SELF-INTERSECTION OF OPTIMAL GEODESICS

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Abstract. Let (X, d, m) be a geodesic metric measure space. Consider a geodesic μ_t in the L^2-Wasserstein space. Then as s goes to t, the support of μ_s and the support of μ_t have to overlap, provided an upper bound on the densities holds. We give a more precise formulation of this self-intersection property. Given a geodesic of (X, d, m) and $t \in [0, 1]$, we consider the set of times for which this geodesic belongs to the support of μ_t. We prove that t is a point of Lebesgue density 1 for this set, in the integral sense. Our result applies to spaces satisfying $\mathsf{CD}(K, \infty)$. The non branching property is not needed.

1. Introduction

Let (X, d, m) be a complete and separable metric measure space, that is (X, d) is a complete and separable metric space and m a Radon measure. Additionally we assume that

- X coincides with the support of m;
- (X, d) is a geodesic space.

Denote the set of all probability measures on X by $\mathcal{P}(X)$. The L^p-Wasserstein distance W_p between two probability measures μ_0 and μ_1 is defined to be

$$W_p^p(\mu_0, \mu_1) := \inf_{q \in \mathsf{Cpl}(\mu_0,\mu_1)} \int d^p(x, y)\ q(dx, dy),$$

where $\mathsf{Cpl}(\mu_0, \mu_1)$ is the set of all couplings between μ_0 and μ_1, i.e. probability measures on $X \times X$ with marginals μ_0 and μ_1. It can be shown that W_p is a metric on the space $\mathcal{P}_p(X) := \{\mu \in \mathcal{P}(X) : \int_X d^p(x_0, x)\ \mu(dx) < \infty\}$, where x_0 is arbitrary. In this note we will focus on the case $p = 2$.

It is then well-known that the associated L^2-Wasserstein space $(\mathcal{P}_2(X), W_2)$ is a geodesic space as well: to any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ we can associate a geodesic $[0, 1] \ni t \mapsto \mu_t$ joining μ_0 to μ_1, see Chapter 7 of [6].

Given μ_t, under some general assumptions on the metric measure space (X, d, m), like $\mathsf{CD}(K, \infty)$, see [2], [4], [5] for their definitions, it is possible to prove that if μ_0 and μ_1 are both absolutely continuous with respect to reference measure m with bounded densities, then the same property holds for the density of μ_t. In particular $\mu_t \ll m$ and its density is bounded uniformly in $t \in [0, 1]$, see for instance [3].

Thanks to this uniform bound on the density, by means of standard arguments in measure theory, one can prove that the support of μ_t has to overlap with μ_0 as t goes to 0, otherwise to much "mass" would be present inside the support of μ_0. The same property holds for any other time $s \in (0, 1]$ as t goes to s. This overlapping property is to our knowledge one of the few qualitative properties of the support of μ_t that has been proved so far.

In this note we want to give a more careful analysis of this overlapping property. We prove a structural property of supp$[\mu_t]$. Let $\mathcal{G}(X) \subset C([0, 1]; X)$ denote the subset of geodesics in (X, d) and $\mathcal{P}(\mathcal{G}(X))$ the space of probability measures over it. Then, it can be shown that to each geodesic $t \mapsto \mu_t \in \mathcal{P}_2(X)$ it is possible to associate $\nu \in \mathcal{P}(\mathcal{G}(X))$, so that

$$(e_t)_\sharp \nu = \mu_t, \qquad e_t : C([0, 1]; X) \to X, \quad e_t(\gamma) := \gamma_t,$$

for all $t \in [0, 1]$, with e_t the evaluation map at time t (e.g. see [6, Theorem 7.21 and Corollary 7.22]). We refer to ν as dynamical optimal plan. Our result will be stated in terms of the support of ν, denoted by $G \subset \mathcal{G}(X)$. For each $t \in [0, 1]$ and $\gamma \in G$ consider the set

$$I_t(\gamma) := \{\tau \in [0, 1] : \gamma_\tau \in e_t(G)\}.$$

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$I_t(\gamma)$ is the set of times for which γ remains inside the support of $e_t(G)$. So clearly $t \in I_t(\gamma)$. We will prove that if there exists a positive constant C so that

$$\mu_\tau = \varrho_\tau m, \quad \varrho_\tau \le C,$$

for all τ in a neighborhood of $t \in (0,1)$, then t is a point of Lebesgue density 1 for $I_t(\gamma)$ in the $L^1(\nu)$-sense that is

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}^1\big(I_t(\gamma) \cap (t-\varepsilon, t+\varepsilon)\big)}{2\varepsilon} = 1, \qquad \text{in } L^1(G,\nu).$$

The problem addressed in this note appeared in the recent paper [1] (Lemma 3.2 and its application to Theorem 7.8). It was used to develop a certain kind of parallel transport along Wasserstein geodesics with application to the problem of globalization for metric measure spaces with lower Ricci curvature bounds in the sense of Lott-Sturm-Villani, in brief $\mathsf{CD}(K,N)$ spaces.

2. The Result

Let (X,d,m) be a metric measure space verifying the assumptions stated at the beginning of the introduction. Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ and $t \mapsto \mu_t \in \mathcal{P}_2(X)$ be a geodesic connecting them. Moreover, denote the dynamical optimal plan associated to μ_t by $\nu \in \mathcal{P}(\mathcal{G}(X))$.

Theorem 2.1. *Fix $t \in (0,1)$ and assume the existence of a positive constant C and a neighborhood $U_t \subset [0,1]$ of t such that $\mu_\tau = \varrho_\tau m$ with $\varrho_\tau \le C$ for each $\tau \in U_t$. Then we have*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}^1\big(I_t(\gamma) \cap (t-\varepsilon, t+\varepsilon)\big)}{2\varepsilon} = 1, \tag{2.1}$$

in $L^1(G,\nu)$.

Proof. Step 1. Suppose by contradiction the claim is false. Then

$$\limsup_{\varepsilon \to 0} \int_G \left| 1 - \frac{\mathcal{L}^1(I_t(\gamma) \cap (t-\varepsilon, t+\varepsilon))}{2\varepsilon} \right| \nu(d\gamma) > 0.$$

Therefore there exist a sequence $\varepsilon_n \to 0$ such that

$$1 > \lim_{n \to \infty} \int_G \frac{\mathcal{L}^1(I_t(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n))}{2\varepsilon_n} \nu(d\gamma).$$

Consider the complement of $I_t(\gamma)$, denoted by $I_t^c(\gamma) := \{\tau \in [0,1] : \gamma_\tau \notin e_t(G)\}$, then from the equality

$$1 - \frac{\mathcal{L}^1(I_t^c(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n))}{2\varepsilon_n} = \frac{\mathcal{L}^1(I_t(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n))}{2\varepsilon_n},$$

we deduce that

$$\lim_{n \to \infty} \int_G \frac{\mathcal{L}^1(I_t^c(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n))}{2\varepsilon_n} \nu(d\gamma) > 0. \tag{2.2}$$

Moreover, by inner regularity, we can assume that $m(e_t(G)) < \infty$.

Step 2. Denote with P_i the projection map on the i-th component, for $i = 1,2$. Let

$$E := \{(\gamma, s) \in G \times (0,1) : s \in I_t(\gamma)^c\} = \{(\gamma,s) \in G \times (0,1) : d(\gamma_s, e_t(G)) > 0\},$$

and

$$E(\gamma) := P_2\Big(E \cap \big(\{\gamma\} \times (0,1)\big)\Big), \qquad E(\tau) := P_1\Big(E \cap \big(G \times \{\tau\}\big)\Big).$$

Then by Fubini's Theorem and (2.2) we obtain that

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{2\varepsilon_n} \int_{(t-\varepsilon_n, t+\varepsilon_n)} \nu(E(\tau)) \mathcal{L}^1(d\tau) &= \lim_{n \to \infty} \frac{1}{2\varepsilon_n} \nu \otimes \mathcal{L}^1\left(E \cap (G \times (t-\varepsilon_n, t+\varepsilon_n))\right) \\ &= \lim_{n \to \infty} \frac{1}{2\varepsilon_n} \int_G \mathcal{L}^1(E(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n)) \nu(d\gamma) > 0. \end{aligned}$$

So there must be a sequence of $\{s_n\}_{n \in \mathbb{N}}$ converging to 0 so that $\nu(E(t+s_n)) \ge \kappa$, for some $\kappa > 0$. Then, since $e_{t+s_n}(G)$ converges to $e_t(G)$ in Hausdorff topology as s_n goes to 0, we have

$$m(e_t(G)^\varepsilon) \ge m(e_t(G) \cup e_{t+s_n}(E(t+s_n))) \ge m(e_t(G)) + m(e_{t+s_n}(E(t+s_n))),$$

where $e_t(G)^\varepsilon := \{z \in X : d(z, e_t(G)) \leq \varepsilon\}$. Since by assumption $\varrho_\tau \leq C$ on $e_\tau(G)$ for all $\tau \in U_t$, it follows that $m(e_{t+s_n}(E(t+s_n)))$ remains uniformly strictly positive as s_n goes to 0. Since

$$m(e_t(G)) = \lim_{\varepsilon \to 0} m(e_t(G)^\varepsilon),$$

we have a contradiction and the claim is proved.

□

Remark 2.2. Arguing similarly we can say something about pointwise convergence: for any sequence $\varepsilon_n \to 0$ there exists H, ν-negligible and depending on the sequence ε_n, such that

$$\limsup_{n\to\infty} \frac{\mathcal{L}^1\big(I_t(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n)\big)}{2\varepsilon_n} = 1,$$

for all $\gamma \in G \setminus H$.

Indeed suppose by contradiction the existence of a set $H \subset G$, with $\nu(H) > 0$ such that for all $\gamma \in H$

$$\limsup_{n\to\infty} \frac{\mathcal{L}^1\big(I_t(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n)\big)}{2\varepsilon_n} < 1.$$

Possibly restricting H, this implies

$$\liminf_{n\to\infty} \frac{\mathcal{L}^1\big(I_t^c(\gamma) \cap (t-\varepsilon_n, t+\varepsilon_n)\big)}{2\varepsilon_n} > \alpha,$$

for some $\alpha > 0$. Then, reasoning as in the proof of Theorem 2.1, we obtain a contradiction.

Remark 2.3. As already mentioned in the introduction the assumption, of the theorem is satisfied in $\mathsf{CD}(K,\infty)$ spaces as shown in [3]. Moreover, in this note we considered the L^2-Wasserstein space. However, the only property we used is that any $\mathcal{P}_2(X)$-geodesic can be represented via a dynamical optimal plan. Hence, for any other cost function inducing the same property on the space of probability measures, the same result applies, e.g. one could take L^p cost for $p > 1$. See also [6, Theorem 7.21] for more general costs.

References

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