

# GEOMETRIC RIGIDITY FOR INCOMPATIBLE FIELDS AND AN APPLICATION TO STRAIN-GRADIENT PLASTICITY

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ABSTRACT. In this paper we show that a strain-gradient plasticity model arises as the  $\Gamma$ -limit of a *nonlinear* semi-discrete dislocation energy. We restrict our analysis to the case of plane elasticity, so that edge dislocations can be modelled as point singularities of the strain field.

A key ingredient in the derivation is the extension of the rigidity estimate [10, Theorem 3.1] to the case of fields  $\beta : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$  with nonzero curl. We prove that the  $L^2$ -distance of  $\beta$  from a single rotation matrix is bounded (up to a multiplicative constant) by the  $L^2$ -distance of  $\beta$  from the group of rotations in the plane, modulo an error depending on the total mass of  $\text{Curl } \beta$ . This reduces to the classical rigidity estimate in the case  $\text{Curl } \beta = 0$ .

KEYWORDS:  $\Gamma$ -convergence, rigidity estimate, nonlinear plane elasticity, edge dislocations, strain-gradient plasticity.

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## 1. INTRODUCTION

The permanent (or plastic) deformations of metals rely on the presence of many types of defects in their atomic structure. Dislocations are one type of such defects and they play a prominent role in the so-called plastic slip, the relative slip of atomic layers that alters permanently the lattice structure of a metal. For this reason there is an increasing interest and effort in the derivation of plasticity models from dislocation models, both in the mathematical and in the mechanical engineering communities (see e.g. [4, 6, 11, 12, 13, 15, 16]). Clearly, the large freedom in the choice of the dislocation model has a strong influence on the method of derivation and on the resulting plasticity theories, and therefore requires some care.

In most of the cases the starting point is a *semi-discrete* (mesoscopic) dislocation model in which the dislocations are modelled individually, while the underlying atomic lattice is averaged out. This simplification is supported by the fact that at low strains the interatomic distance (of the order of few tenth of a nanometer) is much smaller than the typical distance between two dislocations (few microns). For straight and parallel *edge* dislocations the natural setting is that of plane elasticity. Indeed in this case only the two components of the strain on the slip plane are relevant and the positions of the dislocations are completely identified by the intersection of the dislocation lines with an orthogonal plane; *i.e.*, by their trace on a two-dimensional domain  $\Omega$ . Moreover, in semi-discrete models the dislocation energy is usually assumed to be quadratic (see, e.g., [4, 9, 11, 19]). More precisely, the energy is given by

$$\frac{1}{2} \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\beta : \beta \, dx, \tag{1.1}$$

where  $\mathbb{C} \in \mathbb{R}^{4 \times 4}$  is the elasticity tensor,  $\beta: \Omega \rightarrow \mathbb{R}^{2 \times 2}$  denotes the elastic part of the strain of a planar deformation, and  $\Omega_\varepsilon(\mu)$  is obtained from  $\Omega$  by removing discs of radius  $\varepsilon > 0$ , the so-called *core* regions, around each dislocation, on which the measure  $\mu$  (the dislocation density) is supported. The dislocation density  $\mu$  is a measure of the amount of disturbance in the lattice due to the presence of dislocations, and is related to the incompatibility of the strain  $\beta$ ; *i.e.*, to  $\text{Curl}\beta$ . Notice that in this linear setting the  $\varepsilon$ -regularisation of the energy (1.1), although not ideal, is necessary to prevent the blow-up of the energy at the dislocations. Moreover, also the assumption of a linear relation between stress and strain, which is equivalent to assuming small deformations, is debatable. Indeed, few atoms away from a dislocation the use of the quadratic energy (1.1) is justified, since the presence of dislocations causes a very local lattice distortion. However, this description is not satisfactory close to the dislocations, where the strains are too large for the linear approximation to hold. Moreover, in presence of a “large” number of dislocations, the question of reducing to the small-strains case is more subtle. Considering a more general, nonlinear dislocation energy is therefore desirable. This general principle triggered the analysis done in [20], where the authors considered a nonlinear dislocation energy of the form

$$\int_{\Omega} W(\beta) dx \quad (1.2)$$

where the energy density  $W: \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$  satisfies the usual assumptions of nonlinear elasticity (e.g. stress-free reference configuration and frame indifference). In addition,  $W$  is required to satisfy mixed growth conditions (considered also in e.g. [17]) ensuring that far from dislocations the energy is essentially quadratic; *i.e.*,  $W(\beta) \sim \text{dist}^2(\beta, SO(2))$ , whereas close to the defects  $W(\beta) \sim |\beta|^p$ , for some  $p \in (1, 2)$ . Therefore  $W(\beta)$  is integrable also close to the dislocations, thus the  $\varepsilon$ -regularization needed in the linear case is no longer necessary.

In [20] the authors considered the case of a *finite* number  $N$  of fixed edge dislocations located at points  $x_1, \dots, x_N$  with Burgers vectors  $\varepsilon \hat{b}_1, \dots, \varepsilon \hat{b}_N$ , where  $|\hat{b}_i| = 1$  and  $\varepsilon > 0$  is proportional to the interatomic spacing, and analysed the asymptotic behaviour of the scaled energies

$$\frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{\Omega} W(\beta) dx \quad (1.3)$$

in the limit as  $\varepsilon$  tends to zero, by  $\Gamma$ -convergence. In (1.3), the strain  $\beta$  and the dislocation density which is encoded in the measure  $\mu = \sum_{i=1}^N \varepsilon \hat{b}_i \delta_{x_i}$  are coupled via the relation  $\text{Curl}\beta = \mu$ . In [20] it was shown that the energies (1.3) give rise in the limit to the line-tension plasticity model described by

$$\frac{1}{2} \int_{\Omega} \mathbb{C} \nabla u : \nabla u dx + \sum_{i=1}^N \psi(R^T \hat{b}_i), \quad (1.4)$$

where  $\mathbb{C} = \frac{\partial^2 W}{\partial F^2}(I)$ ,  $\psi$  is given in terms of an asymptotic cell formula,  $\nabla u$  is the limit of a sequence of suitably renormalized strains, and  $R$  is a rotation whose presence is characteristic of the nonlinear setting. Hence, although in the  $\varepsilon$ -energy (1.3) the strain  $\beta$  and the dislocation density  $\mu$  are coupled, their limit objects are decoupled in the limit energy (1.4), that depends on a curl-free strain  $\nabla u$ . The decoupling is typical of this *dilute* regime (see also [4, 11]) and is due to the fact that the strains  $\beta$  “live” on a scale  $\varepsilon \sqrt{|\log \varepsilon|}$  while the dislocation densities  $\mu$  on the smaller scale  $\varepsilon$ . As a consequence, in this regime, the limit procedure yields a pair of

macroscopic decoupled variables and therefore two corresponding decoupled terms in the limit energy.

In analogy with the linear case [11], in order to overcome the degeneracy of the dilute regime, in the present paper we consider a scaling of the nonlinear energy (1.2) of order  $\varepsilon^2 |\log \varepsilon|^2$ . Loosely speaking, considering this different scaling corresponds to considering a system of  $|\log \varepsilon|$  dislocations. Then, our aim is to derive in the limit as  $\varepsilon$  tends to zero a *strain-gradient* model for plasticity; *i.e.*, a model in which the energy depends on an incompatible field, and in which elastic energy and dislocation energy are coupled.

From a mathematical point of view the transition between a finite and an “infinite” number of defects is highly nontrivial. Indeed in the linear case it required a rather sophisticated tool, namely a Korn-type inequality for fields with nonzero curl, see [11, Theorem 11]. Analogously, in our nonlinear setting, it requires an extension of the rigidity estimate [10, Theorem 3.1] to the case of incompatible fields. More precisely, in Theorem 3.3 we prove that if  $\Omega \subset \mathbb{R}^2$  is open, bounded, simply connected, and with Lipschitz boundary, then for every  $\beta \in L^2(\Omega; \mathbb{R}^{2 \times 2})$  whose curl is a measure with bounded total variation there exists a constant rotation  $R \in SO(2)$  such that

$$\|\beta - R\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C(\|\text{dist}(\beta, SO(2))\|_{L^2(\Omega)} + |\text{Curl} \beta|(\Omega)), \tag{1.5}$$

for some  $C > 0$  depending only on  $\Omega$ . Notice that (1.5) clearly reduces to the classical rigidity estimate when  $\text{Curl} \beta = 0$ . The above generalised rigidity estimate is one of the main results of this paper and would appear to be widely applicable. The key ingredients of the proof of (1.5) are an  $L^p + L^q$  rigidity estimate recently proved in [5] and a fine regularity result for two-dimensional  $L^1$ -vector fields with divergence in  $H^{-2}$  proved in [3] (see also [2]).

Coming back to our model, in the present paper we treat the case of “infinitely many” dislocations in the nonlinear setting, although under more restrictive coerciveness assumptions on the energy density  $W$  than in [20]. More precisely, the dislocation energy in our model is given by

$$\int_{\Omega_\varepsilon(\mu)} W(\beta) \, dx, \tag{1.6}$$

where the *nonlinear* energy density  $W$  behaves essentially as  $\text{dist}^2(\beta, SO(2))$  (see Section 2 for more details), and the strain  $\beta$  satisfies at any dislocation point  $x_i$  the incompatibility condition

$$\int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \varepsilon \hat{b}_i,$$

where, as above,  $\hat{b}_i$  is the direction of the Burgers vector associated to the dislocation located at  $x_i$ , and  $\varepsilon$  is proportional to the interatomic distance. Unlike the case of fixed dislocations studied in [20], where the dislocation density  $\mu$  was constant (up to an  $\varepsilon$ -scaling) and the energy (1.2) depended only on the strain  $\beta$ , in the present case the distribution of dislocations  $\mu = \sum_{i=1}^N \varepsilon \hat{b}_i \delta_{x_i}$  is a variable of the problem, and therefore the dislocation energy (1.6) depends on both  $\beta$  and  $\mu$ . Notice that, due to the quadratic growth of the energy density, also in this nonlinear setting the  $\varepsilon$ -regularization of the dislocation energy is needed, as in the linear case [11].

In order to obtain in the limit a strain-gradient model we consider the rescaled functionals

$$\frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega_\varepsilon(\mu)} W(\beta) \, dx, \tag{1.7}$$

and analyse their asymptotic behaviour via  $\Gamma$ -convergence as  $\varepsilon$  tends to zero.

As in [20] also here the key idea is to rigorously reduce to the linear setting in the spirit of [7]. To this end it is necessary to show that, in particular, sequences of strains with equibounded energies converge to constant rotations of the plane (minimisers of the nonlinear energy). In the case of an infinite number of dislocations, however, the compactness of the strains does not follow from the corresponding result in [20]. It follows instead from the generalised rigidity estimate (1.5), which allows us to perform a second order Taylor expansion of the energy around a rotation, and to get a quadratic functional in terms of a *renormalised* strain. At this point the final step of our approach is to apply previous results known for linear energies to the linearised functional. Then, as in [11], the  $\Gamma$ -limit is a strain-gradient plasticity energy (see Theorem 4.6) and has the form

$$\frac{1}{2} \int_{\Omega} \mathbb{C} \beta : \beta \, dx + \int_{\Omega} \varphi(R, \text{Curl } \beta) \, dx,$$

where  $\beta$  is the limit of suitably scaled strains and  $R \in SO(2)$  is the limit of the sequence of constant rotations provided by the generalised rigidity estimate (see Proposition 4.3). Concerning the densities of the two terms in the energy, the elasticity tensor  $\mathbb{C}$  equals  $\frac{\partial^2 W}{\partial F^2}(I)$ , while the plastic energy density  $\varphi$  is defined in terms of an asymptotic cell formula and is such that  $\varphi(R, \cdot)$  is positively 1-homogeneous and convex.

This paper is organised as follows: Section 2 is devoted to the introduction of the necessary notation and to the definition of the mesoscopic dislocation model. Then, the two main results, namely the generalised rigidity estimate and the  $\Gamma$ -convergence result, are treated in Sections 3 and 4, respectively.

## 2. NOTATION AND SETTING OF THE PROBLEM

In this section we introduce the *nonlinear mesoscopic* dislocation energy associated to the (elastic part of the) deformation strain in presence of a system of straight and parallel *edge dislocations*. In this setting the dislocations are modelled by points in the plane.

Let  $\Omega \subset \mathbb{R}^2$  be a simply connected, bounded, Lipschitz domain representing a horizontal section of an infinite cylindrical crystal. Let  $S := \{b_1, b_2\}$  be a set of admissible (renormalised) Burgers vectors for the crystal; *i.e.*,  $b_1, b_2 \in \mathbb{R}^2$  are two linearly independent vectors depending on the crystalline structure, *e.g.*, for a square lattice  $S = \{e_1, e_2\}$ . We also consider

$$\mathbb{S} := \text{Span}_{\mathbb{Z}} S,$$

the span of  $S$  with integer coefficients; *i.e.*, the set of (renormalised) Burgers vectors for multiple dislocations. Every dislocation is then characterised by a point  $x_i \in \Omega$  and by a vector  $\xi_i \in \mathbb{S}$ .

For the given crystal, let  $\varepsilon > 0$  denote the interatomic distance. We assume that the distance between two distinct dislocations is bounded from below in terms of an intermediate scale  $\rho_\varepsilon \gg \varepsilon$ , with  $\rho_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This assumption implies that dislocations are well separated (with respect to the atomic spacing  $\varepsilon$ ); *i.e.*, there is a scale separation between  $\varepsilon$ , the scale of the atomic lattice, and the scale of the dislocations distribution, represented by  $\rho_\varepsilon$ . We refer the reader to the recent paper [9] where the assumption that dislocations are well-separated is overcome in the case of a quadratic energy of type (1.1) and for finite number of defects. Here we also require that (cf. [11])

- (1)  $\lim_{\varepsilon \rightarrow 0} \rho_\varepsilon / \varepsilon^s = +\infty$ , for every fixed  $s \in (0, 1)$ ;
- (2)  $\lim_{\varepsilon \rightarrow 0} |\log \varepsilon| \rho_\varepsilon^2 = 0$ .

Under this assumptions on the *hard-core* scale  $\rho_\varepsilon$ , we will show that in the limit the energy can be decomposed into two contributions: a *self energy*, which is concentrated in the hard-core regions  $B_{\rho_\varepsilon}(x_i)$ , and an *interaction energy*, which is essentially all stored outside the union of the hard-core regions.

We define the class  $X_\varepsilon$  of the admissible dislocation densities as

$$X_\varepsilon := \left\{ \mu \in \mathcal{M}(\Omega; \mathbb{R}^2) : \mu = \sum_{i=1}^M \varepsilon \xi_i \delta_{x_i}, M \in \mathbb{N}, B_{\rho_\varepsilon}(x_i) \subset \Omega, \right. \\ \left. |x_j - x_k| \geq 2\rho_\varepsilon \text{ for every } j \neq k, \xi_i \in \mathbb{S} \right\}, \quad (2.1)$$

where  $\mathcal{M}(\Omega; \mathbb{R}^2)$  denotes the space of vector-valued Radon measures on  $\Omega$  and, for every  $i$ ,  $\delta_{x_i}$  denotes the Dirac mass centred at  $x_i$ .

For a given measure  $\mu \in X_\varepsilon$  and  $r > 0$  we define

$$\Omega_r(\mu) := \Omega \setminus \bigcup_{x_i \in \text{supp}(\mu)} \overline{B}_r(x_i). \quad (2.2)$$

The class of admissible strains associated with any  $\mu \in X_\varepsilon$  is given by those  $\beta \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^{2 \times 2})$  satisfying

$$\text{Curl } \beta = 0 \text{ in } \Omega_\varepsilon(\mu) \quad \text{and} \quad \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \varepsilon \xi_i, \text{ for } i = 1, \dots, M,$$

where the equality  $\text{Curl } \beta = 0$  is intended in the sense of distributions.<sup>1</sup> The vector  $t$  above denotes the oriented tangent vector<sup>2</sup> to  $\partial B_\varepsilon(x_i)$  and the integrand  $\beta \cdot t$  is intended in the sense of traces (see [8, Theorem 2, pag. 204]).

Then, in this mathematical setting, an admissible  $\mu$  measures the failure of the condition of being a gradient for the strain  $\beta$  and the presence of dislocations can be detected by looking at the topological singularities of  $\beta$ .

Let  $SO(2) := \{R \in \mathbb{R}^{2 \times 2} : R^T R = I, \det R = 1\}$  be the set of rotations in  $\mathbb{R}^{2 \times 2}$ . Moreover, for  $F \in \mathbb{R}^{2 \times 2}$  we denote the Euclidean nom by  $|F| = \sqrt{\text{tr } F^T F}$ .

The elastic energy density  $W : \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$  satisfies the usual assumptions of nonlinear elasticity, namely

- (i)  $W \in C^0(\mathbb{R}^{2 \times 2})$ ,  $W \in C^2$  in a neighbourhood of  $SO(2)$ ;
- (ii) the reference configuration is stress-free; *i.e.*,  $W(I) = 0$ ;
- (iii)  $W$  is frame indifferent; *i.e.*,  $W(RF) = W(F)$  for every  $F \in \mathbb{R}^{2 \times 2}$  and  $R \in SO(2)$ .

Moreover,  $W$  satisfies the following growth condition:

<sup>1</sup>For a matrix  $\beta \in \mathbb{R}^{2 \times 2}$ ,  $\text{Curl } \beta$  is the vector field of  $\mathbb{R}^2$  defined as  $\text{Curl } \beta = (\partial_1 \beta_{12} - \partial_2 \beta_{11}, \partial_1 \beta_{22} - \partial_2 \beta_{21})$ .

<sup>2</sup>We choose  $t = \nu^\perp$  to be a counterclockwise  $\pi/2$ -rotation of the outward normal  $\nu$  to  $\partial B_\varepsilon$ .

(iv) there exist two constants  $C_1, C_2 > 0$  such that for every  $F \in \mathbb{R}^{2 \times 2}$

$$C_1 \text{dist}^2(F, SO(2)) \leq W(F) \leq C_2 \text{dist}^2(F, SO(2)),$$

where  $\text{dist}(F, SO(2)) := \inf_{R \in SO(2)} |F - R|$ .

The lower bound in (iv) states that the energy wells are non-degenerate and is widely used in nonlinear elasticity. The upper bound, though, is rather restrictive as it rules out the physically relevant case of orientation-preserving deformations as well as the blow-up of the energy for a total compression. Although partially unsatisfactory, the upper bound in (iv) is heavily used in the proof of the  $\Gamma$ -convergence result (Theorem 4.6) to guarantee that the energy along the recovery sequence is linear, up to a small error.

Due to the quadratic growth (iv) the energy associated to an admissible pair  $(\mu, \beta)$  is well defined only away from the dislocations, as in the linear case, namely in the domain  $\Omega_\varepsilon(\mu)$ :

$$\int_{\Omega_\varepsilon(\mu)} W(\beta) dx.$$

In what follows it is useful to extend the admissible strains  $\beta$  to the whole domain  $\Omega$ . There are different possible extensions compatible with our model. Here we decide to consider  $\beta = I$  in the discs  $B_\varepsilon(x_i)$ . Therefore, from now on the class of admissible strains associated with a measure  $\mu \in X_\varepsilon$  is given by

$$\mathcal{AS}_\varepsilon(\mu) := \left\{ \beta \in L^2(\Omega; \mathbb{R}^{2 \times 2}) : \beta \equiv I \text{ in } \cup_{i=1}^M B_\varepsilon(x_i), \text{Curl } \beta = 0 \text{ in } \Omega_\varepsilon(\mu), \right. \\ \left. \int_{\partial B_\varepsilon(x_i)} \beta \cdot t ds = \varepsilon \xi_i, \text{ for } i = 1, \dots, M \right\}. \quad (2.3)$$

By (ii) we can rewrite the energy associated to an (extended) admissible strain  $\beta \in \mathcal{AS}_\varepsilon(\mu)$  as

$$E_\varepsilon(\mu, \beta) := \int_{\Omega} W(\beta) dx.$$

For our purposes, as in the linear case [11], the relevant scaling for the energy is  $\varepsilon^2 |\log \varepsilon|^2$ ; therefore we consider the scaled nonlinear dislocation energy given by

$$\mathcal{E}_\varepsilon(\mu, \beta) := \begin{cases} \frac{1}{\varepsilon^2 |\log \varepsilon|^2} E_\varepsilon(\mu, \beta) & \text{if } \mu \in X_\varepsilon, \beta \in \mathcal{AS}_\varepsilon(\mu), \\ +\infty & \text{otherwise in } \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2}). \end{cases} \quad (2.4)$$

Then, as in [11], we notice that this is the only scaling of the energy for which the strain  $\beta$  and the measure  $\mu$  are of the same order in  $\varepsilon$ . This results into a coupling of their limit rescaled objects, and therefore to a strain-gradient plasticity model (Theorem 4.6).

### 3. RIGIDITY ESTIMATE FOR FIELDS WITH PRESCRIBED CURL

In this section we prove a generalised rigidity estimate for vector fields with nonzero curl. This result provides a quantitative estimate of the distance of a two-dimensional matrix-valued field from a constant rotation in terms of its distance from the set of rotations of the plane, like the classical rigidity estimate [10] in two dimensions, with an additional term depending on the total mass of the curl.

Before proving the desired result, for the reader's convenience we state here a variant of the Rigidity Estimate recently proved in [5]. To this end, we first recall some useful notation.

Let  $U \subset \mathbb{R}^n$  be a measurable set. We denote by  $L^{2,\infty}(U; \mathbb{R}^m)$  the space of weak- $L^2$  functions; i.e.,  $f \in L^{2,\infty}(U; \mathbb{R}^m)$  if and only if  $f$  is measurable and there exists a constant  $C > 0$  such that

$$\mathcal{L}^n(\{x \in U : |f(x)| > \lambda\}) \leq \frac{C^2}{\lambda^2}, \quad \text{for every } \lambda > 0.$$

We also set

$$\|f\|_{L^{2,\infty}(U; \mathbb{R}^m)} := \inf \{C > 0 : \lambda \mathcal{L}^n(\{|f| > \lambda\})^{1/2} \leq C, \forall \lambda > 0\}.$$

Notice that  $\|\cdot\|_{L^{2,\infty}(U; \mathbb{R}^m)}$  is not a norm but only a quasi-norm since the Minkowski Inequality holds only in the following form

$$\|f + g\|_{L^{2,\infty}(U; \mathbb{R}^m)} \leq 2\|f\|_{L^{2,\infty}(U; \mathbb{R}^m)} + 2\|g\|_{L^{2,\infty}(U; \mathbb{R}^m)}.$$

If  $f \in L^2(U; \mathbb{R}^m)$  then clearly  $f \in L^{2,\infty}(U; \mathbb{R}^m)$  and  $\|f\|_{L^{2,\infty}(U; \mathbb{R}^m)} \leq \|f\|_{L^2(U; \mathbb{R}^m)}$ ; but  $L^2(U; \mathbb{R}^m) \subsetneq L^{2,\infty}(U; \mathbb{R}^m)$  as, for example,  $1/|x|^{n/2}$  belongs to  $L^{2,\infty}(U)$  but not to  $L^2(U)$ .

We are now ready to recall the weak rigidity estimate (see [5]).

**Theorem 3.1** ( $L^{2,\infty}$ -rigidity). *Let  $U$  be a bounded Lipschitz domain of  $\mathbb{R}^n$ . There exists a constant  $C = C(U) > 0$  with the following property: For every  $u \in L^1(U; \mathbb{R}^n)$  such that  $\nabla u \in L^{2,\infty}(U; \mathbb{R}^{n \times n})$  there is an associated rotation  $R \in SO(n)$  such that*

$$\|\nabla u - R\|_{L^{2,\infty}(U; \mathbb{R}^{n \times n})} \leq C \|\text{dist}(\nabla u, SO(n))\|_{L^{2,\infty}(U)}. \quad (3.1)$$

We prove a technical result we use in what follows.

**Proposition 3.2.** *Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function such that  $|g(t)| \leq \gamma|t|^\alpha$ , for some  $\gamma > 0$  and for some  $\alpha > 1$ . Let  $U \subset \mathbb{R}^n$  be a measurable set; if  $\theta \in L^{2,\infty}(U)$  then  $g \circ \theta \in L^2(U)$  and*

$$\|g \circ \theta\|_{L^2(U)} \leq M \|\theta\|_{L^{2,\infty}(U)}, \quad (3.2)$$

where  $M := \max\{\|g\|_{L^\infty(U)}, \gamma 2^\alpha / (1 - 4^{(1-\alpha)})^{1/2}\}$ .

*Proof.* We have

$$\int_U |g(\theta)|^2 dx = \int_{\{x: |\theta| > 1\}} |g(\theta)|^2 dx + \int_{\{x: |\theta| \leq 1\}} |g(\theta)|^2 dx. \quad (3.3)$$

The first term in the right hand side of (3.3) can be easily estimated appealing to the boundedness of  $g$ , in fact

$$\int_{\{x: |\theta| > 1\}} |g(\theta)|^2 \leq \|g\|_{L^\infty(U)}^2 \mathcal{L}^n(\{x: |\theta| > 1\}) \leq \|g\|_{L^\infty(U)}^2 \|\theta\|_{L^{2,\infty}(U)}^2. \quad (3.4)$$

For the second term in (3.3) we proceed as follows. Using the growth assumption on  $g$  we find

$$\int_{\{x: |\theta| \leq 1\}} |g(\theta)|^2 dx \leq \gamma^2 \int_{\{x: 0 < |\theta| \leq 1\}} \theta^{2\alpha} dx. \quad (3.5)$$

For  $\delta \in (0, 1/2]$  we have

$$\int_{\{x: \delta < |\theta| \leq 2\delta\}} \theta^{2\alpha} dx \leq 4^\alpha \delta^{2\alpha} \mathcal{L}^n(\{x: |\theta| > \delta\}) \leq 4^\alpha \delta^{2(\alpha-1)} \|\theta\|_{L^{2,\infty}(U)}^2. \quad (3.6)$$

Therefore using (3.6) with  $\delta = 1/2^k$  and  $k \in \mathbb{N}$  we get

$$\begin{aligned} \int_{\{x: 0 < |\theta| \leq 1\}} \theta^{2\alpha} dx &= \sum_{k \geq 1} \int_{\{x: \frac{1}{2^k} < |\theta| \leq \frac{1}{2^{k-1}}\}} \theta^{2\alpha} dx \\ &\leq 4^\alpha \sum_{k \geq 0} \frac{1}{4^{(\alpha-1)k}} \|\theta\|_{L^{2,\infty}(U)}^2 \\ &= \frac{4^\alpha}{1 - 4^{1-\alpha}} \|\theta\|_{L^{2,\infty}(U)}^2. \end{aligned}$$

Finally, combining the last inequality with (3.3)-(3.5) entails the thesis.  $\square$

The following theorem is the main result of this section. It states that in dimension two the rigidity estimate holds true also for vector fields with nonvanishing curl, modulo an error depending on the total mass of the curl. This result is the nonlinear counterpart of the generalised Korn Inequality proved in [11, Theorem 11].

**Theorem 3.3** (Generalised Rigidity Estimate). *Let  $\Omega \subset \mathbb{R}^2$  be open, bounded, simply connected, and Lipschitz. There exists a constant  $C = C(\Omega) > 0$  with the following property: For every  $\beta \in L^2(\Omega; \mathbb{R}^{2 \times 2})$  with  $\mu := \text{Curl} \beta \in \mathcal{M}_b(\Omega; \mathbb{R}^2)$  there is an associated rotation  $R \in SO(2)$  such that*

$$\|\beta - R\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C(\|\text{dist}(\beta, SO(2))\|_{L^2(\Omega)} + |\mu|(\Omega)). \quad (3.7)$$

*Proof.* Set  $\delta := \|\text{dist}(\beta, SO(2))\|_{L^2(\Omega)} + |\mu|(\Omega)$ .

Notice that for  $i = 1, 2$

$$\mu_i = \text{curl}(\beta^T e_i) = -\text{div}(J(\beta^T e_i)), \quad \text{with } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

therefore  $\mu \in H^{-1}(\Omega; \mathbb{R}^2)$  and there exists a unique solution to the following problem:

$$\begin{cases} -\Delta v = \mu & \text{in } \Omega, \\ v \in H_0^1(\Omega; \mathbb{R}^2). \end{cases} \quad (3.8)$$

By classical regularity theory for linear elliptic systems with measure data (see e.g. [18] and references therein) we have

$$\|\nabla v\|_{L^{2,\infty}(\Omega; \mathbb{R}^{2 \times 2})} \leq C|\mu|(\Omega). \quad (3.9)$$

Let  $\tilde{\beta} := \nabla v J$ ; in view of (3.8) we have that  $\text{Curl} \tilde{\beta} = \mu$ . Hence,  $\text{Curl}(\beta - \tilde{\beta}) = 0$  in  $\Omega$ , which implies the existence of  $u \in H^1(\Omega; \mathbb{R}^2)$  such that  $\beta - \tilde{\beta} = \nabla u$  a.e. in  $\Omega$ . Then we have

$$\begin{aligned} \text{dist}(\nabla u, SO(2)) &= \text{dist}(\beta - \tilde{\beta}, SO(2)) \\ &\leq \text{dist}(\beta, SO(2)) + |\tilde{\beta}| \\ &= \text{dist}(\beta, SO(2)) + |\nabla v|. \end{aligned} \quad (3.10)$$

This implies, by (3.9) and by the definition of  $\delta$ , that

$$\|\text{dist}(\nabla u, SO(2))\|_{L^{2,\infty}(\Omega)} \leq C\delta.$$

Then, Theorem 3.1 provides us with a constant  $C > 0$  and a constant rotation  $R \in SO(2)$  such that

$$\|\nabla u - R\|_{L^{2,\infty}(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta,$$



and as a consequence

$$\|\beta - R\|_{L^2, \infty(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta. \quad (3.11)$$

Without loss of generality we may assume that  $R = I$  (otherwise we consider  $R^T \beta$ ).

Let  $\vartheta: \Omega \rightarrow [-\pi, \pi)$  be a measurable function such that the corresponding rotation

$$R(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$$

satisfies

$$|\beta(x) - R(\vartheta(x))| = \text{dist}(\beta(x), SO(2))$$

for a.e.  $x \in \Omega$ . Then, (3.11) yields

$$\|I - R(\vartheta)\|_{L^2, \infty(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta. \quad (3.12)$$

Since  $|I - R(\vartheta)| \geq |\vartheta|/2$  for a.e.  $x \in \Omega$ , by (3.12) we deduce that

$$\|\vartheta\|_{L^2, \infty(\Omega)} \leq C\delta. \quad (3.13)$$

We now consider the linearisation of the rotation  $R(\vartheta)$  around zero, namely

$$R_{\text{lin}}(\vartheta) := \begin{pmatrix} 1 & -\vartheta \\ \vartheta & 1 \end{pmatrix}.$$

Appealing to Proposition 3.2 with  $g(t) = \cos t - 1$ , or  $g(t) = \sin t - t$ , from (3.13) we derive the two following bounds

$$\|\cos \vartheta - 1\|_{L^2(\Omega)} \leq C\delta \quad \text{and} \quad \|\sin \vartheta - \vartheta\|_{L^2(\Omega)} \leq C\delta;$$

therefore  $\|R(\vartheta) - R_{\text{lin}}(\vartheta)\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta$ . Since

$$\|\beta - R_{\text{lin}}(\vartheta)\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq \|\text{dist}(\beta, SO(2))\|_{L^2(\Omega)} + \|R(\vartheta) - R_{\text{lin}}(\vartheta)\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta,$$

we have

$$\beta = R_{\text{lin}}(\vartheta) + h, \quad \text{with} \quad \|h\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta.$$

Then by the definition of  $R_{\text{lin}}$  we deduce

$$\text{Curl } \beta = -\nabla \vartheta + \text{Curl } h,$$

which in its turn implies

$$\text{div}((\text{Curl } \beta)^\perp) = \text{div}((\text{Curl } h)^\perp), \quad (3.14)$$

where, for a vector  $a \in \mathbb{R}^2$  we use the notation  $a^\perp := Ja$ .

Hence we have

$$\|\text{div}((\text{Curl } \beta)^\perp)\|_{H^{-2}(\Omega)} \leq C\|h\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta. \quad (3.15)$$

By [3, Theorem 3.1 and Remark 3.3] (see also [2]) if  $f \in L^1(\Omega; \mathbb{R}^2)$  is a vector field satisfying  $\text{div } f \in H^{-2}(\Omega)$ , then  $f$  also belongs to  $H^{-1}(\Omega; \mathbb{R}^2)$  and the following estimate holds true

$$\|f\|_{H^{-1}(\Omega; \mathbb{R}^2)} \leq C(\|\text{div } f\|_{H^{-2}(\Omega)} + \|f\|_{L^1(\Omega; \mathbb{R}^2)}).$$

This estimate clearly extends by density to measures with bounded total variation. Thus, by applying the previous estimate with  $f = (\text{Curl } \beta)^\perp$ , by virtue of (3.15) we have

$$\|(\text{Curl } \beta)^\perp\|_{H^{-1}(\Omega; \mathbb{R}^2)} \leq \|\text{div}((\text{Curl } \beta)^\perp)\|_{H^{-2}(\Omega)} + \|(\text{Curl } \beta)^\perp\|(\Omega) \leq C\delta. \quad (3.16)$$

Eventually, recalling that  $v$  solves (3.8), by (3.16) we deduce

$$\|\nabla v\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta,$$

and therefore, by (3.10),

$$\|\text{dist}(\nabla u, SO(2))\|_{L^2(\Omega)} \leq C\delta.$$

Hence the classical Rigidity Estimate [10, Theorem 3.1] provides us with a constant  $C > 0$  and with a constant rotation  $R' \in SO(2)$  such that

$$\|\nabla u - R'\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta,$$

thus

$$\|\beta - R'\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C\delta$$

and the thesis is achieved.  $\square$

#### 4. $\Gamma$ -CONVERGENCE OF THE NONLINEAR DISLOCATION ENERGY

In this section we study the asymptotic behaviour of the scaled energies  $\mathcal{E}_\varepsilon$ , defined in (2.4), as  $\varepsilon$  tends to zero. In the spirit of the  $\Gamma$ -convergence analysis performed in [11, 20], we show that a linearisation takes place in the limit and that the limit energy is a macroscopic *strain-gradient* model for plasticity, namely there is a nontrivial interplay between the interaction and the self energy.

**4.1. Cell formula for the limit self energy.** For the definitions and results contained in this subsection we refer the reader to [11, Section 6].

For later reference, it is convenient to introduce a new class of admissible (scaled) strains. For  $0 < r_1 < r_2 < 1$  and  $\xi \in \mathbb{R}^2$  we define

$$\mathcal{AS}_{r_1, r_2}(\xi) := \left\{ \eta \in L^2(B_{r_2} \setminus B_{r_1}) : \text{Curl } \eta = 0 \text{ in } B_{r_2} \setminus B_{r_1}, \int_{\partial B_{r_1}} \eta \cdot t \, ds = \xi \right\},$$

where  $B_r$  denotes the disc of radius  $r$  centred at 0. In the special case  $r_2 = 1$  we will simply write  $\mathcal{AS}_{r_1}(\xi)$  instead of  $\mathcal{AS}_{r_1, 1}(\xi)$ .

We also set

$$\psi(\xi, \delta) := \min \left\{ \frac{1}{2} \int_{B_1 \setminus B_\delta} \mathbb{C} \eta : \eta \, dx, \eta \in \mathcal{AS}_\delta(\xi) \right\}, \quad (4.1)$$

where  $\mathbb{C} = \frac{\partial^2 W}{\partial F^2}(I)$ .

We recall the following fundamental result (see [11, Corollary 6, Remark 7]).

**Proposition 4.1.** *Let  $\xi \in \mathbb{R}^2$  and  $\delta \in (0, 1)$ , and let  $\psi(\xi, \delta)$  be as in (4.1). Then for every  $\xi \in \mathbb{R}^2$*

$$\lim_{\delta \rightarrow 0} \frac{\psi(\xi, \delta)}{|\log \delta|} = \hat{\psi}(\xi),$$

where  $\hat{\psi} : \mathbb{R}^2 \rightarrow [0, +\infty)$  is defined by

$$\hat{\psi}(\xi) := \lim_{\delta \rightarrow 0} \frac{1}{|\log \delta|} \frac{1}{2} \int_{B_1 \setminus B_\delta} \mathbb{C} \eta_0 : \eta_0 \, dx, \quad (4.2)$$

and  $\eta_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$  is a distributional solution to

$$\begin{cases} \text{Curl } \eta = \xi \delta_0 & \text{in } \mathbb{R}^2, \\ \text{Div } \mathbb{C} \eta = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

**Remark 4.2.** Assume that  $\rho_\varepsilon$  satisfies (1) and (2), from [11, Proposition 8] it follows that the function  $\psi_\varepsilon : \mathbb{R}^2 \rightarrow [0, +\infty)$  defined as

$$\psi_\varepsilon(\xi) := \frac{1}{|\log \varepsilon|} \min \left\{ \frac{1}{2} \int_{B_{\rho_\varepsilon} \setminus B_\varepsilon} \mathbb{C} \eta : \eta \, dx, \eta \in \mathcal{AS}_{\varepsilon, \rho_\varepsilon}(\xi) \right\}, \quad (4.3)$$

satisfies

$$\psi_\varepsilon(\xi) = \frac{\psi(\xi, \varepsilon)}{|\log \varepsilon|} (1 + o(1)),$$

where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , uniformly with respect to  $\xi$ . Then, in particular,  $\psi_\varepsilon$  converges pointwise as  $\varepsilon \rightarrow 0$  to  $\hat{\psi}$  given by (4.2).

We are now in a position to define the density  $\varphi : SO(2) \times \mathbb{R}^2 \rightarrow [0, +\infty)$  of the self-energy through the following relaxation procedure:

$$\varphi(R, \xi) := \min \left\{ \sum_{k=1}^M \lambda_k \hat{\psi}(R^T \xi_k) : \sum_{k=1}^M \lambda_k \xi_k = \xi, M \in \mathbb{N}, \lambda_k \geq 0, \xi_k \in \mathbb{S} \right\}. \quad (4.4)$$

It follows from the above definition that the function  $\varphi$  is positively 1-homogeneous and convex (see also [11, Remark 9]).

**4.2. Compactness.** In the next proposition we prove a compactness result for sequences of pairs  $(\mu_\varepsilon, \beta_\varepsilon)$  with equibounded energy  $\mathcal{E}_\varepsilon$  by means of the generalised Rigidity Estimate Theorem 3.3.

**Proposition 4.3** (Compactness). *Let  $\varepsilon_j \rightarrow 0$  and let  $(\mu_j, \beta_j) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  be a sequence such that  $\sup_j \mathcal{E}_{\varepsilon_j}(\mu_j, \beta_j) < +\infty$ . Then there exist a sequence of constant rotations  $(R_j) \subset SO(2)$ , a measure  $\mu \in H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)$ , and a function  $\beta \in L^2(\Omega; \mathbb{R}^{2 \times 2})$  such that, up to subsequences,*

$$\frac{\mu_j}{\varepsilon_j |\log \varepsilon_j|} \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^2), \quad (4.5)$$

$$\frac{R_j^T \beta_j - I}{\varepsilon_j |\log \varepsilon_j|} \rightharpoonup \beta \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}); \quad (4.6)$$

moreover,  $\text{Curl } \beta = R^T \mu$ , where  $R := \lim_{j \rightarrow +\infty} R_j$ .

*Proof.* Let  $(\mu_j, \beta_j) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  be a sequence such that  $\mathcal{E}_{\varepsilon_j}(\mu_j, \beta_j) \leq C$  for some positive constant  $C$  independent of  $j$ , where

$$\mu_j = \sum_{i=1}^{M_j} \varepsilon_j \xi_{i,j} \delta_{x_{i,j}},$$

with  $\xi_{i,j} \in \mathbb{S}$ ,  $x_{i,j} \in \Omega$  such that  $B_{\rho_{\varepsilon_j}}(x_{i,j}) \subset \Omega$  and  $|x_{i,j} - x_{k,j}| \geq 2\rho_{\varepsilon_j}$  for every  $i \neq k$ .

The proof of the compactness is divided into three steps.

**Step 1.** *Weak convergence of the scaled dislocation measures.*

We first show that the sequence  $\mu_j/(\varepsilon_j |\log \varepsilon_j|)$  is uniformly bounded in mass. We claim that

$$\frac{1}{\varepsilon_j |\log \varepsilon_j|} |\mu_j|(\Omega) = \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} |\xi_{i,j}| \leq C. \quad (4.7)$$

Let  $s \in (0, 1)$  and  $\delta \in (0, 1)$  be fixed parameters. From the bound on the energy it follows that, for  $j$  sufficiently large,

$$\begin{aligned} C &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j}(\mu_j)} W(\beta_j) dx \\ &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta \varepsilon_j^s}(x_{i,j})} W(\beta_j) dx, \end{aligned} \quad (4.8)$$

where in the last inequality we used the assumption  $B_{\rho_{\varepsilon_j}}(x_{i,j}) \cap B_{\rho_{\varepsilon_j}}(x_{k,j}) = \emptyset$  for  $i \neq k$ .

For every  $i \in \{1, \dots, M_j\}$  we decompose the annulus  $B_{\rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta \varepsilon_j^s}(x_{i,j})$  centred at  $x_{i,j}$  into dyadic annuli with constant ratio  $\delta \in (0, 1)$  between inner and outer radii. More precisely, the annuli are defined as

$$C_j^{k,i} := B_{\rho_{\varepsilon_j} \delta^{k-1}}(x_{i,j}) \setminus B_{\rho_{\varepsilon_j} \delta^k}(x_{i,j}), \quad (4.9)$$

and we consider only those corresponding to  $k = 1, \dots, \tilde{k}_{\varepsilon_j}$ , where

$$\tilde{k}_{\varepsilon_j} := \lfloor k_{\varepsilon_j} \rfloor + 1 \quad \text{and} \quad k_{\varepsilon_j} := s \frac{|\log \varepsilon_j|}{|\log \delta|} - \frac{|\log \rho_{\varepsilon_j}|}{|\log \delta|}. \quad (4.10)$$

Notice that  $\rho_{\varepsilon_j} \delta^{\tilde{k}_{\varepsilon_j}} \geq \rho_{\varepsilon_j} \delta^{k_{\varepsilon_j}+1} = \delta \varepsilon_j^s$ ; therefore, for every  $i \in \{1, \dots, M_j\}$  we have

$$\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{B_{\rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta \varepsilon_j^s}(x_{i,j})} W(\beta_j) dx \geq \frac{1}{|\log \varepsilon_j|^2} \sum_{k=1}^{\tilde{k}_{\varepsilon_j}} \int_{C_j^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2} dx. \quad (4.11)$$

Arguing as in [20, Proposition 3.11] we deduce that for every  $j, i$ , and  $k$  the following estimate holds true

$$\int_{C_j^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2} dx \geq \psi(R^T \xi_{i,j}, \delta) - \sigma_j, \quad (4.12)$$

where  $\psi(\cdot, \delta)$  is defined as in (4.1) and  $\sigma_j$  is a nonnegative infinitesimal sequence as  $j \rightarrow +\infty$ .

Combining (4.8), (4.11), and (4.12) we find that for every  $\delta \in (0, 1)$

$$\begin{aligned} C &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j}(\mu_j)} W(\beta_j) dx \geq \frac{1}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \sum_{k=1}^{\tilde{k}_{\varepsilon_j}} (\psi(R^T \xi_{i,j}, \delta) - \sigma_j) \\ &\geq \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \frac{\tilde{k}_{\varepsilon_j}}{|\log \varepsilon_j|} (\psi(R^T \xi_{i,j}, \delta) - \sigma_j) \\ &\geq \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \left( s - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \left( \frac{\psi(R^T \xi_{i,j}, \delta)}{|\log \delta|} - \frac{\sigma_j}{|\log \delta|} \right), \end{aligned}$$

where in the last inequality we used the relation  $\tilde{k}_{\varepsilon_j} = \lfloor k_{\varepsilon_j} \rfloor + 1 \geq k_{\varepsilon_j}$ .

Now, as a first step we let  $\delta \rightarrow 0$  in the previous estimate; this leads to

$$C \geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j}(\mu_j)} W(\beta_j) dx \geq \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \left( s - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \hat{\psi}(R^T \xi_{i,j}), \quad (4.13)$$

with  $\hat{\psi}$  defined as in (4.2). By assumption  $\rho_{\varepsilon_j} \gg \varepsilon_j$ , hence (4.13) entails that for  $j$  sufficiently large

$$C \geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j}(\mu_j)} W(\beta_j) dx \geq \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \frac{s}{2} \hat{\psi}(R^T \xi_{i,j}). \quad (4.14)$$

Since the function  $\hat{\psi}$  is 2-homogeneous (being the pointwise limit of 2-homogeneous functions, cf. (4.2)), we have in particular that for every  $i \in \{1, \dots, M_j\}$ ,

$$\hat{\psi}(R^T \xi_{i,j}) = |\xi_{i,j}|^2 \hat{\psi} \left( \frac{R^T \xi_{i,j}}{|R^T \xi_{i,j}|} \right).$$

Set  $\bar{c} := \inf_{|\xi|=1} \hat{\psi}(\xi)$ , from (4.14) we finally deduce

$$\frac{2C}{s} \geq \frac{\bar{c}}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} |\xi_{i,j}|^2 \geq \frac{\bar{c}}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} |\xi_{i,j}|, \quad (4.15)$$

where the last inequality follows from the fact that since  $\xi_{i,j} \in \mathbb{S} = \text{span}_{\mathbb{Z}} S$ ,  $|\xi_{i,j}|$  are bounded away from zero. Therefore the claim (4.7) follows.

**Step 2.** *Weak convergence of the scaled strains.*

In view of the growth condition (iv) we have

$$C \varepsilon_j^2 |\log \varepsilon_j|^2 \geq C \int_{\Omega} W(\beta_j) dx \geq C \int_{\Omega} \text{dist}^2(\beta_j, SO(2)) dx. \quad (4.16)$$

The idea is to apply the generalised rigidity estimate Theorem 3.3 to a suitable modification of  $\beta_j$ . The estimate cannot be applied directly to the strains  $\beta_j$  since it is not clear whether the crucial bound  $|\text{Curl} \beta_j|(\Omega) \leq C \varepsilon_j |\log \varepsilon_j|$  holds true. Indeed, on the one hand the total variation of the measure  $\mu_j$  is bounded by  $C \varepsilon_j |\log \varepsilon_j|$  by Step 1; on the other hand, however,  $\text{Curl} \beta_j$  is related to the measure  $\mu_j$ , but it is not exactly  $\mu_j$  and therefore it does not necessarily satisfy the same bound. To overcome this problem, in what follows we construct new strains  $\tilde{\beta}_j$  satisfying  $|\text{Curl} \tilde{\beta}_j|(\Omega) = |\mu_j|(\Omega)$ , and hence the crucial bound.

For every  $x_{i,j}$  in the support of  $\mu_j$ , set  $C_{i,j} := B_{2\varepsilon_j}(x_{i,j}) \setminus B_{\varepsilon_j}(x_{i,j})$  and consider the function  $K_{i,j}: C_{i,j} \rightarrow \mathbb{R}^{2 \times 2}$  defined as follows

$$K_{i,j}(x) := \frac{\varepsilon_j}{2\pi} \xi_{i,j} \otimes J \frac{x - x_{i,j}}{|x - x_{i,j}|^2},$$

where  $J$ , as above, is the clockwise rotation of  $\pi/2$ . Then we have

$$\int_{C_{i,j}} |K_{i,j}|^2 dx \leq C \int_{C_{i,j}} \text{dist}^2(\beta_j, SO(2)) dx. \quad (4.17)$$

Indeed, a straightforward calculation gives

$$\int_{C_{i,j}} |K_{i,j}|^2 dx = C \varepsilon_j^2 |\xi_{i,j}|^2,$$

while a scaling argument (see [20, Proposition 3.3 and Remark 3.4]) shows that

$$C\varepsilon_j^2|\xi_{i,j}|^2 \leq \int_{C_{i,j}} \text{dist}^2(\beta_j, SO(2)) dx,$$

and hence (4.17).

By construction  $\text{Curl}(\beta_j - K_{i,j}) = 0$  in  $C_{i,j}$  and  $\int_{\partial B_\varepsilon(x_{i,j})} (\beta_j - K_{i,j}) \cdot t ds = 0$ , for every  $i = 1, \dots, M_j$ . Hence, there exist  $M_j$  functions  $u_{i,j} \in H^1(C_{i,j}; \mathbb{R}^2)$  such that  $\beta_j - K_{i,j} = \nabla u_{i,j}$  in  $C_{i,j}$ , for every  $i = 1, \dots, M_j$ .

In view of (4.17) we have

$$\int_{C_{i,j}} \text{dist}^2(\nabla u_{i,j}, SO(2)) dx \leq C \int_{C_{i,j}} \left( \text{dist}^2(\beta_j, SO(2)) + |K_{i,j}|^2 \right) dx \leq C \int_{C_{i,j}} \text{dist}^2(\beta_j, SO(2)) dx,$$

therefore the classical rigidity estimate applied to  $u_{i,j}$  provides us with a constant rotation  $R_{i,j} \in SO(2)$  such that

$$\int_{C_{i,j}} |\nabla u_{i,j} - R_{i,j}|^2 dx \leq C \int_{C_{i,j}} \text{dist}^2(\nabla u_{i,j}, SO(2)) dx \leq C \int_{C_{i,j}} \text{dist}^2(\beta_j, SO(2)) dx,$$

for some  $C > 0$  independent of  $j$ .

By standard extension arguments, there exists a function  $v_{i,j} \in H^1(B_{2\varepsilon_j}(x_{i,j}); \mathbb{R}^2)$  such that  $\nabla v_{i,j} \equiv \nabla u_{i,j} - R_{i,j}$  in  $C_{i,j}$  and

$$\int_{B_{2\varepsilon_j}(x_{i,j})} |\nabla v_{i,j}|^2 dx \leq C \int_{C_{i,j}} |\nabla u_{i,j} - R_{i,j}|^2 dx \leq C \int_{C_{i,j}} \text{dist}^2(\beta_j, SO(2)) dx. \quad (4.18)$$

Now define the field  $\tilde{\beta}_j: \Omega \rightarrow \mathbb{R}^{2 \times 2}$  as

$$\tilde{\beta}_j := \begin{cases} \beta_j & \text{in } \Omega_{\varepsilon_j}(\mu_j), \\ \nabla v_{i,j} + R_{i,j} & \text{in } B_{\varepsilon_j}(x_{i,j}) \text{ for } i = 1, \dots, M_j. \end{cases}$$

By (4.18) we get

$$\begin{aligned} \int_{\Omega} \text{dist}^2(\tilde{\beta}_j, SO(2)) dx &\leq \int_{\Omega} \text{dist}^2(\beta_j, SO(2)) dx + \sum_{i=1}^{M_j} \int_{B_{\varepsilon_j}(x_{i,j})} |\nabla v_{i,j}|^2 \\ &\leq C \int_{\Omega} \text{dist}^2(\beta_j, SO(2)) dx \leq C\varepsilon_j^2 |\log \varepsilon_j|^2. \end{aligned}$$

Moreover, by construction  $|\text{Curl } \tilde{\beta}_j|(\Omega) = |\mu_j|(\Omega)$ ; then we are in a position to apply Theorem 3.3 to  $\tilde{\beta}_j$  to deduce the existence of a sequence of constant rotations  $R_j \in SO(2)$  such that

$$\int_{\Omega} |\tilde{\beta}_j - R_j|^2 \leq C \left( \int_{\Omega} \text{dist}^2(\tilde{\beta}_j, SO(2)) + (|\text{Curl } \tilde{\beta}_j|(\Omega))^2 \right) \leq C\varepsilon_j^2 |\log \varepsilon_j|^2, \quad (4.19)$$

for some  $C > 0$  independent of  $j$ . By the definition of  $\tilde{\beta}_j$  and by (4.19) we deduce that

$$\int_{\Omega_{\varepsilon_j}(\mu_j)} |\beta_j - R_j|^2 = \int_{\Omega_{\varepsilon_j}(\mu_j)} |\tilde{\beta}_j - R_j|^2 \leq \int_{\Omega} |\tilde{\beta}_j - R_j|^2 \leq C\varepsilon_j^2 |\log \varepsilon_j|^2,$$

and therefore

$$\int_{\Omega_{\varepsilon_j}(\mu_j)} \frac{|\beta_j - R_j|^2}{\varepsilon_j^2 |\log \varepsilon_j|^2} dx \leq C,$$

for every  $j$ . Finally, recalling that  $\beta_j \equiv I$  in  $\bigcup_{i=1}^{M_j} B_{\varepsilon_j}(x_{i,j})$  and that from Step 1 we have the bound  $M_j \leq C|\log \varepsilon_j|$ , we deduce that, up to subsequences,

$$R_j^T \frac{\beta_j - R_j}{\varepsilon_j |\log \varepsilon_j|} = \frac{R_j^T \beta_j - I}{\varepsilon_j |\log \varepsilon_j|} \rightharpoonup \beta \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}).$$

**Step 3.** *The limit measure  $\mu$  belongs to  $H^{-1}(\Omega; \mathbb{R}^2)$  and  $\text{Curl } \beta = R^T \mu$ .*

Let  $\phi \in C_0^1(\Omega)$  and let  $(\phi_j) \subset H_0^1(\Omega)$  be a sequence converging to  $\phi$  uniformly and strongly in  $H_0^1(\Omega)$  and such that  $\phi_j \equiv \phi(x_{i,j})$  in  $B_{\varepsilon_j}(x_{i,j})$ , for every  $x_{i,j}$  in the support of  $\mu_j$ . Then we have

$$\begin{aligned} \int_{\Omega} \phi d\mu &= \lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j |\log \varepsilon_j|} \int_{\Omega} \phi_j d\mu_j = \lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j |\log \varepsilon_j|} \langle \text{Curl } \beta_j, \phi_j \rangle \\ &= \lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j |\log \varepsilon_j|} \langle \text{Curl } (\beta_j - R_j), \phi_j \rangle = \lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j |\log \varepsilon_j|} \int_{\Omega} (\beta_j - R_j) J \nabla \phi_j dx \\ &= \int_{\Omega} R \beta J \nabla \phi dx = \langle \text{Curl } (R\beta), \phi \rangle = \langle R \text{Curl } \beta, \phi \rangle, \end{aligned}$$

from which we deduce the admissibility condition  $\text{Curl } \beta = R^T \mu$ . Finally, since in Step 2 we proved that  $\beta \in L^2(\Omega; \mathbb{R}^{2 \times 2})$ , we immediately get that  $\mu$  belongs to  $H^{-1}(\Omega; \mathbb{R}^2)$ .  $\square$

**Remark 4.4.** Notice that in Step 1 of Proposition 4.3, we show that the number of dislocations  $M_j$  corresponding to a pair  $(\mu_j, \beta_j)$  with equibounded energy is such that  $M_j \leq C|\log \varepsilon_j|$  (see (4.15)).

In view of Proposition 4.3, it is convenient to give the following notion of convergence for sequences of pairs  $(\mu_{\varepsilon}, \beta_{\varepsilon})$ .

**Definition 4.5.** *A pair of sequences  $(\mu_{\varepsilon}, \beta_{\varepsilon}) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  is said to converge to a triplet  $(\mu, \beta, R) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  if there exists a sequence of rotations  $(R_{\varepsilon}) \subset SO(2)$  such that*

$$\frac{1}{\varepsilon |\log \varepsilon|} \mu_{\varepsilon} \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^2), \quad (4.20)$$

$$\frac{R_{\varepsilon}^T \beta_{\varepsilon} - I}{\varepsilon |\log \varepsilon|} \rightharpoonup \beta \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}), \quad \text{and } R_{\varepsilon} \rightarrow R. \quad (4.21)$$

**4.3.  $\Gamma$ -convergence result.** We are now in a position to state and prove the main result of this section, namely a  $\Gamma$ -convergence result for the scaled functionals  $\mathcal{E}_{\varepsilon}$ .

In what follows we additionally assume that  $\Omega$  has  $C^{1,1}$  boundary. Notice however that the higher regularity of  $\Omega$  will be used only in the proof of the limsup inequality.

**Theorem 4.6.** *The energy functionals  $\mathcal{E}_\varepsilon$  defined in (2.4)  $\Gamma$ -converge with respect to the convergence of Definition 4.5 to the functional  $\mathcal{E}$  defined on  $\mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  by*

$$\mathcal{E}(\mu, \beta, R) := \begin{cases} \frac{1}{2} \int_{\Omega} \mathbb{C}\beta : \beta \, dx + \int_{\Omega} \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2) \\ & \text{and } \text{Curl } \beta = R^T \mu, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.22)$$

where  $\mathbb{C} = \frac{\partial^2 W}{\partial F^2}(I)$  and  $\varphi$  is as in (4.4). Specifically, the following two inequalities hold true.

**$\Gamma$ -liminf inequality:** *For every  $(\mu, \beta, R) \in (H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  with  $\text{Curl } \beta = R^T \mu$ , and for every sequence  $(\mu_\varepsilon, \beta_\varepsilon) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  converging to  $(\mu, \beta, R)$  in the sense of Definition 4.5, we have*

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \geq \mathcal{E}(\mu, \beta, R).$$

**$\Gamma$ -limsup inequality:** *For every  $(\mu, \beta, R) \in (H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  with  $\text{Curl } \beta = R^T \mu$ , there exists a sequence  $(\mu_\varepsilon, \beta_\varepsilon) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  converging to  $(\mu, \beta, R)$  in the sense of Definition 4.5, such that*

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq \mathcal{E}(\mu, \beta, R).$$

**Remark 4.7.** Notice that (4.4) can be equivalently rewritten as

$$\varphi(R, \xi) = \min \left\{ \sum_{k=1}^M \lambda_k \hat{\psi}(\zeta_k) : \sum_{k=1}^M \lambda_k \zeta_k = R^T \xi, M \in \mathbb{N}, \lambda_k \geq 0, \zeta_k \in R^T \mathbb{S} \right\},$$

hence  $\varphi$  depends on  $\xi$  only in terms of  $R^T \xi$ ; therefore  $\varphi(R, \xi) = \tilde{\varphi}(R, R^T \xi)$  for some function  $\tilde{\varphi}$ . Then, since the limit strains  $\beta$  satisfy the condition  $\text{Curl } \beta = R^T \mu$ , thanks to the 1-homogeneity of  $\varphi(R, \cdot)$  the  $\Gamma$ -limit  $\mathcal{E}$  can be expressed in terms of  $\beta$  and  $\text{Curl } \beta$  in the following way

$$\mathcal{E}(\mu, \beta, R) = \frac{1}{2} \int_{\Omega} \mathbb{C}\beta : \beta \, dx + \int_{\Omega} \tilde{\varphi}\left(R, \frac{d\text{Curl } \beta}{d|\text{Curl } \beta|}\right) d|\text{Curl } \beta|.$$

In particular, if  $\text{Curl } \beta \in L^1(\Omega; \mathbb{R}^2)$  we have

$$\mathcal{E}(\mu, \beta, R) = \frac{1}{2} \int_{\Omega} \mathbb{C}\beta : \beta \, dx + \int_{\Omega} \tilde{\varphi}(R, \text{Curl } \beta) \, dx.$$

*Proof of Theorem 4.6.  $\Gamma$ -liminf inequality.*

Let  $(\mu, \beta, R) \in (H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  and  $(\mu_\varepsilon, \beta_\varepsilon) \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2 \times 2})$  be as in the statement and assume that  $\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon)$ . Suppose moreover that  $\mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq C$  for every  $\varepsilon > 0$  (otherwise there is nothing to prove). This implies in particular that  $(\mu_\varepsilon, \beta_\varepsilon) \in X_\varepsilon \times \mathcal{AS}_\varepsilon(\mu_\varepsilon)$ , where  $X_\varepsilon$  and  $\mathcal{AS}_\varepsilon(\mu_\varepsilon)$  are defined in (2.1) and (2.3), respectively.



Arguing as in [20, Proposition 3.11], we decompose the energy into a contribution far from the dislocations, in  $\Omega_{\rho_\varepsilon}(\mu_\varepsilon) := \Omega \setminus \cup_{i=1}^M B_{\rho_\varepsilon}(x_i)$ , and a contribution close to the dislocations; *i.e.*,

$$\begin{aligned} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) &= \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega_{\rho_\varepsilon}(\mu_\varepsilon)} W(\beta_\varepsilon) dx + \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \sum_{i=1}^M \int_{B_{\rho_\varepsilon}(x_i)} W(\beta_\varepsilon) dx \\ &=: \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; \Omega_{\rho_\varepsilon}(\mu_\varepsilon)) + \sum_{i=1}^M \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; B_{\rho_\varepsilon}(x_i)). \end{aligned} \quad (4.23)$$

We treat the two contributions separately.

*Lower bound far from the dislocations.* For the energy contribution far from the dislocations we perform a linearisation of the energy at scale  $\varepsilon |\log \varepsilon|$  around the identity matrix. By a Taylor expansion of order two we get  $W(I + F) = \frac{1}{2} \mathbb{C}F : F + \sigma(F)$ , where  $\sigma(F)/|F|^2 \rightarrow 0$  as  $|F| \rightarrow 0$ . Then, setting  $\omega(t) := \sup_{|F| \leq t} |\sigma(F)|$ , we have

$$W(I + \varepsilon |\log \varepsilon| F) \geq \frac{1}{2} \varepsilon^2 |\log \varepsilon|^2 \mathbb{C}F : F - \omega(\varepsilon |\log \varepsilon| |F|), \quad (4.24)$$

with  $\omega(t)/t^2 \rightarrow 0$  as  $t \rightarrow 0$ . Now, let

$$G_\varepsilon := \frac{R_\varepsilon^T \beta_\varepsilon - I}{\varepsilon |\log \varepsilon|},$$

and define the characteristic function

$$\chi_\varepsilon := \begin{cases} 1 & \text{if } x \in \Omega_{\rho_\varepsilon}(\mu_\varepsilon) \text{ and } |G_\varepsilon| \leq \varepsilon^{-1/2} \\ 0 & \text{otherwise in } \Omega. \end{cases} \quad (4.25)$$

By the boundedness of  $(G_\varepsilon)$  in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$  and in view of the definition of  $\rho_\varepsilon$  that ensures that  $\Omega_{\rho_\varepsilon}(\mu_\varepsilon)$  has asymptotically full measure, it easily follows that  $\chi_\varepsilon \rightarrow 1$  boundedly in measure. Therefore, from (4.21) we deduce that

$$\tilde{G}_\varepsilon := \chi_\varepsilon G_\varepsilon \rightharpoonup \beta \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \quad (4.26)$$

Using the frame indifference of  $W$  and (4.24) we get

$$\begin{aligned} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; \Omega_{\rho_\varepsilon}(\mu_\varepsilon)) &\geq \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega} \chi_\varepsilon W(\beta_\varepsilon) dx \\ &= \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega} \chi_\varepsilon W(R_\varepsilon^T \beta_\varepsilon) dx \\ &= \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega} \chi_\varepsilon W(I + \varepsilon |\log \varepsilon| G_\varepsilon) dx \\ &\geq \int_{\Omega} \left( \frac{1}{2} \mathbb{C} \tilde{G}_\varepsilon : \tilde{G}_\varepsilon - \chi_\varepsilon \frac{\omega(\varepsilon |\log \varepsilon| |G_\varepsilon|)}{\varepsilon^2 |\log \varepsilon|^2} \right) dx. \end{aligned} \quad (4.27)$$

Then, the first term in (4.27) is lower semicontinuous with respect to the convergence (4.26). On the other hand, the second term converges to zero, which can be easily seen multiplying its numerator and denominator by  $|G_\varepsilon|^2$ . Indeed,  $|G_\varepsilon|^2 \cdot \chi_\varepsilon \omega(\varepsilon |\log \varepsilon| |G_\varepsilon|) / (\varepsilon |\log \varepsilon| |G_\varepsilon|)^2$  is

the product of a bounded sequence in  $L^1(\Omega)$  and a sequence tending to zero in  $L^\infty(\Omega)$ , since  $\varepsilon |\log \varepsilon| |G_\varepsilon| \leq \varepsilon^{1/2} |\log \varepsilon|$  whenever  $\chi_\varepsilon \neq 0$ . Combining these two facts, we eventually obtain

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; \Omega_{\rho_\varepsilon}(\mu_\varepsilon)) \geq \frac{1}{2} \int_{\Omega} \mathbb{C}\beta : \beta \, dx. \quad (4.28)$$

*Lower bound close to the dislocations.* We are going to estimate the energy contribution  $\mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; B_{\rho_\varepsilon}(x_i))$ , for  $i = 1, \dots, M$ . For brevity, we write  $B_r^i$  instead of  $B_r(x_i)$  for every  $r > 0$ . Let  $s \in (0, 1)$  and  $\delta \in (0, 1)$  be fixed and independent of  $\varepsilon$ ; then, for small enough  $\varepsilon$ ,

$$\mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; B_{\rho_\varepsilon}^i) = \int_{B_{\rho_\varepsilon}^i \setminus B_{\delta \varepsilon}^i} W(\beta_\varepsilon) \, dx \geq \int_{B_{\rho_\varepsilon}^i \setminus B_{\delta \varepsilon}^i} W(\beta_\varepsilon) \, dx \geq \sum_{k=1}^{\tilde{k}_\varepsilon} \int_{C_\varepsilon^{k,i}} W(\beta_\varepsilon) \, dx,$$

where  $C_\varepsilon^{k,i}$  and  $\tilde{k}_\varepsilon$  are defined as in (4.9) and (4.10), respectively. Proceeding as in the proof of Proposition 4.3, Step 1, we prove that, as in (4.14),

$$\sum_{i=1}^M \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; B_{\rho_\varepsilon}^i) \geq \frac{1}{|\log \varepsilon|} \sum_{i=1}^M \left( s - \frac{|\log \rho_\varepsilon|}{|\log \varepsilon|} \right) \hat{\psi}(R^T \xi_i). \quad (4.29)$$

By formula (4.4) and by the definition of  $\mu_\varepsilon$  it follows that

$$\frac{1}{|\log \varepsilon|} \sum_{i=1}^M \hat{\psi}(R^T \xi_i) \geq \int_{\Omega} \varphi \left( R, \frac{d\tilde{\mu}_\varepsilon}{d|\tilde{\mu}_\varepsilon|} \right) d|\tilde{\mu}_\varepsilon|, \quad (4.30)$$

where  $\tilde{\mu}_\varepsilon := \mu_\varepsilon / (\varepsilon |\log \varepsilon|)$ . Notice that (4.20) entails that  $\tilde{\mu}_\varepsilon \xrightarrow{*} \mu$  in  $\mathcal{M}(\Omega; \mathbb{R}^2)$ . Since  $\text{Span}_{\mathbb{R}} \mathbb{S} = \mathbb{R}^2$ , the convex 1-homogeneous function  $\varphi$  is finite on  $\mathbb{R}^2$  and therefore continuous. Then, invoking Reshetnyak's lower-semicontinuity Theorem, (4.29) and (4.30) give

$$\liminf_{\varepsilon \rightarrow 0} \sum_{i=1}^M \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon; B_{\rho_\varepsilon}^i) \geq s \int_{\Omega} \varphi \left( R, \frac{d\mu}{d|\mu|} \right) d|\mu|. \quad (4.31)$$

Hence the lower bound for the energy follows from (4.28), (4.31), and from the arbitrariness of  $s$ , which can be taken arbitrarily close to 1.

### $\Gamma$ -limsup inequality.

Let  $(\mu, \beta, R) \in (H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)) \times L^2(\Omega; \mathbb{R}^{2 \times 2}) \times SO(2)$  be such that  $\text{Curl } \beta = R^T \mu$ . By standard density arguments we can assume that  $(\mu, \beta) \in W^{-1, \infty}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ .

We divide the proof into three steps.

**Step 1.**  $\mu = \xi \, dx$  with  $\xi \in \mathbb{R}^2$ .

Given  $\xi \in \mathbb{R}^2$  and  $\beta \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$  with  $\text{Curl } \beta = R^T \xi \, dx$ , we are going to construct a sequence  $(\mu_\varepsilon, \beta_\varepsilon) \subset X_\varepsilon \times \mathcal{AS}_\varepsilon(\mu_\varepsilon)$  converging to  $(\mu, \beta)$  in the sense of Definition 4.5 and such that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq \frac{1}{2} \int_{\Omega} \mathbb{C}\beta : \beta \, dx + \int_{\Omega} \varphi(R, \xi) \, dx. \quad (4.32)$$

Let  $\hat{\psi}$  be as in (4.2); by (4.4) there exist  $M \in \mathbb{N}$ ,  $\xi_1, \dots, \xi_M \in \mathbb{S}$ , and  $\lambda_k \geq 0$ , with  $k = 1, \dots, M$ , such that  $\xi = \sum_{k=1}^M \lambda_k \xi_k$  and

$$\varphi(R, \xi) = \sum_{k=1}^M \lambda_k \hat{\psi}(R^T \xi_k). \quad (4.33)$$

Set

$$\Lambda := \sum_{k=1}^M \lambda_k, \quad r_\varepsilon := \frac{1}{2\sqrt{\Lambda|\log \varepsilon|}};$$

notice that  $r_\varepsilon \gg \rho_\varepsilon$ . By [11, Lemma 14] there exists a sequence of admissible measures defined as

$$\mu_\varepsilon := \sum_{k=1}^M \varepsilon \xi_k \mu_\varepsilon^k, \quad \text{where} \quad \mu_\varepsilon^k := \sum_{i=1}^{M_\varepsilon^k} \delta_{x_{i,\varepsilon}},$$

with the  $\{x_{i,\varepsilon}\}$  such that  $B_{r_\varepsilon}(x_{i,\varepsilon}) \subset \Omega$ ,  $|x_{i,\varepsilon} - x_{j,\varepsilon}| \geq 2r_\varepsilon$  for every  $i \neq j$ , satisfying

$$\frac{\mu_\varepsilon}{\varepsilon|\log \varepsilon|} \xrightarrow{*} \mu \quad \text{weakly in } \mathcal{M}(\Omega; \mathbb{R}^2) \quad (4.34)$$

$$\frac{|\mu_\varepsilon^k|}{|\log \varepsilon|} \xrightarrow{*} \lambda_k dx \quad \text{weakly in } \mathcal{M}(\Omega). \quad (4.35)$$

We show that  $\mu_\varepsilon$  converges to  $\mu$ .

For what follows it is useful to combine the two summations in the definition of  $\mu_\varepsilon$  into just one sum and to rewrite it as

$$\mu_\varepsilon = \sum_{i=1}^{M_\varepsilon} \varepsilon \xi_{i,\varepsilon} \delta_{x_{i,\varepsilon}};$$

we also introduce the auxiliary measures

$$\tilde{\mu}_\varepsilon^{r_\varepsilon} := \frac{1}{\pi r_\varepsilon^2} \sum_{i=1}^{M_\varepsilon} \varepsilon \xi_{i,\varepsilon} \chi_{B_{r_\varepsilon}(x_{i,\varepsilon})} dx, \quad \hat{\mu}_\varepsilon^{r_\varepsilon} := \frac{1}{2\pi r_\varepsilon} \sum_{i=1}^{M_\varepsilon} \varepsilon \xi_{i,\varepsilon} \mathcal{H}^1 \llcorner \partial B_{r_\varepsilon}(x_{i,\varepsilon}).$$

Appealing again to [11, Lemma 14] we get

$$\frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{\varepsilon|\log \varepsilon|} \rightarrow \mu \quad \text{strongly in } H^{-1}(\Omega; \mathbb{R}^2), \quad \frac{\hat{\mu}_\varepsilon^{r_\varepsilon}}{\varepsilon|\log \varepsilon|} \xrightarrow{*} \mu \quad \text{weakly in } \mathcal{M}(\Omega; \mathbb{R}^2). \quad (4.36)$$

To define a recovery sequence for  $\beta$  we first introduce the auxiliary strains

$$\tilde{K}_\varepsilon^{\mu_\varepsilon} := \sum_{i=1}^{M_\varepsilon} \tilde{K}_{i,\varepsilon}^{\xi_{i,\varepsilon}} \chi_{B_{r_\varepsilon}(x_{i,\varepsilon})} \quad \text{with} \quad \tilde{K}_{i,\varepsilon}^{\xi_{i,\varepsilon}}(x) := \frac{\varepsilon}{2\pi r_\varepsilon^2} R^T \xi_{i,\varepsilon} \otimes J(x - x_{i,\varepsilon}),$$

where  $J$  is the clockwise rotation of  $\pi/2$ . Notice that  $\text{Curl } \tilde{K}_\varepsilon^{\mu_\varepsilon} = R^T \tilde{\mu}_\varepsilon^{r_\varepsilon} - R^T \hat{\mu}_\varepsilon^{r_\varepsilon}$ .

Now, for every  $i = 1, \dots, M_\varepsilon$ , let  $\eta_i^\varepsilon: \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$  be a distributional solution of

$$\begin{cases} \text{Curl } \eta = R^T \xi_{i,\varepsilon} \delta_0 & \text{in } \mathbb{R}^2, \\ \text{Div } \mathbb{C}\eta = 0 & \text{in } \mathbb{R}^2, \end{cases}$$

where  $\mathbb{C} := \frac{\partial^2 W}{\partial F^2}(I)$ . In polar coordinates the planar strain  $\eta_i^\varepsilon$  has the form

$$\eta_i^\varepsilon(r, \theta) = \frac{1}{r} \Gamma_{R^T \xi_{i,\varepsilon}}(\theta), \quad (4.37)$$

where the function  $\Gamma_{R^T \xi_{i,\varepsilon}}$  depends on  $R$ ,  $\xi_{i,\varepsilon}$  and on the elasticity tensor  $\mathbb{C}$ , and satisfies the bound  $|\Gamma_{R^T \xi_{i,\varepsilon}}(\theta)| \leq C$  for every  $\theta \in [0, 2\pi)$  (see e.g. [1]). Let  $\hat{\eta}_i^\varepsilon(x) := \eta_i^\varepsilon(x - x_{\varepsilon,i})$ , and let  $\hat{\eta}^\varepsilon$  be defined as

$$\hat{\eta}^\varepsilon := \sum_{i=1}^{M_\varepsilon} \varepsilon \hat{\eta}_i^\varepsilon \chi_{B_{r_\varepsilon}^i}.$$

Notice that  $\text{Curl } \hat{\eta}^\varepsilon = R^T(\mu_\varepsilon - \hat{\mu}_\varepsilon^{r_\varepsilon})$ .

We define the recovery sequence  $\beta_\varepsilon$  as

$$\beta_\varepsilon := R \left( I + \varepsilon |\log \varepsilon| \beta + \hat{\eta}^\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) \chi_{\Omega_\varepsilon(\mu_\varepsilon)} + I \chi_{\cup_{i=1}^{M_\varepsilon} B_\varepsilon^i},$$

where  $\tilde{\beta}_\varepsilon := \nabla w_\varepsilon J$  and  $w_\varepsilon$  is the solution to the following system

$$\begin{cases} -\Delta w_\varepsilon = \varepsilon |\log \varepsilon| R^T \mu - R^T \tilde{\mu}_\varepsilon^{r_\varepsilon} & \text{in } \Omega, \\ w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^2). \end{cases} \quad (4.38)$$

Notice that  $\beta_\varepsilon \in \mathcal{AS}_\varepsilon(\mu_\varepsilon)$ . In fact  $\beta_\varepsilon \equiv I$  in  $\cup_{i=1}^{M_\varepsilon} B_\varepsilon^i$ ; moreover

$$(\text{Curl } \beta_\varepsilon)_\perp \Omega_\varepsilon(\mu_\varepsilon) = \left( \varepsilon |\log \varepsilon| \mu + \mu_\varepsilon - \hat{\mu}_\varepsilon^{r_\varepsilon} - \tilde{\mu}_\varepsilon^{r_\varepsilon} + \hat{\mu}_\varepsilon^{r_\varepsilon} - \varepsilon |\log \varepsilon| \mu + \tilde{\mu}_\varepsilon^{r_\varepsilon} \right)_\perp \Omega_\varepsilon(\mu_\varepsilon) = 0,$$

and by construction  $\beta_\varepsilon$  satisfies the circulation condition on  $\cup_{i=1}^{M_\varepsilon} \partial B_\varepsilon^i$ .

Now we prove that  $\beta_\varepsilon$  converges to  $\beta$  in the sense of Definition 4.5 with  $R_\varepsilon = R$ ; *i.e.*, we show that  $\frac{R^T \beta_\varepsilon - I}{\varepsilon |\log \varepsilon|} \rightharpoonup \beta$  weakly in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$ . To this end we prove the following convergence properties:

- (a)  $\frac{\hat{\eta}^\varepsilon \chi_{\Omega_\varepsilon(\mu_\varepsilon)}}{\varepsilon |\log \varepsilon|} \rightharpoonup 0$  weakly in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$ ;
- (b)  $\frac{\tilde{K}_\varepsilon^{\mu_\varepsilon}}{\varepsilon |\log \varepsilon|} \rightarrow 0$  strongly in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$ ;
- (c)  $\frac{\nabla w_\varepsilon}{\varepsilon |\log \varepsilon|} \rightarrow 0$  strongly in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$ .

Clearly (a), (b), and (c) imply the desired convergence for the sequence  $\beta_\varepsilon$ .

To prove (a), we first notice that the sequence  $\frac{\hat{\eta}_\varepsilon^i}{\varepsilon |\log \varepsilon|}$  has bounded  $L^2$ -norm in  $\Omega_\varepsilon(\mu_\varepsilon)$ . Indeed, since  $|\hat{\eta}_\varepsilon^i| \leq \frac{C}{|x - x_{i,\varepsilon}|}$  for  $i = 1, \dots, M_\varepsilon$ , we have

$$\begin{aligned} \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \int_{\Omega_\varepsilon(\mu_\varepsilon)} |\hat{\eta}_\varepsilon^i|^2 dx &= \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{r_\varepsilon}^i \setminus B_\varepsilon^i} |\hat{\eta}_\varepsilon^i|^2 dx \leq \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{r_\varepsilon}^i \setminus B_\varepsilon^i} \frac{C}{|x - x_{i,\varepsilon}|^2} dx \\ &\leq \frac{C}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_\varepsilon^{r_\varepsilon} \frac{dr}{r} \leq \frac{M_\varepsilon (\log r_\varepsilon - \log \varepsilon)}{|\log \varepsilon|^2} \leq C. \end{aligned} \quad (4.39)$$

Moreover, the  $L^2$ -norm of  $\frac{\hat{\eta}_\varepsilon}{\varepsilon|\log \varepsilon|}$  is concentrated in  $\cup_{i=1}^{M_\varepsilon} (B_{\rho_\varepsilon}^i \setminus B_\varepsilon^i)$ ; in fact, similarly as above,

$$\frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega_{\rho_\varepsilon}(\mu_\varepsilon)} |\hat{\eta}_\varepsilon|^2 dx \leq \frac{C}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{\rho_\varepsilon}^{r_\varepsilon} \frac{dr}{r} \leq \frac{M_\varepsilon(\log r_\varepsilon - \log \rho_\varepsilon)}{|\log \varepsilon|^2} \rightarrow 0, \quad (4.40)$$

as  $\varepsilon \rightarrow 0$ . Then since the measure of the set  $\cup_{i=1}^{M_\varepsilon} (B_{\rho_\varepsilon}^i \setminus B_\varepsilon^i)$  tends to zero as  $\varepsilon \rightarrow 0$ , the two properties (4.39) and (4.40) entail (a).

Concerning (b), we have that

$$\begin{aligned} \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega} |\tilde{K}_\varepsilon^{\mu_\varepsilon}|^2 dx &= \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{r_\varepsilon}^i} \frac{1}{4\pi^2 r_\varepsilon^4} |\xi_{i,\varepsilon}|^2 |x - x_{\varepsilon,i}|^2 dx \\ &\leq C \frac{M_\varepsilon}{|\log \varepsilon|^2} \rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , which proves the claim.

Finally, the convergence (c) follows from the estimate (directly implied by (4.38))

$$\left\| \frac{\nabla w_\varepsilon}{\varepsilon|\log \varepsilon|} \right\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C \left\| \mu - \frac{\tilde{\mu}_\varepsilon^{r_\varepsilon}}{\varepsilon|\log \varepsilon|} \right\|_{H^{-1}(\Omega; \mathbb{R}^2)}$$

and from (4.36).

For what follows it is convenient to notice that, by construction we also have

$$\left( \frac{R^T \beta_\varepsilon - I}{\varepsilon|\log \varepsilon|} - \beta \right) \cdot t \rightarrow 0 \quad \text{strongly in } H^{-1/2}(\partial\Omega). \quad (4.41)$$

Now it remains to prove (4.32). Recalling that  $W(I) = 0$ , we have

$$\begin{aligned} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) &= \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega} W(\beta_\varepsilon) dx = \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega_\varepsilon(\mu_\varepsilon)} W(\beta_\varepsilon) dx \\ &= \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega_{\varepsilon^s}(\mu_\varepsilon)} W(\beta_\varepsilon) dx + \frac{1}{\varepsilon^2|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{\varepsilon^s}^i \setminus B_\varepsilon^i} W(\beta_\varepsilon) dx =: I_\varepsilon^1 + I_\varepsilon^2, \end{aligned}$$

where  $s \in (0, 1)$  is arbitrarily fixed. We are going to perform a linearisation for the term  $I_\varepsilon^1$  around the identity matrix. Appealing to (i), (ii), and to the frame-indifference of  $W$  (iii) we have

$$\begin{aligned} I_\varepsilon^1 &= \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega_{\varepsilon^s}(\mu_\varepsilon)} W(I + \varepsilon|\log \varepsilon|\beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon) \\ &= \frac{1}{\varepsilon^2|\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s}(\mu_\varepsilon)} \mathbb{C} \left( \varepsilon|\log \varepsilon|\beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) : \left( \varepsilon|\log \varepsilon|\beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) dx \\ &\quad + \frac{1}{\varepsilon^2|\log \varepsilon|^2} \int_{\Omega_{\varepsilon^s}(\mu_\varepsilon)} \sigma \left( \varepsilon|\log \varepsilon|\beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) dx, \end{aligned} \quad (4.42)$$

where  $\frac{\sigma(F)}{|F|^2} \rightarrow 0$ , as  $|F| \rightarrow 0$ .

We claim that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)}} \mathbb{C} \left( \varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) : \left( \varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right) dx \\ & \leq \frac{1}{2} \int_{\Omega} \mathbb{C} \beta : \beta dx + \int_{\Omega} \varphi(R, \xi) dx, \end{aligned} \quad (4.43)$$

where  $\varphi$  is defined in (4.4). We trivially have

$$\frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)}} \mathbb{C} \varepsilon |\log \varepsilon| \beta : \varepsilon |\log \varepsilon| \beta dx \leq \frac{1}{2} \int_{\Omega} \mathbb{C} \beta : \beta dx.$$

Then we notice that the mixed products in the left-hand side of (4.43) converge to zero as  $\varepsilon \rightarrow 0$  by (a), (b), and (c), as well as the quadratic terms involving  $\tilde{K}_\varepsilon^{\mu_\varepsilon}$  and  $\tilde{\beta}_\varepsilon$ . Moreover,

$$\begin{aligned} \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)}} \mathbb{C} \hat{\eta}_\varepsilon : \hat{\eta}_\varepsilon dx &= \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)} \setminus \Omega_{\rho_\varepsilon(\mu_\varepsilon)}} \mathbb{C} \hat{\eta}_\varepsilon : \hat{\eta}_\varepsilon dx + o(1) \\ &= \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \frac{1}{2} \int_{B_{\rho_\varepsilon}^i \setminus B_{\varepsilon^s}^i} \mathbb{C} \hat{\eta}_\varepsilon^i : \hat{\eta}_\varepsilon^i dx + o(1), \end{aligned}$$

as  $\varepsilon \rightarrow 0$  since we showed in (4.40) that the  $L^2$ -norm of  $\frac{\hat{\eta}_\varepsilon}{\varepsilon |\log \varepsilon|}$  is concentrated outside  $\Omega_{\rho_\varepsilon(\mu_\varepsilon)}$ . By (4.3) we have that for  $i = 1, \dots, M_\varepsilon$

$$\frac{1}{|\log \varepsilon|^2} \frac{1}{2} \int_{B_{\rho_\varepsilon}^i \setminus B_{\varepsilon^s}^i} \mathbb{C} \hat{\eta}_\varepsilon^i : \hat{\eta}_\varepsilon^i dx \leq \psi_\varepsilon(R^T \xi_{i,\varepsilon})(1 + o(1)),$$

and this leads to

$$\frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)}} \mathbb{C} \hat{\eta}_\varepsilon : \hat{\eta}_\varepsilon dx \leq \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \psi_\varepsilon(R^T \xi_{i,\varepsilon}) + o(1), \quad \text{as } \varepsilon \rightarrow 0. \quad (4.44)$$

Moreover, by (4.33) and (4.35),

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \psi_\varepsilon(R^T \xi_{i,\varepsilon}) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|^2} \sum_{k=1}^M |\mu_\varepsilon^k|(\Omega) \psi_\varepsilon(R^T \xi_k) \\ &= |\Omega| \sum_{k=1}^M \lambda_k \hat{\psi}(R^T \xi_k) = \int_{\Omega} \varphi(R, \xi) dx. \end{aligned} \quad (4.45)$$

Thus, combining (4.44) and (4.45) gives

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \frac{1}{2} \int_{\Omega_{\varepsilon^s(\mu_\varepsilon)}} \mathbb{C} \hat{\eta}_\varepsilon : \hat{\eta}_\varepsilon dx \leq \int_{\Omega} \varphi(R, \xi) dx.$$

We now prove that the remainder term in  $I_\varepsilon^1$  tends to zero as  $\varepsilon \rightarrow 0$ . We notice that, if  $x \in \Omega_{\varepsilon^s(\mu)}$ , then

$$|\hat{\eta}_\varepsilon(x)| \leq \sup_{i=1, \dots, M_\varepsilon} \varepsilon |\chi_{B_{r_\varepsilon}^i}(x) \hat{\eta}_\varepsilon^i(x)| \leq C \varepsilon^{1-s}; \quad (4.46)$$

and

$$|\tilde{K}_\varepsilon^{\mu_\varepsilon}(x)| \leq \sup_{i=1, \dots, M_\varepsilon} \left| \chi_{B_{r_\varepsilon}^i}(x) \frac{\varepsilon}{2\pi r_\varepsilon^2} \xi_{i,\varepsilon} \otimes J(x - x_{i,\varepsilon}) \right| \leq \frac{C\varepsilon}{r_\varepsilon}. \quad (4.47)$$

Moreover, in view of the regularity of  $\Omega$ , by standard regularity theory for elliptic partial differential equations, the following estimate for  $w_\varepsilon$  holds true for every  $1 < p < \infty$  (see, e.g. [14, Lemma 9.17]):

$$\|w_\varepsilon\|_{W^{2,p}(\Omega; \mathbb{R}^2)} \leq C \|\varepsilon |\log \varepsilon| \mu - \tilde{\mu}_\varepsilon^{r_\varepsilon}\|_{L^p(\Omega; \mathbb{R}^2)}, \quad (4.48)$$

where the constant  $C$  depends on  $p$  and  $\Omega$ . Notice also that

$$\|\varepsilon |\log \varepsilon| \mu - \tilde{\mu}_\varepsilon^{r_\varepsilon}\|_{L^p(\Omega; \mathbb{R}^2)} \leq C \varepsilon |\log \varepsilon|,$$

for every  $1 < p < \infty$ . By the Sobolev Imbedding Theorem the estimate above together with (4.48) imply

$$\|w_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega}; \mathbb{R}^2)} \leq C \varepsilon |\log \varepsilon|,$$

for every  $\alpha \in (0, 1)$ . Therefore, in particular,

$$\|\nabla w_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^{2 \times 2})} \leq C \varepsilon |\log \varepsilon|. \quad (4.49)$$

Then (4.46), (4.47), (4.49), and the boundedness of  $\beta$  entail

$$\left| \varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right| \leq C \left( \varepsilon |\log \varepsilon| + \varepsilon^{(1-s)} + \frac{\varepsilon}{r_\varepsilon} + \varepsilon |\log \varepsilon| \right) \quad \text{in } \Omega_{\varepsilon^s}(\mu_\varepsilon). \quad (4.50)$$

Hence, setting  $\chi_\varepsilon := \chi_{\Omega_{\varepsilon^s}(\mu_\varepsilon)}$  and  $\omega(t) := \sup_{|F| \leq t} |\sigma(F)|$ , we have that

$$\begin{aligned} & \left| \int_{\Omega_{\varepsilon^s}(\mu_\varepsilon)} \frac{\sigma \left( \varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right)}{\varepsilon^2 |\log \varepsilon|^2} dx \right| \leq \\ & \int_{\Omega} \chi_\varepsilon \frac{\omega \left( |\varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right)}{|\varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon|^2} \cdot \frac{|\varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon|^2}{\varepsilon^2 |\log \varepsilon|^2} dx, \end{aligned}$$

and the limit for  $\varepsilon \rightarrow 0$  of the expression above is zero as the integrand is the product of a sequence converging to zero uniformly (by (4.50)) and a bounded sequence in  $L^1(\Omega)$ . Therefore, combining this fact with (4.42) and (4.43), we get

$$\limsup_{\varepsilon \rightarrow 0} I_\varepsilon^1 \leq \mathcal{E}(\mu, \beta, R). \quad (4.51)$$

Finally, we deal with  $I_\varepsilon^2$ . Using the quadratic upper bound on  $W$  (iv) we deduce

$$I_\varepsilon^2 \leq \frac{1}{\varepsilon^2 |\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{\varepsilon^s}^i \setminus B_\varepsilon^i} C \left| \varepsilon |\log \varepsilon| \beta + \hat{\eta}_\varepsilon - \tilde{K}_\varepsilon^{\mu_\varepsilon} + \tilde{\beta}_\varepsilon \right|^2 dx. \quad (4.52)$$

Similarly as before, due to (a), (b), and (c) the mixed products in (4.52) converge to zero, as well as the quadratic terms involving  $\tilde{K}_\varepsilon^{\mu_\varepsilon}$  and  $\tilde{\beta}_\varepsilon$ . In addition,

$$\sum_{i=1}^{M_\varepsilon} \int_{B_{\varepsilon^s}^i \setminus B_\varepsilon^i} |\beta|^2 dx \leq \|\beta\|_{L^\infty(\Omega; \mathbb{R}^{2 \times 2})} M_\varepsilon (\varepsilon^{2s} - \varepsilon^2) \longrightarrow 0, \quad \text{as } \varepsilon \rightarrow 0$$

and

$$\frac{1}{|\log \varepsilon|^2} \sum_{i=1}^{M_\varepsilon} \int_{B_{\varepsilon^s}^i \setminus B_\varepsilon^i} |\hat{\eta}_\varepsilon^i|^2 dx \leq \frac{C M_\varepsilon}{|\log \varepsilon|^2} \int_\varepsilon^{\varepsilon^s} \frac{1}{r} dr \leq C(1-s);$$

therefore

$$\limsup_{\varepsilon \rightarrow 0} I_\varepsilon^2 \leq C(1-s). \quad (4.53)$$

Then, gathering (4.51) and (4.53) leads to the final estimate

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq \mathcal{E}(\mu, \beta, R) + C(1-s),$$

which entails the claim (4.32), by the arbitrariness of  $s \in (0, 1)$ .

**Step 2.**  $\mu = \sum_{\ell=1}^L \xi^\ell dx \llcorner \Omega^\ell$ , where  $\xi^\ell \in \mathbb{R}^2$  and  $\Omega^\ell \subset \Omega$  are Lipschitz pair-wise disjoint domains such that  $\Omega = \bigcup_{\ell=1}^L \Omega^\ell$  and  $|\Omega \setminus \bigcup_{\ell=1}^L \Omega^\ell| = 0$ .

For every  $\ell = 1, \dots, L$ , let  $(A^{\ell,k})_{k \in \mathbb{N}}$  be an increasing sequence of regular domains such that  $A^{\ell,k} \subset \Omega^\ell$  and  $|\Omega^\ell \setminus A^{\ell,k}| \leq \frac{1}{k}$ .

For fixed  $k \in \mathbb{N}$  and for every  $\ell = 1, \dots, L$ , we can argue as in Step 1 to construct  $\mu_\varepsilon^{\ell,k}$  and  $\beta_\varepsilon^{\ell,k}$ , recovery sequences for the energy in  $A^{\ell,k}$ , relative to  $\mu^{\ell,k} := \mu \llcorner A^{\ell,k}$  and  $\beta^{\ell,k} := \beta \llcorner_{A^{\ell,k}}$ , respectively. Then, we define the sequences  $(\mu_\varepsilon^k, \tilde{\beta}_\varepsilon^k)$  in the whole domain  $\Omega$  as

$$\mu_\varepsilon^k := \sum_{\ell=1}^L \mu_\varepsilon^{\ell,k} \chi_{A^{\ell,k}}, \quad \tilde{\beta}_\varepsilon^k := \sum_{\ell=1}^L \beta_\varepsilon^{\ell,k} \chi_{A^{\ell,k}} + R \chi_{\Omega \setminus A^k}.$$

where  $A^k := \bigcup_{\ell=1}^L A^{\ell,k}$ . Notice that by construction  $(\mu_\varepsilon^k, \tilde{\beta}_\varepsilon^k)$  converges in  $\Omega$  to  $(\mu^k, \beta^k) := (\mu \chi_{A^k}, \beta \chi_{A^k} + R \chi_{\Omega \setminus A^k})$  in the sense of Definition 4.5. Moreover,  $\mu_\varepsilon^k$  is admissible in  $\Omega$  while, in general, the sequence  $\tilde{\beta}_\varepsilon^k$  does not belong to  $\mathcal{AS}_\varepsilon(\mu_\varepsilon^k)$  since  $\text{Curl } \tilde{\beta}_\varepsilon^k \neq 0$  in  $\Omega_\varepsilon(\mu_\varepsilon^k)$ . In fact,  $\text{Curl } \tilde{\beta}_\varepsilon^k$  may have a contribution concentrated on  $\partial A^k$ .

On the other hand, since by (4.41),  $\left( \frac{R^T \beta_\varepsilon^{\ell,k} - I}{\varepsilon |\log \varepsilon|} - \beta^{\ell,k} \right) \cdot t \rightarrow 0$  strongly in  $H^{-1/2}(\partial A^{\ell,k})$  for every  $\ell = 1, \dots, L$ , we may deduce that

$$\left\| \frac{\text{Curl } \tilde{\beta}_\varepsilon^k \llcorner \Omega_\varepsilon(\mu_\varepsilon^k)}{\varepsilon |\log \varepsilon|} \right\|_{H^{-1}(\Omega; \mathbb{R}^2)} \leq \sum_{\ell=1}^L \left\| \frac{R^T \beta_\varepsilon^{\ell,k} - I}{\varepsilon |\log \varepsilon|} - \beta^{\ell,k} \right\|_{H^{-1/2}(\partial A^{\ell,k})} \rightarrow 0, \quad (4.54)$$

as  $\varepsilon$  tends to zero. Then, invoking the regularity of  $\partial \Omega$  and (4.54) we can argue as in Step 1 and add to  $\tilde{\beta}_\varepsilon^k$  a suitably chosen vanishing sequence, so that the resulting sequence  $\beta_\varepsilon^k$  satisfies  $\text{Curl } \beta_\varepsilon^k \llcorner \Omega_\varepsilon(\mu_\varepsilon^k) = 0$  and it is still a recovery sequence for  $(\mu^k, \beta^k, R)$ . Hence in particular we have

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon^k, \beta_\varepsilon^k) \leq \frac{1}{2} \int_\Omega \mathbb{C} \beta^k : \beta^k dx + \int_\Omega \varphi \left( R, \frac{d\mu^k}{d|\mu^k|} \right) d|\mu^k|.$$

Moreover, taking into account that, for  $k \rightarrow +\infty$ ,  $\beta^k$  converges to  $\beta$  strongly in  $L^2(\Omega; \mathbb{R}^{2 \times 2})$  and  $\mu^k$  converges to  $\mu$  strongly in measure, we get

$$\limsup_{k \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon^k, \beta_\varepsilon^k) \leq \mathcal{E}(\mu, \beta, R).$$



Then, invoking a standard diagonalization argument we can find an increasing sequence  $(k_\varepsilon) \subset \mathbb{N}$ , with  $k_\varepsilon \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ , such that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\mu_\varepsilon^{k_\varepsilon}, \beta_\varepsilon^{k_\varepsilon}) \leq \mathcal{E}(\mu, \beta, R).$$

Finally, it is easy to check that  $(\mu_\varepsilon, \beta_\varepsilon) := (\mu_\varepsilon^{k_\varepsilon}, \beta_\varepsilon^{k_\varepsilon})$  is the desired recovery sequence.

**Step 3.**  $\mu \in W^{-1,\infty}(\Omega; \mathbb{R}^2)$ .

We can argue as in [11, Theorem 12] Step 3 to reduce to the case of locally constant measures, namely to Step 2 in this proof. □

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