ASPECTS OF LOCAL TO GLOBAL RESULTS

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Abstract. We establish local to global results for a function space which is larger than the well known BMO space, and is also introduced by John and Nirenberg.

1. Introduction

The space of functions of bounded mean oscillation, abbreviated to BMO, is introduced by John and Nirenberg [11]. In the same paper, John and Nirenberg introduced a larger space of functions. As opposed to any BMO function, that has exponentially decaying distribution function, a function in this larger space is known to belong to a weak L^p-space, [11, Lemma 3]; the inclusion being strict, see [1, Example 3.5]. We extend this weak-type inequality to the case of John domains. The equivalence of local and global BMO norms is a rather well-known result, due to Reimann and Rychener [15]. We obtain the corresponding local to global result for the mentioned larger space of functions.

Let G be a proper open subset of $\mathbb{R}^n$, $n \geq 1$. The following condition was introduced in [11]: Let $f\colon G \to \mathbb{R}$ be a function in $L^1(G)$ and let us assume that there exists $1 < p < \infty$ such that

$$\mathcal{K}_f^p(G) := \sup_{\mathcal{P}(G)} \sum_{Q \in \mathcal{P}(G)} |Q| \left(\fint_Q |f(x) - f_Q| \, dx \right)^p < \infty \,, \tag{1.1}$$

where the supremum is taken over all partitions $\mathcal{P}(G)$ of G into cubes such that $Q \subset G$ for each $Q \in \mathcal{P}(G)$, the interiors of these cubes are pairwise disjoint, and $G = \bigcup_{Q \in \mathcal{P}(G)} Q$. We call such partitions admissible.

It is shown in [11, Lemma 3] that a function satisfying (1.1), with G being a cube Q in $\mathbb{R}^n$, belongs to a weak $L^p(Q)$-space. More precisely, there exists a positive constant C, depending only on n and p, so that for all $f \in L^1(Q)$,

$$\sigma^p \, |\{x \in Q : \ |f(x) - f_Q| > \sigma\}| \leq C \mathcal{K}_f^p(Q) \tag{1.2}$$

for each $\sigma > 0$. We refer to [6, 17, 1] for other proofs of this result. We also mention papers [4, 5, 14] where a related discrete summability condition is studied. Moreover, in a recent paper [2] its relation to condition (1.1) is discussed.

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Let us localize condition (1.1) in the following way. For a function $f \in L^1_{\mathrm{loc}}(G)$, we define the number

$$\mathcal{K}^p_{f,\mathrm{loc}}(G) := \sup_{\mathcal{P}_{\mathrm{loc}}(G)} \sum_{Q \in \mathcal{P}_{\mathrm{loc}}(G)} |Q| \left(\fint_Q |f(x) - f_Q| \, dx \right)^p ,$$

where the supremum is taken over all partitions $\mathcal{P}_{\mathrm{loc}}(G)$ of G into cubes such that for each $Q \in \mathcal{P}_{\mathrm{loc}}(G)$ a dilated cube $\lambda Q \subset G$, with fixed $\lambda > 1$, and these cubes have bounded overlap, specifically,

$$\sup_{x \in G} \sum_{Q \in \mathcal{P}_{\mathrm{loc}}(G)} \chi_Q(x) \le N ,$$

where $N \ge 1$ is a finite constant depending on n only. We call such partitions local.

We shall prove a Reimann–Rychener-type local to global result. More precisely, in Theorem 3.1, we show that there exists a positive constant C, depending on n, p, and λ, such that for all $f \in L^1(G)$

$$\mathcal{K}^p_f(G) \le C \mathcal{K}^p_{f,\mathrm{loc}}(G) .$$

In the second part of the paper, we consider necessary and sufficient conditions for Euclidean domains to support the weak-type inequality (1.2). Our main results are stated in Theorem 4.1 and Theorem 5.1.

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2. Notation and preliminaries

Throughout the paper, a cube Q in $\mathbb{R}^n$ is a closed cube with sides parallel to the coordinate axes. For a cube Q, with side length $\ell(Q)$, and for $\lambda > 0$, we write the dilated cube, with side length $\lambda \ell(Q)$, as λQ. We write χ_A for the characteristic function of a set A, the boundary of A is written as ∂A, and $|A|$ is the Lebesgue n-measure of a measurable set A in $\mathbb{R}^n$. The integral average of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ over a bounded set A with positive measure is written as f_A, that is,

$$f_A = \fint_A f \, dx = \frac{1}{|A|} \int_A f \, dx .$$

Various constants whose value may change even within a given line are denoted by C.

The family of closed dyadic cubes is written as $\mathcal{D}$. We let $\mathcal{D}_j$ be the family of those dyadic cubes whose side length is 2^{-j}, $j \in \mathbb{Z}$. For a proper open set G we fix its Whitney decomposition $\mathcal{W}(G) \subset \mathcal{D}$, and write $\mathcal{W}_j(G) = \mathcal{D}_j \cap \mathcal{W}(G)$. For a Whitney cube $Q \in \mathcal{W}(G)$ we write $Q^* = \frac{9}{8} Q$. Such dilated cubes have a bounded overlap, with upper bound depending on n only, and they satisfy

$$\frac{3}{4} \operatorname{diam}(Q) \le \operatorname{dist}(x, \partial G) \le 6 \operatorname{diam}(Q), \tag{2.1}$$

whenever $x \in Q^*$. For other properties of Whitney cubes we refer to [16, VI.1].

For a bounded domain G in $\mathbb{R}^n$, we will construct a *chain* of cubes

$$\mathcal{C}(Q) = (Q_0, \dots, Q_k) \subset \mathcal{W}(G)\,,$$

joining Q_0 and $Q = Q_k$, such that $Q_i \neq Q_j$ whenever $i \neq j$, and there exists a positive finite constant $C = C(n)$ for which

$$|Q_j^* \cap Q_{j-1}^*| \geq C \max\{|Q_j^*|, |Q_{j-1}^*|\} \tag{2.2}$$

with each $j \in \{1, \dots, k\}$. A given family $\{\mathcal{C}(Q) : \; Q \in \mathcal{W}(G)\}$ with a fixed Whitney cube Q_0 is a *chain decomposition* of G. A *shadow* of a Whitney cube $R \in \mathcal{W}(G)$ is the set

$$\mathcal{S}(R) = \{Q \in \mathcal{W}(G) : \; R \in \mathcal{C}(Q)\}.$$

Let us recall the definition of John domains. The condition in Definition 2.3 was first used by John in [10].

2.3. **Definition.** A bounded domain G in $\mathbb{R}^n$, $n \geq 2$, is a *John domain*, if there exist a point $x_0 \in G$ and a constant $\beta_G \geq 1$ such that every point x in G can be joined to x_0 by a rectifiable curve $\gamma : [0, \ell] \to G$ parametrized by its arc length for which $\gamma(0) = x$, $\gamma(\ell) = x_0$, $\ell \leq \beta_G \operatorname{diam}(G)$, and for all $t \in [0, \ell]$,

$$\operatorname{dist}(\gamma(t), \partial G) \geq t/\beta_G\,.$$

The point x_0 is called a *John center* of G, and the smallest constant $\beta_G \geq 1$ is called the *John constant* of G.

Bounded Lipschitz domains and bounded domains with the interior cone condition are John domains. Also, the Koch snowflake is a John domain in the plane. Observe that the John constant is invariant under scaling and translation of G.

The following observation concerning a given John domain G will be relevant to us. There exist a positive number $s = s(n, \beta_G) < n$ and a constant $C = C(n, \beta_G) > 0$, such that

$$\int_{B(y,r)} \operatorname{dist}(x, \partial G)^{s-n}\, dx \leq C r^s \tag{2.4}$$

for every $y \in \partial G$ and for every $r > 0$. Inequality (2.4) is essentially covered by [8, Lemma 6], but it is also an immediate consequence of the following three facts:

(1) the boundary ∂G of a John domain is porous in $\mathbb{R}^n$;
(2) the Assouad dimension of a porous set in $\mathbb{R}^n$ is strictly less than n, [13];
(3) the Assouad dimension of ∂G coincides with the Aikawa dimension of ∂G; we refer to a recent paper [12].

Indeed, by (1)–(3), the Aikawa dimension of ∂G is strictly less than n, and inequality (2.4) follows. The fact that both s and C can be chosen, depending on n and β_G only, is straightforward but tedious to verify. We omit the details.

The following proposition provides a chain decomposition of a given John domain. From now on, any reference to a chain decomposition will be to the one presented in Proposition 2.5.

2.5. **Proposition.** *(Chain decomposition) Suppose* $1 < p < \infty$ *and* G *is a John domain in* $\mathbb{R}^n$*. Then there exist constants* $\sigma, \tau \in \mathbb{N}$ *and a chain decomposition* $\{\mathcal{C}(Q) : \ Q \in \mathcal{W}(G)\}$ *of* G *with the following conditions (1)–(3):*

(1) $\ell(Q) \le 2^\tau \ell(R)$ *for each* $R \in \mathcal{C}(Q)$ *and* $Q \in \mathcal{W}(G)$*;*
(2) $\sharp\{R \in \mathcal{W}_j(G) : \ R \in \mathcal{C}(Q)\} \le 2^\tau$ *for each* $Q \in \mathcal{W}(G)$ *and* $j \in \mathbb{Z}$*;*
(3) *The following inequality holds,*

$$\sup_{j\in\mathbb{Z}} \sup_{R\in\mathcal{W}_j(G)} \frac{1}{|R|} \sum_{k=j-\tau}^{\infty} \sum_{\substack{Q\in\mathcal{W}_k(G)\\ Q\in\mathcal{S}(R)}} |Q|(\tau+1+k-j)^p < \sigma\,.$$

Furthermore, the constants σ *and* τ *depend only on* n*,* p*, and the John constant* β_G*.*

Proof. Let us first construct a chain decomposition of G. We fix a Whitney cube Q_0 containing the John center x_0 of G. Let $Q \in \mathcal{W}(G)$ and let us fix a rectifiable curve γ that is parametrized by its arc length and joins the midpoint x_Q of Q and x_0.

First assume that $Q \cap Q_0 \neq \emptyset$. Then, we join x_Q to the midpoint x_{Q_0} of Q_0 by an arc that is contained in $Q \cup Q_0$ and whose length is comparable to $\ell(Q)$. Otherwise there is $r > 0$ such that $\gamma(r)$ lies in the boundary of a Whitney cube P that intersects Q and $\gamma(t)$ belongs to a cube that is not intersecting Q whenever $t \in (r, \ell(\gamma)]$. Join x_Q to x_P by an arc whose length is comparable to $\ell(Q)$ and is in $Q \cup P$. We iterate these steps with Q replaced by P, and we continue until we reach x_{Q_0}. Let γ_Q be this composed curve parametrized by its arc length.

It is straightforward to verify that there is a constant $\rho \ge 1$, depending on n and β_G, such that for every $t \in [0, \ell(\gamma_Q)]$,

$$\operatorname{dist}(\gamma_Q(t), \partial G) \ge t/\rho\,. \tag{2.6}$$

Let $\mathcal{C}(Q)$ be the chain consisting of cubes $R \in \mathcal{W}(G)$ such that the midpoint $x_R = \gamma_Q(t_R)$ for some $t_R \in [0, \ell(\gamma_Q)]$.

We verify that this chain decomposition of G satisfies conditions (1)–(3).

Condition (1): Let $Q \in \mathcal{W}(G)$ and $R \in \mathcal{C}(Q)$. Clearly, we may assume that $R \neq Q$. Hence, if $\gamma_Q(t_R) = x_R$, then by inequalities (2.6) and (2.1),

$$\ell(Q)/2 \le t_R \le \rho \operatorname{dist}(\gamma_Q(t_R), \partial G) = \rho \operatorname{dist}(x_R, \partial G) \le 6\rho\sqrt{n}\,\ell(R)\,.$$

Condition (2): Let $Q \in \mathcal{W}(G)$ and $j \in \mathbb{Z}$. Let $R_1, \dots, R_M \in \mathcal{W}_j(G)$ be cubes such that $R_i \in \mathcal{C}(Q)$ for every $i \in \{1, \dots, M\}$. We number these cubes in the same order as γ_Q hits their midpoints. In particular, if $\gamma_Q(t) = x_{R_M}$, then $\gamma_Q([0,t])$ joins the midpoints of M cubes whose side length is 2^{-j}. By (2.6) and (2.1),

$$(M-1)2^{-j} \le t \le \rho \operatorname{dist}(\gamma_Q(t), \partial G) = \rho \operatorname{dist}(x_{R_M}, \partial G) \le 6\rho\sqrt{n}\,2^{-j}.$$

It follows that $M \le 6\rho\sqrt{n} + 1$, hence we obtain condition (2).

Let us fix $\tau = \tau(n, \beta_G) \in \mathbb{N}$ for which both conditions (1) and (2) are valid.

Condition (3): Let us first prove that there is a constant $C = C(n, \beta_G) > 0$ such that, for each $R \in \mathcal{W}(G)$,

$$\bigcup_{Q \in \mathcal{S}(R)} Q \subset B(y_R, C\ell(R)), \tag{2.7}$$

where $y_R \in \partial G$ is any point satisfying $|x_R - y_R| = \operatorname{dist}(x_R, \partial G)$. Consider any cube $Q \in \mathcal{S}(R)$. Since $R \in \mathcal{C}(Q)$, there is $t_R \in [0, \ell(\gamma_Q)]$ such that $x_R = \gamma_Q(t_R)$. Hence, if $x \in Q$,

$$|x - y_R| \le |x - x_Q| + |x_Q - x_R| + |x_R - y_R| \,.$$

Observe that $|x - x_Q| \le \operatorname{diam}(Q) \le 2^\tau \operatorname{diam}(R)$ and $|x_R - y_R| \le 6 \operatorname{diam}(R)$. By inequality (2.6),

$$|x_Q - x_R| = |\gamma_Q(0) - \gamma_Q(t_R)| \le t_R \le \rho \operatorname{dist}(\gamma_Q(t_R), \partial G) \le 6\rho \operatorname{diam}(R)\,.$$

Relation (2.7) follows from the previous estimates.

Let $\epsilon = n - s > 0$, where $s = s(n, \beta_G)$ is given by (2.4); recall that s is related to the Aikawa dimension of ∂G. Fix $j \in \mathbb{Z}$ and $R \in \mathcal{W}_j(G)$. Then, if $k \ge j - \tau$ and $Q \in \mathcal{W}_k(G)$,

$$\left(\frac{\ell(Q)}{\ell(R)}\right)^\epsilon (\tau + 1 + k - j)^p = 2^{(\tau+1)\epsilon} 2^{-(\tau+1+k-j)\epsilon} (\tau + 1 + k - j)^p \le C 2^{\tau\epsilon}, \tag{2.8}$$

where $C = C(\epsilon, p) > 0$. By inequality (2.8),

$$\sum_{k=j-\tau}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k(G) \\ Q \in \mathcal{S}(R)}} \left(\frac{\ell(Q)}{\ell(R)}\right)^n (\tau + 1 + k - j)^p \le C 2^{\tau\epsilon} \ell(R)^{-(n-\epsilon)} \sum_{Q \in \mathcal{S}(R)} \ell(Q)^{n-\epsilon}.$$

On the other hand, by (2.1), (2.7), and (2.4), we may conclude that

$$\sum_{Q \in \mathcal{S}(R)} \ell(Q)^{n-\epsilon} \le C \int_{B(y_R, C\ell(R))} \operatorname{dist}(x, \partial G)^{s-n} \, dx \le C \ell(R)^{n-\epsilon},$$

where $C = C(n, \epsilon, \beta_G) > 0$, and condition (3) follows. □

3. A local to global result

In this section, we prove the following Reimann–Rychener-type local to global result.

3.1. **Theorem.** *Suppose* G *is a proper open subset of* $\mathbb{R}^n$, $n \ge 2$. *If* $f \in L^1(G)$ *and* $1 < p < \infty$, *then*

$$\mathcal{K}_f^p(G) \le C \mathcal{K}_{f,\mathrm{loc}}^p(G)\,,$$

where a positive constant C *depends on* n, p, *and* λ.

Let us begin with a preliminary lemma, which is useful also in Section 4.

3.2. **Lemma.** *Let* H *be a John domain in* $\mathbb{R}^n$, $f \in L^1(H)$, *and* $1 < p < \infty$. *Then*

$$\left(\fint_H |f(x) - f_{Q_0^*}|\,dx\right)^p + \left(\fint_H |f(x) - f_H|\,dx\right)^p \leq \frac{C}{|H|} \sum_{Q\in\mathcal{W}(H)} |Q^*| \left(\fint_{Q^*} |f(x) - f_{Q^*}|\,dx\right)^p,$$

where Q_0 *is the fixed cube in the chain decomposition of* H. *Moreover, a positive constant* C *depends on* n, p, *and the John constant* β_H.

Proof. Observe that

$$\begin{aligned}\int_H |f(x) - f_H|\,dx &\leq 2\int_H |f(x) - f_{Q_0^*}|\,dx \\ &\leq 2 \sum_{Q\in\mathcal{W}(H)} \int_{Q^*} |f(x) - f_{Q^*}|\,dx + 2 \sum_{Q\in\mathcal{W}(H)} |Q||f_{Q^*} - f_{Q_0^*}|\,. \end{aligned} \tag{3.3}$$

Let us estimate the first term on the right-hand side in (3.3). By Hölder's inequality,

$$\begin{aligned}&\sum_{Q\in\mathcal{W}(H)} \int_{Q^*} |f(x) - f_{Q^*}|\,dx \\ &\leq C \left(\sum_{Q\in\mathcal{W}(H)} |Q|\right)^{1/p'} \left(\sum_{Q\in\mathcal{W}(H)} |Q| \left(\fint_{Q^*} |f(x) - f_{Q^*}|\,dx\right)^p\right)^{1/p} \\ &\leq C|H|^{1/p'} \left(\sum_{Q\in\mathcal{W}(H)} |Q^*| \left(\fint_{Q^*} |f(x) - f_{Q^*}|\,dx\right)^p\right)^{1/p}, \end{aligned} \tag{3.4}$$

where $p' = p/(p-1)$ is the conjugate exponent to p.

To estimate the second term on the right-hand side in (3.3), we use a chain $\mathcal{C}(Q) = (Q_0, \dots, Q_k)$ joining the cube Q_0 to $Q_k = Q \in \mathcal{W}(H)$. Hence,

$$\sum_{Q\in\mathcal{W}(H)} |Q||f_{Q^*} - f_{Q_0^*}| \leq \sum_{Q\in\mathcal{W}(H)} |Q| \sum_{i=1}^{k} |f_{Q_i^*} - f_{Q_{i-1}^*}|. \tag{3.5}$$

Here, by property (2.2), for any $i \in \{1, \dots, k\}$

$$\begin{aligned}|f_{Q_i^*} - f_{Q_{i-1}^*}| &\leq \fint_{Q_i^*\cap Q_{i-1}^*} |f - f_{Q_i^*}|\,dx + \fint_{Q_i^*\cap Q_{i-1}^*} |f - f_{Q_{i-1}^*}|\,dx \\ &\leq C \sum_{j=i-1}^{i} \fint_{Q_j^*} |f(x) - f_{Q_j^*}|\,dx\,. \end{aligned}$$

By the fact that there are no duplicates in $\mathcal{C}(Q)$, i.e., $Q_i \neq Q_j$ if $i \neq j$, we obtain

$$\begin{aligned}\sum_{Q\in\mathcal{W}(H)} |Q||f_{Q^*} - f_{Q_0^*}| &\le C \sum_{Q\in\mathcal{W}(H)} |Q| \sum_{i=1}^{k} \sum_{j=i-1}^{i} \fint_{Q_j^*} |f(x) - f_{Q_j^*}|\,dx \\ &\le C \sum_{Q\in\mathcal{W}(H)} |Q| \sum_{R\in\mathcal{C}(Q)} \fint_{R^*} |f(x) - f_{R^*}|\,dx \\ &\le C \sum_{R\in\mathcal{W}(H)} \sum_{Q\in\mathcal{S}(R)} |Q| \fint_{R^*} |f(x) - f_{R^*}|\,dx \\ &\le C \sum_{R\in\mathcal{W}(H)} \int_{R^*} |f(x) - f_{R^*}|\,dx\,,\end{aligned}$$

where the last inequality is a consequence of inequality (2.7). We may estimate as in connection with (3.4). This completes the proof. □

3.6. *Remark.* The following inequality, interesting as such, follows from Lemma 3.2. Let Q be a cube and $f \in L^1(Q)$. Then, for every $1 < p < \infty$,

$$\left(\fint_Q |f(x) - f_Q|\,dx\right)^p \le \frac{C}{|Q|} \sum_{R\in\mathcal{W}(Q)} |R^*| \left(\fint_{R^*} |f(x) - f_{R^*}|\,dx\right)^p,$$

where $\mathcal{W}(Q)$ refers to Whitney decomposition of the interior of Q and C is a positive constant depending only on n and p.

Proof of Theorem 3.1. Let us fix an admissible partition $\mathcal{P}(G)$ of G into cubes. For each cube $Q \in \mathcal{P}(G)$ we form a local partition $\mathcal{P}_{\mathrm{loc}}(Q) = \{R^* : R \in \mathcal{W}(Q)\}$. We write

$$\mathcal{P}_{\mathrm{loc}}(G) = \bigcup_{Q\in\mathcal{P}(G)} \mathcal{P}_{\mathrm{loc}}(Q).$$

It is straightforward to verify that $\mathcal{P}_{\mathrm{loc}}(G)$ is a local partition of G. In particular, for each $R^* \in \mathcal{P}_{\mathrm{loc}}(Q)$ with $Q \in \mathcal{P}(G)$, the inclusions $\lambda R^* \subset Q \subset G$ are valid for $1 < \lambda < \frac{10}{9}$. By applying Remark 3.6 and observing that for each $R^* \in \mathcal{P}_{\mathrm{loc}}(G)$ there is at most one cube $Q \in \mathcal{P}(G)$ such that $R^* \in \mathcal{P}_{\mathrm{loc}}(Q)$, we obtain

$$\begin{aligned}&\sum_{Q\in\mathcal{P}(G)} |Q| \left(\fint_Q |f(x) - f_Q|\,dx\right)^p \\ &\quad\le C \sum_{Q\in\mathcal{P}(G)} \sum_{R^*\in\mathcal{P}_{\mathrm{loc}}(Q)} |R^*| \left(\fint_{R^*} |f(x) - f_{R^*}|\,dx\right)^p \\ &\quad\le C \sum_{R^*\in\mathcal{P}_{\mathrm{loc}}(G)} |R^*| \left(\fint_{R^*} |f(x) - f_{R^*}|\,dx\right)^p \le C\mathcal{K}^p_{f,\mathrm{loc}}(G)\,.\end{aligned}$$

The proof is completed by taking the supremum over all admissible partitions $\mathcal{P}(G)$. □

3.7. *Remark.* The construction of the Whitney decomposition that is described in Section 2 yields Theorem 3.1 for all $1 < \lambda < \frac{10}{9}$. A simple modification of the definition for dilated cubes Q^* allows one to extend this range to every $1 < \lambda < \frac{5}{4}$. It possible to use the general Whitney decomposition based on Stein [16, pp. 167–170] in order to obtain the result for any $\lambda \geq \frac{5}{4}$.

4. A sufficient condition for a weak-type inequality

In this section, we show that cubes can be replaced by John domains in inequality (1.2).

4.1. **Theorem.** *Suppose that G is a John domain in $\mathbb{R}^n$. If $f \in L^1(G)$ and $1 < p < \infty$, then the following weak-type inequality is valid*

$$\sigma^p |\{x \in G : \ |f(x) - f_G| > \sigma\}| \leq C \mathcal{K}^p_{f,\mathrm{loc}}(G)$$

for all $\sigma > 0$, where a positive constant C depends on n, p, λ, and the John constant β_G.

Proof. Recall that Q_0 is a fixed cube which is used to construct a chain decomposition of G, see Proposition 2.5. By the triangle inequality for each $x \in G$,

$$\begin{aligned} |f(x) - f_G| &\leq |f_{Q_0^*} - f_G| + \left| f(x) - \sum_{Q \in \mathcal{W}(G)} f_{Q^*} \chi_Q(x) \right| + \left| \sum_{Q \in \mathcal{W}(G)} f_{Q^*} \chi_Q(x) - f_{Q_0^*} \right| \\ &=: g_1(x) + g_2(x) + g_3(x)\,. \end{aligned}$$

Hence, for a fixed $\sigma > 0$, we have

$$\sigma^p |\{x \in G : \ |f(x) - f_G| > \sigma\}| \leq \sigma^p \mathbf{F}_1(\sigma) + \sigma^p \mathbf{F}_2(\sigma) + \sigma^p \mathbf{F}_3(\sigma)$$

where we have written

$$\mathbf{F}_j(\sigma) = |\{x \in G : \ g_j(x) > \sigma/3\}|$$

for $j \in \{1, 2, 3\}$. We shall next estimate these three terms.

If $|f_{Q_0^*} - f_G| \leq \sigma/3$, then $\mathbf{F}_1(\sigma) = 0$. Otherwise, by Lemma 3.2,

$$\sigma^p \mathbf{F}_1(\sigma) \leq 3^p |G| \left(\fint_G |f(x) - f_{Q_0^*}|\, dx \right)^p \leq C \sum_{Q \in \mathcal{W}(G)} \mathcal{K}^p_f(Q^*) \leq C \mathcal{K}^p_{f,\mathrm{loc}}(G).$$

Let us focus on the term $\sigma^p \mathbf{F}_2(\sigma)$. By applying inequality (1.2),

$$\begin{aligned} \sigma^p \mathbf{F}_2(\sigma) &= \sum_{Q \in \mathcal{W}(G)} \sigma^p |\{x \in \mathrm{int}(Q) \ : \ g_2(x) > \sigma/3\}| \\ &\leq \sum_{Q \in \mathcal{W}(G)} \sigma^p |\{x \in Q^* : \ |f(x) - f_{Q^*}| > \sigma/3\}| \leq C 3^p \sum_{Q \in \mathcal{W}(G)} \mathcal{K}^p_f(Q^*) \leq C \mathcal{K}^p_{f,\mathrm{loc}}(G). \end{aligned}$$

Let us estimate the remaining term $\sigma^p \mathbf{F}_3(\sigma)$ as follows

$$\sigma^p \mathbf{F}_3(\sigma) = \sigma^p \sum_{Q \in \mathcal{W}(G)} |\{x \in \mathrm{int}(Q) \ : \ |f_{Q^*} - f_{Q_0^*}| > \sigma/3\}|$$

$$= \sum_{\substack{Q\in\mathcal{W}(G)\\ |f_{Q^*}-f_{Q_0^*}|>\sigma/3}} \sigma^p|Q| \le 3^p \sum_{Q\in\mathcal{W}(G)} |Q||f_{Q^*}-f_{Q_0^*}|^p.$$

Estimating as in connection with (3.5), we end up having

$$|f_{Q^*}-f_{Q_0^*}|^p \le C\left(\sum_{R\in\mathcal{C}(Q)} \fint_{R^*} |f(x)-f_{R^*}|\,dx\right)^p.$$

We use condition (1) of the chain $\mathcal{C}(Q)$ in Proposition 2.5. Then we write for $j \le k+\tau$

$$1 = (\tau+1+k-j)^{-1}(\tau+1+k-j),$$

apply Hölder's inequality, and finally use inequality

$$\sup_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k+\tau} (\tau+1+k-j)^{-p'} < \infty\,,$$

to conclude that

$$\begin{aligned}\sigma^p\mathbf{F}_3(\sigma) &\le C\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{W}_k(G)}|Q|\left(\sum_{j=-\infty}^{k+\tau}\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}\fint_{R^*}|f(x)-f_{R^*}|\,dx\right)^p\\ &\le C\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{W}_k(G)}|Q|\sum_{j=-\infty}^{k+\tau}(\tau+1+k-j)^p\left(\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}\fint_{R^*}|f(x)-f_{R^*}|\,dx\right)^p.\end{aligned} \tag{4.2}$$

By condition (2) in Proposition 2.5 and Hölder's inequality, for any $Q\in\mathcal{W}(G)$ and $j\in\mathbb{Z}$,

$$\begin{aligned}\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}\fint_{R^*}|f(x)-f_{R^*}|\,dx &\le \left(\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}1\right)^{1/p'}\left(\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}\left(\fint_{R^*}|f-f_{R^*}|\right)^p\right)^{1/p}\\ &\le C\left(\sum_{\substack{R\in\mathcal{W}_j(G)\\ R\in\mathcal{C}(Q)}}\frac{\mathcal{K}_f^p(R^*)}{|R^*|}\right)^{1/p}.\end{aligned} \tag{4.3}$$

If we substitute the estimate obtained in (4.3) to (4.2), and observe that $R \in \mathcal{C}(Q)$ if and only if $Q \in \mathcal{S}(R)$, we bound $\sigma^p \mathbf{F}_3(\sigma)$ as follows

$$\begin{aligned}
\sigma^p \mathbf{F}_3(\sigma) &\le C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} |Q| \sum_{j=-\infty}^{k+\tau} (\tau + 1 + k - j)^p \sum_{\substack{R \in \mathcal{W}_j(G) \\ R \in \mathcal{C}(Q)}} \frac{\mathcal{K}_f^p(R^*)}{|R^*|} \\
&= C \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j(G)} \frac{\mathcal{K}_f^p(R^*)}{|R|} \sum_{k=j-\tau}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k(G) \\ Q \in \mathcal{S}(R)}} |Q| (\tau + 1 + k - j)^p \\
&\le C \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j(G)} \mathcal{K}_f^p(R^*) \le C \mathcal{K}_{f,\mathrm{loc}}^p(G),
\end{aligned}$$

where we used condition (3) in Proposition 2.5. The claim follows. □

We formulate the preceding theorem for locally integrable functions; the proof is otherwise the same, but term g_1 is omitted and we choose $c = f_{Q_0^*}$.

4.4. **Theorem.** *Suppose that G is a John domain in $\mathbb{R}^n$. If $f \in L^1_{\mathrm{loc}}(G)$ and $1 < p < \infty$, then the following weak-type inequality is valid*

$$\inf_{c \in \mathbb{R}} \sup_{\sigma > 0} \sigma^p |\{x \in G : \ |f(x) - c| > \sigma\}| \le C \mathcal{K}_{f,\mathrm{loc}}^p(G) \,, \tag{4.5}$$

where a positive constant C depends on n, p, λ, and the John constant β_G.

5. Necessary conditions for a weak-type inequality

We study necessary conditions for the validity of weak-type inequality (4.5) on domains. In Theorem 5.1, a necessary condition is formulated in terms of a Poincaré inequality. Corollary 5.6 addresses the necessity of the John condition.

5.1. **Theorem.** *Suppose that $n/(n-1) \le p < \infty$, and that G is a bounded domain in $\mathbb{R}^n$, $n \ge 2$, for which the inequality*

$$\inf_{c \in \mathbb{R}} \sup_{\sigma > 0} \sigma^p |\{x \in G : \ |f(x) - c| > \sigma\}| \le C \mathcal{K}_{f,\mathrm{loc}}^p(G) \tag{5.2}$$

holds for all $f \in L^1_{\mathrm{loc}}(G)$. Then G satisfies the (q^*, q)-Poincaré inequality (5.4) with $p = q^* = nq/(n-q)$, where $1 \le q < n$.

Proof. It is enough to verify that G satisfies the *weak* (q^*, q)-Poincaré inequality. That is, for all locally Lipschitz functions f in G,

$$\inf_{c \in \mathbb{R}} \sup_{\sigma > 0} \sigma^{q^*} |\{x \in G : \ |f(x) - c| > \sigma\}| \le C \left(\int_G |\nabla f(x)|^q \, dx \right)^{q^*/q} . \tag{5.3}$$

By applying inequality (5.3) and [7, Theorem 4], we may conclude that G satisfies the (q^*, q)-Poincaré inequality:

$$\int_G |f(x) - f_G|^{q^*}\,dx \le C\left(\int_G |\nabla f(x)|^q\,dx\right)^{q^*/q}, \tag{5.4}$$

where f is in the Sobolev space $W^{1,q}(G)$.

Therefore, let us prove inequality (5.3). This will be a consequence of the (q^*, q)-Poincaré inequality on cubes in G. Namely, there is a local partition $\mathcal{P}_{\text{loc}}(G)$ such that

$$\begin{aligned}\inf_{c\in\mathbb{R}}\sup_{\sigma>0}\sigma^{q^*}|\{x\in G:\ |f(x)-c|>\sigma\}| &\le C\sum_{Q\in\mathcal{P}_{\text{loc}}(G)}|Q|\left(\fint_Q |f(x)-f_Q|\,dx\right)^{q^*}\\ &\le C\sum_{Q\in\mathcal{P}_{\text{loc}}(G)}\int_Q |f(x)-f_Q|^{q^*}\,dx\\ &\le C\sum_{Q\in\mathcal{P}_{\text{loc}}(G)}\left(\int_Q|\nabla f(x)|^q\,dx\right)^{q^*/q}.\end{aligned}$$

Since $q^*/q = n/(n-q) > 1$, we obtain the desired inequality (5.3). □

5.5. *Remark.* We may also conclude the following weak fractional Sobolev–Poincaré inequality. Suppose that inequality (5.2) holds for all $f \in L^1_{\text{loc}}(G)$ with $n/(n-\delta) < p < \infty$ and $\delta \in (0,1)$. By estimating as in the proof of Theorem 5.1, and applying [9, Theorem 4.10], we find that

$$\inf_{c\in\mathbb{R}}\sup_{\sigma>0}\sigma^{q^{*,\delta}}|\{x\in G\,:\,|f(x)-c|>\sigma\}| \le C\left(\int_G\int_G \frac{|f(x)-f(y)|^q}{|x-y|^{n+\delta q}}\,dy\,dx\right)^{q^{*,\delta}/q}$$

for all $f \in L^1_{\text{loc}}(G)$, where $p = q^{*,\delta} = nq/(n-\delta q)$ and $1 < q < n/\delta$.

We recall from [3, Definition 3.2] that a domain G with a fixed point x_0 satisfies a separation property if there exists a constant C_0 such that for each $x \in G$ there is a curve γ joining x and x_0 in G so that for each t either

$$\gamma([0,t]) \subset B := B(\gamma(t), C_0\,\mathrm{dist}(\gamma(t), \mathbb{R}^n \setminus G))$$

or each $y \in \gamma([0,t]) \setminus B$ belongs to a different component of $G \setminus \partial B$ than x_0. As an example, for simply connected planar domains, the separation property is automatically valid.

The following corollary is a consequence of Theorem 4.4, Theorem 5.1, and [3, Theorem 1.1].

5.6. **Corollary.** *Suppose that G is a bounded domain in $\mathbb{R}^n$, $n \ge 2$, satisfying a separation property. Assume further that $n/(n-1) \le p < \infty$. Then the weak-type inequality*

$$\inf_{c\in\mathbb{R}}\sup_{\sigma>0}\sigma^p|\{x\in G:\ |f(x)-c|>\sigma\}| \le C\mathcal{K}^p_{f,\text{loc}}(G)$$

holds for every $f \in L^1_{\text{loc}}(G)$ if, and only if, G is a John domain.

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