On the classification of entire local minimizers of the Ginzburg-Landau equation

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A Patrizia Pucci con grande stima

Abstract

We study entire solutions of the complex-valued Ginzburg-Landau equation in arbitrary dimension. In particular, in dimension 3 and 4, we prove that entire local minimizers, whose modulus goes to one at infinity, are necessarily constants and of unit modulus.

Keywords: Ginzburg-Landau equation, local minimizers, Liouville-type theorems.

1 Introduction and main results

In this article we study maps $u : \mathbb{R}^N \to \mathbb{R}^2 \simeq \mathbb{C}$, $N \geq 3$, which are solutions of the Ginzburg-Landau system

$$-\Delta u = u(1 - |u|^2) \tag{1.1}$$

subjected to the natural condition at infinity

$$|u(x)| \to 1 \qquad \text{as} \qquad |x| \to +\infty. \tag{1.2}$$

In particular, we are interested to entire local minimizers of the Ginzburg-Landau system (1.1). Let us recall that a weak solution $u \in H^1_{loc}(\mathbb{R}^N, \mathbb{R}^2) \cap L^4_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ of (1.1) is an entire *local minimzer* if, for every bounded domain $\Omega \subset \mathbb{R}^N$ it minimizes the energy functional

$$E(v,\Omega) := \int_\Omega \frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1-|v|^2)^2 \tag{1.3}$$

among all maps $v \in H^1_{loc}(\mathbb{R}^N,\mathbb{R}^2) \cap L^4_{loc}(\mathbb{R}^N,\mathbb{R}^2)$ satisfying $v-u \in H^1_0(\Omega,\mathbb{R}^2)$.

Our main result is

Theorem 1.1. *Assume $N = 3$ or 4 and let $u \in H^1_{loc}(\mathbb{R}^N,\mathbb{R}^2)$ be a local minimizer of (1.1) satisfying (1.2). Then u is constant and $|u| = 1$.*

The above theorem is a consequence of the following Liouville-type result concerning entire solutions (not necessarily locally minimizing) of (1.1) .

Theorem 1.2. *Assume $N \geq 3$ and $0 \leq \alpha < 2\sqrt{N-1}$. Let $u \in H^1_{loc}(\mathbb{R}^N,\mathbb{R}^2) \cap L^3_{loc}(\mathbb{R}^N,\mathbb{R}^2)$ be a distribution solution of (1.1) satisfying (1.2) and such that*

$$\int_{B_R(0)} |\nabla v|^2 \leq CR^\alpha \qquad \forall\, R > R_0 > 0 \tag{1.4}$$

for some positive constants C and R_0 independent of R.
Then u is constant and $|u| = 1$.

Theorem (1.2) is not true when $N = 2$. Indeed (cfr. e.g. [7, 2]) one can construct a non-constant, degree-one solution of (1.1), satisfying (1.2) and (1.4), having the form $u(x) = f(|x|)\frac{x}{|x|}$, for a unique profile f vanishing at zero and increasing to one at infinity. More precisely this particular solution satisfies $\int_{B_R(0)} |\nabla u|^2 \leq C \log R$, for $R >> 1$. Furthermore, it is well-known [1, 8, 9, 10] that this solution is also the unique (up to symmetries) nontrivial local minimizer of (1.1). Hence, in view of the above discussion it is natural to formulate the following :

Question 1. Classify the entire local minimizers of the Ginzburg-Landau equation (1.1), satisfying condition (1.2), when $N \geq 5$.

2 Proofs

Proof of Theorem 1.2. We recall that any distribution solution $u \in L^3_{loc}(\mathbb{R}^N,\mathbb{R}^2)$ of (1.1) is smooth and satisfies the natural bound $\|u\|_{L^\infty} \leq 1$ ([5]). We also recall that in [5] (see Corollary 1.3 therein) we proved that any smooth solution u of (1.1) such that $|\nabla u| \in L^2(\mathbb{R}^N)$ must be constant. So, to conclude, we only need to prove that u has finite Dirichlet energy. To this end, we follow the idea introduced in [3] to prove a quantization effect for the potential energy in the two-dimensional case and further developed in [6] to prove some Liouville-type theorems in dimension $N \geq 3$.

From the condition at infinity (1.2) we can find a real number $R_1 > R_0$ such that $|u(x)|^2 > \delta^2 \in (\frac{\alpha}{2\sqrt{N-1}}, 1)$ for $|x| \geq R_1$. Since $\frac{u}{|u|} \in C^2(\mathbb{R}^N \setminus \overline{B_{R_1}}, \mathbb{S}^1)$

and the set $\mathbb{R}^N \setminus \overline{B_{R_1}}$ is open and simply connected, we can find a real-valued function $\theta \in C^2(\mathbb{R}^N \setminus \overline{B_{R_1}})$ such that :

$$\forall\, x \in \mathbb{R}^N \setminus \overline{B_{R_1}} \qquad u(x) = |u(x)|e^{i\theta(x)} := \rho(x)e^{i\theta(x)}. \tag{2.1}$$

Inserting (2.1) into (1.1) we obtain that ρ and θ satisfy the following system of equations :

$$\begin{cases} \operatorname{div}\left(\rho^2 \nabla\theta\right) = 0 & \text{in } \mathbb{R}^N \setminus \overline{B_{R_1}}, \\ \Delta\rho = \rho(\rho^2 - 1 + |\nabla\theta|^2) & \text{in } \mathbb{R}^N \setminus \overline{B_{R_1}}. \end{cases} \tag{2.2}$$

First we show that, for every $R > R_1$, we have :

$$\int_{\partial B_R} \rho^2 \frac{\partial\theta}{\partial\nu} = 0 \tag{2.3}$$

where ν denotes the outer normal to B_R.

Let us consider the vector-field $V \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ given by :

$$V = \left(u \wedge \frac{\partial u}{\partial x_1}, \ldots, u \wedge \frac{\partial u}{\partial x_N} \right)$$

where, for any couple of maps $u = (u_1, u_2)$ and $v = (v_1, v_2)$ belonging to $C^1(\mathbb{R}^N, \mathbb{R}^2)$, we have let $u \wedge v := u_1 v_2 - u_2 v_1$.

For every $R > R_1$ we have :

$$\int_{\partial B_R} V \cdot \nu = \int_{B_R} \operatorname{div} V = \int_{B_R} u \wedge \Delta u = 0$$

by (1.1). On the other hand, by using (2.1) we find that $V = \rho^2 \nabla\theta$ in $\mathbb{R}^N \setminus \overline{B_{R_1}}$ and (2.3) follows at once.

Next we prove that

$$\int_{\mathbb{R}^N \setminus \overline{B_{R_2}}} \rho^2 |\nabla\theta|^2 < +\infty \tag{2.4}$$

where R_2 is any real number bigger than R_1.

For any $R > R_2$ we denote by θ_R the mean value of θ on the sphere ∂B_R and by Ω_R the open set $B_R \setminus \overline{B_{R_2}}$.

Multiplying the first equation of (2.2) by $\theta - \theta_R$, integrating over Ω_R and using (2.3) we get:

$$
\begin{aligned}
\int_{\Omega_R} \rho^2 |\nabla \theta|^2 &= \int_{\partial \Omega_R} \rho^2 \frac{\partial \theta}{\partial \nu}(\theta - \theta_R) \\
&= \int_{\partial B_R} \rho^2 \frac{\partial \theta}{\partial \nu}(\theta - \theta_R) - \int_{\partial B_{R_2}} \rho^2 \frac{\partial \theta}{\partial \nu}(\theta - \theta_R) \\
&= \int_{\partial B_R} \rho^2 \frac{\partial \theta}{\partial \nu}(\theta - \theta_R) - \int_{\partial B_{R_2}} \rho^2 \frac{\partial \theta}{\partial \nu}\theta + 0 \\
&= \int_{\partial B_R} \rho^2 \frac{\partial \theta}{\partial \nu}(\theta - \theta_R) + C \le \int_{\partial B_R} \left| \frac{\partial \theta}{\partial \nu} \right| \, |(\theta - \theta_R)| + C \\
&\le \left(\int_{\partial B_R} |(\theta - \theta_R)|^2 \right)^{\frac{1}{2}} \left(\int_{\partial B_R} \left| \frac{\partial \theta}{\partial \nu} \right|^2 \right)^{\frac{1}{2}} + C, \qquad (2.5)
\end{aligned}
$$

where C is a constant independent of R.

Since the second eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$ over the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ is $N-1$ (cfr. e.g. [4]), we get

$$
\begin{aligned}
\int_{\Omega_R} \rho^2 |\nabla \theta|^2 &\le \left(\frac{R^2}{N-1} \int_{\partial B_R} |\nabla_T \theta|^2 \right)^{\frac{1}{2}} \left(\int_{\partial B_R} \left| \frac{\partial \theta}{\partial \nu} \right|^2 \right)^{\frac{1}{2}} + C \\
&\le \frac{R}{\sqrt{N-1}} \left(\frac{1}{2} \int_{\partial B_R} |\nabla_T \theta|^2 + \frac{1}{2} \int_{\partial B_R} \left| \frac{\partial \theta}{\partial \nu} \right|^2 \right) + C \\
&= \frac{R}{2\sqrt{N-1}} \int_{\partial B_R} |\nabla \theta|^2 + C \le \frac{R}{2\delta^2 \sqrt{N-1}} \int_{\partial B_R} \rho^2 \, |\nabla \theta|^2 + C,
\end{aligned}
$$

which is equivalent to :

$$
\forall\, R > R_2 \qquad e(R) \le \frac{R}{2\delta^2 \sqrt{N-1}} e^{'}(R) + C, \tag{2.6}
$$

where we have let $e(R) := \int_{\Omega_R} \rho^2 |\nabla \theta|^2$.

Set $\mu := 2\delta^2 \sqrt{N-1}$. Then $\mu > \alpha$ and from (2.6) we obtain :

$$
\forall\, R > R_2 \qquad \left(R^{-\mu}(e(R) - C) \right)^{'} \ge 0.
$$

By integrating the latter we see that either :

$$
\forall\, R > R_2 \qquad e(R) - C \le 0 \tag{2.7}
$$

or there exists $R_3 \ge R_2$ and $\gamma > 0$ such that :

$$
\forall\, R > R_3 \qquad e(R) - C \ge \gamma R^{\mu}.
$$

Since the latter is impossible by assumption (1.4), we deduce that (2.7) must hold. This implies the desired conclusion (2.4).

Now we observe that

$$\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}|\nabla\rho|^2\varphi^2=\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}2\varphi(1-\rho)\nabla\rho\nabla\varphi \\ +\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\rho(1-\rho)(\rho^2-1+|\nabla\theta|^2)\varphi^2-\int_{\partial B_{R_2}}\frac{\partial\rho}{\partial\nu}(1-\rho)\varphi^2 \quad (2.8)$$

for every $\varphi\in C_c^2(\mathbb{R}^N)$. Just multiply the second equation of (2.2) by $\varphi^2(1-\rho)$ and integrate by parts. (2.8) can also be written as

$$\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\left[|\nabla\rho|^2+\rho^2|\nabla\theta|^2\right]\varphi^2=\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}2\varphi(1-\rho)\nabla\rho\nabla\varphi \\ +\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\rho|\nabla\theta|^2\varphi^2+\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\rho(1-\rho)(\rho^2-1)\varphi^2-\int_{\partial B_{R_2}}\frac{\partial\rho}{\partial\nu}(1-\rho)\varphi^2, \quad (2.9)$$

which gives

$$\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\left[|\nabla u|^2+\frac{\rho}{\rho+1}(|u|^2-1)^2\right]\varphi^2=\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}2\varphi(1-\rho)\nabla\rho\nabla\varphi \\ +\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\rho|\nabla\theta|^2\varphi^2-\int_{\partial B_{R_2}}\frac{\partial\rho}{\partial\nu}(1-\rho)\varphi^2. \quad (2.10)$$

Choosing in (2.10) $\varphi=\xi_R(x)=\xi(\frac{x}{R})$, with ξ a fixed smooth function satisfying $0\le\xi\le 1$, $\xi(x)=1$ for $|x|\le 1$ and $\xi(x)=0$ for $|x|\ge 2$, we have for every $R>R_2$

$$\frac{\delta}{2}\int_{B_R\setminus\overline{B_{R_2}}}|\nabla u|^2+(1-|u|^2)^2\le\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}2\xi_R(1-\rho^2)\nabla\rho\nabla\xi_R \\ +\delta^{-1}\int_{\mathbb{R}^N\setminus\overline{B_{R_2}}}\rho^2|\nabla\theta|^2\xi_R^2-\int_{\partial B_{R_2}}\frac{\partial\rho}{\partial\nu}(1-\rho). \quad (2.11)$$

On the other hand, multiplying (1.1) by $u\xi_R$ and integrating by parts yields

$$\int_{\mathbb{R}^N}\left[|\nabla u|^2+(1-|u|^2)^2\right]\xi_R+\frac{1}{2}\int_{\mathbb{R}^N}(1-|u|^2)\Delta\xi_R=\int_{\mathbb{R}^N}(1-|u|^2)\xi_R. \quad (2.12)$$

Since $\|u\|_{L^\infty}\le 1$, standard elliptic estimates imply $\|\nabla u\|_{L^\infty}\le C_N$, where C_N is a positive constant depending only on N. Using this in (2.11), togheter with (2.4), we get for every $R>R_2$

$$\int_{B_R\setminus\overline{B_{R_2}}} |\nabla u|^2 + (1-|u|^2)^2 \leq C^{'}(1+R^{N-1}) \tag{2.13}$$

where $C^{'}$ is a positive constant independent of R. Combining (2.13) and(2.12) we have for every $R > R_2$

$$\int_{B_R} (1-|u|^2) \leq C^{''}(1+R^{N-1}) \tag{2.14}$$

where $C^{''}$ is a positive constant independent of R. Using the latter information into (2.11) yields

$$\int_{B_R\setminus\overline{B_{R_2}}} |\nabla u|^2 + (1-|u|^2)^2 \leq C^{'''}(1+R^{N-2}) \tag{2.15}$$

where again, $C^{'''}$ is a positive constant independent of R. Iterating this procedure, after a finite number of steps, we find the existence of a constant $C > 0$, independent of R, such that

$$\int_{B_R\setminus\overline{B_{R_2}}} |\nabla u|^2 + (1-|u|^2)^2 \leq C \qquad \forall\, R > R_2. \tag{2.16}$$

Thus $|\nabla u| \in L^2(\mathbb{R}^N)$, which concludes the proof.

□

Proof of Theorem 1.1. Let us prove that

$$\int_{B_R} |\nabla u|^2 \leq CR^{N-1} \qquad \forall\, R > 1, \tag{2.17}$$

for some constant $C > 0$ independent of R.

Indeed, let $\psi_R \in C^2_c(\mathbb{R}^N)$ satisfy $0 \leq \psi_R \leq 1$, $\psi_R(x) = 1$ for $|x| \leq R-1$, $\psi_R(x) = 0$ for $|x| \geq R$, $\|\nabla\psi_R\|_{L^\infty} \leq 2$ and consider the map

$$v_R := \psi_R(1-u) + u \tag{2.18}$$

Using the local minimality of u over the ball B_R and v_R as a competitor we have

$$E(u,B_R) \leq E(v_R,B_R) = \int_{B_R} \frac{1}{2}|\nabla v_R|^2 + \frac{1}{4}(1-|v_R|^2)^2$$
$$= \int_{B_R\setminus B_{R-1}} \frac{1}{2}|\nabla v_R|^2 + \frac{1}{4}(1-|v_R|^2)^2 \leq C_N\mathcal{L}_N(B_R\setminus B_{R-1}) = C^{'}_N R^{N-1} \tag{2.19}$$

where $\mathcal{L}_N$ denotes the $N-$dimensional Lebesgue measure and $C_N^{'}$ is a positive constant depending only on the dimension N. This proves (2.17).

Since $N = 3$ or 4, we have that $\alpha := N - 1 < 2\sqrt{N-1}$. This enables us to apply Theorem (1.2) to reach the conclusion.

□

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