SHARP STABILITY THEOREMS FOR THE ANISOTROPIC SOBOLEV AND LOG-SOBOLEV INEQUALITIES ON FUNCTIONS OF BOUNDED VARIATION

A. FIGALLI, F. MAGGI, AND A. PRATELLI

Abstract. Combining rearrangement techniques with Gromov's proof (via optimal mass transportation) of the 1-Sobolev inequality, we prove a sharp quantitative version of the anisotropic Sobolev inequality on $BV(\mathbb{R}^n)$. As a corollary of this result, we also deduce a sharp stability estimate for the anisotropic 1-log-Sobolev inequality.

Keywords: Sobolev inequalities, stability, rearrangement, mass transportation.

1. Introduction

1.1. **Overview.** We present here a sharp stability theorem for the anisotropic Sobolev inequality on functions of bounded variation. Previous contributions to this problem, although providing sharp decay rates, were limited to the isotropic case. In this paper, by a combination of optimal mass transportation methods and rearrangement techniques, we are able to address the anisotropic case, still with sharp decay rates. Further interesting improvements are also obtained: first, the new stability estimates come with explicit constants, a feature of possible interest for numerical applications which was missing so far; second, in the spirit of the celebrated result by Bianchi and Egnell [BE] for the Sobolev inequality on $W^{1,2}(\mathbb{R}^n)$, the distance from the class of optimal functions is also controlled (in a suitable form) at the level of gradients. Finally, by a simple argument, this analysis is extended to the anisotropic 1-log-Sobolev inequality.

1.2. **The anisotropic Sobolev inequality and the Wulff inequality.** The anisotropic Sobolev inequality is a natural extension of the standard Sobolev inequality on $BV(\mathbb{R}^n)$, which is obtained by measuring gradients through the gauge function of a convex set, rather than by the Euclidean norm. Precisely, given an open, bounded convex set K in $\mathbb{R}^n$ ($n \geq 2$), containing the origin, if we define the gauge function of K as

$$\|x\|_* = \sup\{x \cdot y : y \in K\}\,, \qquad x \in \mathbb{R}^n\,,$$

then we have the following anisotropic Sobolev inequality

$$\int_{\mathbb{R}^n} \|-\nabla f(x)\|_* \, dx \geq n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)}\,, \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n)\,.$$

Here, $n' = n/(n-1)$ and $|K|$ denotes the Lebesgue measure of K. By an approximation argument the inequality holds true on $BV(\mathbb{R}^n)$, in the form

$$TV_K(f) \geq n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)}\,, \qquad \forall\, f \in BV(\mathbb{R}^n)\,, \tag{1.1}$$

where $TV_K(f)$ denotes the anisotropic total variation of f,

$$\begin{aligned} TV_K(f) &= \sup\left\{ \sum_{h\in\mathbb{N}} \|-Df(E_h)\|_* : \{E_h\}_{h\in\mathbb{N}} \text{ is a Borel partition of } \mathbb{R}^n \right\} \\ &= \sup\left\{ \int_{\mathbb{R}^n} f(x)\,\mathrm{div}\, T(x)\,dx : T \in C_c^1(\mathbb{R}^n; K) \right\}. \end{aligned}$$

An important particular case of (1.1) is obtained when E is a set of finite perimeter in $\mathbb{R}^n$ with $|E| < \infty$. In this case we have $1_E \in BV(\mathbb{R}^n)$, and $TV_K(1_E)$ agrees with the K-anisotropic perimeter $P_K(E)$ of E, namely,

$$TV_K(1_E) = \int_{\partial^* E} \|\nu_E\|_* \, d\mathcal{H}^{n-1} =: P_K(E) \, .$$

(Here ν_E denotes the (measure theoretic) outer unit normal to E, and $\partial^* E$ is the reduced boundary of E.) Correspondingly, the anisotropic Sobolev inequality reduces to the Wulff inequality

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'} \, , \tag{1.2}$$

which in turn agrees with the Euclidean isoperimetric inequality in the case $K = B$.

1.3. **Equality cases and stability theorems.** Equality holds in (1.2) if and only if E is equivalent (with respect to Lebesgue measure) to $x_0 + r\,K$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$. Sharp quantitative versions of (1.2) have been obtained in [FMP1] concerning the case $K = B$, and in [FiMP] for the general anisotropic case (see also [CL] for an alternative approach to the isotropic case). In particular, in [FiMP] it is proved that, if E is a set of finite perimeter with $|E| = 1$, then there exists $x_0 \in \mathbb{R}^n$ such that

$$P_K(E) \geq n|K|^{1/n} \left\{ 1 + \left(\frac{|E\Delta(x_0 + r_0\,K)|}{C_0(n)} \right)^2 \right\} \, , \qquad r_0 = \frac{1}{|K|^{1/n}} \, , \tag{1.3}$$

where one can take

$$C_0(n) = \frac{181 n^7}{(2 - 2^{1/n'})^{3/2}} \, , \tag{1.4}$$

(in the Euclidean case $K = B$, the factor n^7 may be replaced by n^3). In the case of the anisotropic Sobolev inequality, optimal functions are precisely multiples of characteristic functions of (rescaled and/or translated copies of) K. However, one has to be careful when the sign changes: indeed, the equality $TV_K(1_K) = TV_K(-1_K)$ holds if and only if $K = -K$. If K is not symmetric with respect to the origin, then it turns out that the "prototype" negative optimal function is -1_{-K}, and not -1_K (indeed, it is immediate to check that $TV_K(1_K) = TV_K(-1_{-K})$, and so -1_{-K} is optimal in (1.1)). With this caveat in mind, one sees that the family of (non-zero) optimal functions in (1.1) is

$$g_{a,x_0,r} = a\, 1_{x_0 + ar\,K} \, , \qquad a \neq 0 \, , x_0 \in \mathbb{R}^n \, , r > 0 \, .$$

We are now in the position to look for a quantitative improvement of (1.1), in the spirit of (1.3). Let us agree to work, for the sake of simplicity and without loss of generality, in the class $\mathcal{M}_0$ of those elements $f \in BV(\mathbb{R}^n)$ such that $|f|^{n'}dx$ is a probability measure, i.e.

$$\mathcal{M}_0 = \left\{ f \in BV(\mathbb{R}^n) : \int_{\mathbb{R}^n} |f|^{n'} = 1 \right\} .$$

Correspondingly, let $\{\widehat{g}_{a,x_0}\}_{a\neq 0\,, x_0 \in \mathbb{R}^n}$ be the class of those optimal functions in (1.1) which belong to $\mathcal{M}_0$, that is

$$\int_{\mathbb{R}^n} |\widehat{g}_{a,x_0}|^{n'} = 1 \, , \qquad \widehat{g}_{a,x_0} = g_{a,x_0,r(a)} \, .$$

Finally, let us introduce the "distance" (see Remark 1.4 and Lemma 2.2),

$$d(f,g) = \int_{\mathbb{R}^n} |f - g|^{n'} + d_0(f,g) \, , \qquad f, g \in BV(\mathbb{R}^n) \, , \tag{1.5}$$

where

$$d_0(f,g) = \inf \left\{ \frac{\| - D(f-g)\|_*(\mathbb{R}^n \setminus E)}{n|K|^{1/n}} + \int_E |f|^{n'} + |g|^{n'} \, : \, E \text{ is a Borel set in } \mathbb{R}^n \right\} .$$

Notice that, up to multiplicative factors depending on K only, we could have replaced the anisotropic total variation term $\|-D(f-g)\|_*(\mathbb{R}^n\setminus E)$ with the standard total variation $|D(f-g)|(\mathbb{R}^n\setminus E)$. However, with our definition, we can get a stability estimate with a constant depending on the dimension only. Our main result takes then the following form.

Theorem 1.1. *If $f\in\mathcal{M}_0$, then there exists $a\neq 0$ and $x_0\in\mathbb{R}^n$ such that*

$$TV_K(f)\ge n|K|^{1/n}\left\{1+\left(\frac{d(f,\widehat{g}_{a,x_0})}{C_1(n)}\right)^2\right\}, \tag{1.6}$$

where

$$C_1(n)=1800\,\big(n+C_0(n)\big)\sqrt{n}\,, \tag{1.7}$$

and $C_0(n)$ is given by (1.4).

Remark 1.2. Introducing the scale and translation invariant *Sobolev deficit* functional,

$$\delta(f)=\frac{TV_K(f)}{n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)}}-1\,,\qquad f\in BV(\mathbb{R}^n)\,, \tag{1.8}$$

inequality (1.6) takes the form

$$C_1(n)\,\sqrt{\delta(f)}\ge\inf\left\{d(f,g_{a,x_0,r}):\|f\|_{L^{n'}(\mathbb{R}^n)}=\|g_{a,x_0,r}\|_{L^{n'}(\mathbb{R}^n)}\right\},\quad\forall\, f\in\mathcal{M}_0\,. \tag{1.9}$$

Of course, the restriction $\int_{\mathbb{R}^n}|f|^{n'}=1$ in Theorem 1.1 is easily dropped by applying (1.6) to $f/\|f\|_{L^{n'}(\mathbb{R}^n)}$.

Remark 1.3 (Previous contributions)**.** Theorem 1.1 was proved in [FMP2] in the isotropic case $K=B$, with a non-explicit constant in place of $C_1(n)$ and with $\int_{\mathbb{R}^n}|f-\widehat{g}_{a,x_0}|^{n'}$ in place of $d(f,\widehat{g}_{a,x_0})$. In [Ci], Cianchi presented an argument that, starting from a quantitative version of the Wulff inequality, produces a quantitative version of the anisotropic Sobolev inequality, where the distance between f and a suitable $\widehat{g}_{a,x_0}$ is measured in some Lorentz space instead that in $L^{n'}$. This method produces however a non-sharp decay rate, meaning that the sharp power 2 appearing on the right-hand side of (1.6) has to be replaced by the larger power $1+2n'\in(3,5]$.

Remark 1.4 (Sharpness of the distance)**.** In [BE], Bianchi and Egnell proved the existence of a (non-explicit) constant $C(n)$ with the property that, for every $f\in W^{1,2}(\mathbb{R}^n)$, $f\neq 0$, there exist $a\neq 0$, $x_0\in\mathbb{R}^n$, and $r>0$ such that

$$\int_{\mathbb{R}^n}|\nabla f|^2\ge S(n,2)^2\|f\|^2_{L^{2^\star}(\mathbb{R}^n)}+\frac{1}{C(n)}\int_{\mathbb{R}^n}|\nabla f-\nabla g|^2\,,$$

where $2^\star=2\,n/(n-2)$, $S(n,2)$ is the sharp constant in the Sobolev inequality, and where

$$g(x)=\frac{a}{(1+|r(x-x_0)|^2)^{(n-2)/2}}\,,\qquad x\in\mathbb{R}^n\,.$$

The strong feature of this result, especially in comparison with the stability theorems from [FMP2] and [Ci] for the Sobolev inequality on $BV(\mathbb{R}^n)$, is that the distance from the set of optimal functions is measured by a Lebesgue norm of the gradients. However, it is not clear what should be the correct "gradient distance" one can try to control in a quantitative version of (1.1). A naive candidate distance could be of course the total variation of $f-\widehat{g}_{a,x_0}$, but it is easy to construct a sequence $\{f_h\}_{h\in\mathbb{N}}\subset\mathcal{M}_0$ such that

$$\lim_{h\to\infty}\delta(f_h)=0\,,\qquad\lim_{h\to\infty}\inf_{a\neq0\,,x_0\in\mathbb{R}^n}|D(f_h-\widehat{g}_{a,x_0})|(\mathbb{R}^n)>0\,,$$

see Figure 1.1. Analogously, one cannot expect to control the L^1 norm of the absolutely continuous part of Df, since arguing by approximation and using the lower semicontinuity of the total variation, one would actually be able to control the full total variation of Df, which (as we just observed) is impossible.

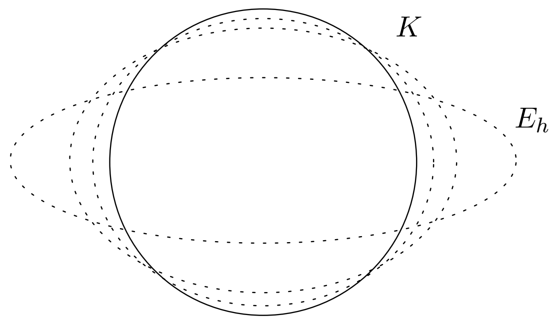

FIGURE 1.1. In the case $K = B$, consider a sequence of ellipsoids $\{E_h\}_{h\in\mathbb{N}}$ converging to a ball B_r with $|E_h| = |B_r| = 1$ and $\mathcal{H}^{n-1}(\partial E_h \cap \partial B) = 0$ for every $h \in \mathbb{N}$, and such that $1_{E_h} \to 1_{B_r}$ in $L^1(\mathbb{R}^n)$. It is clear that $\delta(1_{E_h}) \to 0$, while $|D(1_{E_h} - 1_{B_r})|(\mathbb{R}^n) = P(E_h) + P(B_r) \to 2\,P(B_r)$. However, choosing as test sets C_ε in the definition of $d_0(1_{E_h}, 1_B)$ the complements of ε-neighborhoods of $\partial E_h \cup \partial B$, we easily see that $d_0(1_{E_h}, 1_B) = 0$ and $d(1_{E_h}, 1_B) = |E_h \Delta B| \to 0$.

However, as Theorem 1.1 shows, it is possible to control $d_0(f, \widehat{g}_{a,x_0})$, which amounts to bound the total variation of $f - \widehat{g}_{a,x_0}$ limited to a subset of $\mathbb{R}^n$ whose complement has small measure with respect to both $|f|^{n'}dx$ and $|\widehat{g}_{a,x_0}|^{n'}dx$.

Let us observe that, although d_0 gives no extra informations when f is the characteristic function of a set of finite perimeter (see Lemma 2.5), it provides stronger informations when Df has some absolutely continuous part: for instance, if f is C^1 and has small deficit, then not only f is close in $L^{n'}$ to some optimizer $\hat{g}_{a,x_0}$, but also ∇f is small in L^1 strictly inside $x_0 + ar(a)K$.

1.4. **Strategy of proof and organization of the paper.** The proof of the above result is based on a careful combination of rearrangements techniques applied to Gromov's proof (via optimal transportation) of the anisotropic Sobolev inequality. More precisely, as shown in the proof of Theorem 1.1, we can reduce to the case of a smooth non-negative function f. This case is then adressed in Theorem 2.7. The core in the proof of this latter results is Step I, where we show that a function with small deficit must be close (in a precise quantitative way) to a characteristic function of an isoperimetric set $x_0 + rK$. Once this result is established, we conclude with the help of (1.3).

The paper is organized as follows: in Section 2 we introduce some notation and preliminary results, and we show some basic properties of the "distance" d introduced in (1.5). Then we prove Theorem 1.1 for smooth nonnegative functions (see Theorem 2.7), and we show how the general result of Theorem 1.1 can be deduced from Theorem 2.7. Finally, in Section 3 we observe how Theorem 1.1 implies a stability result for a family of anisotropic 1-log-Sobolev inequalities.

2. Stability for the anistropic Sobolev inequality on BV functions

2.1. **Notation and preliminaries.** We start with some notation and preliminary remarks which reveal useful in the sequel.

2.1.1. *Functions of bounded variation.* We shall work with the space $BV(\mathbb{R}^n)$ of the functions of bounded variation in $\mathbb{R}^n$, referring to the monograph [AFP] for all the needed background. In particular, given $f \in BV(\mathbb{R}^n)$, Df shall denote the distributional gradient of f, which is required to define a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ with finite total variation $|Df|$, and

$$Df = \nabla f\,dx + D^s f\,,$$

shall be the Radon-Nykodim decomposition of Df with respect to the Lebesgue measure. Concerning this decomposition, we shall need the following natural property of regularization by convolution, the proof of which we were not able to track in the literature.

Lemma 2.1. *Let $f \in BV(\mathbb{R}^n)$, and set $f_k = f * \rho_k$, where $\{\rho_k\}_{k\in\mathbb{N}}$ is a sequence of smooth compactly supported convolution kernels. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n\setminus A} |\nabla f_k - \nabla f| = 0\,,$$

whenever A is an open set such that $|D^s f|$ is concentrated on A.

Proof. A truncation argument allows to reduce to the case when f has compact support contained in a closed ball $\overline{B}_R$, $R>0$. Correspondingly, we may assume A to be bounded. If we now consider the compact set $\mathcal{K} = \overline{B}_R \cap (\mathbb{R}^n \setminus A)$, then we want to prove that

$$\lim_{k\to\infty} \int_{\mathcal{K}} |\nabla f_k - \nabla f| = 0\,. \tag{2.1}$$

Since $1_{\mathcal{K}}\nabla f \in L^1(\mathbb{R}^n)$, by standard convolution estimates we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |(1_{\mathcal{K}}\nabla f) * \rho_k - 1_{\mathcal{K}}\nabla f| = 0\,,$$

and thus (2.1) is equivalent to show that

$$\lim_{k\to\infty} \int_{\mathcal{K}} |\nabla f_k - (1_{\mathcal{K}}\nabla f) * \rho_k| = 0\,. \tag{2.2}$$

Since $D^s f = D^s f \llcorner A$, we find that

$$\begin{aligned}\nabla f_k - (1_{\mathcal{K}}\nabla f) * \rho_k &= (Df) * \rho_k - (1_{\mathcal{K}}\nabla f) * \rho_k \\ &= (\nabla f) * \rho_k + (D^s f) * \rho_k - (1_{\mathcal{K}}\nabla f) * \rho_k \\ &= (1_A\,\nabla f) * \rho_k + (D^s f \llcorner A) * \rho_k = (Df \llcorner A) * \rho_k\,,\end{aligned}$$

so that

$$\int_{\mathcal{K}} |\nabla f_k - (1_{\mathcal{K}}\nabla f) * \rho_k| = \int_{\mathcal{K}} |(Df \llcorner A) * \rho_k| \le \int_{\mathcal{K}} (|Df| \llcorner A) * \rho_k\,dx\,.$$

Since $(|Df|\llcorner A) * \rho_k$ weakly* converges to the measure $|Df|\llcorner A$ and $\mathcal{K}$ is compact, by the standard upper semicontinuity of weak* convergence of Radon measures we obtain

$$\limsup_{k\to\infty} \int_{\mathcal{K}} (|Df|\llcorner A) * \rho_k\,dx \le \big(|Df|\llcorner A\big)(\mathcal{K}) = 0\,,$$

where the last equality follows from $\mathcal{K} \cap A = \emptyset$. This concludes the proof of (2.2), as required. □

2.1.2. *Anisotropic total variation.* We will work with a fixed open, bounded and convex set K in $\mathbb{R}^n$, containing the origin. We associate to K two convex and positively 1-homogeneous functions, $\|\cdot\|$ and $\|\cdot\|_*$, by setting for each $x \in \mathbb{R}^n$,

$$\|x\| = \inf\{\lambda > 0 : \lambda^{-1} x \in K\}\,, \qquad \|x\|_* = \sup\{x \cdot y : y \in K\}\,.$$

In this way, $K = \{x \in \mathbb{R}^n : \|x\| < 1\}$, and the Cauchy-Schwartz type-inequality

$$x \cdot y \le \|x\|\,\|y\|_*\,, \qquad \forall\, x, y \in \mathbb{R}^n\,,$$

holds true. Moreover, if $K = B$, the Euclidean unit ball, then $\|x\| = \|x\|_* = |x|$ for every $x \in \mathbb{R}^n$, where, here and in the following, $|\cdot|$ denotes the Euclidean norm. With this notation at hand, the anisotropic total variation of a $\mathbb{R}^n$-valued Radon measure μ defined on $\mathbb{R}^n$ is defined by the formula

$$\|\mu\|_*(E) = \sup\left\{ \sum_{h\in\mathbb{N}} \|\mu(E_h)\|_* \,:\, \{E_h\}_{h\in\mathbb{N}} \text{ is a Borel partition of } E \right\}.$$

Correspondingly, the *anisotropic total variation* $TV_K(f)$ of $f \in BV(\mathbb{R}^n)$ is given by

$$TV_K(f) = \| - Df\|_*(\mathbb{R}^n)\,,$$

where Df denotes the distributional gradient of f. Since K is a bounded open set containing the origin, there exist constants $a_K, b_K > 0$ such that

$$a_K|\nu| \le \|\nu\|_* \le b_K|\nu| \qquad \forall\, \nu \in \mathbb{R}^n\,. \tag{2.3}$$

In particular $f \in BV(\mathbb{R}^n)$ if and only if $TV_K(f) < \infty$, as

$$a_K|Df|(E) \le \| - Df\|_*(E) \le b_K|Df|(E)\,, \qquad \forall\, E \subset \mathbb{R}^n\,.$$

By standard density arguments we see that

$$TV_K(f) = \sup\left\{ \int_{\mathbb{R}^n} f(x)\,\mathrm{div}\, T(x)dx : T \in C^1_c(\mathbb{R}^n; K) \right\}\,.$$

Moreover,

$$TV_K(f) = \int_{\mathbb{R}^n} \| - \nabla f(x)\|_* dx\,, \qquad \forall\, f \in C^1(\mathbb{R}^n)\,, \tag{2.4}$$

so that $TV_B(f) = \|Df\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$. Similarly, if E is a set of finite perimeter with reduced boundary $\partial^* E$ and measure theoretic outer unit normal ν_E, then we have

$$TV_K(1_E) = \int_{\partial^* E} \|\nu_E(x)\|_*\, d\mathcal{H}^{n-1}(x)\,,$$

so that $TV_B(1_E) = P(E)$, the distributional perimeter of E. The anisotropic total variation of 1_E is sometimes called the anisotropic perimeter of E with respect to K, see for instance [FiMP, Section 1.2]. Recalling the definition of deficit introduced in (1.8), given a set of finite perimeter E with $|E| < \infty$ we shall write for simplicity

$$\delta(E) = \delta(1_E)\,.$$

Note that $\delta(1_E) = \delta(a1_E)$ for every $a \neq 0$, since the notion of deficit is scale invariant.

2.2. **Some properties of the "distance" d.** In this short section, we list some simple but important properties of the function d. In the following lemma we start investigating the behavior of d with respect to the axioms of a distance.

Lemma 2.2. *For any $n \ge 2$, one has*

$$d(f,g) \ge 0, \text{ and } d(f,g) = 0 \text{ if and only if } f = g\,,$$

$$\frac{a_K}{b_K}\, d(g,f) \le d(f,g) \le \frac{b_K}{a_K}\, d(g,f)\,,$$

$$d(f,h) \le 4\Big(d(f,g) + d(g,h)\Big)\,,$$

for every $f, g, h \in BV(\mathbb{R}^n)$.

Proof. We only have to check the validity of the "extended" triangle inequality, the first two properties being easily verified. Let $f, g, h \in BV(\mathbb{R}^n)$, and notice that

$$\int_{\mathbb{R}^n} |f-h|^{n'} \le 2^{n'-1}\left(\int_{\mathbb{R}^n} |f-g|^{n'} + \int_{\mathbb{R}^n} |g-h|^{n'} \right)\,. \tag{2.5}$$

Next, consider two Borel sets E_1 and E_2 in $\mathbb{R}^n$. We have

$$\begin{aligned}\|-D(f-h)\|_*(\mathbb{R}^n\setminus(E_1\cup E_2))\leq \\ \|-D(f-g)\|_*(\mathbb{R}^n\setminus E_1)+\|-D(g-h)\|_*(\mathbb{R}^n\setminus E_2)\,,\end{aligned}\tag{2.6}$$

and moreover

$$\begin{aligned}\int_{E_1\cup E_2}|f|^{n'}&=\int_{E_1}|f|^{n'}+\int_{E_2\setminus E_1}|f|^{n'}\\ &\leq\int_{E_1}|f|^{n'}+2^{n'-1}\left(\int_{\mathbb{R}^n}|f-g|^{n'}+\int_{E_2}|g|^{n'}\right).\end{aligned}\tag{2.7}$$

Similarly,

$$\int_{E_1\cup E_2}|h|^{n'}\leq\int_{E_2}|h|^{n'}+2^{n'-1}\left(\int_{\mathbb{R}^n}|g-h|^{n'}+\int_{E_1}|g|^{n'}\right).\tag{2.8}$$

By adding up (2.5), (2.6), (2.7) and (2.8), and by taking into account that $2^{n'-1}\leq 2$, if we use $E_1\cup E_2$ as a test set in the definition of $d_0(f,h)$, then we find

$$\begin{aligned}d(f,h)\leq\ &\frac{\|-D(f-g)\|_*(\mathbb{R}^n\setminus E_1)}{n|K|^{1/n}}+\int_{E_1}|f|^{n'}+2\int_{\mathbb{R}^n}|f-g|^{n'}+2\int_{E_2}|g|^{n'}\\ &+\frac{\|-D(g-h)\|_*(\mathbb{R}^n\setminus E_2)}{n|K|^{1/n}}+\int_{E_2}|h|^{n'}+2\int_{\mathbb{R}^n}|h-g|^{n'}+2\int_{E_1}|g|^{n'}\\ &+2\int_{\mathbb{R}^n}|f-g|^{n'}+2\int_{\mathbb{R}^n}|g-h|^{n'}\,.\end{aligned}$$

Minimizing with respect to E_1 and E_2 separately, we find

$$d(f,h)\leq 4\Big(d(f,g)+d(g,h)\Big)\,,$$

as desired. □

The following two lemmas are essential in reducing the proof of Theorem 1.1 to the case when f is smooth and compactly supported.

Lemma 2.3. *Let $f,g\in BV(\mathbb{R}^n)$, and set $f_k=f*\rho_k$, where $\{\rho_k\}_{k\in\mathbb{N}}$ is a sequence of smooth compactly supported convolution kernels. Then $d(f_k,g)\to d(f,g)$ for $k\to\infty$.*

Proof. Since $f_k\to f$ in $L^{n'}(\mathbb{R}^n)$, we only have to prove that $d_0(f_k,g)\to d_0(f,g)$ when $k\to\infty$. Let us consider the Radon-Nykodim decompositions $Df=\nabla f\,dx+D^sf$ and $Dg=\nabla g\,dx+D^sg$, and let F be a Borel set on which both $|D^sf|$ and $|D^sg|$ are concentrated, with $|F|=0$. Then, given $\varepsilon>0$, we can consider an open set $A_\varepsilon\subset\mathbb{R}^n$ such that $F\subset A_\varepsilon$ and

$$\int_{A_\varepsilon}|f|^{n'}+|g|^{n'}\,dx\leq\varepsilon\,.\tag{2.9}$$

Since $|D^sf|$ and $|D^sg|$ are both concentrated on A_ε, for every Borel set $E\subset\mathbb{R}^n$, we have

$$\|-D(f-g)\|_*(\mathbb{R}^n\setminus(E\cup A_\varepsilon))=\int_{\mathbb{R}^n\setminus(E\cup A_\varepsilon)}\|-(\nabla f-\nabla g)\|_*\,dx\,.$$

Thus, if we restrict the competition class in the definition of $d_0(f,g)$ to the Borel sets of the form $E\cup A_\varepsilon$, taking also (2.9) into account we find that

$$d_0(f,g)\leq\varepsilon+\inf_{E\subset\mathbb{R}^n}\left\{\frac{1}{n|K|^{1/n}}\int_{\mathbb{R}^n\setminus(E\cup A_\varepsilon)}\|-(\nabla f-\nabla g)\|_*\,dx+\int_E|f|^{n'}+|g|^{n'}\right\}.$$

We now remark that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus(E\cup A_\varepsilon)} \|-(\nabla f-\nabla g)\|_*\,dx \\
&\qquad\le \int_{\mathbb{R}^n\setminus(E\cup A_\varepsilon)} \|-(\nabla f_k-\nabla g)\|_*\,dx + \int_{\mathbb{R}^n\setminus(E\cup A_\varepsilon)} \|-(\nabla f_k-\nabla f)\|_*\,dx \\
&\qquad\le \|-D(f_k-g)\|_*(\mathbb{R}^n\setminus E) + \int_{\mathbb{R}^n\setminus A_\varepsilon} \|-(\nabla f_k-\nabla f)\|_*\,dx\,,
\end{aligned}$$

and

$$\int_E |f|^{n'} \le \Big(\|f_k\|_{L^{n'}(E)} + \|f-f_k\|_{L^{n'}(\mathbb{R}^n)}\Big)^{n'}\,.$$

Hence, setting $\alpha_k = \|f-f_k\|_{L^{n'}(\mathbb{R}^n)}$, we conclude that

$$\begin{aligned}
d_0(f,g) \le \varepsilon + \frac{1}{n|K|^{1/n}} \int_{\mathbb{R}^n\setminus A_\varepsilon} \|-(\nabla f_k-\nabla f)\|_*\,dx \\
+ \inf_{E\subset\mathbb{R}^n}\left\{\frac{\|-D(f_k-g)\|_*(\mathbb{R}^n\setminus E)}{n|K|^{1/n}} + \Big(\|f_k\|_{L^{n'}(E)}+\alpha_k\Big)^{n'} + \int_E |g|^{n'}\right\}.
\end{aligned}\tag{2.10}$$

Since $\alpha_k\to 0$ thanks to Lemma 2.1, and since $\|\cdot\|_*$ is comparable to the Euclidean norm by (2.3), letting first $k\to\infty$, and then $\varepsilon\to 0^+$ we obtain

$$d_0(f,g) \le \liminf_{k\to\infty} d_0(f_k,g)\,.$$

If we repeat the above argument exchanging the roles of f and f_k, in place of (2.10) we get

$$\begin{aligned}
d_0(f_k,g) \le \varepsilon + \frac{1}{n|K|^{1/n}} \int_{\mathbb{R}^n\setminus A_\varepsilon} \|-(\nabla f-\nabla f_k)\|_*\,dx \\
+ \inf_{E\subset\mathbb{R}^n}\left\{\frac{\|-D(f-g)\|_*(\mathbb{R}^n\setminus E)}{n|K|^{1/n}} + \Big(\|f\|_{L^{n'}(E)}+\alpha_k\Big)^{n'} + \int_E |g|^{n'}\right\},
\end{aligned}$$

which yields

$$d_0(f,g) \ge \limsup_{k\to\infty} d_0(f_k,g)\,.$$

This concludes the proof. □

Lemma 2.4. *If $f\in BV(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$, then*

$$\lim_{R\to\infty} d(1_{B_R}\,f, g_{a,x_0,r}) = d(f, g_{a,x_0,r})\,,\tag{2.11}$$

for every $a\neq 0$, $x_0\in\mathbb{R}^n$ and $r>0$. Moreover,

$$\delta(f) = \lim_{R\to\infty}\delta(1_{B_R}\,f)\,.\tag{2.12}$$

Proof. First of all, we claim that

$$\lim_{R\to\infty}\int_{\partial B_R} |f|\,d\mathcal{H}^{n-1} = 0\,.\tag{2.13}$$

Indeed, by a simple computation in polar coordinates using the Fundamental Theorem of Calculus, one can easily check that, for any $0<R_1<R_2<\infty$,

$$\left|\int_{\partial B_{R_1}} |f|\,d\mathcal{H}^{n-1} - \int_{\partial B_{R_2}} |f|\,d\mathcal{H}^{n-1}\right| \le \int_{B_{R_2}\setminus B_{R_1}} \left(|\nabla f| + (n-1)\frac{|f|}{R_1}\right)\,.$$

Since $|f|$ and $|\nabla f|$ are both integrable, this implies that the function

$$R\mapsto \int_{\partial B_R} |f|\,d\mathcal{H}^{n-1}$$

is uniformly continuous on $[1,\infty)$. Observing that

$$\int_0^\infty \int_{\partial B_R} |f|\, d\mathcal{H}^{n-1}\, dR = \int_{\mathbb{R}^n} |f| < \infty\,,$$

(2.13) follows easily.

Using (2.13), and taking also into account that $|f|^{n'} \in L^1(\mathbb{R}^n)$ and that $\|-\nabla f\|_* \in L^1(\mathbb{R}^n)$, we conclude that

$$\int_E \|-\nabla f\|_* = \lim_{R\to\infty} \int_{E\cap B_R} \|-\nabla f\|_*\,, \tag{2.14}$$

$$\int_E |f|^{n'} = \lim_{R\to\infty} \int_{E\cap B_R} |f|^{n'}\,, \tag{2.15}$$

$$0 = \lim_{R\to\infty} \int_{E\cap \partial B_R} |f|\, d\mathcal{H}^{n-1}\,, \tag{2.16}$$

uniformly with respect to $E \subset \mathbb{R}^n$. Let us now set for the sake of brevity $K_0 = x_0 + ar\,K$. Since

$$\begin{aligned} D(1_{B_R} f - g_{a,x_0,r}) &= 1_{B_R} \nabla f\, dx - f\,\nu_{B_R}\, \mathcal{H}^{n-1} \llcorner \partial B_R + a\,\nu_{K_0}\, \mathcal{H}^{n-1} \llcorner \partial K_0\,, \\ D(f - g_{a,x_0,r}) &= \nabla f\, dx + a\,\nu_{K_0}\, \mathcal{H}^{n-1} \llcorner \partial K_0\,, \end{aligned}$$

by (2.14), (2.15) and (2.16) we find that

$$\begin{aligned} &\|-D(1_{B_R} f - g_{a,x_0,r})\|_*(E) + n|K|^{1/n'} \int_{\mathbb{R}^n\setminus E} |1_{B_R} f|^{n'} + |g_{a,x_0,r}|^{n'} \\ &\quad= \int_{E\cap B_R} \|-\nabla f\|_* + \int_{E\cap\partial B_R} |f|\|-\nu_{B_R}\|_*\, d\mathcal{H}^{n-1} + |a| \int_{E\cap\partial K_0} \|\nu_{K_0}\|_* d\mathcal{H}^{n-1} \\ &\qquad + n|K|^{1/n'} \int_{\mathbb{R}^n\setminus E} |1_{B_R} f|^{n'} + |g_{a,x_0,r}|^{n'}\,, \end{aligned}$$

as $R \to \infty$ converges, uniformly with respect to $E \subset \mathbb{R}^n$, to

$$\begin{aligned} &\int_E \|-\nabla f\|_* + |a| \int_{E\cap\partial K_0} \|\nu_{K_0}\|_* d\mathcal{H}^{n-1} + n|K|^{1/n'} \int_{\mathbb{R}^n\setminus E} |f|^{n'} + |g_{a,x_0,r}|^{n'} \\ &\qquad = \|-D(f - g_{a,x_0,r})\|_*(E) + n|K|^{1/n'} \int_{\mathbb{R}^n\setminus E} |f|^{n'} + |g_{a,x_0,r}|^{n'}\,. \end{aligned}$$

By the arbitrariness of E we immediately deduce the validity of (2.11). Finally, (2.12) follows by

$$D(1_{B_R} f) = 1_{B_R} \nabla f\, dx - f\,\nu_{B_R}\, d\mathcal{H}^{n-1} \llcorner \partial B_R\,,$$

and by (2.14), (2.15) and (2.16). □

We now prove that, on pairs of characteristic functions, d agrees with the L^1-distance between the corresponding sets.

Lemma 2.5. *If E and F are sets of locally finite perimeter in $\mathbb{R}^n$, then*

$$d(a1_E, b1_F) = \int_{\mathbb{R}^n} |a\,1_E - b\,1_F|^{n'}\,,$$

for every $a, b \in \mathbb{R}$.

Proof. We just have to prove that $d_0(a1_E, b1_F) = 0$. To do this, we use as a test set $G = \mathbb{R}^n \setminus (\partial^* E \cup \partial^* F)$. In this way we find

$$\|D(b1_F - a1_E)\|_*(G) \le |a| \int_{G\cap\partial^* E} \|\nu_E\|_*\, d\mathcal{H}^{n-1} + |b| \int_{G\cap\partial^* F} \|-\nu_F\|_*\, d\mathcal{H}^{n-1} = 0\,,$$

while at the same time, since $|\mathbb{R}^n \setminus G| = 0$,

$$\int_{\mathbb{R}^n \setminus G} |a\, 1_E|^{n'} + |b\, 1_F|^{n'} = 0\,.$$

$\square$

We conclude this section showing the following simple lemma.

Lemma 2.6. *If $f \in BV(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |f|^{n'} = 1$, then*

$$\inf_{a,x_0,r} d(f, g_{a,x_0,r}) \le \inf_{a,x_0} d(f, \widehat{g}_{a,x_0}) \le 8 \inf_{a,x_0,r} d(f, g_{a,x_0,r})\,. \tag{2.17}$$

Proof. The first inequality in (2.17) being trivial, we focus on the second one. Pick any $a \ne 0$, $x_0 \in \mathbb{R}^n$ and $r > 0$, and correspondingly let $b \ne 0$ be such that $|b|^{n'}|x_0 + ar\, K| = 1$, choosing $b > 0$ (resp. $b < 0$) if $a > 0$ (resp. $a < 0$). Then we have

$$\begin{aligned}
\|\widehat{g}_{b,x_0} - g_{a,x_0,r}\|_{L^{n'}(\mathbb{R}^n)} &= |a-b||x_0 + ar\, K|^{1/n'} \\
&= \Big| \|a\, 1_{x_0+ar\, K}\|_{L^{n'}(\mathbb{R}^n)} - \|b\, 1_{x_0+ar\, K}\|_{L^{n'}(\mathbb{R}^n)} \Big| \\
&= \Big| \|a\, 1_{x_0+ar\, K}\|_{L^{n'}(\mathbb{R}^n)} - 1 \Big| = \Big| \|a\, 1_{x_0+ar\, K}\|_{L^{n'}(\mathbb{R}^n)} - \|f\|_{L^{n'}(\mathbb{R}^n)} \Big| \\
&\le \|f - g_{a,x_0,r}\|_{L^{n'}(\mathbb{R}^n)} \le d(f, g_{a,x_0,r})^{1/n'}\,.
\end{aligned}$$

Thus by Lemma 2.2 and by Lemma 2.5 we find that

$$\begin{aligned}
d(f, \widehat{g}_{b,x_0}) &\le 4\Big(d(f, g_{a,x_0,r}) + d(g_{a,x_0,r}, \widehat{g}_{b,x_0}) \Big) \\
&= 4\Big(d(f, g_{a,x_0,r}) + \int_{\mathbb{R}^n} |\widehat{g}_{b,x_0} - g_{a,x_0,r}|^{n'} \Big) \le 8\, d(f, g_{a,x_0,r})\,.
\end{aligned}$$

The conclusion follows by the arbitrariness of a, x_0 and r. $\square$

2.3. **Proof of Theorem 1.1 for smooth nonnegative functions.** We can now enter in the proof of Theorem 1.1. We start dealing with the case of compactly supported, smooth, positive functions. The general case shall then follow by an approximation argument based on Lemma 2.3 and Lemma 2.4.

Theorem 2.7. *If $f \in C^1_c(\mathbb{R}^n)$, $f \ge 0$, $\int_{\mathbb{R}^n} f^{n'} = 1$ and $\delta(f) \le (8\, n)^{-2}$, then*

$$\inf_{a,x_0} d(f, \widehat{g}_{a,x_0}) \le 256\, (n + C_0(n))\, \sqrt{\delta(f)}\,. \tag{2.18}$$

Proof. The proof of this theorem is divided into several steps. The main one is to show that $\delta(f)$ controls the total variation of f on a suitable set $\{f > t_1\}$, and that f has small $L^{n'}$-norm in its complement $\{f \le t_1\}$, see Figure 2.1. Then, we will do a "reduction to sets" argument: we will find a new level set $t_0 \in (t_1/2, t_1)$ such that $t_1 1_{\{f > t_1\}}$ and $t_0 1_{\{f > t_0\}}$ are d-close and, moreover, the Sobolev deficit of f controls the deficit of $\{f > t_0\}$. Finally, we will use the main result of [FiMP] to show that $\{f > t_0\}$ is close to a suitable translated and scaled copy of K. A simple application of Lemma 2.2 will then show that $d(f, \widehat{g}_{a,x_0})$ is controlled by $\sqrt{\delta(f)}$ for suitable values of x_0 and a.

Notice that, by a simple approximation argument, without loss of generality we can assume that

$$\Big| \Big\{ x : f(x) > 0,\, \nabla f(x) = 0 \Big\} \Big| = 0\,. \tag{2.19}$$

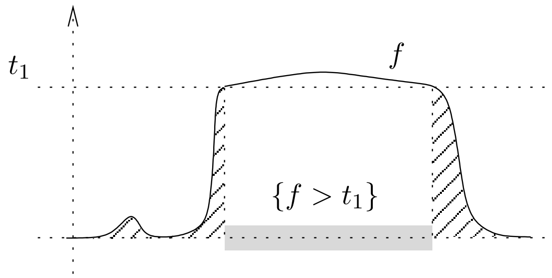

FIGURE 2.1. The key step in the proof of Theorem 2.7. The smooth nonnegative function f is close to a characteristic function, in the sense that there exists a heigth t_1 such that the total variation of f on $\{f > t_1\}$ is small, as well as its $L^{n'}$-norm on $\{f \le t_1\}$. After this step, it remains to prove that $\{f > t_1\}$ is close to $x_0 + r\,K$ for some values of $x_0 \in \mathbb{R}^n$ and $r > 0$.

Step I: There exists $t_1 > 0$ such that

$$\int_{\{f>t_1\}} \| -\nabla f(x)\|_* dx \le 2n|K|^{1/n}\sqrt{\delta(f)}\,, \tag{2.20}$$

$$\int_{\{f\le t_1\}} f^{n'} dx \le n\sqrt{\delta(f)}\,, \tag{2.21}$$

$$d(f, t_1 1_{\{f>t_1\}}) \le 4n\sqrt{\delta(f)}\,. \tag{2.22}$$

Let us first give a brief description of the argument. The starting point consists in applying a "Gromov-type argument" to the Brenier map T between the probability densities $f(x)^{n'}dx$ and $|K|^{-1}1_K dy$. More precisely, $T \in BV(\mathbb{R}^n; K)$ is the gradient of a convex function and satisfies the push-forward condition

$$\frac{1}{|K|}\int_K h(y)dy = \int_{\mathbb{R}^n} h(T(x))f(x)^{n'}dx\,,$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty]$. By the change of variables $y = T(x)$ and through a localization argument, we deduce that

$$\det \nabla T(x) = |K|\, f(x)^{n'} \tag{2.23}$$

(note that $|\det \nabla T| = \det \nabla T$ as ∇T is the Hessian of a convex function). In particular, (2.23) can be rewritten as

$$n\big(\det \nabla T(x)\big)^{1/n} f(x) = n|K|^{1/n} f(x)^{n'}\,.$$

Integrating over $\mathbb{R}^n$ and applying the arithmetic-geometric mean inequality, one finds that

$$\begin{aligned} n|K|^{1/n} &= \int_{\mathbb{R}^n} n|K|^{1/n} f(x)^{n'}\,dx = \int_{\mathbb{R}^n} n\big(\det \nabla T(x)\big)^{1/n} f(x)\,dx \le \int_{\mathbb{R}^n} \operatorname{div} T(x) f(x)\,dx \\ &= \int_{\mathbb{R}^n} T(x)\cdot(-\nabla f(x))dx \le \int_{\mathbb{R}^n} \|T(x)\|\,\|-\nabla f(x)\|_*\,dx \le TV_K(f)\,, \end{aligned}$$

where we used (2.4) and the fact that $\|T(x)\| \le 1$ a.e. in $\mathbb{R}^n$. This argument proves (1.1), and provides a bound on the isoperimetric deficit in terms of T, namely

$$n|K|^{1/n}\delta(f) \ge \int_{\mathbb{R}^n} \big(1 - \|T(x)\|\big)\|-\nabla f(x)\|_*\,dx\,. \tag{2.24}$$

We are going to prove (2.20)–(2.21) starting from this bound, while (2.22) will eventually follow from (2.20) and (2.21).

Indeed, what (2.24) suggests is that the total variation of f is controlled by $\sqrt{\delta(f)}$ on the region $\{1 - \|T\| \geq \sqrt{\delta(f)}\}$, while, at the same time, the mass charged by $f^{n'}dx$ on the the complementary region $\{\|T\| \geq 1 - \sqrt{\delta(f)}\}$ should be controlled by $\sqrt{\delta(f)}$, being this region mapped by T into a $\sqrt{\delta(f)}$-layer of ∂K. Of course, one should expect here some difficulties regarding the regularity of these sets. A key idea is that it does not matter to apply the above remarks directly to f, but rather it suffices to work with its anisotropic radially symmetric decreasing rearrangement $f^\star$. We shall later recover the information on f via the Coarea Formula.

This said, let us go into the details of the proof of Step I. Let us define $f^\star : \mathbb{R}^n \to [0, \infty)$ by

$$f^\star(x) = \sup\Big\{t \geq 0 : |\{f > t\}| \geq |K|\|x\|^n\Big\}, \quad x \in \mathbb{R}^n .$$

Then $\{f^\star > t\} = r(t)K$ for every $t > 0$, where $r(t) > 0$ is so that

$$|\{f^\star > t\}| = |\{f > t\}|, \qquad \forall t > 0. \tag{2.25}$$

It is well known that $f^\star \in W^{1,1}(\mathbb{R}^n)$, and that there exists $u \in AC_{loc}([0, R])$ such that $u' \leq 0$ and $f^\star(x) = u(\|x\|)$ (here $R > 0$ is determined by the relation $|\{f > 0\}| = R^n|K|$). By (2.25), we have $\int_{\mathbb{R}^n} f^{n'} = \int_{\mathbb{R}^n}(f^\star)^{n'} = 1$. Furthermore,

$$\begin{aligned} TV_K(f) &= \int_{\mathbb{R}^n} \| - \nabla f(x)\|_* dx = \int_0^\infty P_K(\{f > t\})dt \geq \int_0^\infty n|K|^{1/n}|\{f > t\}|^{1/n'} dt \\ &\geq n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)} = n|K|^{1/n} , \end{aligned}$$

where in the last inequality we have also applied an elementary inequality on decreasing functions (see, e.g. [LY, Proof of (5.3.3)]). Since the first inequality is in fact an equality if we replace f by $f^\star$, it follows that $\delta(f^\star) \leq \delta(f)$, and in particular

$$n|K|^{1/n}\delta(f) \geq TV_K(f) - TV_K(f^\star) = n|K|^{1/n} \int_0^\infty \delta(\{f > t\})\mu(t)^{1/n'} dt , \tag{2.26}$$

where we have used that

$$\begin{aligned} P_K(\{f > t\}) - n|K|^{1/n}|\{f > t\}|^{1/n'} &= P_K(\{f > t\}) - P_K(\{f^* > t\}) \\ &= n|K|^{1/n}\delta(\{f > t\})\mu(t)^{1/n'} , \end{aligned}$$

and we have set $\mu(t) = |\{f > t\}|$ for the sake of brevity.

We now perform Gromov's argument to derive the inequalities (2.20)–(2.21). More precisely, let $g = |K|^{-1/n'}1_K$. When $\delta(f^\star)$ is small we expect $f^\star$ to be close to g (up to a homothety). For this reason we parameterize $g^{n'}$ with respect to $(f^\star)^{n'}$ by the function $\tau : [0, R] \to [0, 1]$ defined as

$$\int_{rK} (f^\star)^{n'} = \int_{\tau(r)K} g^{n'} .$$

or, equivalently,

$$\tau(r)^n = n|K| \int_0^r u(s)^{n'} s^{n-1} ds , \tag{2.27}$$

(we remark that the Brenier map between $f^\star(x)^{n'}dx$ and $|K|^{-1}1_K(y)dy$ is given by $T^*(x) = \tau(\|x\|)x/\|x\|$). Clearly $\tau \in C^1((0, R); (0, 1))$, with $\tau(R) = 1$, $\tau(0) = 0$, $\tau > 0$ on $(0, R)$, and

$$\tau'(r)\tau(r)^{n-1} = |K|u(r)^{n'} r^{n-1} .$$

Hence, by Young inequality,

$$
\begin{aligned}
u(r)^{n'} &= u(r)^{n'/n}u(r) = \tau'(r)^{1/n}\left(\frac{\tau(r)}{r}\right)^{1/n'}\frac{u(r)}{|K|^{1/n}} \\
&\le \left\{\frac{\tau'(r)}{n} + \frac{(\tau(r)/r)}{n'}\right\}\frac{u(r)}{|K|^{1/n}} = \frac{1}{nr^{n-1}}\left(\tau(r)r^{n-1}\right)'\frac{u(r)}{|K|^{1/n}},
\end{aligned}
$$

which combined with (2.27) gives

$$
1 = \tau(R)^n = \int_0^R n|K|u(r)^{n'}r^{n-1}dr \le |K|^{1/n'}\int_0^R (\tau(r)r^{n-1})'u(r)dr .
$$

Integrating by parts, and recalling that $\tau(0) = u(R) = 0$ and that $0 \le \tau \le 1$, we get

$$
\begin{aligned}
n|K|^{1/n} &\le n|K|\int_0^R |u'(r)|r^{n-1}\tau(r)dr \le n|K|\int_0^R |u'(r)|r^{n-1}dr = \int_{\mathbb{R}^n}\|-\nabla f^\star(x)\|_* dx \\
&= TV_K(f^\star),
\end{aligned}
$$

and so,

$$
n|K|\int_0^R (1-\tau(r))|u'(r)|r^{n-1}dr \le TV_K(f^\star) - n|K|^{1/n} = n|K|^{1/n}\delta(f^\star) \tag{2.28}
$$

(observe that this is just (2.24) for the function $f^\star$ in place of f). We now show how to combine (2.26) and (2.28) to prove the theorem. Let us consider the set

$$
J = \{r \in [0,R] : 1-\tau(r) \ge \sqrt{\delta(f)}\}.
$$

As $\delta(f) < 1$ and τ is increasing, we have that $J = [0, r_1]$, where $r_1 \in (0,R)$ is such that $\tau(r_1) = 1-\sqrt{\delta(f)}$. By (2.28) and the definition of J we easily infer that

$$
\int_{r_1 K}\|-\nabla f^\star\|_* \le n|K|^{1/n}\sqrt{\delta(f)}. \tag{2.29}
$$

Moreover, as $1-(1-\varepsilon)^n \le n\varepsilon$ for every $\varepsilon \in [0,1]$ and minding (2.27), we have

$$
\int_{\mathbb{R}^n\setminus r_1K}(f^\star)^{n'} = 1 - n|K|\int_0^{r_1} u(s)^{n'}s^{n-1}ds = 1-\tau(r_1)^n \le n\sqrt{\delta(f)}.
$$

Set now $t_1 = u(r_1)$, so that $\{f^\star > t_1\} = r_1K$ thanks to (2.19). Thus, (2.21) follows immediately by Fubini Theorem since $|\{f>t\}| = |\{f^\star > t\}|$.

Let us now consider (2.20). We start by noticing that, by the Coarea Formula and keeping in mind (2.19) and (2.29),

$$
\begin{aligned}
\int_{\{f>t_1\}}\|-\nabla f\|_* &= \int_{t_1}^\infty P_K(\{f>t\})\,dt \\
&= \int_{t_1}^\infty \Big(P_K(\{f>t\}) - P_K(\{f^\star>t\})\Big)dt + \int_{t_1}^\infty P_K(\{f^\star>t\})dt \\
&= \int_{t_1}^\infty \Big(P_K(\{f>t\}) - P_K(\{f^\star>t\})\Big)dt + \int_{\{f^*>t_1\}}\|-\nabla f^*\|_* \\
&\le \int_{t_1}^\infty \Big(P_K(\{f>t\}) - P_K(\{f^\star>t\})\Big)dt + n|K|^{1/n}\sqrt{\delta(f)}.
\end{aligned} \tag{2.30}
$$

By the isoperimetric inequality $P_K(\{f>t\}) - P_K(\{f^\star>t\}) \ge 0$, thus by (2.26) we have

$$
\begin{aligned}
\int_{t_1}^\infty P_K(\{f>t\}) - P_K(\{f^\star>t\})dt &\le \int_0^\infty P_K(\{f>t\}) - P_K(\{f^\star>t\})dt \\
&= TV_K(f) - TV_K(f^\star) \le n|K|^{1/n}\delta(f).
\end{aligned}
$$

Inserting this last inequality into (2.30), we conclude the validity of (2.20).

Let us finally prove (2.22). We first claim that

$$\int_{\mathbb{R}^n} |f - t_1 1_{\{f>t_1\}}|^{n'} \le 2n\sqrt{\delta(f)}\,. \tag{2.31}$$

Indeed, by the anisotropic Sobolev inequality (1.1) applied to $\max\{f - t_1, 0\}$, and thanks to (2.20)–(2.21), we have that

$$\begin{aligned}\int_{\mathbb{R}^n} |f - t_1 1_{\{f>t_1\}}|^{n'} &= \int_{\mathbb{R}^n\setminus\{f>t_1\}} f^{n'} + \int_{\{f>t_1\}} |f - t_1|^{n'} \\ &\le n\sqrt{\delta(f)} + \left(\frac{\int_{\{f>t_1\}} \|-\nabla f\|_*}{n|K|^{1/n}}\right)^{n'} \le n\sqrt{\delta(f)} + 2^{n'}\delta(f)^{n'/2} \\ &\le 2n\sqrt{\delta(f)}\,.\end{aligned}$$

Notice that in the last inequality we have used that for every $n \ge 3$ one has $2^{n'} \le n$, while for $n = 2$ one has $4\delta(f) \le 2\sqrt{\delta(f)}$ since by assumption $\delta(f) \le 1/4$. At the same time, if we plug the choice $E = \{f \le t_1\}$ in the definition of $d_0(f, t_1\, 1_{\{f>t_1\}})$, and notice that

$$\|-D(f - t_1\, 1_{\{f>t_1\}})\|_*(\{f > t_1\}) = \int_{\{f>t_1\}} \|-\nabla f\|_* dx\,,$$

then by (2.20)–(2.21) we immediately find

$$d_0(f, t_1\, 1_{\{f>t_1\}}) \le 2n\sqrt{\delta(f)}\,.$$

Combining this estimate with (2.31), we conclude the proof of (2.22), and thus of Step I.

Step II: There exists $t_0 \in (t_1/2, t_1)$ such that

$$t_1^{n'}\big|\{t_0 < f \le t_1\}\big| \le 4n\sqrt{\delta(f)}\,, \tag{2.32}$$

$$\delta\big(\{f > t_0\}\big) < 4\,\delta(f)\,. \tag{2.33}$$

First of all, using the triangle inequality, recalling (2.31) and that $\int_{\mathbb{R}^n} f^{n'} = 1$, and thanks to the assumption $\sqrt{\delta(f)} \le (8n)^{-1}$, we obtain

$$\begin{aligned} t_1|\{f > t_1\}|^{1/n'} &= \|t_1 1_{\{f>t_1\}}\|_{L^{n'}} \ge \|f\|_{L^{n'}} - \|f - t_1 1_{\{f>t_1\}}\|_{L^{n'}} \\ &\ge 1 - \big(2n\sqrt{\delta(f)}\big)^{1/n'} \ge \frac{1}{2}\,.\end{aligned} \tag{2.34}$$

Let us now consider the set

$$I = \Big\{t \in (t_1/2, t_1) : \delta\big(\{f > t\}\big)\big|\{f > t\}\big|^{1/n'} > 2\delta(f)/t_1\Big\}\,.$$

By (2.26) we have $\mathcal{H}^1(I) < t_1/2$, so that there exists $t_0 \in (t_1/2, t_1) \setminus I$. Consequently, by (2.34) we find that

$$2\delta(f) > t_1|\{f > t_0\}|^{1/n'}\delta(\{f > t_0\}) \ge t_1|\{f > t_1\}|^{1/n'}\delta(\{f > t_0\}) \ge \frac{\delta(\{f > t_0\})}{2}\,,$$

hence (2.33) is established. To prove (2.32) it is enough to estimate, also thanks to (2.21),

$$t_1^{n'}\big|\{t_0 < f \le t_1\}\big| \le t_1^{n'}\Big|\Big\{\frac{t_1}{2} < f \le t_1\Big\}\Big| \le 2^{n'}\int_{\{t_1/2<f\le t_1\}} f^{n'} \le 2^{n'}n\sqrt{\delta(f)} \le 4n\sqrt{\delta(f)}\,.$$

Step III: Conclusion.

We are now ready to conclude the proof of the Theorem. First, we claim that there exist $x_0 \in \mathbb{R}^n$ and $r > 0$ such that

$$d(f, g_{t_1,x_0,r}) \le 32\,\big(n + C_0(n)\big)\,\sqrt{\delta(f)}\,, \tag{2.35}$$

where $C_0(n)$ is defined as in (1.4). To show this, observe that thanks to [FiMP, Theorem 1.1], there exist $x_0 \in \mathbb{R}^n$ and $r > 0$ such that

$$\big|\{f > t_0\}\Delta(x_0 + r\,K)\big| \le C_0(n)|\{f > t_0\}|\sqrt{\delta(\{f > t_0\})}\,. \tag{2.36}$$

Let us notice that

$$t_1|\{f > t_0\}|^{1/n'} \le 2\,\frac{t_1}{2}\,\left|\left\{f > \frac{t_1}{2}\right\}\right|^{1/n'} \le 2\Big(\int_{\{f>t_1/2\}} f^{n'}\Big)^{1/n'} \le 2\,,$$

so that $t_1^{n'}|\{f > t_0\}| \le 2^{n'} \le 4$. Hence, by (2.36) and (2.33) we find that

$$t_1^{n'}|\{f > t_0\}\Delta(x_0 + r\,K)| \le 4\,C_0(n)\sqrt{\delta(\{f > t_0\})} \le 8\,C_0(n)\sqrt{\delta(f)}\,.$$

Thus, by applying Lemma 2.2 and Lemma 2.5, and by (2.22) and (2.32), we get

$$\begin{aligned} d(f, g_{t_1,x_0,r}) &\le 4\Big(d(f, t_1\,1_{\{f>t_1\}}) + d(t_1\,1_{\{f>t_1\}}, g_{t_1,x_0,r})\Big) \\ &\le 16n\sqrt{\delta(f)} + 4t_1^{n'}\Big|\{f > t_1\}\Delta(x_0 + rK)\Big| \\ &\le 16n\sqrt{\delta(f)} + 4t_1^{n'}\Big(\big|\{f > t_0\}\Delta(x_0 + rK)\big| + \big|\{t_0 < f \le t_1\}\big|\Big) \\ &\le 32\big(n + C_0(n)\big)\,\sqrt{\delta(f)}\,, \end{aligned}$$

thus (2.35) follows. It is now sufficient to apply Lemma 2.6 to get

$$\inf_{a,x} d(f, \widehat{g}_{a,x}) \le 256\,\big(n + C_0(n)\big)\,\sqrt{\delta(f)}\,,$$

that is, (2.18). □

2.4. **Proof of Theorem 1.1.** We come now to the proof of Theorem 1.1, which follows from Theorem 2.7 by a standard argument, cf. [FMP2].

Proof of Theorem 1.1. We divide for simplicity the proof in three steps.

Step I: Deficit uniformly bounded from below.
In this first step, we consider the situation when

$$\sqrt{\delta(f)} \ge \frac{1}{8\,n}\,.$$

Take any $a \ne 0$ and $x_0 \in \mathbb{R}^n$. Using $E = \mathbb{R}^n$ as a test set it is immediate to observe that $d_0(f, \widehat{g}_{a,x_0}) \le 2$, and then by the triangular inequality

$$d(f, \widehat{g}_{a,x_0}) \le \int_{R^n} |f - \widehat{g}_{a,x_0}|^{n'} + 2 \le 2^{n'} + 2 \le 6\,.$$

Consequently, we find

$$\inf_{a,x_0} d(f, \widehat{g}_{a,x_0}) \le 6 \le 48\,n\sqrt{\delta(f)}\,,$$

which a stronger estimate than (1.9).

Step II: Nonnegative functions with small deficit.
We address now the case

$$f \in BV(\mathbb{R}^n)\,, \qquad f \ge 0 \qquad \int_{\mathbb{R}^n} |f|^{n'} = 1 \qquad \delta(f) < \frac{1}{(8\,n)^2}\,. \tag{2.37}$$

Thanks to Lemma 2.3, up to regularize f with a sequence of smooth compactly supported convolution kernels $\{\rho_k\}_{k\in\mathbb{N}}$ and then let $k \to \infty$, we can directly assume that $f \in C^\infty(\mathbb{R}^n)$. Analogously, by Lemma 2.4, we can now trade the smoothness of f for the compactness of its support, that is to say, we may reduce to the case that $\mathrm{spt}(f)$ is compact and that (2.37) holds true. Then, by a further application of Lemma 2.3, we regain the smoothness of f, without loosing the compactness of its support. Summarizing, it suffices to consider the case

$$f \in C_c^\infty(\mathbb{R}^n)\,, \qquad f \geq 0 \qquad \int_{\mathbb{R}^n} |f|^{n'} = 1 \qquad \delta(f) < \frac{1}{(8n)^2}\,.$$

As this is exactly the situation covered in Theorem 2.7, we have finally proved that, whenever f satisfies (2.37), then

$$\inf_{a,x_0} d(f, \widehat{g}_{a,x_0}) \leq 256\,(n + C_0(n))\,\sqrt{\delta(f)}\,.$$

In turn, also this inequality is stronger than (1.9).

Step III: Generic functions with small deficit.
We finally drop the sign condition. Thus, we now have

$$f \in BV(\mathbb{R}^n)\,, \qquad \int_{\mathbb{R}^n} |f|^{n'} = 1 \qquad \delta(f) < \frac{1}{(8n)^2}\,. \tag{2.38}$$

By Lemma 2.3 we may further assume that $f \in C^\infty(\mathbb{R}^n)$, so to have $Df \llcorner \{f = 0\} = 0$. Moreover, up to switch between $f(x)$ and $-f(-x)$, we may directly consider the case that

$$s = \int_{\{f<0\}} f^{n'} \leq \frac{1}{2}\,.$$

Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. By the Sobolev inequality,

$$TV_K(f) = TV_K(f^+) + TV_K(f^-) \geq n|K|^{1/n}\big(s^{1/n'} + (1-s)^{1/n'}\big)\,.$$

In particular, from the elementary concavity inequality (see [FiMP, Figure 7])

$$s^{1/n'} + (1-s)^{1/n'} - 1 \geq \big(2 - 2^{1/n'}\big)s^{1/n'}\,, \quad s \in [0, 1/2]\,,$$

we get

$$\delta(f) \geq (2 - 2^{1/n'})\|f - f^+\|_{L^{n'}(\mathbb{R}^n)}\,. \tag{2.39}$$

We now notice that, since

$$e \geq 2, \qquad 1/2 \leq \log(2) \leq 1\,, \qquad e^x \leq 1 + \left(1 - \frac{1}{e}\right)x \quad \forall\, x \in [-1, 0]\,,$$

we have

$$2 - 2^{1/n'} = 2\left(1 - e^{-\log(2)/n}\right) \geq 2\left(1 - \frac{1}{e}\right)\frac{\log(2)}{n} \geq \frac{1}{2\,n}\,, \tag{2.40}$$

Hence, by the triangle inequality and the fact that $\|f\|_{L^{n'}(\mathbb{R}^n)} = 1$, we conclude that

$$1 - 2\,n\,\delta(f) \leq \|f^+\|_{L^{n'}(\mathbb{R}^n)} \leq 1\,.$$

Since $TV_K(f^+) \leq TV_K(f)$, we deduce that

$$\delta(f^+) \leq \frac{TV_K(f)}{n|K|^{1/n}\,(1 - 2n\delta(f))} - 1 = \frac{2n+1}{1 - 2n\delta(f)}\,\delta(f)\,.$$

By (2.38) we have

$$\frac{1}{1 - 2n\delta(f)} \leq \frac{1}{1 - (1/32n)} \leq \frac{64}{63}\,, \tag{2.41}$$

so that in conclusion, for the sake of writing a neat estimate, we may say that

$$\delta(f^+) \leq \frac{64}{63}\,\big(2n + 1\big)\,\delta(f) \leq 3\,n\,\delta(f)\,.$$

Set now

$$f_0^+ = \frac{f^+}{\|f^+\|_{L^{n'}(\mathbb{R}^n)}}\,,$$

so that $\delta(f_0^+) = \delta(f^+)$. Evidently, f_0^+ satisfies the assumptions (2.37) considered in Step II. Therefore, there exist $a > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$\begin{aligned} d(f_0^+, \widehat{g}_{a,x_0}) &\le 256\,(n + C_0(n))\,\sqrt{\delta(f^+)} \le 256\,(n + C_0(n))\,\sqrt{3n}\,\sqrt{\delta(f)} \\ &\le 448\,(n + C_0(n))\,\sqrt{n}\,\sqrt{\delta(f)}\,. \end{aligned} \tag{2.42}$$

So, we are left to estimate $d(f, f_0^+)$. To this end, let us first notice that, by (2.39) and (2.40),

$$\|f - f_0^+\|_{L^{n'}(\mathbb{R}^n)} \le \Big(\|f - f^+\|_{L^{n'}(\mathbb{R}^n)} + \Big|\|f^+\|_{L^{n'}(\mathbb{R}^n)} - 1\Big|\Big) \le 2\|f - f^+\|_{L^{n'}(\mathbb{R}^n)} \le 4n\delta(f)\,.$$

Since, by the small deficit assumption in (2.38), we have $(4n\delta(f))^{n'} \le 4n\delta(f) \le \sqrt{\delta(f)}/2$, we conclude that

$$d(f, f_0^+) \le \frac{\sqrt{\delta(f)}}{2} + d_0(f, f_0^+)\,. \tag{2.43}$$

We now use as test set in the definition (1.5) of $d_0(f, f_0^+)$ the Borel set $E = \{f < 0\}$. First of all we notice that, by definition and thanks to (2.39) and (2.40)

$$\int_E (f_0^+)^{n'} = 0\,, \qquad \int_E |f|^{n'} \le 2n\,\delta(f) \le \frac{\sqrt{\delta(f)}}{4}\,. \tag{2.44}$$

Moreover, since $Df \llcorner \{f = 0\} = 0$, we have

$$D(f - f_0^+) = Df - \frac{Df \llcorner \{f > 0\}}{\|f^+\|_{L^{n'}(\mathbb{R}^n)}} = \left(1 - \frac{1}{\|f^+\|_{L^{n'}(\mathbb{R}^n)}}\right) Df \llcorner \{f > 0\} + Df \llcorner \{f < 0\}\,.$$

Taking into account that $\|-Df\|_*(\mathbb{R}^n \setminus E) \le TV_K(f) = n|K|^{1/n}(1 + \delta(f)) \le 2n|K|^{1/n}$, we find that

$$\begin{aligned} \frac{\|-D(f - f_0^+)\|_*(\mathbb{R}^n \setminus E)}{n|K|^{1/n}} &\le 2\left|1 - \frac{1}{\|f^+\|_{L^{n'}(\mathbb{R}^n)}}\right| \le 2\,\frac{\|f - f^+\|_{L^{n'}(\mathbb{R}^n)}}{\|f^+\|_{L^{n'}(\mathbb{R}^n)}} \\ &\le 4\,n\,\delta(f)\,\frac{64}{63} \le 5\,n\,\delta(f) \le \frac{5}{8}\,\sqrt{\delta(f)}\,. \end{aligned} \tag{2.45}$$

where we have applied again (2.39), (2.40), (2.41), and the small deficit assumption in (2.38). Hence, by (2.43), (2.44) and (2.45), we see that

$$d(f, f_0^+) \le 2\,\sqrt{\delta(f)}\,.$$

Combining this last estimate with Lemma 2.2 and (2.42) we conclude that

$$d(f, \widehat{g}_{a,x_0}) \le 4\Big(d(f, f_0^+) + d(f_0^+, \widehat{g}_{a,x_0})\Big) \le 1800\,\big(n + C_0(n)\big)\,\sqrt{n}\,\sqrt{\delta(f)}\,,$$

from which (1.6) and (1.7) immediately follow. □

3. From Sobolev to log-Sobolev

We finally remark that Theorem 1.1 immediately gives a stability result for (a family of) anisotropic 1-log-Sobolev inequalities. Let us recall that if $n \ge 2$ and $\alpha \in (0, n')$, then for every $f \in BV(\mathbb{R}^n)$ we have

$$\frac{\alpha\, n'}{n' - \alpha} \log\left(\frac{TV_K(f)}{n|K|^{1/n}\|f\|_{L^\alpha(\mathbb{R}^n)}}\right) \ge \int_{\mathbb{R}^n} \log\left(\frac{|f|^\alpha}{\int_{\mathbb{R}^n}|f|^\alpha\,dx}\right)\frac{|f|^\alpha}{\int_{\mathbb{R}^n}|f|^\alpha\,dx}\,dx\,, \tag{3.1}$$

which, for $K = B$ and $\alpha = 1$ amounts to the classical 1-log-Sobolev inequality on $\mathbb{R}^n$. The family of inequalities (3.1) follows immediately from the anisotropic Sobolev inequality (1.1) by the following argument:

$$\begin{aligned}\int_{\mathbb{R}^n} \log\left(\frac{|f|^\alpha}{\int_{\mathbb{R}^n}|f|^\alpha\,dx}\right)\frac{|f|^\alpha}{\int_{\mathbb{R}^n}|f|^\alpha\,dx}\,dx &= \frac{\alpha}{n'-\alpha}\int_{\mathbb{R}^n}\log\left(\frac{|f|^{n'-\alpha}}{\left(\int_{\mathbb{R}^n}|f|^\alpha\,dx\right)^{\frac{n'-\alpha}{\alpha}}}\right)\frac{|f|^\alpha}{\int_{\mathbb{R}^n}|f|^\alpha\,dx}\,dx\\
(\text{Jensen}) &\le \frac{\alpha}{n'-\alpha}\log\left(\int_{\mathbb{R}^n}\frac{|f|^{n'-\alpha+\alpha}}{\left(\int_{\mathbb{R}^n}|f|^\alpha\,dx\right)^{\frac{n'-\alpha}{\alpha}+1}}\,dx\right)\\
&= \frac{\alpha\,n'}{n'-\alpha}\log\left(\frac{\|f\|_{L^{n'}(\mathbb{R}^n)}}{\|f\|_{L^\alpha(\mathbb{R}^n)}}\right)\\
(\text{Sobolev}) &\le \frac{\alpha\,n'}{n'-\alpha}\log\left(\frac{TV_K(f)}{n|K|^{1/n}\|f\|_{L^\alpha(\mathbb{R}^n)}}\right).\end{aligned}$$

A quick inspection of this chain of inequalities shows that, if we set

$$\begin{aligned}\delta_{LS,\alpha}(f) = &\frac{TV_K(f)}{n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)}}\\
&- \frac{\|f\|_{L^\alpha(\mathbb{R}^n)}}{\|f\|_{L^{n'}(\mathbb{R}^n)}}\exp\left(\frac{n-\alpha(n-1)}{n}\int_{\mathbb{R}^n}\log\left(\frac{f}{\|f\|_{L^\alpha(\mathbb{R}^n)}}\right)\frac{|f|^\alpha}{\|f\|^\alpha_{L^\alpha(\mathbb{R}^n)}}\,dx\right),\end{aligned}$$

then we have

$$\delta_{LS,\alpha}(f) \ge \delta(f).$$

Observe that the formula defining $\delta_{LS,\alpha}(f)$ makes sense also for $\alpha = n'$, and reduces to anisotropic Sobolev inequality (1.1) (since $\delta_{LS,n'}(f) = \delta(f)$). Moreover, if E is a set of finite perimeter and measure, then a simple calculation ensures

$$\delta_{LS,\alpha}(1_E) = \delta(1_E), \qquad \forall\,\alpha \in (0, n']. \tag{3.2}$$

In particular, $\delta_{LS,\alpha}(f) = 0$ if and only if $f = a1_{x_0+arK}$ for some $a \neq 0$, $x_0 \in \mathbb{R}^n$ and $r > 0$. It is now easy to infer from Theorem 1.1 the following sharp quantitative versions of these inequalities (the sharpness follows from (3.2) combined with the sharpness of the quantitative anisotropic isoperimetric inequality proved in [FiMP]):

Theorem 3.1. *If $f \in BV(\mathbb{R}^n)$, with $\|f\|_{L^{n'}(\mathbb{R}^n)} = 1$, then*

$$C_1(n)\sqrt{\delta_{LS,\alpha}(f)} \ge \inf_{a,x_0} d(f, \widehat{g}_{a,x_0}), \qquad \forall\,\alpha \in (0, n'],$$

with $C_1(n)$ as in Theorem 1.1.

Acknowledgment: We thank Matteo Bonforte for pointing out to us the relation between the anistropic Sobolev inequality (1.1) and the family of 1-log-Sobolev inequalities (3.1). This work has been partially supported by the NSF Grant DMS-0969962, the ERC Starting Grant n. 258685 ANOPTSETCON, and the ERC Advanced Grant n. 226234 ANTEGEFI.

References

[AFP] L. Ambrosio, N. Fusco & D. Pallara, Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[BE] G. Bianchi & H. Egnell, A note on the Sobolev inequality, J. Funct. Anal. **100** (1991), no. 1, 18–24.

[Ci] A. Cianchi, A quantitative Sobolev inequality in BV, J. Funct. Anal. **237** (2006), no. 2, 466–481.

[CL] M. Cicalese, G. P. Leonardi, A selection principle for the sharp quantitative isoperimetric inequality, preprint.

[FiMP] A. Figalli, F. Maggi & A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities, Invent. Math., **182** (2010), no. 1, 167–211.

[FMP1] N. Fusco, F. Maggi & A. Pratelli, The sharp quantitative isoperimetric inequality, Ann. of Math. 168 (2008), 941-980.

[FMP2] N. Fusco, F. Maggi & A. Pratelli, The sharp quantitative Sobolev inequality for functions of bounded variation. J. Funct. Anal. 244 (2007), no. 1, 315–341.

[LY] F. Lin & X. Yang, Geometric measure theory – an introduction. Advanced Mathematics (Beijing/Boston), 1. Science Press, Beijing; International Press, Boston, MA, 2002. x+237 pp.

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin TX 78712 USA

E-mail address: figalli@math.utexas.edu

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, ITALY

E-mail address: maggi@math.unifi.it

Dipartimento di Matematica, Università di Pavia, via Ferrata 1, 27100 Pavia, ITALY

E-mail address: aldo.pratelli@unipv.it