STABILITY RESULTS FOR DOUBLY NONLINEAR DIFFERENTIAL INCLUSIONS BY VARIATIONAL CONVERGENCE

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ABSTRACT. We present a stability result for a wide class doubly nonlinear equations, featuring general maximal monotone operators, and (possibly) nonconvex and nonsmooth energy functionals. The limit analysis resides on the reformulation of the differential evolution as a scalar energy-conservation equation with the aid of the so-called Fitzpatrick theory for the representation of monotone operators. In particular, our result applies to the vanishing viscosity approximation of rate-independent systems.

1. Introduction

This note is concerned with a convergence result for doubly nonlinear differential inclusions of the type

$$\alpha_n\left(\dot{u}_n(t)\right) + \partial\mathcal{E}_t(u_n(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.a. } t \in (0,T). \tag{1.1}$$

Here, (α_n) is a sequence of maximal monotone (and possibly multivalued) operators $\alpha_n : X \rightrightarrows X^*$, $(X, \|\cdot\|)$ is a (separable) reflexive Banach space, and $\mathcal{E} : [0,T] \times X \to (-\infty, \infty]$ is a (proper) time-dependent *energy* functional. We will prove that for $\alpha_n \to \alpha$ in the graph sense any limit point u of the sequence $(u_n)_{n\in\mathbb{N}}$ is a solution to

$$\alpha\left(\dot{u}(t)\right) + \partial\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.a. } t \in (0,T)$$

Throughout the paper, we write $\mathcal{E}_t(u)$ in place of $\mathcal{E}(t,u)$. We will understand the multivalued operator $\partial\mathcal{E} : (0,T)\times X \rightrightarrows X^*$ to be a suitable notion of subdifferential for the possibly nonsmooth and nonconvex map $u \mapsto \mathcal{E}_t(u)$, namely the so-called the *Fréchet subdifferential*, defined at $(t,u) \in \mathrm{dom}(\mathcal{E})$ by

$$\xi \in \partial\mathcal{E}_t(u) \quad \text{ if and only if } \mathcal{E}_t(v) \geq \mathcal{E}_t(u) + \langle \xi, v-u\rangle + \mathrm{o}(\|v-u\|) \quad \text{as } v \to u. \tag{1.2}$$

Observe that, as soon as the mapping $u \mapsto \mathcal{E}_t(u)$ is *convex*, the Fréchet subdifferential $\partial\mathcal{E}_t(u)$ coincides with the subdifferential of $u \mapsto \mathcal{E}_t(u)$ in the sense of convex analysis.

Doubly nonlinear equations as in (1.1) arise in a variety of different applications, ranging from Thermomechanics, to phase change, to magnetism. As such, they have attracted a substantial deal of attention in recent years. Correspondingly, the related literature is quite rich. Being beyond our scope to attempt here a comprehensive review, we limit ourselves to recording the seminal observations by MOREAU [56, 57] and GERMAIN [37], as well as the early existence results by ARAI [9], SENBA [71], COLLI & VISINTIN [26], and COLLI [25]. The reader can find a selection of more recent results in [5, 2, 3, 7, 33, 41, 51, 65, 69]. Without going into details, let us mention that, over the last decade, the convexity requirement on the map $u \mapsto \mathcal{E}_t(u)$ in the pioneering papers [9, 71, 26, 25] has been progressively weakened: in particular, in [51] a quite broad class of nonsmooth and nonconvex energy functionals has been considered. Nonetheless, in all of the aforementioned contributions the operator α is assumed to fulfill some coercivity property, namely to have at least linear growth at infinity. We will refer to this case as *viscous*.

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The case of 0-homogeneous operators α has been recently investigated as well, for it connects with the modeling of so-called rate-independent systems. We shall hence refer to this situation as *rate-independent*. Some references in this direction are to be found in the papers [28, 29, 36, 43, 48, 47, 53, 54, 55].

Additionally, relation (1.1) has been considered in connection with the study of the long-time behavior of solutions in [4, 31, 68, 70, 69], and their variational characterization in [6, 8, 72, 75].

The focus of this paper is on the study of the stability of the doubly nonlinear flows (1.1). Namely, we investigate the convergence of solutions u_n to equations (1.1), under the assumption

$$\alpha_n \to \alpha \text{ in the graph sense in } X \times X^*, \tag{1.3}$$

viz. for all $(\xi, \xi^*) \in \mathrm{graph}(\alpha)$ there exists $(\xi_n, \xi_n^*) \in \mathrm{graph}(\alpha_n)$ such that $\xi_n \to \xi$ in X and $\xi_n^* \to \xi^*$ in X^* as $n \to \infty$. The main result of this paper, Theorem 4.5, states that cluster points u of the curves (u_n) are in fact solutions to the limiting equation

$$\alpha\left(\dot{u}(t)\right) + \partial\mathcal{E}_t(u(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.a. } t \in (0,T). \tag{1.4}$$

We have to mention that stability results for the doubly nonlinear flows (1.1) are already available in the literature. For *viscous* graphs α_n, a first convergence theorem in the case of convex energies has been obtained by Aizicovici & Yan [1] (see also [72]), whereas stability results for doubly nonlinear equations with nonconvex energies have been proved in [51] . This issue has been recently reconsidered by Visintin [80, 81, 82], who has remarkably extended the reach of the theory to treat subdifferential inclusions of the type

$$\beta(\dot{u}(t)) + \gamma(u(t)) \ni 0 \quad \text{in } X^* \qquad \text{for a.a. } t \in (0,T),$$

with β, $\gamma : X \rightrightarrows X^*$ maximal monotone operators, β cyclically monotone and γ noncyclic monotone, by resorting to the so-called Fitzpatrick theory [34].

Let us briefly recall that an operator $\alpha : X \rightrightarrows X^*$ is *cyclically monotone* if α is the *generalized gradient* of some potential. Namely, if $\alpha = \partial\psi$ for some proper, convex, and lower semicontinuous function $\psi : X \to (-\infty, \infty]$, where the symbol ∂ here denotes the subdifferential in the sense of convex analysis. In the cyclic-monotone case $\alpha = \partial\psi$, it is well known that the relation $y \in \partial\psi(x)$ can be equivalently reformulated as $\langle y, x\rangle = \psi(x) + \psi^*(y)$, where ψ^* is the Legendre-Fenchel conjugate of ψ and $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. The use of this variational fact for the aim of variationally reformulating evolution equations dates back to Brezis-Ekeland [18, 17] and Nayroles [59, 60]. Among the many contributions stemming from this idea, the reader is especially referred to the existence proofs by Auchmuty [11] and Roubicek [67], and to the recent monograph by Ghoussoub [39] on self-dual variational principles (see also the references in [72]).

The Fitzpatrick theory allows us to extend this variational view to subdifferential inclusions of the type (1.4), with α possibly *noncyclic* monotone, by introducing *representative* functions $f_\alpha : X \times X^* \to (-\infty, \infty]$ for the operator α. These are convex functions f_α with the property

$$\begin{gathered} \forall (x,y) \in X \times X^*,\ \langle y, x\rangle \leq f_\alpha(x,y) \text{ and} \\ y \in \alpha(x) \text{ iff } \langle y, x\rangle = f_\alpha(x,y). \end{gathered} \tag{1.5}$$

The reader is referred to Section 2 below for a selection of relevant results within this theory. In particular, in [80] these tools are used in order to reformulate variationally relations (1.1) for noncyclic monotone operators. This reformulation opens the way to devise a suitable Γ-convergence analysis toward structural stability of the flows.

As for the *rate-independent* case, one shall mention the stability results for hysteresis operators from the classical monographs [19, 42, 78] (see also [74]). Another stability result in the rate-independent setting is in [72]. Moreover, we record Visintin [81, 82], which exploits the Fitzpatrick idea in the rate-independent context, but by taking perturbations in $\partial\mathcal{E}_n$ (again, in a possibly noncyclic monotone frame).

Finally, the approximation of rate-independent flows by viscous flows (in the cyclic-monotone case) has been recently attracted a great deal of attention. This is especially critical as viscous and rate-independent evolutions usually call for different analytical treatments. The *vanishing viscosity* approach to abstract rate-independent systems has been in particular developed in [32, 49, 50]. More specifically, in the latter two papers it has been shown that the vanishing viscosity limit leads to the notion of BV solution to a rate-independent system. In the recent [52], still within the cyclic-monotone frame, the $p_n \to 1$ limit, where p_n is the homogeneity of the potential ψ_n of α_n, has been addressed, and it has been proved that BV solutions arise in the limit. A stability result with respect to variational convergence for the latter solution concept has also been obtained.

Our result. The focus here is that of obtaining a stability result for the differential inclusions (1.1) by allowing for maximal generality on the perturbations α_n and on the functional $\mathcal{E}$. In particular, we shall neither assume super-linear equi-coercivity in α_n, nor cyclic monotonicity. As for the energy $\mathcal{E}$, we do not require neither smoothness nor convexity with respect to u, but still ask for lower semicontinuity and some coercivity, see Assumption 3.9 below.

This generality sets our result aside from the available contributions on this topic. In particular, our analysis also encompasses the passage from viscous to rate-independent doubly nonlinear evolution. Indeed, we are able to treat here the $p_n \to 1$ case for noncyclic monotone operators α_n (in this setting, p_n is the coercivity exponent for α_n). In our general context, we prove that the so-called *local solutions* [49, 50] to a rate-independent system arise in the $p_n \to 1$ limit.

The basic idea for handling the noncyclic monotone case, is to resort to a variational reformulation of the flows (1.1) which is well suited for discussing limits. By letting f_{α_n} *represent* the monotone operator α_n in the sense of (1.5) and assuming the validity of a suitable chain rule for the energy $\mathcal{E}$, relation (1.1) is proved to be equivalent (see Proposition 3.12) to an *energy conservation* identity, namely

$$\underbrace{\mathcal{E}_t(u_n(t))}_{\text{energy at } t} + \underbrace{\int_0^t f_{\alpha_n}\Big(\dot{u}_n(s), -\partial\mathcal{E}_s(u(s))\Big)dt}_{\text{dissipated energy on } [0,t]} = \underbrace{\mathcal{E}_0(u(0)}_{\text{initial energy}} + \underbrace{\int_0^t \partial_t\mathcal{E}_s(u(s))ds}_{\text{work of ext. actions}} \quad \text{for all } t \in [0,T]. \tag{1.6}$$

The strategy is then to prove that, by passing to the lim inf in (1.6), the structure of the relation is preserved. In particular, we provide sufficient conditions under which the lim inf of the integral of the representative functions f_{α_n} is a representative function of the limit graph α. Care here is given to developing such a lower semicontinuity argument for functions which are only BV in time. This allows us to directly include the case of rate-independent flows.

Let us once more emphasize that we can encompass in our analysis a broad class of time-dependent energies $\mathcal{E} : [0,T] \times X \to (-\infty, +\infty]$, (possibly) nonsmooth and nonconvex with respect to the state variable u, but still satisfying a suitable set of coercivity and regularity type conditions mutuated from [51]. Note that in such a general setting existence of absolutely continuous solutions $u_n : [0,T] \to X$ to (1.1) is presently not known. A possibility in order to overcome this would be to strengthen our assumptions on the energy functional $\mathcal{E}$, for instance assuming it to be a suitable perturbation of a (λ-)convex functional, as in [66, 65, 51]. We however refrain from this, for the sake of keeping maximal generality for the convergence result. In this respect, our result should be regarded purely as a stability analysis, with focus on the convergence properties of the operators (α_n). A stability result with respect to suitable convergence of the energy functionals could also be obtained, closely following the lines of [51, Thm. 4.8]. Again, we have chosen not to detail this, in order to highlight the usage of the Fitzpatrick theory to deal with the *noncyclic* operators (α_n). This very generality will allow us, for instance, to address in the upcoming Section 4.3 the quasistatic limits of a class of *dynamical* problems, which can be in fact reformulated as doubly nonlinear equations of the form (1.1).

Structure of the paper. Section 2 contains some background material on Fitzpatrick theory, and on the notions of variational convergence for functionals and operators which will be relevant for the subsequent analysis. In Section 3.1, some further preliminaries of measure theory and convex analysis are provided, whereas in Section 3.2 the basic assumptions on the energy functional $\mathcal{E}$ are stated in detail, and suitable reformulations of (1.1) are discussed. In Section 4 we state our main stability result Theorem 4.5 and thoroughly comment it. We also give two corollaries (i.e.

Theorem 4.8 and Proposition 4.9) in two particular cases: specifically, Prop. 4.9 deals with the $p_n \to 1$ vanishing-viscosity limit. We conclude Sec. 4 by discussing classes of energy functionals to which our results apply (cf. Sec. 4.2), and developing applications to rate-independent limits of Hamiltonian systems (in Sec. 4.3). The proof of Theorem 4.5 is developed throughout Section 5, also exploiting some results from Young measure theory which are contained in Appendix A.

2. Fitzpatrick theory

Within this section, we shall systematically use the notation

$$\pi(\xi, \xi^*) := \langle \xi^*, \xi \rangle \quad \text{for all } (\xi, \xi^*) \in X \times X^*$$

for the duality pairing between the reflexive space X and X^*, and identify possibly multivalued operators $\alpha : X \rightrightarrows X^*$ with the corresponding graphs $\alpha \subset X \times X^*$ without changing notation. We recall that $\alpha : X \rightrightarrows X^*$ is *monotone* if

$$\langle \xi^* - \xi_0^*, \xi - \xi_0 \rangle \geq 0 \quad \text{for all } \xi^* \in \alpha(\xi),\ \xi_0^* \in \alpha(\xi_0)$$

(where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X), and that it is *maximal monotone,* if it is maximal for set inclusion within the class of monotone operators.

We shall provide here a minimal aside on the Fitzpatrick theory, essentially mutuated from [79]. The reader is however referred to [20, 21, 22, 23, 38, 39, 45, 46, 62, 63] for additional material and a collection of related results and applications to PDEs.

Representative functionals. We denote by $\mathcal{F}(X)$ the set of functionals $\varphi : X \times X^* \to (-\infty, \infty]$ such that

$$\varphi \ \text{ is convex, lower semicontinuous, and } \ \varphi(\xi, \xi^*) \geq \pi(\xi, \xi^*) \quad \forall (\xi, \xi^*) \in X \times X^*.$$

We associate with $\varphi \in \mathcal{F}(X)$ the set $\alpha \subset X \times X^*$ given by

$$(\xi, \xi^*) \in \alpha \quad \Leftrightarrow \quad \varphi(\xi, \xi^*) = \pi(\xi^*, \xi). \tag{2.1}$$

Whenever (2.1) holds we say that φ *represents* α, that φ is *representative*, and that α is *representable*. A representable operator can be represented by different representative functionals (cf. Example 2.4 below). On the contrary, each representative functional represents only one operator.

Example 2.1. In the cyclically monotone case of $\alpha = \partial\psi$ for some (proper) convex and lower semicontinuous potential $\psi : X \to (-\infty, \infty]$, a representative functional for α is given by the *bipotential* (according to the terminology of [20])

$$\varphi(\xi, \xi^*) = \psi(\xi) + \psi^*(\xi^*). \tag{2.2}$$

We have the following *strict* set inclusions

$$\{\text{maximal monotone operators}\} \subsetneqq \{\text{representable operators}\} \subsetneqq \{\text{monotone operators}\}.$$

Namely, representable operators are intermediate between monotone and maximal monotone. One may wonder how to translate maximality at the level of representative functionals. The following result provides a useful criterium for the representability of a maximal monotone operator.

Proposition 2.2 (Representative of a maximal monotone operator [76])**.** *A functional $\varphi \in \mathcal{F}(X)$ represents a maximal monotone operator iff $\varphi^* \in \mathcal{F}(X^*)$. In this case, if φ represents α then φ^* represents $\alpha^{-1} = \{(\xi, \xi^*) \,:\, (\xi^*, \xi) \in \alpha\}$.*

The Fitzpatrick and the Penot functions. Given $\alpha \subset X \times X^*$ with $\alpha \neq \emptyset$ we define the *Fitzpatrick function* (associated with α) $f_\alpha : X \times X^* \to (-\infty, \infty]$ by

$$\begin{aligned} f_\alpha(\xi, \xi^*) &:= \pi(\xi, \xi^*) + \sup\{\pi(\xi_0 - \xi, \xi^* - \xi_0^*) \ : \ (\xi_0, \xi_0^*) \in \alpha\} \\ &= \sup\{\pi(\xi_0, \xi^*) - \pi(\xi_0 - \xi, \xi_0^*) \ : \ (\xi_0, \xi_0^*) \in \alpha\} \quad \text{for all } (\xi, \xi^*) \in X \times X^* \end{aligned} \tag{2.3}$$

and the *Penot function* (associated with α) $\rho_\alpha : X \times X^* \to (-\infty, \infty]$ by

$$\rho_\alpha := (\pi + I_\alpha)^{**}. \tag{2.4}$$

Both f_α and ρ_α represent α. Moreover f_α and ρ_α are respectively the minimal and maximal element (with respect to pointwise ordering) of the functional *interval*

$$\mathcal{I}(\alpha) = \{\varphi \in \mathcal{F}(X) \; : \; \varphi \text{ represents } \alpha\}. \tag{2.5}$$

In particular, in the cyclically monotone case of $\alpha = \partial\psi$, there holds

$$f_\alpha(\xi,\xi^*) \le \psi(\xi) + \psi^*(\xi^*) \le \rho_\alpha(\xi,\xi^*) \quad \text{for all } (\xi,\xi^*) \in X \times X^*. \tag{2.6}$$

Let us also point out that, in view of the definitions (2.3) and (2.4) of f_α and ρ_α the following formulae hold

$$f_\alpha(\xi,\xi^*) = \sup\{\pi(\xi_0,\xi^*) + \pi(\xi,\xi_0^*) - \rho_{\alpha^{-1}}(\xi_0^*,\xi_0) \; : \; (\xi_0,\xi_0^*) \in X{\times}X^*\} \quad \forall\,(\xi,\xi^*) \in X{\times}X^*, \tag{2.7}$$

$$\rho_\alpha(\xi,\xi^*) = \sup\{\pi(\xi_0,\xi^*) + \pi(\xi,\xi_0^*) - f_{\alpha^{-1}}(\xi_0^*,\xi_0) \; : \; (\xi_0,\xi_0^*) \in X{\times}X^*\} \quad \forall\,(\xi,\xi^*) \in X{\times}X^*, \tag{2.8}$$

namely f_α (ρ_α, resp.) is the convex conjugate of the Penot (Fitzpatrick, resp.) function of the inverse operator α^{-1}.

Finally, for later use we observe that

$$0 \in \alpha(0) \;\Rightarrow\; f_\alpha(\xi,\xi^*) \ge 0 \quad \text{for all } (\xi,\xi^*) \in X \times X^*. \tag{2.9}$$

Self-dual representatives. Recall that a function $\varphi : X \times X^* \to (-\infty,\infty]$ is called *self-dual* iff

$$\varphi(\xi,\xi^*) = \varphi^*(\xi^*,\xi) \quad \text{for all } (\xi,\xi^*) \in X \times X^*.$$

The interval $\mathcal{I}(\alpha)$ from (2.5) includes *self-dual* representative functions [63, Thm. 3.3]. In the cyclically monotone case of $\alpha = \partial\psi$, an example in this direction is given by the *bipotential* (2.2). Out of the cyclically monotone realm, an example of a self-dual representative in the case $\alpha = \gamma + \partial\psi$ with γ skew adjoint is $(\xi,\xi^*) \mapsto \psi(\xi) + \psi^*(-\gamma\xi + \xi^*)$ [38].

In the general case, the indirect proof of the existence of self-dual representative functions is due to Penot [61, 62] and Svaiter [76], whereas direct constructions have been firstly provided by Penot & Zălinescu [64] under some restriction on α. An explicit self-dual representative function in the general maximal monotone case has been recently obtained by Bauschke & Wang [15] and reads

$$(\xi,\xi^*) \mapsto \frac{1}{2} \inf_{(\zeta,\zeta^*) \in X \times X^*} \big\{ f_\alpha(\xi+\zeta,\xi^*+\zeta^*) + f_\alpha(\xi-\zeta,\xi^*-\zeta^*) + \|\zeta\|^2 + \|\zeta\|_*^2 \big\}. \tag{2.10}$$

Note that neither the Fitzpatrick function f_α nor the Penot function ρ_α are self-dual in general.

Let us now recast the characterization of maximal monotonicity of Proposition 2.2 in the following.

Proposition 2.3 (Self-dual representatives = maximality)**.** *An operator $\alpha : X \rightrightarrows X^*$ is maximal monotone iff it is represented by a self-dual functional φ.*

Proof. By [15], if α is maximal monotone, then it admits the self-dual representative (2.10).

As for the converse implication, note that by self-duality of φ and π we get

$$\varphi^*(\xi^*,\xi) = \varphi(\xi,\xi^*) \ge \pi(\xi,\xi^*) = \pi^*(\xi^*,\xi).$$

Thus, $\varphi^* \in \mathcal{F}(X^*)$ and Proposition 2.2 applies. □

Self-dual representatives vs. Fitzpatrick and Penot functions. Let $\alpha \subset X \times X^*$ be a cyclically monotone operator with $\alpha = \partial\psi$ for some convex and lower semicontinuous potential $\psi : X \to (-\infty,\infty]$. As already observed, a self-dual representative of α is the sum of ψ and its convex conjugate. However in general the Fitzpatrick functional f_α may differ from $\psi + \psi^*$, as shown by the following.

Example 2.4. Consider $X = \mathbb{R} = X^*$ and set $\alpha =$ identity, namely $\alpha = \partial\psi$ with $\psi(\xi) = \frac{1}{2}\xi^2$. The Fitzpatrick function of $\partial\psi$ is $f_{\partial\psi}(\xi,\xi^*) = \xi^2/4 + (\xi^*)^2/4 + \xi\cdot\xi^*/2$, which is not self-dual.

Fitzpatrick and Penot functions in the case of 1-positively homogeneous potentials. Our next result reveals that, when $\alpha = \partial\psi$ and $\psi : X \to (-\infty, +\infty]$ is positively homogeneous of degree 1, then also the Fitzpatrick functional f_α coincides with the bipotential (2.2).

Proposition 2.5. *Let $\psi : X \to (-\infty, \infty]$ be convex, lower semicontinuous, and positively homogeneous of degree 1, viz. $\psi(\lambda\xi) = \lambda\psi(\xi)$ for all $\xi \in X$ and $\lambda \geq 0$.*

Then, the Fitzpatrick function of the subdifferential of ψ coincides with the sum of ψ and its convex conjugate, i.e.

$$f_{\partial\psi}(\xi, \xi^*) = \psi(\xi) + \psi^*(\xi^*) \qquad \text{for all } (\xi, \xi^*) \in X \times X^*. \tag{2.11}$$

Before developing the *proof*, we recall that, for all ψ convex, lower semicontinuous, and 1-homogeneous there exists a closed convex set $0 \in K \subset X$ such that ψ coincides with the *Minkowski functional* of K, viz.

$$\psi(\xi) = M_K(\xi) = \inf\left\{\sigma > 0 \, : \, \frac{\xi}{\sigma} \in K\right\}. \tag{2.12a}$$

Furthermore,

$$\begin{aligned} &\psi^*(\xi^*) = I_{K^*}(\xi^*) \text{ where } K^* \subset X^* \text{ is the } \textit{polar} \text{ set of } K, \text{ i.e.} \\ &\qquad K^* = \{\xi^* \in K^* \, : \, \pi(\xi, \xi^*) \leq 1 \; \forall \, \xi \in K\}, \end{aligned} \tag{2.12b}$$

so that

$$\forall \, (\xi, \xi^*) \in \partial\psi \qquad M_K(\xi) + I_{K^*}(\xi^*) = \pi(\xi, \xi^*). \tag{2.12c}$$

For later convenience, we also recall that

$$\psi(\xi) = \sup_{\xi^* \in K^*} \pi(\xi, \xi^*) \quad \text{for all } \xi \in X. \tag{2.12d}$$

Proof of Proposition 2.5. It follows from the definition of the Fitzpatrick function (2.3) and from (2.12c) that

$$f_{\partial\psi}(\xi, \xi^*) = \sup\{\underbrace{\pi(\xi_0, \xi^*) - M_K(\xi_0)}_{\leq I_{K^*}(\xi^*)} + \underbrace{\pi(\xi, \xi_0^*) - I_{K^*}(\xi_0^*)}_{\leq M_K(\xi)} : \; (\xi_0, \xi_0^*) \in \alpha\}$$

Hence, we obtain that $f_{\partial\psi}(\xi, \xi^*) \leq M_K(\xi) + I_{K^*}(\xi^*)$ for all $(\xi, \xi^*) \in X \times X^*$.

For the opposite inequality assume first that $\xi^* \notin K^*$. Then there exists $\xi_0 \in K$ such that $\pi(\xi_0, \xi^*) > 1$. Choose an arbitrary $\xi_0^* \in \partial M_K(\xi_0) = \partial M_K(\lambda\xi_0)$ for any positive $\lambda > 0$. Then, taking into account that $M_K(\lambda\xi_0) + I_{K^*}(\xi_0^*) = \lambda\pi(\xi_0, \xi_0^*) \leq \lambda$, we get

$$\begin{aligned} &\pi(\lambda\xi_0, \xi^*) - M_K(\lambda\xi_0) + \pi(\xi, \xi_0^*) - I_{K^*}(\xi_0^*) \\ &\quad \geq \quad \lambda\,(\pi(\xi_0, \xi^*) - 1) + \pi(\xi, \xi_0^*) \to +\infty \qquad \text{as } \lambda \to \infty. \end{aligned}$$

Therefore $f_{\partial\psi}(\xi, \xi^*) \geq I_{K^*}(\xi^*)$. On the other hand, taking into account that $\partial M_K(0) = K^*$, we deduce that

$$f_{\partial\psi}(\xi, \xi^*) \geq \sup\{\pi(\xi, \xi_0^*) : \xi_0^* \in K^*\} = M_K(\xi).$$

Eventually, we get that $f_{\partial\psi}(\xi, \xi^*) \geq M_K(\xi) + I_{K^*}(\xi^*)$ for all $(\xi, \xi^*) \in X \times X^*$, which concludes the proof. □

Corollary 2.6. *Let $\psi : X \to (-\infty, \infty]$ be convex, lower semicontinuous and positively homogeneous of degree 1. Then*

$$f_{\partial\psi}(\xi, \xi^*) = \psi(\xi) + \psi^*(\xi^*) = \rho_{\partial\psi^*}(\xi, \xi^*) \qquad \text{for all } (\xi, \xi^*) \in X \times X^*. \tag{2.13}$$

Proof. To prove (2.13) we observe that

$$f_{\partial\psi} = \psi + \psi^* = (\psi + \psi^*)^* = f_{\partial\psi}^* = \rho_{\partial\psi^*}$$

where the first identity is due to Proposition 2.5, the second one to the fact that $\psi + \psi^*$ is self-dual, and the last one to (2.8). □

2.1. **Approximation of maximal monotone operators.** In the next lines, the symbol X may stand for the space X, for X^*, or for $X \times X^*$. Let f_n, $\mathsf{f} : \mathsf{X} \to (-\infty, \infty]$ be convex, proper, and l.s.c. functionals and let α_n, $\alpha \subset X \times X^*$ be maximal monotone operators. We introduce the notation

$$\Gamma\text{–}\liminf_{n\to\infty} \mathsf{f}_n(\mathsf{x}) := \min\{\liminf_{n\to\infty} \mathsf{f}_n(\mathsf{x}_n),\ \mathsf{x}_n \rightharpoonup \mathsf{x} \text{ in } \mathsf{X}\},$$
$$\Gamma\text{–}\limsup_{n\to\infty} \mathsf{f}_n(\mathsf{x}) := \min\{\limsup_{n\to\infty} \mathsf{f}_n(\mathsf{x}_n),\ \mathsf{x}_n \to \mathsf{x} \text{ in } \mathsf{X}\}.$$

These correspond to the classical Γ–liminf and Γ–limsup constructions (cf. e.g. [27]), with respect to the weak and the strong topology of X, respectively. We will use the following convergence notions, for which the reader is referred to [10]:

$$\mathsf{f}_n \xrightarrow{\mathrm{M}} \mathsf{f} \quad \Leftrightarrow \quad \Gamma\text{–}\limsup_{n\to\infty} \mathsf{f}_n \leq \mathsf{f} \leq \Gamma\text{–}\liminf_{n\to\infty} \mathsf{f}_n$$
$$\alpha_n \xrightarrow{\mathrm{g}} \alpha \quad \Leftrightarrow \quad \forall (\xi, \xi^*) \in \alpha \ \ \exists (\xi_n, \xi_n^*) \in \alpha_n \ : \ \xi_n \to \xi,\ \xi_n^* \to \xi^*.$$

where the symbol $\xrightarrow{\mathrm{M}}$ stands for *Mosco convergence* in X and $\xrightarrow{\mathrm{g}}$ is usually referred to as *graph convergence*. In particular, Mosco convergence corresponds to Γ-convergence with respect to both the strong and the weak topology of X and can be made more explicit by

$$\mathsf{f}_n \xrightarrow{\mathrm{M}} \mathsf{f} \quad \Leftrightarrow \quad \begin{cases} \forall \mathsf{x}_n \rightharpoonup \mathsf{x}, \quad \mathsf{f}(\mathsf{x}) \leq \liminf_{n\to\infty} \mathsf{f}_n(\mathsf{x}_n) \\ \forall \mathsf{x} \in \mathsf{X},\ \exists \mathsf{x}_n \to \mathsf{x} \ : \ \mathsf{f}_n(\mathsf{x}_n) \to \mathsf{f}(\mathsf{x}) \end{cases}$$

In the case of cyclically monotone operators, graph convergence is known to be equivalent to the Mosco convergence of the respective potentials (up to some normalization condition), viz. we have the following result.

Theorem 2.7 ([10, Thm. 3.66, p. 373])**.** *Let $\phi_n,\ \phi : X \to (-\infty, +\infty]$ be proper, convex and lower semicontinuous functionals. The following are equivalent:*

i) $\partial\phi_n \xrightarrow{\mathrm{g}} \partial\phi$ in $X \times X^*$ and there exist $(\xi_n, \xi_n^*) \in \partial\phi_n$ such that $\xi_n \to \xi$ in X, $\xi_n^* \to \xi^*$ in X^*, $\phi_n(\xi_n) \to \phi(\xi)$, and $(\xi, \xi^*) \in \partial\phi$;
ii) $\phi_n \xrightarrow{\mathrm{M}} \phi$ in X.

The importance of graph convergence is revealed by the following identification lemma, basically consisting of the *approximation version* of [16, Prop. 2.5, p. 27].

Lemma 2.8. *Let $\alpha_n \xrightarrow{\mathrm{g}} \alpha$, $(\xi_n, \xi_n^*) \in \alpha_n$, $\xi_n \rightharpoonup \xi$, $\xi_n^* \rightharpoonup \xi$, and $\liminf_{n\to\infty} \pi(\xi_n, \xi_n^*) \leq \pi(\xi, \xi^*)$. Then $(\xi, \xi^*) \in \alpha$.*

In order to prove this, quite classical, approximation lemma, in the particular case of cyclically monotone operators, what is actually needed is just the implication $i) \Rightarrow ii)$ in Theorem 2.7 above. Indeed, let $\partial\phi_n \xrightarrow{\mathrm{g}} \partial\phi$, $(\xi_n, \xi_n^*) \in \partial\phi_n$, with $\xi_n \rightharpoonup \xi$, $\xi_n^* \rightharpoonup \xi^*$, and $\liminf_{n\to\infty} \pi(\xi_n, \xi_n^*) \leq \pi(\xi, \xi^*)$ be given. We readily have that

$$0 \leq \phi(\xi) + \phi^*(\xi^*) - \pi(\xi, \xi^*) \leq \liminf_{n\to\infty} \big(\phi_n(\xi_n) + \phi_n^*(\xi_n^*) - \pi(\xi_n, \xi_n^*)\big) = 0$$

where we have used the fact that $\phi_n^* \xrightarrow{\mathrm{M}} \phi^*$ iff $\phi_n \xrightarrow{\mathrm{M}} \phi$ [10, Thm. 3.18, p. 295], i.e. the bicontinuity of the Legendre-Fenchel transformation with respect to the topology induced by the Mosco convergence. Therefore we conclude that $(\xi, \xi^*) \in \alpha$.

Remark 2.9. As indeed one just needs

$$\partial\phi_n \xrightarrow{\mathrm{g}} \partial\phi \quad \Rightarrow \quad \phi \leq \Gamma\text{–}\liminf_{n\to\infty} \phi_n \ \text{ in } \ V \ \text{ and } \ \phi^* \leq \Gamma\text{–}\liminf_{n\to\infty} \phi_n^* \ \text{ in } \ V^*$$

in order to check for Lemma 2.8, one may wonder if asking directly the two Γ–liminf conditions above would weaken the convergence requirements on the functionals. This is however not the case. Indeed, under some very general condition of *equi-properness* type, we have that the two separate Γ–liminf conditions are indeed equivalent to $\phi_n \xrightarrow{\mathrm{M}} \phi$ [72, Lemma 4.1] and hence entail $\partial\phi_n \xrightarrow{\mathrm{g}} \partial\phi$.

Our next aim is that of extending the above arguments to the case of noncyclically maximal monotone operators. In particular, we present an extension of Theorem 2.7 in terms of representative functions, and in particular of the Fitzpatrick and of the Penot functionals.

Theorem 2.10. *Let (α_n), α be maximal monotone operators $\alpha_n,\, \alpha : X \rightrightarrows X^*$. The following are equivalent:*

i) $\alpha_n \xrightarrow{\mathrm{g}} \alpha$ *in* $X \times X^*$,
ii) $f_\alpha \leq \Gamma\text{–}\liminf\limits_{n\to\infty} f_{\alpha_n}$ *in* $X \times X^*$,
iii) $\Gamma\text{–}\limsup_{n\to\infty} \rho_{\alpha_n} \leq \rho_\alpha$ *in* $X \times X^*$.

Exactly as above, the proof of Lemma 2.8 follows just from implication $i) \Rightarrow ii)$. We shall however give a full equivalence proof for the sake of completeness and comparison with Theorem 2.7. In particular, note that Condition $ii)$ above is weaker than $f_{\alpha_n} \xrightarrow{\mathrm{M}} f_\alpha$. That is to say that the former Theorem 2.7 does not follow directly from Theorem 2.10.

Proof. **Claim 1: $i) \Rightarrow ii)$.** Fix $(\xi_0, \xi_0^*) \in \alpha$ and let $(\xi_{0n}, \xi_{0n}^*) \in \alpha_n$ be such that $\xi_{0n} \to \xi_0$ in X and $\xi_{0n}^* \to \xi_0^*$ in X^*. Moreover, let $\xi_n \rightharpoonup \xi$ in X and $\xi_n^* \rightharpoonup \xi^*$ in X^*. We have that

$$\begin{aligned}\liminf_{n\to\infty} f_{\alpha_n}(\xi_n, \xi_n^*) &\geq \liminf_{n\to\infty} \left(\pi(\xi_{0n}, \xi_n^*) - \pi(\xi_{0n} - \xi_n, \xi_{0n}^*)\right)\\ &= \pi(\xi_0, \xi^*) - \pi(\xi_0 - \xi, \xi_0^*).\end{aligned}$$

In particular, by passing to the supremum with respect to $(\xi_0, \xi_0^*) \in \alpha$, we conclude that $f_\alpha \leq \Gamma\text{–}\liminf_{n\to\infty} f_{\alpha_n}$.

Claim 2: $i) \Rightarrow iii)$. Observe that $\alpha_n \xrightarrow{\mathrm{g}} \alpha$ implies $\alpha_n^{-1} \xrightarrow{\mathrm{g}} \alpha^{-1}$, hence Claim 1 yields $f_{\alpha^{-1}} \leq \Gamma\text{–}\liminf\limits_{n\to\infty} f_{\alpha_n^{-1}}$. By convex conjugation and taking into account (2.8) and [10, Thm. III.3.7, p. 271], we then have

$$\rho_\alpha = (f_{\alpha^{-1}})^* \geq \left(\Gamma\text{–}\liminf_{n\to\infty} f_{\alpha_n^{-1}}\right)^* = \Gamma\text{–}\limsup_{n\to\infty} \rho_{\alpha_n}.$$

Claim 3: $iii) \Rightarrow i)$. Fix $(\xi, \xi^*) \in \alpha$ and let $(\xi_n, \xi_n^*) \in \alpha_n$ fulfill $\xi_n \to \xi$ in X, $\xi_n^* \to \xi^*$ in X^*, and

$$\limsup_{n\to\infty} \rho_{\alpha_n}(\xi_n, \xi_n^*) \leq \rho_\alpha(\xi, \xi^*) = \pi(\xi, \xi^*)$$

(such sequences exist as $\Gamma\text{–}\limsup_{n\to\infty} \rho_{\alpha_n} \leq \rho_\alpha$). In particular, we have that

$$\rho_{\alpha_n}(\xi_n, \xi_n^*) < \pi(\xi, \xi^*) + \varepsilon_n$$

for some sequence $\varepsilon_n \to 0$. By exploiting the extension of the *Brønsted-Rockafellar* approximation Lemma from [44, Thm. 3.4], we have that there exist $(\tilde{\xi}_n, \tilde{\xi}_n^*) \in \alpha_n$ such that for all $n \in \mathbb{N}$

$$\|\xi_n - \tilde{\xi}_n\|^2 \leq \varepsilon_n, \quad \|\xi_n^* - \tilde{\xi}_n^*\|_*^2 \leq \varepsilon_n.$$

Then, a classical diagonal-extraction argument yields $\tilde{\xi}_n \to \xi$ in X and $\tilde{\xi}_n^* \to \xi^*$ in X^*.

Claim 4: $ii) \Rightarrow i)$. Again by convex conjugation, and (2.8), we deduce from *ii)* that

$$\rho_{\alpha^{-1}} \geq \Gamma\text{–}\limsup_{n\to\infty} \rho_{\alpha_n^{-1}}.$$

Therefore, in view of Claim 3 we have that $\alpha_n^{-1} \xrightarrow{\mathrm{g}} \alpha^{-1}$, whence $\alpha_n \xrightarrow{\mathrm{g}} \alpha$. □

3. **Setup and preliminary results**

Before stating our working assumptions in Sec. 3.2, in the upcoming Sec. 3.1 we recall all the basic definitions, and tools of measure theory and convex analysis, which we will use in the following.

3.1. **Preliminaries of measure theory, BV functions, and convex analysis.** We start with the notion of measure with values in a Banach space X, which later on will either coincide with the reflexive space X, or with $\mathbb{R}$.

Definition 3.1 (Vector measure)**.** Let (Ω, Σ) be a measurable space. A function $\mu : \Sigma \to \mathsf{X}$ is called a (Banach-space valued) vector measure, if

$$\forall (A_i)_{i\in\mathbb{N}}, A_i \in \Sigma \text{ with } (i \neq j \Rightarrow A_i \cap A_j = \emptyset) \quad \text{it holds} \quad \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) . \tag{3.1}$$

Here the convergence of the series on the right-hand side has to be understood in terms of the norm of X.

Definition 3.2 (Variation of a measure)**.** Let (Ω, Σ) be a measurable space and $\mu : \Sigma \to \mathsf{X}$ a vector measure. Then the variation of μ, denoted by $\|\mu\| : \Sigma \to [0, \infty]$, is given by

$$\|\mu\|(A) := \sup\left\{\sum_{i=1}^{\infty} \|\mu(A_i)\| \, : (A_i)_{i\in\mathbb{N}} \subset \Sigma\,, \ \bigcup_{i\in\mathbb{N}} A_i = A\,, \ \forall i \neq j : A_i \cap A_j = \emptyset\right\} \tag{3.2}$$

for all $A \in \Sigma$. If $\|\mu\|(\Omega) < \infty$ then we say μ is of bounded variation.

Indeed, $\|\mu\|$ itself is a (positive) measure on (Ω, Σ), see [30, Prop. 9, p. 3].

Definition 3.3 (Absolute continuity and singularity of measures)**.** Let (Ω, Σ) be a measurable space, $\mu : \Sigma \to \mathsf{X}$ a vector measure, and $\nu : \Sigma \to [0, \infty]$ be a (real-valued, positive) measure. We say that μ is absolutely continuous w.r.t. to ν, and write $\mu \ll \nu$, if

$$\forall A \in \Sigma : \Big[\ \nu(A) = 0 \quad \Longrightarrow \quad \mu(A) = 0\ \Big] . \tag{3.3}$$

Moreover, we say that two real-valued, positive measures μ and ν are singular, and write $\mu \perp \nu$, if there exist $B_1, B_2 \in \Sigma$ with $B_1 \cup B_2 = \Omega$ and $B_1 \cap B_2 = \emptyset$ such that

$$\forall A \in \Sigma : \qquad \mu(A) = \mu(A \cap B_1) \quad \text{and} \quad \nu(A) = \nu(A \cap B_2) \tag{3.4}$$

We recall the following generalization of the Lebesgue decomposition theorem, see e.g. [30, Thm. 9, p. 31].

Theorem 3.4 (Lebesgue decomposition theorem)**.** *Let (Ω, Σ) be a measure space, σ be a Banach space-valued measure of bounded variation and λ a real valued, positive measure. Then there exist two unique vector measures σ_{ac}, σ_{sin} on (Ω, Σ), which are of bounded variation, such that*

$$\|\sigma_{\mathrm{ac}}\| \ll \lambda\,, \ \|\sigma_{\mathrm{sin}}\| \perp \lambda \quad \textit{and} \quad \sigma = \sigma_{\mathrm{ac}} + \sigma_{\mathrm{sin}} . \tag{3.5}$$

BV **functions.** We fix here some definitions and notation concerning BV-functions on $[0, T]$ with values in a Banach space X, referring e.g. to [58] for a comprehensive introduction to the topic. We denote by $BV([0, T]; \mathsf{X})$ the space of the measurable, pointwise defined at every time $t \in [0, T]$, functions $v : [0, T] \to \mathsf{X}$ such that their *pointwise* total variation on $[0, T]$ is finite, i.e.

$$\mathrm{Var}(v; [0, T]) = \sup\left\{\sum_{m=1}^{M} \|v(t_m) - v(t_{m-1})\| \ : \ 0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T\right\} < \infty .$$

More in general, given a convex, lower semicontinuous, 1-positively homogeneous functional $\psi : X \to [0, +\infty)$, we denote by Var_ψ the induced total variation, i.e.

$$\mathrm{Var}_\psi(v; [0, T]) = \sup\left\{\sum_{m=1}^{M} \psi(v(t_m) - v(t_{m-1})) \ : \ 0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T\right\} < \infty . \tag{3.6}$$

It is well known that the distributional derivative $\mathrm{d}v$ of a curve $v \in BV(0, T; X)$ is a vector measure in $\mathcal{M}(0, T; \mathsf{X})$, where

$$\mathcal{M}(0, T; \mathsf{X}) = \{ \text{ Radon vector measures } \mu : (0, T) \to \mathsf{X} \text{ with bounded variation } \},$$

which we will endow with the weak*-topology.

Notation 3.5. Let $u \in BV([0,T];X)$. Applying Thm. 3.4 with the choices $\sigma = \mathrm{d}u$ and $\lambda = \mathcal{L}$ (where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure on $[0,T]$), we find that there exist vector measures $(\mathrm{d}u)_{\mathrm{ac}}$, $(\mathrm{d}u)_{\mathrm{sin}} \in \mathcal{M}(0,T;X)$ such that

$$\|(\mathrm{d}u)_{\mathrm{ac}}\| \ll \mathcal{L}, \quad \|(\mathrm{d}u)_{\mathrm{sin}}\| \perp \mathcal{L} \quad \text{and} \quad \mathrm{d}u = (\mathrm{d}u)_{\mathrm{ac}} + (\mathrm{d}u)_{\mathrm{sin}} \,. \tag{3.7}$$

Thanks to the Radon-Nikodým property of the *reflexive* space X, the Radon-Nikodým derivatives

$$\dot{u}_{\mathrm{ac}}(t) := \frac{(\mathrm{d}u)_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}\,,\ \dot{u}_{\mathrm{sin}}(t) := \frac{(\mathrm{d}u)_{\mathrm{sin}}}{\|(\mathrm{d}u)_{\mathrm{sin}}\|} \qquad \text{exist for a.a. } t \in (0,T). \tag{3.8}$$

For later use, we remark that for any convex, lower semicontinuous, 1-positively homogeneous $\psi : X \to [0,+\infty)$ there holds

$$\mathrm{Var}_{\psi}(u;[0,T]) = \int_0^T \psi(\dot{u}_{\mathrm{ac}}(t))\,\mathrm{d}t + \int_0^T \psi(\dot{u}_{\mathrm{sin}}(t))\|(\mathrm{d}u)_{\mathrm{sin}}\|(t)\,. \tag{3.9}$$

The recession function. Finally, we recall the concept of *recession function* (see [35, Chap. 4]). Note that the following definitions and results, which are stated in [35] for convex functions on $\mathbb{R}^m$, in fact extend to an infinite-dimensional setting, as it can be easily checked.

Definition 3.6 (Recession function)**.** Let X be a vector space and $g : \mathsf{X} \to (-\infty,\infty]$ be a convex functional. Its recession function g^∞ is defined as

$$g^\infty(z) := \sup\left\{g(y+z) - g(y) : y \in D(g)\right\}\,. \tag{3.10}$$

Trivially adapting the argument from [35, Thm. 4.70, p. 290], it can be shown that that g^∞ is positively homogeneous of degree 1 and convex. Moreover, if g is lower semicontinuous, so is g^∞. Furthermore, there holds

$$g^\infty(z) = \lim_{t\to\infty} \frac{g(y+tz) - g(y)}{t} = \sup_{t>0} \frac{g(y+tz)-g(y)}{t} \quad \text{for every } y \in D(g)\,. \tag{3.11}$$

In what follows, we will denote by f_α^∞ the recession function of the Fitzpatrick function f_α, viz.

$$f_\alpha^\infty(\xi,\xi^*) = \sup\left\{f_\alpha(\xi+x,\xi^*+x^*) - f_\alpha(x,x^*) : (x,x^*) \in D(f_\alpha)\right\}\,. \tag{3.12}$$

We now prove a useful representation formula for f_α^∞, cf. [35, Prop. 4.77, p. 294].

Lemma 3.7. *There holds*

$$f_\alpha^\infty(\xi,\xi^*) = \sup\{\langle\xi^*,\xi_0\rangle + \langle\xi_0^*,\xi\rangle \ :\ (\xi_0,\xi_0^*) \in D(\rho_{\alpha^{-1}})\} \quad \forall\,(\xi,\xi^*) \in X \times X^*\,. \tag{3.13}$$

Proof. Following the proof of [35, Prop. 4.77], from (3.11) and (2.7) we infer

$$\begin{aligned}
f_\alpha^\infty(\xi,\xi^*) &= \sup_{t>0} \frac{f_\alpha(x+t\xi,x^*+t\xi^*) - f_\alpha(x,x^*)}{t} \\
&\geq \sup_{t>0} \frac{1}{t}\Big(t\,\langle\xi^*,\xi_0\rangle + t\,\langle\xi_0^*,\xi\rangle \\
&\qquad\qquad + \langle\xi_0^*,x\rangle + \langle x^*,\xi_0\rangle - \rho_{\alpha^{-1}}(\xi_0,\xi_0^*) - f_\alpha(x,x^*)\Big) \\
&\geq \langle\xi^*,\xi_0\rangle + \langle\xi_0^*,\xi\rangle \\
&\quad + \frac{1}{t}\left(\langle\xi_0^*,x\rangle + \langle x^*,\xi_0\rangle - \rho_{\alpha^{-1}}(\xi_0,\xi_0^*) - f_\alpha(x,x^*)\right)\ \forall\,(\xi_0,\xi_0^*) \in X\times X^*,\ t>0\,.
\end{aligned}$$

In view of (2.7), we thus conclude that

$$f_\alpha^\infty(\xi,\xi^*) \geq \sup\{\langle\xi^*,\xi_0\rangle + \langle\xi_0^*,\xi\rangle \ :\ (\xi_0,\xi_0^*) \in D(\rho_{\alpha^{-1}})\}\,.$$

The converse inequality may be proved arguing along the very same lines, cf. also the proof of [35, Prop. 4.77]. □

As a direct consequence of Lemma 3.7, we have the following representation formula for the recession function of f_α, in the case α is the subdifferential of a 1-positively homogeneous potential.

Corollary 3.8. *Let $\psi : X \to \mathbb{R}$ be convex, lower semicontinuous and positively homogeneous of degree 1, and let $K^* \subset X^*$ be the associated polar set, cf.* (2.12)*. Then,*

$$f^{\infty}_{\partial\psi}(\xi, \xi^*) = \sup\{\langle \xi_0^*, \xi\rangle + \langle \xi^*, \xi_0\rangle \; : \; (\xi_0, \xi_0^*) \in X \times K^*\}\,. \tag{3.14}$$

Proof. Formula (3.14) follows from (3.13), taking into account that

$$\rho_{\alpha^{-1}} = \rho_{\partial\psi^*} = \psi + \psi^* = \psi + I_{K^*},$$

and that $D(\psi) = X$ by assumption. □

3.2. **Basic assumptions.** In what follows, we will suppose that

$$X \text{ is a reflexive Banach space} \tag{3.15}$$

and that

$$\alpha : X \rightrightarrows X^* \text{ is a maximal monotone operator with } 0 \in \alpha(0). \tag{3.α_0}$$

As for the energy functional $\mathcal{E}$, along the lines of [51] we require the following *coercivity* and *regularity* type conditions. Recall that $\partial\mathcal{E}$ denotes the *Fréchet subdifferential* of the map $u \mapsto \mathcal{E}_t(u)$, cf. (1.2).

Assumption 3.9 (Assumptions on the energy)**.** *We assume that the pair $(\mathcal{E}, \partial\mathcal{E})$ has the following properties:*

Lower semicontinuity: *The domain of $\mathcal{E}$ is of the form $D(\mathcal{E}) = [0,T] \times D$ for some $D \subset X$, and $\partial\mathcal{E} : [0,T] \times D \rightrightarrows X^*$. Furthermore, we ask that*

$$\begin{gathered} u \mapsto \mathcal{E}_t(u) \textit{ is l.s.c. for all } t \in [0,T], \quad \exists C_0 > 0 : \; \forall (t,u) \in [0,T] \times D : \mathcal{E}_t(u) \geq C_0 \textit{ and} \\ \mathrm{graph}(\partial\mathcal{E}) \textit{ is a Borel set of } [0,T] \times X \times X^*\,. \end{gathered} \tag{3.$\mathcal{E}_0$}$$

Coercivity: *Set $\mathcal{G}(u) := \sup_{t\in[0,T]} \mathcal{E}_t(u)$ for every $u \in D$. We require that*

$$u \mapsto \mathcal{G}(u) \text{ has compact sublevels.} \tag{3.$\mathcal{E}_1$}$$

Time-differentiability: *For any $u \in D$ the map $t \mapsto \mathcal{E}_t(u)$ is differentiable with derivative $\partial_t\mathcal{E}_t(u)$ and it holds*

$$\exists C_1 > 0 : \forall u \in D : \quad |\partial_t\mathcal{E}_t(u)| \leq C_1 \mathcal{E}_t(u) \tag{3.$\mathcal{E}_2$}$$

Weak closedness: *For all $t \in [0,T]$ and for all sequences $(u_n)_{n\in\mathbb{N}} \subset X$, $\xi_n \in \partial\mathcal{E}_t(u_n)$, $E_n = \mathcal{E}_t(u_n)$ and $p_n = \partial_t\mathcal{E}_t(u_n)$ with*

$$u_n \to u \textit{ in } X, \quad \xi_n \rightharpoonup \xi \textit{ in } X^*, \quad p_n \to p, \quad \textit{and} \quad E_n \to E \textit{ in } \mathbb{R}$$

it holds

$$(t,u) \in D(\partial\mathcal{E}), \; \xi \in \partial\mathcal{E}_t(u), \; p \leq \partial_t\mathcal{E}_t(u) \quad \textit{and} \quad E = \mathcal{E}_t(u)\,. \tag{3.$\mathcal{E}_3$}$$

Remark 3.10. In fact, up to a translation, we may always suppose that the constant involved in (3.$\mathcal{E}_0$) is strictly positive. As in [51], combining (3.$\mathcal{E}_2$) with the Gronwall Lemma we observe that

$$\exists\, C > 0 \qquad \forall\, (t,u) \in [0,T] \times D \qquad \mathcal{G}(u) \leq C \inf_{t\in[0,T]} \mathcal{E}_t(u). \tag{3.16}$$

Later on, Assumption 3.9 will be complemented by a suitable version of the chain rule for $\mathcal{E}$, cf. Assumption 4.4 below. As already mentioned, in order to investigate the stability properties of the doubly nonlinear equation

$$\alpha(\dot{u}(t)) + \partial\mathcal{E}_t(u(t)) \ni 0 \text{ in } X^* \quad \text{for a.a. } t \in (0,T)\,, \tag{3.17}$$

under graph convergence of α, it is essential to resort to the Fitzpatrick function f_α associated with α. In the following lines, we will therefore shed light on how (3.17) can be in fact reformulated in terms of an *energy identity* (cf. (3.20) below) featuring f_α. At first, we will confine the discussion to the case of *absolutely continuous* solutions u to (3.17).

Reformulations of (3.17) in the *absolutely continuous* case. Preliminarily, let us precisely define what we understand by an *absolutely continuous* solution to (3.17).

Definition 3.11 (Absolutely continuous solution)**.** In the framework of (3.15), (3.α_0), and (3.$\mathcal{E}_0$), we say that a curve $u \in W^{1,1}(0,T;X)$ is a solution to (3.17), if there exists $\xi \in L^1(0,T;X^*)$ with

$$\xi(t) \in (-\alpha(\dot u(t))) \cap \partial\mathcal{E}_t(u(t)) \qquad \text{for a.a.}\, t \in (0,T)\,. \tag{3.18}$$

In what follows, with a slight abuse of notation we will sometimes say that (u,ξ) is a solution to (3.17), meaning that (3.18) holds.

In Proposition 3.12, we reformulate (3.18) by means of an energy identity involving the Fitzpatrick function f_α. In the proof, a key role is played by the chain-rule condition (3.19) below on the energy $\mathcal{E}$, whereas note that not all of the conditions collected in Assumption 3.9 are needed.

Proposition 3.12 (Variational reformulation)**.** *In the framework of* (3.15)*, let* $\alpha : X \rightrightarrows X^*$ *fulfill* (3.α_0) *and suppose that* $\mathcal{E} : [0,T]\times X \to (-\infty,+\infty]$ *complies with* (3.$\mathcal{E}_0$)*,* (3.$\mathcal{E}_1$)*,* (3.$\mathcal{E}_2$)*, and the following chain rule: for every* $u \in W^{1,1}(0,T;X)$ *and* $\xi \in L^1(0,T;X^*)$ *such that*

$$\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty, \quad \xi(t) \in \partial\mathcal{E}_t(u(t)) \text{ for a.a. } t \in (0,T)\,, \quad \int_0^T f_\alpha(\dot u(t), -\xi(t))\,\mathrm{d}t < \infty,$$

(observe that, thanks to (3.$\mathcal{E}_2$)*, the first of the conditions above guarantees* $\int_0^T |\partial_t\mathcal{E}_t(u(t))|\,\mathrm{d}t < \infty$ *as well), there holds*

$$\begin{aligned} &\text{the map } t \mapsto \mathcal{E}_t(u(t)) \text{ is absolutely continuous and} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_t(u(t)) = \langle \xi(t), \dot u(t)\rangle + \partial_t\mathcal{E}_t(u(t)) \qquad \text{for a.a.}\, t \in (0,T). \end{aligned} \tag{3.19}$$

Then, the following implications hold:

(1) if $(u,\xi) \in W^{1,1}(0,T;X)\times L^1(0,T;X^*)$ *fulfills the energy identity*

$$\mathcal{E}_t(u(t)) + \int_0^t f_\alpha\left(\dot u(s), -\xi(s)\right)\,\mathrm{d}s = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \qquad \text{for all}\ t \in (0,T], \tag{3.20}$$

then (u,ξ) *is a solution to* (3.17) *in the sense of Def. 3.11.*

(2) every solution (u,ξ) *to* (3.17) *(in the sense of Def. 3.11) fulfilling*

$$\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty, \qquad \int_0^T |\,\langle \xi(t), \dot u(t)\rangle\,|\,\mathrm{d}t < \infty, \tag{3.21}$$

complies in addition with the energy identity (3.20)*.*

Observe that, for every solution (u,ξ) to (3.17), since $-\xi \in \alpha(\dot u)$ a.e. in $(0,T)$ and $0 \in \alpha(0)$, we have $\langle -\xi, \dot u\rangle \geq 0$ a.e. in $(0,T)$. Hence the second of (3.21) in fact reduces to $\int_0^T \langle -\xi, \dot u\rangle\,\mathrm{d}t < \infty$.

Proof. Let (u,ξ) fulfill (3.20). Taking into account that $f_\alpha(\dot u, -\xi) \geq 0$ a.e. in $(0,T)$ thanks to (2.9), and exploiting (3.$\mathcal{E}_2$) we gather

$$\mathcal{E}_t(u(t)) \leq \mathcal{E}_0(u(0)) + C_1\int_0^t \mathcal{E}_s(u(s))\,\mathrm{d}s \qquad \text{for all}\ t \in (0,T], \tag{3.22}$$

whence $\sup_{t\in[0,T]}\mathcal{E}_t(u(t)) < \infty$. Therefore, a fortiori (3.20) yields $f_\alpha(\dot u, -\xi) \in L^1(0,T)$. Hence the pair (u,ξ) fulfills the conditions for the chain rule (3.19), which yields for all $t \in (0,T]$

$$\begin{aligned} \int_0^t f_\alpha\left(\dot u(s), -\xi(s)\right)\,\mathrm{d}s &\leq \mathcal{E}_0(u(0)) - \mathcal{E}_t(u(t)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \\ &\leq \int_0^T \langle -\xi(s), \dot u(t)\rangle\,\mathrm{d}s\,. \end{aligned} \tag{3.23}$$

Using that f_α represents α, it is immediate to deduce from the above inequality that $-\xi(t) \in \alpha\left(\dot u(t)\right)$ for almost all $t \in (0,T)$, thus (u,ξ) is a solution to (3.17) in the sense of Def. 3.11.

Conversely, let $(u,\xi) \in W^{1,1}(0,T;X) \times L^1(0,T;X^*)$ be a solution to (3.17) (in the sense of Def. 3.11) fulfilling in addition (3.21). Then, since $f_\alpha(\dot{u},-\xi) = \langle -\xi, \dot{u}\rangle$, the chain rule (3.19) applies, yielding, for all $t \in [0,T]$, the energy identity

$$\begin{aligned}\int_0^t f_\alpha\left(\dot{u}(s),-\xi(s)\right) \mathrm{d}s &= \int_0^t \langle -\xi(s), \dot{u}(t)\rangle \, \mathrm{d}s \\ &= \mathcal{E}_0(u(0)) - \mathcal{E}_t(u(t)) + \int_0^T \partial_t \mathcal{E}_s(u(s)), \xi(s))\, \mathrm{d}s\,.\end{aligned}$$

□

Remark 3.13. A few comments on Proposition 3.12 are in order.

(1) It is not difficult to check that in Proposition 3.12 the Fitzpatrick function f_α could be replaced by *any* representative functional for α.
(2) Observe that, in the chain of inequalities (3.23) leading to the proof of part (1) of Proposition 3.12, it is in principle necessary for (3.20) and for the chain rule (3.19) to hold as inequalities, only. The proof of part (2) requires (3.19) to hold as an equality, instead.

4. **Main results**

Before stating Thm. 4.5, let us precise our hypothesis on the sequence (α_n) of maximal monotone operators.

Assumption 4.1. *Let $\alpha_n : X \rightrightarrows X^*$ fulfill $(3.\alpha_0)$ for all $n \in \mathbb{N}$ and*

$$\begin{aligned}\exists\, c_1, c_2, c_3 > 0, \quad p \geq 1, \quad q > 1 \qquad & \forall\, n \in \mathbb{N} \quad \forall (x,y) \in \alpha_n \ : \\ & \langle y, x\rangle \geq c_1\|x\|^p + c_2\|y\|_*^q - c_3\,.\end{aligned} \tag{3.α_1}$$

Furthermore, there exists $\alpha : X \rightrightarrows X^*$ fulfilling $(3.\alpha_0)$ such that $\alpha_n \xrightarrow{\text{g}} \alpha$.

Remark 4.2. Combining $(3.\alpha_1)$ with the graph convergence of (α_n) to α, it is immediate to conclude

$$\langle y, x\rangle \geq c_1\|x\|^p + c_2\|y\|_*^q - c_3 \qquad \text{for all } (x,y) \in \alpha\,. \tag{4.1}$$

The following example guarantees that our analysis encompasses the $p_n \to 1$ vanishing-viscosity limit.

Example 4.3. Let $(p_n) \subset [1,+\infty)$ fulfill $p_n \downarrow 1$ as $n \to \infty$, and let us set

$$\psi_n(x) = \frac{1}{p_n}\|x\|^{p_n}, \qquad \alpha_n = \partial\psi_n : X \rightrightarrows X^*\,.$$

Clearly, (ψ_n) Mosco-converges to $\psi(x) = \|x\|$, hence (α_n) converges in the sense of graphs to $\alpha = \partial\psi$. Observe that $\psi_n^*(y) = \frac{1}{q_n}\|y\|_*^{q_n}$ with $q_n = p_n/(p_n-1) \in [2,\infty]$ for all $n \in \mathbb{N}$, and that

$$\langle y, x\rangle = \frac{1}{p_n}\|x\|^{p_n} + \frac{1}{q_n}\|y\|_*^{q_n} = \|x\|^{p_n} = \|y\|_*^{q_n} \quad \text{for all } (x,y) \in \alpha_n\,.$$

Therefore, Assumption 4.1 is satisfied.

The main result of this section addresses the passage to the limit as $n \to \infty$ in the doubly nonlinear equations

$$\alpha_n(\dot{u}(t)) + \partial\mathcal{E}_t(u(t)) \ni 0 \text{ in } X^* \quad \text{for a.a. } t \in (0,T)\,. \tag{4.2}$$

In particular, we will assume to be given a sequence (u_n) of absolutely continuous solutions to (4.2) and we will show that, if the sequence (α_n) complies with Assumption 4.1, up to a subsequence (u_n) converges to a curve u fulfilling a suitable generalized formulation of (3.17).

Observe that, $(3.\alpha_1)$ in principle only allows for a bound of the type $\|\dot{u}_n\|_{L^1(0,T;X)} \leq C$. That it why, we can only expect a $BV([0,T];X)$-regularity for the limiting curve u, and (3.17) has to be weakly formulated accordingly. This will be done through an energy *inequality* akin to (3.20), cf. (4.5) below. Therein, suitable replacements of the "time-derivative" of u are suitably handled in terms of the Fitzpatrick function f_α and of its recession function f_α^∞ (cf. Definition 3.6), and of the absolutely continuous and singular parts of the Radon derivative $\mathrm{d}u$ of u. Having in mind the role of the chain rule (3.19) relating (3.17) and the energy identity (3.20), it is to be expected that a suitable BV *version* of (3.19) will play a relevant role. We state it in the following:

Assumption 4.4. *Let $u \in BV([0,T];X)$ and $\xi \in L^1(0,T;X^*)$ fulfill*

$$\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty, \quad \xi(t) \in \partial\mathcal{E}_t(u(t)) \text{ for a.a. } t \in (0,T)\,, \quad \int_0^T f_\alpha(\dot u(t), -\xi(t))\,\mathrm{d}t < \infty,$$

and suppose that the map $t \mapsto \mathcal{E}_t(u(t))$ is almost everywhere equal on $(0,T)$ to a function $E \in BV([0,T])$. Furthermore let $\mathrm{d}u$ and $\mathrm{d}E$ denote the Radon derivatives of u and E.

Then, for almost all Lebesgue points t_0 of the absolutely continuous parts $\dot u_{\mathrm{ac}}$ and $\dot E_{\mathrm{ac}}$ of $\mathrm{d}u$ and $\mathrm{d}E$ there holds

$$\dot E_{\mathrm{ac}}(t_0) \geq \langle \xi(t_0), \dot u_{\mathrm{ac}}(t_0)\rangle + \partial_t \mathcal{E}_{t_0}(u(t_0)) \quad \text{for all } \xi(t_0) \in \partial\mathcal{E}_{t_0}(u(t_0))\,. \tag{3.$\mathcal{E}_4$}$$

Observe that, since X has the Radon-Nikodým property, the set of Lebesgue points of $\dot u_{\mathrm{ac}}$ and $\dot E_{\mathrm{ac}}$ has full Lebesgue measure in $(0,T)$.

As it will be clear from the proof of Thm. 4.5 below, Assumption 4.4 does not only provide a motivation for the energy inequality (4.5), but it also has a key role in the proof of the passage to the limit as $n \to \infty$ in (4.2).

Theorem 4.5. *Assume* (3.15)*. Let $\alpha_n,\, \alpha : X \rightrightarrows X^*$ fulfill Assumption* 4.1*, and suppose that $\mathcal{E} : [0,T] \times X \to (-\infty, +\infty]$ complies with Assumptions* 3.9 *and* 4.4*. Let us consider a sequence $(u_0^n) \subset D$ of initial data such that*

$$u_0^n \rightharpoonup u_0 \quad \text{in } X, \qquad \mathcal{E}_0(u_0^n) \to \mathcal{E}_0(u_0), \tag{4.3}$$

and let $(u_n, \xi_n) \subset W^{1,1}(0,T;X) \times L^1(0,T;X^*)$ be solutions to (4.2) *in the sense of Definition 3.11, fulfilling the initial conditions $u_n(0) = u_0^n$. Suppose that, in addition, for all $n \in \mathbb{N}$ the functions (u_n, ξ_n) comply with the energy identity* (3.20)*.*

Then, there exist functions $u \in BV([0,T];X)$ and $\xi \in L^q(0,T;X^*)$ (with $q > 1$ from (3.α_1)) satisfying $u(0) = u_0$, $\xi(t) \in \partial\mathcal{E}_t(u(t))$ for almost all $t \in (0,T)$, and such that up to a (not relabeled) subsequence

$$u_n(t) \to u(t) \ \forall t \in [0,T]\,, \ \ \mathrm{d}u_n = (\dot u_n)_{\mathrm{ac}} \cdot \mathcal{L}|_{[0,T]} \overset{*}{\rightharpoonup} \mathrm{d}u \in \mathcal{M}(0,T;X), \tag{4.4}$$

and (u, ξ) satisfies the energy inequality

$$\begin{aligned} \mathcal{E}_t(u(t)) + \int_0^t f_\alpha\left(\dot u_{\mathrm{ac}}(s), -\xi(s)\right)\mathrm{d}s + \int_0^t f_\alpha^\infty\left(\dot u_{\sin}(s), 0\right)\|(\,\mathrm{d}u)_{\sin}\|(s) \\ \leq \mathcal{E}_0(u(0)) + \int_0^t \partial_t \mathcal{E}_s(u(s))\,\mathrm{d}s \qquad \text{for all } t \in [0,T], \end{aligned} \tag{4.5}$$

as well as

$$\xi(t) \in (-\alpha(\dot u_{\mathrm{ac}}(t))) \cap \partial\mathcal{E}_t(u(t)) \qquad \text{for a.a.}\, t \in (0,T)\,. \tag{4.6}$$

Furthermore, there exists $E \in BV([0,T])$ such that

$$E(t) = \mathcal{E}_t(u(t)) \quad \text{for a.a.}\, t \in (0,T), \qquad E(t) \geq \mathcal{E}_t(u(t)) \quad \text{for all}\, t \in [0,T], \tag{4.7}$$

and we have the pointwise energy identity

$$\dot E_{\mathrm{ac}}(t) + f_\alpha\left(\dot u_{\mathrm{ac}}(t), -\xi(t)\right) = \partial_t \mathcal{E}_t(u(t)) \qquad \text{for a.a.}\, t \in (0,T). \tag{4.8}$$

Remark 4.6. In view of Proposition 3.12, a sufficient condition for functions (u_n, ξ_n) solving (4.2) to comply with the energy identity (3.20), is that they fulfill

$$\sup_{t\in(0,T)} \mathcal{E}_t(u_n(t)) < \infty \qquad \text{and} \qquad \langle -\xi_n, \dot u_n\rangle \in L^1(0,T).$$

This, provided that the *absolutely continuous* version (3.19) of the chain rule holds.

In Section 4.2, we will discuss some sufficient conditions on $\mathcal{E}$ for both the chain rule (3.19) and its BV-version of Assumption 4.4 to hold.

4.1. **Further results.** We conclude this section with some results which shed light on the interpretation of the energy identities (4.5) and (4.8) satisfied by the pair (u, ξ). More precisely:

- Proposition 4.7 focuses on the case in which we have the additional information that u is absolutely continuous. For instance, this is granted whenever u occurs as limiting curve of a sequence $(u_n) \subset W^{1,1}(0, T : X)$ of solutions to the differential inclusions (4.2), driven by operators (α_n) which fulfill a stronger version of condition $(3.\alpha_1)$, cf. Thm. 4.8 ahead.
- In Proposition 4.9 we address the special case in which $\alpha = \partial\psi$, with $\psi : X \to [0, +\infty)$ a convex, lower semicontinuous, and 1-homogeneous dissipation potential. We show that in this case any $u \in BV([0, T]; X)$ complying with the energy inequality (4.5) is a *local solution* (cf. [49, 50]) to the rate-independent system $(X, \mathcal{E}, \psi)$.

The absolutely continuous case. Under a slightly stronger version of the chain rule of Assumption 4.4, Proposition 4.7 shows that, if in addition we have that the curve u is absolutely continuous on $(0, T)$, then $f_\alpha^\infty(\dot{u}_{\rm sin}(t), 0) = 0$ for $\|(\mathrm{d}u)_{\rm sin}\|$-a.a. $t \in (0, T)$, and (4.5) holds on every sub-interval $[s, t] \subset [0, T]$. Furthermore, the pair (u, ξ) solves (3.17) in the sense of Definition 3.11, cf. (4.11) below.

Proposition 4.7. *In the framework of* (3.15)*, let* $\alpha : X \rightrightarrows X^*$ *fulfill* $(3.\alpha_0)$*, and* $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ *comply with Assumption* 3.9 *and with the following chain rule: for every* $u \in W^{1,1}(0, T; X)$ *and* $\xi \in L^1(0, T; X^*)$ *such that*

$$\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty, \quad \xi(t) \in \partial\mathcal{E}_t(u(t)) \text{ for a.a. } t \in (0, T)\,, \quad \int_0^T f_\alpha(\dot{u}(t), -\xi(t))\,\mathrm{d}t < \infty,$$

then

$$(\mathrm{d}u)_{\rm sin} = 0 \quad \Rightarrow \quad (\mathrm{d}E)_{\rm sin} = 0 \tag{4.9}$$

and the chain rule inequality $(3.\mathcal{E}_4)$ *holds.*

Let $(u, \xi, E) \in BV([0, T]; X) \times L^1(0, T; X^*) \times BV([0, T])$ *fulfill* (4.7) *and* (4.8)*. Suppose in addition that* $u \in W^{1,1}(0, T; X)$*. Then,*

$$E \in W^{1,1}(0, T). \tag{4.10}$$

Furthermore, the pair (u, ξ) *fulfills*

$$-\xi(t) \in \alpha(\dot{u}(t)) \qquad \text{for a.a.}\, t \in (0, T), \tag{4.11}$$

and there holds the improved *energy inequality*

$$\begin{aligned} &\mathcal{E}_t(u(t)) + \int_s^t f_\alpha(\dot{u}(r), -\xi(r))\,\mathrm{d}r \\ &\qquad \le \mathcal{E}_s(u(s)) + \int_s^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r \text{ for all } t \in (0, T], \text{ for a.a. } s \in (0, t) \text{ and for } s = 0. \end{aligned} \tag{4.12}$$

Finally, if $\mathcal{E}$ *also fulfills the enhanced chain rule* (3.19)*, then* (4.12) *holds as an* equality *for every* $0 \le s \le t \le T$*.*

Proof. Since $u \in W^{1,1}(0, T; X)$, its distributional derivative $\mathrm{d}u$ has zero singular part, viz. $\mathrm{d}u = \dot{u}_{\rm ac}\mathcal{L}$. Then, it follows from (4.9) that $\mathrm{d}E = \dot{E}_{\rm ac}\mathcal{L}$, viz. E is absolutely continuous. Therefore, (4.8) becomes

$$\dot{E}(t) + f_\alpha(\dot{u}(t), -\xi(t)) = \partial_t\mathcal{E}_t(u(t)) \qquad \text{for a.a.}\, t \in (0, T). \tag{4.13}$$

Now, combining this with the chain rule inequality $(3.\mathcal{E}_4)$, we conclude that $f_\alpha(\dot{u}(t), -\xi(t)) \le \langle -\xi(t), \dot{u}(t)\rangle$ for almost all $t \in (0, T)$, hence (4.11) holds. Then, to prove (4.12) we integrate (4.13), thus obtaining

$$E(t) + \int_s^t f_\alpha(\dot{u}(r), -\xi(r))\,\mathrm{d}r = E(s) + \int_s^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r \qquad \text{for all } 0 \le s \le t \le T, \tag{4.14}$$

and we use (4.7).

If moreover $\mathcal{E}$ complies with the chain rule (3.19), then $E(t) = \mathcal{E}_t(u(t))$ for all $t \in [0,T]$, since both functions $t \mapsto E(t)$ and $t \mapsto \mathcal{E}_t(u(t))$ are continuous on $[0,T]$ and coincide on a set of full Lebesgue measure. Therefore from (4.14) we get (4.12) for $t \mapsto \mathcal{E}_t(u(t))$. This concludes the proof. □

As a straightforward consequence of Prop. 4.7 we have the following result, showing that, under a stronger coercivity assumption on the sequence of maximal monotone operators (α_n) (cf. (4.15) below), any sequence (u_n) of solutions to (4.2) converges up to a subsequence to a curve complying with (4.10)–(4.12). In particular, observe that, unlike in $(3.\alpha_1)$, in (4.15) we do not allow the "degenerate" value 1 for exponent p. Indeed, Theorem 4.8 below for instance applies to a sequence of operators $\alpha_n = \partial\psi_n$, with $\psi_n(v) = 1/p_n\|v\|^{p_n}$ and $p_n \downarrow p > 1$ as $n \to \infty$. In this way, we obtain a stability result for doubly nonlinear differential inclusions driven by *viscous* dissipation potentials, which generalizes the results in [1, Thms. 3.1, 3.2].

Theorem 4.8. *In the frame of* (3.15)*, suppose that* $\mathcal{E} : [0,T] \times X \to (-\infty, +\infty]$ *complies with Assumptions* 3.9 *and* 4.4*. Let* $\alpha_n : X \rightrightarrows X^*$ *fulfill* $(3.\alpha_0)$ *for all* $n \in \mathbb{N}$ *and*

$$\begin{aligned} \exists\, c_1, c_2, c_3 > 0, \quad p > 1, \quad q > 1 \qquad & \forall\, n \in \mathbb{N} \quad \forall (x,y) \in \alpha_n : \\ & \langle y, x \rangle \geq c_1\|x\|^p + c_2\|y\|_*^q - c_3\,. \end{aligned} \tag{4.15}$$

Suppose that there exists $\alpha : X \rightrightarrows X^*$ *fulfilling* $(3.\alpha_0)$ *such that* $\alpha_n \xrightarrow{\mathrm{g}} \alpha$*. Let* $(u_0^n) \subset D$ *be a sequence of initial data fulfilling* (4.3) *and let* $(u_n, \xi_n) \subset W^{1,1}(0,T;X) \times L^1(0,T;X^*)$ *be solutions to* (4.2)*, fulfilling* $u_n(0) = u_0^n$ *and* (3.21) *for every* $n \in \mathbb{N}$*.*

Then, there exists $u \in W^{1,p}(0,T;X)$ *with* $u(0) = u_0$ *such that up to a (not relabeled) subsequence*

$$u_n(t) \to u(t) \quad \text{for all } t \in [0,T]\,, \quad u_n \rightharpoonup u \quad \text{in } W^{1,p}(0,T;X), \tag{4.16}$$

and there exists $\xi \in L^q(0,T;X^*)$ *such that the pair* (u,ξ) *is a solution to* (3.17) *in the sense of Definition 3.11, fulfilling the improved energy inequality* (4.12)*.*

The *proof* is outlined at the end of Sec. 5.

The rate-independent case. Let us now focus on the case in which

$$\alpha = \partial\psi \text{ with } \psi : X \to [0,+\infty) \text{ convex, lower semicontinuous and 1-positively homogeneous} \tag{4.17}$$

with associated polar set $K^* \subset X^*$. In this case, the energy inequality (4.5) rephrases in a more explicit way.

Proposition 4.9. *Assume* (3.15)*. Let* α *fulfill* (4.17) *and let* $(u,\xi) \in BV(0,T;X) \times L^1(0,T:X^*)$ *satisfy the energy inequality* (4.5)*. Then,* (u,ξ) *fulfill*

$$-\,\xi(t) \in K^* \quad \text{for a.a. } t \in (0,T), \tag{4.18}$$

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_\psi(u;[0,t]) \leq \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]\,, \tag{4.19}$$

with Var_ψ *from* (3.6)*.*

In the frame of rate-independent evolution, (4.18) is interpreted as a *local stability condition*, while the energy inequality (4.19) balances the stored energy $\mathcal{E}_t(u(t))$ and the dissipated energy $\mathrm{Var}_\psi(u;[0,t])$, with the initial energy and the work of the external forces $\int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s$. In fact, the local stability (4.18) and the energy inequality (4.19) yield (a slightly weaker version of) the notion of *local solution* to the rate-independent system $(X,\mathcal{E},\psi)$ from [49, 50]. Therein, it was observed that this concept is the weakest among all notions of rate-independent evolution, in that it yields the least precise information on the behavior of the solution at jump points. On the other hand, *local solutions* arise in the limit of a very broad class of approximations of rate-independent systems. This is in the same spirit as the stability results of this work. In particular, notice that the maximal monotone operators α_n converging in the sense of graphs to $\alpha = \partial\psi$ need not be cyclically monotone. An example in this direction in the plane $X = \mathbb{R}^2$ is given by the graphs $\alpha_n = \partial\psi + (1/n)Q$ where Q is a rotation of $\pi/2$. In this case $\alpha_n \xrightarrow{\mathrm{g}} \partial\psi$ but each α_n is noncyclic.

We now proceed with the

Proof of Proposition 4.9. Let $(u,\xi) \in BV(0,T;X) \times L^1(0,T:X^*)$ fulfill (4.5). Now, in view of Proposition 2.5 and of formula (2.12b), we have

$$f_\alpha\left(\dot u_{\mathrm{ac}}(t), -\xi(t)\right) = \psi(\dot u_{\mathrm{ac}}(t)) + \psi^*(-\xi(t)) = \psi(\dot u_{\mathrm{ac}}(t)) + I_{K^*}(-\xi(t)) \quad \text{for a.a. } t \in (0,T)\,. \tag{4.20}$$

Furthermore, we have that

$$f_\alpha^\infty\left(\dot u_{\sin}(t), 0\right) \geq \sup_{\xi_0^* \in K^*} \langle \xi_0^*, \dot u_{\sin}(t)\rangle = \psi(\dot u_{\sin}(t)) \quad \text{for } \|(\,\mathrm{d}u)_{\sin}\|\text{-a.a. } t \in (0,T) \tag{4.21}$$

where the first inequality is due to (3.14) and the second identity to (2.12d).

Then, taking into account formula (3.9) for Var_ψ, (4.5) yields

$$\mathcal{E}_t(u(t)) + \mathrm{Var}_\psi(u;[0,t]) + \int_0^t I_{K^*}(-\xi(s))\,\mathrm{d}s \leq \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_s(u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T],$$

which is equivalent to (4.18)–(4.19). □

4.2. **Sufficient conditions for closedness and chain rule.** Following [51], we now show that conditions of λ-convexity type on the energy functional $\mathcal{E}$ ensure the validity of the closedness property (3.$\mathcal{E}_3$), of the chain rules (3.19), (3.$\mathcal{E}_4$), and of property (4.9).

More precisely, in [51, Sec. 2] the following *subdifferentiability property* was introduced.

Definition 4.10. Let $\mathcal{E} : [0,T] \times X \to (-\infty,+\infty]$ fulfill (3.$\mathcal{E}_0$). For every $R > 0$, set

$$D_R = \{u \in D \,:\, \mathcal{G}(u) \leq R\}\,.$$

We say that $\mathcal{E}$ is *uniformly subdifferentiable* (w.r.t. the variable u) if for all $R > 0$ there exists a *modulus of subdifferentiability* $\omega^R : [0,T] \times D_R \times D_R \to [0,+\infty)$ such that for all $t \in [0,T]$:

$$\begin{gathered} \omega_t^R(u,u) = 0 \text{ for every } u \in D_R, \\ \text{the map } (t,u,v) \mapsto \omega_t^R(u,v) \text{ is upper semicontinuous, and} \\ \mathcal{E}_t(v) - \mathcal{E}_t(u) - \langle \xi, v-u\rangle \geq -\omega_t^R(u,v)\|v-u\| \quad \text{for all } u,\, v \in D_R \text{ and } \xi \in \partial\mathcal{E}_t(u). \end{gathered} \tag{4.22}$$

It was shown in [51, Sec. 2] that, a sufficient condition for (4.22) is that the map $u \mapsto \mathcal{E}_t(u)$ is λ*-convex* uniformly in $t \in [0,T]$, namely

$$\begin{aligned} &\exists\,\lambda \in \mathbb{R} \quad \forall\, t \in [0,T]\ \forall\, u_0,\, u_1 \in D\ \forall\,\theta \in [0,1]\ : \\ &\qquad \mathcal{E}_t((1-\theta)u_0 + \theta u_1) \leq (1-\theta)\mathcal{E}_t(u_0) + \theta\mathcal{E}_t(u_1) - \frac{\lambda}{2}\theta(1-\theta)\|u_0 - u_1\|^2\,. \end{aligned} \tag{4.23}$$

Suitable perturbations of λ-convex functionals also fulfill the closedness and the chain rule properties: we refer to [66, 51, 65] for more details and explicit examples.

We have the following

Proposition 4.11. *Let* $\mathcal{E} : [0,T] \times X \to (-\infty,+\infty]$ *fulfill* (3.$\mathcal{E}_0$), (3.$\mathcal{E}_2$), *and the* uniform subdifferentiability *condition* (4.22). *Then,* $\mathcal{E}$ *complies with the closedness condition* (3.$\mathcal{E}_3$), *with the chain rules* (3.19) *and* (3.$\mathcal{E}_4$), *and with property* (4.9).

Proof. In [51, Prop. 2.4], it was proved that condition (4.22) implies (3.$\mathcal{E}_3$) and (3.19). The validity of (3.$\mathcal{E}_4$) and (4.9) can be checked trivially adapting the arguments developed for the proof of [51, Prop. 2.4], to which the reader is referred. □

4.3. **Examples of quasistatic limits.** Our approach to the approximation of doubly nonlinear evolution equations in particular allows us to discuss quasistatic limits of dynamical problems. Indeed, the flexibility in the choice of the approximating graphs α_n, possibly noncyclic monotone, makes it possible to take rate-independent limits of Hamiltonian systems. We shall provide here some examples of ODEs and PDEs that can be reformulated within our frame.

Let us start by considering the case of a *nonlinearly damped oscillator*. In particular, let $q = q(t)$ represent the set of generalized coordinates of the system, M be the mass matrix, and $U = U(q)$ its smooth and coercive potential energy. Assume moreover that the system dissipates energy in terms of a positively 1-homogeneous and nondegenerate dissipation potential $D = D(\dot{q})$. By rescaling time t as εt, the quasistatic limit of the system corresponds to the limit as $\varepsilon \to 0$ in the equation

$$\varepsilon^2 M\ddot{q} + \partial D(\dot{q}) + \nabla U(q) \ni 0. \tag{4.24}$$

The latter can be rephrased as a single doubly nonlinear Hamiltonian system in the pair $v = (p, q)$, by introducing the Hamiltonian $H(p, q) = U(p) + q{\cdot}M^{-1}q/2$, the symplectic operator

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the dissipation potential $\widehat{D}(\dot{p}, \dot{q}) = D(\dot{p})$. Then, (4.24) reads

$$\partial\widehat{D}(\dot{p}, \dot{q}) + \varepsilon J(\dot{p}, \dot{q}) + \nabla H(p, q) \ni (0, 0), \tag{4.25}$$

which can be equivalently rewritten as

$$\begin{aligned} \partial D(\dot{p}) + \varepsilon\dot{q} + \nabla U(p) &\ni 0, \\ -\varepsilon\dot{p} + M^{-1}q &= 0, \end{aligned}$$

Taking the quasistatic limit $\varepsilon \to 0$ in relation (4.25) requires to deal with the graphs $\alpha_\varepsilon = \partial\widehat{D} + \varepsilon J$, which are noncyclic monotone for all $\varepsilon > 0$. Apart from the coercivity assumption $(3.\alpha_1)$ (which can however be relaxed in this case), this situation fits into our theory. In particular, solution trajectories to the dynamic problem (4.24) converge to solutions of the corresponding quasistatic limit. By generalizing the choice of the graphs α_ε, convergence can be obtained for a large class of different approximating problems.

The nonlinear oscillator example can be turned into a first PDE example by considering the nonlinearly damped semilinear wave equation

$$\varepsilon^2 u_{tt} + \partial D(u_t) - \Delta u + f(u) = 0. \tag{4.26}$$

This is to be posed in the cylinder $\Omega \times (0, T)$ for some smoothly bounded open set $\Omega \subset \mathbb{R}^n$, along with the positively 1-homogeneous and nondegenerate dissipation potential D, the smooth and polynomially bounded function f, and suitable initial and homogeneous Dirichlet boundary conditions (for simplicity). Equation (4.26) can be variationally reformulated in terms of a first-order system as

$$\partial\mathcal{D}(u_t, v_t) + \varepsilon\mathcal{J}(u_t, v_t) + \partial\mathcal{H}(u, v) \ni (0, 0) \quad \text{in } \mathcal{U}^*{\times}\mathcal{V}^* \quad \text{for a.a. } t \in (0, T), \tag{4.27}$$

where $\mathcal{U} = H^1_0(\Omega)$, $\mathcal{V} = L^2(\Omega)$, the functionals $\mathcal{D} : \mathcal{V}^2 \to [0, \infty]$, and $\mathcal{H} : \mathcal{U}{\times}\mathcal{V} \to (-\infty, \infty]$ are given by

$$\mathcal{D}(u_t, v_t) = \int_\Omega D(u_t)\,\mathrm{d}x, \quad \mathcal{H}(u, v) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \widehat{f}(u) + \frac{1}{2}|v|^2\right)\mathrm{d}x$$

for $\widehat{f}' = f$, and $\mathcal{J}(u_t, v_t)(x) = J(u_t(x), v_t(x))$ for almost every $x \in \Omega$. Equation (4.27) fits in our frame along with the choice $\alpha_\varepsilon = \partial\mathcal{D} + \varepsilon\mathcal{J}$, which are noncyclic monotone for all $\varepsilon > 0$. In particular, owing to our analysis we can take the quasistatic limit $\varepsilon \to 0$ in the latter (again by suitably circumventing the lack of coercivity, which is inessential here).

Let us now provide a second PDE example by considering the quasistatic limit in *linearized elastoplasticity* with linear kinematic hardening [40]. We let $\Omega \subset \mathbb{R}^3$ be the reference configuration of an elastoplastic body which is subject to a displacement $u : \Omega \to \mathbb{R}^3$ and a *plastic strain*

$p:\Omega\to\mathbb{R}^{3\times 3}_{\rm dev}$ (traceless or *deviatoric* symmetric 3×3 tensors). Then, the evolution of the elasto-plastic medium is described by the system of the (time-rescaled) momentum balance (in $\mathbb{R}^3$) and constitutive equation (in $\mathbb{R}^{3\times 3}_{\rm dev}$) as

$$\varepsilon^2\rho u_{tt} - \nabla\cdot(\mathbb{C}(\varepsilon(u){-}p)) = b,$$
$$\partial D(p_t) + \mathbb{H}p = \mathbb{C}(\varepsilon(u){-}p)$$

in $\Omega\times(0,T)$, where $\rho=\rho(x)$ stands for the material density, $\mathbb{C}$ is the elasticity tensor (symmetric, positive definite), $\varepsilon(u) = (\nabla u{+}\nabla u^\top)/2$ is the symmetrized strain gradient, $b=b(t,x)$ denotes some body force density, $\mathbb{H}$ is the hardening tensor, and D is a positively 1-homogeneous and nondegenerate dissipation potential. The choice $D(p_t) = R|p_t|$ for some $R>0$ corresponds to the classical *Von Mises* plasticity. We shall close the latter elastoplasticity system by imposing homogeneous Dirichlet conditions on u and no-traction conditions at the boundary (for simplicity). Then, the system can be recast in the form of a first-order system by augmenting the variables, including the momentum $v_t=\rho u_t$. In particular, we can variationally reformulate the system as

$$\partial\mathcal{D}(u_t,v_t,p_t) + \varepsilon\mathcal{J}(u_t,v_t,p_t) + \partial\mathcal{H}(u,v,p) \ni (b,0,0) \quad \text{in } \mathcal{U}^*{\times}\mathcal{V}^*{\times}\mathcal{P}^* \quad \text{for a.a. } t\in(0,T), \tag{4.28}$$

where now the spaces are defined as $\mathcal{U}=\{u\in H^1(\Omega;\mathbb{R}^3) : u=0 \text{ in } \partial\Omega\}$, $\mathcal{V}=L^2(\Omega;\mathbb{R}^3)$, $\mathcal{P}=L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev})$. The functionals and the operator are given by

$$\mathcal{D}(u_t,v_t,p_t) = \int_\Omega D(p_t)\,\mathrm{d}x \quad \forall p_t\in L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}),$$
$$\mathcal{H}(u,v,p) = \int_\Omega\left(\frac12(\varepsilon(u){-}p){:}\mathbb{C}(\varepsilon(u){-}p) + \frac12 p{:}\mathbb{H}p + \frac{1}{2\rho}|v|^2\right)\mathrm{d}x \quad \forall(u,v,p)\in\mathcal{U}{\times}\mathcal{V}{\times}\mathcal{P},$$
$$\mathcal{J}(u_t,v_t,p_t) = \begin{pmatrix} v_t\\ -u_t\\ 0\end{pmatrix} \quad \forall(u,v,p)\in\mathcal{U}{\times}\mathcal{V}{\times}\mathcal{P}.$$

Once again, the operators $\alpha_\varepsilon = \partial\mathcal{D}+\varepsilon\mathcal{J}$ are noncyclic monotone for all $\varepsilon>0$. In particular, our analysis is suited in order to analyze the quasistatic limit $\varepsilon\to0$ in the elastoplastic system (4.28). This clearly distinguishes our frame from the former variational principle from [73], which is of no use in the dynamical case.

5. **Proof of Theorem 4.5**

Outline. Our starting point is the fact that, the functions (u_n,ξ_n) fulfill for every $n\in\mathbb{N}$ the energy identity

$$\mathcal{E}_t(u_n(t)) + \int_0^t f_{\alpha_n}\left(\dot u_n(s),-\xi_n(s)\right)\mathrm{d}s = \mathcal{E}_0(u_0^n) + \int_0^t \partial_t\mathcal{E}_s(u_n(s))\,\mathrm{d}s \qquad \text{for all } t\in(0,T]. \tag{5.1}$$

From (5.1), we will deduce a priori estimates on the sequence (u_n,ξ_n). Relying on well-known strong and weak compactness results, we will then prove the convergence (up to a subsequence) of (u_n,ξ_n) to a limit pair $(u,\hat\xi)$. Hence we will pass to the limit as $n\to\infty$ in (5.1), following the lines of the proof of [51, Thm. 4.4]. Namely, we will combine the finite-dimensional lower semicontinuity theorem [35, Theorem 5.27], with tools from infinite-dimensional Young measure theory (see Appendix A for some basic recaps), and refined selection arguments mutuated from the proof of [51, Thm. 4.4]. Such arguments will yield the existence of a function $\xi\in L^1(0,T;X^*)$ such that the pair (u,ξ) fulfill the energy inequality (4.5).

Notation 5.1. Hereafter we will denote by the symbols C, C' various positive constants, which may change from line to line, only depending on known quantities and in particular independent of $n\in\mathbb{N}$. We will also use the place-holders

$$E_n(t) := \mathcal{E}_t(u_n(t)), \qquad P_n(t) := \partial_t\mathcal{E}_t(u_n(t)). \tag{5.2}$$

Step 1 - A priori estimates and compactness: It follows from (5.1) and (3.$\mathcal{E}_2$) (cf. also estimate (3.22)) that $E_n(t) \leq E_n(0) + C_1 \int_0^t E_n(s)\,\mathrm{d}s$ for all $t \in [0,T]$. Since $\sup_{n\in\mathbb{N}} E_n(0) \leq C$ by (4.3), applying the Gronwall Lemma we deduce $\sup_{t\in[0,T]}\{E_n(t) : t \in [0,T]\} \leq C$. Therefore, in view of assumption (3.$\mathcal{E}_2$) and property (3.16), we conclude that

$$\exists\, C>0 \ \ \forall\, n \in \mathbb{N}: \quad \sup_{t\in[0,T]} \left(\mathcal{G}(u_n(t)) + |P_n(t)|\right) \leq C. \tag{5.3}$$

Thanks to (3.$\mathcal{E}_1$) we then infer that

$$\exists\, K \Subset X \quad \forall\, n \in \mathbb{N}\, \forall\, t \in [0,T]: \quad u_n(t) \in K. \tag{5.4}$$

Then, taking into account that $f_{\alpha_n}(\dot{u}_n, -\xi_n) \geq 0$ a.e. in $(0,T)$ in view of (2.9), (5.1) yields

$$\exists\, C>0 \ \ \forall\, n \in \mathbb{N}: \qquad \|f_{\alpha_n}(\dot{u}_n, -\xi_n)\|_{L^1(0,T)} \leq C. \tag{5.5}$$

In view of assumption (3.α_1), from (5.5) we conclude

$$\int_0^T c_1\|\dot{u}_n(s)\| + c_2\|\xi_n(s)\|_*^q \,\mathrm{d}s \leq C.$$

Also due to (5.4), we ultimately deduce that

$$\exists\, C>0 \ \ \forall\, n \in \mathbb{N}: \quad \|u_n\|_{BV([0,T];X)} + \|\xi_n(s)\|_{L^q(0,T;X^*)} \leq C\,. \tag{5.6}$$

Furthermore, from the energy identity (5.1) we immediately infer that, setting $h_n(t) := E_n(t) - \int_0^t P_n(s)\,\mathrm{d}s$, there holds

$$h_n(t) - h_n(s) = -\int_s^t f_{\alpha_n}(\dot{u}_n(r), -\xi_n(r))\,\mathrm{d}s \leq 0 \quad \forall\, 0 \leq s \leq t \leq T.$$

Therefore we have $\mathrm{Var}(h_n;[0,T]) = E_n(0) - E_n(T)) + \int_0^T P_n(s)\,\mathrm{d}s \leq C$ thanks to (5.3) and (4.3). Since (P_n) is uniformly bounded in $L^\infty(0,T)$, we conclude that

$$\exists\, C>0 \ \ \forall\, n \in \mathbb{N}: \quad \mathrm{Var}(E_n;[0,T]) \leq C\,. \tag{5.7}$$

Estimates (5.4), (5.6), (5.7), and the Helly principle guarantee that there exists a subsequence (n_k) and functions $u \in BV([0,T];X)$ and $E \in BV([0,T])$ such that, as $k \to \infty$,

$$(u_{n_k}(t), \mathcal{E}_t\left(u_{n_k}(t)\right)) \to (u(t), E(t)) \ \text{ in } X \times \mathbb{R} \text{ for all } t \in [0,T], \tag{5.8}$$

$$\mathrm{d}u_{n_k} = \dot{u}_{n_k} \cdot \mathcal{L} \overset{*}{\rightharpoonup} \mathrm{d}u \ \text{ in } \mathcal{M}(0,T;X). \tag{5.9}$$

Exploiting Thm. 3.4, we decompose $\mathrm{d}u$ as

$$\mathrm{d}u = (\,\mathrm{d}u)_{\mathrm{ac}} + (\,\mathrm{d}u)_{\mathrm{sin}} = \dot{u}_{\mathrm{ac}}\,\mathcal{L} + \dot{u}_s\,\|(\,\mathrm{d}u)_{\mathrm{sin}}\|.$$

Observe that, by the lower semicontinuity (3.$\mathcal{E}_0$),

$$E(t) \geq \mathcal{E}_t(u(t)) \qquad \text{for all } t \in [0,T]. \tag{5.10}$$

Further, in view of estimate (5.5), there exists $\mu \in \mathcal{M}(0,T)$ such that (up to not relabeled a subsequence)

$$f_{\alpha_k}\left(\dot{u}_k(\cdot), -\xi_k(\cdot)\right) \cdot \mathcal{L} \overset{*}{\rightharpoonup} \mu \ \text{ in } \mathcal{M}(0,T) \tag{5.11}$$

Moreover, by an infinite-dimensional version of the fundamental compactness theorem of Young measure theory (cf. Thm. A.3 in Appendix A), we can associate with (possibly a subsequence of) (ξ_{n_k}, P_{n_k}) a limiting Young measure $(\sigma_t)_{t\in(0,T)} \in \mathscr{Y}(0,T;X\times\mathbb{R})$ such that, for almost all $t \in (0,T)$ it holds $\sigma_t(X\times\mathbb{R}) = 1$ and σ_t is supported on the set of the limit points of $(\xi_{n_k}(t), P_{n_k}(t))$ w.r.t. the weak topology on $X^* \times \mathbb{R}$, viz.

$$\mathrm{supp}\,(\sigma_t) \subset \bigcap_{j\in\mathbb{N}} \overline{\{(\xi_{n_k}(t), P_{n_k}(t)) : k \geq j\}}^{\mathrm{weak}} \tag{5.12}$$

(where with $\overline{B}^{\text{weak}}$ we denote the closure of a set $B \subset X^* \times \mathbb{R}$ w.r.t. the weak topology). Furthermore, it holds

$$\xi_{n_k} \rightharpoonup \int_{X^* \times \mathbb{R}} \zeta \, \mathrm{d}\sigma_t(\zeta, p) =: \hat{\xi} \quad \text{in } L^q(0,T;X^*) \text{ and} \tag{5.13}$$

$$P_{n_k} \overset{*}{\rightharpoonup} \int_{X^* \times \mathbb{R}} p \, \mathrm{d}\sigma_t(\zeta, p) =: \hat{P} \quad \text{in } L^\infty(0,T)\,. \tag{5.14}$$

Step 2 - Nonemptyness of admissible sets: From now on, for the sake of simplicity, we shall write k instead of n_k . There exists a negligible set $N \subset (0,T)$ such that for every $t \in (0,T) \setminus N$ convergences (5.8) and the support property (5.12) hold. Taking into account the closedness condition (3.$\mathcal{E}_3$), it can be easily checked (cf. also [51, Sec. 6]), that for almost all $t \in (0,T)$ there holds

$$\begin{array}{rcl} (t, u(t)) & \in & D(\partial\mathcal{E}), \\ \mathcal{E}_t(u(t)) & = & E(t), \qquad \mathcal{E}_0(u(0)) = E(0), \\ \operatorname{supp}(\sigma_t) & \subset & \{(\zeta,p) \in X^* \times \mathbb{R} : \zeta \in \partial\mathcal{E}_t(u(t))\,,\ p \leq \partial_t \mathcal{E}_t(u(t))\}\,. \end{array} \tag{5.15}$$

In particular, from (5.14) and the third of (5.15) it follows that

$$\hat{P}(t) \leq \partial_t \mathcal{E}_t(u(t)) \qquad \text{for a.a. } t \in (0,T)\,. \tag{5.16}$$

Step 3 - $\liminf$ **result for the Fitzpatrick function:** In the next lines, we are going to prove that

$$\begin{aligned} &\liminf_{k\to\infty} \int_0^t f_{\alpha_k}\left(\dot{u}_k(r), -\xi_k(r)\right) \mathrm{d}r \\ &\geq \int_0^t \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(r), -\zeta\right) \mathrm{d}\sigma_r(\zeta,p)\,\mathrm{d}r + \int_0^t f_\alpha^\infty\left(\dot{u}_s(r), 0\right) \|(\,\mathrm{d}u)_{\sin}\|(r)\,. \end{aligned} \tag{5.17}$$

In order to do so, employing [35, Corollary 1.116, p. 75], we decompose the measure μ from (5.11) as follows: there exist μ_{ac}, $\mu_{\sin}$, $\mu_\perp$ in $\mathcal{M}(0,T)$ such that

$$\begin{gathered} \mu_{\mathrm{ac}} \ll \|(\,\mathrm{d}u)_{\mathrm{ac}}\|\,,\ \mu_{\sin} \ll \|(\,\mathrm{d}u)_{\sin}\|\,,\ \mu_\perp \perp \|(\,\mathrm{d}u)_{\mathrm{ac}}\| + \|(\,\mathrm{d}u)_{\sin}\| \text{ and} \\ \mu = \mu_{\mathrm{ac}} + \mu_{\sin} + \mu_\perp \end{gathered} \tag{5.18}$$

In particular, μ_{ac} is absolutely continuous w.r.t. the Lebesgue measure $\mathcal{L}$. Since $f_{\alpha_k}(\dot{u}_k, -\xi_k) \geq 0$ a.e. in $(0,T)$, we obtain $\mu_\perp \geq 0$. We will split the proof of (5.17) in two steps.

<u>First step:</u> Now, it follows from (3.8) and (5.18) and the Radon-Nikodým property of X that the set of the points $t_0 \in (0,T)$ such that $\sigma_{t_0}(X^* \times \mathbb{R}) = 1$ and

$$\begin{array}{rcl} \dot{u}_{\mathrm{ac}}(t_0) & = & \displaystyle\lim_{\varepsilon\to 0} \frac{(\,\mathrm{d}u)\left([t_0-\varepsilon, t_0+\varepsilon] \cap [0,T]\right)}{\varepsilon}\,, \\ \hat{\xi}(t_0) & = & \displaystyle\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{t_0-\varepsilon/2}^{t_0+\varepsilon/2} \hat{\xi}(t)\,\mathrm{d}t, \text{ and} \\ \displaystyle\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}(t_0) & = & \displaystyle\lim_{\varepsilon\to 0} \frac{\mu\left([t_0-\varepsilon, t_0+\varepsilon] \cap [0,T]\right)}{\varepsilon} < \infty \end{array} \tag{5.19}$$

has full Lebesgue measure. From now on, we shall use the notation

$$Q_\varepsilon(t_0) := [t_0 - \varepsilon, t_0 + \varepsilon] \cap [0,T] \qquad \text{with } t_0 \in (0,T) \text{ such that (5.19) holds.} \tag{5.20}$$

We then prove that

$$\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}(t_0) \geq \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(t_0), -\zeta\right) \mathrm{d}\sigma_{t_0}(\zeta,p) \tag{5.21}$$

with $t_0 \in (0,T)$ such that (5.19) holds. For any such t_0, it is also possible to choose a vanishing sequence $(\varepsilon_m)_m$ such that for all $m \in \mathbb{N}$ there holds

$$\mu\left(\{t_0-\varepsilon_m, t_0+\varepsilon_m\} \cap [0,T]\right) = (\,\mathrm{d}u)\left(\{t_0-\varepsilon_m, t_0+\varepsilon_m\} \cap [0,T]\right) = 0\,. \tag{5.22}$$

In order to show (5.21), we will use (2.7), which yields

$$f_\alpha\left(\dot{u}_{\mathrm{ac}}(t_0), -\zeta\right) = \sup\{\langle x^*, \dot{u}_{\mathrm{ac}}(t_0)\rangle - \langle \zeta, x\rangle - \rho_{\alpha^{-1}}(x^*, x) \,:\, (x,x^*) \in X \times X^*\}\,. \tag{5.23}$$

In view of (5.23), we thus confine ourselves to showing that

$$\begin{aligned}\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}(t_0) \geq \langle x^*, \dot{u}_{\mathrm{ac}}(t_0)\rangle - \int_{X^*\times\mathbb{R}} \langle \zeta, x\rangle \, \mathrm{d}\sigma_{t_0}(\zeta,p) - \rho_{\alpha^{-1}}(x,x^*) \\ \text{for all } (x,x^*) \in X \times X^* \text{ with } \rho_{\alpha^{-1}}(x,x^*) < \infty.\end{aligned} \tag{5.24}$$

Now, to check (5.24) we observe that, since $\alpha_k \xrightarrow{\mathrm{g}} \alpha$ also $\alpha_k^{-1} \xrightarrow{\mathrm{g}} \alpha^{-1}$ in the graph sense, we can apply Theorem 2.10 to $\rho_{\alpha_k^{-1}} = f^*_{\alpha_k}$. Therefore, for any $(x,x^*) \in X \times X^*$ there exists a sequence $(x_k, x_k^*)_{k\in\mathbb{N}}$ such that

$$(x_k, x_k^*) \to (x,x^*) \text{ and } \limsup_{n\to\infty} \rho_{\alpha_k^{-1}}(x_k^*, x_k) \leq \rho_{\alpha^{-1}}(x^*, x). \tag{5.25}$$

Combining (5.11) with the third of (5.19) (for the sequence $(\varepsilon_m)_m$ fulfilling (5.22)), we have that

$$\begin{aligned}&\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}(t_0) \\ &= \lim_{m\to\infty}\lim_{k\to\infty} \varepsilon_m^{-1} \int_{Q_{\varepsilon_m}(t_0)} f_{\alpha_k}(\dot{u}_k(t), -\xi_k(t))\,\mathrm{d}t \\ &\geq \liminf_{m\to\infty}\liminf_{k\to\infty} \varepsilon_m^{-1} \int_{Q_{\varepsilon_m}(t_0)} \Big(\langle x_k^*, \dot{u}_k(t)\rangle + \langle -\xi_k(t), x_k\rangle - \rho_{\alpha_k^{-1}}(x_k^*, x_k)\Big)\,\mathrm{d}t,\end{aligned} \tag{5.26}$$

where in the latter inequality we have plugged in the sequence (x_k, x_k^*) from (5.25) and applied formula (5.23) for f_{α_k}. On account of convergences (5.9) and (5.13), and of the fact that $(x_k, x_k^*) \to (x,x^*)$, we have for every $m \in \mathbb{N}$

$$\int_{Q_{\varepsilon_m}(t_0)} \langle x_k^*, \dot{u}_k(t)\rangle\,\mathrm{d}t \to \int_{Q_{\varepsilon_m}(t_0)} \langle x^*, \mathrm{d}u(t)\rangle\,. \tag{5.27}$$

$$\begin{aligned}\int_{Q_{\varepsilon_m}(t_0)} \langle \xi_k(t), x_k\rangle\,\mathrm{d}t \to &\int_{Q_{\varepsilon_m}(t_0)} \Big\langle \hat{\xi}(t), x\Big\rangle\,\mathrm{d}t \\ &= \int_{Q_{\varepsilon_m}(t_0)} \left(\int_{X^*\times\mathbb{R}} \langle\zeta, x\rangle \,\mathrm{d}\sigma_t(\zeta,p)\right)\mathrm{d}t\,.\end{aligned} \tag{5.28}$$

Inserting (5.27)–(5.28) into (5.26) and using (5.25), we thus get

$$\begin{aligned}&\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\mathcal{L}}(t_0) \\ &\geq \liminf_{m\to\infty} \frac{1}{\varepsilon_m}\left(\int_{Q_{\varepsilon_m}(t_0)} \langle x^*, \mathrm{d}u(t)\rangle + \int_{Q_{\varepsilon_m}(t_0)} \left(\int_{X^*\times\mathbb{R}} \langle -\zeta, x\rangle \,\mathrm{d}\sigma_t(\zeta,p) - \rho_{\alpha^{-1}}(x,x^*)\right)\mathrm{d}t\right)\end{aligned}$$

and in view of (5.19) we infer (5.24), whence the desired (5.21).

<u>*Second step:*</u> choose $t_0 \in (0,T)$ such that it satisfies

$$\begin{aligned}\dot{u}_{\mathrm{sin}}(t_0) &= \lim_{\varepsilon\to 0} \frac{(\mathrm{d}u)([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])}{\|(\mathrm{d}u)_{\mathrm{sin}}\|([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])}\,, \\ 0 &= \lim_{\varepsilon\to 0} \frac{\mathcal{L}([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])}{\|(\mathrm{d}u)_{\mathrm{sin}}\|([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])}, \quad \text{and} \\ \frac{\mathrm{d}\mu_{\mathrm{sin}}}{\|(\mathrm{d}u)_{\mathrm{sin}}\|}(t_0) &= \lim_{\varepsilon\to 0} \frac{\mu([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])}{\|(\mathrm{d}u)_{\mathrm{sin}}\|([t_0-\varepsilon, t_0+\varepsilon]\cap[0,T])} < \infty\,.\end{aligned} \tag{5.29}$$

The set of all t_0 failing any of (5.29) is a $\|(\mathrm{d}u)_{\mathrm{sin}}\|$-null set. We are now going to prove that

$$\frac{\mathrm{d}\mu_{\mathrm{sin}}}{\|(\mathrm{d}u)_{\mathrm{sin}}\|}(t_0) \geq f_\alpha^\infty(\dot{u}_{\mathrm{sin}}(t_0), 0)\,. \tag{5.30}$$

for any $t_0 \in (0,T)$ complying with (5.29). As before, we will use the notation (5.20) for the set $Q_\varepsilon(t_0)$ with any such t_0, and we choose correspondingly a vanishing sequence (ε_m) such that

(5.22) is satisfied. In order to show (5.30), in view of the representation formula (3.13) for f_α^∞ it is sufficient to show that

$$\frac{\mathrm{d}\mu_{\mathrm{sin}}}{\|(\mathrm{d}u)_{\mathrm{sin}}\|}(t_0) \geq \langle x^*, \dot{u}_{\mathrm{sin}}(t_0)\rangle \text{ for all } (x,x^*) \text{ such that } \rho_{\alpha^{-1}}(x,x^*) < \infty. \tag{5.31}$$

With the same argument as in the previous lines, we see that

$$\begin{aligned}
&\frac{\mathrm{d}\mu_{\mathrm{sin}}}{\|(\mathrm{d}u)_{\mathrm{sin}}\|}(t_0) \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \frac{1}{\|(\mathrm{d}u)_{\mathrm{sin}}\|\,(Q_{\varepsilon_m}(t_0))} \int_{Q_{\varepsilon_m}(t_0)} f_{\alpha_k}(\dot{u}_k(t), -\xi_k(t))\,\mathrm{d}t \\
&\geq \liminf_{m\to\infty}\liminf_{k\to\infty} \frac{1}{\|(\mathrm{d}u)_{\mathrm{sin}}\|\,(Q_{\varepsilon_m}(t_0))} \int_{Q_{\varepsilon_m}(t_0)} \left(\langle x_k^*, \dot{u}_k(t)\rangle + \langle -\xi_k(t), x_k\rangle - \rho_{\alpha_k^{-1}}(x_k^*, x_k)\right)\,\mathrm{d}t\,,
\end{aligned} \tag{5.32}$$

where (x_k, x_k^*) as in (5.25) approximates (x, x^*) from (5.31). Once again, due to (5.9) and (5.13) we have for every fixed $m \in \mathbb{N}$ that

$$\begin{aligned}
&\int_{Q_{\varepsilon_m}(t_0)} \langle x_k^*, \dot{u}_k(t)\rangle\,\mathrm{d}t \to \int_{Q_{\varepsilon_m}(t_0)} \langle x^*, \mathrm{d}u(t)\rangle \text{ and} \\
&\int_{Q_{\varepsilon_m}(t_0)} \langle \xi_k(t), x_k\rangle\,\mathrm{d}t \to \int_{Q_{\varepsilon_m}(t_0)} \left\langle \hat{\xi}(t), x\right\rangle\,\mathrm{d}t\,.
\end{aligned}$$

By construction (cf. (5.29)), there holds

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{\|(\mathrm{d}u)_{\mathrm{sin}}\|\,(Q_{\varepsilon_m}(t_0))} \int_{Q_{\varepsilon_m}(t_0)} \langle x^*, \mathrm{d}u(t)\rangle = \langle x^*, \dot{u}_{\mathrm{sin}}(t_0)\rangle\,, \\
&\lim_{m\to\infty} \frac{1}{\|(\mathrm{d}u)_{\mathrm{sin}}\|\,(Q_{\varepsilon_m}(t_0))} \int_{Q_{\varepsilon_m}(t_0)} \left\langle -\hat{\xi}(t), x\right\rangle\,\mathrm{d}t = 0, \\
&\lim_{m\to\infty} \frac{\mathcal{L}(Q_{\varepsilon_m}(t_0))}{\|(\mathrm{d}u)_{\mathrm{sin}}\|\,(Q_{\varepsilon_m}(t_0))} \rho_{\alpha^{-1}}(x,x^*) = 0\,.
\end{aligned}$$

We thus conclude (5.31), whence (5.30).

In conclusion, passing to the limit as $n_k \to \infty$ in (5.1) and relying on the initial data convergence (4.3), the energy convergence (5.8) joint with (5.10), (5.14), and the lower semicontinuity (5.17), we have obtained

$$\begin{aligned}
&\mathcal{E}_t(u(t)) + \int_0^t \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(s), -\zeta\right)\,\mathrm{d}\sigma_s(\zeta,p)\,\mathrm{d}s + \int_0^t f_\alpha^\infty\left(\dot{u}_{\mathrm{sin}}(s), 0\right)\|(\mathrm{d}u)_{\mathrm{sin}}\|(s) \\
&\leq E(t) + \int_0^t \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(s), -\zeta\right)\,\mathrm{d}\sigma_s(\zeta,p)\,\mathrm{d}s + \int_0^t f_\alpha^\infty\left(\dot{u}_{\mathrm{sin}}(s), 0\right)\|(\mathrm{d}u)_{\mathrm{sin}}\|(s) \\
&\leq \mathcal{E}_0(u(0)) + \int_0^t \int_{X^*\times\mathbb{R}} p\,\mathrm{d}\sigma_s(\zeta,p)\,\mathrm{d}s \qquad \text{for all } t \in (0,T]\,.
\end{aligned} \tag{5.33}$$

Step 4 - Enhanced support properties of the Young measure $(\sigma_t)_{t\in(0,T)}$: We can now improve the third of (5.15), showing that indeed

$$\operatorname{supp}(\sigma_t) \subset \{(\zeta,p) \in X^*\times\mathbb{R} : \zeta \in \partial\mathcal{E}_t(u(t))\,,\ -\zeta \in \alpha(u(t))\,,\ p \leq \partial_t\mathcal{E}_t(u(t))\} \quad \text{for a.a. } t \in (0,T). \tag{5.34}$$

To this end, observe that, passing to the limit as $n_k \to \infty$ in (5.1) (written on the interval (s,t)), yields, in view of convergences (5.8), (5.11), and (5.14), the following energy identity

$$E(t) + \mu([s,t]) = E(s) + \int_s^t p(r)\,\mathrm{d}r \qquad \text{for all } 0 \leq s \leq t \leq T \qquad \text{with} \tag{5.35a}$$

$$\mu([s,t]) \geq \int_s^t \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(r), -\zeta\right)\,\mathrm{d}\sigma_r(\zeta,p)\,\mathrm{d}r \tag{5.35b}$$

the latter inequality due to (5.21). In particular, observe that

$$\int_0^T \int_{X^*\times\mathbb{R}} f_\alpha\left(\dot{u}_{\mathrm{ac}}(t),-\zeta\right)\,\mathrm{d}\sigma_t(\zeta,p)\,\mathrm{d}t < \infty\,. \tag{5.36}$$

Let $\mathcal{T}\subset[0,T]$ be the set of all Lebesgue points t_0 of $\hat{P}$ (5.14), such that relations (5.19) and (5.21) hold, and

$$\dot{E}_{\mathrm{ac}}(t_0) = \lim_{\varepsilon\to 0}\frac{E\left(t_0+\frac{\varepsilon}{2}\right)-E\left(t_0-\frac{\varepsilon}{2}\right)}{\varepsilon}\,.$$

Then $\mathcal{T}$ has full measure. Let us now choose a sequence (ε_m), $\varepsilon_m\downarrow 0$, such that (5.22) holds. Then, passing to the limit as $\varepsilon_m\downarrow 0$ in (5.35) (written for $s=t_0-\varepsilon_m/2$ and $t=t_0+\varepsilon_m/2$), we obtain

$$\dot{E}_{\mathrm{ac}}(t_0)+\int_{X^*\times\mathbb{R}} f_\alpha(\dot{u}_{\mathrm{ac}}(t_0),-\zeta)\,\mathrm{d}\sigma_{t_0}(\zeta,p)\leq\int_{X^*\times\mathbb{R}} p\,\mathrm{d}\sigma_{t_0}(\zeta,p)\quad\text{for all } t_0\in\mathcal{T} \tag{5.37}$$

(up to removing from $\mathcal{T}$ a set of zero Lebesgue measure). Now, observe that thanks to the third of (5.15) and (5.36), the Young measure $(\sigma_t)_{t\in(0,T)}$ satisfies the assumptions of the forthcoming Lemma A.4. Therefore, in view of the Young measure version of the chain rule inequality (3.$\mathcal{E}_4$) therein, we find that

$$-\dot{E}_{\mathrm{ac}}(t_0)+\int_{X^*\times\mathbb{R}} p\,\mathrm{d}\sigma_{t_0}(\zeta,p)\leq\int_{X^*\times\mathbb{R}}\langle-\zeta,\dot{u}_{\mathrm{ac}}(t_0)\rangle\,\mathrm{d}\sigma_{t_0}(\zeta,p)\quad\text{for almost all } t_0\in\mathcal{T}\,. \tag{5.38}$$

Combining (5.37) and (5.38), we deduce that for almost all $t_0\in\mathcal{T}$ (and hence for almost all $t_0\in(0,T)$) it holds

$$\int_{X^*\times\mathbb{R}}\left(f_\alpha(\dot{u}_{\mathrm{ac}}(t_0),-\zeta)-\langle-\zeta,\dot{u}_{\mathrm{ac}}(t_0)\rangle\right)\,\mathrm{d}\sigma_{t_0}(\zeta,p)\leq 0\,. \tag{5.39}$$

Since f_α is a representative function for α, we easily see that (5.39) holds as an equality, and that in fact

$$-\zeta\in\alpha(\dot{u}_{\mathrm{ac}}(t_0))\qquad\text{for } \sigma_{t_0}\text{-a.a. } (\zeta,p)\in\mathrm{supp}(\sigma_{t_0}).$$

Since $t_0\in(0,T)$ is arbitrary out of a Lebesgue-null set, we have ultimately proved the desired support property (5.34). Furthermore, as a by-product of (5.37)–(5.39) holding as equalities, we infer the following pointwise energy equality

$$\dot{E}_{\mathrm{ac}}(t)+\int_{X^*\times\mathbb{R}} f_\alpha(u_{\mathrm{ac}}(t),-\zeta)\,\mathrm{d}\sigma_t(\zeta,p)=\int_{X^*\times\mathbb{R}} p\,\mathrm{d}\sigma_t(\zeta,p)\qquad\text{for a.a. } t\in(0,T). \tag{5.40}$$

Step 5 - Selection argument and conclusion of the proof: We can now apply Lemma A.5 and deduce that there exist measurable functions $\xi:(0,T)\to X$ and $P:(0,T)\to\mathbb{R}$ such that

$$(\xi(t),p(t))\in\operatorname{argmin}\left\{f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta)-p:(\zeta,p)\in\mathcal{S}\left(t,u(t),\dot{u}_{\mathrm{ac}}(t)\right)\right\}\quad\text{for a.a. } t\in(0,T), \tag{5.41}$$

with $\mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t)):=\{(\zeta,p)\in X^*\times\mathbb{R}:\zeta\in\partial\mathcal{E}_t(u(t)),\,-\zeta\in\alpha(\dot{u}_{\mathrm{ac}}(t)),\,p\leq\partial_t\mathcal{E}_t(u(t))\}$. In particular,

$$\xi(t)\in\partial\mathcal{E}_t(u(t))\,,\qquad -\xi(t)\in\alpha(\dot{u}_{\mathrm{ac}}(t)),\qquad\text{and } P(t)\leq\partial_t\mathcal{E}_t(u(t))\quad\text{for a.a. } t\in(0,T), \tag{5.42}$$

We then have the following chain of inequalities for almost all $t\in(0,T)$

$$\begin{aligned}-\dot{E}_{\mathrm{ac}}(t)&=\int_{X^*\times\mathbb{R}}\left(f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta)-p\right)\,\mathrm{d}\sigma_t(\zeta,p)\\&\geq f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\xi(t))-P(t)\\&\geq\langle-\xi(t),\dot{u}_{\mathrm{ac}}(t)\rangle-\partial_t\mathcal{E}_t(u(t))\geq-\dot{E}_{\mathrm{ac}}(t),\end{aligned} \tag{5.43}$$

where the first identity follows from (5.40), the second inequality from (5.41), the third one from the fact that f_α is a representative function for α and from (5.42), and the last one from the chain rule inequality $(3.\mathcal{E}_4)$. Therefore we infer that all inequalities in (5.43) hold as equalities, which proves (4.8). In particular, we have that for almost all $t \in (0,T)$

$$
\begin{aligned}
&P(t) = \int_{X^*\times\mathbb{R}} p \,\mathrm{d}\sigma_t(\zeta,p) = \partial_t \mathcal{E}_t(u(t)),\\
&f_\alpha(\dot{u}_{\mathrm{ac}}(t), -\xi(t)) = \int_{X^*\times\mathbb{R}} f_\alpha(\dot{u}_{\mathrm{ac}}(t), -\zeta)\,\mathrm{d}\sigma_t(\zeta,p) = \langle -\xi(t), \dot{u}_{\mathrm{ac}}(t)\rangle \,.
\end{aligned}
\tag{5.44}
$$

Combining (5.44) with (5.33) we ultimately deduce (4.5). Now, from (4.5) with (4.6) we have that $\int_0^T |\langle -\xi(t), \dot{u}_{\mathrm{ac}}(t)\rangle|\,\mathrm{d}t < \infty$. Since α complies with (4.1), we then conclude that $\xi \in L^q(0,T;X^*)$. This completes the proof. □

Remark 5.2 (The role of the Fitzpatrick function). As pointed out in Remark 3.13, the variational reformulation of the doubly nonlinear differential inclusion (3.17) could be given in terms of *any* representative functional for α. The distinguished role of the Fitzpatrick function f_α is apparent in the passage to the limit argument developed in Step 3 of the proof of Thm. 4.5. Therein (cf. (5.21)–(5.28)), we exploit the duality formula (2.7) for f_α, as well as Theorem 2.10.

Remark 5.3 (Refinement of the measurable selection argument). A close perusal of Step 5 in the above proof reveals that, in principle, it should be sufficient to select the functions $t \mapsto (\xi(t), p(t))$ in the set $\tilde{\mathcal{S}}(t,u(t)) := \{(\zeta,p) \in X^*\times\mathbb{R} : \zeta \in \partial\mathcal{E}_t(u(t)),\ p \le \partial_t\mathcal{E}_t(u(t))\}$, i.e. dropping the requirement $-\zeta \in \alpha(\dot{u}_{\mathrm{ac}}(t))$. Indeed, if we were in the position of applying Lemma A.5 to the set $\tilde{\mathcal{S}}$, from the chain of inequalities (5.43) the second of (5.44) would still follow, yielding $-\xi(t) \in \alpha(\dot{u}_{\mathrm{ac}}(t))$ for almost all $t \in (0,T)$, i.e. (4.6).

Nonetheless, the extension of Lemma A.5 to the set $\tilde{\mathcal{S}}$ seems to be an open problem, at the moment, cf. the upcoming Remark A.6.

We conclude this section with the

Proof of Theorem 4.8. Repeating the calculations from Step 1 of the proof of Thm. 4.5, we prove that the sequence (u_n) is bounded in $W^{1,p}(0,T;X)$ and in addition fulfills estimate (5.3). Therefore, convergence (4.16) holds. We use the arguments from the above Steps 1 and 4 to infer that there exist $(\xi, E) \in L^1(0,T;X^*) \times BV([0,T])$ complying with (4.7) and (4.8). Since $u \in W^{1,p}(0,T;X)$, Proposition 4.7 applies, and we conclude the proof. □

Appendix A. **Young measure results**

We fix here some definitions and results on parameterized (or Young) measures (see e.g. [12, 13, 14, 77]) with values in a reflexive Banach space $\mathcal{Y}$. In particular, in Section 5 the upcoming results are applied to the space $\mathcal{Y} = X^* \times \mathbb{R}$.

Notation A.1. In what follows, we will denote by $\mathscr{L}_{(0,T)}$ the σ-algebra of the Lebesgue measurable subsets of $(0,T)$ and by $\mathscr{B}(\mathcal{Y})$ the Borel σ-algebra of $\mathcal{Y}$. We use the symbol $\otimes$ for product σ-algebrae. We recall that a $\mathscr{L}_{(0,T)} \otimes \mathscr{B}(\mathcal{Y})$-measurable function $h : (0,T)\times\mathcal{Y} \to (-\infty,+\infty]$ is a *normal integrand* if for a.a. $t \in (0,T)$ the map $y \mapsto h_t(y) = h(t,y)$ is lower semicontinuous on $\mathcal{Y}$.

We consider the space $\mathcal{Y}$ endowed with the *weak* topology, and say that a $\mathscr{L}_{(0,T)} \otimes \mathscr{B}(\mathcal{Y})$–measurable functional $h : (0,T)\times\mathcal{Y} \to (-\infty,+\infty]$ is a *weakly-normal integrand* if for a.a. $t \in (0,T)$ the map

$$
y \mapsto h(t,y) \text{ is sequentially lower semicontinuous on } \mathcal{Y} \text{ w.r.t. the weak topology.} \tag{A.1}
$$

We denote by $\mathscr{M}(0,T;\mathcal{Y})$ the set of all $\mathscr{L}_{(0,T)}$-measurable functions $y : (0,T) \to \mathcal{Y}$. A sequence $(y_n) \subset \mathscr{M}(0,T;\mathcal{Y})$ is said to be *weakly-tight* if there exists a weakly-normal integrand $h : (0,T)\times\mathcal{Y} \to (-\infty,+\infty]$ such that the map

$$
y \mapsto h_t(y) \text{ has compact sublevels w.r.t. the weak topology of } \mathcal{Y}, \text{ and } \sup_n \int_0^T h(t, y_n(t))\,\mathrm{d}t < \infty.
$$

Definition A.2 (**Young measures with values in $\mathcal{Y}$**)**.** A (time-dependent) *Young measure* in the space $\mathcal{Y}$ is a family $\boldsymbol{\sigma} := \{\sigma_t\}_{t\in(0,T)}$ of Borel probability measures on $\mathcal{Y}$ parameterized by $t \in (0,T)$, such that the map on $(0,T)$

$$t \mapsto \sigma_t(B) \quad \text{is} \quad \mathscr{L}_{(0,T)}\text{-measurable} \quad \text{for all } B \in \mathscr{B}(\mathcal{Y}). \tag{A.2}$$

We denote by $\mathscr{Y}(0,T;\mathcal{Y})$ the set of all Young measures in $\mathcal{Y}$.

The following result is taken from [51] (cf. Thms. A.2 and A.3 therein). It is a generalization of the so-called *Fundamental Theorem of Young measures* (cf. the classical results [12, Thm. 1], [13, Thm. 2.2], [14], [77, Thm. 16]), to the case of Young measures with values in $\mathcal{Y}$ endowed with the weak topology (see also [66, Thm. 3.2] for the case in which $\mathcal{Y}$ is a Hilbert space endowed with the weak topology).

Theorem A.3 (The Fundamental Theorem for weak topologies)**.** *Let $(y_n) \subset \mathscr{M}(0,T;\mathcal{Y})$ be a weakly-tight sequence. Then,*

(1) there exists a subsequence (y_{n_k}) and a Young measure $\boldsymbol{\sigma} = (\sigma_t)_{t\in(0,T)} \in \mathscr{Y}(0,T;\mathcal{Y})$ such that

$$\limsup_{k\uparrow\infty} \|y_{n_k}(t)\|_{\mathcal{Y}} < \infty \quad \text{and} \quad \operatorname{supp}(\sigma_t) \subset \bigcap_{j=1}^{\infty} \overline{\{y_{n_k}(t) : k \geq j\}}^{\text{weak}} \quad \text{for a.a.}\, t \in (0,T), \tag{A.3}$$

(where $\overline{B}^{\text{weak}}$ denotes the closure of a set $B \subset \mathcal{Y}$ w.r.t. the weak topology), and such that for every weakly-normal integrand $h : [0,T] \times \mathcal{Y} \to (-\infty,\infty]$ such that $h^-(\cdot, y_{n_k}(\cdot))$ is uniformly integrable it holds

$$\liminf_{k\to\infty} \int_0^T h(t, y_{n_k}(t))\,\mathrm{d}t \geq \int_0^T \int_{\mathcal{Y}} h(t,y)\,\mathrm{d}\sigma_t(y)\,\mathrm{d}t\,. \tag{A.4}$$

(2) In particular, let $(y_n) \subset L^q(0,T;\mathcal{Y})$ be a bounded sequence, with $q \in (1,+\infty]$. Then, there exists a further (not relabeled) subsequence (y_{n_k}) and a Young measure $\boldsymbol{\sigma} = \{\sigma_t\}_{t\in(0,T)} \in \mathscr{Y}(0,T;\mathcal{Y})$ such that for a.a. $t \in (0,T)$ properties (A.3) *hold. Setting $\mathrm{y}(t) := \int_{\mathcal{Y}} y\,\mathrm{d}\sigma_t(y)$ for almost all $t \in (0,T)$, there holds*

$$y_{n_k} \overset{*}{\rightharpoonup} \mathrm{y} \quad \text{in } L^p(0,T;\mathcal{Y}). \tag{A.5}$$

A.1. **A Young-measure version of the chain rule.** In what follows, we will work with Young measures with values in the space $\mathcal{Y} = X^* \times \mathbb{R}$. Our first result, a small variation of [51, Prop. B.1], provides the version of the chain rule inequality (3.$\mathcal{E}_4$) in terms of Young measures used in Step 4 of the proof of Thm. 4.5.

Lemma A.4. *In the frame of* (3.15)*, let $\alpha : X \rightrightarrows X^*$ fulfill* (3.α_0) *and the coercivity condition* (4.1)*, and let $\mathcal{E} : [0,T] \times X \to (-\infty,+\infty]$ comply with Assumption* 4.4*. Let $u \in BV([0,T];X)$ satisfy*

$$\begin{aligned} &\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty\,, \quad (t,u(t)) \in \operatorname{dom}(\partial\mathcal{E}) \text{ for a.a. } t \in (0,T), \quad \int_0^T |\partial_t \mathcal{E}_t(u(t))|\,\mathrm{d}t < \infty, \\ &\exists E \in BV([0,T]) \text{ such that } E(t) = \mathcal{E}_t(u(t)) \text{ for a.a.}\, t \in (0,T), \end{aligned} \tag{A.6}$$

and let $(\sigma_t)_{t\in(0,T)} \in \mathscr{Y}(0,T;X^* \times \mathbb{R})$ be a Young measure such that

$$\forall (\xi,p) \in \operatorname{supp}(\sigma_t) : \xi \in \partial\mathcal{E}_t(u(t))\,,\ p \leq \partial_t \mathcal{E}_t(u(t)) \text{ for a.a. } t \in (0,T) \tag{A.7}$$

$$\int_0^T \int_{X^*\times\mathbb{R}} f_\alpha(\dot{u}_{\mathrm{ac}}(s), -\zeta)\,\mathrm{d}\sigma(\zeta,p)ds < \infty \tag{A.8}$$

Then, for almost all $t \in (0,T)$ such that t is Lebesgue point of $\dot{E}_{\mathrm{ac}}$ and $\dot{u}_{\mathrm{ac}}$ there holds

$$\dot{E}_{\mathrm{ac}}(t) \geq \int_{X^*\times\mathbb{R}} (\langle \zeta, \dot{u}_{\mathrm{ac}}(t)\rangle + p)\,\mathrm{d}\sigma_t(\zeta,p)\,. \tag{A.9}$$

Proof. We consider the set $K(t,u(t)) := \{(\xi,p) \in X^* \times \mathbb{R} \,:\, \xi \in \partial\mathcal{E}_t(u(t)),\, p \leq \partial_t\mathcal{E}_t(u(t))\}$. Repeating the very same arguments as in the proof of [51, Prop. B.1], we can show that there exists a sequence (ξ_n,p_n) of strongly measurable functions $(\xi_n,p_n) : (0,T) \to X^* \times \mathbb{R}$ such that

$$\{(\xi_n(t),p_n(t)) \,:\, n \in \mathbb{N}\} \subset K(t,u(t)) \subset \overline{\{(\xi_n(t),p_n(t)) \,:\, n \in \mathbb{N}\}} \quad \text{for a.a.}\ t \in (0,T) \tag{A.10}$$

(where $\overline{B}$ denotes the closure of $B \subset X^* \times \mathbb{R}$ w.r.t. the strong topology of $X^* \times \mathbb{R}$).

We now claim that the sequence (ξ_n,p_n) can be chosen such that

$$\forall n \in \mathbb{N} : \xi_n \in L^1(0,T;X^*) \ \text{ and } \sup_{n\in\mathbb{N}} \int_0^T f_\alpha(\dot{u}_{\mathrm{ac}}(t), -\xi_n(t))\,\mathrm{d}t < \infty\,. \tag{A.11}$$

To this aim, we define the function $g(t) := \inf\{f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta) : (\zeta,p) \in K(t,u(t))\}$ for almost all $t \in (0,T)$. Notice that due to (A.10) it holds

$$g(t) := \inf_{n\in\mathbb{N}}\{f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\xi_n(t))\} \text{ for a.a. } t \in (0,T) \tag{A.12}$$

and hence g is measurable on $(0,T)$. Moreover,

$$\int_0^T g(t)\,\mathrm{d}t \leq \int_0^T \int_{X\times\mathbb{R}} f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta)\,\mathrm{d}\sigma_t(\zeta,p)\,\mathrm{d}t < \infty. \tag{A.13}$$

With a straightforward adaptation of the argument of [51, Prop. B.1] (see also [66, Lemma 3.4]), from (A.12) and (A.13) we deduce (A.11).

In view of the obtained (A.10) and (A.11), we are in the position to apply the chain rule inequality $(3.\mathcal{E}_4)$ to the pair (u,ξ_n) for every $n \in \mathbb{N}$. Therefore for every $n \in \mathbb{N}$ there exists a set $\mathcal{T}_n \subset (0,T)$ of full measure such that $\dot{E}_{\mathrm{ac}}(t) \geq \langle \xi_n(t), \dot{u}_{\mathrm{ac}}(t)\rangle + p_n(t)$ for all $t \in \mathcal{T}_n$, where we have also used that $p_n(t) \leq \partial_t\mathcal{E}_t(u(t))$. The set $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$, has still full measure, and there holds for all $t \in \mathcal{T}$

$$\dot{E}_{\mathrm{ac}}(t) \geq \langle \zeta, \dot{u}_{\mathrm{ac}}(t)\rangle + p \quad \text{for all } (\zeta,p) \in \overline{\mathrm{conv}\ K(t,u(t))}, \tag{A.14}$$

the latter set denoting the closed convex hull of $K(t,u(t))$. Integrating (A.14) w.r.t. the measure σ_t we obtain (A.9). □

We conclude with the measurable selection result exploited in Step 5 of the proof of Thm. 4.5.

Lemma A.5. *In the framework of* (3.15), *let* $\alpha : X \rightrightarrows X^*$ *fulfill* $(3.\alpha_0)$ *and the coercivity condition* (4.1), *and let* $\mathcal{E} : [0,T] \times X \to (-\infty,+\infty]$ *comply with Assumptions* 3.9 *and* 4.4. *Furthermore, let* $u \in BV([0,T];X)$ *fulfill* (A.6). *Suppose that for almost all* $t \in (0,T)$

$$\mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t)) := \{(\zeta,p) \in X^* \times \mathbb{R} : \zeta \in \partial\mathcal{E}_t(u(t))\,,\ -\zeta \in \alpha(\dot{u}_{\mathrm{ac}}(t))\,, p \leq \partial_t\mathcal{E}_t(u(t))\} \neq \emptyset\,. \tag{A.15}$$

Then, there exist measurable functions $\xi : (0,T) \to X^*$ *and* $P : (0,T) \to \mathbb{R}$ *such that*

$$(\xi(t),P(t)) \in \operatorname{argmin}\{f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta) - p : (\zeta,p) \in \mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t))\} \quad \textit{for a.a. } t \in (0,T)\,. \tag{A.16}$$

Proof. The argument follows the very same lines of [51, Lemma B.2]. First of all, we observe that

$$\operatorname{argmin}\{f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta) - p : (\zeta,p) \in \mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t))\} \neq \emptyset \ \text{ for a.a. } t \in (0,T)\,. \tag{A.17}$$

To this aim, let $(\zeta_n,p_n) \subset \mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t))$ be an infimizing sequence: then there exist constants C, $C' > 0$ such that for every $n \in \mathbb{N}$

$$C \geq f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta_n) - p_n = \langle -\zeta_n, \dot{u}_{\mathrm{ac}}(t)\rangle - p_n \geq c_1\|\dot{u}_{\mathrm{ac}}(t)\|^p + c_2\|\zeta_n\|_*^q - c_3 - C' \tag{A.18}$$

where we have used that $-\zeta_n \in \dot{u}_{\mathrm{ac}}(t)$, the coercivity property (4.1) of α, and that $p_n \leq \partial_t\mathcal{E}_t(u(t)) \leq C$ due to the fact that $\sup_{t\in[0,T]} \mathcal{E}_t(u(t)) < \infty$ and to $(3.\mathcal{E}_2)$. Therefore, we infer that $\sup_{n\in\mathbb{N}}(\|\zeta_n\|_*^q + |p_n|) < \infty$. Hence, there exist $(\zeta,p) \in X^* \times \mathbb{R}$ such that, up to a not relabeled subsequence, $\zeta_n \rightharpoonup \zeta$ in X^* and $p_n \to p$. Thanks to the closedness condition $(3.\mathcal{E}_3)$ and to the weak closedness of $\alpha(\dot{u}_{\mathrm{ac}}(t))$, we have $(\zeta,p) \in \mathcal{S}(t,u(t),\dot{u}_{\mathrm{ac}}(t))$. Using that $\zeta \mapsto f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta)$ is (sequentially) weakly-lower semicontinuous, we conclude that

$$\liminf_{n\to\infty}\left(f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta_n) - p_n\right) \geq f_\alpha(\dot{u}_{\mathrm{ac}}(t),-\zeta) - p$$

and (A.17) ensues.

Once obtained (A.17), the argument for (A.16) is a straightforward adaptation of the proof of [51, Lemma B.2], to which we refer for all details. Let us only mention here that the existence of (ξ, P) as in (A.16) is a consequence of the measurable selection results [24, Cor. III.3, Thm. III.6]. $\square$

Remark A.6. Let us stress that the requirement $\zeta \in \alpha(\dot{u}_{\rm ac}(t))$ in the definition (A.15) of the set $\mathcal{S}(t, u(t), \dot{u}_{\rm ac}(t))$ has a crucial role in proving that

$$\operatorname{argmin}\{f_\alpha(\dot{u}_{\rm ac}(t), -\zeta) - p : (\zeta, p) \in S(t, u(t), \dot{u}_{\rm ac}(t))\}$$

is nonempty. In fact, it ensures the estimates in (A.18) for any infimizing sequence (ζ_n, p_n).

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