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VARIATIONAL APPROXIMATION OF A FUNCTIONAL OF MUMFORD-SHAH TYPE IN CODIMENSION HIGHER THAN ONE

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Abstract. In this paper we consider a new kind of Mumford-Shah functional $E(u,\Omega)$ for maps $u:\mathbb{R}^m\to\mathbb{R}^n$ with $m\geq n$. The most important novelty is that the energy features a singular set S_u of codimension greater than one, defined through the theory of distributional jacobians. After recalling the basic definitions and some well established results, we prove an approximation property for the energy $E(u,\Omega)$ via $\Gamma-$convergence, in the same spirit of the work by Ambrosio and Tortorelli [12].

Résumé. Dans cet article on considère une nouvelle fonctionnelle du type de Mumford-Shah $E(u,\Omega)$ pour des applications $u:\mathbb{R}^m\to\mathbb{R}^n$ avec $m\geq n$. La nouveauté principale est que l'énergie présente un ensemble singulier S_u de codimension supérieure à un, défini par la théorie des déterminant au sense de distributions. Après avoir rappelé les définitions de base et certains résultats classiques, nous prouvons une propriété d'approximation pour l'énergie $E(u,\Omega)$ par Γ-convergence, dans le même esprit de Ambrosio et Tortorelli [12].

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1. Introduction

The objects and the results of this paper belong to a larger research project on the fundamental properties of distributional jacobians. In this work we continue the study of a new functional in the calculus of variations of Mumford-Shah type [6, 19, 38] started in [10], where the minimization involves an unknown function as well as a set:

$$\mathcal{A}(u,K;\Omega)=\int_{\Omega\setminus K} f(x,u,M\nabla u)\,dx+\int_{\Omega\cap K} g\,d\mathscr{H}^{m-n}.$$

Here $\Omega\subset\mathbb{R}^m$ is a bounded open set of class C^1, $u\in C^1(\Omega\setminus K,\mathbb{R}^n)$, $M\nabla u$ is the vector of minors of ∇u of every rank, $\mathscr{H}^{m-n}$ is the $(m-n)$-dimensional Hausdorff measure and K is a sufficiently regular closed set. The main novelty in this type of energies with respect to the classical Mumford-Shah energies [8, 18, 38] is the presence of a "free discontinuity" set of codimension higher than one.

In this paper we are concerned with the model case

$$E(u,\Omega)=\int_\Omega |\nabla u|^p+|M_n\nabla u|^\gamma dx+\sigma\mathscr{H}^{m-n}(\Omega\cap S_u) \tag{1.1}$$

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which we present already in the weak formulation: this energy features a new class of vector valued maps called $GSB_nV(\Omega)$, whose definition is related to the concept of distributional jacobian Ju. Here u is a Sobolev map, S_u is the singular set of its distributional jacobian Ju, see section 2 for the precise definitions. The simplified idea, in the special case $m = n$, is that u is a vector-valued map regular outside a finite number of points where the map covers a set of positive measure, thus imposing a singularity to its jacobian. The functional penalizes maps with an excessively large area factor $M_n\nabla u = \det \nabla u$ as well as the creation of too large singular sets S_u. Note that the p-th power of the gradient helps smoothing possible wild oscillations of u, however if $p < n$ the map might still have a singular jacobian. Moreover like in [38] a lower semicontinuous fidelity term $\int_\Omega |u - g|^r\,dx$, forcing u to be close to a given map g, can be added to (1.1).

The class GSB_nV can be interpreted as a generalization of the well known function spaces SBV and B_nV (see [8,31]), where the differential Du is replaced by Ju and where the jacobian is allowed to have infinite mass. Its definition takes deeply advantage of the slicing procedure available for flat currents, as it is well documented in [9,11,20,22,24,42]. GSB_nV consists of Sobolev functions u whose jacobian can be written as a sum of a finite mass flat current R_u whose total variation is absolutely continuous with respect to Lebesgue measure, and a flat current T_u of finite size. We devote part of section 2 to describe this construction and compare it with the finite mass space B_nV.

Note that similarly to the codimension one case [8,18] the finiteness of the energy does not imply any boundedness of the multiplicity density $\Theta^{m-n}(\|Ju\|,\cdot)$ with respect to the Hausdorff measure $\mathscr{H}^{m-n}\llcorner S_u$: therefore $E(\cdot,\Omega)$ demands an adapted compactness and lower semicontinuity Theorem to show the existence of minimizers of suitable Dirichlet and Neumann problems. This result was obtained in [10], along with several examples and phenomenologies.

Recall the centrality of the distributional jacobian in the literature of Ginzburg-Landau problems, where the defects of constrained Sobolev maps are detected via the appearance of a singularity in Ju, and where approximation results similar to ours have been obtained, see [3,4,15,23,28,39]. Another research field revolving around weak notions of area deformation is nonlinear elasticity, where the deformation u of a material is driven by the energy minimization of a functional depending on the minors of ∇u. The groundbreaking work [14] has been followed by a rich literature, where several theories treating possible formation of fractures and cavitations are described, see [1,27,28,37,40].

In this paper we discuss a variational approximation of E via Γ-convergence by (degenerate) elliptic functionals E_ε, in the spirit of [12,13]. These densities, being absolutely continuous, are easier to handle from the numerical viewpoint. Similarly to the scalar Mumford-Shah functional, we are able to approximate the defect measure, which is singular, via a family of bulk functionals (although not uniformly elliptic), a phenomenon already outlined in the pioneering papers by Modica and Mortola [33,34].

We want to approximate the maps $u \in GSB_nV$ with functions u_ε possessing "better regularity", namely having absolutely continuous jacobian. Our choice of approximating functionals is

$$E_\varepsilon(u,v,\Omega) = \int_\Omega |\nabla u|^p + (v + k_\varepsilon)|M_n\nabla u|^\gamma dx + \int_\Omega \varepsilon^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon^n}dx, \tag{1.2}$$

and the limit takes place for $\varepsilon \to 0$. In (1.2) v is a control function for the pointwise determinant $M_n\nabla u$, ranging in the interval $[0,1]$, and depends on the singular set S_u; k_ε is an infinitesimal number apt to guarantee coercivity of E_ε. The second integral, referred to as the Modica-Mortola term because of the similarity with the phase transition energies contained in [12], contains a nonnegative convex potential W vanishing in 0.

After a brief analysis on the existence of minimizers for E_ε we proceed to show the main convergence result. The approximation of E via E_ε takes place in the sense of Γ-convergence, whose main properties are summarized at the beginning of section 3. In particular the fundamental Theorem for such convergence yields:

$$(u_\varepsilon, v_\varepsilon) \text{ minimizes } E_\varepsilon,\ \ (u_\varepsilon, v_\varepsilon) \to (u,v) \quad \Rightarrow \quad (u,v) \text{ minimizes } E.$$

As ε goes to 0, the potential term $W(1-v)$ forces v_ε to converge to 1 in measure; on the contrary v_ε becomes closer to 0 where the jacobian of the functions u_ε tends to form a singularity, and compensates the loss of energy due to this damping with the Modica-Mortola term. Because of the scaling property of the Modica-Mortola part the transition from $v_\varepsilon \sim 0$ to $v_\varepsilon \sim 1$ happens in a set of width of order ε, and up to a rescaling v_ε converges to a precise profile w_0 analysed in section 4. In particular this transition energy concentrates around the singular set S_u proportionally to its $\mathscr{H}^{m-n}$-measure.

The proof of the approximation will be carried out in two steps: first we show

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) \geq E(u,\Omega)$$

whenever $(u_\varepsilon, v_\varepsilon) \to (u,1)$. This step is achieved first in codimension $m-n=0$, where S_u is a discrete set, and then generalized to every codimension with the help of the slicing Theorem. The second part of the proof concerns the upper limit: here we construct (u_ε) truncating the function u around the singularity S_u and we use the optimal profile w_0 to build functions v_ε such that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ and

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) \leq E(u,\Omega).$$

In order to make this construction we will assume a mild regularity assumption on the singular set, namely

$$\limsup_{r\downarrow 0} \frac{\mathscr{L}^m(\{x\in\Omega : \mathrm{dist}(x,S_u)\leq r\})}{\mathscr{L}^n(B_1^n)r^n} = \mathscr{H}^{m-n}(S_u). \tag{1.3}$$

In order to conclude the proof of the Γ-convergence of E_ε to E we would need to know the density in energy of the set of GSB_nV maps satisfying (1.3). In the codimension 1 case this property was deduced by the regularity of minimizers of the Mumford-Shah energy, for which a lower bound on the $(m-1)$-dimensional density of the singular set is available. The analogous density property as well as a regularity result for minimizers of E will be subject to further investigation.

In section 7 we prove an analog approximation result where we impose a fixed boundary condition to both u and the approximating sequence (u_ε). In the case $S_u \cap \partial\Omega \neq \emptyset$ then the transition made by v takes place partially outside the domain Ω, which translates in a loss of mass in the limit energy.

Finally in the last section we discuss a possible generalization to general Lagrangians, featuring a polyconvex integrand for the bulk part and where the size term is weighted by a continuous density. Growth and convexity assumptions will be crucial to extend the results of the previous sections to this broader class of energies.

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2. Distributional jacobians

We begin by fixing some basic notions and recalling some properties of distributional jacobians: we will assume, if not otherwise specified, that Ω is a bounded open subset of $\mathbb{R}^m$ with boundary of class C^1, that $m \geq n$ are positive integers and that p and s are positive exponents satisfying

$$\frac{1}{s} + \frac{n-1}{p} \leq 1, \qquad s < \infty : \tag{2.1}$$

observe that this limitation allows p to be smaller than the critical exponent n.

As customary the symbol $\Lambda_k\mathbb{R}^m$ will denote the space of k-vectors of $\mathbb{R}^m$. We will let

$$\mathbf{O}_k = \{L : \mathbb{R}^m \to \mathbb{R}^m : L = L^t,\, L^2 = L,\, \mathrm{rk}(L) = k\}$$

be the space of orthogonal projections of rank k, for $1 \le k \le m$. Given a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ we adopt for the collection $M_k L$ of determinants of $k \times k$ minors of L the following sign convention:

$$M_k L := \left(e_1 \wedge \cdots \wedge e_m \llcorner L^{i_1} \wedge \cdots \wedge L^{i_k}\right)_{\{i_1 < \cdots < i_k\} \subset \{1,\ldots,n\}}.$$

In this way we group the minors with the same rows in a single element of $\Lambda_{m-k}\mathbb{R}^m$. We let $ML = (M_1 L, \ldots, M_n L)$ be the vector minors of every rank; $\kappa := \sum_{k=1}^n \binom{m}{k}\binom{n}{k}$ will be its dimension. Given $w \in \mathbb{R}^\kappa$ we let w_k be the variables relative to $k \times k$ minors. For our purposes we will need to measure the length of $\nu \in \Lambda_k \mathbb{R}^m$ so that

$$|\nu| = \sup_{\pi \in \mathbf{O}_k} |\nu \llcorner d\pi| = \sup_{\pi \in \mathbf{O}_k} |\langle d\pi, \nu\rangle| : \tag{2.2}$$

it can be proved that the Euclidean norm satisfies this property, see [10, 24].

Weak convergence in the L^p spaces will be customarily denoted with the symbol $\rightharpoonup$: in particular in the non-reflexive case $p = 1$ this is the convergence against fixed L^∞ functions. Sobolev maps $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ are known to possess an approximate differential $\nabla u(x) \in \mathbb{R}^{n\times m}$ at almost every point $x \in \Omega$, see [43, Theorem 3.4.2].

We will denote by $\mathbf{F}_k(\Omega)$ and $\mathbf{M}_k(\Omega)$ the spaces respectively of flat and finite mass k-dimensional currents in an open subset $\Omega \subset \mathbb{R}^m$ (see [9,24,26]). The action of a current T on a differential form ψ will be denoted by $\langle T, \psi\rangle$, and weak* convergence (that is: pointwise convergence of the functionals $\langle T_h, \psi\rangle \to \langle T, \psi\rangle$ for every fixed compactly supported smooth differential form ψ) will be denoted by $\overset{*}{\rightharpoonup}$. The same notation is adopted for weak* convergence of measures. The top dimensional m-current representing the Lebesgue integration with the standard orientation on $\mathbb{R}^m$ will be denoted by $\mathbf{E}^m$:

$$\mathbf{E}^m(\varphi dx^1 \wedge \cdots \wedge dx^m) = \int_{\mathbb{R}^m} \varphi(x) d\mathscr{L}^m(x).$$

To our knowledge the notions of distributional jacobian and $B_n V$ function were defined first in [31]: the basic necessary assumption on u to give this definition is membership to $\dot{W}^{1,p} \cap L^s = \{u \in L^s, \nabla u \in L^p\}$.

Definition 2.1 (Distributional jacobian and $B_n V$ functions)**.** *Let $u \in \dot{W}^{1,p} \cap L^s(\Omega, \mathbb{R}^n)$. We denote by $j(u)$ the $(m - n + 1)$-current*

$$\langle j(u), \omega\rangle := (-1)^n \int_\Omega u^1 du^2 \wedge \cdots \wedge du^n \wedge \omega, \tag{2.3}$$

where ω is a smooth $(m - n + 1)$-form with compact support in Ω; we define the distributional jacobian of u as the $(m - n)$-dimensional flat current

$$Ju := \partial j(u) \in \mathbf{F}_{m-n}(\Omega).$$

We say that a map $u \in \dot{W}^{1,p} \cap L^s$ belongs to $B_n V$ if its distributional jacobian Ju has finite mass (and hence it can be represented by a Radon measure).

Few observation are in order: first of all the integrability assumption $u \in \dot{W}^{1,p} \cap L^s$ ensures that (2.3) is well-defined; observe that for $p \ge \frac{mn}{m+1}$ Ju is defined for $u \in W^{1,p}$, since in this case $W^{1,p} \subset L^s$ for some sufficiently large exponent s satisfying (2.1), by Sobolev embedding. Notice also that this constraint allows the case $p < n$. Since $j(u)$ is explicitly represented as the integration against a Λ_{m-n}-valued L^1 function, Ju belongs to the space of flat currents $\mathbf{F}_{m-n}(\Omega)$ (see [24, 4.1.18] and [10] for a proof of this fact). Regarding the convergence properties of jacobians, we consider the following flat norm on k-currents

$$\mathbf{F}(T) := \sup\left\{\langle T, \psi\rangle : \psi \in C_c^\infty(\Omega, \Lambda^k\mathbb{R}^m),\ \max\{\|\psi\|_\infty, \|d\psi\|_\infty\} \le 1\right\} :$$

given $u_h, u \in \dot W^{1,p} \cap L^s(\Omega, \mathbb{R}^n)$ we have

$$u_h \to u \text{ in } L^s(\Omega,\mathbb{R}^n), \quad \nabla u_h \rightharpoonup \nabla u \text{ in } L^p(\Omega,\mathbb{R}^{n\times m}) \quad \Rightarrow \quad \mathbf{F}(Ju_h - Ju) \to 0, \tag{2.4}$$

hence weakly* in the sense of currents, compare [10]. If moreover $(u_h) \subset B_nV$ and $\mathbf{M}(Ju_h) = \|Ju_h\|(\Omega) \le C < \infty$ then $u \in B_nV$ and the convergence takes place in the sense of measures. In particular if $p \ge n$, by convolution every function $u \in \dot W^{1,p} \cap L^s$ has a sequence $(u_h) \subset C^\infty$ approximating u strongly in the Sobolev space $W^{1,n}_{\mathrm{loc}}$: since ω has compact support passing to the limit in the integration by parts formula

$$\langle Ju_h, \psi\rangle = (-1)^n \int_\Omega u_h^1 du_h^2 \wedge \cdots \wedge du_h^n \wedge d\psi = \int_\Omega du_h^1 \wedge du_h^2 \wedge \cdots \wedge du_h^n \wedge \psi$$

we obtain that $Ju = \mathbf{E}^m \llcorner (-1)^{n(m-n)} du^1 \wedge \cdots \wedge du^n$. In particular if the gradient ∇u has a sufficiently high summability, then Ju is an absolutely continuous measure. On the other hand when $p < n$ there are several examples of functions whose jacobian is not in L^1: for instance the "monopole" function $u(x) := \frac{x}{|x|}$ satisfies $Ju = \mathscr{L}^n(B_1)[\![0]\!]$, where $[\![0]\!]$ is the Dirac mass in the origin. More complicated examples, including maps such that Ju has infinite mass or such that Ju is not even a Radon measure, are presented in [3, 10, 31, 37]. We finally remark that in our paper membership to B_nV, or to any other space whose definition involves Ju, implicitly assumes $u \in \dot W^{1,p} \cap L^s$, for p, s as in (2.1).

Distributional jacobians of B_nV functions, being Λ_{m-n}-valued measures, satisfy a decomposition in three mutually singular parts (see [8, 20, 31]):

$$Ju = \nu \cdot \mathscr{L}^m + J^c u + \theta \cdot \mathscr{H}^{m-n} \llcorner S_u$$

where

- $\nu = \frac{dJu}{d\mathscr{L}^m} \in L^1(\Omega, \Lambda_{m-n}(\mathbb{R}^m))$ is the Radon Nikodym derivative of Ju with respect to $\mathscr{L}^m$;
- $\theta \in L^1(\Omega, \Lambda_{m-n}(\mathbb{R}^m), \mathscr{H}^{m-n})$ is a measurable function and S_u is a $\mathscr{H}^{m-n}$ σ-finite subset of Ω;
- $\|J^c u\|(F) = 0$ whenever $\mathscr{H}^{m-n}(F) < \infty$.

It can be proved that

$$\nu(x) = M_n \nabla u(x) = e_1 \wedge \cdots \wedge e_m \llcorner du^1 \wedge \cdots \wedge du^n \in \Lambda_{m-n}(\mathbb{R}^m)$$

at $\mathscr{L}^m$-almost every point $x \in \Omega$ (see [36] and [21]). The set S_u is unique up to $\mathscr{H}^{m-n}$-negligible sets, by intersecting it with $\{|\theta| > 0\}$; moreover S_u is $\mathscr{H}^{m-n}$-countably rectifiable (see [20]). In analogy with the codimension one case we denote $SB_nV(\Omega)$ the subset of $B_nV(\Omega)$ of functions such that $J^c u = 0$. This space enjoys a closure property proved in [20]:

Theorem 2.2 (Closure Theorem for SB_nV)**.** *Let us consider $u, u_k \in B_nV(\Omega, \mathbb{R}^n)$ with $\Omega \subset \mathbb{R}^m$ and suppose that*

(a) *$u_h \to u$ strongly in $L^s(\Omega, \mathbb{R}^n)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$,*
(b) *if we write*

$$Ju_h = \nu_h \cdot \mathscr{L}^m + \theta_h \cdot \mathscr{H}^{m-n} \llcorner S_{u_h}$$

then $|\nu_h|$ are equiintegrable in Ω and $\mathscr{H}^{m-n}(S_{u_h}) \le C < \infty$.

Then $u \in SB_nV(\Omega, \mathbb{R}^n)$ and

$$\nu_h \rightharpoonup \nu \text{ weakly in } L^1(\Omega, \Lambda_{m-n}(\mathbb{R}^m)), \quad \mathscr{H}^{m-n}(S_u) \le \liminf_h \mathscr{H}^{m-n}(S_{u_h}).$$

As explained in [24, 4.2] and [9], every flat current $T \in \mathbf{F}_k(\Omega)$ can be sliced via a Lipschitz map $\pi \in \mathrm{Lip}(\Omega, \mathbb{R}^\ell)$, $\ell \le k$: the result is a collection of currents

$$\langle T, \pi, x\rangle \in \mathbf{F}_{k-\ell}(\Omega) \qquad \text{defined for } \mathscr{L}^\ell\text{-a.e. } x \in \mathbb{R}^\ell$$

satisfying several properties. Amongst them we recall

$$
\begin{aligned}
&T \llcorner d\pi = \int_{\mathbb{R}^\ell} \langle T, \pi, x\rangle \, d\mathscr{L}^\ell(x),\\
&\langle T, \pi, x\rangle \text{ is concentrated on } \pi^{-1}(x) \text{ for } \mathscr{L}^\ell\text{-a.e. } x \in \mathbb{R}^\ell,\\
&\int_{\mathbb{R}^\ell} \mathbf{F}(\langle T, \pi, x\rangle)\, d\mathscr{L}^\ell(x) \le \mathrm{Lip}(\pi)^\ell \mathbf{F}(T),
\end{aligned}
$$

and we refer to [24, 4.2.1] and to [11] for a general account in the Euclidean and general metric setting. We aim to apply this operation to $Ju \in \mathbf{F}_{m-n}(\Omega)$ in the special case $\ell = m - n$, thus reducing it to 0-dimensional slices; moreover we want to relate those slices to the jacobian of the restriction $J(u|_{\pi^{-1}(x)})$. Let therefore $\pi \in \mathbf{O}_{m-n}$: for each $x \in \pi(\mathbb{R}^m)$ we let $i^x : \mathbb{R}^n \to \mathbb{R}^m$ be the orthogonal injection of $\mathbb{R}^n$ onto $\pi^{-1}(x)$. In [20], the author proved the following slicing Theorem for jacobians:

Theorem 2.3 (Slicing). *Let $u \in \dot{W}^{1,p} \cap L^s(\Omega, \mathbb{R}^n)$ and let $\pi \in \mathbf{O}_{m-n}$. Then for $\mathscr{L}^{m-n}$-almost every $x \in \mathbb{R}^{m-n}$*

$$\langle Ju, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x), \tag{2.5}$$

where $u^x = u \circ i^x$. Moreover $u \in B_nV(\Omega, \mathbb{R}^n)$ if and only if for every $\pi \in \mathbf{O}_{m-n}$ the following two conditions hold:

$$
\begin{aligned}
&\text{(i)} \qquad u^x \in B_nV(\Omega^x, \mathbb{R}^n) \quad \textit{for} \quad \mathscr{L}^{m-n}\textit{-almost every } x \in \mathbb{R}^{m-n},\\
&\text{(ii)} \qquad \int_{\pi(\Omega)} \|\langle Ju, \pi, x\rangle\|(\Omega^x)\, d\mathscr{L}^{m-n}(x) < \infty,
\end{aligned}
$$

where $\Omega^x = \Omega \cap \pi^{-1}(x)$. If $u \in B_nV(\Omega, \mathbb{R}^n)$ the slicing property (2.5) holds separately for the absolutely continuous part, the Cantor part and the jump part of Ju, namely:

- $\langle J^a u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^a u^x)$,
- $\langle J^c u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^c u^x)$,
- $\langle J^j u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^j u^x)$.

Since we will work with functions whose jacobian does not have a Cantor part, it is useful to notice that in order to check that some function u belongs to SB_nV it is sufficient to check that, along with the integrability assumption (ii), for almost every slice Ju^x has no Cantor part. Moreover in the general theory of current in metric spaces the bound (ii) would be required uniform in π, see [7, 10]; in the Euclidean space it is of course enough to check such property for $\binom{m}{n}$ linearly independent projections.

2.1. A new class of maps related to Size

In order to study a minimization problem it is necessary to consider, along with the topology, the natural domain of the functional, and to understand the potential limit points of energy-bounded sequences. As anticipated in the introduction our functional E penalizes the size of the singular set of Ju, regardless of the multiplicity function θ. This lack of control on the mass of Ju, which already appears in Theorem 2.2 when we require $u \in B_nV$, forces us to extend the notion of admissible maps beyond B_nV, through the concept of size. In general it is possible to define a measure-theoretic quantity $\mathbf{S}(T)$, called size of T, for flat currents $T \in \mathbf{F}_k(\Omega)$ with possibly infinite mass. This quantity was introduced in [9], borrowing some ideas already used by Hardt and Rivière in [30] and Almgren [5] and agrees with the classical notion of size for finite mass currents, namely

$$\mathbf{S}(T) = \mathscr{H}^{m-n}\big(\{\Theta^{m-n}(\|T\|, \cdot) > 0\}\big), \qquad T \in \mathbf{M}_k(\Omega) \cap \mathbf{F}_k(\Omega)$$

as in [25]. The main idea behind this to detect the support of the 0-dimensional slices of T and then to optimize the choice of projection π.

Definition 2.4 (Size of a flat current)**.** *We say that $T \in \mathbf{F}_k(E)$ has finite size if there exists a positive Borel measure μ such that*

$$\mathscr{H}^0 \llcorner \operatorname{spt}(T) \leq \mu \qquad \textit{for } k = 0,$$
$$\mu_{T,\pi} := \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \operatorname{spt}\langle T, \pi, x\rangle \, d\mathscr{L}^k(x) \leq \mu \quad \forall \pi \in \mathbf{O}_k \qquad \textit{for } k \geq 1.$$

The choice of μ can be optimized by choosing the least upper bound of the family $\{\mu_{T,\pi}\}$ in the lattice of nonnegative measures:

$$\mu_T := \bigvee_{\pi \in \mathbf{O}_k} \mu_{T,\pi} = \bigvee_{\pi \in \mathbf{O}_k} \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \operatorname{spt}\langle T, \pi, x\rangle \, d\mathscr{L}^k(x).$$

We set $\mathbf{S}(T) := \mu_T(\Omega)$.

It can be proved (see [9]) that every flat k-current with finite size has a unique (up to null sets) countably $\mathscr{H}^k$-rectifiable set called $set(T)$ where μ_T is concentrated, which satisfies $\mathscr{H}^k(set(T)) = \mathbf{S}(T)$. The reader can find an example of flat current having finite size but infinite mass in [10, 35].

The natural space for our problem is the set of Sobolev functions u with the integrability expressed in (2.1), whose jacobian can be split in the sum of two parts:

- one is an m-dimensional current R of finite mass, such that the measure $\|R\|$ is absolutely continuous with respect to $\mathscr{L}^m$;
- the other one is an $(m-n)$-dimensional flat chain T of finite size.

Definition 2.5 (Functions of Special jacobian)**.** *The space of function of Special jacobian is*

$$GSB_nV(\Omega) = \left\{ u \in \dot{W}^{1,p} \cap L^s(\Omega, \mathbb{R}^n) : Ju = R + T,\, \mathbf{M}(R) + \mathbf{S}(T) < \infty,\, \|R\| \ll \mathscr{L}^m \right\}.$$

This space is clearly meant to mimic the aforementioned SB_nV class. Thanks to the relation between the slices of Ju and the jacobian of the restrictions expressed by (2.5), if $u \in GSB_nV(\Omega)$ and $\pi \in \mathbf{O}_{m-n}$ we can observe that for almost every x the slice $\langle R_u, \pi, x\rangle$ has finite mass and is absolutely continuous with respect to $\mathscr{H}^n \llcorner \pi^{-1}(x)$, while by Definition 2.4 $\mathbf{S}(\langle T_u, \pi, x\rangle) < \infty$. Therefore $u^x \in GSB_nV(\Omega^x)$ for $\mathscr{L}^{m-n}$-almost every $x \in \mathbb{R}^{m-n}$. In the following propositions we describe some useful properties of the class $GSB_nV(\Omega)$.

Proposition 2.6 ([10, Lemma 3.0.5])**.** *If $m = n$ then $GSB_nV(\Omega) = SB_nV(\Omega)$.*

The last Proposition shows that the difference between the spaces $GSB_nV(\Omega)$ and $SB_nV(\Omega)$ relies on the failure of the integrability condition (ii) in Theorem 2.3. Moreover, since the Radon-Nikodym decomposition of a measure into the sum of an absolutely continuous and a singular part is unique, by slicing also R and T are uniquely determined in the decomposition. Therefore we can write $Ju = R_u + T_u$, so that $set(T_u)$ is a well defined countably $\mathscr{H}^{m-n}$-rectifiable set. In agreement with the scalar case $n = 1$ we let

$$S_u := set(T_u)$$

be the singular set of the map u. Moreover the pointwise characterization of R_u also holds for GSB_nV maps.

Proposition 2.7 ($Det = det$ in the GSB_nV class, [10, Proposition 3.2.1])**.** *Let $u \in GSB_nV(\Omega)$ and write $Ju = R_u + T_u$ as in Definition 2.5. Then $\mathscr{L}^m$-almost everywhere*

$$\frac{dR_u}{d\mathscr{L}^m} = M_n \nabla u.$$

The fundamental Theorem on the space GSB_nV is the following compactness result:

Theorem 2.8 (Compactness for the class GSB_nV, [10, Theorem 4.0.3])**.** *Let $\Psi : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying $\lim_{t\to\infty} \Psi(t)/t = \infty$. Let $(u_h) \subset GSB_nV(\Omega)$ satisfy $u_h \to u$ in $L^s(\Omega,\mathbb{R}^n)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega,\mathbb{R}^{n\times m})$. Assume that the Jacobians $Ju_h = R_{u_h} + T_{u_h}$ satisfy*

$$\sup_h \int_\Omega \Psi\left(\left|\frac{dR_{u_h}}{d\mathscr{L}^m}\right|\right) d\mathscr{L}^m + \mathbf{S}(T_{u_h}) < \infty.$$

Then $u \in GSB_nV(\Omega)$ and, writing $Ju = R_u + T_u$,

$$\frac{dR_{u_h}}{d\mathscr{L}^m} \rightharpoonup \frac{dR_u}{d\mathscr{L}^m} \qquad \text{weakly in } L^1(\Omega,\Lambda_{m-n}\mathbb{R}^m), \tag{2.6}$$
$$\mathbf{S}(T_u) \le \liminf_h \mathbf{S}(T_{u_h}).$$

In the sequel it will be handier to have a name for the space of function of bounded n-variation with absolutely continuous jacobian:

Definition 2.9 (Regular maps)**.** *We let*

$$R_n(\Omega) := \{u \in B_nV(\Omega) : \|Ju\| \ll \mathscr{L}^m\}$$

be the space of regular maps.

We have now all the elements to define our Mumford-Shah energy of codimension higher than one.

Definition 2.10. *Let $\gamma > 1$ and $\sigma > 0$. For every $u \in GSB_nV(\Omega)$ we set*

$$E(u,\Omega) = \int_\Omega |\nabla u|^p + |M_n \nabla u|^\gamma dx + \sigma \mathscr{H}^{m-n}(\Omega \cap S_u).$$

It has been proved in [10] the following existence theorem, even for a broader class of Lagrangians, and for several notions of boundary conditions. Here we report the version most suitable to the scope of this paper.

Theorem 2.11 (Existence of minimizers for the Dirichlet and Neumann problems)**.** *Let Ω be a regular open and bounded subset of $\mathbb{R}^m$ and let U be an open neighborhood of $\overline{\Omega}$. Let $\phi \in GSB_nV(U)$ be a given function and suppose $p^* = \frac{mp}{m-p} > s$. Then the minimum problem*

$$\inf\left\{E(u,\overline{\Omega}) : u \in GSB_nV(U),\ u = \phi \text{ in } U \setminus \Omega\right\} \tag{2.7}$$

has a solution. Similarly for the Neumann problem if $r > s$ and $g \in L^r(\Omega,\mathbb{R}^n)$ is given, then

$$\inf\left\{E(u,\Omega) + \int_\Omega |u-g|^r\, dx : u \in GSB_nV(\Omega)\right\} \tag{2.8}$$

has a solution.

2.2. Minkowski content

As Theorem 3.7 below involves the concept of Minkowski content, we here briefly review its definition and main properties.

Definition 2.12. *Let $S \subset \mathbb{R}^m$ and let $k \in [0,m]$ be and integer. The lower and upper Minkowski contents of S in Ω are defined respectively as*

$$\mathcal{M}^k_{*\Omega}(S) = \liminf_{r\downarrow 0} \frac{\mathscr{L}^m(\{x \in \Omega : dist(x,S) \le r\})}{\mathscr{L}^{m-k}(B_1) r^{m-k}}, \tag{2.9}$$

$$\mathcal{M}^{*k}_{\Omega}(S) = \limsup_{r\downarrow 0} \frac{\mathscr{L}^m(\{x \in \Omega : dist(x,S) \le r\})}{\mathscr{L}^{m-k}(B_1) r^{m-k}}, \tag{2.10}$$

where $\mathscr{L}^{m-k}(B_1)$ is the measure of the unit ball in $\mathbb{R}^{m-k}$. We omit the subscript when $\Omega = \mathbb{R}^m$. If $\mathcal{M}^k_*(S) = \mathcal{M}^{*k}(S)$ we define the Minkowski content of S as this common value.

We must observe that neither $\mathcal{M}^k_*$ nor $\mathcal{M}^{*k}$ is a measure, and that they both give the same value to a set and its closure. It is natural to compare the upper and lower Minkowski contents with the k-dimensional Hausdorff measure: it can be proved (see [24, 3.2.37-39], [8, 2.101]) that for every countably $\mathscr{H}^k$-rectifiable and closed set S

$$\mathcal{M}^k_*(S) \geq \mathscr{H}^k(S).$$

By inner regularity of the Hausdorff measure the last inequality holds also relative to Ω. Various assumptions on S besides rectifiability are possible in order to have that $\mathcal{M}^k(S) = \mathscr{H}^k(S)$. One of the most general is the following:

Proposition 2.13 ([8, Proposition 2.104])**.** *Let S be a countably $\mathscr{H}^k$-rectifiable set such that*

$$\nu(B_\rho(x)) \geq c\rho^k \quad \forall x \in S \quad \forall \rho \in (0, \rho_0) \tag{2.11}$$

for a suitable Radon measure $\nu \ll \mathscr{H}^k$ and $c, \rho_0 > 0$. Then

$$\mathcal{M}^k(S) = \mathscr{H}^k(S).$$

Note that the equality implies that $\mathscr{H}^k(S) = \mathscr{H}^k(\overline{S})$. To ease the notation we will denote $S_r = \{x \in \Omega : 0 < \operatorname{dist}(x, S) \leq r\}$ and $V(r) = \mathscr{L}^m(S_r)$. Let $S \subset \mathbb{R}^m$ be a closed set, and consider the distance function from it. Then (see [24, 3.2.34])

$$|\nabla \operatorname{dist}(\cdot, S)| = 1 \quad \mathscr{L}^m\text{-a.e. in } \{\operatorname{dist}(\cdot, S) > 0\}. \tag{2.12}$$

Moreover the following property holds:

Lemma 2.14. *The function $V(t) = \mathscr{L}^m(\{0 < \operatorname{dist}(\cdot, S) \leq t\})$ is absolutely continuous and*

$$V'(t) = \mathscr{H}^{m-1}(\{x \in \Omega : \operatorname{dist}(x, S) = t\})$$

for $\mathscr{L}^1$-almost every $t > 0$.

Proof. Recall the Coarea formula [24, 3.2.11-12]: if $f : \Omega \to \mathbb{R}$ is a Lipschitz function and $g : \Omega \to \mathbb{R}$ is a non-negative Borel function, then

$$\int_\Omega g(x)|\nabla f(x)|\, dx = \int_0^{+\infty} \int_{\{f=t\}} g \, d\mathscr{H}^{m-1}\, dt. \tag{2.13}$$

In particular taking $f(x) = \operatorname{dist}(x, S)$ and g the characteristic function of the set $\{\operatorname{dist}(\cdot, S) \leq t\}$ we obtain that for every $t > 0$

$$V(t) = \int_0^t \mathscr{H}^{m-1}(\Omega \cap \{\operatorname{dist}(\cdot, S) = s\}) ds.$$

Therefore $V(t)$ is an absolutely continuous function with

$$V'(t) = \mathscr{H}^{m-1}(\{x \in \Omega : \operatorname{dist}(x, S) = t\})$$

$\mathscr{L}^1$-almost everywhere. □

3. Variational approximation

In this section we state our main approximation theorem. We start by recalling the fundamental features of the variational convergence we will use, the Γ-convergence, and we refer to [16, 17] for a thorough presentation. Let X be a separable metric space and let a sequence of functions $F_h : X \to [0,\infty]$ be given. We define the upper and the lower Γ-limits as follows:

$$\underline{F}(x) = (\Gamma - \liminf_{h\to\infty} F_h)(x) = \inf\{\liminf_{h\to\infty} F_h(x_h) : x_h \to x\}, \tag{3.1}$$

$$\overline{F}(x) = (\Gamma - \limsup_{h\to\infty} F_h)(x) = \inf\{\limsup_{h\to\infty} F_h(x_h) : x_h \to x\}. \tag{3.2}$$

Both $\underline{F}$ and $\overline{F}$ are lower semicontinuous by construction, and we say that F_h Γ-converges to F if $\underline{F} = \overline{F}$. The statement $\Gamma - \lim_h F_h = F$ is equivalent to the fulfillment of the following two conditions: for every $x \in X$

$$\forall x_h \to x \text{ we have } \liminf_h F_h(x_h) \geq F(x), \tag{3.3}$$

$$\exists x_h \to x \text{ such that } \limsup_h F_h(x_h) \leq F(x).$$

The following Theorem describes the fundamental properties of this type of convergence, in particular the behaviour of sequences of minima:

Theorem 3.1. *Assume F_h Γ-converges to F.*

(a) *Let $t_h \downarrow 0$. Then any cluster point of the sequence of sets*

$$\{x \in X : F_h(x) \leq \inf_X F_h + t_h\}$$

minimizes F.

(b) *Assume also that F_h are lower semicontinuous, and that for every $t \geq 0$ there exists a compact set $K_t \subset X$ such that*

$$\{F_h \leq t\} \subset K_t.$$

Then every function F_h has a minimizer, and any sequence of minimizers admits a subsequence converging to some minimizer of F.

(c) *Given a continuous function $G : X \to [0,\infty]$ we have*

$$\Gamma - \liminf_h (F_h + G) = (\Gamma - \liminf_h F_h) + G,$$

$$\Gamma - \limsup_h (F_h + G) = (\Gamma - \limsup_h F_h) + G.$$

The following remark recalls a useful tool in proving Γ-convergence results.

Remark 3.2. Let $X' \subset X$ and $F, F_h : X \to \mathbb{R}$ as above: we say that X' is dense in energy in X if for every $x \in X$ there exists a sequence $(x'_h) \subset X'$ such that $x'_h \to x$ and $F(x'_h) \to F(x)$. A simple diagonal argument shows that in order to prove $\Gamma - \lim F_h = F$, whilst already knowing the $\Gamma - \liminf$ inequality $F \leq \underline{F}$ (namely the validity of (3.3)), it is enough to prove that for every $\delta > 0$ and $x \in X'$ there exists $x_h \to x$ such that $\limsup_h F_h(x_h) \leq F(x) + \delta$.

3.1. Main Theorem

We introduce now the function spaces involved in our approximation Theorem. Given an open set $U \subset \mathbb{R}^n$ we let $B(U)$ be the space of Borel functions ranging in $[0,1]$:

$$B(U) = \{v : U \to [0,1] : v \text{ is a Borel function}\},$$

endowed with a distance that induces the convergence in measure, namely:

$$d(v,v') = \int_\Omega \frac{|v-v'|}{1+|v-v'|}dx.$$

We want to approach the energy $E(u,\Omega)$ by a sequence $E_h(u_h,v_h,\Omega)$ where the functions u_h belong to $R_n(\Omega)$, namely $Ju_h = R_{u_h} = M_n\nabla u_h \mathscr{L}^m$. Our function spaces will be the following:

Definition 3.3. *We define the space* $X(\Omega) := L^s(\Omega,\mathbb{R}^n)\times B(\Omega)$ *with the following convergence notion:*

$$(u_h,v_h)\to(u,v) \iff u_h\to u \text{ in } L^s(\Omega,\mathbb{R}^n), \quad v_h\to v \text{ in measure.} \tag{3.4}$$

The subspace $Y(\Omega)$ *will be:*

$$Y(\Omega) := R_n(\Omega)\times B(\Omega)\subset X(\Omega),$$

endowed with the same topology.

The convergence (3.4) is clearly metrizable. We also introduce two subspaces of $X(\Omega)$ and $Y(\Omega)$ where the trace is fixed in a strong sense:

Definition 3.4. *Given* $U\Supset\Omega$ *open and* $\phi\in L^s(U)$ *we let*

$$X^\phi = \{(u,v)\in X(U) : u=\phi \text{ in } U\setminus\Omega\},$$
$$Y^\phi = \{(u,v)\in Y(U) : u=\phi \text{ in } U\setminus\Omega\}.$$

Following [12, 13, 33], we introduce a Modica-Mortola type energy to approximate the size term $\mathbf{S}(T_u) = \mathscr{H}^{m-n}(S_u\cap\Omega)$. Observe that the parameter ε is present with suitable exponents in order for the energy to concentrate on $(m-n)$-dimensional sets: in particular it concentrates on points if $m=n$.

Definition 3.5. *Let* $W\in C^1(\mathbb{R})$ *be a nonnegative convex potential vanishing only at* 0 *and let* $q>n$ *be a given exponent. If* $v\in B(\Omega)$ *we set*

$$MM_\varepsilon(v,\Omega) = \int_\Omega \varepsilon^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon^n}dx.$$

Note in particular that W is increasing in the positive real axis. We are now ready to introduce our family of energies:

Definition 3.6. *Let* $\gamma>1$ *and* $q>n$ *be fixed exponents. We set, for* $(u,v)\in X(\Omega)$*:*

$$E(u,v,\Omega) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p + |M_n(\nabla u)|^\gamma dx + \sigma\mathscr{H}^{m-n}(S_u\cap\Omega) & \text{if } u\in GSB_nV(\Omega) \text{ and } v=1,\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$E_\varepsilon(u,v,\Omega) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p + (v+k_\varepsilon)|M_n(\nabla u)|^\gamma dx + MM_\varepsilon(v,\Omega) & \text{for } (u,v)\in Y(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where the constant σ *is defined by the minimum problem 4.1 and* k_ε *is an infinitesimal faster than* ε^γ*.*

The first functional $E(u,v,\Omega)$ is clearly a trivial extension to $X(\Omega)$ of Definition 2.10, as $E(u,1,\Omega)=E(u,\Omega)$. We fix once and for all a sequence ε_h of positive numbers converging to zero and to simplify the notation we write E_h instead of E_{ε_h}. We will also write

$$F(u,1,\Omega)=F(u,\Omega)=\int_\Omega |\nabla u|^p+|M_n\nabla u|^\gamma\,dx,$$
$$F_\varepsilon(u,v,\Omega)=\int_\Omega |\nabla u|^p+(v+k_\varepsilon)|M_n\nabla u|^\gamma\,dx$$

for the part of the energy explicitly depending on u.

We can now state our main Theorem: we prefer to present separately the lower and upper limit part of the Γ-convergence, since it is more clear where the hypotheses are used.

Theorem 3.7. *Let Ω be a bounded open subset of class C^1 of $\mathbb{R}^m$ and suppose (2.1) and*

$$s\geq \frac{np}{n-p},\qquad 1<\gamma\leq \frac{1}{\frac{n-1}{p}+\frac{1}{s}},\qquad q>n.$$

(a) *For every sequence $\big((u_h,v_h)\big)\subset Y(\Omega)$ such that $(u_h,v_h)\to(u,v)$ in $X(\Omega)$ we have*

$$\liminf_{h\to\infty} E_h(u_h,v_h,\Omega)\geq E(u,v,\Omega);$$

moreover

$$\liminf_{h\to\infty} E_h(u_h,v_h,\Omega)<\infty \quad\Rightarrow\quad u\in GSB_nV(\Omega)\ \text{and}\ v=1.$$

(b) *For every $u\in GSB_nV(\Omega)$ such that $E(u,1,\Omega)<\infty$ and $\mathcal{M}^{*m-n}(S_u)=\mathscr{H}^{m-n}(S_u)$, there exists a sequence $\big((u_h,v_h)\big)\subset Y(\Omega)$ such that $(u_h,v_h)\to(u,1)$ in $X(\Omega)$ and*

$$\limsup_{h\to\infty} E_h(u_h,v_h,\Omega)\leq E(u,1,\Omega).$$

Note that in particular the restrictions of E_h and E to the subspace

$$Z(\Omega)=\{u\in GSB_nV(\Omega):\mathcal{M}^{*m-n}_\Omega(S_u)=\mathscr{H}^{m-n}(S_u)\}\times B(\Omega)$$

satisfy (with the convergence (3.4))

$$\Gamma-\lim_h E_h\big|_{Z(\Omega)}=E\big|_{Z(\Omega)}.$$

We start the analysis on the whole family of energies (E_h) by proving that at a fixed positive scale ε_h the functional E_h has a minimizer in $Y(\Omega)$, once we assign suitable Dirichlet or Neumann boundary conditions.

Theorem 3.8. *Let $C\geq 0$ and $h\in\mathbb{N}$ be fixed. The sets*

$$\big\{(u,v)\in Y(U): u=\phi \text{ in } U\setminus\Omega,\ E_h(u,v,U)\leq C\big\};\tag{3.5}$$

with U a neighborhood of $\overline{\Omega}$, $p^*>s$ and $\phi\in GSB_nV(\Omega)$; and

$$\Big\{(u,v)\in Y(\Omega): E_h(u,v,\Omega)+\int_\Omega |u-g|^r\,dx\leq C\Big\}\tag{3.6}$$

with $g\in L^r(\Omega,\mathbb{R}^n)$ and $r>s$, are compact subsets of $X(\Omega)$.

Proof. Recall that it is sufficient to check sequential compactness, since (3.5) and (3.6) are subsets of the metric space $X(\Omega)$. As the product of two precompact spaces is precompact, we can examine separately the bounds on u and v:

$$\int_U |\nabla u|^p \, dx \le C, \qquad MM_h(v,\Omega) \le C.$$

Concerning u the gradients ∇u are bounded in L^p, and since

$$\|\nabla u - \nabla\phi\|_{L^p(U,\mathbb{R}^{n\times m})} \quad \text{and} \quad \|u-\phi\|_{W^{1,p}(U,\mathbb{R}^n)}$$

are equivalent, by Sobolev embedding the set of $u-\phi$'s is precompact in L^s, and so is the set of u's since $\phi \in L^s$. Similarly in the Neumann problem the L^p gradient bound and the L^r bound on u give precompactness in every Lebesgue space of exponent strictly smaller than $\max\{r,p^*\}$, in particular in L^s. Clearly the constraint $u=\phi$ outside Ω in (3.5) is preserved. To get compactness for v we can apply Young's inequality $ab \le \frac{a^s}{s} + \frac{b^t}{t}$ with $s=\frac{q}{n}$ and $t=\frac{q}{q-n}$ to the two integrand addenda:

$$\begin{aligned} MM_h(v,\Omega) &= \int_\Omega \varepsilon_h^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon_h^n}\,dx \ge \\ &\ge \int_\Omega \left(\frac{q}{n}\varepsilon_h^{q-n}|\nabla v|^q\right)^{\frac{n}{q}} \left(\frac{q}{q-n}\varepsilon_h^{-n}W(1-v)\right)^{\frac{q-n}{q}} \, dx = \\ &= c_{n,q}\int_\Omega |\nabla v|^n W(1-v)^{\frac{q-n}{q}}\,dx = c'_{n,q}\int_\Omega \left|\nabla[F(1-v)]\right|^n dx, \end{aligned} \tag{3.7}$$

with $F(t)=\int_0^t W^{\frac{q-n}{qn}}(s)ds$. Since Ω is bounded and $0\le v\le 1$ we can use the compact embedding $W^{1,n}(\Omega)\hookrightarrow L^n(\Omega)$ to deduce that the set of $F(1-v_h)$'s is precompact in L^n, hence the set of v's is precompact for the convergence in measure topology since F has a continuous inverse. It remains to prove the closedness of (3.5) and (3.6): this is equivalent to show the respective energies being lower semicontinuous. Suppose then $\big((u_i,v_i)\big)\subset X(\Omega)$ a convergent sequence and h fixed. The phase transition term MM_h is clearly lower semicontinuous (see the proof of Proposition 4.1); so are also $\int_\Omega |\nabla u|^p$ and $\int_\Omega |u-g|^r$. Moreover since $k_h>0$

$$\int_\Omega |M_n\nabla u_i|^\gamma\,dx \le \frac{C}{k_h} < \infty,$$

therefore up to subsequences we have $Ju_i \overset{*}{\rightharpoonup} Ju$, and by Theorem 2.8 $Ju \ll \mathscr{L}^m$, thus $u\in R_n(\Omega)$. Furthermore $M_n\nabla u_i \rightharpoonup M_n\nabla u$ weakly in L^1: we claim that

$$\int_\Omega (v(x)+k_h)|M_n\nabla u(x)|^\gamma\,dx \le \liminf_i \int_\Omega (v_i(x)+k_h)|M_n\nabla u_i(x)|^\gamma\,dx.$$

In fact following [29, Theorem 4.4], since $v_i\to v$ in measure for every $\delta>0$ there exists $G\Subset\Omega$ compact such that $v_i\to v$ uniformly in G, v and $M_n\nabla u$ are continuous in G and $\int_G(v+k_h)|M_n\nabla u|^\gamma dx \ge \int_\Omega(v+k_h)|M_n\nabla u|^\gamma dx - \delta$. Therefore

$$\begin{aligned} \liminf_i \int_\Omega (v_i(x)+k_h)|M_n\nabla u_i(x)|^\gamma dx &\ge \liminf_i \int_G (v_i+k_h)|M_n\nabla u|^\gamma dx \\ &+ \liminf_i \int_G \gamma(v+k_h)|M_n\nabla u|^{\gamma-2}\langle M_n\nabla u, M_n\nabla u_h - M_n\nabla u\rangle dx \\ &+ \liminf_i \int_G \gamma(v_i-v)|M_n\nabla u|^{\gamma-2}\langle M_n\nabla u, M_n\nabla u_h - M_n\nabla u\rangle dx : \end{aligned}$$

The first integral tends to $\int_G (v+k_h)|M_n\nabla u|^\gamma dx$ by uniform convergence; the second integral is infinitesimal by weak convergence, the term $\gamma(v+k_h)|M_n\nabla u|^{\gamma-2}M_n\nabla u$ being bounded; finally the last addendum can be bounded by

$$\gamma\|M_n\nabla u_h - M_n\nabla u\|_{L^1(\Omega)}\|M_n\nabla u\|^{\gamma-1}_{L^\infty(G)}\sup_G |v_i - v|$$

which is infinitesimal by uniform convergence. Therefore we can bound below the lower limit with $\int_\Omega (v+k_h)|M_n\nabla u|^\gamma dx - \delta$: letting $\delta\downarrow 0$ we obtain the claimed property. □

In particular the previous Theorem guarantees that the energies (E_h) are equicoercive, because by (3.7) the compactness of the set of v's is obtained independently of h. As a consequence the functionals satisfy condition (b) of Proposition 3.1, validating the choice of the topology 3.3 in the Γ-limit.

4. Optimal profile

In order to investigate the asymptotic behaviour of the functionals E_ε it is useful to understand the behaviour of the Modica-Mortola term, to single out the optimal profile and to study its properties. We consider the fixed scale $\varepsilon = 1$.

Proposition 4.1. *We define, for* $f\in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$I(f) = \int_{\mathbb{R}^n} |\nabla f|^q + W(f)\,dx.$$

The infimum

$$\sigma = \inf\left\{I(f)\,\middle|\,I(f)<\infty,\ f(0)=1\right\} \tag{4.1}$$

is meaningful, positive and attained by a unique radial function $w_0\in B(\mathbb{R}^n)\cap C^{0,\alpha}(\mathbb{R}^n)$, *with* $\alpha = 1-\frac{n}{q}$, *satisfying:*

$$\lim_{x\to\infty} w_0(x) = 0. \tag{4.2}$$

Proof. First of all it is important to specify that we implicitly set $I(f)=\infty$ whenever f does not possess weak derivatives in L^1_{loc}; moreover since $q>n$ the constraint requirement $f(0)=1$ in the minimization problem is meaningful, because the Sobolev embedding Theorem (see [2], 4.12) ensures that a function f with $I(f)<\infty$ has a pointwise continuous representative. We will always consider the continuous representative, without specifying it anymore. Observe furthermore that since W is increasing in $\mathbb{R}^+$ and nonnegative, by truncation we can reduce to minimize the energy among functions in $B(\mathbb{R}^n)$ which are ranging in the interval $[0,1]$. Take a minimizing sequence (f_h): again by Sobolev embedding Theorem the functions (f_h) are uniformly Hölder continuous, and equibounded on every compact subset thanks to the constraint $f_h(0)=1$. Hence by the Ascoli-Arzelà Theorem the sequence is precompact in the topology of the local uniform convergence, and we can extract a subsequence converging to $w_0\in C^{0,\alpha}$ locally uniformly. Hence $w_0(0)=1$, $W(f_h)\to W(w_0)$ locally uniformly and it is not difficult to check that $\nabla f_h \rightharpoonup \nabla w_0$ in L^q_{loc}. By lower semicontinuity w_0 achieves the infimum. Moreover a radial monotone rearrangement decreases the energy (see [32, 41]) and by the strict convexity of the gradient part there is only one minimizer, w_0, and it is radial. Hölder continuity forces w_0 to be positive on a small ball around 0 implying that the minimum energy σ is strictly positive; for the same reason, since $\int W(w_0)<\infty$, equation (4.2) must be satisfied. □

Observe that $I(f) = MM_1(1-f,\mathbb{R}^n)$ for $f\in B(\mathbb{R}^n)$. As our optimal function w_0 is radial it is worth investigating its one dimensional profile. Setting $w:[0,\infty)\to\mathbb{R}$, $w(|x|)=w_0(x)$ we have:

$$\sigma = \int_{\mathbb{R}^n} |\nabla w_0|^q + W(w_0)\,dx = \mathscr{H}^{n-1}(S^{n-1})\int_0^\infty t^{n-1}\left[|w'(t)|^q + W(w(t))\right]dt, \tag{4.3}$$

and the Euler-Lagrange equation in $\mathbb{R}^n \setminus \{0\}$ is

$$-q\Delta_q w_0 + W'(w_0) := -q\,\mathrm{div}(|\nabla w_0|^{q-2}\nabla w_0) + W'(w_0) = 0.$$

In radial coordinates it becomes

$$-\frac{q}{t^{n-1}}\left(t^{n-1}|w'(t)|^{q-2}w'(t)\right)' + W'(w) = 0 \tag{4.4}$$

outside the origin. We have the following Lemma:

Lemma 4.2. *Let $w : [0,\infty) \to \mathbb{R}$ be the profile of the minimizer of (4.1). Then w is convex, belongs to $C^1(0,+\infty) \cap C^2(\{0 < w < 1\})$ and the following two properties hold:*

$$\lim_{t\to 0} t^n |w'(t)|^q = 0, \tag{4.5}$$

$$\lim_{t\to +\infty} t^n[|w'(t)|^q + W(w(t))] = 0. \tag{4.6}$$

Proof. Since w is nonnegative and decreasing, and $W' \geq 0$ by convexity, the Euler equation implies that

$$0 \leq t^{n-1}W'(w) = q\left(t^{n-1}|w'(t)|^{q-2}w'(t)\right)' = -q(t^{n-1}|w'(t)|^{q-1})'.$$

Both the functions $t^{n-1}|w'(t)|^{q-1}$ and $\frac{1}{t^{n-1}}$ are positive and decreasing. Hence multiplying them we get that $|w'|$ decreases, and since w' is negative we obtain that w is convex. By monotonicity of $|w'|$ and the finiteness of the energy (4.3),

$$\limsup_{t\to 0} t^n|w'(t)|^q \leq \limsup_{t\to 0} n\int_0^t s^{n-1}|w'(s)|^q ds = 0.$$

Furthermore, since $Z(t) := |w'(t)|^q + W(w(t))$ is decreasing, we have

$$\limsup_{t\to\infty} \frac{1}{n}\left(1 - \frac{1}{2^n}\right)t^n Z(t) \leq \limsup_{t\to\infty} \int_{\frac{t}{2}}^t s^{n-1}Z(s)\,ds \leq \limsup_{t\to\infty} \int_{\frac{t}{2}}^\infty s^{n-1}Z(s)\,ds = 0$$

by the finiteness of the energy (4.3), which proves (4.6). Finally in every interval $(a,b) \Subset \{0 < w < 1\}$ we have that $-\infty < w' < w'(b) < 0$, otherwise w would be a positive constant in the half line $(b,+\infty)$. Hence we can extract the $(q-1)$-st root without loosing any smoothness and bootstrap (4.4):

$$w \in C^{0,\alpha}(a,b) \quad \Rightarrow \quad W'(w) \in C^0(a,b) \quad \Rightarrow \quad w \in C^2(a,b).$$

The same argument shows that $w \in C^1(0,+\infty)$, since $|\cdot|^{\frac{1}{q-1}}$ is continuous. □

In general if $W \in C^k$ and $w \in C^m(a,b)$ then $W'(w) \in C^{m\wedge(k-1)}(a,b)$, which gives $w \in C^{(m\wedge(k-1))+2}(a,b)$: therefore starting from $m = 1$ we obtain $w \in C^{k+1}(\{0 < w < 1\})$.

5. Γ-lower limit

In this section we aim to prove the first part of Theorem 3.7, regarding the Γ-lower limit of the sequence (E_h):

Theorem 5.1. *Let Ω be an open subset of $\mathbb{R}^m$. For every sequence $\big((u_h,v_h)\big) \subset Y(\Omega)$ such that $(u_h,v_h) \to (u,v)$ we have*

$$\liminf_{h\to\infty} E_h(u_h,v_h,\Omega) \geq E(u,v,\Omega);$$

moreover

$$\liminf_{h\to\infty} E_h(u_h,v_h,\Omega) < \infty \quad \Rightarrow \quad u \in GSB_nV(\Omega) \text{ and } v = 1.$$

The proof will be achieved through a slicing argument, by first proving that in codimension $m - n = 0$ the jacobians Ju_h concentrate around a finite number of points. Our definition of size outlined in the introduction is well-suited to this slicing procedure, and a final localization result yields the proof.

5.1. **Proof in $\mathbb{R}^n$**

Let A be an open subset of $\mathbb{R}^n$: to ease the exposition for any $(u,v) \in Y(A)$ we let

$$G_h(u,v,A) = \int_A (v+k_h)|\det \nabla u|^\gamma dx + \int_A \varepsilon_h^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon_h^n}dx$$

be the part of energy depending explicitly on v.

Theorem 5.2. *Let A be an open subset of $\mathbb{R}^n$ and let $((u_h,v_h)) \subset Y(A)$, $(u,v) \in X(\Omega)$ satisfy $(u_h,v_h) \to (u,v)$ and $\|\nabla u_h\|_p \le C$. Assume also*

$$\liminf_{h\to\infty} G_h(u_h,v_h,A) < \infty. \tag{5.1}$$

Then $u \in GSB_nV(A,\mathbb{R}^n)$, $v = 1$ and

$$\liminf_{h\to\infty} G_h(u_h,v_h,A) \ge \int_A |\det \nabla u|^\gamma dx + \sigma \mathscr{H}^0(A \cap S_u).$$

First of all we extract a subsequence, not relabeled, that achieves the lower limit in (5.1) and such that $\nabla u_h \rightharpoonup \nabla u$ weakly in L^p. We notice right away that $u \in W^{1,p}$ and $v = 1$; also by (2.4) we know that $\mathbf{F}(Ju_h - Ju) \to 0$, hence $Ju_h \overset{*}{\rightharpoonup} Ju$ as currents. We begin with the regular part, disregarding the positive infinitesimal k_h:

Lemma 5.3. *Assume that A is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then*

$$\liminf_{h\to\infty} \int_A v_h|\det \nabla u_h|^\gamma dx \ge \int_A |\det \nabla u|^\gamma dx. \tag{5.2}$$

Proof. Since A is regular and bounded, $q > n$ and the norms $\|\nabla v_h\|_q$ are equibounded by Sobolev embedding Theorem

$$[v_h]_{C^\alpha(A)} \le C(A)\varepsilon_h^{-\alpha},$$

where $\alpha = 1 - \frac{n}{q}$ and $C(A)$ depends on the energy and on the regularity of A. We also fix a threshold $t \in (0,1)$: by Hölder continuity there exists $c_0 = c_0(C,t) > 0$ independent of h such that for every $x \in A \cap \{v_h < t\}$

$$A \cap B(x, c_0\varepsilon_h) \subset A \cap \left\{ v_h < \frac{1+t}{2} \right\}. \tag{5.3}$$

We can then cover $A \cap \{v_h < t\}$ with balls centered at every point having radius $\frac{c_0\varepsilon_h}{5}$: by Vitali's covering Lemma there is a countable disjoint subfamily $\mathcal{F} = \left\{B(x_i, \frac{c_0\varepsilon_h}{5})\right\}$ such that

$$\bigcup_i B(x_i, c_0\varepsilon_h) \supset A \cap \{v_h < t\}.$$

Thanks to (5.3) we can estimate from below MM_{ε_h} of every such small ball:

$$\int_{A\cap B\left(x_i,\frac{c_0\varepsilon_h}{5}\right)} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx \ge W\left(\frac{1-t}{2}\right) \frac{\mathscr{L}^n\left(A \cap B\left(x_i, \frac{c_0\varepsilon_h}{5}\right)\right)}{\varepsilon_h^n}.$$

The latter quantity is bounded below independently of h because the Lipschitz boundary condition on A ensures that $\mathscr{L}^n\left(A \cap B\left(x_i, \frac{c_0\varepsilon_h}{5}\right)\right) \ge c\varepsilon_h^n$. The family $\mathcal{F}$ being disjoint, by the finiteness of the energy we argue that there can be only a finite number N, independent of h, of such balls. Let us then extract a subsequence, not relabeled, along which the balls are in constant number N and the centers $\{x_i^h\}, i = 1,\dots,N$ converge to points $x_i^\infty \in \overline{A}$. For every open set

$$A' \Subset A \setminus \bigcup_i \{x_i^\infty\}$$

we have that for h sufficiently large:

$$A' \cap \bigcup_i B(x_i^h, c_0\varepsilon_h) = \emptyset \qquad \text{and} \qquad v_h\big|_{A'} \geq t.$$

The energy bound (5.1) allows to bound a superlinear power of the jacobians in A'

$$\int_{A'} |\det \nabla u_h|^\gamma \, dx \leq \frac{C}{t+1}$$

hence Theorem 2.8 gives

$$\det \nabla u_h \rightharpoonup \det \nabla u \quad \text{weakly in } L^1(A'). \tag{5.4}$$

By lower semicontinuity

$$\liminf_{h\to\infty} \int_A v_h |\det \nabla u_h|^\gamma dx \geq \liminf_{h\to\infty} \int_{A'} v_h |\det \nabla u_h|^\gamma dx \geq$$
$$\geq \liminf_{h\to\infty} t \int_{A'} |\det \nabla u_h|^\gamma dx \geq t \int_{A'} |\det \nabla u|^\gamma dx. \tag{5.5}$$

Finally letting $A' \uparrow A \setminus \bigcup_i \{x_i^\infty\}$ and then $t \uparrow 1$ we obtain the result. □

Remark 5.4. The same result of Lemma 5.3 holds without the regularity hypothesis on A. In fact it is sufficient to consider a sequence of nested regular open subsets $A_j \subset A$ invading A, apply the Lemma to A_j and then let $A_j \uparrow A$: the left hand side of (5.2) clearly decreases when restricted to each A_j, and the right hand side by Monotone convergence Theorem increases to $\int_A |\det \nabla u|^\gamma dx$.

Now we analyze the MM_ε term, and prove that around the potentially singular points of the limit function u this energy concentrates. Observe that we still do not know that $u \in GSB_nV$: Ju so far is only a flat current, nevertheless chosen a fixed point x_0 for almost every radius ρ the restriction $Ju \llcorner B_\rho(x_0)$ is meaningful and furthermore $\mathbf{F}(Ju_h \llcorner B_\rho(x_0) - Ju \llcorner B_\rho(x_0)) \to 0$ (see [9, Section 2.2]). With a slight abuse of notation we indicate by $Ju \llcorner B_\rho \ll \mathscr{L}^n$ the fact that $\mathbf{M}(Ju \llcorner B_\rho) < \infty$ and $\|Ju \llcorner B_\rho\| \ll \mathscr{L}^n$: by definition this is satisfied if $u \in R_n$.

Lemma 5.5. *Let $((u_h, v_h))$, u and A as in Theorem 5.2, and fix $x_0 \in A$. Suppose $Ju \llcorner B_\rho(x_0) \not\ll \mathscr{L}^n$ for every $\rho > 0$ such that $B_\rho(x_0) \subset A$. Then*

$$\liminf_{h\to\infty} MM_h(v_h, B_\rho(x_0)) \geq \sigma \quad \forall \rho > 0, \tag{5.6}$$

where σ is defined as in (4.1).

Proof. Fix an arbitrary ρ as in the hypotheses and let us suppose for simplicity that $x_0 = 0$: since $Ju \llcorner B_\rho \not\ll \mathscr{L}^n$ we must have

$$\lim_{h\to\infty} \inf_{B_\rho} v_h = 0 \quad \forall \rho > 0.$$

In fact if there were a radius $\bar\rho$ and a subsequence $(v_{\bar h})$ bounded below by some $\delta > 0$ in $B_{\bar\rho}$, we would have the uniform bound $\sup_{\bar h} \int_{B_{\bar\rho}} |\det \nabla u_{\bar h}|^\gamma \, dx \leq C\delta^{-1}$. Therefore we could apply Theorem 2.8 with $\Psi(t) = |t|^\gamma$: since $Ju_{\bar h} = \det \nabla u_{\bar h} \mathbf{E}^m \ll \mathscr{L}^m$ by the weak L^1 convergence (2.6) the limit $Ju \llcorner B_{\bar\rho}$ would otherwise be a current in $\mathbf{M}_0(B_{\bar\rho})$ with absolutely continuous mass. The finiteness of the energy (5.1) guarantees that $v_h \to 1$ in measure in B_ρ. In order to show (5.6) we modify in B_ρ the asymptotic profiles v_h and we relate them to problem (4.1). Let us perform the following radial monotone rearrangement of v_h, denoted v_h^*, which preserve the measure of *sublevels*:

$$v_h^*(x) := \inf \left\{ t : |\{v_h < t\} \cap B_\rho| > \mathscr{L}^n(B_1)|x|^n \right\} \quad \forall x \in B_\rho.$$

This rearrangement preserves the integral $\int_{B_\rho} W(1-v)$ by the Coarea formula (2.13), while the L^q norm of the gradient decreases, see [32, 41]. We immediately have that

$$MM_h(v_h, B_\rho) \geq MM_h(v_h^*, B_\rho) \quad \text{and} \quad v_h^* \to 1 \text{ in measure in } B_\rho.$$

In particular $\lambda_h := v_h^*\big|_{\partial B_\rho} \to 1$, and $\mu_h := \inf_{B_\rho} v_h^* = v_h^*(0) \to 0$, hence we can extend v_h^* equal to λ_h for $|x| \geq \rho$. The functions $f_h : \mathbb{R}^n \to \mathbb{R}$,

$$f_h(y) := \frac{1}{\lambda_h - \mu_h}\big(v_h^*(\varepsilon_h y) - \mu_h\big) \tag{5.7}$$

satisfy

(a) $f_h(0) = 0$,
(b) spt $(1 - f_h) \subset \overline{B_\rho}$,
(c) $1 - v_h^*(x) \geq \lambda_h - v_h^*(x) = (\lambda_h - \mu_h)\big(1 - f_h(\frac{x}{\varepsilon_h})\big)$.

Let us now evaluate the MM_h energy (recall W is monotone increasing):

$$\begin{aligned} MM_h(v_h, B_\rho) \geq MM_h(v_h^*, B_\rho) &= \int_{B_\rho} \varepsilon_h^{q-n}|\nabla v_h^*(x)|^q + \frac{W(1 - v_h^*(x))}{\varepsilon_h^n}\,dx \\ &\overset{(c)}{\geq} \int_{B_\rho} \varepsilon_h^{q-n}|\nabla v_h^*(x)|^q + \frac{W(\lambda_h - v_h^*(x))}{\varepsilon_h^n}\,dx \\ &\geq \int_{\mathbb{R}^n} (\lambda_h - \mu_h)^q|\nabla f_h|^q + W((\lambda_h - \mu_h)(1 - f_h))dx. \end{aligned} \tag{5.8}$$

By properties (a) and (b) the functions $1 - f_h$ are competitors for problem (4.1) and $\lambda_h - \mu_h \to 1$, hence the last integral is asymptotically greater or equal than the infimum σ. □

Proof of Theorem 5.2. Let $\Sigma = \{x \in A : Ju \llcorner B_\rho(x) \not\ll \mathscr{L}^n \text{ for all } B_\rho(x) \subset A\}$. Then the superadditivity of the lim inf together with (5.1) and Lemma 5.5 gives

$$\mathscr{H}^0(\Sigma) \leq \frac{1}{\sigma} \liminf_h G_h(u_h, v_h, A).$$

Moreover Lemma 5.3 showed the existence of another finite set Υ such that $Ju \llcorner (A \setminus \Upsilon) \ll \mathscr{L}^n$. Hence necessarily $\Sigma \subset \Upsilon$ and the flat defect current

$$T := (Ju - \det \nabla u\, \mathbf{E}^m) \llcorner A$$

is supported in Υ. By the general theory of flat currents (see [9, Theorem 3.4], [24, 4.1.18]) $\mathbf{M}(T) < \infty$ and $T = \sum_{x_i \in \Sigma} a_i [\![x_i]\!]$. In particular $u \in GSB_nV(A)$ and $S_u \subset \Sigma$, so

$$\mathscr{H}^0(S_u \cap A) \leq \mathscr{H}^0(\Sigma).$$

Taking $B \Subset A \setminus \Sigma$ open, and applying the superadditivity of the lower limit on open disjoint sets, as well as (5.2) to B we obtain for some ρ sufficiently small (so that $B \cap \bigcup_{x\in\Sigma} B_\rho(x) = \emptyset$)

$$\begin{aligned} \liminf_h G_h(u_h, v_h, A) &\geq \liminf_h \int_B v_h |\det \nabla u_h|^\gamma dx \\ &\quad + \sum_{x \in S_u \cap A} \liminf_h \int_{B_\rho(x)\cap A} \varepsilon_h^{q-2}|\nabla v_h|^q + \frac{W(1 - v_h)}{\varepsilon_h^2} dx \\ &\geq \int_B |\det \nabla u|^\gamma dx + \sigma \mathscr{H}^0(S_u \cap A). \end{aligned}$$

Letting $B \uparrow A$ concludes the proof. □

5.2. Reduction argument and proof of Theorem 5.1 for general m, n

In this paragraph we prove Theorem 5.1 from the results obtained in the previous paragraph in dimension n. We will first use the slicing properties of the jacobians to reduce to the n-dimensional case discussed above, and then we will optimize the choices of the slicing directions to conclude.

Proof. As a preliminary step let us extract a subsequence out of $((u_h, v_h))$ such that the lower limit $\liminf_h E_h(u_h, v_h, \Omega)$ is attained and such that $(u_h, v_h) \to (u, 1)$ rapidly in $X(\Omega)$:

$$\sum_h \|u_h - u\|_{L^s} + d(v_h, 1) < \infty.$$

This implies that given an orthogonal projection $\pi \in \mathbf{O}_{m-n}$, for $\mathscr{L}^{m-n}$-almost every $x \in \pi(\Omega)$

$$u_h(x,\cdot) \to u(x,\cdot) \text{ in } L^s(\Omega^x, \mathbb{R}^n) \qquad \text{and} \qquad v_h(x,\cdot) \to 1 \text{ in measure in } \Omega^x,$$

where we put $\Omega^x := \Omega \cap \pi^{-1}(x)$. Let us consider an arbitrary open subset $A \subset \Omega$ and let us fix a projection π as above. Observe that the energy E_h is bounded along (u_h, v_h): using Fatou's Lemma we obtain

$$\lim_{h\to\infty} E_h(u_h, v_h, \Omega) \geq \liminf_{h\to\infty} E_h(u_h, v_h, A) \geq$$
$$\geq \int_{\pi(A)} \liminf_{h\to\infty} \left\{ \int_{A^x} |\nabla_y u_h|^p + v_h |\det \nabla_y u_h|^\gamma + \varepsilon_h^{q-n} |\nabla_y v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n} dy \right\} dx.$$

In particular for $\mathscr{L}^{m-n}$-almost every $x \in \pi(A)$

$$\liminf_h \int_{A^x} |\nabla_y u_h(x,y)|^p \, dy + G_h(u_h, v_h, A^x) \leq C(x) < \infty.$$

For these x we can extract a subsequence $(u_{h(k)})$, a priori depending on the point x, along which both the L^p norm of $\nabla u_{h(k)}$ and the n-dimensional energy G_h are bounded:

$$\sup_k \int_{A^x} |\nabla_y u_{h(k)}|^p \, dy + G_h(u_{h(k)}, v_{h(k)}, A^x) < \infty. \tag{5.9}$$

This implies that

$$\nabla_y u_{h(k)}(x,\cdot) \rightharpoonup \nabla_y u(x,\cdot) \text{ in } L^p(A^x). \tag{5.10}$$

Finally observe that the slicing Theorem 2.3 immediately gives that $(u_h(x,\cdot), v_h(x,\cdot)) \in Y(A^x)$ almost everywhere. Theorem 5.2 implies that $u(x,\cdot) \in GSB_nV(A^x)$ and that

$$\liminf_h G_h(u_h, v_h, A^x) \geq \int_{A^x} |\det \nabla_y u(x,\cdot)|^\gamma dy + \sigma \mathscr{H}^0(A^x \cap S_{u(x,\cdot)});$$

integrating and applying Fatou's Lemma on the left hand side we have

$$\liminf_h E_h(u_h, v_h, A) \geq \int_A |\nabla u|^p + \int_{\pi(A)} \liminf_h G_h(u_h(x,\cdot), v_h(x,\cdot), A^x) \, dx \geq$$
$$\geq \int_A |\nabla u|^p + \int_A |\det \nabla_y u(x,\cdot)|^\gamma dy dx + \sigma \int_{\pi(A)} \mathscr{H}^0(A^x \cap S_{u(x,\cdot)}) dx.$$

Let us call

$$\tau_\pi(A) := \int_A |\det \nabla_y u(x,y)|^\gamma dy dx + \sigma \int_{\pi(A)} \mathscr{H}^0(A^x \cap S_{u(x,\cdot)}) dx.$$

the right hand side and $\underline{E}(A) = \liminf_h E_h(u_h, v_h, A)$. $\underline{E}(\cdot)$ is a superadditive set function on open sets such that $\underline{E}(A) \leq \underline{E}(\Omega) < \infty$ and each single τ_π is a finite Borel measure; therefore taking disjoint open sets $A_1, \ldots, A_k$ and orthogonal projections $\pi_1, \ldots, \pi_k$ we have that

$$\sum_i \tau_{\pi_i}(A_i) \leq \sum_i \underline{E}(A_i) \leq \underline{E}(\Omega). \tag{5.11}$$

By inner and outer regularity of τ_{π_i} inequality (5.11) holds for generic disjoint Borel sets B_i instead of A_i, hence the supremum

$$\tau := \bigvee_{\pi} \tau_{\pi} \tag{5.12}$$

is a finite Borel measure. In particular $M_n\nabla u \in L^{\gamma}$ and since for every projection π slice and jacobian commute according to Theorem 2.3, we have that the current $T = (Ju - M_n\nabla u\,\mathbf{E}^m)\llcorner\Omega$ satisfies $\operatorname{spt}(\langle T,\pi,x\rangle) \subset S_{u(x,\cdot)}$ almost everywhere, so its size is finite. Hence $u \in GSB_nV(\Omega)$. Finally since by Definition 2.4 the measures μ_T and $|M_n\nabla u|^{\gamma}\mathscr{L}^m$ are mutually singular and $|M_nL| = \sup_{\pi}|M_nL\llcorner d\pi|$, it is not difficult to prove that the supremum τ equals

$$\tau = |M_n\nabla u|^{\gamma}\mathscr{L}^m + \sigma\mathscr{H}^{m-n}\llcorner S_u,$$

which concludes the proof. □

6. Γ-upper limit

This section is devoted to the proof of the upper limit inequality: our construction of the recovery sequence will mimic the truncation argument presented in [12] and [13]. Note that we only assume a mild geometric property on the singular set S_u expressed in terms of its Minkowski content. We provide an interior statement as well as boundary statement, where differently from [12] we need to take care of any possible accumulation of the singular set at the boundary. The limit energy must account for the possible loss of mass in the Modica-Mortola term, due to the transition of v happening partially outside the domain. We finally generalize the form of the functional in which the size term is weighted by a continuous density.

Theorem 6.1. *Suppose $\Omega \subset \mathbb{R}^m$ is a bounded set of class C^1 and $u \in GSB_nV(\Omega)$ with constraints*

$$s \geq \frac{np}{n-p}, \qquad 1 < \gamma \leq \frac{1}{\frac{n-1}{p} + \frac{1}{s}}.$$

Let also (k_h) be a positive sequence such that $k_h = o(\varepsilon_h^{\gamma})$. If

$$E(u,1,\Omega) < \infty, \qquad \mathcal{M}^{*m-n}_{\Omega}(S_u) = \mathscr{H}^{m-n}(S_u) \tag{6.1}$$

then there exists a sequence $\big((u_h,v_h)\big) \subset Y(\Omega)$ such that

$$(u_h,v_h) \to (u,1) \quad \text{and} \quad \limsup_{h\to\infty} E_h(u_h,v_h,\Omega) \leq E(u,1,\Omega).$$

6.1. Proof of Theorem 6.1

We start by setting the approximating sequence (v_h) for a generic $\mathscr{L}^m$-null set $S \subset \Omega$ satisfying

$$\mathcal{M}^{*m-n}_{\Omega}(S) = \mathscr{H}^{m-n}(S) < \infty. \tag{6.2}$$

Let $w(t)$ be the optimal profile of problem 4.1 and choose $\delta_h \downarrow 0$ such that $k_h(\varepsilon_h\delta_h)^{-\gamma} \to 0$. Let

$$w_h(t) := \min\left\{\frac{w(t)}{w(\delta_h)}, 1\right\}$$

so that $w_h(|x|) = 1$ in $B_{\delta_h}(0)$. Clearly $w_h'(\delta_h)$ is finite and $I(w_h) \to I(w)$ as $h \to \infty$. Moreover by the proof of Lemma 4.2

$$|w'(t)|^q + W(w(t))$$

is C^1 and decreases to 0 for $t \to \infty$: these properties hold true in $(\delta_h + \infty)$ for w_h. Set

$$v_h(x) = 1 - w_h\left(\frac{d(x,S)}{\varepsilon_h}\right), \tag{6.3}$$

where $d(x,S) = \mathrm{dist}(x,S)$: note that $v_h \to 1$ in measure and that by equation (2.12)

$$|\nabla v_h(x)| = \frac{1}{\varepsilon_h}\left|w_h'\left(\frac{d(x,S)}{\varepsilon_h}\right)\right|$$

at almost every point x. Recall the notation $S_r = \{x \in \Omega : 0 < \mathrm{dist}(x,S) \leq r\}$ and $V(r) = \mathscr{L}^m(S_r)$.

Proposition 6.2. *The functions* (v_h) *satisfy*

$$v_h = 0 \quad on \quad S_{\varepsilon_h\delta_h}$$

and

$$\limsup_{h\to\infty} MM_h(v_h,\Omega) \leq \sigma \mathcal{M}_\Omega^{*m-n}(S).$$

Proof. The first statement is true by the definition (6.3). Looking at the energy

$$MM_h(v_h,\Omega) = \int_\Omega \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx$$

we observe right away that the integration on the set $S_{\varepsilon_h\delta_h}$ in infinitesimal, since there v_h is identically 0 and so

$$\int_{S_{\varepsilon_h\delta_h}} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx = W(1)\frac{V(\varepsilon_h\delta_h)}{\varepsilon_h^n} \to 0.$$

Applying the Coarea formula (2.13) on the level sets of the distance function $d(\cdot,S)$ we can write

$$\begin{aligned} MM_h(v_h,\Omega) &= o(1) + \int_{\Omega\setminus S_{\varepsilon_h\delta_h}} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx \\ &= o(1) + \int_{\varepsilon_h\delta_h}^{+\infty} \left[\left|w_h'\left(\tfrac{t}{\varepsilon_h}\right)\right|^q + W\left(w_h\left(\tfrac{t}{\varepsilon_h}\right)\right)\right]\frac{V'(t)}{\varepsilon_h^n}dt \\ &= o(1) + \int_{\delta_h}^{+\infty} \left[|w_h'(s)|^q + W(w_h(s))\right]\frac{[V(\varepsilon_h s)]'}{\varepsilon_h^n}ds. \end{aligned}$$

Since $Z_h(s) := |w_h'(s)|^q + W(w_h(s))$ is C^1 we can integrate by parts

$$\int_{\delta_h}^{+\infty} Z_h(s)\frac{[V(\varepsilon_h s)]'}{\varepsilon_h^n}ds = -\int_{\delta_h}^{+\infty} Z_h'(s)\frac{V(\varepsilon_h s)}{\varepsilon_h^n}ds + \frac{Z_h(+\infty)V(+\infty) - Z_h(\delta_h)V(\varepsilon_h\delta_h)}{\varepsilon_h^n}.$$

As previously outlined $Z_h(+\infty) = 0$ and $V(+\infty) = \mathscr{L}^m(\Omega)$, hence the second addendum is null; moreover $\frac{V(\varepsilon_h\delta_h)}{\varepsilon_h^n} \leq (\mathcal{M}_\Omega^{*m-n}(S)+1)\mathscr{L}^n(B_1)\delta_h^n$ and $Z_h(\delta_h) = |w_h'(\delta_h)|^q + W(w_h(\delta_h))$, so also the third term goes to 0 by Lemma 4.2. The basic assumption (6.2) on the Minkowski content of S implies that there exist infinitesimal numbers ξ_h such that

$$V(s) \leq \mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S)s^n + \xi_h s^n \quad \forall s \in [0, \varepsilon_h \mathrm{diam}(\Omega)]. \tag{6.4}$$

Recall that $Z_h'(s) \le 0$ in $[\delta_h, \infty)$ and $I(w_h) \to I(w) = \sigma$. Then, integrating by parts

$$\begin{aligned}
MM_h(v_h, \Omega) &= o(1) - \int_{\delta_h}^{+\infty} Z_h'(s) \frac{V(\varepsilon_h s)}{\varepsilon_h^n} ds \\
&\le o(1) - \int_{\delta_h}^{+\infty} Z_h'(s)(\mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S) + \xi_h) s^n ds \qquad (6.5)\\
&\overset{(4.5)(4.6)}{=} o(1) + n(\mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S) + \xi_h) \int_{\delta_h}^{+\infty} s^{n-1} Z_h(s) ds \\
&= o(1) + (\mathscr{H}^{n-1}(S^{n-1})\mathcal{M}_\Omega^{*m-n}(S) + n\xi_h) \int_{\delta_h}^{+\infty} s^{n-1} Z_h(s) ds \\
&= o(1) + \mathcal{M}_\Omega^{*m-n}(S) \cdot I(w_h) = o(1) + \sigma \mathcal{M}_\Omega^{*m-n}(S).
\end{aligned}$$

□

Remark 6.3. Observe that the same Proposition proves something more general, that will be useful in the sequel: if $\bar{w}$ is a radial profile such that $\bar{Z}(t) := |\bar{w}'(t)|^q + W(\bar{w}(t))$ is decreasing, then the sequence (v_h) constructed from $\bar{w}$ as in (6.3) satisfies:

$$\limsup_h MM_h(v_h, \Omega) \le I(\bar{w}(|x|))\mathcal{M}_\Omega^{*m-n}(S).$$

We now show how to construct the sequence (u_h). Outside $S_{\varepsilon_h \delta_h}$ the jacobian Ju is absolutely continuous, hence there is no need to modify u there. We will only change u inside $S_{\varepsilon_h \delta_h}$ with the scope of keeping

$$\int_\Omega |\nabla(u - u_h)|^p \, dx$$

infinitesimal, and letting

$$\int_{S_{\varepsilon_h \delta_h}} |M_n \nabla u_h|^\gamma \, dx$$

diverge at a controlled rate, independently of the function u. Note that this is equivalent to show

$$E_h(u_h, v_h, S_{\varepsilon_h \delta_h}) = \int_{S_{\varepsilon_h \delta_h}} |\nabla u_h|^p + k_h |M_n \nabla u_h|^\gamma \, dx + W(1) \frac{\mathscr{L}^m(S_{\varepsilon_h \delta_h})}{\varepsilon_h^n} \to 0$$

for suitable k_h, because the last term is infinitesimal by (6.1). Suppose ϕ^1 is a smooth function. If we multiply only the first coordinate by ϕ^1 and compute the jacobian determinant we obtain

$$\nabla(\phi^1 u^1, u^2, \dots, u^n) = (\phi^1 \nabla u^1, \nabla u^2, \dots, \nabla u^n) + (u^1 \nabla \phi^1, \nabla u^2, \dots, \nabla u^n),$$

hence

$$J(\phi^1 u^1, u^2, \dots, u^n) = \phi^1 Ju + u^1 J(\phi^1, u^2, \dots, u^n) \qquad (6.6)$$

in the sense of currents; also the following pointwise estimate holds for $1 \le k \le n$:

$$\begin{aligned}
&|M_k \nabla(\phi^1 u^1, u^2, \dots, u^n)| \\
&\quad \le \left| \binom{n-1}{k} |M_k \nabla u|^2 + \binom{n-1}{k-1} (\|\phi^1\|_{L^\infty} |M_k \nabla u| + \|\nabla \phi^1\|_{L^\infty} |u^1| |M_{k-1} \nabla u|)^2 \right|^{\frac{1}{2}} \\
&\qquad \le c_{n,k} \big((1 + \|\phi^1\|_{L^\infty}) |M_k \nabla u| + \|\nabla \phi^1\|_{L^\infty} |u^1| |M_{k-1} \nabla u| \big).
\end{aligned}$$

Therefore if we truncate u by multiplying each component u^i by smooth functions ϕ^i which satisfy $\mathrm{spt}(\nabla\phi^i) \cap \mathrm{spt}(\nabla\phi^j) = \emptyset$ for $i \neq j$, we obtain that

$$\begin{aligned} &\phi \bowtie u := (\phi^1 u^1, \phi^2 u^2, \dots, \phi^n u^n) = 0 \qquad \text{in } \{\phi = 0\} = \bigcap_i \{\phi^i = 0\}, \\ &|M_k \nabla(\phi \bowtie u)| \leq c_{n,k}\big((1 + \|\phi\|_{L^\infty})|M_k \nabla u| + \|\nabla\phi\|_{L^\infty}|u||M_{k-1}\nabla u|\big) \end{aligned} \tag{6.7}$$

because at each point for only one index j the gradient row $\nabla(\phi^j u^j)$ will present the non zero extra term $u^j \nabla\phi^j$. Observe also that (6.6) implies that

$$\text{if} \quad S_u \Subset \{\phi = 0\} \qquad \text{then} \quad J(\phi^1 u^1, \dots, \phi^n u^n) \ll \mathscr{L}^m.$$

Finally note that if the supports of the gradients $\mathrm{spt}(\nabla\phi^j)$ overlap then the jacobian of $u \bowtie \phi$ will in general be bounded by the full vector of minors $M\nabla u$; however the particular choice where all ϕ^i's are equal restores the dependence of the bound only on the precedent order minor, since the choice of $\nabla\phi$ in two rows annihilates the minor.

Choose functions $\phi_h = (\phi_h^1, \dots, \phi_h^n)$ such that

- $0 \leq \phi_h^i \leq 1$;
- $\phi_h^i = 1$ outside $S_{(2^{-1}+2^{-i})\varepsilon_h\delta_h}$;
- $\phi_h^i = 0$ inside $S_{(2^{-1}+2^{-i-1})\varepsilon_h\delta_h}$;
- $|\nabla\phi_h^i| \leq 2^{i+2}(\varepsilon_h\delta_h)^{-1}$

and set $u_h := \phi_h \bowtie u$. Then clearly $(u_h, v_h) \in Y(\Omega)$; note also that $u_h \to u$ in L^s by dominated convergence. Moreover by the conditions on (ϕ_h^i), estimate (6.7) applied to $k = 1$ (with the convention $M_0\nabla u = 1$) reduces to $|\nabla u_h| \leq c_n(|\nabla u| + (\varepsilon_h\delta_h)^{-1}|u|)$ and yields

$$\begin{aligned} \int_\Omega |\nabla(u_h - u)|^p dx &\leq c_{n,p} \int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + c_{n,p}(\varepsilon_h\delta_h)^{-p} \int_{S_{\varepsilon_h\delta_h}} |u|^p dx \leq \\ &\leq c_{n,p} \int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + c_{n,p}(\varepsilon_h\delta_h)^{-p} \left(\int_{S_{\varepsilon_h\delta_h}} |u|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{n}} \mathscr{L}^m(S_{\varepsilon_h\delta_h})^{\frac{p}{n}} \\ &\leq c_{n,p} \int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + c_{n,p}\|u\|^p_{L^{\frac{np}{n-p}}(S_{\varepsilon_h\delta_h})} (1 + \mathcal{M}^{*m-n}(S_u))^{\frac{p}{n}}. \end{aligned} \tag{6.8}$$

Therefore u_h is close to u in $W^{1,p}$. Regarding the jacobian term, we have:

Proposition 6.4. *If $k_h(\varepsilon_h\delta_h)^{-\gamma} \to 0$ then*

$$\limsup_{h\to\infty} \int_\Omega (v_h + k_h)|M_n \nabla u_h|^\gamma \, dx = \int_\Omega |M_n \nabla u|^\gamma \, dx. \tag{6.9}$$

Proof. By construction $u_h = u$ outside $S_{\varepsilon_h\delta_h}$ and by Lebesgue dominated convergence Theorem

$$\int_{\Omega \setminus S_{\varepsilon_h\delta_h}} v_h |M_n \nabla u_h|^\gamma \, dx \to \int_\Omega |M_n \nabla u|^\gamma \, dx.$$

On the other hand inside $S_{\varepsilon_h\delta_h}$ v_h is identically zero, hence we are left with the estimate of $k_h \int_{S_{\varepsilon_h\delta_h}} |M_n \nabla u_h|^\gamma dx$. Thanks to (6.7) we know that

$$\begin{aligned} \int_{S_{\varepsilon_h\delta_h}} |M_n \nabla u_h|^\gamma dx \leq c_{m,n,\gamma}(1 + \|\phi_h\|_{L^\infty})^\gamma \int_{S_{\varepsilon_h\delta_h}} |M_n \nabla u|^\gamma \, dx \\ + c_{m,n,\gamma}\|\nabla\phi_h\|^\gamma_{L^\infty} \int_{S_{\varepsilon_h\delta_h}} |u|^\gamma |M_{n-1}\nabla u|^\gamma \, dx. \end{aligned}$$

The first term is infinitesimal by the absolute continuity of the integral. The second one can be estimated applying Hölder's inequality with exponents $\frac{s}{\gamma}$ and $\frac{p}{\gamma(n-1)}$: this can be done because

$$\frac{\gamma(n-1)}{p} + \frac{\gamma}{s} \leq 1.$$

Recalling Hadamard's inequality $|M_k \nabla u| \leq c_k |\nabla u|^k$ we get

$$\int_{S_{\varepsilon_h \delta_h}} |u|^\gamma |M_{n-1} \nabla u|^\gamma \, dx \leq c_k \|u\|^\gamma_{L^s(S_{\varepsilon_h \delta_h})} \|\nabla u\|^{\gamma(n-1)}_{L^p(S_{\varepsilon_h \delta_h})}.$$

Since $\|\nabla \phi_h\|^\gamma_{L^\infty} \leq c(\varepsilon_h \delta_h)^{-\gamma}$ our assumption on k_h allows to conclude. □

Putting Propositions 6.2, 6.4 and (6.8) together we conclude the proof of Theorem 6.1.

Remark 6.5. From the proof of Theorem 3.7 we deduce that

$$\liminf_h F_h(u_h, v_h, A) \geq F(u, A)$$

for every open set $A \subset \Omega$, and

$$\limsup_h F_h(u_h, v_h, \Omega) \leq F(u, \Omega).$$

This entails that $F_h(u_h, v_h, A) \to F(u, A)$ whenever $F(u, \partial A) = 0$ and $(u_h, v_h) \to (u, 1)$ with equibounded energies. Note that $A \mapsto F(u, A)$ is the restriction to open sets of an absolutely continuous measure, hence it does not charge the boundary of any regular open set.

6.2. Further observations

It is interesting to notice that the exponent γ is bounded above by $\frac{p}{n-1}$, in order for Theorem 6.1 to hold. There is however a trick allowing to overcome this bound, if we assume the Lagrangian to contain a nonlinear power of the full vector of minors $M\nabla u$:

$$|M \nabla u| = \left(\sum_{k=1}^n |M_k \nabla u|^2 \right)^{\frac{1}{2}}.$$

Retaining the structure of the size and phase transition terms as in Definition 3.6, the bulk energy

$$\tilde{F}(u, \Omega) = \int_\Omega |\nabla u|^p + |M \nabla u|^\gamma \, dx \tag{6.10}$$

can be approximated by

$$\tilde{F}_\varepsilon(u, v, \Omega) = \int_\Omega |\nabla u|^p + \Big| \sum_{k=1}^{n-1} |M_k \nabla u|^2 + (v + k_\varepsilon) |M_n \nabla u|^2 \Big|^{\frac{\gamma}{2}} \, dx. \tag{6.11}$$

Although $p > n - 1$ guarantees that the same approximation holds, we can observe the following: applying Minkowski's inequality to (6.11) we have

$$\begin{aligned} k_h^{\frac{\gamma}{2}} \int_{S_{\varepsilon_h \delta_h}} |M_n \nabla u_h|^\gamma &\leq C_\gamma k_h^{\frac{\gamma}{2}} \int_{S_{\varepsilon_h \delta_h}} |M_n \nabla u|^\gamma + |\nabla \phi_h|^\gamma |u|^\gamma |M_{n-1} \nabla u|^\gamma \, dx \\ &\leq C_\gamma (1 + k_h^{\frac{\gamma}{2}} \|\nabla \phi_h\|^\gamma_\infty \|u\|^\gamma_\infty) \tilde{F}(u, S_{\varepsilon_h \delta_h}). \end{aligned}$$

Again if k_h goes to 0 sufficiently fast then $k_h^{\frac{\gamma}{2}}\|\nabla\phi_h\|_\infty^\gamma \to 0$ and we get the Γ-upper limit statement, at least when $u \in L^\infty$. More generally the Lagrangian can feature different summability exponents on every order of the minors considered. In the model case

$$\tilde{\tilde{F}}(u,\Omega) := \int_\Omega |\nabla u|^p + \sum_{k=2}^n |M_k \nabla u|^{p_k}\, dx$$

Theorem 3.7 can be proved if we assume $p_n > 1$ and (here $p_1 = p$)

$$\frac{1}{s} + \frac{n-1}{p} \leq 1, \qquad \frac{1}{s} + \frac{1}{p_{k-1}} \leq \frac{1}{p_k}.$$

In particular if we impose $p < n$ to retain the possibility of Ju having a singular part, for the price of a very large s we can take the p_k's arbitrarily close to the threshold n.

7. Boundary constraints

In this section we analyse the behaviour of the previous Γ-convergence Theorems first when we compute the energy on subsets of the domain and then when we impose a boundary condition for u at $\partial\Omega$ to be preserved by the approximating sequence. We start by applying the "free" version of the Theorem and combine it with Remark 3.2. If we want to prescribe a fixed trace at $\partial\Omega$ as observed in [10] the Sobolev trace constraint is not sufficient to properly set our problem, due to possible dependence of Ju on the exterior extension. We therefore set $U \Supset \Omega$ open and fix $\phi \in W^{1,n}(U,\mathbb{R}^n)$ such that $\phi|_{\partial\Omega} \in W^{1,p}(\partial\Omega,\mathbb{R}^n)$: our approximating sequences will enjoy $u_h = \phi$ in $U \setminus \Omega$. Recall the previous result establishes the variational approximation of the energy on open sets: potential losses of mass due to presence of singular set at the boundary are disregarded in the lower limit, and a priori excluded in the upper limit by the hypothesis $\mathcal{M}^{*m-n}_\Omega(S_u) = \mathscr{H}^{m-n}(S_u)$.

The following proposition is an easy consequence of Theorem 3.7.

Proposition 7.1. *Suppose $\big((u_h,v_h)\big) \subset Y^\phi$ such that $(u_h,v_h) \to (u,v)$ and $\liminf_{h\to\infty} E_h(u_h,v_h,U) < \infty$. Then $v = 1$ and*

$$\liminf_{h\to\infty} F_h(u_h,v_h,\Omega) + MM_h(v_h,U) \geq E(u,1,\overline{\Omega}) = F(u,1,\Omega) + \sigma\mathscr{H}^{m-n}(S_u \cap \overline{\Omega}). \tag{7.1}$$

Proof. The statement follows straightforward from Theorem 3.7 applied to the domain U, since $S_u \subset \overline{\Omega} \Subset U$, hence $\mathscr{H}^{m-n}(S_u \cap U) = \mathscr{H}^{m-n}(S_u \cap \overline{\Omega})$. Moreover Remark 6.5 entails that

$$\liminf_h F_h(u_h,v_h,\Omega) \geq F(u,1,\Omega),$$

thus the proof is complete. □

Similarly we can prove the upper limit analog:

Proposition 7.2. *Suppose that Ω is of class C^2, $E(u,1,U) < \infty$ and $\mathcal{M}^{*m-n}(S_u) = \mathscr{H}^{m-n}(S_u)$. Then there exists $\big((u_h,v_h)\big) \subset Y^\phi$ such that*

$$\limsup_h E_h(u_h,v_h,\Omega) \leq E(u,1,\overline{\Omega}).$$

Proof. Denote $\Omega_s = \{x \in U : \mathrm{sgndist}(x,\partial\Omega) \leq s\}$, where sgndist is the signed distance from $\partial\Omega$, positive outside Ω and negative inside. The C^2 regularity of Ω ensures the existence of a tubular neighborhood of $\partial\Omega$, namely there exists s_0 (depending on the C^2 norm of $\partial\Omega$) and a C^1 diffeomorphism

$$\partial\Omega \times (-s_0,s_0) \ni (y,t) \mapsto x = y + t\nu(y) \in (\partial\Omega)_{s_0}$$

built up via the normal map ν to $\partial\Omega$. With the help of this map one can construct, for any given $s \in (-s_0, s_0)$, Lipschitz diffeomorphisms $T_s : U \to U$ deforming Ω_s to Ω and satisfying $T_0 = id$ and

$$\|T_s - T_{s'}\|_{W^{1,\infty}(U,U)} + \|T_s^{-1} - T_{s'}^{-1}\|_{W^{1,\infty}(U,U)} \le C|s - s'| \tag{7.2}$$

for every s, s'. We also point out that the existence of the tubular neighborhood gives a reflection map

$$\Pi_{s_0} : (\partial\Omega)_{s_0} \ni (y,t) \mapsto (y,-t) \in (\partial\Omega)_{s_0}$$

of class C^1 such that $\lim_{s_0\to 0} \|\Pi_{s_0} - id\|_{C^1} = 0$. Since the energy is finite $u \in GSB_nV(U)$: given $\eta > 0$ we let

$$u_\eta = \begin{cases} u \circ T_{-\eta} & \text{in } \Omega_{-\eta}, \\ u \circ T_s & \text{on } \partial\Omega_s,\ -\eta < s < 0, \\ \phi & \text{in } U \setminus \Omega. \end{cases} \tag{7.3}$$

Notice that $u_\eta = \phi$ outside Ω and $u_\eta \in W^{1,n}(U \setminus \Omega_{-\eta}, \mathbb{R}^n)$, hence $u_\eta \in X^\phi$ and $S_{u_\eta} \subset \overline{\Omega_{-\eta}} \Subset \Omega$. Moreover it is not difficult to use (7.2) to show $E(u_\eta, 1, U) \to E(u, 1, U)$ for $\eta \downarrow 0$: in fact the energy in $U \setminus \Omega$ is fixed, the one in $\overline{\Omega_{-\eta}}$ after a change of variables equals to

$$\int_\Omega \Big\{ |\nabla u \cdot (DT_{-\eta} \circ T_{-\eta}^{-1})|^p + \Big| \sum_{|I|=|J|=n} \Big| \sum_{|K|=n} \det(\nabla u)^I_K \det(DT_{-\eta} \circ T_{-\eta}^{-1})^K_J \Big|^2 \Big|^{\frac{\gamma}{2}} \Big\} |\det DT_{-\eta}^{-1}|\, dx + \int_{\overline{\Omega} \cap S_u} |\langle \Lambda_{m-n} DT_{-\eta}^{-1}, \tau_{S_u} \rangle| d\mathscr{H}^{m-n}, \tag{7.4}$$

which is asymptotically equal to $E(u, \overline{\Omega})$ thanks to (7.2); finally in the annulus $\Omega \setminus \overline{\Omega_{-\eta}}$, u_η being a constant extension along the trajectories $s \mapsto T_s(x)$, enjoys

$$\int_{\Omega \setminus \overline{\Omega_{-\eta}}} |\nabla u_\eta|^p\, dx \le C(\partial\Omega)\eta \int_{\partial\Omega} |\nabla_\tau \phi|^p\, dx$$

and $M_n \nabla u_\eta = 0$, hence $E(u, 1, \Omega \setminus \overline{\Omega_{-\eta}}) \to 0$. Thanks to Remark 3.2 and Proposition 7.1 it is sufficient to prove the $\Gamma - \limsup$ for u_η. Theorem 6.1 ensures the existence of $(u_h, v_h) \in Y^\phi$ satisfying

$$\limsup_h E_h(u_h, v_h, U) \le E(u, 1, U) :$$

subtracting the constant term $F(\phi, 1, U \setminus \Omega)$ we have the thesis. □

Propositions 7.1 and 7.2 are only in part satisfactory, since in (7.1) we took into account some energy outside Ω. We want to refine these results assessing the quantitative loss of energy due to exterior phase transition in MM_h.

Proposition 7.3. *With the same hypotheses of Proposition 7.1 it holds:*

$$\liminf_h MM_h(v_h, \Omega) \ge \sigma \mathscr{H}^{m-n}(S_u \cap \Omega) + \frac{1}{2} \sigma \mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

Proof. Let us start from the codimension zero case $m = n$. The proof stems from Lemma 5.5, applied to the larger domain U, whose argument we here retrace. Since we are evaluating the energy $MM_h(v_h, \Omega \cap B_{2\rho}(x_0))$ we can suppose $x_0 = 0 \in S_u \cap \partial\Omega$, as the interior case is already contained in Lemma 5.5. Recall the proof showed that in every ball $B_\rho(0) \subset U$ the sequence satisfies $\lim_h \inf_{B_\rho} v_h = 0$. We actually know that

$$\lim_h \inf_{B_\rho \cap \overline{\Omega}} v_h = 0,$$

because every u_h equals ϕ in $U \setminus \Omega$ and $J\phi \ll \mathscr{L}^n$. Let (x_h) be one of the minimum points of v_h in B_ρ: we have two cases.
Case 1: $(\liminf_h \frac{|x_h|}{\varepsilon_h}) < \infty$. In this case scaling back v_h by a factor ε_h we obtain

$$MM_h(v_h, \Omega \cap B_{2\rho}) = MM_1(v_h(\varepsilon_h x), \tfrac{\Omega \cap B_{2\rho}}{\varepsilon_h}).$$

Using a diagonal argument and reasoning as in Proposition 4.1 we produce a limit f_∞ such that

$$v_h(\varepsilon_h x) \to 1 - f_\infty(x) \qquad \text{locally uniformly in } \mathbb{R}^n$$

and

$$\min_{\mathbb{R}^n}\{1 - f_\infty\} = 0.$$

Fix a compact $K \Subset H := \{\langle x, \nu(0)\rangle < 0\}$: by C^1 regularity $\frac{\Omega}{\varepsilon_h} \to H$ locally in the Hausdorff metric and $K \subset \frac{1}{\varepsilon_h}(\Omega \cap B_{2\rho})$ for h large enough. By lower semicontinuity

$$\liminf_h MM_1(v_h(\varepsilon_h x), \tfrac{\Omega \cap B_{2\rho}}{\varepsilon_h}) \geq \liminf_h MM_1(v_h(\varepsilon_h x), K) \geq MM_1(1 - f_\infty, K) = I(f_\infty, K)$$

and letting $K \uparrow H$ we entail

$$\liminf_h MM_h(v_h, \Omega \cap B_{2\rho}) \geq I(f_\infty, H).$$

Therefore if we redefine f_∞ in $\mathbb{R}^n \setminus H$ by reflection with respect to ∂H we obtain $I(f_\infty, H) = \frac{1}{2} I(f_\infty, \mathbb{R}^n)$. A radial rearrangement f_∞^* of f_∞ decreases the energy and gives $f_\infty^*(0) = 1$, hence by Proposition 4.1 $I(f_\infty, H) \geq \frac{1}{2}\sigma$.
Case 2: $\lim_h \frac{|x_h|}{\varepsilon_h} = \infty$. In this situation we blow-up around x_h and obtain that

$$\frac{\Omega \cap B_{2\rho}(0) - x_h}{\varepsilon_h} \supset B_{\frac{\rho}{\varepsilon_h}}(0) \to \mathbb{R}^n$$

in the same sense as before. The limit f_∞ of the translated sequence $(v_h(x_h + \varepsilon_h y))$ will now satisfy $f_\infty(0) = 0$, hence by lower semicontinuity

$$\liminf_h MM_1(v_h(x_h + \varepsilon_h y), \tfrac{\Omega \cap B_{2\rho}(0) - x_h}{\varepsilon_h}) \geq I(f_\infty, \mathbb{R}^n) \geq \sigma.$$

The case $m > n$ can be treated as in (5.12), where now the projection measures τ_π contain the extra term $\frac{1}{2}\sigma \mathscr{H}^{m-n} \llcorner (S_u \cap \partial\Omega)$. □

Similarly we have a statement for the upper limit:

Proposition 7.4. *For every $u \in X^\phi$ such that $E(u, 1, \Omega) < \infty$, $\mathcal{M}^*(S_u) = \mathscr{H}^{m-n}(S_u)$ and $\mathscr{H}^{m-n}(\overline{S_u \cap \Omega} \cap \partial\Omega) = 0$ there exists a sequence $\big((u_h, v_h)\big) \subset Y^\phi$ such that $(u_h, v_h) \to (u, 1)$ and*

$$\limsup_{h \to \infty} E_h(u_h, v_h, \Omega) \leq E(u, 1, \Omega) + \frac{1}{2}\sigma \mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

In order to prove this result we begin with a Lemma:

Lemma 7.5. *Let $\tau > 0$ be a given positive number: there exists a profile $\bar{w} : [0, \infty) \to [0, 1]$ such that*

1. *$|I(\bar{w}(|x|)) - \sigma| < \tau$;*
2. *$\bar{Z}(t) := |\bar{w}'(t)|^q + W(\bar{w}(t))$ is decreasing;*
3. *$\bar{w} \in \mathrm{Lip}([0, \infty))$ and $\bar{w} = 0$ in $[R, \infty)$ for some R;*

Proof. Using the optimal profile w given by Proposition 4.1, it is sufficient to take into account the continuity of I along the family of profiles

$$\frac{w(t+\lambda)}{w(\lambda)}, \qquad \lambda \geq 0 \tag{7.5}$$

and choose a $\lambda > 0$ satisfying $|I(\frac{w(|x|+\lambda)}{w(\lambda)}) - \sigma| < \tau$. We name $\bar{w}$ the profile (7.5) relative to such choice: $\bar{w}$ is clearly Lipschitz by Lemma 4.2. The second property follows from the fact that both $w(t)$ and $|w'(t)|$ are decreasing. The third one can be obtained again by dilating the new profile around 1 and truncate it to 0 changing the energy I only of a small amount. ☐

We will also use the following fact, whose proof we leave to the reader:

Lemma 7.6. *If $S \subset \Omega$ is countably $\mathscr{H}^k$-rectifiable and satisfies $\mathcal{M}^{*k}_{\Omega}(S) = \mathscr{H}^k(S)$ then the same is true for every $S' \subset S$ such that $\mathscr{H}^k(S \cap (\overline{S'} \setminus S')) = 0$.*

We can now prove Proposition 7.4.

Proof. By the finiteness of the energy $u \in GSB_nV(U)$ and $S_u \subset \overline{\Omega}$. Let $\eta_h \downarrow 0$ to be chosen later. We can consider the tilted sequence u_{η_h} described in (7.3): we have

$$\lim_h F(u_{\eta_h}, 1, \Omega) = F(u, 1, \Omega).$$

For an arbitrary τ let $\bar{w}$ be a function as in Lemma 7.5: by Proposition 6.2 and Remark 6.3 we can construct a sequence (v_h) of approximating functions such that

$$\limsup_h MM_h(v_h, \Omega) = I(\bar{w}(|x|))\mathcal{M}^{*m-n}_{\Omega}(S_u) < (\sigma+\tau)\mathcal{M}^{*m-n}_{\Omega}(S_u).$$

Denote $v_{\eta_h,h} = v_h \circ T_{-\eta_h}$: recall that $v_h = 0$ in $S_{\varepsilon_h\delta_h}$ and because of (7.2) we have that

$$||T^{-1}_{-\eta_h}(x) - T^{-1}_{-\eta_h}(y)| - |x-y|| \leq \mathrm{Lip}(T^{-1}_{-\eta_h} - id)|x-y| \leq C\eta_h|x-y|,$$

therefore $v_{\eta_h,h} = 0$ on $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h(1-C\eta_h)}$, thus eventually in $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h/2}$.

Let us analyse the bulk part first. Since the null set of $v_{\eta_h,h}$ has width at least $\varepsilon_h\delta_h/2$ we can apply Theorem 6.1 relative to the limit u_{η_h} in the domain U and define $u_{\eta_h,h}$ such that

- $(u_{\eta_h,h}, v_{\eta_h,h}) \in Y(U)$,
- $|u_{\eta_h,h}| \leq |u_{\eta_h}|$ pointwise almost everywhere,
- $u_{\eta_h,h} = u_{\eta_h}$ outside $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h/2} \subset \{v_{\eta_h,h} = 0\}$.

In particular $u_{\eta_h,h} \to u$ in L^s. Moreover the construction guarantees that

$$\left| F_h(u_{\eta_h,h}, v_h, U) - \int_U |\nabla u_{\eta_h}|^p + v_h|M_n\nabla u_{\eta_h}|^\gamma\,dx \right| \leq \int_U k_h|M_n\nabla u_{\eta_h,h}|^\gamma\,dx + \int_{(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h/2}} |\nabla u_{\eta_h,h}|^p + |\nabla u_{\eta_h}|^p\,dx.$$

The same estimates yielding (6.8) and (6.9) show that the right hand side is infinitesimal. Furthermore the constraint $(u_{\eta_h,h}, v_{\eta_h,h}) \in Y^\phi$ is satisfied once we choose $\eta_h = \varepsilon_h\delta_h$. Observing that

$$\int_U |\nabla u_{\eta_h}|^p + v_h|M_n\nabla u_{\eta_h}|^\gamma\,dx \leq F(u_{\eta_h}, 1, U) \to F(u, 1, U)$$

the previous two equations entail

$$\limsup_h F_h(u_{\eta_h,h}, v_h, U) \leq F(u, 1, U).$$

Subtracting the constant term $F(u,1,U\setminus\Omega)$ we remain with

$$\begin{aligned} F(u,1,\Omega) &\geq \limsup_h F_h(u_{\eta_h,h},v_h,U) - F(u,1,U\setminus\Omega) \\ &= \limsup_h F_h(u_{\eta_h,h},v_h,\Omega) + F_h(u_{\eta_h,h},v_h,U\setminus\Omega) - F(u,1,U\setminus\Omega) \\ &= \limsup_h F_h(u_{\eta_h,h},v_h,\Omega) + F_h(u,v_h,U\setminus\Omega) - F(u,1,U\setminus\Omega) \\ &= \limsup_h F_h(u_{\eta_h,h},v_h,\Omega) + \int_{U\setminus\Omega} (v_h+k_h-1)|M_n\nabla u|^\gamma\,dx \\ &= \limsup_h F_h(u_{\eta_h,h},v_h,\Omega). \end{aligned}$$

It remains to evaluate the asymptotic of $MM_h(v_{\eta_h,h},\Omega)$. First of all changing back variables we have that

$$MM_h(v_{\eta_h,h},\Omega) = \int_{T_{\eta_h}(\Omega)} \left\{ \varepsilon_h^{q-n}|(DT_{-\eta_h}\circ T_{-\eta_h}^{-1})\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}\,dx \right\} |\det DT_{-\eta_h}^{-1}|\,dx$$

and by (7.2) and Lemma 7.5 this is asymptotic to $MM_h(v_h,T_{\eta_h}(\Omega))$: we now show that if $\frac{\eta_h}{\varepsilon_h}=\delta_h\to 0$ sufficiently fast then the last energy is asymptotically equal to $MM_h(v_h,\Omega)$, namely $MM_h(v_h,T_{\eta_h}(\Omega)\setminus\Omega)\to 0$.

Fix a radius R such that $\mathrm{spt}(\bar w)\subset B_R^n$ and $\mathscr{L}^m(B_{2\varepsilon_h R}\cap(T_{\eta_h}(\Omega)\setminus\Omega))\leq C(\varepsilon_h R)^{m-1}\eta_h$. We can cover $S_{\varepsilon_h R}$ with (closed) balls of radius $\varepsilon_h R$ centered at $x_0\in S_u$:

$$S_{\varepsilon_h R}\subset\bigcup_{x_0\in S_u}\overline{B_{\varepsilon_h R}(x_0)}.$$

By Besicovitch's covering Lemma there are N disjoint subfamilies $\mathcal{F}_i$ that still cover the set of old centers, namely S_u: by triangle inequality

$$S_{\varepsilon_h R}\subset\bigcup_{i=1}^N\bigcup_{\mathcal{F}_i}B_{2\varepsilon_h R},$$

and the assumption of $\mathcal{M}^{*m-n}(S_u)$ implies that $\#\mathcal{F}_i\leq C(\varepsilon_h R)^{n-m}$; as a consequence the family of double balls $\{B_{2\varepsilon_h R}\}$ has bounded overlap. Without loss of generality we can also assume that $\mathcal{M}^{*m-n}(S_u\cap B_{2\varepsilon_h R}(x_0))=\mathscr{H}^{m-n}(S_u\cap B_{2\varepsilon_h R}(x_0))$, recalling that this is true at almost every radius. For any of such double ball

$$\frac{MM_h(v_h,B_{2\varepsilon_h R}(x_0)\cap(T_{\eta_h}(\Omega)\setminus\Omega))}{\varepsilon_h^{m-n}} = \int_{B_{2R}(0)\cap\frac{(T_{\eta_h}(\Omega)\setminus\Omega)-x_0}{\varepsilon_h}} |\nabla\psi_h|^q + W(\psi_h)\,dy \tag{7.6}$$

with

$$\psi_h(y)=\bar w_h\left(d\big(y,\frac{S_u-x_0}{\varepsilon_h}\big)\right).$$

The integral can be simply bounded by

$$(\mathrm{Lip}(\bar w_h)^q+\|W\|_\infty)\mathscr{L}^m\left(B_{2R}(0)\cap\frac{(T_{\eta_h}(\Omega)\setminus\Omega)-x_0}{\varepsilon_h}\right)\leq C(\mathrm{Lip}(\bar w_h)^q+\|W\|_\infty)R^{m-1}\frac{\eta_h}{\varepsilon_h}.$$

Recall the construction of $\bar w_h$ from $\bar w$ in (6.3) gives that $\mathrm{Lip}(\bar w_h)\leq C\mathrm{Lip}(\bar w)$. Summing on the number of balls we have

$$\begin{aligned} MM_h(v_h,\Omega_{\eta_h}\setminus\Omega) &\leq C\varepsilon_h^{m-n}(\varepsilon_h R)^{n-m}(\mathrm{Lip}(\bar w_h)^q+\|W\|_\infty)R^{m-1}\frac{\eta_h}{\varepsilon_h} \\ &= C(\mathrm{Lip}(\bar w)^q+\|W\|_\infty)R^{n-1}\frac{\eta_h}{\varepsilon_h}\to 0. \end{aligned}$$

Suppose now that $\mathscr{H}^{m-n}(\overline{S_u \cap \Omega} \cap \partial\Omega) = 0$: then $\mathcal{M}_U^{*m-n}(S_u \cap \Omega) = \mathscr{H}^{m-n}(S_u \cap \Omega)$ by Lemma 7.6 (applied to $S' = S_u \cap \Omega$). Moreover:

$$\mathcal{M}_\Omega^{*m-n}(S_u) \leq \mathcal{M}_U^{*m-n}(S_u \cap \Omega) + \mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega) = \mathscr{H}^{m-n}(S_u \cap \Omega) + \mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega).$$

Regarding the last term, by (7.2) the reflection map Π_{s_0} that swaps Ω and $U \setminus \Omega$ has a jacobian uniformly close to 1 as we move close to $\partial\Omega$ and therefore

$$\mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega) = \frac{1}{2}\mathcal{M}_U^{*m-n}(S_u \cap \partial\Omega) = \frac{1}{2}\mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

In conclusion

$$\limsup_h MM_h(v_h, \Omega) \leq (\sigma + \tau)(\mathscr{H}^{m-n}(S_u \cap \Omega) + \frac{1}{2}\mathscr{H}^{m-n}(S_u \cap \partial\Omega))$$

and the assertion follows by letting $\tau \to 0$. □

8. General Lagrangians

The Γ-convergence Theorem 3.7 proved in the previous sections can be extended, always in the setting of higher codimension singular sets, to polyconvex Lagrangians of more general form than Definition (2.10).

Indeed the key ingredients for the $\Gamma - \liminf$ are again the compactness Theorem 2.8, which is at the heart of Theorem 2.11, as well as the lower semicontinuity of the energy for the convergence provided by it. Regarding the $\Gamma - \limsup$ in order to approximate the size term we rely on the same Modica-Mortola approximation of before. The recovery sequence is obtained via an approximation in measure of the limit function u, with regular functions $u_\varepsilon \in R_n$ coinciding with u outside the narrow sets S_ε. The proof of Proposition 6.4 amounts to show that the contribution to the bulk energy in S_ε is infinitesimal.

Both these arguments can be adapted to a broader class of Lagrangians that we now present (see [10] for the study of the relative Dirichlet and Neumann problems). We denote $\mathscr{L}_m$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$ and $\mathscr{B}(\mathbb{R}^{n+\kappa})$ the σ-algebra of Borel subsets of $\mathbb{R}^{n+\kappa}$. Assume the following hypotheses on the functions $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^\kappa \to [0, +\infty)$ and $g : \Omega \to [0, \infty)$ are satisfied:

(a) f is $\mathscr{L}_m \times \mathscr{B}(\mathbb{R}^{n+\kappa})$-measurable;
(b) for $\mathscr{L}^m$-a.e. $x \in \Omega$, $(u, w) \mapsto f(x, u, w)$ is lower semicontinuous;
(c) for $\mathscr{L}^m$-a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^n$ the map $w \mapsto f(x, u, w)$ is convex in $\mathbb{R}^\kappa$;
(d) $c\big(|w_1|^p + \Psi(|w_n|)\big) \leq f(x, u, w) \leq C\big(1 + |u|^s + |w_1|^p + |w_n|^\gamma\big)$ for Ψ convex and superlinear at infinity and for some constants $\gamma > 1$, $c, C > 0$;

and $g \in C^0(\overline{\Omega})$, $g \geq c > 0$. Our energy is:

$$\mathcal{E}(u, \Omega) = \int_\Omega f(x, u, M\nabla u)\, dx + \sigma \int_{\Omega \cap S_u} g\, d\mathscr{H}^{m-n}. \tag{8.1}$$

Thanks to the Theorem 2.8 the energy (8.1) is lower semicontinuous along sequences converging strongly in L^s and with equibounded energies. The upper bound on f on the other side allows to prove the upper limit statement. The approximating energies will be

$$\mathcal{E}_\varepsilon(u, v, \Omega) := \int_\Omega f(x, u, \nabla u, \dots, M_{n-1}\nabla u, (v + k_\varepsilon)M_n\nabla u)\, dx + \int_\Omega g(x) \left(\varepsilon^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon^n}\right) dx.$$

We therefore have:

Theorem 8.1. *Let Ω be a bounded open subset of class C^1 of $\mathbb{R}^m$ and suppose*

$$s \geq \frac{np}{n-p}, \qquad 1 < \gamma \leq \frac{1}{\frac{n-1}{p} + \frac{1}{s}}, \qquad q > n, \qquad k_\varepsilon = o(\varepsilon).$$

Suppose the integrands f, g satisfy the assumptions above. Then:

(a) *For every sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $\liminf_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) < \infty$ and $(u_h, v_h) \to (u, v)$ in $X(\Omega)$ we have*

$$u \in GSB_nV(\Omega),\ v = 1 \ \text{and} \ \liminf_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) \geq \mathcal{E}(u, \Omega).$$

(b) *For every $u \in GSB_nV(\Omega)$ such that $\mathcal{E}(u, 1, \Omega) < \infty$ and $\mathcal{M}^{*m-n}_\Omega(S_u) = \mathscr{H}^{m-n}(S_u)$ there exists a sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $(u_h, v_h) \to (u, 1)$ in $X(\Omega)$ and*

$$\limsup_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) \leq \mathcal{E}(u, \Omega).$$

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