

# UNIQUENESS FOR KELLER-SEGEL-TYPE CHEMOTAXIS MODELS

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ABSTRACT. We prove uniqueness in the class of integrable and bounded nonnegative solutions in the energy sense to the Keller-Segel (KS) chemotaxis system. Our proof works for the fully parabolic KS model, it includes the classical parabolic-elliptic KS equation as a particular case, and it can be generalized to nonlinear diffusions in the particle density equation as long as the diffusion satisfies the classical McCann displacement convexity condition. The strategy uses Quasi-Lipschitz estimates for the chemoattractant equation and the above-the-tangent characterizations of displacement convexity. As a consequence, the displacement convexity of the free energy functional associated to the KS system is obtained from its evolution for bounded integrable initial data.

## 1. INTRODUCTION

The classical Keller-Segel (KS) model for chemotaxis is the system

$$\begin{cases} \partial_t n &= \kappa \Delta n - \chi \operatorname{div}(n \nabla c), \\ \partial_t c &= \eta \Delta c + \theta n - \gamma c. \end{cases}$$

Here,  $n$  is the number/mass density of a bacteria/cell population and  $c$  represents the concentration of a chemical attractant that can suffer chemical degradation and that is produced by the cells themselves due to chemotactic interaction. The parameters  $\kappa, \chi, \eta, \theta, \gamma$  might be suitable functions, assumed to be constant in this simplified model. We can perform a time scaling and a suitable change of variables, that is  $\tau = \kappa t$ ,  $\rho(x, \tau) = \frac{\theta \chi}{\eta \kappa} n(x, \tau/\kappa)$ ,  $v(x, \tau) = \frac{\chi}{\kappa} c(x, \tau/\kappa)$ . The system is therefore reduced to

$$\begin{cases} \partial_t \rho &= \Delta \rho - \operatorname{div}(\rho \nabla v), \\ \varepsilon \partial_t v &= \Delta v + \rho - \alpha v, \end{cases} \quad (1.1)$$

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where  $\alpha \geq 0$  and  $\varepsilon \geq 0$  are constants ( $\alpha = \gamma/\eta$ ,  $\varepsilon = \kappa/\eta$ ). In case  $\varepsilon = 0$ , it restricts to the classical parabolic-elliptic Patlak-KS model

$$\begin{cases} \partial_t \rho = \Delta \rho - \operatorname{div}(\rho \nabla v), \\ -\Delta v + \alpha v = \rho. \end{cases} \quad (1.2)$$

For  $\varepsilon > 0$ , the natural free energy functional associated to the dynamics of the system (1.1) is

$$\mathcal{F}_{\varepsilon, \alpha}(\rho, v) := \int_{\mathbb{R}^d} (\rho \log \rho - v \rho) dx + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla v|^2 + \alpha v^2) dx. \quad (1.3)$$

In the case  $\varepsilon = 0$ , corresponding to (1.2), this Liapunov functional is at least formally equivalent to

$$\mathcal{F}_{0, \alpha}(\rho) := \int_{\mathbb{R}^d} (\rho \log \rho - \frac{1}{2} v \rho) dx \quad (1.4)$$

with the convention that  $v$  is obtained from the density  $\rho$  by  $v = \mathcal{B}_{\alpha, d} * \rho$ . Here,  $\mathcal{B}_{\alpha, d}$  denotes the Bessel kernel for  $\alpha > 0$  or the Newtonian kernel for  $\alpha = 0$ , for any dimension  $d$ . Therefore the role of the parameter  $\varepsilon$  is to discriminate between parabolic-parabolic and parabolic-elliptic system. Note that the Liapunov functionals (1.3) and (1.4) are just formally equivalent since the  $L^2$ -integrability of  $\nabla \mathcal{B}_{\alpha, d} * \rho$  fails if  $d = 1, 2$  and  $\alpha = 0$ . Thus, even if the classical free energy writing and valid for all cases when  $\varepsilon = 0$  is the one in (1.4), we will prefer to work with the functional as in (1.3) even if  $\varepsilon = 0$ , with a suitable renormalization for the cases  $d = 1, 2$  and  $\alpha = 0$  discussed in Section 3.

Our main objective is the uniqueness of certain solutions, for both systems (1.1) and (1.2). Let us introduce the notion of solution for the Cauchy problems associated to (1.1) and (1.2) that we will consider in this work. We denote by  $\mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$  the set of nonnegative densities over  $\mathbb{R}^d$  with mass  $\mathbf{m}$  and finite second moment, i.e.,

$$\mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) := \left\{ \rho \in L^1(\mathbb{R}^d) : \rho \geq 0, \int_{\mathbb{R}^d} \rho(x) dx = \mathbf{m}, \int_{\mathbb{R}^d} |x|^2 \rho(x) dx < +\infty \right\}.$$

**Definition 1.1.** We say that a weakly continuous map  $\rho \in C_w([0, T]; \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}))$  is a bounded solution to the Cauchy problem for (1.2), with initial datum  $\rho^0 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ , if

- i)  $\rho \in L^\infty((0, T) \times \mathbb{R}^d)$  and  $|x|^2 \rho_t(x) \in L^\infty((0, T), L^1(\mathbb{R}^d))$ ,
- ii)  $\rho_0 = \rho^0$  and the first equation of (1.2) holds in the sense of distributions on  $(0, T) \times \mathbb{R}^d$ , where  $v_t = \mathcal{B}_{\alpha, d} * \rho_t$  for all  $t \in [0, T]$ ,
- iii)  $\rho_t \in W^{1,1}(\mathbb{R}^d)$  for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$  and

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_t(x)|^2}{\rho_t(x)} dx dt < +\infty. \quad (1.5)$$

**Definition 1.2.** We say that the couple  $(\rho, v)$ , satisfying  $\rho \in C_w([0, T]; \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}))$  and  $v \in L^2((0, T); W^{1,2}(\mathbb{R}^d))$ , is a bounded solution to (1.1) with initial datum  $(\rho^0, v^0) \in (\mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)) \times W^{1,2}(\mathbb{R}^d)$ , if

- I)  $\rho \in L^\infty((0, T) \times \mathbb{R}^d)$  and  $|x|^2 \rho_t(x) \in L^\infty((0, T), L^1(\mathbb{R}^d))$ ,
- II)  $\rho_0 = \rho^0$ , the first equation of (1.1) holds in the sense of distributions on  $(0, T) \times \mathbb{R}^d$ , and  $v$  is the unique solution to the Cauchy problem for the forced parabolic equation  $\varepsilon \partial_t v - \Delta v + \alpha v = \rho$  over  $(0, T) \times \mathbb{R}^d$  in the standard sense, with initial datum  $v^0$ ,
- III) the property iii) of Definition 1.1 holds.

Let us emphasize that the main properties we need to get uniqueness of solution are the boundedness of the densities and the Fisher information (1.5). They together imply that the velocity field of the continuity equation for the density  $\rho$  is a well defined object belonging to the right functional space, as we will see later on. Moreover, the boundedness of the density implies that we have a uniform in bounded time intervals estimate on the quasi-Lipschitz constant of part of the velocity field. These are the basic properties that imply the uniqueness for bounded solutions. Let us finally mention that part of the strategy is related to the uniqueness of solutions to fluid and aggregation equations developed in [35, 28, 5, 29, 19, 7, 30]. The main novelty here is the interplay between the diffusive and the aggregation parts. The main results of this work are:

**Theorem 1.3.** *Let  $T > 0$  and let  $\rho^0 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ . Let  $\rho_1, \rho_2$  be two bounded solutions on  $[0, T] \times \mathbb{R}^d$  to the Cauchy problem associated to (1.2), with initial datum  $\rho^0$ . Then  $\rho_1 = \rho_2$ .*

**Theorem 1.4.** *Let  $T > 0$  and let  $\rho^0 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ ,  $v^0 \in W^{1,2}(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ . Let  $(\rho_1, v_1)$  and  $(\rho_2, v_2)$  be two bounded solutions on  $[0, T] \times \mathbb{R}^d$  of the Cauchy problem associated to (1.1), with initial datum  $(\rho^0, v^0)$ . Then  $(\rho_1, v_1) = (\rho_2, v_2)$ .*

The proof of uniqueness as stated in Theorems 1.3 and 1.4 will be a consequence of a more general property: we will show that bounded solutions satisfy a strong gradient flow formulation by means of a family of evolution variational inequalities. This formulation is similar to the one for semi-convex functionals and implies a non-expansivity property of the distance between two solutions. This non-expansivity property yields uniqueness. All these results will be stated in Theorems 3.1 and 5.1. Theorem 1.4 is stated under the assumption  $v^0 \in W^{2,\infty}(\mathbb{R}^d)$ , but it still holds true assuming that  $v^0$  belongs to suitable Zygmund spaces, which will be introduced in the next sections. Moreover the evolution variational inequality formulation leads to a relaxed convexity property of the energy functional as stated in Theorem 4.1.

There is a huge literature about the KS system and their variations, so we just restrict here to discuss the main results concerning bounded solutions. In the classical parabolic-elliptic KS equation  $\varepsilon = \alpha = 0$  and  $d = 2$ , global in time bounded solutions in the subcritical case  $\mathbf{m} < 8\pi$  have been obtained joining the results in [12, 25, 14]. Actually, the global existence of weak solutions satisfying all properties in Definition 1.1 except the  $L^\infty$  bound was obtained in [12] while  $L^\infty$ -bounds in bounded time intervals can be obtained from the results in [25, 14]. The same techniques could eventually be used to get local in time bounded solutions for all masses, although such a result is not present in the literature. Let us also mention the recent paper [17] in which the authors actually show that the  $L^\infty$ -norm of the solution decays in time like for the heat equation in the subcritical case  $\mathbf{m} < 8\pi$  for more restricted initial data.  $L^\infty$ -a priori estimates were obtained in the classical parabolic-elliptic KS equation  $\varepsilon = 0$  with  $d \geq 2$  and  $\alpha \geq 0$  for small  $L^{d/2}$  initial data in [20, 21]. These results together with similar arguments as in [12] to get the free energy dissipation property and thus the Fisher information bounds, could lead to the existence of bounded solutions in these cases. We emphasize that these  $L^\infty$  estimates show that the solution in bounded time intervals is bounded by a constant that depends only on the  $L^\infty$ -norm of the initial data, the initial free energy, and the final time. In particular, existence of bounded solutions is expected if  $\rho^0 \in L^\infty(\mathbb{R}^d)$ , and this explains the presence of such an assumption in the previous definitions.

Concerning the fully parabolic KS system, we find global in time solutions satisfying all properties stated in Definition 1.2 except the  $L^\infty$  bounds in [15] for  $d = 2$  and the subcritical

mass case  $m < 8\pi$ .  $L^\infty$ -apriori estimates were obtained in [26] for the fully parabolic case but in bounded domains. It is reasonable to expect that this strategy should work for the whole space case, although it is not written as such in the literature. Results in higher dimensions concerning solutions with  $L^\infty$  estimates for small initial data can be found in [8] but estimates on the free energy dissipation are missing there. We finally refer to [11, 13, 14, 23] for different results concerning the existence of solutions satisfying the boundedness of the Fisher information and/or the uniform bounds of the solutions for particular choices of  $\varepsilon \geq 0$ ,  $\alpha \geq 0$ , and nonlinear diffusions.

As mentioned before, Theorems 1.3 and 1.4 are based on the derivation of quasi-Lipschitz estimates for the chemoattractant  $v$  (this is the reason behind the additional assumption on the initial datum  $v^0$ ). We will clarify the use of quasi-Lipschitz estimates of the chemoattractant in Section 2 together with a quick summary of the main properties of optimal transport that we need in this work. Section 3 is devoted to show that bounded solutions for the Keller-Segel model satisfy suitable evolution variational inequalities that imply, among the other properties, the main uniqueness results. In Section 4 we show that the same evolution variational inequalities lead to certain convexity of the associated free energy functional. In Section 5 we give the derivation of the quasi-Lipschitz estimates of the parabolic equations for  $v$ . In the same section, we will also prove a strengthening of Theorems 1.4 and 3.1, with more general initial data. Finally, Section 6 is devoted to show how to adapt these arguments to Keller-Segel models with nonlinear diffusion.

## 2. PRELIMINARY NOTIONS

**2.1. Some elliptic and parabolic regularity estimates.** The proofs of our results are based on the technique used by Yudovich [35] for treating uniqueness in the case of incompressible Euler equations for fluidodynamics. In particular, we exploit a quasi-Lipschitz property for the velocity field of the continuity equation for  $\rho$  in (1.1) and (1.2). This property comes from the regularity that  $v$  gains being solution to the second equation in (1.1) and (1.2).

Suppose first that  $v = \mathcal{B}_{0,d} * \rho$ . If  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$ , by exploiting some estimates of the Newtonian potential,  $\nabla v$  satisfies the following log-Lipschitz property (see [6] and [31, Chapter 8], [33] and also [35]),

$$|\nabla v(x) - \nabla v(y)| \leq C|x - y|(1 + \log^- |x - y|),$$

where  $C$  is a suitable positive constant, depending only on  $\|\rho\|_{L^1}$  and  $\|\rho\|_{L^\infty}$  and  $\log^-$  denotes the negative part of the natural logarithm function. As a consequence, we get the estimate

$$|\nabla v(x) - \nabla v(y)|^2 \leq C^2 \varphi(|x - y|^2) \tag{2.1}$$

for some new positive constant  $C$ , where  $\varphi$  is the concave function on  $[0, \infty)$  defined as

$$\varphi(x) := \begin{cases} x \log^2 x & \text{if } x \leq e^{-1-\sqrt{2}}, \\ x + 2(1 + \sqrt{2})e^{-1-\sqrt{2}} & \text{if } x > e^{-1-\sqrt{2}}. \end{cases} \tag{2.2}$$

Indeed, for large values of  $|x - y|$  the estimate (2.1) is quite obvious, since it is immediate to show that  $\nabla \mathcal{B}_{0,d} * \rho$  is a bounded function in the whole space with a direct estimate using the fact that  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$ .

Analogous facts hold if we consider the equation  $-\Delta v + \alpha v = \rho$ , appearing in (1.2), or more general uniformly elliptic operators, so that we have the following

**Proposition 2.1.** *Suppose that  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$  and  $\alpha \geq 0$ . Then  $v = \mathcal{B}_{\alpha,d} * \rho$  satisfies the estimate (2.1), where  $C$  is a suitable positive constant, depending only on  $\alpha, d, \|\rho\|_{L^1(\mathbb{R}^d)}$ , and  $\|\rho\|_{L^\infty(\mathbb{R}^d)}$ .*

The log-Lipschitz property in general can be justified through standard elliptic regularity, requiring the introduction of Zygmund spaces. These classes of functions were introduced in [36], and they belong to the more general framework of Besov spaces. The basic Zygmund class  $\Lambda_1(\mathbb{R}^d)$  is the set of continuous bounded functions  $f$  over  $\mathbb{R}^d$  such that

$$[f]_{\Lambda_1(\mathbb{R}^d)} := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - 2f((x+y)/2) + f(y)|}{|x-y|} < +\infty.$$

It is well known that functions in the Zygmund class  $\Lambda_1(\mathbb{R}^d)$  are in general not Lipschitz, possibly nowhere differentiable, but enjoy a log-Lipschitz modulus of continuity. Indeed, for any  $f \in \Lambda_1(\mathbb{R}^d)$  there exists a positive constant  $C$  such that

$$|f(x) - f(y)| \leq C|x-y|(1 + \log^- |x-y|) \quad \forall x, y \in \mathbb{R}^d.$$

we refer for instance to [37, Chapter 2, §3]. We say that  $f \in \Lambda_2(\mathbb{R}^d)$  if  $f \in W^{1,\infty}(\mathbb{R}^d)$  and all the partial derivatives of  $f$  belong to  $\Lambda_1(\mathbb{R}^d)$  (see for instance [34, Chapter 5]). In the usual notation of Besov spaces,  $\Lambda_2$  corresponds to  $B_{\infty,\infty}^2$ . The vector spaces  $\Lambda_1$  and  $\Lambda_2$  can be endowed with the norms

$$\begin{aligned} \|f\|_{\Lambda_1(\mathbb{R}^d)} &= \|f\|_{L^\infty(\mathbb{R}^d)} + [f]_{\Lambda_1(\mathbb{R}^d)}, \\ \|f\|_{\Lambda_2(\mathbb{R}^d)} &= \|f\|_{L^\infty(\mathbb{R}^d)} + \|\nabla f\|_{\Lambda_1(\mathbb{R}^d)} \end{aligned}$$

and they become complete.

*Proof of Proposition 2.1.* If  $\alpha > 0$ , from the general theory on Bessel potentials (see for instance [34, Chapter 5, §3-6]) we learn that by convolution with the Bessel kernel  $\mathcal{B}_{\alpha,d}$  we indeed get two indices of regularity in Zygmund spaces. Therefore, if  $\rho \in L^\infty(\mathbb{R}^d)$ , we indeed get that  $v = \mathcal{B}_{\alpha,d} * \rho$  belongs to  $\Lambda_2(\mathbb{R}^d)$ , and thus  $\nabla v \in \Lambda_1(\mathbb{R}^d)$  and, since  $\nabla v$  is bounded, (2.1) follows. For the case  $\alpha = 0$  we address to the references mentioned at the beginning of this section (it is also possible to directly check that  $\nabla v \in L^\infty(\mathbb{R}^d)$ , and then the Newtonian potential behaves like the Bessel potential near the origin so that  $\nabla v$  is also log-Lipschitz).  $\square$

About the parabolic equation for  $v$  in (1.1), the quasi-Lipschitz property also carries over, since formally inequality (2.1) translates in terms of the parabolic metric to

$$|\nabla v(t, x) - \nabla v(s, y)|^2 \leq C^2 \varphi((|x-y| + |s-t|^{1/2})^2) \quad \forall x, y \in \mathbb{R}^d, s, t \in [0, T]. \quad (2.3)$$

Indeed, we have the following

**Proposition 2.2.** *Suppose that  $\rho \in L^\infty((0, T) \times \mathbb{R}^d)$ ,  $v^0 \in \Lambda_2(\mathbb{R}^d)$  and  $\alpha \geq 0$ . If  $v$  is the unique solution to the Cauchy problem for the parabolic equation  $\partial_t v = \Delta v - \alpha v + \rho$  (in the standard sense of convolution with fundamental solution), then  $v$  satisfies (2.3), where  $C$  is a suitable positive constant, depending only on  $\alpha, d, \|v^0\|_{\Lambda_2(\mathbb{R}^d)}$ , and  $\|\rho\|_{L^\infty((0,T) \times \mathbb{R}^d)}$ .*

In order not to introduce some not really necessary notation before the proof of our main results, we prefer to postpone the proof of Proposition 2.2 to Section 5. Indeed, in Section 5 we will develop a discussion about log-Lipschitz estimates for parabolic equations, and we will also prove a strengthening of Theorem 1.4, considering the initial datum  $v^0$  in  $\Lambda_1(\mathbb{R}^d)$  instead of  $W^{2,\infty}(\mathbb{R}^d)$ .

**2.2. Elementary notions of optimal transport.** Given  $\rho_0, \rho_1 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$ , we define the Wasserstein distance between  $\rho_0$  and  $\rho_1$  as

$$W_2(\rho_0, \rho_1) = \left( \int_{\mathbb{R}^d} |x - \mathcal{T}(x)|^2 \rho_0(x) dx \right)^{\frac{1}{2}},$$

where  $\mathcal{T}$  is the unique optimal transport map between  $\rho_0$  and  $\rho_1$ , that is, the map  $\mathcal{T} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  which minimizes  $\int_{\mathbb{R}^d} |x - \mathcal{S}(x)|^2 \rho_0(x) dx$  among all the Borel maps  $\mathcal{S} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying  $\mathcal{S}_\# \rho_0 = \rho_1$ . We recall that  $\mathcal{S}_\# \rho_0 = \rho_1$  means that  $\int_{\mathbb{R}^d} \varphi(x) \rho_1(x) dx = \int_{\mathbb{R}^d} \varphi(\mathcal{S}(x)) \rho_0(x) dx$  for every continuous and bounded function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ .

The Wasserstein geodesic between  $\rho_0$  and  $\rho_1$  is the curve  $s \in [0, 1] \mapsto \rho^s \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$  defined by the so-called displacement interpolation along the optimal transport map  $\mathcal{T}$  between  $\rho_0$  and  $\rho_1$ , that is,  $\rho^s := ((1-s)\mathbf{i} + s\mathcal{T})_\# \rho_0$ . In particular, for any  $s$ ,  $\mathcal{T}_s := (1-s)\mathbf{i} + s\mathcal{T}$  is the optimal map between  $\rho_0$  and  $\rho^s$  and there holds  $W_2(\rho^r, \rho^s) = |s-r|W_2(\rho_0, \rho_1)$ .

We recall a formula for the differentiation of the squared Wasserstein distance along solutions of the continuity equation. Let  $t \in [0, T] \mapsto \rho_t \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$  be a weakly continuous curve which is distributional solution of

$$\partial_t \rho_t + \operatorname{div}(\xi_t \rho_t) = 0,$$

for some Borel velocity field  $\xi_t$  such that  $\int_0^T \|\xi_t\|_{L^2(\mathbb{R}^d, \rho_t; \mathbb{R}^d)}^2 dt < +\infty$ . Then the curve is absolutely continuous with respect to the Wasserstein distance, [3, Theorem 8.3.1]. Then, for any  $\bar{\rho} \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$ , it holds

$$\frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) = \int_{\mathbb{R}^d} \langle \xi_t(x), x - \mathcal{T}_t(x) \rangle \rho_t(x) dx, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T), \quad (2.4)$$

where  $\mathcal{T}_t$  is the optimal map between  $\rho_t$  and  $\bar{\rho}$  (see [3, Theorem 8.4.7, Remark 8.4.8]).

Finally, let us recall an estimate relating the 2-Wasserstein distance and the  $H^{-1}$  norm proved in [28, Proposition 2.8]. Given two nonnegative densities with the same mass  $\rho_1, \rho_2 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ , there holds

$$\|\rho_1 - \rho_2\|_{\dot{H}^{-1}(\mathbb{R}^d)} \leq \max\{\|\rho_1\|_\infty, \|\rho_2\|_\infty\}^{1/2} W_2(\rho_1, \rho_2). \quad (2.5)$$

Here  $\dot{H}^1(\mathbb{R}^d)$  denotes the space of Lebesgue measurable functions  $v : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\|\nabla v\|_{L^2(\mathbb{R}^d)} < +\infty$ , so that  $\dot{H}^{-1}(\mathbb{R}^d)$  is defined by duality with functions having finite  $L^2(\mathbb{R}^d)$  norm of the gradient only. By the way, we can also consider the space  $H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$ . In fact, from the proof in [28, Proposition 2.8] it is not difficult to see that the same estimate holds considering the  $H^{-1}(\mathbb{R}^d)$  space given by duality with the full norm  $(\|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2)^{1/2}$ .

### 3. BOUNDED SOLUTIONS AS GRADIENT FLOWS: EVI AND UNIQUENESS

The uniqueness Theorems 1.3 and 1.4 are consequences of a general result interpreting bounded solutions to (1.1) (resp. (1.2)) as the trajectory of the *gradient flow* of the functional (1.3) (resp. (1.4)) in the appropriate metric setting. We prove that bounded solutions satisfy a family of evolution variational inequalities (EVI). Among different notions of gradient flow in metric sense, the EVI formulation is stronger than other formulations and typically corresponding to a convex structure, as in [3, Theorem 11.2.1] for the theory in the Wasserstein setting.

**Notation for the energy functional.** Before giving the proof, we introduce some uniform notation for working with the full functional (1.3) even in the parabolic-elliptic case. Let  $\rho \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ . We are considering the free energy functional

$$\mathcal{F}_{\varepsilon, \alpha}(\rho, v) := \int_{\mathbb{R}^d} (\rho \log \rho - v\rho) dx + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla v|^2 + \alpha v^2) dx,$$

defined for  $v$  being any  $W^{1,2}(\mathbb{R}^d)$  function if  $\varepsilon > 0$ . On the other hand, if  $\varepsilon = 0$  it is understood that  $v$  is given by  $\mathcal{B}_{\alpha, d} * \rho$ . Therefore the parameter  $\varepsilon$  only indicates if we are considering problem (1.1) or (1.2). In particular, this writing of the functional as in (1.3) is valid in general, even for  $\varepsilon = 0$ , except for two particular cases:  $\varepsilon = \alpha = 0$  and  $d = 1, 2$ , as discussed in the introduction. In these two cases, we need to renormalize the free energy functional. Given  $\rho^* \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$  a smooth and compactly supported density and  $v^* = \mathcal{B}_{0, d} * \rho^*$ , we redefine (1.3) for  $\varepsilon = \alpha = 0$  and  $d = 1, 2$  as

$$\mathcal{F}_{0,0}(\rho, v) := \int_{\mathbb{R}^d} [\rho \log \rho - v(\rho - \rho^*)] dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla(v - v^*)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^d} \rho^* v^* dx. \quad (3.1)$$

Notice that  $\nabla(v - v^*) \in L^2(\mathbb{R}^d)$ , as  $\rho - \rho^*$  has zero mean, see [4, 33] for more details.

In the rest of this work, when referring to the free energy functional  $\mathcal{F}_{\varepsilon, \alpha}(\rho, v)$ , we will be using (1.3) for any  $\varepsilon \geq 0, \alpha \geq 0$ , except for  $\varepsilon = \alpha = 0$  and  $d = 1, 2$  where the free energy functional is given by (3.1).

Let us observe that now all the integrals involved in the definition of  $\mathcal{F}_{\varepsilon, \alpha}$  are well defined and finite for  $\varepsilon \geq 0, \alpha \geq 0$  and  $\rho, v$  as above. The negative part of the entropy term can be classically treated by the Carleman inequality, see for instance [9, Lemma 2.2] where the second moment bound on the density is used. The boundedness of the density controls the positive contribution of the entropy term together with the integrability of  $v\rho$  in case  $\varepsilon > 0$  since  $v \in W^{1,2}(\mathbb{R}^d)$ . For  $\varepsilon = 0$  the integrability of  $v\rho$  in case  $\alpha > 0$  is implied by the Newtonian potential case  $\alpha = 0$  since the singularity of the Bessel potential at the origin is the same. The integrability for  $\alpha = \varepsilon = 0$  and  $d \geq 3$  results directly from the Hardy-Littlewood-Sobolev inequality for the Newtonian potential. For  $\alpha = \varepsilon = 0$  and  $d = 1, 2$  we use the behavior at infinity of the density  $\rho$ . Actually,  $\alpha = \varepsilon = 0$  and  $d = 1$  is a trivial case since the Newtonian potential is given by  $\mathcal{B}_{0,1}(x) = |x|$ . For  $\alpha = \varepsilon = 0$  and  $d = 2$  since  $\log(e + |x|^2)\rho \in L^1(\mathbb{R}^d)$  then  $v\rho \in L^1(\mathbb{R}^d)$  using the logarithmic HLS inequality, see for instance [10].

**Notation for the ambient metric space.** We let  $X_\varepsilon := \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \times L^2(\mathbb{R}^d)$  endowed with the distance

$$D^2(z_1, z_2) = D^2((\rho_1, v_1), (\rho_2, v_2)) = W_2^2(\rho_1, \rho_2) + \varepsilon \|v_1 - v_2\|_{L^2(\mathbb{R}^d)}^2,$$

with the convention that  $X_0 = \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$  and  $D_0(z_1, z_2) = W_2(\rho_1, \rho_2)$ . Moreover, for  $z = \rho \in X_0 \times L^\infty(\mathbb{R}^d)$ ,  $\mathcal{F}_{0, \alpha}(z)$  will be understood to be  $\mathcal{F}_{0, \alpha}(\rho, v)$  with  $v = \mathcal{B}_{\alpha, d} * \rho$ , as usual when  $\varepsilon = 0$ .

In the space  $X_\varepsilon$  the metric derivative of an absolutely continuous curve  $t \mapsto z_t$  is denoted and defined by

$$|z'|_D(t) = \lim_{h \rightarrow 0} \frac{D(z_{t+h}, z_t)}{h},$$

and it exists for  $\mathcal{L}^1$ -a.e.  $t > 0$ . The local metric slope of the functional  $\mathcal{F}_{\varepsilon, \alpha}$  is defined by

$$|\partial \mathcal{F}_{\varepsilon, \alpha}|_D(z) := \limsup_{D(\zeta, z) \rightarrow 0} \frac{(\mathcal{F}_{\varepsilon, \alpha}(z) - \mathcal{F}_{\varepsilon, \alpha}(\zeta))^+}{D(\zeta, z)}.$$

These two abstractly defined objects are used to give the notion of *curves of maximal slope* in general metric setting, see [2, §3], [3, Chapter 1]. The main consequences of this gradient flow structure are summarized in the following result.

Before stating the Theorem we define the function  $\omega : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\omega(x) = \sqrt{mx\varphi(\mathbf{m}^{-1}x)}, \quad (3.2)$$

where  $\varphi$  is defined in (2.2). Moreover, given a fixed  $s_0 > 0$ , we define a strictly monotone continuous function  $G : [0, +\infty) \rightarrow [-\infty, +\infty)$  by  $G(s) := \int_{s_0}^s \frac{1}{\omega(r)} dr$  for  $s > 0$  and  $G(0) = -\infty$  (we observe that  $G^{-1} : [-\infty, +\infty) \rightarrow [0, +\infty)$  is surjective).

**Theorem 3.1.** *Let  $t \mapsto z_t = (\rho_t, v_t)$  be a bounded solution of problem (1.1) for  $\varepsilon > 0$ , starting from  $z^0 = (\rho^0, v^0) \in X_\varepsilon \cap (L^\infty(\mathbb{R}^d) \times (W^{1,2}(\mathbb{R}^d) \cap \Lambda_2(\mathbb{R}^d)))$ , according to Definition 1.2. If  $\varepsilon = 0$ , let  $z_t = \rho_t$  be a bounded solution to problem (1.2), starting from  $z^0 = \rho^0 \in X_0 \cap L^\infty(\mathbb{R}^d)$ , according to Definition 1.1. Then the three following properties hold:*

- i) *The evolution variational inequality (EVI) formulation: for any  $\bar{z} = (\bar{\rho}, \bar{v}) \in X_\varepsilon \cap (L^\infty(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d))$  (reduced to  $\bar{z} = \bar{\rho} \in X_0 \cap L^\infty(\mathbb{R}^d)$  if  $\varepsilon = 0$ ), the map  $t \mapsto D^2(z_t, \bar{z})$  is absolutely continuous and there exists a constant  $C$  depending on  $\|\rho\|_{L^\infty((0,T) \times \mathbb{R}^d)}$ ,  $\|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)}$  and  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}$ , such that*

$$\frac{1}{2} \frac{d}{dt} D^2(z_t, \bar{z}) \leq \mathcal{F}_{\varepsilon, \alpha}(\bar{z}) - \mathcal{F}_{\varepsilon, \alpha}(z_t) + C\omega(D^2(z_t, \bar{z})) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T). \quad (3.3)$$

- ii) *The energy dissipation equality (EDE) in metric sense: the map  $t \mapsto \mathcal{F}_{\varepsilon, \alpha}(z_t)$  is locally Lipschitz continuous and*

$$\frac{d}{dt} \mathcal{F}_{\varepsilon, \alpha}(z_t) = -\frac{1}{2} |\partial \mathcal{F}_{\varepsilon, \alpha}|_D^2(z_t) - \frac{1}{2} |z'|_D^2(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T). \quad (3.4)$$

- iii) *The following expansion control property: given another bounded solution  $t \mapsto \zeta_t$ , with initial datum  $\zeta^0$  in the same space of  $z^0$  above, there exists a constant  $C$ , depending on  $\|\rho\|_{L^\infty((0,T) \times \mathbb{R}^d)}$  and  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}$  (and the same quantities associated to  $\zeta$ ), such that there holds*

$$D^2(z_t, \zeta_t) \leq G^{-1}(G(D^2(z^0, \zeta^0)) + 4Ct) \quad \text{for every } t \in [0, T]. \quad (3.5)$$

We explicitly observe that all the constants in Theorem 3.1 clearly depend also on the parameters of the problem:  $\varepsilon, \mathbf{m}, d, \alpha$ . Often we omit to mention this dependence and we only stress the more relevant dependence on the norms of the data.

*Proof.* We first introduce the auxiliary functional

$$\Phi_{\varepsilon, \alpha}(\rho, v) := \int_{\mathbb{R}^d} (\rho \log \rho - v\rho) dx,$$

for  $\rho$  and  $v$  being as in the definition of  $\mathcal{F}_{\varepsilon, \alpha}$  at the beginning of this section, so that

$$\mathcal{F}_{\varepsilon, \alpha}(\rho, v) = \Phi_{\varepsilon, \alpha}(\rho, v) + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla v|^2 + \alpha v^2) dx$$

and, for  $d = 1, 2$ ,

$$\Phi_{0,0}(\rho, v) = \mathcal{F}_{0,0}(\rho, v) - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla(v - v^*)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} \rho^* v^* dx - \int_{\mathbb{R}^d} \rho^* v dx. \quad (3.6)$$

The proof is organized in four steps.



*Step 1. Quasi-Lipschitz Estimate implies control of the evolution of the Wasserstein distance.*-  
 Thanks to the assumption (1.5), we learn that the Fisher information  $\int_{\mathbb{R}^d} \frac{|\nabla \rho_t(x)|^2}{\rho_t(x)} dx$  is finite for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$ . Let  $\bar{\rho} \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$ . Exploiting the differentiability properties of the entropy functional, we can use the above-the-tangent formulation of displacement convexity to get for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$

$$\int_{\mathbb{R}^d} \bar{\rho}(x) \log \bar{\rho}(x) dx - \int_{\mathbb{R}^d} \rho_t(x) \log \rho_t(x) dx \geq \int_{\mathbb{R}^d} \langle \nabla \rho_t(x), \mathcal{T}_t(x) - x \rangle dx, \quad (3.7)$$

where  $\mathcal{T}_t$  denotes the optimal transport map between  $\rho_t$  and  $\bar{\rho}$ . We refer to [2, §3.3.1] for an intuitive proof of this fact, and to [3, Chapter 10] for the theory in full generality. In particular, the finiteness of the Fisher information of  $\rho_t$  implies that the second term is finite, so that this differentiation formula is meaningful. If  $\varepsilon > 0$  (resp.  $\varepsilon = 0$ ), let  $\bar{v} \in W^{1,2}(\mathbb{R}^d)$  (resp.  $\bar{v} = \mathcal{B}_{\alpha,d} * \bar{\rho}$ ). Take

$$I_t := \Phi_{\varepsilon,\alpha}(\bar{\rho}, \bar{v}) - \Phi_{\varepsilon,\alpha}(\rho_t, v_t) + \int_{\mathbb{R}^d} (\bar{v}(x) - v_t(x)) \bar{\rho}(x) dx.$$

Using the notation  $x_t^s := (1-s)x + s\mathcal{T}_t(x)$ ,  $s \in [0, 1]$ , and taking into account that

$$\begin{aligned} \int_{\mathbb{R}^d} v_t(x)(\bar{\rho}(x) - \rho_t(x)) dx &= \int_{\mathbb{R}^d} (v_t(\mathcal{T}_t(x)) - v_t(x)) \rho_t(x) dx \\ &= \int_{\mathbb{R}^d} (v_t(x_t^1) - v_t(x_t^0)) \rho_t(x) dx \\ &= \int_0^1 \frac{d}{ds} \int_{\mathbb{R}^d} v_t(x_t^s) \rho_t(x) dx ds \end{aligned}$$

and (3.7), we obtain for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$

$$\begin{aligned} I_t &\geq \int_{\mathbb{R}^d} \langle \nabla \rho_t(x), \mathcal{T}_t(x) - x \rangle dx - \int_{\mathbb{R}^d} v_t(x)(\bar{\rho}(x) - \rho_t(x)) dx \\ &= \int_{\mathbb{R}^d} \langle \nabla \rho_t(x), \mathcal{T}_t(x) - x \rangle dx - \int_0^1 \int_{\mathbb{R}^d} \langle \nabla v_t(x_t^s), \mathcal{T}_t(x) - x \rangle \rho_t(x) dx ds \\ &= \int_{\mathbb{R}^d} \langle \nabla \rho_t(x) - \rho_t(x) \nabla v_t(x), \mathcal{T}_t(x) - x \rangle dx \\ &\quad - \int_0^1 \int_{\mathbb{R}^d} \langle \nabla v_t(x_t^s) - \nabla v_t(x), \mathcal{T}_t(x) - x \rangle \rho_t(x) dx ds. \end{aligned}$$

Let us denote by  $II_t$  the last term in the right hand side above. The crucial point is to treat such term using the log-Lipschitz property of  $\nabla v$ . Notice that, if  $\varepsilon = 0$ , we are in the assumptions of Proposition 2.1 and we apply (2.1), where the constant  $C$  depends in principle only on  $(\mathbf{m}, \alpha, d)$  and the  $L^\infty$  norm of  $\rho_t$ , which we are assuming to be uniformly bounded on  $(0, T)$ . In the case  $\varepsilon > 0$ , still by the uniform space-time  $L^\infty$  assumption on  $\rho_t$  and the  $\Lambda_2$  assumption on  $v^0$ , we are in the framework of Proposition 2.2, so that we can apply the estimate (2.3). In this case the constant will depend also on  $(\varepsilon)$  and  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}$ . Since  $\varphi$  is concave, we can also use the Jensen inequality, and letting  $\rho_t^s = x_{t\#}^s \rho_t$  be the Wasserstein

geodesic connecting  $\rho_t$  and  $\bar{\rho}$  we have

$$\begin{aligned}
|II_t| &\leq W_2(\rho_t, \bar{\rho}) \int_0^1 \left( \int_{\mathbb{R}^d} |\nabla v_t(x_t^s) - \nabla v_t(x)|^2 \rho_t(x) dx \right)^{1/2} ds \\
&\leq CW_2(\rho_t, \bar{\rho}) \int_0^1 \left( \int_{\mathbb{R}^d} \varphi(|x_t^s - x|^2) \rho_t(x) dx \right)^{1/2} ds \\
&\leq \sqrt{m} CW_2(\rho_t, \bar{\rho}) \int_0^1 \sqrt{\varphi(m^{-1}W_2^2(\rho_t, \rho_t^s))} ds \\
&\leq \sqrt{m} CW_2(\rho_t, \bar{\rho}) \sqrt{\varphi(m^{-1}W_2^2(\rho_t, \bar{\rho}))}.
\end{aligned} \tag{3.8}$$

The last inequality holds since geodesic interpolation ensures

$$\int_{\mathbb{R}^d} |x - x_t^s|^2 \rho_t(x) dx = W_2^2(\rho_t, \rho_t^s) = s^2 W_2^2(\rho_t, \bar{\rho})$$

for all  $s \in [0, 1]$  and since  $\varphi$  is non decreasing. We recall that the constant  $C$  in (3.8) depends only on  $(\varepsilon, \alpha, d, \text{the mass } m \text{ and})$  the  $L^\infty((0, T) \times \mathbb{R}^d)$  norm of  $\rho$  and, in the case  $\varepsilon > 0$ , the  $\Lambda_2(\mathbb{R}^d)$  norm of  $v^0$ . Inserting this in the estimate for  $I_t$ , we have for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$

$$I_t \geq \int_{\mathbb{R}^d} \langle \nabla \rho_t(x) - \rho_t(x) \nabla v_t(x), \mathcal{T}_t(x) - x \rangle dx - C\omega(W_2^2(\rho_t, \bar{\rho})), \tag{3.9}$$

where  $\omega$  is the function defined in (3.2). Since  $\rho_t$  satisfies the continuity equation

$$\partial_t \rho_t + \text{div}(\xi_t \rho_t) = 0 \quad \text{with} \quad \rho_t \xi_t = -\nabla \rho_t + \rho_t \nabla v_t$$

and (1.5), the uniform  $L^\infty$  bound of  $\rho_t$  implies that  $\int_0^T \|\xi_t\|_{L^2(\mathbb{R}^d, \rho_t; \mathbb{R}^d)}^2 dt < +\infty$ . Therefore  $t \mapsto \rho_t$  is absolutely continuous with respect to  $W_2$  and by (2.4)

$$\frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) = \int_{\mathbb{R}^d} \langle \nabla \rho_t(x) - \rho_t(x) \nabla v_t(x), \mathcal{T}_t(x) - x \rangle dx \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T).$$

Inserting this into (3.9), and recalling the definition of  $I_t$ , we finally obtain

$$\frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) \leq \Phi_{\varepsilon, \alpha}(\bar{\rho}, \bar{v}) - \Phi_{\varepsilon, \alpha}(\rho_t, v_t) + \int_{\mathbb{R}^d} (\bar{v} - v_t) \bar{\rho} dx + C\omega(W_2^2(\rho_t, \bar{\rho})) \tag{3.10}$$

for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$ .

*Step 2: EVI for the parabolic-parabolic case.*- Recalling that  $\bar{v} \in W^{1,2}(\mathbb{R}^d)$ , observing that  $\Delta v_t \in L^2(\mathbb{R}^d)$  for a.e.- $t \in (0, T)$  and using the elementary identity  $|a|^2 - |b|^2 = |a - b|^2 + 2\langle b, a - b \rangle$  for every  $a, b \in \mathbb{R}^k$ , the variation of the second part of the functional (1.3) (that is,  $\mathcal{F}_{\varepsilon, \alpha} - \Phi_{\varepsilon, \alpha}$ ) can be written as

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^d} [|\nabla \bar{v}|^2 - |\nabla v_t|^2 + \alpha(\bar{v}^2 - v_t^2)] dx \\
&= \int_{\mathbb{R}^d} (\alpha v_t - \Delta v_t)(\bar{v} - v_t) dx + \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 \\
&= \int_{\mathbb{R}^d} (\rho_t - \varepsilon \partial_t v_t)(\bar{v} - v_t) dx + \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 \\
&= \int_{\mathbb{R}^d} \rho_t(\bar{v} - v_t) dx + \frac{\varepsilon}{2} \frac{d}{dt} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned} \tag{3.11}$$

Therefore, we deduce

$$\begin{aligned} \mathcal{F}_{\varepsilon,\alpha}(\bar{\rho}, \bar{v}) - \mathcal{F}_{\varepsilon,\alpha}(\rho_t, v_t) &= \Phi_{\varepsilon,\alpha}(\bar{\rho}, \bar{v}) - \Phi_{\varepsilon,\alpha}(\rho_t, v_t) + \int_{\mathbb{R}^d} \rho_t(\bar{v} - v_t) dx \\ &+ \frac{\varepsilon}{2} \frac{d}{dt} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (3.12)$$

Now, we use again (3.10), leading to

$$\begin{aligned} &\frac{\varepsilon}{2} \frac{d}{dt} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) \\ &\leq \mathcal{F}_{\varepsilon,\alpha}(\bar{\rho}, \bar{v}) - \mathcal{F}_{\varepsilon,\alpha}(\rho_t, v_t) + C\omega(W_2^2(\rho_t, \bar{\rho})) \\ &+ \int_{\mathbb{R}^d} (\bar{\rho} - \rho_t)(\bar{v} - v_t) dx - \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (3.13)$$

By using the duality between  $\dot{H}^1$  and  $\dot{H}^{-1}$ , the Young inequality, and (2.5) we have

$$\begin{aligned} \int_{\mathbb{R}^d} (\bar{\rho} - \rho_t)(\bar{v} - v_t) dx &\leq \|\bar{\rho} - \rho_t\|_{\dot{H}^{-1}(\mathbb{R}^d)} \|\bar{v} - v_t\|_{\dot{H}^1(\mathbb{R}^d)} \\ &\leq \frac{1}{2} \|\bar{\rho} - \rho_t\|_{\dot{H}^{-1}(\mathbb{R}^d)}^2 + \frac{1}{2} \|\nabla(\bar{v} - v_t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \frac{1}{2} QW_2^2(\bar{\rho}, \rho_t) + \frac{1}{2} \|\nabla(\bar{v} - v_t)\|_{L^2(\mathbb{R}^d)}^2, \end{aligned} \quad (3.14)$$

where  $Q$  is the largest of the  $L^\infty$  norms of  $\bar{\rho}$  and  $\rho_t$  over the time interval  $(0, T)$ . Taking into account that  $\omega$  is given by (3.2) and that  $\sqrt{\mathfrak{m}\varphi(\mathfrak{m}^{-1}x^2)} \geq x$  for every  $x > 0$ , combining (3.13) and (3.14) we get, up to introducing a new constant  $C$ ,

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) &\leq \mathcal{F}_{\varepsilon,\alpha}(\bar{\rho}, \bar{v}) - \mathcal{F}_{\varepsilon,\alpha}(\rho_t, v_t) \\ &+ C\omega(W_2^2(\rho_t, \bar{\rho})) - \frac{\alpha}{2} \|v_t - \bar{v}\|_{L^2(\mathbb{R}^d)}^2 \end{aligned} \quad (3.15)$$

for a.e.  $t \in (0, T)$ . The new constant  $C$  depends as usual on  $(\varepsilon, \alpha, d, \mathfrak{m})$  and  $\|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d)}$ ,  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}$ ,  $\|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)}$ .

*Step 3: EVI for the parabolic-elliptic case.*— When either  $d \geq 3$  or  $\alpha > 0$ , we can repeat the proof of the parabolic-parabolic case, letting  $\varepsilon = 0$  therein and recalling that  $\bar{v}$  is no more an arbitrary  $W^{1,2}(\mathbb{R}^d)$  function but is given by convolution with  $\bar{\rho}$ . In particular we arrive to the corresponding of (3.13), and the second line therein can now be estimated as follows. Using the inequality  $\|v\|_{H_\alpha^1(\mathbb{R}^d)} \leq \|\rho\|_{H_\alpha^{-1}(\mathbb{R}^d)}$  for  $-\Delta v + \alpha v = \rho$ ,  $\alpha > 0$ , where the notation is  $\|v\|_{H_\alpha^1(\mathbb{R}^d)}^2 := \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \alpha \|v\|_{L^2(\mathbb{R}^d)}^2$  (and using  $\dot{H}^1$  if  $\alpha = 0$ ), we get

$$\begin{aligned} \int_{\mathbb{R}^d} (\bar{v} - v_t)(\bar{\rho} - \rho_t) dx &\leq \|\bar{v} - v_t\|_{H_\alpha^1(\mathbb{R}^d)} \|\bar{\rho} - \rho_t\|_{H_\alpha^{-1}(\mathbb{R}^d)} \\ &\leq \frac{1}{2} \|\bar{v} - v_t\|_{H_\alpha^1(\mathbb{R}^d)}^2 + \frac{1}{2} \|\bar{\rho} - \rho_t\|_{H_\alpha^{-1}(\mathbb{R}^d)}^2. \end{aligned}$$

Moreover, recalling the estimate (2.5) (which works both in  $\dot{H}^{-1}$  and  $H_\alpha^{-1}$ ) we have

$$\|\bar{\rho} - \rho_t\|_{H_\alpha^{-1}(\mathbb{R}^d)} \leq QW_2^2(\bar{\rho}, \rho_t),$$

for all  $t \in [0, T]$ , where  $Q$  is the largest of the  $L^\infty$  norms of  $\bar{\rho}$  and  $\rho_t$  over the time interval  $[0, T]$ . Inserting these estimates in (3.13) we obtain

$$\frac{1}{2} \frac{d}{dt} W_2^2(\rho_t, \bar{\rho}) \leq \mathcal{F}_{0,\alpha}(\bar{\rho}) - \mathcal{F}_{0,\alpha}(\rho_t) + C\omega(W_2^2(\rho_t, \bar{\rho})), \quad (3.16)$$

for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$ , where the constant  $C$  depends only on  $\varepsilon$ ,  $\alpha$ ,  $d$ ,  $\mathbf{m}$ ,  $\|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)}$ ,  $\|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d)}$ .

In the case  $\alpha = 0$ ,  $d = 1, 2$ , we have to consider the functional in (3.1). By using the identity

$$\frac{1}{2} \|\nabla(\bar{v} - v^*)\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{2} \|\nabla(v_t - v^*)\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (\rho_t - \rho^*)(\bar{v} - v_t) dx + \frac{1}{2} \|\nabla(v_t - \bar{v})\|_{L^2(\mathbb{R}^d)}^2,$$

with similar computations as in (3.11), this time considering  $\mathcal{F}_{0,0}(\rho, v) - \Phi_{0,0}(\rho, v)$  as obtained from (3.6), we can still find (3.12) and conclude obtaining again (3.16).

*Step 4: Conclusion.*- We are ready to prove the three points in the statement of the theorem. The proof of *i*) is a consequence of (3.15) for the case  $\varepsilon > 0$ , and (3.16) for the case  $\varepsilon = 0$ , taking into account that  $\alpha \geq 0$  and that  $\omega(D^2(z_t, \bar{z})) \geq \omega(W_2^2(\rho_t, \bar{\rho}))$  being  $\omega$  increasing.

It is a standard fact that the gradient flow formulation in EVI sense implies the one in EDE sense in (3.4). Indeed, the proof of *ii*) follows from (3.3) and (3.5) and can be exactly carried out as in [2, Proposition 3.6].

The proof of (3.5) still follows from (3.3). Indeed we can apply [3, Lemma 4.3.4] (see also the argument of [3, Theorem 11.1.4]) and obtain that for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$

$$\frac{1}{2} \frac{d}{ds} D^2(z_s, \zeta_s) \Big|_{s=t} \leq \frac{1}{2} \frac{d}{ds} D^2(z_s, \zeta_t) \Big|_{s=t} + \frac{1}{2} \frac{d}{ds} D^2(z_t, \zeta_s) \Big|_{s=t} \leq 2C\omega(D^2(z_t, \zeta_t)). \quad (3.17)$$

Here,  $C = \max\{C_1, C_2\}$ , where  $C_1$  is the supremum on  $s \in (0, T)$  of the constant in (3.3) for  $z_t$  with  $\bar{z} = \zeta_s$ , which is finite since the first component of  $\zeta$  belongs to  $L^\infty((0, T) \times \mathbb{R}^d)$ , and  $C_2$  is the same inverting  $z$  and  $\zeta$ . The estimate (3.17) implies

$$\frac{d}{dt} D^2(z_t, \zeta_t) \leq 4C\omega(D^2(z_t, \zeta_t)), \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T).$$

Since the inequality

$$y(t) \leq y(0) + 4C \int_0^t \omega(y(s)) ds$$

entails that  $y(t) \leq G^{-1}(G(y(0)) + 4Ct)$ , we conclude.  $\square$

*Proof of Theorems 1.3 and 1.4.* The main theorems in the introduction are now a straightforward consequence of the expansion control *iii*) in Theorem 3.1. Both Theorems follow from the inequality (3.5), observing that  $G^{-1}(G(0) + 4Ct) = G^{-1}(-\infty) = 0$ , and recalling that  $W^{2,\infty}(\mathbb{R}^d) \subset \Lambda_2(\mathbb{R}^d)$ .  $\square$

#### 4. $\omega$ -CONVEXITY OF THE FUNCTIONAL

In this section we show another consequence of the EVI formulation of bounded solutions. For the functional  $\mathcal{F}_{\varepsilon,\alpha}$  a relaxed  $\omega$ -convexity along geodesics holds, see [18] for  $\omega$ -convexity of functionals on measures. The proof of the geodesic convexity as a consequence of the EVI was introduced in [22].

We take  $Z_\varepsilon := X_\varepsilon \cap (L^\infty(\mathbb{R}^d) \times (W^{1,2}(\mathbb{R}^d) \cap \Lambda_2(\mathbb{R}^d)))$  if  $\varepsilon > 0$  (resp.  $Z_0 := X_0 \cap L^\infty(\mathbb{R}^d)$ ) if  $\varepsilon = 0$ ) as the set of initial data for the evolution equation (1.1) (resp. (1.2)), as in theorem

**3.1.** We remark that such set is geodesically convex in  $(X_\varepsilon, D)$ . This is trivial for the part concerning  $v$  while for the density  $\rho$  we use the classical displacement convexity of all the  $L^p$  norms [32] (moreover, along a geodesic the  $L^\infty(\mathbb{R}^d)$  norm of  $\rho$  and the energy are uniformly bounded in terms of the same quantities at the endpoints).

Here we also assume that bounded solutions to (1.1) (resp. (1.2) for  $\varepsilon = 0$ ), with initial data in  $Z_\varepsilon$ , verify that for some  $T > 0$

$$\|\rho_t\|_{L^\infty(\mathbb{R}^d)} \leq R(\rho^0, v^0) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T), \quad (4.1)$$

where  $R$  is a bound only depending (increasingly) on  $\|\rho^0\|_{L^\infty(\mathbb{R}^d)}$  and the initial energy  $\mathcal{F}_{\varepsilon, \alpha}(\rho^0, v^0)$  (and possibly, if  $\varepsilon > 0$ , on  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}$ ). This assumption has been proved in several cases, see the introduction for more details.

**Theorem 4.1.** *Assume that bounded solutions for the evolutions (1.1) (resp. (1.2) for  $\varepsilon = 0$ ), with initial data in  $Z_\varepsilon$ , exist and verify (4.1). Then, for every  $z^0, z^1 \in Z_\varepsilon$  there exists a constant  $\bar{C}$  (depending only on  $\|\rho^i\|_{L^\infty(\mathbb{R}^d)}$ ,  $\mathcal{F}_{\varepsilon, \alpha}(z^i)$ ,  $i = 0, 1$ , and, in the case  $\varepsilon > 0$ , also on  $\|v^i\|_{\Lambda_2(\mathbb{R}^d)}$ ,  $i = 0, 1$ ) such that for every geodesic  $s \in [0, 1] \rightarrow z^s$  of the space  $(X_\varepsilon, D)$  connecting  $z^0$  to  $z^1$  there holds*

$$\begin{aligned} \mathcal{F}_{\varepsilon, \alpha}(z^s) &\leq (1-s)\mathcal{F}_{\varepsilon, \alpha}(z^0) + s\mathcal{F}_{\varepsilon, \alpha}(z^1) \\ &\quad + \bar{C} [(1-s)\omega(s^2 D^2(z^0, z^1)) + s\omega((1-s)^2 D^2(z^0, z^1))] \quad \forall s \in [0, 1]. \end{aligned}$$

*Proof.* Let  $z^0, z^1 \in Z_\varepsilon$ , let  $s \in [0, 1] \rightarrow z^s = (\rho^s, v^s)$  be a geodesic of the space  $(X_\varepsilon, D)$  connecting  $z^0$  to  $z^1$ . Let  $t \mapsto z_t^s$  be the bounded solution of (1.1) or (1.2) in  $[0, T] \times \mathbb{R}^d$  starting from the initial datum  $z^s$ ,  $T$  being chosen according to (4.1).

Consider any  $\bar{z} = (\bar{\rho}, \bar{v}) \in X_\varepsilon \cap (L^\infty(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d))$  (reduced to  $\bar{z} = \bar{\rho} \in X_0 \cap L^\infty(\mathbb{R}^d)$  if  $\varepsilon = 0$ ). Taking into account that  $t \mapsto z_t^s$  is a bounded solution,  $t \mapsto D^2(z_t^s, \bar{z})$  is absolutely continuous and  $t \mapsto \mathcal{F}_{\varepsilon, \alpha}(z_t^s)$  is decreasing by (3.4), so that using (3.3) we obtain

$$\frac{1}{2}D^2(z_t^s, \bar{z}) - \frac{1}{2}D^2(z^s, \bar{z}) \leq t(\mathcal{F}_{\varepsilon, \alpha}(\bar{z}) - \mathcal{F}_{\varepsilon, \alpha}(z_t^s)) + C \int_0^t \omega(D^2(z_r^s, \bar{z})) dr \quad (4.2)$$

for all  $t \in [0, T]$ . We claim that (4.2) holds with a constant  $C$  depending only on the  $L^\infty(\mathbb{R}^d)$  norms of  $\rho^0, \rho^1, \bar{\rho}$ , on  $\mathcal{F}_{\varepsilon, \alpha}(z^0), \mathcal{F}_{\varepsilon, \alpha}(z^1)$  and, if  $\varepsilon > 0$ , on the  $\Lambda_2(\mathbb{R}^d)$  norms of  $v^0, v^1$ . Indeed, since  $T$  is chosen as in (4.1), the constant  $C$  in (4.2), coming from (3.3), depends only on  $\|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)}$ ,  $\|\rho^s\|_{L^\infty(\mathbb{R}^d)}$ ,  $\mathcal{F}_{\varepsilon, \alpha}(z^s)$  and  $\|v^s\|_{\Lambda_2(\mathbb{R}^d)}$ . But the last three quantities are uniformly bounded with respect to  $s$  as remarked at the beginning of this section and the claim follows. Therefore, multiplying by  $(1-s)$  the inequality (4.2) for  $\bar{z} = z^0$  and by  $s$  the inequality (4.2) for  $\bar{z} = z^1$ , then summing up, we may conclude that

$$\begin{aligned} &\frac{1}{2}((1-s)D^2(z_t^s, z^0) + sD^2(z_t^s, z^1)) - \frac{1}{2}((1-s)D^2(z^s, z^0) + sD^2(z^s, z^1)) \\ &\leq t((1-s)\mathcal{F}_{\varepsilon, \alpha}(z^0) + s\mathcal{F}_{\varepsilon, \alpha}(z^1) - \mathcal{F}_{\varepsilon, \alpha}(z_t^s)) \\ &\quad + \bar{C} \left( (1-s) \int_0^t \omega(D^2(z_r^s, z^0)) dr + s \int_0^t \omega(D^2(z_r^s, z^1)) dr \right), \end{aligned}$$

where  $\bar{C}$  depends only on  $\|\rho^0\|_{L^\infty(\mathbb{R}^d)}$ ,  $\|\rho^1\|_{L^\infty(\mathbb{R}^d)}$ ,  $\mathcal{F}_{\varepsilon, \alpha}(z^0)$ ,  $\mathcal{F}_{\varepsilon, \alpha}(z^1)$  and, in the case  $\varepsilon > 0$ , on  $\|v^0\|_{\Lambda_2(\mathbb{R}^d)}, \|v^1\|_{\Lambda_2(\mathbb{R}^d)}$ .

Using the fact that  $s \mapsto z^s$  is a geodesic, the left hand side is nonnegative, thus

$$\begin{aligned} & \mathcal{F}_{\varepsilon, \alpha}(z_t^s) - (1-s)\mathcal{F}_{\varepsilon, \alpha}(z^0) - s\mathcal{F}_{\varepsilon, \alpha}(z^1) \\ & \leq \bar{C} \left( (1-s) \frac{1}{t} \int_0^t \omega(D^2(z_r^s, z^0)) dr + s \frac{1}{t} \int_0^t \omega(D^2(z_r^s, z^1)) dr \right). \end{aligned}$$

The lower semi continuity of  $t \mapsto \mathcal{F}_{\varepsilon, \alpha}(z_t^s)$  and the continuity of  $r \mapsto D^2(z_r^s, z^i)$ ,  $i = 0, 1$  yield

$$\mathcal{F}_{\varepsilon, \alpha}(z^s) \leq (1-s)\mathcal{F}_{\varepsilon, \alpha}(z^0) + s\mathcal{F}_{\varepsilon, \alpha}(z^1) + \bar{C}((1-s)\omega(D^2(z^s, z^0)) + s\omega(D^2(z^s, z^1))).$$

Since  $s \mapsto z^s$  is a geodesic we have  $D^2(z^s, z^0) = s^2 D^2(z^1, z^0)$  and  $D^2(z^s, z^1) = (1-s)^2 D^2(z^0, z^1)$  and we conclude.  $\square$

**Remark 4.2.** In general, geodesical  $\lambda$ -convexity for some  $\lambda \in \mathbb{R}$  is not expected for functional  $\mathcal{F}_{\varepsilon, \alpha}$  on  $Z_\varepsilon$  (with respect to the distance  $D$ ), because of the presence of the term  $-\int_{\mathbb{R}^d} v \rho dx$ . For instance in the case  $\varepsilon > 0$  we may fix  $\tilde{v} \in \Lambda_2(\mathbb{R}^d) \setminus W^{2, \infty}(\mathbb{R}^d)$  such that  $-\tilde{v}$  is not a  $\lambda$ -convex function. With this choice of  $\tilde{v}$ , the functional  $\mathcal{F}_{\varepsilon, \alpha}$  is not  $\lambda$ -convex along geodesics of the form  $s \mapsto z^s = (\rho^s, v^s)$  with  $v^s \equiv \tilde{v}$ . We may recover the  $\lambda$ -convexity of  $\mathcal{F}_{\varepsilon, \alpha}$  in the case  $\varepsilon > 0$  by restricting to the set  $X_\varepsilon \cap (L^\infty(\mathbb{R}^d) \times (W^{1,2}(\mathbb{R}^d) \cap W^{2, \infty}(\mathbb{R}^d)))$ .

## 5. A REFINED RESULT IN ZYGMUND SPACES

This section is devoted to give a rigorous justification of the estimates of Section 2 in the parabolic case. We will also give a slight improvement of Theorem 3.1 and Theorem 1.4 by guaranteeing a suitable quasi-Lipschitz estimate under a more general condition on the initial datum  $v^0$ .

*Zygmund estimates and log-Lipschitz regularity in the parabolic case.* Let  $T > 0$ . Let us denote  $Q_T := (0, T) \times \mathbb{R}^d$  and then  $\bar{Q}_T := [0, T] \times \mathbb{R}^d$ . In the half  $d + 1$  dimensional space, we consider the standard parabolic metric

$$\delta((x, t), (y, s)) := \max\{|x - y|, \sqrt{|t - s|}\}.$$

With respect to the parabolic metric, the definition of Zygmund spaces adapts as follows. We have  $\Lambda_0(\bar{Q}_T) := L^\infty(Q_T)$ , and  $\Lambda_1(\bar{Q}_T)$  is the space of continuous bounded functions  $f$  over  $\bar{Q}_T$  such that there hold

$$\begin{aligned} & \sup_{\substack{x, y \in \mathbb{R}^d, x \neq y \\ t \in [0, T]}} \frac{|f(x, t) - 2f((x+y)/2, t) + f(y, t)|}{|x - y|} < +\infty, \\ & \sup_{\substack{x \in \mathbb{R}^d \\ 0 \leq s < t \leq T}} \frac{|f(x, t) - 2f(x, (t+s)/2) + f(x, s)|}{|t - s|^{1/2}} < +\infty. \end{aligned} \tag{5.1}$$

Moreover, we say that  $f \in L^\infty(Q_T)$  belongs to  $\Lambda_2(\bar{Q}_T)$  if

$$\sup_{x \in \mathbb{R}^d, 0 \leq s < t \leq T} \frac{|f(x, t) - 2f(x, (t+s)/2) + f(x, s)|}{|t - s|} < +\infty$$

and  $\nabla f \in \Lambda_1(\bar{Q}_T)$ . We see that  $f \in \Lambda_2(\bar{Q}_T)$  implies  $f \in L^\infty((0, T); W^{1, \infty}(\mathbb{R}^d))$ , with  $\nabla f$  satisfying (5.1), so that finally  $f$  satisfies also (2.3).

When dealing with parabolic equations, it is suitable to consider spaces of functions defined with respect to the parabolic metric, since it is natural to deal with functions which have derivative up to order  $k$  with respect to time and  $2k$  with respect to space. For classic results,

we refer for instance to [16] or to the monograph [27], where estimates are derived in Sobolev and Hölder spaces of this kind, see Chapter 4 therein.

In [16] we find that if the forcing term of the heat equation has bounded mean oscillation (BMO), still with respect to the parabolic metric, than the same holds true for second order space derivatives and first order time derivatives of the solution. This would be enough for deducing that first derivatives in space are in the Zygmund class with respect to the parabolic metric and that therefore they satisfy a log-Lipschitz estimate. The results in [16] deal only with null initial datum, but they can be generalized to more general data with suitable regularity requirements. Some extensions involving initial data in Zygmund classes are found in [1, 24], based on direct estimates on fundamental solutions. Summing up, we have

*Proof of Proposition 2.2.* Suppose that  $v$  is the solution (convolution with fundamental operator) of the forced heat equation  $\partial_t v = \Delta v + \rho$ . Suppose  $\rho \in \Lambda_0(\bar{Q}_T)$  and  $v^0 \in \Lambda_2(\mathbb{R}^d)$ . Then we have  $v \in \Lambda_2(Q_T)$ . See [16] for the case  $v^0 = 0$ , see [24, Theorem 4] for a general result. As already observed, if  $v \in \Lambda_2(Q_T)$  then (2.3) follows. If we consider the second equation of (1.1) with  $\alpha > 0$ , the fundamental solution is just multiplied by a decaying exponential at infinity and the same result carries over.  $\square$

This gives a rigorous justification of the assumptions on the initial datum of Theorem 3.1. However a refined analysis shows that this assumption can be weakened, as we do next.

*Initial datum in  $\Lambda_1(\mathbb{R}^d)$ .* We have to consider the weighted Zygmund space  $\Lambda_2^{-1}(Q_T)$ , defined as the corresponding space  $\Lambda_2(Q_T)$ , with the addition of a time weight which is divergent as  $t \rightarrow 0$ . In particular, locally in  $Q_T$  functions in  $\Lambda_2^{-1}(Q_T)$  have the same smoothness as the ones in  $\Lambda_2(Q_T)$ , but this regularity does no more extend to the closure of  $Q_T$ . More precisely, by definition  $f \in \Lambda_2^{-1}(Q_T)$  means that  $f \in \Lambda_1(\bar{Q}_T)$ ,

$$\sup_{\substack{x, y \in \mathbb{R}^d, x \neq y \\ t \in [0, T]}} \sqrt{t} \frac{|\nabla f(x, t) - 2\nabla f((x+y)/2, t) + \nabla f(y, t)|}{|x-y|} < +\infty \quad (5.2)$$

and the second finite differences of  $f$  and  $\nabla f$  with respect to time verify the corresponding estimates, as in the definition of  $\Lambda_2(Q_T)$ , still with the addition of the weight  $t^{1/2}$ .

**Theorem 5.1.** *Let  $T > 0$ . Let  $\rho^0 \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$  and  $v^0 \in \Lambda_1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$ . Let  $z_t = (\rho_t, v_t)$  be a bounded solution on  $[0, T] \times \mathbb{R}^d$  to the Cauchy problem for (1.1), according to Definition 1.2, with initial datum  $z^0 = (\rho^0, v^0)$ . For any reference point  $\bar{z} = (\bar{\rho}, \bar{v}) \in (\mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)) \times W^{1,2}(\mathbb{R}^d)$ , the general EVI holds*

$$\frac{1}{2} \frac{d}{dt} D^2(z_t, \bar{z}) \leq \mathcal{F}_{\varepsilon, \alpha}(\bar{z}) - \mathcal{F}_{\varepsilon, \alpha}(z_t) + Ct^{-1/2} \omega(D^2(z_t, \bar{z})) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T), \quad (5.3)$$

for a constant  $C$  depending on  $\|\rho\|_{L^\infty((0, T) \times \mathbb{R}^d)}$ ,  $\|v^0\|_{\Lambda^1(\mathbb{R}^d)}$ ,  $\|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)}$ .

Moreover the EDE (3.4) holds, and the expansion control property holds in this form: given another bounded solution  $t \mapsto \zeta_t$  as above with initial datum  $\zeta^0 \in (\mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)) \times (\Lambda_1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d))$  there is

$$D^2(z_t, \zeta_t) \leq G^{-1}(G(D^2(z^0, \zeta^0)) + 8C\sqrt{t}) \quad \text{for every } t \in [0, T], \quad (5.4)$$

where  $C$  is a constant depending on  $\|\rho\|_{L^\infty((0, T) \times \mathbb{R}^d)}$  and  $\|v^0\|_{\Lambda^1(\mathbb{R}^d)}$  (and the same quantities associated to  $\zeta$ ). In particular,  $z = \zeta$  if  $z^0 = \zeta^0$ .

*Proof.* Since we are in the hypotheses of [24, Theorem 4],  $v$  belongs to  $\Lambda_2^{-1}(Q_T)$ , so that (5.2) above holds for  $v$  and then, due to the log-Lipschitz regularity in the Zygmund class, we deduce

$$|\nabla v_t(x) - \nabla v_t(y)| \leq Kt^{-1/2}|x - y|(1 + \log^- |x - y|), \quad (5.5)$$

for all  $x, y \in \mathbb{R}^d$ ,  $t \in (0, T)$ , where  $K$  is a suitable constant depending only on  $T$  and the data. Notice that from the definition of  $\Lambda_2^{-1}(Q_T)$ , it does not follow that  $\nabla v \in L^\infty(Q_T)$ . Thus we deduce the weighted analogous of (2.3), that is

$$|\nabla v_t(x) - \nabla v_t(y)|^2 \leq \frac{C^2}{t} \varphi(|x - y|^2), \quad (5.6)$$

where  $C$  is a new suitable positive constant depending on the data and  $\varphi$  is defined in (2.2). Following the line of the proof Theorem 3.1 we reach the estimate (3.8) for  $II_t$ , which now has to be changed because we have to use (5.6), obtaining

$$|II_t| \leq Ct^{-1/2}W_2(\rho_t, \bar{\rho})\sqrt{\mathfrak{m}\varphi(\mathfrak{m}^{-1}W_2^2(\rho_t, \bar{\rho}))} = Ct^{-1/2}\omega(W_2^2(\rho_t, \bar{\rho})).$$

We can repeat all the other steps which lead to (3.14), obtaining the corresponding EVI with the additional weight  $t^{-1/2}$ , which directly lead to (5.3). We conclude as in Step 4 of the proof of Theorem 3.1: from (5.3), the EDE formulation (3.4) follows, still referring to [2, Proposition 3.6]. Moreover, (5.4) follows by (5.3) by

$$\frac{d}{dt}D^2(z_t, \zeta_t) \leq 4Ct^{-1/2}\omega(D^2(z_t, \zeta_t)), \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, T).$$

Indeed the inequality

$$y(t) \leq y(0) + 4C \int_0^t s^{-1/2}\omega(y(s)) ds$$

implies that  $y(t) \leq G^{-1}(G(y(0)) + 8C\sqrt{t})$  as desired. Finally, the uniqueness result follows since  $G(0) = -\infty$  and  $G^{-1}(-\infty) = 0$ .  $\square$

## 6. THE CASE OF NONLINEAR DIFFUSION

We show next how to adapt our techniques to more general aggregation diffusion equations in a quite straightforward way. Let us consider the problem

$$\begin{cases} \partial_t \rho &= \operatorname{div}(\rho \nabla P(\rho)) - \operatorname{div}(\rho \nabla v), \\ \varepsilon \partial_t v &= \Delta v + \rho - \alpha v, \end{cases} \quad (6.1)$$

to which we associate the functional

$$\mathcal{G}_{\varepsilon, \alpha}(\rho, v) := \int_{\mathbb{R}^d} (\Psi(\rho) - v\rho) dx + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla v|^2 + \alpha v^2) dx, \quad (6.2)$$

for all  $\varepsilon > 0$ ,  $\alpha \geq 0$ ,  $\rho \in \mathcal{M}_2(\mathbb{R}^d; \mathfrak{m}) \cap L^\infty(\mathbb{R}^d)$ ,  $v \in W^{1,2}(\mathbb{R}^d)$ , where

$$\Psi(\rho) := \int_0^\rho P(r) dr.$$

We give the same restrictions as [3, §9.3], the first one being

$$\lim_{r \rightarrow 0} \frac{\Psi(r)}{r^q} > -\infty \quad \text{for some } q > \frac{d}{d+2},$$

a property ensuring that  $\int_{\mathbb{R}^d} \Psi(\rho) \neq -\infty$ . Moreover, the crucial property to be satisfied by the new nonlinearity is the displacement convexity, that is the map  $r \mapsto r^d \Psi(r^{-d})$  is convex



and nondecreasing on  $(0, +\infty)$ . This notion, introduced in [32], is stronger than convexity and corresponds for  $C^2$  functions to the inequality

$$r^{-1}\Psi(r) - \Psi'(r) + r\Psi''(r) \geq -\frac{1}{d-1}r\Psi''(r) \quad \forall r \in (0, +\infty).$$

The more relevant cases correspond to nonlinear diffusion of power kind. Indeed, if

$$\Psi(\rho) = \frac{1}{m-1}\rho^m, \quad m \geq \frac{d-1}{d}$$

the displacement convexity property holds. The case  $m > 1$  (resp.  $m < 1$ ) correspond to a slow diffusion (resp. fast diffusion) in the equation. On the other hand, the linear diffusion is recovered taking  $P(\rho) = \log \rho$ , it is seen that in this case functional (6.2) is reduced, up to a constant, to (1.3). Finally, let us mention that the free-energy functional in the parabolic-elliptic case is similar to (1.4) and given by

$$\mathcal{G}_{0,\alpha}(\rho, v) := \int_{\mathbb{R}^d} (\Psi(\rho) - \frac{1}{2}v\rho) dx, \quad (6.3)$$

for  $\rho \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$  and  $v = \mathcal{B}_{\alpha,d} * \rho$ . It can be written as (6.2), taking into account the same renormalization as in (3.1), to be done in the pathological cases  $\varepsilon = \alpha = 0$  and  $d = 1, 2$ .

The notion of bounded solution is completely analogous to Definitions 1.1 and 1.2, both for the parabolic-elliptic and the parabolic-parabolic case. Indeed, the only point to adapt is the finiteness of the Fisher information, now rewritten into the generalized version

$$\int_0^T \int_{\mathbb{R}^d} |\nabla P(\rho_t(x))|^2 \rho_t(x) dx dt < +\infty. \quad (6.4)$$

**Corollary 6.1.** Theorem 1.3, Theorem 1.4, Theorem 3.1 and Theorem 5.1 hold for bounded solutions to (6.1).

*Proof.* The displacement convexity property makes the internal energy functional

$$\rho \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \mapsto \int_{\mathbb{R}^d} \Psi(\rho(x)) dx$$

convex along Wasserstein geodesics, as shown in [3, §9.3]. This in turn gives the possibility to write down a subdifferential inequality in Wasserstein sense (for a definition see [3, §10.1.1]) as follows. Let  $\rho \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$  be such that  $\int_{\mathbb{R}^d} |\nabla P(\rho)|^2 \rho dx$  is finite. Then

$$\int_{\mathbb{R}^d} \Psi(\bar{\rho}(x)) dx - \int_{\mathbb{R}^d} \Psi(\rho(x)) dx \geq \int_{\mathbb{R}^d} \langle \nabla P(\rho(x)), \mathcal{T}(x) - x \rangle \rho(x) dx, \quad (6.5)$$

for any  $\bar{\rho} \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m})$ , where  $\mathcal{T}$  is the optimal transport map from  $\rho$  to  $\bar{\rho}$ . Convexity and differentiability of functionals defined on probability densities, as the internal energy, are standard elements in the theory of Wasserstein gradient flows. For the proof of inequality (6.5), which characterizes the vector  $\nabla P(\rho)$  as the Wasserstein subdifferential of the internal energy functional, we refer to [2, §3.3.1] or to the general theory in [3, §10.4.3].

On the other hand, (6.5) can be used to generalize the proof of Theorem 3.1. Indeed, if  $(\rho_t, v_t)$  solves (6.1) according to our notion of solution, thanks to (6.4)  $\rho_t$  satisfies the identity (6.5) for almost any  $t$ . From this inequality, all the rest of the proof of Theorem 3.1 can

be carried out. Indeed, with the same notation therein, we obtain the  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$  inequality

$$I_t := \mathcal{G}_{\varepsilon, \alpha}(\bar{\rho}, \bar{v}) - \mathcal{G}_{\varepsilon, \alpha}(\rho_t, v_t) + \int_{\mathbb{R}^d} (\bar{v}(x) - v_t(x)) \bar{\rho}(x) dx \geq \int_{\mathbb{R}^d} \langle \rho_t(x) \nabla P(\rho_t(x)) - \rho_t(x) \nabla v(x), \mathcal{T}_t(x) - x \rangle dx - C\omega(W_2^2(\rho_t, \bar{\rho})),$$

for any  $\bar{\rho} \in \mathcal{M}_2(\mathbb{R}^d; \mathbf{m}) \cap L^\infty(\mathbb{R}^d)$  and any  $\bar{v} \in W^{1,2}(\mathbb{R}^d)$  if  $\varepsilon > 0$  or  $\bar{v} = \mathcal{B}_{\alpha, d} * \bar{\rho}$  if  $\varepsilon = 0$ . This estimate substitutes (3.9) in the proof of Theorem 3.1. The rest of the proofs is completely analogous.  $\square$

Notice that the corresponding of Theorem 4.1 also holds in the nonlinear diffusion case. However, here we have less knowledge about existence of bounded solutions.

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