p(\Omega) \,:\, \|u\|_{L^p(\Omega)} = 1\},$$

and we indicate with $W_0^{1,p}(\Omega)$ the usual Sobolev space, given by the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}.$$

Recall that λ a *Dirichlet eigenvalue* of $-\Delta_p$ in Ω if there exists a non trivial $u \in W_0^{1,p}(\Omega)$ satisfying

$$-\Delta_p u = \lambda |u|^{p-2}\, u, \quad \text{in } \Omega \tag{5.0.5}$$

in a weak sense, i.e.

$$\int_\Omega \langle |\nabla u(x)|^{p-2}\, \nabla u(x), \nabla\varphi(x)\rangle \, dx = \lambda \int_\Omega |u(x)|^{p-2}\, u(x)\, \varphi(x)\, dx, \qquad \text{for every } \varphi \in W_0^{1,p}(\Omega).$$

Correspondingly u will be a *Dirichlet eigenfunction* of $-\Delta_p$. In particular, observe that for every such a pair (λ, u) there results

$$\int_\Omega |\nabla u(x)|^p \, dx = \lambda \int_\Omega |u(x)|^p \, dx.$$

Remark 5.0.5. Observe that even in this model equation general solutions of (5.0.5) are just in $C^{1,\alpha}$. In fact, the second derivatives cannot exist in a weak sense either, unless $1 < p \leq 2$ (see [**6**]). Then eigenfunctions in general are not classical solutions of the equation (5.0.5).

Remark 5.0.6. If Ω is connected, then for all eigenfunctions $u \geq 0$ the following alternative holds: either $u \equiv 0$ or $u > 0$. This follows by a Harnack inequality, see Theorem 2.2.1.

Recall Theorem 1.4.1. It is possible to show the existence of a diverging sequence of variational eigenvalues of $-\Delta_p$

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots \leq \lambda_n(\Omega) \to \infty$$

In the following, a particular class of shape optimization problems involving this variational spectrum of $-\Delta_p$ is considered. The variational eigenvalues are monotone decreasing with respect to set inclusion in the sense that

$$\Omega_1 \subset \Omega_2 \qquad \Longrightarrow \qquad \lambda_n(\Omega_1) \geq \lambda_n(\Omega_2).$$

Their scaling properties read as follows:

$$\lambda_n(t\,\Omega) = t^{-p}\lambda_n(\Omega), \quad t > 0,\ n = 1, 2, \ldots$$

In words, the eigenvalue scales as a length to the power p. In particular, the shape functional $\Omega \mapsto |\Omega|^{p/N}\,\lambda_n(\Omega)$ is scaling invariant. Thus the two shape optimization problems

$$\min\{\lambda_n(\Omega)\,:\,|\Omega| = c\} \qquad \text{and} \qquad \min|\Omega|^{p/N}\,\lambda_n(\Omega), \qquad n = 1, 2, \ldots$$

are equivalent, in the sense that they both provide the same optimal shapes, up to a scaling.

Remark 5.0.7. Concerning the *a priori* regularity of the shape optimizing the n-th eigenvalue, one should expect significantly much weaker results than the classical ones for minimal surfaces. Indeed, at variance with the shape functional given by the (distributional) perimeter, the eigenvalues are affected by dilations rather than oscillations concentrated on the boundary.

Remark 5.0.8. In shape optimization problems, the dependance of the cost functional on the admissible shapes is often given through a variational problem. That is the case for the Dirichlet p-eigenvalues $\lambda_n(\Omega)$. A comment on the way how $\lambda_n(\Omega)$ depends on the shape Ω is appropriate. The continuity of the eigenvalue with respect to Ω can be made rigorously justified up to restricting to a suitable class $\mathcal{A}_0$ of open sets near to a given reference Ω_0. Namely, it is possible to define a notion of distance d_0 such that

$$\left|\lambda_n(\Omega_1) - \lambda_n(\Omega_2)\right| \lesssim d_0(\Omega_1, \Omega_2)$$

provided that the right hand side is sufficiently small. The interested reader is referred to [**24**]. If both of Ω_1 and Ω_2 are contained in a small ε-neighborhood of the other, then such estimates read

$$\left|\lambda_n(\Omega_1) - \lambda_n(\Omega_2)\right| \lesssim \varepsilon^\alpha$$

provided that Ω_1, Ω_2 are choosen in the subclass $\mathcal{A}_0^\alpha$ of shapes having $C^{0,\alpha}$ boundaries. In the linear case ($p = 2$) similar estimates become sharp, see [**25**] for details on this topic. We mention the nice inequality

$$\big|\lambda_n(\Omega_1) - \lambda_n(\Omega_2)\big| \lesssim |\Omega_1 \Delta \Omega_2|$$

which holds for all pairs of shapes sufficiently close to a reference Ω_0.

The purpose of this chapter is to adress some issues related rather to the stability of the eigenvalues than to their continuity. The two viewpoints are opposite in a sense. If this chapter was the answer, then the question would be the following:

$$\lambda_n(\Omega) \cong \text{ optimal} \qquad \overset{?}{\Longrightarrow} \qquad \Omega \cong \text{ optimal.}$$

A positive answer is given for $n = 1, 2$.

Remark 5.0.9. In the case of low eigenvalues of the Dirichlet p-Laplacian, the optimal shapes are known, see the discussion on Faber-Krahn and Hong-Krahn-Szego inequality below. For a more general shape optimization problem, existence has to be proved before adressing the stability issue, and regularity has to be discussed. Flexible tools are provided in the framework of γ-convergence, for which the reader is referred to [**54**]. Concerning the n-th Dirichlet eigenvalue of the (linear) Laplacian, the existence of an optimal shape under volume constrained was recently established independentely by Mazzoleni and Pratelli in [**76**] and Bucur [**21**]. The variational direct methods of spectral optimization provide as an optimizer a capacitary measure rather than an open set. To prove that a general spectral optimization problem is solvable in a set of (possibly smooth) open sets is an nowadays an outstanding problem. This is again not necessary once the optimal shape is known.

1. Faber-Krahn inequality

The optimal shape for the first eigenvalue under volume constraint is well-known. As one may expect, it is given by any ball. This is the celebrated *Faber-Krahn inequality*. The classical proof combines the Schwarz symmetrization with the so called Pólya-Szegő principle (see [**54**, Chapter 3], for example). Basically, if u is a first eigenfunction on a set Ω of unit volume, by denoting u^* its spherical rearrangement on the ball B of radius 1 then Pólya-Szegő inequality yields

$$\lambda_1(\Omega) = \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx} \geq \frac{\displaystyle\int_B |\nabla u^*|^p \, dx}{\displaystyle\int_B |u^*|^p \, dx} \geq \lambda_1(B).$$

A finer analysis shows that the only possible equality case is achieved by choosing $\Omega = B$.

Faber-Krahn Inequality. *Let $1 < p < \infty$. For every open set $\Omega \subset \mathbb{R}^N$ having finite measure, we have*

$$|\Omega|^{p/N} \lambda_1(\Omega) \geq \omega_N^{p/N} \lambda_1(B), \tag{5.1.1}$$

where B is the $N-$dimensional ball of radius 1 and $\omega_N := |B|$. Moreover, equality sign in (5.1.1) holds if and only if Ω is a ball.

In other words, for every $c > 0$ the unique solutions of the following spectral optimization problem

$$\min\{\lambda_1(\Omega) \,:\, |\Omega| = c\},$$

are given by balls having measure c.

To make this section complete, it should be mentioned that a quantitative version of Faber-Krahn inequality is at disposal. The *Fraenkel asymmetry* of an open set $\Omega \subset \mathbb{R}^N$ having finite measure is defined by

$$\mathcal{A}(\Omega) = \inf\left\{\frac{\|1_\Omega - 1_B\|_{L^1}}{|\Omega|} \,:\, B \text{ is a ball such that } |B| = |\Omega|\right\}.$$

This is a scaling invariant quantity such that $0 \leq \mathcal{A}(\Omega) < 2$, with $\mathcal{A}(\Omega) = 0$ if and only if Ω coincides with a ball, up to a set of measure zero. Then we recall the following quantitative improvement of the Faber-Krahn inequality, proven in [**14**] (case $N = 2$) and [**44**] (general case). For every $\Omega \subset \mathbb{R}^N$ open set with $|\Omega| < \infty$, we have

$$|\Omega|^{p/N} \lambda_1(\Omega) \geq \omega_N^{p/N} \lambda_1(B) \left[1 + \gamma_{N,p} \mathcal{A}(\Omega)^{\kappa_1}\right], \tag{5.1.2}$$

where $\gamma_{N,p}$ is a constant depending only on N and p and the exponent $\kappa_1 = \kappa_1(N,p)$ is given by

$$\kappa_1(N,p) = \begin{cases} 3, & \text{if } N = 2, \\ 2 + p, & \text{if } N \geq 3. \end{cases}$$

Remark 5.1.1. One may ask wheter the exponent κ_1 in (5.1.2) is sharp or not. By introducing the *deficit*

$$FK(\Omega) := \frac{|\Omega|^{p/N} \lambda_1(\Omega)}{\omega_N^{p/N} \lambda_1(B)} - 1,$$

one would like to prove the existence of suitable deformations $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of a ball B, such that

$$\lim_{\varepsilon\to 0} FK(\Omega_\varepsilon) = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \frac{\mathcal{A}(\Omega_\varepsilon)^{\kappa_1}}{FK(\Omega_\varepsilon)} = \ell \neq \{0, +\infty\}.$$

i.e. the asymmetry to the power κ_1 and the deficit have the same decay rate to 0. At least in the case $p = 2$, the answer should be *no*, since the conjectured sharp exponent is 2 (see [**15**, pag. 56]), while $\kappa_1(N, 2) \geq 3$. At present, a proof of this fact still lacks.

2. The Hong-Krahn-Szego inequality

In this section, we are going to prove that the disjoint unions of equal balls are the only sets minimizing λ_2 under volume constraint, i.e. we will prove the Hong-Krahn-Szego inequality for the $p-$Laplacian. A key step in the proof is provided by the following technical lemma. That is an extension of a similar result which holds in the linear case $p = 2$ (see [**19**, Lemma 3.1], for example).

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set with $|\Omega| < \infty$. Then there exists Ω_+, Ω_- disjoint subsets of Ω such that*

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\}. \tag{5.2.1}$$

Proof. Let us take $u_1, u_2 \in M_p(\Omega) \cap W_0^{1,p}(\Omega)$ a first and second eigenfunction, respectively: notice that if $\lambda_1(\Omega)$ is not simple, we mean that u_1 and u_2 are two linearly independents eigenfunctions corresponding to $\lambda_1(\Omega)$. We can distinguish two alternatives:

(i) u_2 is sign-changing;
(ii) u_2 has constant sign in Ω.

Let us start with (i): in this case, u_2 has exactly two nodal domains

$$\Omega_+ = \{x \in \Omega \,:\, u_2(x) > 0\} \qquad \text{and} \qquad \Omega_- = \{x \in \Omega \,:\, u_2(x) < 0\},$$

which by definition are connected sets. The restriction of u_2 to Ω_+ is an eigenfunction of constant sign for Ω_+, then Theorem 4.2.2 implies that u_2 must be a first eigenfunction for it. Replacing Ω_+ with Ω_-, the previous observation leads to

$$\lambda_2(\Omega) = \lambda_1(\Omega_-) = \lambda_1(\Omega_+).$$

which implies in particular (5.2.1) in this case.

In case (ii), let us set

$$\Omega_+ = \{x \in \Omega \,:\, |u_1(x)| > 0\} \qquad \text{and} \qquad \Omega_- = \{x \in \Omega \,:\, |u_2(x)| > 0\}.$$

Using Theorem 4.2.2, we have that Ω_+ and Ω_- have to be two distinct connected components of Ω: in addition u_1, u_2 are eigenfunctions (with constant sign) of Ω_+ and Ω_-, respectively. Then

$$\lambda_1(\Omega_-) = \int_{\Omega_-} |\nabla u_2(x)|^p \, dx = \int_\Omega |\nabla u_2(x)|^p = \lambda_2(\Omega).$$

Clearly, one also has $\lambda_1(\Omega_+) = \lambda_1(\Omega) \le \lambda_2(\Omega)$, which finally gives (5.2.1) also in this case. □

We are now ready for the main result of this section.

Theorem 5.2.2 (HKS inequality for the $p-$Laplacian)**.** *For every $\Omega \subset \mathbb{R}^N$ open set having finite measure, we have*

$$|\Omega|^{p/N}\, \lambda_2(\Omega) \ge 2^{p/N}\, \omega_N^{p/N}\, \lambda_1(B), \tag{5.2.2}$$

where B is the $N-$dimensional ball of radius 1 and $\omega_N := |B|$. Moreover, equality sign in (5.2.2) holds if and only if Ω is the disjoint union of two equal balls.

In other words, for every $c > 0$ the unique solutions of the following spectral optimization problem

$$\min\{\lambda_2(\Omega) \,:\, |\Omega| = c\},$$

are given by disjoint unions of two balls, both having measure $c/2$.

PROOF. With the notation of Lemma 5.2.1, an application of the Faber-Krahn inequality yields

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\} \geq \max\{\lambda_1(B_+), \lambda_1(B_-)\}, \tag{5.2.3}$$

where B_+, B_- are balls such that $|B_+| = |\Omega_+|$ and $|B_-| = |\Omega_-|$. Thanks to the scaling properties of λ_1, we have

$$\lambda_1(B_+) = \left(\frac{\omega_N}{|\Omega_+|}\right)^{p/N} \lambda_1(B) \qquad \text{and} \qquad \lambda_1(B_-) = \left(\frac{\omega_N}{|\Omega_-|}\right)^{p/N} \lambda_1(B),$$

so that from (5.2.3) we obtain

$$\lambda_2(\Omega) \geq \omega_N^{p/N} \lambda_1(B) \, \max\{|\Omega_+|^{-p/N}, |\Omega_-|^{-p/N}\}.$$

Finally, observe that since $|\Omega_+| + |\Omega_-| \leq |\Omega|$, we get

$$\max\{|\Omega_+|^{-p/N}, |\Omega_-|^{-p/N}\} \geq \left(\frac{|\Omega|}{2}\right)^{-p/N}, \tag{5.2.4}$$

which concludes the proof of the inequality.

As for the equality cases, we start observing that we just used two inequalities, namely (5.2.3) and (5.2.4). On the one hand, equality in (5.2.3) holds if and only if at least one among the two subsets is a ball, say $\Omega_+ = B_+$, with $\lambda_1(B_+) \geq \lambda_1(\Omega_-)$; on the other hand, if equality holds in (5.2.4) then we must have $|\Omega_+| = |\Omega_-| = |\Omega|/2$. Since Ω_+ and Ω_- have the same measure and the one with the greatest λ_1 is a ball, we can conclude that both have to be a ball, thanks to the equality cases in the Faber-Krahn inequality. □

3. The stability issue

We now come to the question of stability for optimal shapes of λ_2 under measure constraint. In particular, we will enforce the lower bound on $|\Omega|^{2/N} \lambda_2(\Omega)$ provided by the Hong-Krahn-Szego inequality, by adding a remainder terms in the right-hand side of (5.2.2). At this aim, we need to introduce some further tools. In the case of the Hong-Krahn-Szego inequality, the relevant notion of asymmetry is the *Fraenkel* $2-$*asymmetry*, introduced in [**19**]

$$\mathcal{A}_2(\Omega) = \inf\left\{\frac{\|1_\Omega - 1_{B_1\cup B_2}\|_{L^1}}{|\Omega|} \,:\, B_1, B_2 \text{ balls such that } |B_1 \cap B_2| = 0,\, |B_i| = \frac{|\Omega|}{2},\, i = 1,2\right\}.$$

The main result of this section is the following quantitative version of Theorem 5.2.2.

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, with $|\Omega| < \infty$ and $p \in (1, \infty)$. Then*

$$|\Omega|^{p/N} \lambda_2(\Omega) \geq 2^{p/N}\, \omega_N^{p/N}\, \lambda_1(B)\; [1 + C_{N,p}\, \mathcal{A}_2(\Omega)^{\kappa_2}]\,, \tag{5.3.1}$$

with $C_{N,p} > 0$ constant depending on N and p only and $\kappa_2 = \kappa_2(N, p)$ given by

$$\kappa_2(N, p) = \kappa_1(N, p) \cdot \frac{N+1}{2}.$$

Proof. Thanks to Lemma 5.2.1, we have existence of two disjoint sets $\Omega_+, \Omega_- \subset \Omega$ such that (5.2.1) holds. We then set

$$\delta_+ = |\Omega_+| - \frac{|\Omega|}{2}, \qquad\qquad \delta_- = |\Omega_-| - \frac{|\Omega|}{2}\,,$$

and we observe that it must be $\delta_+ + \delta_- \leq 0$, since $|\Omega_+| + |\Omega_-| \leq |\Omega|$. To simplify a bit formulas, let us introduce the *deficit functional*

$$HKS(\Omega) := \frac{|\Omega|^{p/N} \lambda_2(\Omega)}{2^{p/N}\, \omega_N^{p/N}\, \lambda_1(B)} - 1.$$

In order to prove (5.3.1), we just need to show that

$$HKS(\Omega) \geq C_{N,p} \max\left\{ \mathcal{A}(\Omega_+)^{\kappa_1} + \left|\frac{\delta_+}{|\Omega|}\right|,\, \mathcal{A}(\Omega_-)^{\kappa_1} + \left|\frac{\delta_-}{|\Omega|}\right| \right\}\,, \tag{5.3.2}$$

then a simple application of Lemma 5.3.2 below will conclude the proof. To obtain (5.3.2), it will be useful to distinguish between the case $p \leq N$ and the case $p > N$. For both of them, we will in turn treat separately the case where both δ_+ and δ_- are non positive and the case where they have opposite sign. Finally, since the quantities appearing in the right-hand side of (5.3.2) are all universally bounded, it is not restrictive to prove (5.3.2) under the additional assumption

$$HKS(\Omega) \leq \frac{1}{4}. \tag{5.3.3}$$

Indeed, it is straightforward to see that if $HKS(\Omega) > 1/4$ then (5.3.2) trivially holds with constant

$$C_{N,p} = \frac{1}{2}\, \frac{1}{2^{\kappa_1+1} + 1} > 0.$$

Case A: $p \leq N$. In this case the proof runs very similarly to the linear case $p = 2$ treated in [**19**]. We start applying the quantitative Faber-Krahn inequality (5.1.2) to Ω_+. If we indicate with B the ball of unit radius, recalling (5.2.1) and using the definition of δ_+, we find

$$\gamma_{N,p}\, \mathcal{A}(\Omega_+)^{\kappa_1} \leq \frac{|\Omega_+|^{p/N}\, \lambda_1(\Omega_+)}{\omega_N^{p/N}\, \lambda_1(B)} - 1 \leq \frac{(|\Omega| + 2\,\delta_+)^{p/N}\, \lambda_2(\Omega)}{2^{p/N} \omega_N^{p/N}\, \lambda_1(B)} - 1$$

Since $p \le N$, the power function $t \mapsto (|\Omega| + t)^{p/N}$ is concave, thus we have

$$(|\Omega| + 2\delta_+)^{p/N} \le |\Omega|^{p/N} + \frac{2p}{N}\,|\Omega|^{p/N}\,\frac{\delta_+}{|\Omega|}.$$

Using this information in the previous inequality, we get

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega) + \frac{2p}{N}\,\frac{\delta_+}{|\Omega|}\,\frac{|\Omega|^{p/N}\,\lambda_2(\Omega)}{2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)},$$

that we can rewrite as follows

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega) + \frac{2p}{N}\,\frac{\delta_+}{|\Omega|}\,(HKS(\Omega) + 1)\,. \tag{5.3.4}$$

Replacing Ω_+ with Ω_-, one obtains a similar estimate for Ω_-.

Case A.1: δ_+ and δ_- are both non-positive. In this case, it is enough to observe that $HKS(\Omega) \ge 0$ while $\delta_+ \le 0$, thus from (5.3.4) we get

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} + \frac{2p}{N}\,\frac{|\delta_+|}{|\Omega|} \le HKS(\Omega).$$

The same computations with Ω_- in place of Ω_+ yield (5.3.2).

Case A.2: δ_+ and δ_- have opposite sign. Let us assume for example that $\delta_+ \ge 0$ and $\delta_- \le 0$: the main difference with the previous case is that now the larger piece Ω_+ could be so large that the information provided by (5.2.1) is useless. However, estimate (5.3.4) still holds true for both Ω_+ and Ω_-. Using this and the fact that $\delta_+ + \delta_- \le 0$, we can thus infer

$$HKS(\Omega) \ge -\frac{2p}{N}\frac{\delta_-}{|\Omega|} \ge \frac{2p}{N}\frac{\delta_+}{|\Omega|},$$

i.e. the deficit is controlling the error term $|\delta_+|/|\Omega|$. To finish, we still have to control the asymmetry of the larger piece Ω_+ in terms of the deficit: it is now sufficient to introduce the previous information into (5.3.4), thus getting

$$\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \le HKS(\Omega)(2 + HKS(\Omega)).$$

Since we are assuming $HKS(\Omega) \le 1/4$, the previous implies that $HKS(\Omega)$ controls $\mathcal{A}(\Omega_+)^{\kappa_1}$, modulo a constant depending only on N and p. These estimates on Ω_+, together with the validity of (5.3.4) for Ω_- and with the fact that $\delta_- \le 0$, ensure that (5.3.2) holds also in this case.

Case B: $p > N$. Let us start once again with Ω_+. Using (5.2.1) and the quantitative Faber-Krahn (5.1.2) as before, we get

$$HKS(\Omega) \ge \frac{|\Omega|^{p/N}\,\lambda_1(\Omega_+)}{2^{p/N}\,\omega_N^{p/N}\,\lambda_1(B)} - 1 \ge \left[\left(\frac{|\Omega|}{2\,|\Omega_+|}\right)^{p/N}(1 + \gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1}) - 1\right].$$

Then using the definition of δ_+ and the convexity of the function $t \mapsto (1+t)^{p/N}$ (since $p > N$), we have

$$\left(\frac{|\Omega|}{2\,|\Omega_+|}\right)^{p/N} = \left(1 - \frac{\delta_+}{|\Omega_+|}\right)^{p/N} \geq 1 - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}.$$

Inserted in the previous estimate, this yields

$$HKS(\Omega) \geq \left[\gamma_{N,p}\left(1 - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}\right)\mathcal{A}(\Omega_+)^{\kappa_1} - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}\right]. \tag{5.3.5}$$

In the same way, using Ω_- in place of Ω_+, we obtain a similar estimate for Ω_-.

Case B.1: δ_+ and δ_- are both non positive. In this case, in (5.3.5) we can drop the terms

$$-\frac{p}{N}\frac{\delta_+}{|\Omega_+|}\gamma_{N,p}\,\mathcal{A}(\Omega_+)^{\kappa_1} \qquad \text{and} \qquad -\frac{p}{N}\frac{\delta_-}{|\Omega_-|}\gamma_{N,p}\,\mathcal{A}(\Omega_-)^{\kappa_1},$$

since these are positive, thus we arrive once again at (5.3.2), keeping into account that

$$-\frac{\delta_+}{|\Omega_+|} \geq -\frac{\delta_+}{|\Omega|} \qquad \text{and} \qquad -\frac{\delta_-}{|\Omega_-|} \geq -\frac{\delta_-}{|\Omega|}.$$

Case B.2: δ_+ and δ_- have opposite sign. Let us suppose as before that $\delta_+ \geq 0$ and $\delta_- \leq 0$. Now the main problem is the term in front of the asymmetry $\mathcal{A}(\Omega_+)$ in (5.3.5), which could be negative. Since $\delta_+ + \delta_- \leq 0$, applying (5.3.5) to Ω_- we obtain

$$\frac{\delta_+}{|\Omega|} \leq -\frac{\delta_-}{|\Omega|} \leq \frac{N}{p}\,HKS(\Omega). \tag{5.3.6}$$

We then observe that if

$$\delta_+ \leq \frac{N}{p}\frac{|\Omega|}{4}, \tag{5.3.7}$$

we have

$$\left(1 - \frac{p}{N}\frac{\delta_+}{|\Omega_+|}\right) \geq 1 - \frac{1}{4}\frac{|\Omega|}{|\Omega_+|} \geq \frac{1}{2},$$

thanks to the fact that $|\Omega| \leq 2\,|\Omega_+|$, which easily follows from the assumption that $\delta_+ \geq 0$. From (5.3.5) we can now infer

$$HKS(\Omega) \geq \frac{\gamma_{N,p}}{2}\,\mathcal{A}(\Omega_+)^{\kappa_1} - \frac{p}{N}\frac{\delta_+}{|\Omega_+|},$$

then (5.3.2) follows as before, using (5.3.6) and the fact that

$$-\frac{\delta_+}{|\Omega_+|} \geq -2\,\frac{\delta_+}{|\Omega|}.$$

This would prove the thesis under the additional hypothesis (5.3.7): however, if this is not satisfied, then (5.3.6) would imply $HKS(\Omega) > 1/4$, which is in contrast with our assumption (5.3.3). $\square$

The following technical Lemma of geometrical content completes the proof of Theorem 5.3.1. This is the same as [**19**, Lemma 3.3] and we omit the proof.

Lemma 5.3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set, with finite measure. For every $\Omega_+, \Omega_- \subset \Omega$ such that $|\Omega_+ \cap \Omega_-| = 0$, we have*

$$\mathcal{A}_2(\Omega) \le C_N \left(\mathcal{A}(\Omega_+) + \left| \frac{1}{2} - \frac{|\Omega_+|}{|\Omega|} \right| + \mathcal{A}(\Omega_-) + \left| \frac{1}{2} - \frac{|\Omega_-|}{|\Omega|} \right| \right)^{2/(N+1)}, \tag{5.3.8}$$

for a suitable dimensional constant $C_N > 0$.

4. Extremal cases: $p = 1$ and $p = \infty$

In the above a stability analysis was performed on the isoperimetric bound for the second Dirichlet p–eigenvalue. To make this analysis more complete, is seems to be natural to give a brief look at the asymptotics as p approaches the extrema of its range of admissible values, namely $p = 1$ and $p = \infty$. In these cases some shape functionals of geometric flavour appear in place of the eigenvalues of an elliptic operator.

To enter more in this question, some definitions are needed. For $\Omega \subset \mathbb{R}^N$ open set with $|\Omega| < \infty$, $\mathcal{C}_1(\Omega)$ and $\mathcal{C}_2(\Omega)$ stand for the *first two Cheeger constants*, which are defined respectively by

$$\mathcal{C}_1(\Omega) = \inf_{E \subset \Omega} \frac{P(E)}{|E|} \quad \text{and} \quad \mathcal{C}_2(\Omega) = \inf \left\{ t : \begin{array}{l} \text{there exist } E_1, E_2 \subset \Omega \\ \text{such that } E_1 \cap E_2 = \emptyset \end{array} \text{ and } \max_{i=1,2} \frac{P(E_i)}{|E_i|} \le t \right\}.$$

Here $P(E)$ equals the distributional perimeter of a set E if this is a finite perimeter sets and is $+\infty$ otherwise. Also, if $|E| = 0$ we use the convention $P(E)/|E| = +\infty$.

By $\Lambda_1(\Omega)$ one denotes the inverse of the radius r_1 of the largest ball inscribed in Ω. By $\Lambda_2(\Omega)$ will denote the inverse of the largest positive number r_2 such that there exist two disjoint balls of radius r_2 contained in Ω. It is remarkable to notice that Λ_1 and Λ_2 are indeed two eigenvalues. Namely, they coincide with the *first two eigenvalues of the $\infty-$Laplacian*, cf. Chapter 8.

Our interest in these quantities is motivated by the following Theorem, collecting various results about the asymptotic behaviour of λ_1 and λ_2.

Limiting behaviour of eigenvalues. *For every set $\Omega \subset \mathbb{R}^N$, there holds*

$$\lim_{p \to 1^+} \lambda_i(\Omega) = \mathcal{C}_i(\Omega), \qquad i = 1, 2 \qquad \textit{and} \qquad \lim_{p \to \infty} \lambda_i(\Omega)^{1/p} = \Lambda_i(\Omega), \qquad i = 1, 2. \tag{5.4.1}$$

Proof. The first fact is proven in [**46**] and [**82**], respectively. For the second, one can consult [**58**] and the references therein. □

Remark 5.4.1. At this point, one could be tempted to use the previous results for λ_1, in order to improve inequality (5.1.2). For example, using the subadditivity of the function

$t \mapsto (1+t)^{1/p}$, it is not difficult to see that

$$\lim_{p\to\infty} FK(\Omega)^{1/p} \geq \frac{|\Omega|^{1/N}\,\Lambda_1(\Omega)}{\omega_N^{1/N}} - 1 \geq \frac{1}{2N}\,\mathcal{A}(\Omega),$$

where in the last inequality we used the (sharp) quantitative stability estimate[1] for Λ_1 (see [**58**], equation (2.6)). Then one could bravely guess that for p "very large", inequality (5.1.2) has to hold with the exponent $\kappa_1(N,p)$ replaced by p, which is strictly small if $N \geq 3$. This would prove that (5.1.2) is not sharp, at least for $N \geq 3$ and p going to ∞. Needless to say, this argument (and the related one for $p \to 1$) is only a heuristic one, since these limits are not uniform with respect to the sets Ω.

In these extremal cases, the analogues of problem

$$\min\Big\{\lambda_2(\Omega)\colon |\Omega| = c\Big\}$$

are

$$\min\Big\{\mathcal{C}_2(\Omega)\ :\ |\Omega| = c\Big\} \qquad \text{and} \qquad \min\Big\{\Lambda_2(\Omega)\ :\ |\Omega| = c\Big\}.$$

Once again, they both have (unique) solution, that is given by any disjoint union of two equal balls. For a proof of that in the first case, the reader can see [**82**, Proposition 3.14]. For the other problem, this statement plainly follows by the geometrical meaning of Λ_2. Therefore, the above rewrite scaling invariant form as

$$|\Omega|^{1/N}\,\mathcal{C}_2(\Omega) \geq 2^{1/N}\,N\omega_N^{1/N} \qquad \text{and} \qquad |\Omega|^{1/N}\,\Lambda_2(\Omega) \geq 2^{1/N}\,\omega_N^{1/N},$$

respectively. Both the inequalities can be improved by a reminder term making them quantitative. This is proved in the following theorem.

Theorem 5.4.2. *Let Ω be an open subset of $\mathbb{R}^N$ having finite measure. Then*

$$|\Omega|^{1/N}\,\mathcal{C}_2(\Omega) \geq 2^{1/N}\,N\,\omega_N^{1/N}\,\left[1 + h_N\,\mathcal{A}_2(\Omega)^{N+1}\right], \tag{5.4.2}$$

where the constant $h_N > 0$ only depens on the dimension N. Moreover, for Λ_2 we have

$$|\Omega|^{1/N}\,\Lambda_2(\Omega) \geq 2^{1/N}\,\omega_N^{1/N}\,\left[1 + \frac{1}{2N}\,\mathcal{A}_2(\Omega)\right]. \tag{5.4.3}$$

Proof. To prove (5.4.2), we start defining

$$T_\Omega = \left\{t > 0\ :\ \text{ there exist } \Omega_1, \Omega_2 \subset \Omega \text{ disjoint and s.t. } \max_{i=1,2} \frac{P(\Omega_i)}{|\Omega_i|} \leq t\right\}.$$

[1] The relation between the Fraenkel asymmetry $\alpha(\Omega)$ as defined in [**58**] and our definition is given by $\mathcal{A}(\Omega) = 2\,\alpha(\Omega)$. This explains the discrepancy between our constant $1/(2N)$ and the constant $1/N$ that can be found in [**58**], equation (2.6).

It is not difficult to see that if Ω is open, then $T_\Omega \neq \emptyset$, since Ω contains at least two disjoint small balls, which are in particular two sets with positive measure and finite perimeter. Then let us pick up a $t \in T_\Omega$. Correspondingly, there exist $\Omega^t_+, \Omega^t_- \subset \Omega$ disjoint and such that

$$t \geq \max\left\{\frac{P(\Omega^t_+)}{|\Omega^t_+|}, \frac{P(\Omega^t_-)}{|\Omega^t_-|}\right\} \geq \max\{\mathcal{C}_1(\Omega^t_+), \mathcal{C}_1(\Omega^t_-)\}, \tag{5.4.4}$$

where we used the straightforward estimate $\mathcal{C}_1(E) \leq P(E)/|E|$, which is valid for every finite perimeter set E. Now, we introduce the following quantity

$$D_\Omega(t) := \frac{|\Omega|^{1/N}\, \max\{\mathcal{C}_1(\Omega^t_+), \mathcal{C}_1(\Omega^t_-)\}}{2^{1/N}\, N\, \omega_N^{1/N}} - 1,$$

and we proceed exactly as in Case A of the proof of Theorem 5.3.1. We only need to replace $HKS(\Omega)$ by $D_\Omega(t)$ and the quantitative Faber-Krahn inequality by the following (sharp) quantitative Cheeger inequality (see [**45**]),

$$|\Omega|^{1/N}\mathcal{C}_1(\Omega) \geq N\, \omega_N^{1/N}\, \left[1 + \gamma_N\, \mathcal{A}(\Omega)^2\right], \tag{5.4.5}$$

where $\gamma_N > 0$ is a constant depending only on the dimension N. In this way, one arrives at the estimate

$$D_\Omega(t) \geq h_N\, \mathcal{A}_2^{N+1}(\Omega), \qquad \text{for every } t \in T_\Omega,$$

that is

$$\frac{|\Omega|^{1/N}\, t}{2^{1/N}\, N\, \omega_N^{1/N}} - 1 \geq h_N\, \mathcal{A}_2^{N+1}(\Omega), \qquad \text{for every } t \in T_\Omega,$$

thanks to (5.4.4). Taking the infimum on T_Ω on both sides and using the definition of second Cheeger constant, we eventually prove the thesis.

In order to prove (5.4.3), let us take a pair of optimal disjoint balls $B(x_0, r), B(x_1, r) \subset \Omega$, whose common radius r is given by

$$\Lambda_2(\Omega) = r^{-1},$$

and set for simplicity $\mathcal{O}_1 := B(x_0, r) \cup B(x_1, r)$, then obviously we have

$$|\Omega \setminus \mathcal{O}_1| = |\Omega| - 2\,\omega_N\, r^N.$$

Up to a rigid movement, we can assume that $x_0 = (M, 0, \ldots, 0)$ and $x_1 = (-M, 0, \ldots, 0)$, for some $M \geq r$, then for every $t \geq 1$ we define the new centers $x_0(t) := (M + (t-1)\, r, 0, \ldots, 0)$ and $x_1(t) := ((1-t)\, r - M, 0, \ldots, 0)$: observe that $x_i(1) = x_i$, $i = 0, 1$. Finally, we set

$$\mathcal{O}_t := B(x_0(t), t\, r) \cup B(x_1(t), t\, r), \quad t \geq 1,$$

i.e. for every $t \geq 1$ this is a disjoint union of two balls of radius $t\, r$ and moreover $\mathcal{O}_t \subset \mathcal{O}_s$ if $t < s$.The latter fact implies that the function $t \mapsto |\Omega \cap \mathcal{O}_t|$ is increasing, thus $t \mapsto |\Omega \setminus \mathcal{O}_t|$ is decreasing. We exploit this fact by taking $t_0 > 1$ such that $|\mathcal{O}_{t_0}| = |\Omega|$: then we have

$$|\Omega| - 2\,\omega_N\, r^N = |\Omega \setminus \mathcal{O}_1| \geq |\Omega \setminus \mathcal{O}_{t_0}| \geq \frac{1}{2}\, \mathcal{A}_2(\Omega)\, |\Omega|,$$

where in the last inequality we used that $\mathcal{O}_{t_0}$ is admissible for the problem defining $\mathcal{A}_2(\Omega)$. From the previous, we easily obtain

$$\frac{|\Omega|}{r^N} \geq \frac{2\,\omega_N}{(1-1/2\,\mathcal{A}_2(\Omega))},$$

which finally gives (5.4.3), just by raising both members to the power $1/N$, using the elementary inequality $(1-t)^{-1/N} \geq 1 + 1/N\,t$ for $t < 1$ and recalling that $r = \Lambda_2(\Omega)^{-1}$. □

5. Sharpness of the estimates: examples and open problems

The estimates of Theorem 5.3.1 and 5.4.2 show that for every set the *deficit* dominates a certain power κ of the *asymmetry* $\mathcal{A}_2$. In addition to this, the reader could ask the question whether or not for some sets converging to the optimal shape (i.e. a disjoint union of two equal balls) the deficit and $\mathcal{A}_2^\kappa$ have the same decay rate. A positive answer to the question would imply the sharpness of those estimates. The last section of the chapter is devoted to discuss this interesting topic.

5.1. Quantitative Hong-Krahn-Szego inequality. This is quite a delicate issue. First of all, observe that in contrast with the case of the Faber-Krahn inequality, the exponent of the asymmetry κ_2 blows up with N. For this reason, one could automatically guess that κ_2 is not the sharp exponent. However, it has to be noticed that this dependence on N is directly inherited from the geometrical estimate (5.3.8), *which is indeed sharp*. Let us fix a small parameter $\varepsilon > 0$ and consider the following set

$$\Omega^\varepsilon = \{(x_1, x') \in \mathbb{R}^N \,:\, (x_1 + 1 - \varepsilon)^2 + |x'|^2 < 1\} \cup \{(x_1, x') \in \mathbb{R}^N \,:\, (x_1 - 1 + \varepsilon)^2 + |x'|^2 < 1\},$$

which is just the union of two balls of radius 1, with an overlapping part whose area is of order $\varepsilon^{(N+1)/2}$. We set

$$\Omega^\varepsilon_+ = \{(x_1, x') \in \Omega^\varepsilon \,:\, x_1 \geq 0\} \qquad \text{and} \qquad \Omega^\varepsilon_- = \{(x_1, x') \in \Omega^\varepsilon \,:\, x_1 \leq 0\},$$

and it is not difficult to see that $\mathcal{A}(\Omega^\varepsilon_+) = O(\varepsilon^{(N+1)/2})$, while on the contrary $\mathcal{A}_2(\Omega^\varepsilon) = O(\varepsilon)$ which means

$$\mathcal{A}_2(\Omega^\varepsilon)^{(N+1)/2} \simeq \mathcal{A}(\Omega^\varepsilon).$$

i.e. both sides in (5.3.8) are asymptotically equivalent, as the area of the overlapping region goes to 0 (see [**19**, Example 3.4], for more details on this example). And in fact one can use these sets Ω^ε to show that the *sharp exponent in* (5.3.1) *has to blow-up with the dimension*. Also observe that in the proof of Theorem 5.3.1, the precise value of κ_1 plays no role, so the same proof actually gives (5.3.1) with

$$\kappa_2 = (\text{sharp exponent for (5.1.2)}) \times \frac{N+1}{2}.$$

Though we strongly suspect this κ_2 not to provide the right decay rate, currently we are not able to solve this issue, which seems to be quite a changelling one even for $p = 2$.

5.2. Second Cheeger constant. Also in this case, the exponent $N+1$ in (5.4.2) seems not to be sharp in the decay rate of the deficit. In order to shed some light on this fact, we estimate the deficit for $\mathcal{C}_2$ of the same set Ω^ε as before. First of all, thanks to the symmetries of Ω^ε, it is not difficult to see that $\mathcal{C}_2(\Omega^\varepsilon) = \mathcal{C}_1(\Omega^\varepsilon_+) = \mathcal{C}_1(\Omega^\varepsilon_-)$. Then we have

$$h_N\,\mathcal{A}_2(\Omega^\varepsilon)^{N+1} \le \frac{|\Omega^\varepsilon|^{1/N}\,\mathcal{C}_2(\Omega^\varepsilon)}{2^{1/N}\,N\,\omega_N^{1/N}} - 1 = \frac{|\Omega^\varepsilon|^{1/N}\,\mathcal{C}_1(\Omega^\varepsilon_+)}{2^{1/N}\,N\,\omega_N^{1/N}} - 1 \le \frac{|\Omega^\varepsilon_+|^{1/N-1}\,P(\Omega^\varepsilon_+)}{N\,\omega_N^{1/N}} - 1,$$

so that the deficit of this inequality is controlled from above by the isoperimetric deficit of one of the two cut balls. We then estimate the right-hand side in the previous expression: observe that setting $\vartheta = \arccos(1-\varepsilon)$, we have for instance

$$P(\Omega^\varepsilon_+) = N\omega_N + \omega_{N-1}\left[(\sin\vartheta)^{N-1} - (N-1)\int_0^{\sin\vartheta} \frac{t^{N-2}}{\sqrt{1-t^2}}\,d\varrho\right],$$

and

$$|\Omega^\varepsilon_+| = \omega_N - \omega_{N-1}\int_{\cos\vartheta}^1 (1-t^2)^{\frac{N-1}{2}}\,dt,$$

then

$$P(\Omega^\varepsilon_+) \simeq N\,\omega_N - \frac{N-1}{N+1}\,\frac{\omega_{N-1}}{2}\,\vartheta^{N+1} \quad \text{and} \quad |\Omega^\varepsilon_+|^{1/N-1} \simeq \omega_N^{\frac{1-N}{N}}\left(1 + \frac{N-1}{N(N+1)}\,\frac{\omega_{N-1}}{\omega_N}\,\vartheta^{N+1}\right),$$

from which we can infer

$$|\Omega^\varepsilon_+|^{1/N-1}\,P(\Omega^\varepsilon_+) - N\,\omega^{1/N} \simeq \frac{N-1}{N+1}\,\omega_{N-1}\,\omega_N^{\frac{1-N}{N}}\,\vartheta^{N+1} \simeq c_N\,\varepsilon^{\frac{N+1}{2}}.$$

In the end, we get

$$C_1\,\mathcal{A}_2(\Omega^\varepsilon)^{N+1} \le |\Omega^\varepsilon|^{1/N}\,\mathcal{C}_2(\Omega^\varepsilon) - 2^{1/N}\,N\,\omega_N^{1/N} \le C_2\,\mathcal{A}_2(\Omega^\varepsilon)^{\frac{N+1}{2}}, \tag{5.5.1}$$

where we used that $\mathcal{A}_2(\Omega^\varepsilon) \simeq \varepsilon$. Notice that this estimate implies in particular that, also in this case, *the sharp exponent is dimension-dependent and it blows up as N goes to ∞.*

We point out that the previous computations give the correct decay rate to 0 of the quantity $\mathcal{C}_2(\Omega^\varepsilon) - \mathcal{C}_2(B)$, which is $O(\varepsilon^{(N+1)/2}) = O(\mathcal{A}_2(\Omega_\varepsilon)^{(N+1)/2})$. Indeed, from the right-hand side of (5.5.1) we can promptly infer that

$$\mathcal{C}_2(\Omega^\varepsilon) = \mathcal{C}_1(\Omega^\varepsilon_+) \le N + c\,\varepsilon^{\frac{N+1}{2}} = \mathcal{C}_1(B) + c\,\varepsilon^{\frac{N+1}{2}}.$$

Now assume that $\mathcal{C}_1(\Omega^\varepsilon_+) \le \mathcal{C}_1(B) + \omega(\varepsilon)$ for some modulus of continuity ω such that $\omega(\varepsilon) = o(\varepsilon^{(N+1)/2})$ as ε goes to 0, in this case we would obtain

$$0 \le |\Omega^\varepsilon|^{1/N}\,\mathcal{C}_1(\Omega^\varepsilon_+) - 2^{1/N}\,N\,\omega_N^{1/N} \le -K\,\varepsilon^{\frac{N+1}{2}},$$

for some constant $K > 0$ independent of ε. This gives a contradiction, thus proving that

$$\mathcal{C}_2(\Omega^\varepsilon) - \mathcal{C}_2(B) \simeq \varepsilon^{\frac{N+1}{2}}.$$

5.3. Second eigenvalue of $-\Delta_\infty$. On the contrary, it is not difficult to see that the quantitative estimate (5.4.3) is sharp. By still taking the set Ω^ε as before, we observe that

$$\Lambda_2(\Omega^\varepsilon) = \Lambda_1(\Omega^\varepsilon_+) = \frac{2}{2-\varepsilon} \simeq 1 + \frac{\varepsilon}{2} \quad \text{and} \quad |\Omega_\varepsilon|^{1/N} \simeq \omega_N^{1/N} \left(1 - \frac{\omega_{N-1}}{\omega_N} \frac{2^{\frac{N+1}{2}}}{N\,(N+1)}\, \varepsilon^{\frac{N+1}{2}}\right),$$

while $\mathcal{A}_2(\Omega_\varepsilon) = O(\varepsilon)$ as already observed. Then

$$|\Omega|^{1/N}\,\Lambda_2(\Omega) - \omega_N^{1/N} \simeq \mathcal{A}_2(\Omega),$$

proving the sharpness of (5.4.3). We remark that in this case the sharp exponent does not depend on the dimension, in contrast with the cases $p \in [1,\infty)$.

CHAPTER 6

Optimization of a nonlinear anisotropic Stekloff p-eigenvalue

Let Ω be a bounded Lipschitz open set in $\mathbb{R}^N$. Then each function in $W^{1,p}(\Omega)$ has a trace belonging to the fractional Sobolev spaces $W^{1-1/p,p}(\partial\Omega)$. Recall that the embedding (trace operator)

$$W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$$

is compact.

Next lemma is really elementary. The Sobolev space $W^{1,p}(\Omega)$ is endowed with the usual norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega |\nabla u(x)|^p\right)^{\frac{1}{p}} \tag{6.0.1}$$

but the following holds.

Lemma 6.0.1. *Let $1 < p < \infty$ and Ω be an open bounded Lipschitz set in $\mathbb{R}^N$. Then*

$$\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\partial\Omega)}, \qquad u \in W^{1,p}(\Omega),$$

defines a norm on the Sobolev space $W^{1,p}(\Omega)$, which is equivalent to (6.0.1)*.*

Proof. It is straightforward to check that the above quantity defines a norm. Then the conclusion follows by combining the trace inequality

$$\|u\|_{L^p(\partial\Omega)} \le c_\Omega \|u\|_{W^{1,p}(\Omega)},$$

and the following Poincaré inequality

$$\|u\|_{L^p(\Omega)} \le \widetilde{c}_\Omega \Big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\partial\Omega)}\Big).$$

The latter in turn follows by a standard contradiction argument, exploiting the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. $\square$

1. The Stekloff spectrum of the pseudo p-Laplacian

Let Ω be a bounded Lipschitz open set in $\mathbb{R}^N$ and $\varrho : \partial\Omega \to \mathbb{R}$ be a measurable function satisfying

$$0 < c_1 \le \varrho(x) \le c_2 < \infty, \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega. \tag{6.1.1}$$

For every $1 < p < \infty$, we consider the pseudo p-Laplacian, i.e. the nonlinear operator

$$\widetilde{\Delta}_p u = \sum_{j=1}^{N} \left(|u_{x_j}|^{p-2} u_{x_j} \right)_{x_j}.$$

Definition 6.1.1. A real number σ is said to be a *Stekloff eigenvalue of the pseudo p-Laplacian* in Ω if the boundary value problem

$$\begin{cases} -\widetilde{\Delta}_p u = 0, & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} |u_{x_i}|^{p-2} u_{x_i} \nu_\Omega^i = \sigma |u|^{p-2} u \varrho, & \text{on } \partial\Omega, \end{cases} \tag{6.1.2}$$

admits a nontrivial solution u. If this is the case, we say that u is a *Stekloff eigenfunction* corresponding to σ. We also set

$$\mathfrak{S}_p(\Omega) = \{\sigma \in \mathbb{R} : \sigma \text{ is a Stekloff eigenvalue}\},$$

to denote the Stekloff spectrum of the pseudo $p-$Laplacian on Ω.

Remark 6.1.2. Since the behaviour of the spectrum under varying weights is not investigated here, the notation does not account for the choice of the function $\varrho : \partial\Omega \to \mathbb{R}$.

The solutions u of the problem (6.1.2) are always understood in the weak sense, i.e. $u \in W^{1,p}(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega |u_{x_i}|^{p-2} u_{x_i} \varphi_{x_i} \, dx = \sigma \int_{\partial\Omega} |u|^{p-2} u \, \varphi \; \varrho \, d\mathcal{H}^{N-1}, \qquad \text{for every } \varphi \in W^{1,p}(\Omega). \tag{6.1.3}$$

Observe that the integral on the right-hand side is well-defined, since the trace of a function in $W^{1,p}(\Omega)$ belongs to $L^p(\partial\Omega)$.

We start with the following basic result.

Lemma 6.1.3. *Let $1 < p < \infty$, Ω be a bounded open Lipschitz set and $\varrho : \partial\Omega \to \mathbb{R}$ be such that* (6.1.1) *holds. There exists a least eigenvalue, given by $\sigma = 0$ and corresponding to constant eigenfunctions. Moreover, any other eigenfunction whose trace does not change sign on $\partial\Omega$, is constant in Ω.*

Proof. By testing $\varphi = u$, equation (6.1.3) implies

$$\int_\Omega \|\nabla u\|_{\ell^p} \, dx = \sigma \int_{\partial\Omega} |u|^p \, \varrho \, d\mathcal{H}^{N-1},$$

so that every eigenvalue must be positive. Moreover, it is easily seen that $\sigma = 0$ is an eigenvalue and by the previous equality any corresponding eigenfunction is constant.

Let us now prove the second part of the statement. Let $u \neq 0$ have a constant sign on the boundary and assume, arguing by contradiction, that it corresponds to an eigenvalue $\sigma \neq 0$. Inserting a constant test function in (6.1.3) we then obtain

$$\int_{\partial\Omega} |u|^{p-1} \varrho \, d\mathcal{H}^{N-1} = 0,$$

where we also used that u does not change sign on $\partial\Omega$. Thus, u has a null trace on $\partial\Omega$ and it solves in a weak sense the problem

$$\left\{ \begin{array}{rcll} \tilde{\Delta}_p u & = & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Solutions to the latter problem are minimizers of the strictly convex energy

$$v \mapsto \int_{\Omega} \|\nabla v\|_{\ell^p}^p \, dx,$$

on $W_0^{1,p}(\Omega)$. Since the unique minimizer is given by the zero constant function, there must hold $u \equiv 0$, a contradiction. Therefore, $\sigma = 0$ and u is a constant eigenfunction. □

Definition 6.1.4. If u is a Stekloff eigenfunction, call *nodal domains* the connected components of $\{x \in \Omega \, : \, u(x) \neq 0\}$. Observe that the latter is an open set, since each pseudo $p-$harmonic function is locally Hölder continuous, as a local minimizer of $\int_\Omega \|\nabla v\|_{\ell^p}^p$ (see [**51**, Theorem 7.6]). We also observe that each nodal domain is itself an open set. This follows from the fact that the connected components of an open sets in $\mathbb{R}^N$ are open as well.

The following property of eigenfunctions will be useful in the next section.

Lemma 6.1.5. *Let $u \in W^{1,p}(\Omega)$ be a Stekloff eigenfunction, with eigenvalue $\sigma > 0$. Then u has at least two nodal domains, both touching the boundary.*

Proof. The fact that u has to change sign follows from Lemma 3.3. Now take

$$u_+(x) = \max\{u(x), 0\} \qquad \text{and} \qquad u_-(x) = \max\{0, -u(x)\},$$

and let $\Omega_1, \Omega_2, \dots,$ be the nodal domains of u. Suppose that for some j we have $\Omega_j \Subset \Omega$. Then the restriction of u to Ω_j belongs to $W_0^{1,p}(\Omega_j)$ and it solves

$$-\widetilde{\Delta}_p u = 0, \qquad \text{on } \Omega_j,$$

in the weak sense. This implies that $u \equiv 0$ on Ω_j, hence contradicting the definition of nodal domain. □

The whole collection $\mathfrak{S}_p(\Omega)$ of Stekloff eigenvalues forms a closed set.

Proposition 6.1.6. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain and $\varrho : \partial\Omega \to \mathbb{R}$ be a function such that* (6.1.1) *holds. Then $\mathfrak{S}_p(\Omega)$ is a non empty closed subset of $[0, \infty)$.*

PROOF. This is a standard proof. The fact that the collection of all the Stekloff eigenvalues is non empty and consists of nonnegative numbers is due to Lemma 6.1.3. In order to prove the second part of the statement, we take a sequence of eigenvalues $\{\sigma^k\}_{k\in\mathbb{N}} \subset \mathfrak{S}_p(\Omega)$ converging to some positive number σ and we let $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)$ be a sequence of corresponding eigenfunctions, normalized by the condition

$$\int_{\partial\Omega} |u^k|^p \,\varrho\, d\mathcal{H}^{N-1}(x) = 1, \qquad k \in \mathbb{N}.$$

This implies in particular that

$$\sum_{i=1}^{N} \int_{\Omega} |u^k_{x_i}|^p \, dx = \sigma^k, \qquad k \in \mathbb{N},$$

so that the sequence $\{u^k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, thanks to Lemma 6.0.1. Thus, by the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, the sequence weakly converges (up to a subsequence) to some limit function u in $W^{1,p}(\Omega)$. Moreover, this convergence is strong in $L^p(\partial\Omega)$. We have to show that u is an eigenfuction with eigenvalue σ: testing the equations solved by u^k with $\varphi = u^k - u$, we then obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} &\left(|u^k_{x_i}|^{p-2}\, u^k_{x_i} - |u_{x_i}|^{p-2}\, u_{x_i}\right)\, (u^k_{x_i} - u_{x_i})\, dx \\ &= \sigma_k \int_{\partial\Omega} \left(\, |u^k|^{p-2}\, u^k - |u|^{p-2}\, u\right)\, \varrho\, d\mathcal{H}^{N-1} - \sum_{i=1}^{N} \int_{\Omega} |u_{x_i}|^{p-2}\, u_{x_i}\, \left(u^k_{x_i} - u_{x_i}\right)\, dx.\end{aligned}$$

Then, by the strong convergence of $\{u^k\}_{k\in\mathbb{N}} \subset L^p(\partial\Omega)$, sending k to infinity yields

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_{\Omega} \left(|u^k_{x_i}|^{p-2}\, u^k_{x_i} - |u_{x_i}|^{p-2}\, u_{x_i}\right)\, (u^k_{x_i} - u_{x_i})\, dx = 0.$$

Thanks to Proposition A.3.2, the previous gives the strong convergence of ∇u^k to ∇u in $L^p(\Omega)$. Since $\{u^k\}_{k\in\mathbb{N}}$ converges to u strongly in $L^p(\partial\Omega)$, we also have

$$\lim_{k\to\infty} \int_{\partial\Omega} |u^k - u|^p \,\varrho\, d\mathcal{H}^{N-1} = 0. \tag{6.1.4}$$

Thanks to these informations, we can now pass to the limit in the equation (6.1.3) satisfied by u^k, so to obtain that u is an eigenfunction as well, with eigenvalue σ. This shows that $\sigma \in \mathfrak{S}_p(\Omega)$, which is then closed. $\square$

2. Existence of an unbounded sequence

In this section, we will show that the spectrum $\mathfrak{S}_p(\Omega)$ contains an infinite sequence of eigenvalues, diverging at ∞. The proof just amounts to apply Theorem 1.4.1 to the variational integral

$$\mathcal{F}(u,\Omega)=\int_\Omega \|\nabla u(x)\|_{\ell^p}^p\,dx,\qquad u\in W^{1,p}(\Omega),\tag{6.2.1}$$

restricted to the manifold M defined by

$$M=\left\{u\in W^{1,p}(\Omega)\,:\,\mathcal{G}(u,\Omega)=1\right\},\qquad\text{where}\quad \mathcal{G}(u,\Omega)=\int_{\partial\Omega}|u|^p\,\varrho\,d\mathcal{H}^{N-1}.\tag{6.2.2}$$

By $C_o(\mathbb{S}^{n-1};M)$ we denote the set of all odd continuous mappings from the unit sphere $\mathbb{S}^{n-1}$ to M.

Theorem 6.2.1. *Given $1<p<\infty$, let $\Omega\subset\mathbb{R}^N$ be an open bounded connected set, having Lipschitz boundary. Let also $\varrho:\partial\Omega\to\mathbb{R}$ be a function such that* (6.1.1) *holds. For every $k\in\mathbb{N}$, we define*

$$\sigma_{n,p}(\Omega)=\inf_{f\in C_o(\mathbb{S}^{k-1};M)}\max_{u\in f(\mathbb{S}^{n-1})}\int_\Omega\|\nabla u\|_{\ell^p}^p\,dx.\tag{6.2.3}$$

Then each $\sigma_{n,p}(\Omega)$ is a Stekloff eigenvalue of the pseudo p-Laplacian on Ω. Moreover,

$$0=\sigma_{1,p}(\Omega)<\sigma_{2,p}(\Omega)\le\ldots\le\sigma_{n,p}(\Omega)\le\ldots\tag{6.2.4}$$

and $\sigma_{n,p}(\Omega)\to\infty$ as $n\to\infty$.

Proof. Taking into account Remark 1.4.2, the proof is a straghtforward adaptation of the one of Theorem 1.4.1 to this case. One is just left to prove the following two facts.

The first element is zero. To see that, note that any continuous odd mapping from $\mathbb{S}^0=\{1,-1\}$ to M can be identified with the choice of an antipodal pair $u,-u$ on the symmetric manifold M. This and the fact that the functional is even imply that if $n=1$ formula (6.2.3) gives the minimum of (6.2.1) on M. The latter is of course zero, corresponding to constant functions.

The existence of a gap. The gap inequality $0<\sigma_{2,p}(\Omega)$ can be proved as in Theorem 4.3.2. □

Remark 6.2.2. If Ω has m connected components $\Omega_1,\ldots,\Omega_m$, equation (6.2.3) still defines an infinite sequence of eigenvalues, diverging at ∞, but in this case one has

$$\sigma_{1,p}(\Omega)=\cdots=\sigma_{m,p}(\Omega)=0.$$

They correspond to the (normalized) piecewise constant eigenfunctions, which are given by

$$c_i=\left(\int_{\partial\Omega_i}\varrho(x)\,d\mathcal{H}^{N-1}(x)\right)^{-\frac{1}{p}}\cdot 1_{\Omega_i}(x)$$

for $i=1\ldots,m$.

3. The first nontrivial eigenvalue

Number $\sigma_{2,p}(\Omega)$ is actually the first nontrivial Stekloff eigenvalue of $-\widetilde{\Delta}_p$. In other words, the first eigenvalue $\sigma = 0$ is always isolated in the spectrum and any other eigenvalue has to be greater than $\sigma_{2,p}(\Omega)$. Then the quantity $\sigma_{2,p}(\Omega)$ can also be seen as the *fundamental gap* of the pseudo $p-$Laplacian, with Stekloff boundary conditions.

Theorem 6.3.1. *Let $u \in W^{1,p}(\Omega)$ be a Stekloff eigenfunction, with eigenvalue $\sigma > 0$. Then we have $\sigma \geq \sigma_{2,p}(\Omega)$.*

Proof. The proof is inspired to [**58**, Theorem 3.4]. First observe that the positive and negative parts u_+ and u_- of u are both not identically zero, due to Lemma 6.1.5. Also, they belong to $W^{1,p}(\Omega)$, hence they have a trace on the boundary $\partial\Omega$. Moreover

$$\mathrm{trace}_{|\partial\Omega}(u_+) = (\mathrm{trace}_{|\partial\Omega} u)_+ \qquad \text{and} \qquad \mathrm{trace}_{|\partial\Omega}(u_-) = (\mathrm{trace}_{|\partial\Omega} u)_-.$$

Thus, using u_+ and u_- as test functions in (6.1.3) it follows that

$$\int_\Omega \|\nabla u_+(x)\|_{\ell^p}^p\,dx = \sigma \int_{\partial\Omega} |u_+(x)|^p\,\varrho(x)\,d\mathcal{H}^{N-1}(x)$$

and

$$\int_\Omega \|\nabla u_-(x)\|_{\ell^p}^p\,dx = \sigma \int_{\partial\Omega} |u_-(x)|^p\,\varrho(x)\,d\mathcal{H}^{N-1}(x).$$

Consider now the odd and continuous mapping $\widetilde{f} : \mathbb{S}^1 \to M$ defined by

$$\widetilde{f}_\omega(x) = \frac{\omega_1\,u_+(x) - \omega_2\,u_-(x)}{|\omega_1|^p \displaystyle\int_{\partial\Omega} |u_+|^p\,\varrho\,d\mathcal{H}^{N-1} + |\omega_2|^p \int_{\partial\Omega} |u_-|^p\,\varrho\,d\mathcal{H}^{N-1}}, \qquad \omega = (\omega_1,\omega_2) \in \mathbb{S}^1.$$

Choosing $f = \tilde{f}$ in the definition of $\sigma_{2,p}(\Omega)$ yields

$$\sigma_{2,p}(\Omega) \leq \max_{\omega\in\mathbb{S}^{k-1}} \frac{|\omega_1|^p \displaystyle\int_\Omega \|\nabla u_+\|_{\ell^p}^p\,dx + |\omega_2|^p \int_\Omega \|\nabla u_-\|_{\ell^p}^p\,dx}{|\omega_1|^p \displaystyle\int_{\partial\Omega} |u_+|^p\,\varrho\,d\mathcal{H}^{N-1} + |\omega_2|^p \int_{\partial\Omega} |u_-|^p\,\varrho\,d\mathcal{H}^{N-1}} = \sigma,$$

and this concludes the proof. □

The rest of this section is devoted to providing alternative characterizations of $\sigma_{2,p}(\Omega)$. The first one is a mountain pass theorem. Given a pair of functions $u, v \in M$, we denote by $\Gamma_\Omega(u, v)$ the set of all continuous paths in M, parametrized on $[0, 1]$ and connecting u to v, i.e.

$$\Gamma_\Omega(u,v) = \{\gamma : [0,1] \to M \,:\, \gamma \text{ is continuous } \text{ and } \gamma(0) = u,\, \gamma(1) = v\},$$

where continuity is understood in the norm topology of $W^{1,p}$. Then we have the following alternative characterization for $\sigma_{2,p}(\Omega)$.

Theorem 6.3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bouned connected Lipschitz set. Let us define the constant function*

$$c = \left(\int_{\partial\Omega} \varrho(x)\, d\mathcal{H}^{N-1}(x) \right)^{-\frac{1}{p}} \in M.$$

Then the first nontrivial Stekloff eigenvalue has the following Mountain Pass characterization

$$\sigma_{2,p}(\Omega) = \inf_{\gamma \in \Gamma_\Omega(c,-c)} \max_{u \in \gamma} \int_\Omega \|\nabla u\|_{\ell^p}^p \, dx. \tag{6.3.1}$$

A slightly different version of Theorem 6.3.2 was proved in Section 1 of Chapter 3, see Theorem 4.1.3. In the case of Stekloff boundary the proof requires a minor adjustment. Basically, one should prove the following modified version of Lemma 4.1.2.

Lemma 6.3.3. *Let $u, v \in M$, with $v \geq 0$ on Ω and u satisfying one of the following assumptions:*

(i) $u \geq 0$ on Ω;

(iii) the positive and negative parts of u are both not identically zero and

$$u_+ \not\equiv 0 \text{ on } \partial\Omega \qquad \text{and} \qquad \frac{\displaystyle\int_\Omega \|\nabla u_+\|_{\ell^p}^p \, dx}{\displaystyle\int_{\partial\Omega} u_+^p \, \varrho \, d\mathcal{H}^{N-1}} \leq \frac{\displaystyle\int_\Omega \|\nabla u_-\|_{\ell^p}^p \, dx}{\displaystyle\int_{\partial\Omega} u_-^p \, \varrho \, d\mathcal{H}^{N-1}}, \tag{6.3.2}$$

with the convention that (6.3.2) *is satisfied if $u_- \equiv 0$ on $\partial\Omega$.*

Then there exists a continuous curve $\gamma : [0,1] \to M$, such that

$$\int_\Omega \|\nabla \gamma_t(x)\|_{\ell^p}^p \, dx \leq \max \left\{ \int_\Omega \|\nabla u(x)\|_{\ell^p}^p \, dx, \int_\Omega \|\nabla v(x)\|_{\ell^p}^p \, dx \right\}, \qquad t \in [0,1].$$

Remark 6.3.4. Of course, the positivity of the function v in the previous Lemma can be dropped and replaced by condition (6.3.2). We kept it just for ease of exposition.

In what follows, we will use the shortcut notation

$$\mathcal{R}_\Omega(u) = \frac{\displaystyle\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\displaystyle\int_{\partial\Omega} |u|^p \varrho \, d\mathcal{H}^{N-1}}, \qquad u \in W^{1,p}(\Omega) \setminus \{0\}, \tag{6.3.3}$$

where it is understood that $\mathcal{R}(u) = +\infty$ whenever u has zero trace on the boundary. The following is the main result of this section. It gives a simpler variational description of $\sigma_{2,p}(\Omega)$ just in terms of a minimization, rather than through a minimax procedure.

Theorem 6.3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set. Then the infimum*

$$\inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \left\{ \mathcal{R}_\Omega(u) \, : \, \int_{\partial\Omega} |u|^{p-2} \, u \, \varrho \, d\mathcal{H}^{N-1} = 0 \right\}, \tag{6.3.4}$$

is attained and coincides with $\sigma_{2,p}(\Omega)$. Moreover, every minimizer of (6.3.4) is a Stekloff eigenfunction.

PROOF. If Ω is not connected, then the infimum in (6.3.4) is zero. In that case $\sigma_{2,p}(\Omega) = 0$ as well, see Remark 6.2.2. This concludes the proof in the case of a disconnected open set.

The suppose that Ω is connected. The infimum (6.3.4) is attained. Indeed, a standard contradiction argument exploiting the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ leads to the existence of a constant $C_{p,\Omega}$ such that

$$\int_{\partial\Omega} |v|^p \, \varrho \, d\mathcal{H}^{N-1} \le C_{p,\Omega} \int_\Omega |\nabla v|^p \, dx,$$

for all $v \in W^{1,p}(\Omega)$ verifying

$$\int_{\partial\Omega} |v|^{p-2} v \, \varrho \, d\mathcal{H}^{N-1} = 0. \tag{6.3.5}$$

Then, by the equivalence of all norms in $\mathbb{R}^N$, it is not difficult to deduce that

$$\mathcal{R}_\Omega(u) \ge C_{p,\Omega} > 0, \qquad \text{for all } \varphi \in W^{1,p}(\Omega) \text{ satisfying (6.3.5)},$$

possibly for a different constant $C_{p,\Omega}$. This shows that the infimum (6.3.4) is strictly positive. The existence of a minimizer is again a straightforward consequence of Lemma 6.0.1 and of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

Now denote by σ^* the minimum value (6.3.4) and take a function $u \in W^{1,p}(\Omega)$ realizing it. Then u minimizes the functional

$$v \mapsto \int_\Omega \|\nabla v\|_{\ell^p}^p \, dx - \sigma^* \int_{\partial\Omega} |v|^p \, \varrho \, d\mathcal{H}^{N-1},$$

as well, among functions $v \in W^{1,p}(\Omega)$ satisfying the zero-mean condition (6.3.5). The Euler-Lagrange equation corresponding to this problem is precisely given by (6.1.3), with $\sigma = \sigma^*$, since the Lagrange multiplier corresponding to (6.3.5) is zero[1]. This in turn implies that σ^* is a Stekloff eigenvalue.

Eventually, let $u \in W^{1,p}(\Omega)$ be an eigenfuction for some eigenvalue $\sigma \neq 0$. Testing equation (6.1.3) with $\varphi = u$ shows that $\mathcal{R}_\Omega(u) = \sigma$. Similarly, by taking a constant test function in φ in (6.1.3), it follows that u verifies (6.3.5). Therefore each nontrivial Stekloff eigenfunction u is admissible for problem (6.3.4) and

$$\sigma^* \le \sigma_{2,p}(\Omega).$$

By Theorem 6.3.1, the reverse inequality holds as well, since $\sigma^* > 0$. $\square$

[1]As a matter of fact, some care is needed for the singular case $1 < p < 2$, see Lemma 6.3.7.

Remark 6.3.6. The value (6.3.4) coincides with the best constant in the following Poincaré-Wirtinger trace inequality

$$c_\Omega \left[\min_{t\in\mathbb{R}} \int_{\partial\Omega} |u+t|^p \,\varrho\, d\mathcal{H}^{N-1}\right] \le \int_\Omega \|\nabla u\|_{\ell^p}^p \,dx, \qquad u\in W^{1,p}(\Omega).$$

It is sufficient to observe that for every $u\in W^{1,p}(\Omega)$, the function $t\mapsto \|u+t\|^p_{L^p(\partial\Omega;\varrho)}$ is C^1 strictly convex and coercive (see below), then the value

$$\min_{t\in\mathbb{R}} \int_{\partial\Omega} |u+t|^p \,\varrho\, d\mathcal{H}^{N-1},$$

is uniquely realized and one has

$$t \text{ minimizes } \int_{\partial\Omega} |u+t|^p \,\varrho\, d\mathcal{H}^{N-1} \qquad \Longleftrightarrow \qquad u+t \text{ is admissible in (6.3.4)}.$$

This section ends with the following technical result, which was used to deduce the characterization of $\sigma_{2,p}$ given by Theorem 6.3.5.

Lemma 6.3.7 (Euler-Lagrange equation)**.** *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, having Lipschitz boundary. Let $u\in W^{1,p}(\Omega)$ be a minimizer of the functional*

$$\mathfrak{F}_p(v) = \frac{1}{p}\int_\Omega \|\nabla v\|_{\ell^p}^p\,dx - \frac{\sigma}{p}\int_{\partial\Omega} |v|^p\,\varrho\,d\,\mathcal{H}^{N-1}, \qquad v\in W^{1,p}(\Omega),$$

on the set of admissible functions $\mathcal{A} = \left\{v\in W^{1,p}(\Omega) : \int_{\partial\Omega} |v|^{p-2}\,v\,\varrho\,d\mathcal{H}^{N-1} = 0\right\}$. Then u is a Stekloff eigenfunction with eigenvalue σ.

Proof. For $p\ge 2$, observe that $\mathcal{A}$ is a C^1 manifold, thus the thesis is a plain consequence of the Lagrange Multipliers Theorem. Indeed, in this case u has to satisfy

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |u_{x_i}|^{p-2}\,u_{x_i}\,\varphi_{x_i}\,dx - \sigma\int_{\partial\Omega} |u|^{p-2}\,u\,\varphi\,\varrho\,d\mathcal{H}^{N-1}&\\ +\mu\int_\Omega |u|^{p-2}\,\varphi\,\varrho\,d\mathcal{H}^{N-1} = 0,&\qquad \text{for every } \varphi\in W^{1,p}(\Omega),\end{aligned}$$

for some $\mu\in\mathbb{R}$. By choosing as φ any constant function and by using that $u\in\mathcal{A}$, we can then easily conclude that $\mu=0$, i.e. u satisfies (6.1.3).

For $1<p<2$, some care is needed, since the constraint $\mathcal{A}$ is no more a C^1 manifold and we can not directly conclude as before. In this case, we modify the argument in [**33**], the only difference being the fact that we are not assuming u to be in $L^\infty(\Omega)$. Let $\varphi\in \mathrm{Lip}(\Omega)$ and $n\in\mathbb{N}\setminus\{0\}$, then the C^1 convex function

$$h_n(c) = \int_{\partial\Omega} \left|u+\frac{1}{n}\varphi + c\right|^p \varrho\,d\mathcal{H}^{N-1}, \qquad c\in\mathbb{R},$$

is coercive, since we have

$$h_n(c) \geq 2^{1-p}\,|c|^p \int_{\partial\Omega} \varrho\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} \left|u + \frac{1}{n}\,\varphi\right|^p \varrho\, d\mathcal{H}^{N-1}.$$

In particular, for every $n \in \mathbb{N} \setminus \{0\}$, h_n admits a minimum point c_n, which thus satisfies $h_n'(c_n) = 0$, that is

$$\int_{\partial\Omega} \left|u + \frac{1}{n}\,\varphi + c_n\right|^{p-2} \left(u + \frac{1}{n}\,\varphi + c_n\right) \varrho\, d\mathcal{H}^{N-1} = 0,$$

i.e. $u + 1/n\,\varphi + c_n \in \mathcal{A}$. Moreover, as n goes to ∞, we can guarantee that the quantity $n\,c_n$ stays uniformly bounded. More precisely, for every $n \in \mathbb{N}$, there must exist $x_n \in \partial\Omega$ such that

$$\varphi(x_n) + n\,c_n = 0. \tag{6.3.6}$$

Indeed, if this would not be true, then either $\varphi(x) + n\,c_n > 0$ for every $x \in \partial\Omega$ or $\varphi(x) + n\,c_n < 0$, thanks to the continuity of φ on $\partial\Omega$. Since the function $\tau \mapsto |u + \tau|^{p-2}\,\tau$ is strictly increasing, we would obtain

$$0 = \int_{\partial\Omega} \left|u + \frac{1}{n}\,\varphi + c_n\right|^{p-2} \left(u + \frac{1}{n}\,\varphi + c_n\right) \varrho\, d\mathcal{H}^{N-1} > \int_{\partial\Omega} |u|^{p-2}\,u\,\varrho\, d\mathcal{H}^{N-1} = 0$$

or

$$0 = \int_{\partial\Omega} \left|u + \frac{1}{n}\,\varphi + c_n\right|^{p-2} \left(u + \frac{1}{n}\,\varphi + c_n\right) \varrho\, d\mathcal{H}^{N-1} < \int_{\partial\Omega} |u|^{p-2}\,u\,\varrho\, d\mathcal{H}^{N-1} = 0.$$

In both cases, we would get a contradiction, so (6.3.6) must be true. This in turn implies that, possibly passing to a subsequence, the sequence $\{n\,c_n\}_{n\in\mathbb{N}}$ converges to some real number C, as n goes to ∞. Using the minimality of u and the fact that $u + 1/n\,\varphi + c_n$ is admissible, we then get

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \frac{\mathfrak{F}_p\left(u + \dfrac{1}{n}\,(\varphi + n\,c_n)\right) - \mathfrak{F}_p(u)}{\dfrac{1}{n}} &= \sum_{i=1}^{N} \int_{\Omega} |u_{x_i}|^{p-2}\,u_{x_i}\,\varphi_{x_i}\,dx \\ &\quad - \sigma \int_{\partial\Omega} |u|^{p-2}\,u\,(\varphi + C)\,\varrho\, d\mathcal{H}^{N-1}. \end{aligned}$$

Since $u \in \mathcal{A}$, the previous is equivalent to

$$0 \leq \sum_{i=1}^{N} \int_{\Omega} |u_{x_i}|^{p-2}\,u_{x_i}\,\varphi_{x_i}\,dx - \sigma \int_{\partial\Omega} |u|^{p-2}\,u\,\varphi\,\varrho\, d\mathcal{H}^{N-1}, \qquad \varphi \in \mathrm{Lip}(\Omega).$$

The same argument with $-\varphi$ in place of φ shows that u satisfies equation (6.1.3), for every Lipschitz test function. The conclusion then follows by exploiting the density of Lipschitz functions in $W^{1,p}(\Omega)$, which is true since Ω has Lipschitz boundary (see [**51**, Theorem 3.6]). $\square$

4. Halving pairs

The next result concerns some nodal properties of the first nontrivial eigenvalue. The proof is inspired to the linear case (see [**7, 65**]).

Proposition 6.4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and connected bounded set, having Lipschitz. There exists a first nontrivial Stekloff eigenfunction $w \in W^{1,p}(\Omega)$ with exactly two nodal domains.*

PROOF. Let us take $u \in W^{1,p}(\Omega)$ a first nontrivial eigenfuction, thanks to Lemma 6.1.5 we have that u has at least two nodal domains.

Let us now suppose that u has $n \geq 3$ nodal domains, $\Omega_1, \dots, \Omega_n \subset \Omega$. We then take the functions

$$v_k = u \cdot 1_{\Omega_k}, \qquad k = 1, 2,$$

i.e. the restrictions of u to Ω_1 and Ω_2, respectively and we define

$$w = \alpha\, v_1 + \beta\, v_2.$$

This is a function in $W^{1,p}(\Omega)$ and observe that we can always choose $\alpha, \beta \in \mathbb{R}$ such that

$$\int_{\partial\Omega} |w|^{p-2}\, w\, \varrho\, d\mathcal{H}^{N-1}(x) = 0.$$

By construction w is admissible for the variational problem (6.3.4) which gives $\sigma_{2,p}(\Omega)$. Moreover, we can infer

$$\begin{aligned}
\int_\Omega \|\nabla w\|_{\ell^p}^p\, dx &= \alpha^p \int_{\Omega_1} \|\nabla v_1\|^p\, dx + \beta^p \int_{\Omega_2} \|\nabla v_2\|^p\, dx \\
&= \sigma_{2,p}(\Omega) \left[\alpha^p \int_{\partial\Omega_1 \cap \partial\Omega} |v_1|^p\, \varrho\, d\mathcal{H}^{N-1} + \beta^p \int_{\partial\Omega_2 \cap \partial\Omega} |v_2|^p\, \varrho\, d\mathcal{H}^{N-1}\right] \\
&= \sigma_{2,p}(\Omega) \int_{\partial\Omega} |w|^p\, \varrho\, d\mathcal{H}^{N-1}.
\end{aligned}$$

Owing to the characterization of Theorem 6.3.5 for $\sigma_{2,p}(\Omega)$, we then get that w is a first nontrivial Stekloff eigenfuction of Ω, having exactly two nodal domains. $\square$

Remark 6.4.2. Very likely, the previous property is verified by *every* Stekloff eigenfunction corresponding to $\sigma_{2,p}(\Omega)$, i.e. every first nontrivial eigenfunction should have exactly two nodal domains. The main obstruction to the proof is the lack of a *unique continuation principle* for pseudo $p-$harmonics functions. Indeed, observe that in the previous proof we constructed a function w which satisfies $\widetilde{\Delta} w = 0$ and identically vanishes on a open subset of Ω, but we can not get a contradiction from this. We also like to point out that Harnack's inequality is of not use here, since we can not guarantee that $\partial\Omega_1 \cap \Omega$ does not coincide with

$\partial\Omega_2 \cap \Omega$. This is linked to the existence of the so-called *Lakes of Wada*, i.e. triples of open connected sets in the plane, which share the same boundaries.

In the case of the second Dirichlet eigenvalue of the $p-$Laplacian, the use of the unique continuation property can be avoided, as proved in [**31**]. However, this proof can not be applied here either, since our eigenfunctions are not known to be in C^1, as required by the argument in [**31**].

Definition 6.4.3. Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with Lipschitz boundary. Let us consider two open connected Lipschitz subsets $\Omega_1, \Omega_2 \subset \Omega$, then (Ω_1, Ω_2) is said a *halving pair for* Ω if the following conditions are satisfied:

$$|\Omega_1 \cup \Omega_2| \le |\Omega|, \qquad \Omega_1 \cap \Omega_2 = \emptyset \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial\Omega_i \cap \partial\Omega) > 0,\ i = 1, 2. \tag{6.4.1}$$

We also set

$$\mathrm{Hal}(\Omega) = \{(\Omega_1, \Omega_2) \text{ halving pair of } \Omega\}.$$

If $\Sigma \subset \Omega$ is such that $\Gamma := \partial\Sigma \cap \Omega \ne \emptyset$ and this is a Lipschitz surface, we also introduce the following quantity

$$\Lambda_p(\Sigma; \Omega) = \min_{u \in W^{1,p}(\Sigma)\setminus\{0\}} \Big\{ \mathcal{R}_\Sigma(u) \ :\ u = 0 \text{ on } \Gamma \Big\}. \tag{6.4.2}$$

An optimal function in (6.4.2) is a weak solution of the following mixed Dirichlet-Stekloff eigenvalue problem

$$\begin{cases} -\widetilde{\Delta}_p u = 0, & \text{in } \Sigma \\ u = 0, & \text{on } \Gamma, \\ \displaystyle\sum_{i=1}^N |u_{x_i}|^{p-2}\, u_{x_i}\, \nu_\Omega^i = \lambda\, |u|^{p-2}\, u\, \varrho, & \text{on } \partial\Omega \cap \partial\Sigma, \end{cases} \tag{6.4.3}$$

with $\lambda = \Lambda_p(\Sigma; \Omega)$, i.e. a minimizer of (6.4.2) satisfies

$$\sum_{i=1}^N \int_\Omega |u_{x_i}|^{p-2}\, u_{x_i}\, \varphi_{x_i}\, dx = \Lambda_p(\Sigma; \Omega) \int_{\partial\Omega\cap\partial\Sigma} |u|^{p-2}\, u\, \varphi\, \varrho\, d\mathcal{H}^{N-1},$$

for every $\varphi \in W^{1,p}(\Sigma)$ with $\varphi = 0$ on Γ.

Lemma 6.4.4. *With the previous notation, for every $p \in (1, \infty)$ problem* (6.4.2) *admits a unique positive solution $u \in W^{1,p}(\Sigma)$ satisfying the normalization condition*

$$\int_{\partial\Omega\cap\partial\Sigma} |u(x)|^p\, \varrho\, d\mathcal{H}^{N-1}(x) = 1.$$

Moreover, the boundary value problem (6.4.3) *admits a positive (weak) solution if and only if $\lambda = \Lambda_p$.*

Proof. Existence of a solution for this problem is straightforward. Positivity follows as always by observing that for every admissible u, the function $|u|$ is still admissible and

$$\mathcal{R}_\Sigma(|u|) = \mathcal{R}_\Sigma(u).$$

Uniqueness can be proved using the device of Belloni and Kawohl, that we already used in Lemma 4.1.2. Suppose to have two distinct strictly positive[2] solutions u_0 and u_1 such that

$$\int_{\partial\Omega\cap\partial\Sigma} |u_i(x)|^p \,\varrho\, d\mathcal{H}^{N-1}(x) = 1, \qquad i = 0, 1. \tag{6.4.4}$$

As in Lemma 4.1.2, we set $\gamma_t(x) = [(1-t)\,u_0(x)^p + t\,u_1(x)^p]^{1/p}$, for a given $0 < t < 1$. This still satisfies the normalization condition (6.4.4) and

$$t \mapsto \mathcal{R}_\Sigma(\gamma_t) \text{ is strictly convex on } [0,1]. \tag{6.4.5}$$

Then γ_t is still a solution and we must have

$$\mathcal{R}_\Sigma(\gamma_t) = \mathcal{R}_\Sigma(u_0) = \mathcal{R}_\Sigma(u_1), \qquad t \in [0,1].$$

This can hold if and only if $u_0 = \mu\, u_1$ for some $\mu > 0$ (see [**11**] for more details). By using (6.4.4), we get $\mu = 1$ and thus we obtain a contradiction.

The second part of the statement can be proved along the same lines of [**F2**, Theorem 3.1], still using property (6.4.5). One just needs to observe that every λ such that (6.4.3) has a solution is a crititical value of $\int_\Omega \|\nabla u\|_{\ell^p}^p$ on the manifold

$$\left\{ v \in W^{1,p}(\Omega) \,:\, v = 0 \text{ on } \Gamma \quad \text{and} \quad \int_{\partial\Omega} |v|^p \,\varrho\, d\mathcal{H}^{N-1} = 1 \right\}.$$

This concludes the proof. □

Using problem (6.4.3), we have yet another minimax characterization of $\sigma_{2,p}(\Omega)$, this time in terms of the eigenvalues Λ_p. For this, we assume some smoothness on the nodal domains.

Proposition 6.4.5. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, having Lipschitz boundary. Suppose that the nodal domains Ω_+ and Ω_- of a first nontrivial eigenfunction u belongs to* $\mathrm{Hal}(\Omega)$. *Then there holds*

$$\sigma_{2,p}(\Omega) = \min\left\{ \max\{\Lambda_p(\Omega_1;\Omega), \Lambda_p(\Omega_2;\Omega)\} \,:\, (\Omega_1,\Omega_2) \in \mathrm{Hal}(\Omega) \right\}. \tag{6.4.6}$$

The minimum above is realized by the pair (Ω_+, Ω_-) and

$$\Lambda_p(\Omega_+;\Omega) = \mathcal{R}_{\Omega_+}(u) = \mathcal{R}_{\Omega_-}(u) = \Lambda_p(\Omega_-;\Omega). \tag{6.4.7}$$

[2]Strict positivity is a consequence of Harnack's inequality. Indeed, as already observed, a pseudo $p-$harmonic function is a local minimizer of the Dirichlet energy $\int_\Omega \|\nabla u\|_{\ell^p}^p\, dx$. Then Harnack's inequality for these functions is a consequence of [**51**, Theorem 7.11].

Proof. Let us take a halving pair (Ω_1, Ω_2) and $u_i \in W^{1,p}(\Omega_i)$ such that $u_i = 0$ on $\partial\Omega_i \cap \Omega$, with

$$\int_{\Omega_i} \|\nabla u_i(x)\|_{\ell^p}^p \, dx = \Lambda_p(\Omega_i; \Omega) \quad \text{and} \quad \int_{\partial\Omega_i \cap \partial\Omega} |u_i(x)|^p \, \varrho \, d\mathcal{H}^{N-1}(x) = 1, \ i = 1, 2.$$

Then we can choose two parameters $\alpha_1, \alpha_2 \in \mathbb{R}$ in such a way that

$$v(x) = \sum_{i=1}^{2} \alpha_i \, u_i(x) \cdot 1_{\Omega_i}(x), \qquad x \in \Omega,$$

satisfies the zero-mean condition (6.3.5). Thus, we can infer

$$\begin{aligned}\sigma_{2,p}(\Omega) &\leq \frac{\alpha_1^p \displaystyle\int_{\Omega_1} \|\nabla u_1(x)\|_{\ell^p}^p \, dx + \alpha_2^p \int_{\Omega_2} \|\nabla u_2(x)\|_{\ell^p}^p \, dx}{\alpha_1^p + \alpha_2^p} \\ &= \frac{\alpha_1^p \, \Lambda_p(\Omega_1; \Omega) + \alpha_2^p \, \Lambda_p(\Omega_2; \Omega)}{\alpha_1^p + \alpha_2^p} \leq \max\{\Lambda_p(\Omega_1; \Omega), \Lambda_p(\Omega_2; \Omega)\},\end{aligned}$$

and since this is true for every halving pair (Ω_1, Ω_2), this remains true taking the infimum over $\mathrm{Hal}(\Omega)$.

Let us now take an eigenfunction $u \in W^{1,p}(\Omega)$ relative to $\sigma_{2,p}(\Omega)$, i.e. a minimizer of (6.3.4). By Proposition 6.4.1, we can choose it in such a way that it has two nodal domains Ω_+ and Ω_-, both touching the boundary of Ω. Using the equation, we then have

$$\mathcal{R}_{\Omega_+}(u) = \mathcal{R}_{\Omega_-}(u) = \sigma_{2,p}(\Omega).$$

By definition of Λ_p and the hypothesis on Ω_+, Ω_-, we then get

$$\Lambda_p(\Omega_+; \Omega) \leq \mathcal{R}_{\Omega_+}(u) \qquad \text{and} \qquad \Lambda_p(\Omega_-; \Omega) \leq \mathcal{R}_{\Omega_-}(u),$$

so that

$$\max\{\Lambda_p(\Omega_+; \Omega), \, \Lambda_p(\Omega_-; \Omega)\} \leq \sigma_{2,p}(\Omega).$$

This concludes the proof of (6.4.6) and shows that the minimum is realized by the pair (Ω_+, Ω_-). In order to prove (6.4.7), it is sufficient to observe that u restricted to Ω_+ is a positive solution of (6.4.3), with $\lambda = \sigma_{2,p}(\Omega)$. By the second part of Lemma 6.4.4, we can infer that $\Lambda_p(\Omega_+; \Omega) = \mathcal{R}_{\Omega_+}(u)$. The same observation applies to Ω_-, thus leading to (6.4.7). □

5. An upper bound for $\sigma_{2,p}$

In this section we prove an upper bound for $\sigma_{2,p}(\Omega)$, in terms of geometric quantities. For this, we need the following simple result. It guarantees that the coordinate functions $\varphi_j(x) = x_j$, $j = 1, \dots, N$ are always admissible in (6.3.4), modulo a translation.

Lemma 6.5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, having Lipschitz boundary. Let $\varrho : \partial\Omega \to \mathbb{R}$ be a function satisfying* (6.1.1)*. Then there exists $z \in \mathbb{R}^N$ such that the translated set $\Omega' = \Omega - z$ satisfies*

$$\int_{\partial\Omega'} |x_i|^{p-2}\, x_i\, \varrho(x+z)\, d\mathcal{H}^{N-1}(x) = 0, \tag{6.5.1}$$

for all $i = 1, \dots, N$.

Proof. It is not difficult to see that the function

$$g(y) = \sum_{i=1}^{N} \frac{1}{p} \int_{\partial\Omega} |x_i - y_i|^p\, \varrho(x)\, d\mathcal{H}^{N-1}(x), \qquad y = (y_1, \dots, y_N) \in \Omega,$$

is C^1 and that it admits a global minimum point. Thus there exists z such that

$$\int_{\partial\Omega} |x_i - z_i|^{p-2}\, (x_i - z_i)\, \varrho(x)\, d\mathcal{H}^{N-1}(x) = 0, \qquad i = 1, \dots, N.$$

Let us now make the change of variable $y = x - z$. By defining $\Omega' = \Omega - z$, the above reads

$$\int_{\partial\Omega'} |y_i|^{p-2}\, y_i\, \varrho(y+z)\, d\mathcal{H}^{N-1}(y) = 0, \qquad i = 1, \dots, N,$$

which concludes the proof. □

The following is the main result of this section, dealing with the case of a general weight ϱ. This is the nonlinear counterpart of Brock's inequality for the first nontrivial Stekloff eigenvalue of the Laplacian (compare with [**20**, Theorem 1]). Its proof crucially exploits the weighted Wulff inequality derived in Theorem 7.3.4 and Corollary 7.4.2 of the Appendix.

Theorem 6.5.2. *Let $1 < p < \infty$ and $p' = p/(p-1)$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, having Lipschitz boundary and ϱ a function satifying* (6.1.1)*. Then there holds*

$$\sigma_{2,p}(\Omega) \le \left(\frac{\displaystyle\int_{\partial\Omega} \varrho(x)^{-\frac{1}{p-1}}\, \|\nu_\Omega(x)\|_{\ell^{p'}}^{p'}\, d\mathcal{H}^{N-1}(x)}{N\,|\Omega|} \right)^{p-1}. \tag{6.5.2}$$

Proof. Let $z \in \mathbb{R}^N$ be as in Lemma 6.5.1 and let us set $\Omega' = \Omega - z$. By the characterization (6.3.4) of $\sigma_{2,p}(\Omega)$, we obtain

$$\sigma_{2,p}(\Omega) \le \frac{\displaystyle\int_{\Omega'} \|\nabla\varphi_i(x)\|_{\ell^p}^p\, dx}{\displaystyle\int_{\partial\Omega'} |\varphi_i(x)|^p\, \varrho(x+z)\, d\mathcal{H}^{N-1}(x)} = \frac{|\Omega|}{\displaystyle\int_{\partial\Omega'} |x_i|^p\, \varrho(x+z)\, d\mathcal{H}^{N-1}(x)}, \quad i = 1, \dots, N,$$

where $\varphi_i(x) = x_u$, as before. Taking the sum over $i = 1, \dots, N$, we obtain

$$\sigma_{2,p}(\Omega) \le \frac{N\,|\Omega|}{\displaystyle\int_{\partial\Omega'} \|x\|_{\ell^p}^p\, \varrho(x+z)\, d\mathcal{H}^{N-1}(x)},$$

then we observe that by Hölder inequality, we have

$$\int_{\partial\Omega'} \|x\|_{\ell^p}^p\, \varrho(x+z)\, d\mathcal{H}^{N-1}(x) \ge \frac{\left(\displaystyle\int_{\partial\Omega'} \|x\|_{\ell^p}\, \|\nu_{\Omega'}(x)\|_{\ell^{p'}}\, d\mathcal{H}^{N-1}(x)\right)^p}{\left(\displaystyle\int_{\partial\Omega'} \varrho(x+z)^{-\frac{1}{p-1}}\, \|\nu_{\Omega'}(x)\|_{\ell^{p'}}^{p'} d\mathcal{H}^{N-1}(x)\right)^{p-1}}.$$

Note that $\nu_{\Omega'}(x) = \nu_\Omega(x+z)$. Set

$$P_{p,\beta}(\Omega') = \int_{\partial\Omega'} \|x\|_{\ell^p}^\beta\, \|\nu_{\Omega'}(x)\|_{\ell^{p'}}\, d\mathcal{H}^{N-1}(x),$$

The weighted Wulff inequality

$$P_{p,\beta}(\Omega') \ge N\,|B_p|^{\frac{1-\beta}{N}}\,|\Omega'|^{\frac{N+\beta-1}{N}},$$

is proved in Corollary 7.4.2 of next chapter. Taking $\beta = 1$, one gets

$$\begin{aligned}
\sigma_{2,p}(\Omega) &\le \frac{N\,|\Omega|}{\left(\displaystyle\int_{\partial\Omega'} \|x\|_{\ell^p}\, \|\nu_{\Omega'}(x)\|_{\ell^{p'}}\, d\mathcal{H}^{N-1}(x)\right)^p} \left(\int_{\partial\Omega} \varrho(x)^{-\frac{1}{p-1}}\, \|\nu_\Omega(x)\|_{\ell^{p'}}^{p'} d\mathcal{H}^{N-1}(x)\right)^{p-1} \\
&\le \frac{N\,|\Omega|}{N^p\,|\Omega|^p} \left(\int_{\partial\Omega} \varrho(x)^{-\frac{1}{p-1}}\, \|\nu_\Omega(x)\|_{\ell^{p'}}^{p'} d\mathcal{H}^{N-1}(x)\right)^{p-1},
\end{aligned}$$

which gives the desired estimate. □

A significant and intrinsic instance of weight function ϱ verifying (6.1.1) is given by

$$\varrho(x) = \|\nu_\Omega(x)\|_{\ell^{p'}}, \qquad x \in \partial\Omega.$$

In this case, a more elegant and simpler bound is possible, that should be compared with the Brock-Weinstock inequality

$$\sigma_2(\Omega) \le \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{1}{N}}. \tag{6.5.3}$$

Theorem 6.5.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, having Lipschitz boundary. Then there holds*

$$\sigma_{2,p}(\Omega) \le \left(\frac{|B_p|}{|\Omega|}\right)^{\frac{p-1}{N}}, \tag{6.5.4}$$

where $B_p = \{x \in \mathbb{R}^N : \|x\|_{\ell^p} < 1\}$.

Proof. Again, we take $\varphi_i(x) = x_i$, $i = 1, \dots, N$, then up to a translation of Ω (which does not affect $\sigma_{2,p}(\Omega)$), we can suppose that (6.5.1) is satisfied. We again obtain

$$\sigma_{2,p}(\Omega) \le \frac{\displaystyle\int_\Omega \|\nabla \varphi_i(x)\|_{\ell^p}^p \, dx}{\displaystyle\int_{\partial\Omega} |\varphi_i(x)|^p \, \|\nu_\Omega(x)\|_{\ell^{p'}} \, d\mathcal{H}^{N-1}(x)} = \frac{|\Omega|}{\displaystyle\int_{\partial\Omega} |x_i|^p \, \|\nu_\Omega(x)\|_{\ell^{p'}} \, d\mathcal{H}^{N-1}(x)}, \quad i = 1, \dots, N.$$

that is, summing up over $i = 1, \dots, N$, we have

$$\sigma_{2,p}(\Omega) \le \frac{N\,|\Omega|}{\displaystyle\int_{\partial\Omega} \|x\|_{\ell^p}^p \, \|\nu_\Omega(x)\|_{\ell^{p'}} \, d\mathcal{H}^{N-1}(x)}.$$

Using the isoperimetric property of B_p given by Corollary 7.4.2, this time with $\beta = p$, we eventually obtain the thesis. $\square$

Remark 6.5.4. We conjecture the bounds (6.5.2) and (6.5.4) to be “isoperimetric” as in the linear case, that corresponds to the Brock-Weinstock inequality (6.5.3). In other words, we conjecture that equality holds in (6.5.4) if and only if $\Omega = B_p$, up to dilations and translations. For (6.5.2) one also needs to require

$$\varrho(x) = c\,\|\nu_\Omega(x)\|_{\ell^{p'}}, \qquad x \in \partial\Omega.$$

To prove this conjecture, one would need to show that $\sigma_{2,p}(B_p) = 1$, i.e. the coordinate functions $\varphi_i(x) = x_i$, $i = 1, \dots, N$ are first nontrivial eigenfuctions of $-\widetilde{\Delta}_p$ on B_p. It is easily seen that $x_1, \dots, x_N$ are indeed Stekloff eigenfunctions on B_p, with corresponding eigenvalue 1. Of course, it could happen that $\sigma_{2,p}(B_p) < 1$. To conclude, it would be sufficient to prove the existence of a first nontrivial eigenfunction having $\{x_j = 0\}$ as nodal line, for some $j = 1, \dots, N$. The thesis would then follows from Lemma 6.4.4 and Proposition 6.4.5.

CHAPTER 7

Anisotropic weighted Wulff inequalities

1. Basics on convex bodies

For more details on this topic, the reader may consult [**85**]. Let $K \subset \mathbb{R}^N$ be a bounded convex set containing the origin as an interior point. Consider the subadditive and $1-$positively homogeneous function defined by

$$\|x\| = \inf\{\lambda > 0 \,:\, x \in \lambda\, K\},$$

for all $x \in \mathbb{R}^N$. The *convex body* K turns out to be the unit ball for this "norm", called the *Minkowski gauge* associated with K. Actually, this is a norm if and only if K is symmetric with respect to the origin in $\mathbb{R}^N$. Otherwise $\|-x\|$ may happen to be different from $\|x\|$ for some $x \in \mathbb{R}^N$. The dual "norm" is defined by setting

$$\|\xi\|_* = \max_{x \in K} \langle x, \xi \rangle,$$

for all $\xi \in \mathbb{R}^N$. That is sometimes called *support function* of K. Then the polar set K^* is usually defined as the unit ball for $\|\cdot\|_*$, i.e.

$$K^* = \{\xi \in \mathbb{R}^N \,:\, \|\xi\|_* \leq 1\}$$

and it is often referred to as the *Wulff shape* associated with K. By definition, we have the following general version of the Cauchy-Schwarz inequality

$$|\langle x, \xi \rangle| \leq \|x\|\, \|\xi\|_*, \qquad x, \xi \in \mathbb{R}^N, \tag{7.1.1}$$

with equality if and only if ξ belongs to the normal cone $N_K(x/\|x\|)$ to K at the point $x/\|x\|$. In particular, if K is C^1, equality holds if and only if $\xi = t\,\nu_K(x/\|x\|)$, for some $t \geq 0$.

2. Differentiation of norms

For the convenience of the reader, some basic facts of convex analysis are recalled. If

$$F : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$$

is a convex lower semicontinuous proper function, then for all $x, \xi \in \mathbb{R}^N$

$$\xi \in \partial F(x) \qquad \text{if and only if} \qquad F(x) + F^*(\xi) = \langle x, \xi \rangle$$

where F^* denotes the Legendre-Fenchel conjugate of F and $\partial F(x)$ is the subdifferential of F at the point x.

Lemma 7.2.1. *The map $x \mapsto \|x\|$ is convex and thus differentiable almost everywhere. Namely*

$$\nabla\|x\| = \frac{\nu_K\left(\dfrac{x}{\|x\|}\right)}{\left\|\nu_K\left(\dfrac{x}{\|x\|}\right)\right\|_*} \qquad \text{and} \qquad \left\langle \nu_K\left(\frac{x}{\|x\|}\right), x\right\rangle = \left\|\nu_K\left(\frac{x}{\|x\|}\right)\right\|_* \|x\|, \tag{7.2.1}$$

almost everywhere.

Proof. Choosing $F(x) = \|x\|$, it is easy to see that its Legendre-Fenchel conjugate function is given by $F^*(\xi) = \delta_{K^*}(\xi)$, i.e. the indicator function of the polar set K^*. This yields

$$\xi \in \partial\|x\| \qquad \text{if and only if} \qquad \|\xi\|_* \leq 1 \text{ and } \langle \xi, x\rangle = \|x\|.$$

In particular, if $x \neq 0$ and $\xi \in \partial\|x\|$, by (7.1.1) we get

$$\|x\| = \langle \xi, x\rangle \leq \|x\|\,\|\xi\|_* \leq \|x\|,$$

i.e. $\|\xi\|_* = 1$ and equality holds in (7.1.1). This implies that if $x \neq 0$, the subdifferential of $\partial\|x\|$ is characterized by

$$\xi \in \partial\|x\| \qquad \text{if and only if} \qquad \|\xi\|_* = 1 \text{ and } \xi \in N_K\left(\frac{x}{\|x\|}\right). \tag{7.2.2}$$

Since for almost every $x \in \mathbb{R}^N$ we have

$$\partial\|x\| = \{\nabla\|x\|\} \qquad \text{and} \qquad N_K\left(\frac{x}{\|x\|}\right) = \left\{z \in \mathbb{R}^N \,:\, z = t\,\nu_K\left(\frac{x}{\|x\|}\right) \text{ for some } t \geq 0\right\},$$

the characterization (7.2.2) gives the first relation in (7.3.4).

Observe that the second relation in (7.3.4) comes again from the cases of equality in the Cauchy-Schwarz inequality, by simply noticing that

$$\left\langle \nu_K\left(\frac{x}{\|x\|}\right), x\right\rangle = \left\langle \nu_K\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \|x\|.$$

This concludes the proof. □

3. Weighted Wulff Inequalities

Let $\|\cdot\|$ denote the Minkowski gauge associated with a convex body K in $\mathbb{R}^N$ and denote by $\|\cdot\|_*$ its support function. Some preparatory facts about the theory of convex bodies are recalled in Chapter 7. Given $\Omega \subset \mathbb{R}^N$ a bounded open Lipschitz set, if we define its *anisotropic perimeter* by

$$P_K(\Omega) = \int_{\partial\Omega} \|\nu_\Omega(x)\|_*\, d\mathcal{H}^{N-1},$$

we have the classical Wulff inequality

$$P_K(\Omega) \geq N\,|K|^{\frac{1}{N}}\,|\Omega|^{\frac{N-1}{N}}. \tag{7.3.1}$$

Recalling that $P_K(K) = N\,|K|$, the previous is equivalent to say that K minimizes P_K, among sets with given measure. Moreover, strict equality holds in (7.3.1), if Ω is not a scaled and translated copy of K. See for example [**44**] for a detailed study of Wulff inequality.

Definition 7.3.1. Let $V : [0, +\infty) \to [0, +\infty)$ a Borel function such that $V(0) = 0$. For every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, we define its *weigthed anisotropic perimeter* by

$$P_{V,K}(\Omega) = \int_{\partial\Omega} V(\|x\|)\,\|\nu_\Omega(x)\|_*\,d\mathcal{H}^{N-1}(x).$$

Remark 7.3.2. When K coincides with the unit ball of the Euclidean norm $|\cdot|$, it easily seen that $\|x\| = \|x\|_* = |x|$ and $P_{V,K}$ coincides with the weighted perimeter

$$\int_{\partial\Omega} V(|x|)\,d\mathcal{H}^{N-1}(x),$$

already studied in [**13, 18**].

Let us now further suppose that $V \in C^1([0,\infty))$, $V(t) > 0$ for $t > 0$ and it satisfies the following condition

$$v(t) := V'(t) + (N-1)\,\frac{V(t)}{t}, \qquad \text{is non decreasing on } (0,+\infty). \tag{7.3.2}$$

We consider the vector field

$$W(x) = V(\|x\|)\,\frac{x}{\|x\|}, \qquad x \in \mathbb{R}^N,$$

with the convention that $W(0) = 0$. The crucial property of W is expressed by the following Lemma, which extends to the anisotropic case a straightforward calculation of the Euclidean one.

Lemma 7.3.3. *With the previous notations, there holds*

$$\operatorname{div} W(x) = v(\|x\|), \qquad x \in \mathbb{R}^N \setminus \{0\}. \tag{7.3.3}$$

In particular, $\operatorname{div} W$ *is a non decreasing function of* $\|\cdot\|$.

Proof. First of all, we observe that $x \mapsto \|x\|$ is convex and thus differentiable almost everywhere. Namely,

$$\nabla\|x\| = \frac{\nu_K\left(\dfrac{x}{\|x\|}\right)}{\left\|\nu_K\left(\dfrac{x}{\|x\|}\right)\right\|_*} \qquad \text{and} \qquad \left\langle \nu_K\left(\frac{x}{\|x\|}\right), x\right\rangle = \left\|\nu_K\left(\frac{x}{\|x\|}\right)\right\|_* \|x\|, \tag{7.3.4}$$

where these relations hold almost everywhere. We refer to Section 2 for a proof of the identify (7.3.4) Observe that (7.3.3) is a simple consequence of (7.3.4). Indeed, using these we get

$$\begin{aligned}\operatorname{div} W(x) &= V'(\|x\|)\left\langle \nabla\|x\|, \frac{x}{\|x\|}\right\rangle + N\,\frac{V(\|x\|)}{\|x\|} - V(\|x\|)\,\frac{\langle\nabla\|x\|, x\rangle}{\|x\|^2}\\ &= V'(\|x\|)\,\|x\| + (N-1)\,\frac{V(\|x\|)}{\|x\|} = v(\|x\|), \qquad \text{for a.e. } x\in\mathbb{R}^N,\end{aligned}$$

which gives the desired result. □

Next theorem is the main result of the chapter. The idea of the proof is completely borrowed from the paper [**18**] by Brasco, De Philippis and Ruffini, who studied the isotropic case.

Theorem 7.3.4 (Weighted Wulff inequality)**.** *Let $\Omega\subset\mathbb{R}^N$ be an open bounded Lipschitz set. Then we have*

$$P_{V,K}(\Omega) \geq N\,|K|^{\frac{1}{N}}\,|\Omega|^{1-\frac{1}{N}}\,V\left(\left(\frac{|\Omega|}{|K|}\right)^{\frac{1}{N}}\right), \tag{7.3.5}$$

with equality if and only if Ω coincides with K, up to dilations. In other words, K is the only minimizer of $P_{K,V}$, under measure constraint, i.e.

$$P_{V,K}(K) = \min\{P_{V,K}(\Omega)\,:\,|\Omega| = |K|\}. \tag{7.3.6}$$

Proof. It is easily seen that (7.3.5) and (7.3.6) are equivalent, so let us suppose that $|\Omega| = |K|$. We divide the proof in two steps: first we prove the inequality, then we detect the cases of equality.

Inequality. By using the Divergence Theorem and Lemma 7.3.3 we get

$$\begin{aligned}\int_\Omega v(\|x\|)\,dx = \int_\Omega \operatorname{div} W(x) &= \int_{\partial\Omega} V(\|x\|)\,\left\langle \frac{x}{\|x\|}, \nu_\Omega(x)\right\rangle\,d\mathcal{H}^{N-1}(x)\\ &= \int_{\partial\Omega} V(\|x\|)\,\left[\left\langle \frac{x}{\|x\|}, \nu_\Omega(x)\right\rangle - \|\nu_\Omega(x)\|_*\right]\,d\mathcal{H}^{N-1}(x)\\ &\quad + P_{V,K}(\Omega),\end{aligned}$$

while integrating v over K yields

$$\begin{aligned}\int_K v(\|x\|)\,dx = \int_K \operatorname{div} W(x) &= \int_{\partial K} V(\|x\|)\,\langle x, \nu_K(x)\rangle\,d\mathcal{H}^{N-1}(x)\\ &= \int_{\partial K} V(\|x\|)\,\|\nu_\Omega(x)\|_*\,d\mathcal{H}^{N-1}(x) = P_{V,K}(K),\end{aligned}$$

since by definition $\|x\| = 1$ on ∂K. Subtracting the two equalities, we get

$$P_{V,K}(\Omega) - P_{V,K}(K) = \mathcal{I}_1(\Omega) + \mathcal{I}_2(\Omega) \tag{7.3.7}$$

where we set

$$\mathcal{I}_1(\Omega) = \int_{\partial\Omega} V(\|x\|)\, \left[\|\nu_\Omega(x)\|_* - \left\langle \frac{x}{\|x\|}, \nu_\Omega(x)\right\rangle\right]\, d\mathcal{H}^{N-1}(x),$$

and

$$\mathcal{I}_2(\Omega) = \int_\Omega v(\|x\|)\, dx - \int_K v(\|x\|)\, dx.$$

It is not difficult to see that both quantities are positive. For the first, this is a simple consequence of the Cauchy-Schwarz inequality (7.1.1); for the second, we just observe that

$$\mathcal{I}_2(\Omega) = \int_{\Omega\setminus K} [v(\|x\|) - v(1)]\, dx + \int_{K\setminus\Omega} [v(1) - v(\|x\|)]\, dx \tag{7.3.8}$$

thanks to the fact that $|K \setminus \Omega| = |\Omega \setminus K|$, since K and Ω have the same measure. On the other hand, there holds

$$\Omega \setminus K \subset \{x : \|x\| \geq 1\} \qquad \text{and} \qquad K \setminus \Omega \subset \{x : \|x\| \leq 1\},$$

then by using the monotone behaviour of v, we can infer $\mathcal{I}_2(\Omega) \geq 0$. Thus (7.3.7) shows that K minimizes $P_{V,K}$ among sets with given measure.

Cases of equality. Let us suppose that $P_{V,K}(\Omega) = P_{V,K}(K)$. Again by (7.3.7) we can infer

$$\mathcal{I}_1(\Omega) = 0 = \mathcal{I}_2(\Omega).$$

If the function v is strictly increasing, then the previous and (7.3.8) easily imply that $|\Omega\Delta K| = 0$, i.e. Ω has to coincide with K. On the contrary, if v is simply a non decreasing functions, the proof is a bit more complicated. In this case, the information $\mathcal{I}_2(\Omega) = 0$ is useless and we need to exploit the first one i.e. $\mathcal{I}_1(\Omega) = 0$. Keeping into account that $V(t) > 0$ for $t > 0$, from the latter we can infer that

$$\|\nu_\Omega(x)\|_* = \left\langle \frac{x}{\|x\|}, \nu_\Omega(x)\right\rangle, \qquad \text{for } \mathcal{H}^{N-1}\text{–a.e. } x \in \partial\Omega. \tag{7.3.9}$$

This implies that the standard anisotropic perimeter of Ω can be written as

$$P_K(\Omega) = \int_{\partial\Omega} \|\nu_\Omega(x)\|_*\, d\mathcal{H}^{N-1}(x) = \int_\Omega \operatorname{div}\left(\frac{x}{\|x\|}\right)\, dx = \int_\Omega \frac{N-1}{\|x\|}\, dx,$$

where we used the computations of Lemma 7.3.3, with $V \equiv 1$. We now observe that the last integrand is a strictly decreasing function of $\|\cdot\|$. Then using that $K = \{x : \|x\| \leq 1\}$ and that $|\Omega| = |K|$, we have

$$\begin{aligned}\int_\Omega \frac{N-1}{\|x\|}\, dx &\leq \int_{\Omega\cap K} \frac{N-1}{\|x\|}\, dx + (N-1)\, |\Omega \setminus K| \\ &= \int_{\Omega\cap K} \frac{N-1}{\|x\|}\, dx + (N-1)\, |K \setminus \Omega| \leq \int_K \frac{N-1}{\|x\|}\, dx,\end{aligned}$$

with strict inequality if $|\Omega\Delta K| \neq 0$. This implies that $P_K(\Omega) \leq P_K(K)$ and $P_K(\Omega) < P_K(K)$ as soon as $|\Omega\Delta K| \neq 0$. Appealing to the Wulff inequality (7.3.1). Therefore $|\Omega\Delta K| = 0$, that is Ω coincides with K also in this case. □

4. Stability issues

The results of previous section nay be enforced under the additional assumptions that V is a C^2 function and

$$v'(t) > 0, \qquad \text{for all } t > 0, \tag{7.4.1}$$

where $v(t)$ is as in (7.3.2). If that is the case, then the weighted Wulff estimate holds in the following stronger version.

Theorem 7.4.1 (Quantitative weighted Wulff inequality)**.** *The minimizer K of $P_{V,K}$ with volume constraint is stable. Namely, let Ω be a bounded Lipschitz open sets in $\mathbb{R}^N$, set $\omega_{K,N} = |K|$ and denote by $T_\Omega K$ the dilation of K whose volume $T_\Omega^N \omega_{K,N}$ equals $|\Omega|$. Then*

$$P_{V,K}(\Omega) \geq N\omega_{K,N}^{\frac{1}{N}}|\Omega|^{1-\frac{1}{N}}\left[V\left(\left(\frac{|\Omega|}{\omega_{K,N}}\right)\right)^{\frac{1}{N}} + C_{N,V,|\Omega|}\left(\frac{|\Omega\Delta(T_\Omega K)|}{|\Omega|}\right)^2\right].$$

Proof. The proof is essentially the same as in the isotropic case treated in [**18**]. Consider the term denoted by

$$\mathcal{I}_2(\Omega) = \int_{\Omega\setminus K}[v(\|x\|) - v(T_\Omega)]\,dx + \int_{K\setminus\Omega}[v(T_\Omega) - v(\|x\|)]\,dx$$

in the first step of the proof above. Besides being non-negative, it can be further estimated from below. The notations $\omega_{K,N} = |K|$

$$|T_\Omega \cdot K| = |\Omega|$$

and

$$T = \left(T_\Omega^N + \frac{|\Omega\setminus K|}{\omega_{K,N}}\right)^{\frac{1}{N}}$$

were introduced. The *anisotropic annulus*

$$A = \{x \in \mathbb{R}^N : T_\Omega < \|x\| < T\}$$

has the same volume as the sets $\Omega \setminus K$ and $K \setminus \Omega$, by construction. Thus,

$$\mathcal{I}_2(\Omega) \geq \int_{\Omega\setminus K}[v(\|x\|) - v(T_\Omega)]\,dx \geq \int_A [v(\|x\|) - v(T_\Omega)]\,dx,$$

since $v(t)$ is decreasing and $|A \setminus \Omega| = |\Omega \setminus (A \cup K)|$. Changing variables

$$\begin{aligned}\int_A [v(\|x\|) - v(T_\Omega)]\,dx &= \int_{\partial K} \|\nu_K(\omega)\|_* d\mathcal{H}^{N-1} \int_{T_\Omega}^{T} t^{N-1}[v(t) - v(T_\Omega)]\,dt \\ &= \int_{\partial K} \langle \nu_K, \omega\rangle d\mathcal{H}^{N-1} \int_{T_\Omega}^{T} t^{N-1}[v(t) - v(T_\Omega)]\,dt \\ &= N|K| \int_{T_\Omega}^{T} t^{N-1}[v(t) - v(T_\Omega)]\,dt\end{aligned}$$

where in the last passage the divergence theorem was used. In the last integral, the difference may be estimated from below by elementary mean value theorem. This is made possible by assumption (7.4.1). Therefore, to conclude, one can argue as done in the paper [**18**]. □

Some significant instances of functions V satisfying our hypothesis (7.3.2) are given by convex powers, i.e.

$$V(t) = t^\beta, \qquad t \geq 0,$$

for every $\beta \geq 1$. In particular, choosing as K the unit ball B_p of the ℓ^p norm centered at the origin, i.e.

$$B_p = \left\{x \in \mathbb{R}^N \,:\, \|x\|_{\ell^p} < 1\right\}$$

and using the distinguished notation

$$P_{p,\beta}(\Omega) = \int_{\partial\Omega} \|x\|_{\ell^p}^{\beta}\, \|\nu_\Omega(x)\|_{\ell^{p'}}\, d\mathcal{H}^{N-1}(x), \tag{7.4.2}$$

we have the following particular case of Theorem 7.3.4, that we enunciate as a separate result.

Corollary 7.4.2. *Let $p \geq 1$ and $\beta \geq 1$, for every $\Omega \subset \mathbb{R}^N$ open bounded Lipschitz set, we have*

$$P_{p,\beta}(\Omega) \geq N\,|B_p|^{\frac{1-\beta}{N}}\,|\Omega|^{\frac{N+\beta-1}{N}},$$

with equality if and only if Ω coincides with B_p, up to dilations.

CHAPTER 8

An eigenvalue problem with variable exponents

1. Preliminaries

An expedient feature of many eigenvalue problems is that the eigenfunctions may be multiplied by constants. That is the case for the non-linear problem in this chapter. Consider the problem of minimizing the "Rayleigh quotient"

$$\frac{\|\nabla u\|_{p(x),\Omega}}{\|u\|_{p(x),\Omega}} \tag{8.1.1}$$

among all functions belonging to the Sobolev space $W_0^{1,p(x)}(\Omega)$ with variable exponent $p(x)$. Here Ω is a bounded domain in $\mathbb{R}^N$ and the variable exponent $p(x)$ is a smooth function with

$$1 < p^- \leq p(x) \leq p^+ < \infty.$$

The norm is the so-called *Luxemburg norm.*

If $p(x) = p$, a constant in the range $1 < p < \infty$, the problem reduces to the minimization of the Rayleigh quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^p\, dx}{\displaystyle\int_\Omega |u|^p\, dx} \tag{8.1.2}$$

among all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. It is decisive that homogeneity holds: if u is a minimizer, so is cu for any non-zero constant c. At variance with that, that is not the case for the quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\, dx}{\displaystyle\int_\Omega |u|^{p(x)}\, dx} \tag{8.1.3}$$

with variable exponent, in general. Therefore its infimum over all $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$, is often zero and no minizer appears in the space $W_0^{1,p(x)}(\Omega)$. An explicit example is discussed in [**42**, pp. 444–445]. Unfortunately, even if imposing the constraint

$$\int_\Omega |u|^{p(x)}\, dx = \text{constant}$$

avoids this collapse, the minimizers obtained for different normalization constants are difficult to compare in any reasonable way, except, of course, when $p(x)$ is constant. For a suitable $p(x)$, it can even happen that any positive λ is an eigenvalue for some suitable choice of the normalizing constant. Thus (8.1.3) is not a proper generalization of (8.1.2), which has a well defined spectrum.

A way to avoid this situation is to use the Rayleigh quotient (8.1.1), where the notation

$$\|f\|_{p(x),\Omega} = \inf\left\{\gamma > 0 : \int_\Omega \left|\frac{f(x)}{\gamma}\right|^{p(x)} \frac{dx}{p(x)} \le 1\right\} \tag{8.1.4}$$

was used for the Luxemburg norm. This restores the homogeneity. In the integrand, the use of $p(x)^{-1}\,dx$, rather than dx, has no bearing, but it has the advantage of simplifying the equations a little.

Remark 8.1.1. Needless to say, many open problems remain. To mention one, for a finite variable exponent $p(x)$ it is not clear whether or not the first eigenvalue (the minimum of the Rayleigh quotient) is simple. The methods of chapter 3 do not work well, except fot the case of a constant exponent. There are also many annoying gaps in the theory available at present: due to the lack of a proper Harnack inequality, it is not possible to assure that the limit of the $jp(x)$-eigenfunctions is strictly positive. A discussion about analogous difficulties can be found in [**1**]. In the present chapter only positive eigenfunctions are considered.

Throughout the chapter, Ω denotes a given bounded domain in $\mathbb{R}^N$ and that the variable exponent $p(x)$ is in the range

$$1 < p^- \le p(x) \le p^+ < \infty, \tag{8.1.5}$$

Moreover, it is assumed that $p(x)$ belongs to $C^1(\Omega) \cap W^{1,\infty}(\Omega)$. Thus $\|\nabla p\|_{\infty,\Omega} < \infty$. Such assumptions are not sharp, but they make the exposition easier.

Definition 8.1.2. A measurable function $u\colon \Omega \to \mathbb{R}$ is said to belong to the space $L^{p(x)}(\Omega)$ if

$$\int_\Omega |u(x)|^{p(x)}\,dx < +\infty.$$

One says that $u \in W^{1,p(x)}(\Omega)$ if u and its distributional gradient ∇u are measurable functions satisfying

$$\int_\Omega |u|^{p(x)}\,dx < \infty, \qquad \int_\Omega |\nabla u|^{p(x)}\,dx < \infty.$$

The reader is referred to [**42**] and the monograph [**35**] about these spaces. The norm of the space $L^{p(x)}(\Omega)$ is defined by (8.1.4). This is a Banach space. So is $W^{1,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{p(x),\Omega} + \|\nabla u\|_{p(x),\Omega}.$$

Smooth functions are dense in $W^{1,p(x)}(\Omega)$, and so one can define the space $W_0^{1,p(x)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the above norm.

The following properties are used later.

Lemma 8.1.3 (Sobolev)**.** *The inequality*

$$\|u\|_{p(x),\Omega} \leq C\|\nabla u\|_{p(x),\Omega}$$

holds for all $u \in W_0^{1,p(x)}(\Omega)$*; the constant is independent of* u*.*

In fact, even a stronger inequality is valid.

Lemma 8.1.4 (Rellich-Kondrachev)**.** *Given a sequence* $u_j \in W_0^{1,p(x)}(\Omega)$ *such that* $\|\nabla u_j\|_{p(x),\Omega} \leq M$, $j = 1, 2, 3, \ldots$, *there exists a* $u \in W_0^{1,p(x)}(\Omega)$ *such that* $u_{j_\nu} \to u$ *strongly in* $L^{p(x)}(\Omega)$ *and* $\nabla u_{j_\nu} \rightharpoonup \nabla u$ *weakly in* $L^{p(x)}(\Omega)$ *for some subsequence.*

Eventually, from now on notation $\|\cdot\|_{p(x)}$ will be used in place of $\|\cdot\|_{p(x),\Omega}$, provided that this causes no confusion.

One has to identify the space

$$\bigcap_{j=1}^{\infty} W^{1,jp(x)}(\Omega).$$

According to the next lemma, it turns out to be independent of the variable exponent $p(x)$. Actually, this limit space is nothing else than the familiar $W^{1,\infty}(\Omega)$.

Lemma 8.1.5. *If* u *is a measurable function in* Ω*, then*

$$\lim_{j\to\infty} \|u\|_{jp(x)} = \|u\|_\infty.$$

Proof. The proof is elementary. We use the notation

$$M = \|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$
$$M_j = \|u\|_{jp(x)},$$

and we claim that

$$\lim_{j\to\infty} M_j = M.$$

If $M = 0$, then there is nothing to prove. Then, assume $M > 0$.

To show that $\limsup_{j\to\infty} M_j \leq M$, one only has to consider those indices j for which $M_j > M$. Then, since $p(x) > 1$,

$$1 = \left(\int_\Omega \left|\frac{u(x)}{M_j}\right|^{jp(x)} \frac{dx}{jp(x)}\right)^{\frac{1}{j}} \leq \frac{M}{M_j}\left(\int_\Omega \frac{dx}{jp(x)}\right)^{\frac{1}{j}},$$

and the inequality follows.

To show that $\liminf_{j\to\infty} M_j \ge M$, assume first that $M < \infty$. Given $\varepsilon > 0$, there is a set $A_\varepsilon \subset \Omega$ such that $\operatorname{meas}(A_\varepsilon) > 0$ and $|u(x)| > M - \varepsilon$ in A_ε. We claim that $\liminf_{j\to\infty} M_j \ge M - \varepsilon$. Ignoring those indices for which $M_j \ge M - \varepsilon$, we have

$$1 = \left(\int_\Omega \left| \frac{u(x)}{M_j} \right|^{jp(x)} \frac{dx}{jp(x)} \right)^{\frac{1}{j}} \ge \left(\int_{A_\varepsilon} \left| \frac{u(x)}{M_j} \right|^{jp(x)} \frac{dx}{jp(x)} \right)^{\frac{1}{j}} \ge \frac{M-\varepsilon}{M_j} \left(\int_{A_\varepsilon} \frac{dx}{jp^+} \right)^{\frac{1}{j}},$$

and the claim follows. Since ε was arbitrary the Lemma follows. The case $M = \infty$ requires a minor modification in the proof. □

2. The Euler Lagrange equation

Define

$$\Lambda_1 = \inf_v \frac{\|\nabla v\|_{p(x)}}{\|v\|_{p(x)}}, \tag{8.2.1}$$

where the infimum is taken over all $v \in W_0^{1,p(x)}(\Omega)$, $v \not\equiv 0$. One gets the same infimum by requiring that $v \in C_0^\infty(\Omega)$. The Sobolev inequality (Lemma 8.1.3)

$$\|v\|_{p(x)} \le C \|\nabla v\|_{p(x)},$$

where C is independent of v, shows that $\Lambda_1 > 0$.

To establish the existence of a non-trivial minimizer, we select a minimizing sequence of admissible functions v_j normalized so that $\|v_j\|_{p(x)} = 1$. Then

$$\Lambda_1 = \lim_{j\to\infty} \|\nabla v_j\|_{p(x)}.$$

Recall the Rellich-Kondrachev Theorem for Sobolev spaces with variable exponents (Lemma 8.1.4). Hence, we can extract a subsequence v_{j_ν} and find a function $u \in W_0^{1,p(x)}(\Omega)$ such that $v_{j_\nu} \to u$ strongly in $L^{p(x)}(\Omega)$ and $\nabla v_{j_\nu} \rightharpoonup \nabla u$ weakly in $L^{p(x)}(\Omega)$. The norm is weakly sequentially lower semicontinuous. Thus,

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \le \lim_{\nu\to\infty} \frac{\|\nabla v_{j_\nu}\|_{p(x)}}{\|v_{j_\nu}\|_{p(x)}} = \Lambda_1.$$

This shows that u is a minimizer. Notice that if u is a minimizer, so is $|u|$. We have proved the following proposition.

Proposition 8.2.1. *There exists a non-negative minimizer $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, of the Rayleigh quotient* (8.1.1).

In order to derive the Euler-Lagrange equation for the minimizer(s), we fix an arbitrary test function $\eta \in C_0^\infty(\Omega)$ and consider the competing function

$$v(x) = u(x) + \varepsilon\eta(x),$$

and write

$$k = k(\varepsilon) = \|v\|_{p(x)}, \quad K = K(\varepsilon) = \|\nabla v\|_{p(x)}.$$

A necessary condition for the inequality

$$\Lambda_1 = \frac{K(0)}{k(0)} \leq \frac{K(\varepsilon)}{k(\varepsilon)}$$

is that

$$\frac{d}{d\varepsilon}\left(\frac{K(\varepsilon)}{k(\varepsilon)}\right) = \frac{K'(\varepsilon)k(\varepsilon) - K(\varepsilon)k'(\varepsilon)}{k(\varepsilon)^2} = 0, \quad \text{for } \varepsilon = 0,$$

provided that the derivative does exist. Thus the necessary condition of minimality reads

$$\frac{K'(0)}{K(0)} = \frac{k'(0)}{k(0)}. \tag{8.2.2}$$

The existence of the derivatives here is understood. The proof of this fact is postponed in Lemma 8.2.5. We claim that

$$\frac{K'(0)}{K(0)} = \frac{\displaystyle\int_\Omega K^{-p(x)}|\nabla u|^{p(x)-2}\langle \nabla u, \nabla\eta\rangle\, dx}{\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)}\, dx}. \tag{8.2.3}$$

To see that formally one differentiates the identity

$$\int_\Omega \left|\frac{\nabla u(x) + \varepsilon\nabla\eta(x)}{K(\varepsilon)}\right|^{p(x)} \frac{dx}{p(x)} = 1$$

with respect to ε. Differentiation under the integral sign is justifiable. Therefore

$$\int_\Omega \frac{|\nabla u + \varepsilon\nabla\eta|^{p(x)-2}\langle \nabla u + \varepsilon\nabla\eta, \nabla\eta\rangle}{K(\varepsilon)^{p(x)}}\, dx = \int_\Omega \frac{|\nabla u + \varepsilon\nabla\eta|^{p(x)}}{K(\varepsilon)^{p(x)+1}} K'(\varepsilon)\, dx$$

and the conclusion follows by taking $\varepsilon = 0$. A similar calculation yields

$$\frac{k'(0)}{k(0)} = \frac{\displaystyle\int_\Omega k^{-p(x)}|u|^{p(x)-2}u\,\eta\, dx}{\int_\Omega \left|\frac{u}{k}\right|^{p(x)}\, dx}. \tag{8.2.4}$$

For a rigorous proof of (8.2.3) and (8.2.4) the reader is referred to Lemma 8.2.5. Inserting the results into (8.2.2), one arrives at equation

$$\operatorname{div}\left(\left|\frac{\nabla u}{K}\right|^{p(x)-2}\frac{\nabla u}{K}\right) + \frac{K}{k} S \left|\frac{u}{k}\right|^{p-2}\frac{u}{k} = 0,$$

in weak form, viz.

$$\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)-2}\left\langle \frac{\nabla u}{K}, \nabla\eta\right\rangle dx = \Lambda_1 S \int_\Omega \left|\frac{u}{k}\right|^{p(x)-2}\frac{u}{k}\,\eta\, dx, \qquad \eta \in C_0^\infty(\Omega),$$

where $K = \|\nabla u\|_{p(x)}$, $k = \|u\|_{p(x)}$ and

$$S = \frac{\displaystyle\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)} dx}{\displaystyle\int_\Omega \left|\frac{u}{k}\right|^{p(x)} dx}. \tag{8.2.5}$$

Here $\Lambda_1 = K/k$.

The weak solutions with zero boundary values are called *eigenfunctions*, except $u \equiv 0$. The reader is referred to [**7, 3, 40, 41, 53**] about regularity theory.

Definition 8.2.2. A function $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, is an *eigenfunction* if the equation

$$\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla\eta \right\rangle dx = \Lambda S \int_\Omega \left|\frac{u}{k}\right|^{p(x)-2} \frac{u}{k}\,\eta\, dx \tag{8.2.6}$$

holds whenever $\eta \in C_0^\infty(\Omega)$. Here $K = K_u$, $k = k_u$ and $S = S_u$. The corresponding Λ is the *eigenvalue.*

Remark 8.2.3. According to [**3, 41, 40**], the weak solutions of equations like (8.2.6) are continuous if the variable exponent $p(x)$ is Hölder continuous. Thus the eigenfunctions are continuous.

If Λ_1 is the minimum of the Rayleigh quotient in (8.2.1), we must have

$$\Lambda \geq \Lambda_1,$$

in (8.2.6), thus Λ_1 is called the *first eigenvalue* and the corresponding eigenfunctions are said to be *first eigenfunctions.* To see this, take $\eta = u$ in the equation, which is possible by approximation. Then we obtain, upon cancellations, that

$$\Lambda = \frac{K}{k} = \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \geq \Lambda_1.$$

We shall restrict ourselves to positive eigenfunctions.

Theorem 8.2.4. *There exists a continuous strictly positive first eigenfunction. Moreover, any non-negative eigenfunction is strictly positive.*

Proof. The existence of a first eigenfunction was clear, since minimizers of (8.2.1) are solutions of (8.2.6). But if u is a minimizer, so is $|u|$, and $|u| \geq 0$. Thus we have a non-negative one. By Remark 8.2.3 the eigenfunctions are continuous. The strict positivity then follows by the strong minimum principle for weak supersolutions in [**53**]. $\square$

2.1. Regularity of the $p(x)-$norm. This subsection is devoted to prove that the quantities computed above are meaningful. Note that formula

$$\frac{d}{d\varepsilon}\frac{K(\varepsilon)}{k(\varepsilon)}\bigg|_{\varepsilon=0}=0 \tag{8.2.7}$$

takes sense if and only if both the functions $K(\varepsilon), k(\varepsilon)$ are differentiable at $\varepsilon=0$, where

$$K := K(0) > 0, \qquad k := k(0) > 0.$$

Conversely, if this is the case then there exists the partial derivative at u of the $p(x)-$norm along the direction η. Recall that

$$K(\varepsilon)=\|\nabla u+\varepsilon\nabla\eta\|_{p(x)}, \qquad k(\varepsilon)=\|u+\varepsilon\eta\|_{p(x)}$$

for a given $\eta\in C_0^\infty(\Omega)$. Then (8.2.7) follows as a necessary condition for the minimality of u along the line passing through u with the same direction as η.

Lemma 8.2.5. *Let $p(x)>1$. Then K and k are differentiable at $\varepsilon=0$, and their derivatives are respectively given by*

$$K'(0)=K(0)\frac{\displaystyle\int_\Omega\left|\frac{\nabla u}{K}\right|^{p(x)-2}\left\langle\frac{\nabla u}{K},\frac{\nabla\eta}{K}\right\rangle dx}{\displaystyle\int_\Omega\left|\frac{\nabla u}{K}\right|^{p(x)}dx} \tag{8.2.8}$$

and

$$k'(0)=k(0)\frac{\displaystyle\int_\Omega\left|\frac{u}{k}\right|^{p(x)-2}\frac{u}{k}\,\frac{\eta}{k}dx}{\displaystyle\int_\Omega\left|\frac{u}{k}\right|^{p(x)}dx}. \tag{8.2.9}$$

PROOF. For all $a,b\in\mathbb{R}$ and all $p>1$ one has

$$|b|^p-|a|^p=\int_0^1\frac{d}{dt}\big|a+t(b-a)\big|^p\,dt=p(b-a)\int_0^1|a+t(b-a)|^{p-2}\big[a+t(b-a)\big]\,dt.$$

By the definition of the $p(x)$-norm $k(\varepsilon)$ of $u+\varepsilon\eta$, it follows that

$$\int_\Omega\left|\frac{u+\varepsilon\eta}{k(\varepsilon)}\right|^{p(x)}\frac{dx}{p(x)}=1, \qquad \int_\Omega\left|\frac{u}{k}\right|^{p(x)}\frac{dx}{p(x)}=1.$$

Subtracting the last two identities and using $a=(u+\varepsilon\eta)/k(\varepsilon)$, $b=u/k$, $p=p(x)$ gives

$$\begin{aligned}0 &= \int_\Omega\left|\frac{u+\varepsilon\eta}{k(\varepsilon)}\right|^{p(x)}\frac{dx}{p(x)}-\int_\Omega\left|\frac{u}{k}\right|^{p(x)}\frac{dx}{p(x)}\\ &= \int_\Omega\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\int_0^1\left|\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right|^{p(x)-2}\left[\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right]dtdx\end{aligned}$$

After dividing out ε and moving a term one gets

$$\frac{k(\varepsilon)-k}{\varepsilon}\,\frac{1}{k(\varepsilon)}\int_\Omega \frac{u}{k}\int_0^1 \left|\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right|^{p(x)-2}\left[\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right]dtdx$$
$$=\int_\Omega \frac{\eta}{k(\varepsilon)}\int_0^1 \left|\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right|^{p(x)-2}\left[\frac{u+\varepsilon\eta}{k(\varepsilon)}+t\left(\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}\right)\right]dtdx$$

By the (Lipschitz) continuity of the norm, the quantity $k(\varepsilon)$ depends continuously on ε. Thus the quantity

$$\frac{u+\varepsilon\eta}{k(\varepsilon)}-\frac{u}{k}$$

converges to zero pointwise a.e. in Ω, as $\varepsilon\to 0$. Hence, by dominated convergence theorem the integrals have their obvious limits as $\varepsilon\to 0$, and the derivative of k at zero exists, taking the value

$$\lim_{\varepsilon\to 0}\frac{k(\varepsilon)-k}{\varepsilon}=\frac{\displaystyle\int_\Omega \left|\frac{u}{k}\right|^{p(x)-2}\frac{u}{k}\,\eta\,dx}{\displaystyle\int_\Omega \left|\frac{u}{k}\right|^{p(x)}dx}.$$

This proves that the formula for $k'(0)$ is valid. The proof for $K'(0)$ runs at the same way. □

2.2. The equation in non-divergence form. The procedure to consider the asymptotic case when the variable exponent approaches ∞ via the sequence $p(x)$, $2p(x)$, $3p(x)\dots$ will require viscosity solutions. Thus we first verify that the weak solutions of the equation (8.2.6), formally written as

$$\operatorname{div}\left(\left|\frac{\nabla u}{K}\right|^{p(x)-2}\frac{\nabla u}{K}\right)+\Lambda\, S\left|\frac{u}{k}\right|^{p-2}\frac{u}{k}=0,$$

are viscosity solutions. Given $u\in C(\Omega)\cap W_0^{1,p(x)}(\Omega)$, *we fix the parameters* $k=\|u\|_{p(x)}$, $K=\|\nabla u\|_{p(x)}$ *and* S. Replacing u by a function $\phi\in C^2(\Omega)$, but keeping k,K,S unchanged, we formally get

$$\Delta_{p(x)}\phi-|\nabla\phi|^2\log(K)\big\langle\nabla\phi,\nabla p(x)\big\rangle+\Lambda^{p(x)}\,S|\phi|^{p(x)-2}\phi=0,$$

where

$$\Delta_{p(x)}\phi=\operatorname{div}\big(|\nabla\phi|^{p(x)-2}\nabla\phi\big)=|\nabla\phi|^{p(x)-4}\Big\{|\nabla\phi|^2\Delta\phi+\big(p(x)-2\big)\Delta_\infty\phi$$
$$+|\nabla\phi|^2\ln\big(|\nabla\phi|\big)\big\langle\nabla\phi,\nabla p(x)\big\rangle\Big\},$$

and

$$\Delta_\infty\phi=\sum_{i,j=1}^n\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\frac{\partial^2\phi}{\partial x_i\partial x_j}$$

is the ∞-Laplacian. The relation $\Lambda=K/k$ was used in the simplifications.

Let us abbreviate the expression as

$$
\begin{aligned}
F(x,\phi,\nabla\phi,&D^2\phi) = \\
&|\nabla\phi|^{p(x)-4}\Big\{|\nabla\phi|^2\Delta\phi + \big(p(x)-2\big)\Delta_\infty\phi + |\nabla\phi|^2\ln\big(|\nabla\phi|\big)\big\langle\nabla\phi,\nabla p(x)\big\rangle \\
&\qquad - |\nabla\phi|^2\log(K)\big\langle\nabla\phi,\nabla p(x)\big\rangle\Big\} + \Lambda^{p(x)}\, S|\phi|^{p(x)-2}\phi = 0. \qquad (8.2.10)
\end{aligned}
$$

where we deliberately take $p(x) \geq 2$. Notice that

$$F(x,\phi,\nabla\phi,D^2\phi) < 0$$

exactly when

$$\Delta_{p(x)}\left(\frac{\phi}{K}\right) + \Lambda\, S\left|\frac{\phi}{k}\right|^{p(x)-2}\frac{\phi}{k} < 0.$$

Recall that k, K, S where dictated by u.

Let $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$. We say that $\phi \in C^2(\Omega)$ touches u from below at the point x_0, if $\phi(x_0) = u(x_0)$ and $\phi(x) < u(x)$ when $x \neq x_0$.

Definition 8.2.6. Suppose that $u \in C(\Omega)$. We say that u is a *viscosity supersolution* of the equation

$$F(x,u,\nabla u,D^2u) = 0$$

if, whenever ϕ touches u from below at a point $x_0 \in \Omega$, we have

$$F(x_0,\phi(x_0),\nabla\phi(x_0),D^2\phi(x_0)) \leq 0.$$

We say that u is a *viscosity subsolution* if, whenever $\psi \in C^2(\Omega)$ touches u from above at a point $x_0 \in \Omega$, we have

$$F(x_0,\psi(x_0),\nabla\psi(x_0),D^2\psi(x_0)) \geq 0.$$

Finally, we say that u is a *viscosity solution* if it is both a viscosity super- and subsolution.

Several remarks are appropriate. Notice that the operator F is evaluated for the test function and only at the touching point. If the family of test functions is empty at some point, then there is no requirement on F at that point. The definition makes sense for a merely continuous function u, provided that the parameters k, K, S, Λ have been assigned values. We always have ∇u available for this in our problem.

Theorem 8.2.7. *The eigenfunctions u are viscosity solutions of the equation*

$$F(x,u,\nabla u,D^2u) = 0.$$

PROOF. This is a standard proof. The equation

$$\int_\Omega\left|\frac{\nabla u}{K}\right|^{p(x)-2}\left\langle\frac{\nabla u}{K},\nabla\eta\right\rangle dx = \Lambda S\int_\Omega\left|\frac{u}{k}\right|^{p(x)-2}\frac{u}{k}\,\eta\,dx \qquad (8.2.11)$$

holds for all $\eta \in W_0^{1,p(x)}(\Omega)$. We first claim that u is a viscosity supersolution. Our proof is indirect. The *antithesis* is that there exist a point $x_0 \in \Omega$ and a test function $\phi \in C^2(\Omega)$, touching u from below at x_0, such that $F(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) > 0$. By continuity,

$$F(x, \phi(x), \nabla\phi(x), D^2\phi(x)) > 0$$

holds when $x \in B(x_0, r)$ for some radius r small enough. Then also

$$\Delta_{p(x)}\left(\frac{\phi(x)}{K}\right) + \Lambda\, S \left|\frac{\phi(x)}{k}\right|^{p-2} \frac{\phi(x)}{k} > 0, \tag{8.2.12}$$

in $B(x_0, r)$. Denote

$$\varphi = \phi + \frac{m}{2}, \qquad m = \min_{\partial B(x_0,r)} (u - \phi).$$

Then $\varphi < u$ on $\partial B(x_0, r)$ but $\varphi(x_0) > u(x_0)$, since $m > 0$. Define

$$\eta = \big[\varphi - u\big]_+ \chi_{B(x_0,r)}.$$

Now $\eta \geq 0$. If $\eta \not\equiv 0$, we multiply (8.2.12) by η and we integrate by parts to obtain the inequality

$$\int_\Omega \left|\frac{\nabla\phi}{K}\right|^{p(x)-2} \left\langle \frac{\nabla\phi}{K}, \nabla\eta \right\rangle dx < \Lambda S \int_\Omega \left|\frac{\phi}{k}\right|^{p(x)-2} \frac{\phi}{k}\, \eta\, dx$$

We have $\nabla\eta = \nabla\phi - \nabla u$ in the subset where $\varphi \geq u$. Subtracting equation (8.2.11) by the above inequality, we arrive at

$$\begin{aligned}\int_{\{\varphi>u\}} \left\langle \left|\frac{\nabla\phi}{K}\right|^{p(x)-2} \frac{\nabla\phi}{K} - \left|\frac{\nabla u}{K}\right|^{p(x)-2} \frac{\nabla u}{K}, \frac{\nabla\phi}{K} - \frac{\nabla u}{K} \right\rangle dx \\ < S \int_{\{\varphi>u\}} \left(\left|\frac{\phi}{k}\right|^{p(x)-2} \frac{\phi}{k} - \left|\frac{u}{k}\right|^{p(x)-2} \frac{u}{k}\right)\left(\frac{\varphi - u}{k}\right) dx,\end{aligned}$$

where the domain of integration is comprised in $B(x_0, r)$. The last integral is negative since $\phi < u$. The first one is non-negative due to the elementary inequality

$$\left\langle |b|^{p-2}b - |a|^{p-2}a, b - a \right\rangle \geq 0,$$

which holds for all $p > 1$ because of the convexity of the p-th power. We can take $p = p(x)$. It follows that $\varphi \leq u$ in $B(x_0, r)$. This contradicts $\varphi(x_0) > u(x_0)$. Thus the antithesis was false and u is a viscosity supersolution.

In a similar way we can prove that u is also a viscosity subsolution. $\square$

3. Passage to infinity

In the limit procedure as $jp(x) \to \infty$, the distance function

$$\delta(x) = \mathrm{dist}(x, \partial\Omega)$$

plays a crucial role. Write

$$\Lambda_\infty = \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty} = \frac{1}{R} \tag{8.3.1}$$

where R is the radius of the largest ball inscribed in Ω, the so-called *inradius*. Recall that δ is Lipschitz continuous and $|\nabla\delta| = 1$ a.e. in Ω.

In fact, Λ_∞ is the minimum of the Rayleigh quotient in the ∞-norm:

$$\Lambda_\infty = \min_u \frac{\|\nabla u\|_\infty}{\|u\|_\infty}, \tag{8.3.2}$$

where the minimum is taken among all $u \in W_0^{1,\infty}(\Omega)$. To see this, let $\xi \in \partial\Omega$ be the closest boundary point to $x \in \Omega$. By the mean value theorem

$$|u(x)| = |u(x) - u(\xi)| \le \|\nabla u\|_\infty |x - \xi| = \|\nabla u\|_\infty \delta(x).$$

It follows that

$$\Lambda_\infty = \frac{1}{\|\delta\|_\infty} \le \frac{\|\nabla u\|_\infty}{\|u\|_\infty}.$$

Consider

$$\Lambda_{jp(x)} = \min_v \frac{\|\nabla v\|_{jp(x)}}{\|v\|_{jp(x)}}, \qquad (j = 1, 2, 3 \ldots) \tag{8.3.3}$$

where the minimum is taken over all v in $C(\overline{\Omega}) \cap W_0^{1,jp(x)}(\Omega)$. When j is large, the minimizer u_j (we do mean $u_{jp(x)}$) is continuous up to the boundary and $u_{j|\partial\Omega} = 0$. This is a property of the Sobolev space.

Proposition 8.3.1.

$$\lim_{j\to\infty} \Lambda_{jp(x)} = \Lambda_\infty. \tag{8.3.4}$$

Proof. Assume for simplicity that

$$\int_\Omega \frac{dx}{p(x)} = 1.$$

The Hölder inequality implies that

$$\|f\|_{jp(x)} \le \|f\|_{lp(x)}, \qquad l \ge j.$$

Let u_j be the minimizer of the Rayleigh quotient with the $jp(x)$-norm normalized so that $\|u_j\|_{jp(x)} = 1$. Thus,

$$\Lambda_{jp(x)} = \|\nabla u_j\|_{jp(x)}.$$

Since $\Lambda_{jp(x)}$ is the minimum, we have

$$\Lambda_{jp(x)} \leq \frac{\|\nabla\delta\|_{jp(x)}}{\|\delta\|_{jp(x)}},$$

for all $j = 1, 2, 3 \ldots$ Then, by Lemma 8.1.5,

$$\limsup_{j\to\infty} \Lambda_{jp(x)} \leq \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty} = \Lambda_\infty.$$

It remains to prove that

$$\liminf_{j\to\infty} \Lambda_{jp(x)} \geq \Lambda_\infty.$$

To this end, observe that the sequence $\|\nabla u_j\|_{jp(x)}$ is bounded. Using a diagonalization procedure one can extract a subsequence u_{j_ν} such that u_{j_ν} converges strongly in each fixed $L^q(\Omega)$ and ∇u_{j_ν} converges weakly in each fixed $L^q(\Omega)$. In other words,

$$u_{j_\nu} \to u_\infty, \qquad \nabla u_{j_\nu} \rightharpoonup \nabla u_\infty, \qquad\qquad \text{as } \nu \to \infty,$$

for some $u_\infty \in W_0^{1,\infty}(\Omega)$. By the lower semicontinuity of the norm under weak convergence

$$\|\nabla u_\infty\|_q \leq \liminf_{\nu\to\infty} \|\nabla u_{j_\nu}\|_q$$

For large indices ν, we have

$$\|\nabla u_{j_\nu}\|_q \leq \|\nabla u_{j_\nu}\|_{j_\nu p(x)} = \Lambda_{j_\nu p(x)}.$$

Therefore,

$$\|\nabla u_\infty\|_q \leq \liminf_{\nu\to\infty} \Lambda_{j_\nu p(x)}$$

Finally, letting $q \to \infty$ and taking the normalization into account (by Ascoli's Theorem, $\|u_\infty\|_\infty = 1$) we obtain

$$\frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty} \leq \liminf_{\nu\to\infty} \Lambda_{j_\nu p(x)},$$

but, since u_∞ is admissible, Λ_∞ is less than or equal to the above ratio. This implies that

$$\lim_{\nu\to\infty} \Lambda_{j_\nu p(x)} = \Lambda_\infty.$$

By possibly repeating the above, starting with an arbitrary subsequence of variable exponents, it follows that the limit (8.3.4) holds for the full sequence. This concludes the proof. □

Using Ascoli's theorem it is assured that the convergence $u_{j_\nu} \to u_\infty$ is uniform in Ω. Thus the limit of the normalized first eigenfunctions is continuous and

$$u_\infty \in C(\overline{\Omega}) \cap W_0^{1,\infty}(\Omega),$$

with $u_{\infty|\partial\Omega} = 0$, $u_\infty \geq 0$, $u_\infty \not\equiv 0$. However, *the function u_∞ might depend on the particular sequence extracted.*

Theorem 8.3.2. *The limit of the normalized first eigenfunctions is a viscosity solution of the equation*

$$\max\left\{\Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)}\left(\frac{u}{K}\right)\right\} = 0, \tag{8.3.5}$$

where $K = \|\nabla u\|_\infty$.

Remark 8.3.3. The limit u of the normalized first eigenfunctions is a non-negative function. At the points where $u > 0$, the equation above means that the largest of the two quantities is zero. At the points where $u = 0$, we agree that there is no requirement[1].

Proof of Theorem 8.3.2. One begins with the case of viscosity supersolutions. If $\phi \in C^2(\Omega)$ touches u_∞ from below at $x_0 \in \Omega$, we claim that

$$\Lambda_\infty \leq \frac{|\nabla\phi(x_0)|}{\phi(x_0)}, \quad \text{and} \quad \Delta_{\infty(x_0)}\left(\frac{\phi(x_0)}{K}\right) \leq 0,$$

where $K = K_{u_\infty}$. We know that u_j is a viscosity (super)solution of the equation

$$\Delta_{jp(x)}u - |\nabla u|^{jp(x)-2}\ln K_j\big\langle\nabla u, j\nabla p(x)\big\rangle + \Lambda_{jp(x)}^{jp(x)} S_{jp(x)}|u|^{jp(x)-2}u = 0$$

where $K_j = \|\nabla u_j\|_{jp(x)}$ and

$$S_{jp(x)} = \frac{\displaystyle\int_\Omega \left|\frac{\nabla u_j}{K_j}\right|^{jp(x)} dx}{\displaystyle\int_\Omega \left|\frac{u_j}{k_j}\right|^{jp(x)} dx}.$$

We have the trivial estimate

$$\frac{p^-}{p^+} \leq S_{jp(x)} \leq \frac{p^+}{p^-}.$$

We need a test function ψ_j touching u_j from below at a point x_j very near x_0. To construct it, let $B(x_0, 2R) \subset \Omega$. Obviously,

$$\inf_{B_R\setminus B_r}\big\{u_\infty - \phi\big\} > 0,$$

when $0 < r < R$. By the uniform convergence,

$$\inf_{B_R\setminus B_r}\big\{u_\infty - \phi\big\} > u_j(x_0) - u_\infty(x_0) = u_j(x_0) - \phi(x_0),$$

provided j is larger than an index large enough, depending on r. For such large indices, $u_j - \phi$ attains its minimum in $B(x_0, R)$ at a point $x_j \in B(x_0, r)$, and letting $j \to \infty$, we see that $x_j \to x_0$, as $j \to \infty$. Actually, $j \to \infty$ via the subsequence j_ν extracted, but we drop this notation. Define

$$\psi_j = \phi + \big(u_j(x_j) - \phi(x_j)\big).$$

[1] When $u < 0$ this is not the right equation, but we keep $u \geq 0$.

This function touches u_j from below at the point x_j. Therefore ψ_j will do as a test function for u_j. We arrive at

$$
\begin{aligned}
&|\nabla\phi(x_j)|^{jp(x_j)-4}\Big\{|\nabla\phi(x_j)|^2\Delta\phi(x_j)\\
&\qquad + (jp(x_j)-2)\,\Delta_\infty\phi(x_j) + |\nabla\phi(x_j)|^2\ln\big(|\nabla\phi(x_j)|\big)\Big\langle\nabla\phi(x_j), j\nabla p(x_j)\Big\rangle\Big\}\\
&\qquad \le -\Lambda_{jp(x_j)}^{jp(x_j)} S_{jp(x_j)}|\phi(x_j)|^{jp(x_j)-2}\phi(x_j)\\
&\qquad\qquad + |\nabla\phi(x_j)|^{jp(x_j)-2}\ln K_j\Big\langle\nabla\phi(x_j), j\nabla p(x_j)\Big\rangle.
\end{aligned}
\tag{8.3.6}
$$

First, we consider the case $\nabla\phi(x_0)\neq 0$. Then $\nabla\phi(x_j)\neq 0$ for large indices. Dividing by

$$(jp(x_j)-2)|\nabla\phi(x_j)|^{jp(x_j)-2}$$

one obtains

$$
\begin{aligned}
&\frac{|\nabla\phi(x_j)|^2\Delta\phi(x_j)}{jp(x_j)-2} + \Delta_\infty\phi(x_j) + |\nabla\phi(x_j)|^2\,\ln|\nabla\phi(x_j)|\,\left\langle\nabla\phi(x_j), \frac{\nabla p(x_j)}{p(x_j)-2/j}\right\rangle\\
&\qquad \le \ln K_j\left\langle\nabla\phi(x_j), \frac{\nabla p(x_j)}{p(x_j)-2/j}\right\rangle - \left(\frac{\Lambda_{jp(x_j)}\phi(x_j)}{|\nabla\phi(x_j)|}\right)^{jp(x_j)-4}\Lambda_{jp(x)}^4 S_{jp(x_j)}\phi(x_j)^3.
\end{aligned}
$$

In this inequality, all terms have a limit except possibly the last one. Thus, in order to avoid a contradiction

$$\limsup_{j\to\infty}\frac{\Lambda_{jp(x_j)}\phi(x_j)}{|\nabla\phi(x_j)|}\le 1. \tag{8.3.7}$$

Therefore

$$\Lambda_\infty\phi(x_0) - |\nabla\phi(x_0)|\le 0, \tag{8.3.8}$$

as desired. Taking the limit it follows that

$$\Delta_\infty\phi(x_0) + |\nabla\phi(x_0)|^2\ln\left|\frac{\nabla\phi(x_0)}{K_\infty}\right|\Big\langle\nabla\phi(x_0), \nabla\ln p(x_j)\Big\rangle\le 0.$$

Second, consider the case $\nabla\phi(x_0)=0$. Then the last inequality above is evident. Now the inequality

$$\Lambda_\infty\phi(x_0) - |\nabla\phi(x_0)|\le 0$$

reduces to $\phi(x_0)\le 0$. But, if $\phi(x_0)>0$, then $\phi(x_j)\neq 0$ for large indices. According to inequality (8.3.6) we must have $|\nabla\phi(x_j)|\neq 0$ and so we can divide by $(jp(x_j)-2)|\nabla\phi(x_j)|^{jp(x_j)-2}$ and conclude from (8.3.7) that $\phi(x_0)=0$, in fact. This shows that u_∞ is a viscosity supersolution.

In the case of a subsolution one has to show that for a test function ψ touching u_∞ from above at x_0 at least one of the inequalities

$$\Lambda_\infty \psi_\infty(x_0) - |\nabla\psi(x_0)| \geq 0$$

or

$$\Delta_\infty \psi(x_0) + |\nabla\psi(x_0)|^2 \ln\left|\frac{\nabla\psi(x_0)}{K_\infty}\right| \Big\langle \nabla\psi(x_0), \nabla \ln p(x_0)\Big\rangle \geq 0$$

is valid. We omit this case, since the proof is pretty similar to the one for supersolutions. $\square$

4. Local uniqueness

The *existence* of a viscosity solution to the equation

$$\max\left\{\Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)}\left(\frac{u}{\|\nabla u\|_\infty}\right)\right\} = 0$$

was established in section 3. The question of *uniqueness* is a more delicate one.

In the special case of a constant exponent, say $p(x) = p$, there is a recent counterexample in [**57**] of a domain (a dumb-bell shaped one) in which there are several linearly independent solutions in $C(\overline{\Omega}) \cap W_0^{1,\infty}(\Omega)$ of the equation

$$\max\left\{\Lambda - \frac{|\nabla u|}{u}, \Delta_\infty u\right\} = 0, \qquad \Lambda = \Lambda_\infty.$$

It is decisive that they have boundary values zero. According to [**59**, Theorem 2.3], this cannot happen for strictly positive boundary values, which excludes eigenfunctions. This partial uniqueness result implied that there are no positive eigenfunctions for $\Lambda \neq \Lambda_\infty$, cf. [**59**, Theorem 3.1].

Let us return to the variable exponents. Needless to say, one cannot hope for more than in the case of a constant exponent. Actually, a condition involving the quantities $\min u$, $\max u$, $\max|\nabla \ln p|$ taken over subdomains enters. This complicates the matter and restricts the result.

Consider first a normalized strictly positive viscosity solution u of the equation

$$\max\left\{\Lambda_\infty - \frac{|\nabla u|}{u},\ \Delta_{\infty(x)} u\right\} = 0. \tag{8.4.1}$$

Now $K = \|\nabla u\|_\infty = 1$. The normalization is used in no other way than that the constant K is erased. This equation is not a "proper" one[2] and the first task is to find the equation for $v = \ln(u)$.

Lemma 8.4.1. *Let $C > 0$. The function*

$$v = \ln(Cu)$$

[2]A term used in the viscosity theory for second order equations

is a viscosity solution of the equation

$$\max\left\{\Lambda - |\nabla v|, \Delta_\infty v + |\nabla v|^2 \ln\left(\frac{|\nabla v|}{C}\right)\langle \nabla v, \nabla \ln p\rangle + v|\nabla v|^2\langle \nabla v, \nabla \ln p\rangle\right\} = 0. \tag{8.4.2}$$

We need a *strict* supersolution (this means that the 0 in the right hand side has to be replaced by a negative quantity) which approximates v uniformly. To this end we use the *approximation of unity* introduced in [**59**]. Let

$$g(t) = \frac{1}{\alpha}\ln\bigl(1 + A(e^{\alpha t} - 1)\bigr), \qquad A > 1,\ \alpha > 0,$$

and keep $t > 0$. The function

$$w = g(v)$$

will have the desired properties, provided that $v \geq 0$. This requires that

$$Cu(x) \geq 1,$$

which cannot hold globally for an eigenfunction, because $u = 0$ on the boundary. This obstacle restricts the method to local considerations and forces to limit the following constructions to subdomains.

We use a few elementary results:

$$\begin{aligned}
&0 < g(t) - t < \frac{A-1}{\alpha},\\
&A^{-1}(A-1)e^{-\alpha t} < g'(t) - 1 < (A-1)e^{-\alpha t},\\
&g(t) - t < \frac{A}{\alpha}(e^{\alpha t} - 1)(g'(t) - 1),\\
&g''(t) = -\alpha(g'(t) - 1)g'(t),\\
&0 < \ln g'(t) < g'(t) - 1.
\end{aligned} \tag{8.4.3}$$

In particular, $g'(t) - 1$ will appear as a decisive factor in the calculations. The formula

$$\ln g'(t) = \ln A - \alpha(g(t) - t) \tag{8.4.4}$$

is helpful.

In the next lemma our choice of the parameter α is not optimal, but it is necessary to take $\alpha > 1$, at least. For convenience, one sets $\alpha = 2$.

Lemma 8.4.2. *Take $\alpha = 2$ and assume that $1 < A < 2$. If $v > 0$ is a viscosity supersolution of equation* (8.4.2)*, then $w = g(v)$ is a viscosity supersolution of the equations*

$$\Lambda - \frac{|\nabla w|}{g'(v)} = 0,$$

and

$$\Delta_\infty w + |\nabla w|^2 \ln\left(\frac{\nabla w}{C}\right)\langle \nabla w, \nabla \ln p\rangle + w|\nabla w|^2\langle \nabla w, \nabla \ln p\rangle + |\nabla w|^4 = -\mu,$$

where

$$\mu = A^{-1}(A-1)|\nabla w|^3 e^{-2v}\Big\{\Lambda - \|e^{2v}\nabla \ln p\|_\infty\Big\},$$

provided that

$$\|e^{2v}\nabla \ln p\|_\infty < \Lambda.$$

Remark 8.4.3. One can further estimate μ and replace it by a constant, viz.

$$A^{-1}\Lambda^3(A-1)e^{-2\|v\|_\infty}\{\Lambda - \|e^{2v}\nabla \ln p\|_\infty\},$$

but the pointwise estimate is favourable.

Proof. The proof below is only formal and should be rewritten in terms of test functions. One only has to observe that an arbitrary test function φ touching w from below can be represented as $\varphi = g(\phi)$ where ϕ touches v from below.

First one computes the expressions

$$\begin{aligned}
&\nabla w = g'(v)\nabla v,\\
&\Delta_\infty w = g'(v)^2 g''(v)|\nabla v|^4 + g'(v)^3\Delta_\infty v,\\
&|\nabla w|^2 \ln\left(\frac{|\nabla w|}{C}\right)\langle \nabla w, \nabla \ln p\rangle\\
&\quad = g'(v)^3\Big\{|\nabla v|^2\ln\left(\frac{|\nabla v|}{C}\right)\langle \nabla v, \nabla \ln p\rangle + |\nabla v|^2\ln(g'(v))\langle\nabla v, \nabla \ln p\rangle.
\end{aligned}$$

Then, using that v is a supersolution, it follows that

$$\begin{aligned}
&\Delta_\infty w + |\nabla w|^2\ln\left(\frac{|\nabla w|}{C}\right)\big\langle \nabla w, \nabla \ln p\,\big\rangle\\
&\quad = g'(v)^2 g''(v)|\nabla v|^4 + g'(v)^3\Big\{\Delta_\infty v + |\nabla v|^2\ln\left(\frac{|\nabla v|}{C}\right)\big\langle\nabla v, \nabla \ln p\,\big\rangle\Big\}\\
&\qquad\quad + g'(v)^3|\nabla v|^2\ln(g'(v))\Big\langle \nabla v, \nabla \ln p\,\Big\rangle\\
&\quad \le g'(v)^2 g''(v)|\nabla v|^4 + g'(v)^3\Big\{-v|\nabla v|^2\big\langle\nabla v, \nabla \ln p\,\big\rangle - |\nabla v|^4\Big\}\\
&\qquad\quad + g'(v)^3|\nabla v|^2\ln(g'(v))\Big\langle \nabla v, \nabla \ln p\,\Big\rangle.
\end{aligned}$$

Let us collect the terms appearing on the left-hand side of the equation for w. Using the formulas (8.4.3) for $g''(v)$ and $\ln\big(g'(v)\big)$ one arrives at

$$\begin{aligned}
&\Delta_\infty w + |\nabla w|^2\ln\left(\frac{|\nabla w|}{C}\right)\big\langle\nabla w, \nabla \ln p\,\big\rangle + |\nabla w|^4 + w|\nabla w|^2\Big\langle \nabla w, \nabla \ln p\Big\rangle,\\
&\quad \le g'(v)^3|\nabla v|^3\big(g'(v)-1\big)\Big\{-|\nabla v| + |\nabla \ln p|\Big\} + g'(v)^3|\nabla v|^3\big(g(v)-v\big)|\nabla \ln p|,
\end{aligned}$$

after some arrangements. Since

$$g(t)-t<\frac{A}{2}(e^{2t}-1)(g'(t)-1)\leq(e^{2t}-1)(g'(t)-1),$$

collecting all the terms with the factor $|\nabla\ln p|$ separately and observing that $1+(e^{2t}-1)=e^{2t}$, one sees that the right-hand side is less than

$$g'(v)^3|\nabla v|^3\big(g'(v)-1\big)\{-|\nabla v|+|e^{2v}\nabla\ln p|\}\leq|\nabla w|^3A^{-1}(A-1)e^{-2v}\{-\Lambda+|e^{2v}\nabla\ln p|\},$$

since the expression in braces is negative. $\square$

We abandon the requirement of zero boundary values. Thus Ω below can represent a proper subdomain. Eigenfunctions belong to a Sobolev space but we cannot ensure this for an arbitrary viscosity solution. This requirement is therefore included in our next theorem.

Theorem 8.4.4. *Suppose that $u_1\in C(\overline{\Omega})$ is a viscosity subsolution and that $u_2\in C(\overline{\Omega})$ is a viscosity supersolution of equation* (8.4.1)*. Assume that at least one of them belongs to $W^{1,\infty}(\Omega)$. If $u_1(x)>0$ and $u_2(x)\geq m_2>0$ in Ω, and*

$$3\left\|\left(\frac{u_2}{m_2}\right)^2\nabla\ln p\right\|_\infty\leq\Lambda, \tag{8.4.5}$$

then the following comparison principle holds:

$$u_1\leq u_2\quad\text{on }\partial\Omega\quad\Longrightarrow\quad u_1\leq u_2\quad\text{in }\Omega.$$

Proof. Define

$$v_1=\ln(Cu_1),\qquad v_2=\ln(Cu_2),$$

with $C=1/m_2$. Then $v_2>0$, but v_1 may take negative values. We define

$$w_2=g(v_2),\quad\alpha=2,\quad 1<A<2.$$

If $v_2\geq v_1$, we are done. If not, consider the open subset $\{v_2<v_1\}$ and denote

$$\sigma=\sup\{v_1-v_2\}>0.$$

Note that σ is independent of C. (The antithesis was that $\sigma>0$.) Then, taking $A=1+\sigma$,

$$v_2<w_2<v_2+\frac{A-1}{2}=v_2+\frac{\sigma}{2}.$$

Note that $v_1-w_2=v_1-v_2+v_2-w_2\geq v_1-v_2-\sigma/2$. Taking the supremum on the subdomain $\mathcal{U}=\{w_2<v_1\}$ we have

$$\sup_{\mathcal{U}}\{v_1-w_2\}\geq\frac{\sigma}{2}>0=\max_{\partial\mathcal{U}}\{v_1-w_2\}$$

and $\mathcal{U}\Subset\Omega$, i.e. $\mathcal{U}$ is strictly interior. Moreover,

$$\sup\{v_1-w_2\}\leq\frac{3\sigma}{2}. \tag{8.4.6}$$

In order to obtain a contradiction, we double the variables and write

$$\mathsf{M}_j = \max_{\overline{\mathcal{U}}\times\overline{\mathcal{U}}} \left\{ v_1(x) - w_2(y) - \frac{j}{2}|x-y|^2 \right\}.$$

If the index j is large, the maximum is attained at some interior point (x_j, y_j) in $\mathcal{U}\times\mathcal{U}$. The points converge to some interior point, say $x_j \to \hat{x}$, $y_j \to \hat{x}$, and

$$\lim_{j\to\infty} j|x_j - y_j|^2 = 0.$$

This is a standard procedure. According to the "Theorem of Sums", cf. [**29**] or [**63**], there exist symmetric $n \times n$-matrices $\mathbb{X}_j$ and $\mathbb{Y}_j$ such that

$$\begin{aligned}
&\Big(j(x_j - y_j), \mathbb{X}_j\Big) \in \overline{J_{\mathcal{U}}^{2,+}} v_1(x_j),\\
&\Big(j(x_j - y_j), \mathbb{Y}_j\Big) \in \overline{J_{\mathcal{U}}^{2,-}} w_2(y_j),\\
&\Big\langle \mathbb{X}_j\xi\,,\,\xi\Big\rangle \le \Big\langle \mathbb{Y}_j\xi\,,\,\xi\Big\rangle, \quad \text{when } \xi \in \mathbb{R}^n.
\end{aligned}$$

The definition of the semijets and their closures $\overline{J_{\mathcal{U}}^{2,+}}$, $\overline{J_{\mathcal{U}}^{2,-}}$ can be found in the above mentioned references[3]. The equations have to be written in terms of jets.

We exclude one alternative from the equations. In terms of jets

$$\Lambda - \frac{|\nabla w_2|}{g'(v_2)} \le 0 \qquad \text{reads} \qquad \Lambda - \frac{j|x_j - y_j|}{g'(v_2(y_j))} \le 0$$

and, since $v_2 > 0$, $g'(v_2(y_j)) > 1$, and so

$$\Lambda < j|x_j - y_j|.$$

This rules out the alternative $\Lambda - |\nabla v_1(x_j)| \ge 0$ in the equation for v_1, which reads $\Lambda - j|x_j - y_j| \ge 0$. Therefore we must have that $\Delta_\infty v_1 + \cdots + |\nabla v_1|^4 \ge 0$, i.e.

$$\begin{aligned}
&\Big\langle \mathbb{X}_j\, j(x_j - y_j), j(x_j - y_j)\Big\rangle + j^2|x_j - y_j|^2 \ln\left(\frac{j|x_j - y_j|}{C}\right) \Big\langle j(x_j - y_j), \nabla \ln p(x_j)\Big\rangle\\
&\qquad + v_1(x_j) j^2 |x_j - y_j|^2 \ln\left(\frac{j|x_j - y_j|}{C}\right) \Big\langle j(x_j - y_j), \nabla \ln p(x_j)\Big\rangle + j^4|x_j - y_j|^4 \ge 0.
\end{aligned}$$

[3]Symbolically the interpretation is: $j(x_j - y_j)$ means $\nabla v_1(x_j)$ and $\nabla w_2(y_j)$, $\mathbb{X}_j$ means $D^2 v_1(x_j)$, and $\mathbb{Y}_j$ means $D^2 w_2(y_j)$.

The equation for w_2 reads

$$\begin{aligned}
&\Big\langle \mathbb{Y}_j\, j(x_j-y_j), j(x_j-y_j)\Big\rangle + j^2|x_j-y_j|^2 \ln\left(\frac{j|x_j-y_j|}{C}\right)\Big\langle j(x_j-y_j), \nabla \ln p(y_j)\Big\rangle \\
&\quad + w_2(y_j) j^2|x_j-y_j|^2 \ln\left(\frac{j|x_j-y_j|}{C}\right)\Big\langle j(x_j-y_j), \nabla \ln p(y_j)\Big\rangle + j^4|x_j-y_j|^4 \\
&\quad \le -A^{-1}\sigma\, j^3|x_j-y_j|^3 e^{-2v_2(y_j)}\Big\{\Lambda - \big\|e^{2v_2}\nabla \ln p\big\|_{\infty,\mathcal{U}}\Big\}.
\end{aligned}$$

Subtracting the last two inequalities, we notice that the terms $j^4|x_j-y_j|^4$ cancel. The result is

$$\begin{aligned}
&\Big\langle \big(\mathbb{Y}_j - \mathbb{X}_j\big)\, j(x_j-y_j), j(x_j-y_j)\Big\rangle \\
&\quad + j^2|x_j-y_j|^2 \ln\left(\frac{j|x_j-y_j|}{C}\right)\Big\langle j(x_j-y_j), \nabla \ln p(y_j) - \nabla \ln p(x_j)\Big\rangle \\
&\quad + j^2|x_j-y_j|^2\Big\langle j(x_j-y_j), w_2(y_j)\, \nabla \ln p(y_j) - v_1(x_j)\nabla \ln p(x_j)\Big\rangle \\
&\quad \le -A^{-1}\sigma\, j^3|x_j-y_j|^3\, e^{-2v_2(y_j)}\Big\{\Lambda - \big\|e^{2v_2}\nabla \ln p\big\|_{\infty,\mathcal{U}}\Big\}.
\end{aligned}$$

The first term, the one with matrices, is non-negative and can be omitted from the inequality. Then we move the remaining terms and divide by $j^3|x_j-y_j|^3$ to get

$$\begin{aligned}
&A^{-1}\,\sigma\, e^{-2v_2(y_j)}\Big\{\Lambda - \big\|e^{2v_2}\nabla \ln p\big\|_{\infty,\mathcal{U}}\Big\} \\
&\le \left|\ln \frac{j|x_j-y_j|}{C}\right|\, \big|\nabla \ln p(y_j) - \nabla \ln p(x_j)\big| + \big|w_2(y_j)\nabla \ln p(y_j) - v_1(x_j)\nabla \ln p(x_j)\big|
\end{aligned}$$

A uniform bound

$$\Lambda \le j|x_j-y_j| \le L$$

is needed. The inequality with Λ was already clear. Using the definition of M_j, one can take

$$L = 2\|v_1\|_{\infty,\mathcal{U}} \qquad \text{or} \qquad L = \|w_2\|_{\infty,\mathcal{U}} \le 4\|v_2\|_{\infty,\mathcal{U}}.$$

Thus, taking the limit as $j \to \infty$ we use the continuity of $\nabla \ln p$ to arrive at

$$A^{-1}\,\sigma\left\{\Lambda - \big\|e^{2v_2}\nabla \ln p\big\|_{\infty,\mathcal{U}}\right\} \le e^{2v_2(\hat{x})}\big|w_2(\hat{x})\nabla \ln p(\hat{x}) - v_1(\hat{x})\nabla \ln p(\hat{x})\big|.$$

Recall (8.4.6). Since $A = 1+\sigma$, the above implies that

$$A^{-1}\,\sigma\left\{\Lambda - \big\|e^{2v_2}\nabla \ln p\big\|_{\infty,\mathcal{U}}\right\} \le \|e^{2v_2}\nabla \ln p\|_{\infty,\mathcal{U}}\,\frac{3\sigma}{2}.$$

Divide out σ. Now $A^{-1} \ge 1/2$. The final inequality is

$$\Lambda \le 3\|e^{2v_2}\nabla \ln p\|_{\infty,\mathcal{U}}.$$

Thus there is a contradiction, if the opposite inequality is assumed to be valid. Recall that

$$e^{2v_1} = \left(\frac{u_2}{m_2}\right)^2$$

to finish the proof.

$\square$

Corollary 8.4.5. *Local uniqueness holds. In other words, in a sufficiently small interior subdomain we cannot perturb the eigenfunction continuously.*

Proof. We can make

$$\frac{\max\limits_{\mathcal{U}} u}{\min\limits_{\mathcal{U}} u}$$

as small as we please, by shrinking the domain $\mathcal{U}$. Thus condition (8.4.5) is valid with the L^∞ norm taken over $\mathcal{U}$. $\square$

5. Discussion about the one-dimensional case

In the one-dimensional case an explicit comparison of the minimization problem for the two Rayleigh quotients (8.1.1) and (8.1.3) is possible. Let $\Omega = (0,1)$ and consider the limits of the problem coming from minimizing either

$$\text{(I)} \qquad \frac{\|u'\|_{jp(x)}}{\|u\|_{jp(x)}}$$

or

$$\text{(II)} \qquad \frac{\displaystyle\int_0^1 |v'(x)|^{jp(x)}\,dx}{\displaystyle\int_0^1 |v(x)|^{jp(x)}\,dx}, \qquad \text{with} \int_0^1 |v(x)|^{jp(x)}\,dx = C,$$

as $j \to \infty$. In the second case the equation is

$$\min\left\{\Lambda - \frac{|v'|}{v}, (v')^2 v'' + (v')^3 \ln(|v'|)\frac{p}{p'}\right\} = 0$$

for $v > 0$ ($v(0) = 0$, $v(1) = 0$, $\|v^p\|_\infty = C$).

The Luxemburg norm leads to the same equation, but with

$$v(x) = \frac{u(x)}{\|u'\|_\infty} = \frac{u(x)}{K}$$

as in equation (8.3.5). Thus all the solutions violating the condition $\|v'\|_\infty = 1$ are ruled out. This is the difference between the two problems.

Let us return to (II). The equation for v (without any normalization) can be solved. Upon separation of variables, we obtain

$$v(x) = \begin{cases} \displaystyle\int_0^x e^{\frac{A}{p(t)}}\,dt, & \text{when } 0 \le x \le x_0, \\ \\ \displaystyle\int_x^1 e^{\frac{A}{p(t)}}\,dt, & \text{when } x_0 \le x \le 1, \end{cases}$$

where the constant A is at our disposal and the point x_0 is determined by the continuity condition

$$\int_0^{x_0} e^{\frac{A}{p(t)}}\,dt = \int_{x_0}^1 e^{\frac{A}{p(t)}}\,dt.$$

Clearly, $0 < x_0 < 1$. Now Λ is determined from

$$\frac{v'(x_0^-)}{v(x_0)} = \Lambda = -\frac{v'(x_0^+)}{v(x_0)}.$$

Provided that the inequality

$$\frac{|v'(x)|}{v(x)} \ge \Lambda \quad (0 < x < 1, x \ne x_0)$$

holds, the number Λ is an eigenvalue for the non-homogeneous problem. What about the value of A? Given C, we can determine A from

$$\max_{0<x<1} v(x)^{p(x)} = C.$$

At least for a suitable $p(x)$, we can this way reach any real number A and therefore Λ can take all positive values, as C varies.

The problem in the Luxemburg norm is different. If u is an eigenfunction and

$$v = \frac{u}{\|u'\|_\infty},$$

then $0 \le v'(x) \le 1$ in some interval $(0, x_0)$. But the equation leads to

$$\frac{u'(x)}{\|u'\|_\infty} = e^{-\frac{A_1}{p(x)}}, \quad A_1 \ge 0,$$

in $(0, x_0)$ and

$$-\frac{u'(x)}{\|u'\|_\infty} = e^{-\frac{A_2}{p(x)}}, \quad A_2 \ge 0,$$

in $(x_0, 1)$. (In fact, $A_1 = A_2$). But this is impossible at points where the left-hand side is ± 1, unless at least one of the constants A_1, A_2 is zero, say that $A_1 = 0$. Then $u(x) = x$ when

$0 \le x \le x_0$. The determination of Λ from the equation

$$\frac{1}{x_0} = \Lambda = \frac{e^{-A_2/p(x_0)}}{x_0},$$

forces also $A_2 = 0$. It follows that

$$u(x) = \delta(x), \quad \Lambda = \Lambda_\infty = 2$$

is the only positive solution of the equation (0.0.13). In this problem Λ is unique. Recall that δ is the distance function.

APPENDIX A

Elementary inequalities in $\mathbb{R}^N$

1. The case $p \geq 2$

Lemma A.1.1.

$$|z-w|^p \leq 2^{p-2}\langle |z|^{p-2}z - |w|^{p-2}w, z-w\rangle \tag{A.1.1}$$

$$\left||z|^{\frac{p-2}{2}}z - |w|^{\frac{p-2}{2}}w\right|^2 \leq \frac{p^2}{4}\langle |z|^{p-2}z - |w|^{p-2}w, z-w\rangle \tag{A.1.2}$$

$$\left||z|^{p-2}z - |w|^{p-2}w\right| \leq (p-1)\left(|z|^{\frac{p-2}{2}} + |w|^{\frac{p-2}{2}}\right)\left||z|^{\frac{p-2}{2}}z - |w|^{\frac{p-2}{2}}w\right| \tag{A.1.3}$$

Remark A.1.2. By means of (A.1.1) and Cauchy-Schwartz inequality, we obtain

$$|z-w|^p \leq 2^{p-2}\left||z|^{\frac{p-2}{2}}z - |w|^{\frac{p-2}{2}}w\right|^2. \tag{A.1.4}$$

Indeed, (A.1.1) implies

$$|z-w|^p \leq 2^{p-2}|z-w|\left||z|^{p-2}z - |w|^{p-2}w\right|,$$

that is

$$|z-w|^{p-1} \leq 2^{p-2}\left||z|^{p-2}z - |w|^{p-2}w\right|,$$

and replacing p with $(p+2)/2$ and raising at the power 2, we obtain (A.1.4).

2. The case $1 < p < 2$

Lemma A.2.1. *Let $1 < p < 2$, then for every $z, w \in \mathbb{R}^N$ with $|z| + |w| > 0$ we have*

$$|z-w|^2\,(|z|+|w|)^{p-2} \leq \frac{1}{p-1}\,\langle |z|^{p-2}z - |w|^{p-2}w, z-w\rangle. \tag{A.2.5}$$

PROOF. Given $z, w \in \mathbb{R}^N$ and $t \in [0,1]$, we set $z_t = (1-t)w + tz$, then we have

$$\begin{aligned}|z|^{p-2}z - |w|^{p-2}w &= \int_0^1 \frac{d}{dt}\left(|z_t|^{p-2}\,z_t\right)\,dt \\ &= (p-2)\int_0^1 |z_t|^{p-4}\,\langle z_t, z-w\rangle\, z_t\,dt + \int_0^1 |z_t|^{p-2}\,(z-w)\,dt\end{aligned} \tag{A.2.6}$$

then taking the scalar product with $z-w$ and using that $p-2<0$, we get

$$\langle |z|^{p-2}z - |w|^{p-2}w, z-w\rangle \geq (p-1)|z-w|^2 \int_0^1 |z_t|^{p-2}\,dt.$$

We can conclude by simply using that

$$|z_t| \leq |z| + |w|,$$

and then raising to the power $p-2$ and integrating. □

Lemma A.2.2. *Let $1<p<2$, then for every $z, w \in \mathbb{R}^N$ with $|z|+|w|>0$ we have*

$$\big||z|^{p-2}z - |w|^{p-2}w\big| \leq 2^{2-p}\,\frac{3-p}{p-1}\,|z-w|\,(|z|+|w|)^{p-2}. \tag{A.2.7}$$

Proof. Observe that from (A.2.6) using Cauchy-Schwarz inequality we obtain

$$\big||z|^{p-2}z - |w|^{p-2}w\big| \leq (3-p)\,|z-w|\,\int_0^1 |z_t|^{p-2}\,dt.$$

We can conclude the proof by showing the validity of the following inequality

$$\int_0^1 |z_t|^{p-2}\,dt \leq \frac{2^{2-p}}{p-1}\,(|z|+|w|)^{p-2}. \tag{A.2.8}$$

In order to prove (A.2.8), we first observe that for every $z, w \in \mathbb{R}^N \setminus \{0\}$ we have

$$|z_t| = |(1-t)w + tz| \geq \left|(1-t)w - t\frac{|z|}{|w|}w\right|,$$

which is true by simple computations: indeed, we have

$$\begin{aligned}\left|(1-t)w - t\frac{|z|}{|w|}w\right|^2 &= (1-t)^2|w|^2 + t^2|z|^2 - 2t(1-t)|w||z| \\ &\leq (1-t)^2|w|^2 + t^2|z|^2 + 2t(1-t)\langle z, w\rangle = |(1-t)w + tz|^2.\end{aligned}$$

Recalling that $p-2<0$, this implies that we have

$$|z_t|^{p-2} \leq \left|(1-t)w - t\frac{|z|}{|w|}w\right|^{p-2}.$$

Let us now set $\lambda = |z|/|w|$, then we have

$$\int_0^1 |z_t|^{p-2}\,dt \leq \int_0^1 |(1-t)w - t\lambda w|^{p-2}\,dt = |w|^{p-2}\int_0^1 |1-(\lambda+1)t|^{p-2}\,dt.$$

and we only need to compute the integral in the right-hand side:

$$\int_0^1 |1-(\lambda+1)t|^{p-2}\,dt = \int_0^{\frac{1}{\lambda+1}} (1-(\lambda+1)t)^{p-2}\,dt + \int_{\frac{1}{\lambda+1}}^1 ((\lambda+1)t-1)^{p-2}\,dt$$
$$= \frac{1}{p-1}\frac{\lambda^{p-1}+1}{\lambda+1} = \frac{|w|^{2-p}}{p-1}\,\frac{|z|^{p-1}+|w|^{p-1}}{|z|+|w|}.$$

Finally, it is only left to observe that we have

$$\frac{|z|^{p-1}+|w|^{p-1}}{|z|+|w|} \le 2^{2-p}\,\frac{(|z|+|w|)^{p-1}}{|z|+|w|} = 2^{2-p}\,(|z|+|w|)^{p-2},$$

where we used the concavity of the function $t \mapsto t^{p-1}$. Putting all together, we have concluded the proof. □

Lemma A.2.3. *Given $1 < p < 2$, we set $q = p/(p-1)$. Then for every $z, w \in \mathbb{R}^N$ with $|z| + |w| > 0$ we have*

$$\big||z|^{p-2}z - |w|^{p-2}w\big|^q \le C\,|z-w|^2\,(|z|+|w|)^{p-2}, \tag{A.2.9}$$

for a suitable constant C which only depends on p.

Proof. Let us set $C = 2^{q(2-p)}\,(3-p)^q\,(p-1)^{-q}$, then taking (A.2.7) and raising to the power q we have

$$\begin{aligned}\big||z|^{p-2}z - |w|^{p-2}w\big|^q &\le C\,|z-w|^q\,(|z|+|w|)^{q(p-2)}\\ &= C\,|z-w|^2\,|z-w|^{q-2}\,(|z|+|w|)^{q(p-2)}\\ &\le C\,|z-w|^2(|z|+|w|)^{q(p-2)+(q-2)} = C\,|z-w|^2\,(|z|+|w|)^{p-2},\end{aligned}$$

which concludes the proof, where we used the simple computations $q(p-2)+q-2 = p-2$. □

Lemma A.2.4. *Let $1 < p < 2$ and set $q = p/(p-1)$. Then for every $z, w \in \mathbb{R}^N$ there holds*

$$\big||z|^{p-2}z - |w|^{p-2}w\big|^q \le C\,\langle |z|^{p-2}z - |w|^{p-2}w, z-w\rangle, \tag{A.2.10}$$

for a suitable constant C which only depends on p.

Proof. It is enough to combine (A.2.9) with (A.2.5) and observe that when $|z|+|w| = 0$ there is nothing to prove. □

3. A useful compactness criterion

We apply the inequalities above so as to provide a sufficient condition for the strong convergence in the Sobolev space $W_0^{1,p}(\Omega)$. Here Ω is some given open set of finite measure. The first lemma is elementary.

Lemma A.3.1. *If u_ν is a bounded sequence in $W^{1,p}(\Omega)$ and $u_\nu \rightharpoonup u$ weakly in $L^p(\Omega)$ then $\partial_{x_i} u_\nu \rightharpoonup \partial_{x_i} u$ weakly in $L^p(\Omega)$ for all $i \in \{1, \dots, N\}$.*

Proof. The weak convergence of the sequence in $L^p(\Omega)$ implies

$$\lim_{\nu\to\infty} \int_\Omega \partial_{x_i} u_\nu \, \varphi \, dx = -\lim_{\nu\to\infty} \int_\Omega u_\nu \, \partial_{x_i} \varphi \, dx = -\int_\Omega u \, \partial_{x_i} \varphi \, dx = \int_\Omega \partial_{x_i} u \, \varphi \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. The fact that $\partial_{x_i} f$ is the weak derivative of f was used. On the other hand, given a function $\varphi \in L^{p'}(\Omega)$ and a sequence $\{\varphi_k\}_k$ converging to φ in $L^{p'}(\Omega)$, one has

$$\int_\Omega \big(\partial_{x_i} u_\nu - \partial_{x_i} u\big)\varphi \, dx = \int_\Omega \big(\partial_{x_i} u_\nu - \partial_{x_i} u\big)\varphi_k \, dx + \int_\Omega \big(\partial_{x_i} u_\nu - \partial_{x_i} u\big)\big(\varphi - \varphi_k\big) \, dx.$$

Since φ_k is smooth the first summand in the right hand side goes to zero. The second one can be made suitably small by Hölder inequality, since the sequence u_ν is bounded. This concludes the proof. $\square$

The following sufficient condition for the strong L^p-convergence of weakly converging sequences is due to the elementary inequalities (A.1.1), (A.2.5), in turn related to the convexity of the mapping $t \mapsto |t|^p$,

Proposition A.3.2. *Let $\{f_\nu\}_{\nu\in\mathbb{N}} \subset L^p(\Omega)$ be such that*

$$\lim_{\nu\to\infty} \int_\Omega \big(|f_\nu|^{p-2} f_\nu - |f|^{p-2} f\big) \, (f_\nu - f) \, dx = 0, \qquad \text{for every } i = 1, \dots, N, \tag{A.3.11}$$

for some function $f \in L^p(\Omega)$. Then $\{f_\nu\}_{\nu\in\mathbb{N}}$ converges in $L^p(\Omega)$ to f.

Proof. We have two distinguish between two cases. If $p \geq 2$, then by A.1.1, it follows directly that

$$\lim_{\nu\to\infty} \int_\Omega |f_\nu - f|^p \, dx = 0, \qquad \text{for every } i = 1, \dots, N.$$

For $1 < p < 2$, we start observing that the hypothesis implies

$$\int_\Omega |f_\nu|^p \, dx \leq C, \qquad \text{for every } \nu \in \mathbb{N}. \tag{A.3.12}$$

Indeed, by means of Young inequality we can infer

$$\begin{aligned}\int_\Omega \big(|f_\nu|^{p-2} f_n u - |f|^{p-2} f\big) \, (f_\nu - f) \, dx \geq \left(1 - \frac{\varepsilon^p}{p} - \frac{\varepsilon^q}{q}\right) \int_\Omega |f_\nu|^p \, dx \\ + \left(1 - \frac{\varepsilon^{-p}}{p} - \frac{\varepsilon^{-q}}{q}\right) \int_\Omega |f|^p \, dx,\end{aligned}$$

where we set $q = p/(p-1)$. Then, by taking ε small enough, (A.3.11) implies (A.3.12). Now use inequality (A.2.5) raised to the power $p/2$ to get

$$\int_\Omega |f_\nu - f|^p \, dx \le C \int_\Omega (1 + |f_\nu|^2 + |f|^2)^{\frac{(2-p)\,p}{4}} \left[\left(|f_\nu|^{p-2} f_\nu - |f|^{p-2} f\right)(f_\nu - f)\right]^{\frac{p}{2}} \, dx.$$

An application of Hölder inequality with exponents $2/(2-p)$ and $2/p$ yields

$$\begin{aligned}\int_\Omega |f_\nu - f|^p \, dx \le C \left[\int_\Omega (1 + |f_\nu|^2 + |f|^2)^{\frac{p}{2}} \, dx\right]^{\frac{2-p}{2}} \\ \times \left[\int_\Omega \left(|f_\nu|^{p-2} f_\nu - |f|^{p-2} f\right)(f_\nu - f) \, dx\right]^{\frac{p}{2}}.\end{aligned}$$

The conclusion is now an easy consequence of (A.3.11) and (A.3.12). $\square$

Combining Proposition A.3.2 and Lemma A.3.1 one can prove the following:

Proposition A.3.3. *Let u_ν be a sequence converging weakly to u in $W_0^{1,p}(\Omega)$. If*

$$\lim_{\nu\to\infty} \int_\Omega \left(|\partial_{x_i} u_\nu|^{p-2} \partial_{x_i} u_\nu - |\partial_{x_i} u|^{p-2} \partial_{x_i} u\right)(\partial_{x_i} u_\nu - \partial_{x_i} u) \, dx = 0$$

then u_ν converges to u strongly in $W_0^{1,p}(\Omega)$.

The above Proposition entails that the maps $H(z) = \|z\|^p_{\ell^p}$ satisfies the assumption (A.3.13)

$$\lim_{\nu\to\infty} \int_\Omega \left\langle \nabla_z H(\nabla u_\nu) - H(\nabla u), \nabla u_\nu - \nabla u \right\rangle dx = 0 \quad \Rightarrow \quad \lim_{\nu\to\infty} \int_\Omega |\nabla u_\nu - \nabla u|^p \, dx = 0$$

for all weakly convergent sequences $\{u_\nu\}_{\nu\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$. A similar condition can be proved if $H(z) = |z|^p$.

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The Thesis is based on the following papers

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