STRESS CONSTRAINTS IN SIMPLE BODIES UNDERGOING LARGE STRAINS: A VARIATIONAL APPROACH

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ABSTRACT. We consider a simple body that is hyperelastic in large strain regime until the 3−covector defining the first Piola-Kirchhoff stress, once projected on the appropriate space of second-rank tensor, reaches a threshold indicating critical states. No information is given on the post-critical behavior. We determine the existence of equilibrium configurations according to the constraint. Such configurations can have concentration of strain in regions with vanishing volume. The related stress appears naturally as a measure over the deformation graph. Once restricted to the regular part of the deformation, such a measure determines the first Piola-Kirchhoff stress tensor and may also be concentrated over sets with vanishing volume projections on the reference and current placements. These projections can be interpreted as dislocations or dislocation walls. We analyze explicitly specific cases.

1. INTRODUCTION

1.1. **The problem.** We consider a body with hyperelastic behavior until the stress reaches a certain threshold. Nothing is assumed about the material behavior beyond the stress barrier. The problem we tackle is whether in large strain regime ground states for such a body exist under boundary conditions expressed in terms of deformation.

In small strain regime and linear elastic behavior, it is natural to discuss the problem in terms of convex duality: the energy density is a convex function of the strain measure, its Legendre transform with respect to the small-strain tensor (the complementary energy) is well defined, and the analysis reduces to a saddle-point problem. In finite strain regime the strategy does not apply: the physical incompatibility between the objectivity (frame indifference under the action of the Euclidean group over the physical space) of the elastic energy and its convexity with respect to the deformation gradient (see [6]) excludes a complementary energy, defined to be the Legendre transform of the elastic energy with respect to the deformation gradient.

1.2. **A digression.** In large strain elasticity, with u a deformation, the elastic energy density e is assumed to be a polyconvex function of the deformation gradient $F := Du(x)$ [4], i.e. a convex function of a three-plet composed by F, its determinant, $\det F$, and its cofactor, $\mathrm{cof} F$. Lower semicontinuity of the energy, relevant growth conditions and weak convergence of *independent* sequences of its entries are the tools that allow us to find existence of the energy minimizers by using the De Giorgi-Ioffe semicontinuity theorem. Summability conditions over the

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independent minimizing sequences of the energy entries assure that the limits form a three-plet $(D\bar{u}, \mathrm{cof} D\bar{u}, \det D\bar{u})$, with $\bar{u}$ the minimizer. When we take the minimizing sequences in the Sobolev space $W^{1,p}$, with $p \geq 3$ (see [4], [17], and also [5]), the lower semicontinuity implies that the minimizers are functions describing configurations that are free of fractures and holes (unless we do not foresee the initial presence of a hole that can be enlarged elastically along the deformation, but that hole is *not* nucleated anew).

For $p < 3$, an additional condition is necessary to assure that the weak limits of the sequences of the energy entries are linked to a minimizing deformation. To get it, we start considering that F, $\mathrm{cof} F$ and $\det F$ can be included in a third-rank contravariant tensor, $M(F)$, which is completely skew-symmetric and describes at a point x the tangent hyperplane to the deformation graph, the collection of points $(x, u(x))$, a three-dimensional surface in a six-dimensional space when the body under scrutiny is three-dimensional.

Hence, the polyconvexity of the energy density with respect to F can be naturally interpreted as as convexity with respect to $M(Du)$. Other conditions are dictated by the physics: the energy density blows up to infinity when (1) $\det F$ goes to zero or, else, (2) $|M(F)|$ tends to infinity. These two physical conditions translate into an analytical property: the energy is coercive. It is also so when it is evaluated over the inverse map, that is when it is referred to the actual shape of the body.

Third-rank covariant tensors ω, which are completely skew-symmetric, are dual to $M(Du)$. Once we fix a deformation u, defined over a fit region $\mathcal{B}$ in $3D-$space, say, we can define a functional G_u by

$$G_u(\omega) := \int_{\mathcal{B}} \omega \cdot M(Du) \ dx,$$

the dot indicating the duality pairing. G_u is the **current** of u. For any map $x \longmapsto \bar{\omega}(x, u(x))$, second-rank skew symmetric tensor valued, we write ∂G_u for the functional defined by

$$\partial G_u(\bar{\omega}) = G_u(d\bar{\omega}),$$

with d the exterior derivative. ∂G_u is the **boundary** of G_u.

The condition

$$\partial G_u(\bar{\omega}) = 0 \tag{1.1}$$

defines a functional class modeled on $W^{1,1}$. Its elements are called **weak diffeomorphisms**. A subclass with summability $p > 1$ includes minimizers of the non-linear elastic energy in large strain regime, as shown in [10] and [11].

The condition (1.1) has a geometrical interpretation: the graph of the deformation is free of vertical components – in other words, a weak diffeomorphism is not multi-valued in any part of its domain, as it occurs, for example, when a crack and/or a hole are nucleated and, eventually, open and/or close along a deformation.

Questions emerging from this view are the physical significance of G_u and the mechanical reasons for accepting low summability of maps describing deformations.

In what follows we discuss possible answers to these questions.

1.3. **Back to the problem.** For $p \geq 3$, there is no evident reason to prefer acting in terms of currents. However, when we analyze the problem tackled here, we find naturally the occurrence of low summability of maps describing equilibrium configurations, and we are in a sense 'forced' to adopt a view based on weak diffeomorphisms.

The first problem is the definition of a functional able to select possible deformations and compatible stresses with the imposed threshold. In small strain regime a functional made of the sum of the elastic and complementary energies plays a role.

We construct an analogous functional in large strain regime by exploiting the convexity of the energy density e with respect to $M(F)$. We do not get the saddle-point result appearing in small-strain regime, rather we find an existence result for the minimizers of such a functional. The related stress is a vector-valued measure on a three-dimensional surface in a six dimensional space. Such a surface is determined by the reference and the actual places of the body. Looking at it appears natural when the deformation is no more a one-to-one map everywhere, as it occurs in presence of stress constraints.

The resulting stress measure corresponds to the first Piola-Kirchhoff stress where the surface just mentioned is the graph of a one-to-one map and may also be concentrated over sets of zero volume measure in the reference and actual places. Hence, it may describe localized actions along single dislocations and/or dislocation walls, which can be nucleated once the stress reaches the admissibility threshold. Their presence is the prodrome for a possible phase transition toward the elastic-plastic behavior.

Along the path we follow, we show that a (functional) current can be intended as an **inner work** in a **generalized** sense, for the natural inclusion of incompatible strains, which can be considered pertinent to plastic behavior, at least in appropriate circumstances.

1.4. **A way of considering stress constraints.** Given a polyconvex elastic energy density $e = \hat{e}(F, \mathrm{cof}F, \det F)$, the associated first Piola-Kirchhoff stress is given by

$$\begin{aligned} P &= \frac{de}{dF} = \frac{\partial e}{\partial F} I + \frac{\partial e}{\partial \mathrm{cof}F}\frac{\partial \mathrm{cof}F}{\partial F} + \frac{\partial e}{\partial \det F}\frac{\partial \det F}{\partial F} = \\ &= \frac{\partial e}{\partial F} I + \frac{\partial e}{\partial \mathrm{cof}F}\frac{\partial \mathrm{cof}F}{\partial F} + \frac{\partial e}{\partial \det F}\mathrm{cof}F, \end{aligned}$$

where I is the second-rank $1-$contravariant, $1-$covariant identity, and the derivative $\frac{\partial \mathrm{cof}F}{\partial F}$ depends on $\det F$ and $\mathrm{cof}F$, the latter factor appearing in the product $\mathrm{cof}F \otimes \mathrm{cof}F$.

The prescription that P belongs to a convex set of the stress space at a point of the body is the common assignment of a stress constraint, a natural generalization of the inequality $|P| \leq k$.

Differently from the common view, here, we prescribe constraints not directly on P, but on the coefficients of the polynomial above, namely

$$\frac{\partial e}{\partial F}, \quad \frac{\partial e}{\partial \mathrm{cof}F}, \quad \frac{\partial e}{\partial \det F}.$$

The choice allows us to treat the problem at hands in a setting where these factors, or their more general counterparts, can vary independently and/or are not linked with each another by a unique linear operator F or a unique deformation u in case of compatible strain. Such an independence could appear, in fact, along minimizing sequences of the functional that we consider and is lost in the limit.

Our approach is based on the possibility of expressing the first Piola-Kirchhoff stress as the product of a thrird-rank skew-symmetric tensor and a fifth-rank tensor that is skew-symmetric in the first three indices and depends only on the deformation and not the energy, as shown in Proposition 1 below. In what we propose the constraint may involve cases of incompatible strain where linear and surface stresses, and the volume pressure, might be independent with each other, due to the incompatibility of the strain.

1.5. **An additional remarks.** The last section of this paper contains a number of specific examples with physical interest, such as the description of slip systems, which have a key rôle in crystal plasticity.

With the techniques presented here, among such special cases we can include the blow-up of a point into a ball or a line into a surface or a cylinder, or the opposite processes in the precise sense emerging from the treatment. These two effects cannot be described by means of techniques based, for example, on functions with bounded variation. Their physical meaning emerges when we consider every material point as a piece of matter, with a certain non-zero diameter, an internal length scale in the continuum modeling. This way, the continuum scheme becomes implicitly multiscale and the blow-up of a point into a sphere reduces to the swelling of the matter below the internal length-scale up to the continuum (macroscopic) spatial scale. An analogous remark holds also for the blow-up of a line into a surface or a cylinder.

2. A commentary on large strain kinematics

We have already mentioned that, at a point x in $\mathcal{B}$, the elements of the three-plet $(F, \mathrm{cof}F, \det F)$, appearing as entries of the elastic energy density, contribute to the construction of a third-rank, completely skew-symmeric tensor $M(F)$, which can be determined just by kinematic arguments, once we accept the possibility of having incompatible strain.

Incidentally, in constructing $M(F)$, we find a natural expression of standard instances in continuum mechanics in terms of **forms**. Recalling introductory geometric notions appears then useful.

2.1. **Preparatory notions.** Let $\mathcal{L}$ be a linear space, with basis $(\mathbf{e}_1, ..., \mathbf{e}_m)$. The symbol $\wedge$ indicates a map $\wedge : \mathcal{L} \times \mathcal{L} \to Skw(\mathcal{L}^*, \mathcal{L})$, with $Skw(\mathcal{L}^*, \mathcal{L})$ the space of skew-symmetric tensors from the dual of $\mathcal{L}$, indicated by $\mathcal{L}^*$, to $\mathcal{L}$. The linear space $\mathcal{L}^*$ is endowed with basis $\left(\mathbf{e}^1, ..., \mathbf{e}^m\right)$.

For $v = v^\alpha \mathbf{e}_\alpha \in \mathcal{L}$ and $\bar{v} = \bar{v}^\beta \mathbf{e}_\beta \in \mathcal{L}$, $v \wedge \bar{v}$ is of the form $\xi^{\alpha\beta}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta$ with $\xi^{\alpha\beta} = -\xi^{\beta\alpha}$.

The space of skew-symmetric tensors of type $(k, 0)$, namely tensor with k contravariant components, also called k-**vectors,** is then indicated by $\Lambda_k(\mathcal{L})$. It is a linear space, so is its dual[1] $\Lambda^k(\mathcal{L})$, a linear space containing tensors of the type $(0, k)$: they have k covariant components. In particular, $\Xi \in \Lambda_k(\mathcal{L})$ has the form $\Xi = \xi^{i_1 \dots i_k}\mathbf{e}_i \wedge ... \wedge \mathbf{e}_k$, while any $\omega \in \Lambda^k(\mathcal{L})$ is of the type $\omega = \omega_{i_1 \dots i_k}\mathbf{e}^i \wedge ... \wedge \mathbf{e}^{i_k}$.

Maps $x \longmapsto \omega(x) \in \Lambda^k(\mathcal{L})$, with $x \in \mathcal{B}$, are called $k-$**forms** over $\mathcal{B}$.

To recognize their possible role in continuum mechanics, it is useful to think briefly of the way we describe standard kinematics of continuous bodies.

[1] It is the space of all linear maps over $\Lambda_k(\mathcal{L})$.

2.2. **Deformations.** Deformation is a *relative* concept. A body in a certain shape is deformed *with respect to* another shape that we select as a reference and is typically described by a fit region $\mathcal{B}$, a regularly open, connected and bounded subset of an Euclidean point space (dimension depending on the problem at hands), with surface-like boundary oriented everywhere by the normal n, to within a finite number of corners and edges. It is unessential that $\mathcal{B}$ is occupied by the body at any instant along any motion. It is important just that it *could* be occupied by the body. For this reason we take first two isomorphic oriented copies ($\mathcal{E}^3$ and $\tilde{\mathcal{E}}^3$) of the three-dimensional Euclidean point space, the isomorphism being an isometry preserving the orientation. We then imagine that $\mathcal{B}$ is in $\mathcal{E}^3$, while the *actual* or *current* shapes are in $\tilde{\mathcal{E}}^3$. Reasons for adopting such a distinction are related with the action of changes in observers, above all when the body undergoes macroscopic irreversible mutations in its material structure – a topic not discussed here, however (see [14] for details) – and for questions connected with the evaluation of the energy first variation, discussed later.

We equip $\mathcal{B}$ with a metric $x \longmapsto g(x) \in \mathrm{Sym}\left(\mathbb{R}^3, \mathbb{R}^{3*}\right)$ and write $\tilde{g}(y) \in \mathrm{Sym}\left(\tilde{\mathbb{R}}^3, \tilde{\mathbb{R}}^{3*}\right)$ for a metric assigned in the physical space $\tilde{\mathcal{E}}^3$. **Deformations** $u : \mathcal{B} \to \tilde{\mathcal{E}}^3$ select all current places $u(\mathcal{B})$. Reasons of physical plausibility suggest that the generic u be

***(i)*:** one-to-one and differentiable,
***(ii)*:** orientation preserving, and
***(iii)*:** able to allow self contact of the body boundary and to exclude self-penetration[2].

D indicates the spatial derivative with respect to coordinates in $\mathcal{B}$. The derivative of the deformation, indicated by $F := Du(x) \in \mathrm{Hom}\left(T_x\mathcal{B}, T_{u(x)}u(\mathcal{B})\right)$, is commonly called **deformation gradient**, although the difference between $Du(x)$ and the gradient $\nabla u(x)$ is clear for we have $\nabla u(x) = Du(x)\, g^{-1}$.

Assumption (ii) imposes equal orientation to $\mathcal{E}^3$ and $\tilde{\mathcal{E}}^3$, once we have selected frames in the two spaces. In particular, we commonly choose the isomorphism between $\mathcal{E}^3$ and $\tilde{\mathcal{E}}^3$ to be an isometry. The orientation preserving restriction reads $\det F > 0$, the determinant being computed after mapping the basis of the frame in $\mathcal{E}^3$ onto the one in $\tilde{\mathcal{E}}^3$ – the mapping is not associated with the deformation: it is just implicit in the calculation of the determinant. By using common notations we write F^T and F^* for the transpose and the formal adjoint of F. The link between these two operators is given by the metrics: $F^T = g^{-1}F^*\tilde{g}$.

2.3. **Volume and area forms.** Select a vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and an origin O, rendering this way $\mathcal{E}^3$ to be coincident with $\mathbb{R}^3$. These three vectors determine a prism – denote it by $prism(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ – that we assume with *unitary* volume. Take then three linearly independent vectors $a_1, a_2, a_3 \in \mathbb{R}^3$ at $x \in \mathcal{B}$. They determine another prism, $prism(a_1, a_2, a_3)$. The volume of such a prism, indicated by $vol(a_1, a_2, a_3)$, is given by $(a_1 \times a_2) \cdot a_3$, a triple product equal to the determinant of a matrix having as columns the three vectors considered. The map $(a_1, a_2, a_3) \longmapsto vol(a_1, a_2, a_3)$ is then three-linear and skew-symmetric. Hence, we

[2]Although orientation preserving diffeomorphisms satisfy all conditions (i)-(iii), the analysis of the existence of ground states for elastic bodies undergoing large strain requires an enlarged functional setting to develop consistent analyses. The enlargement implies then a constitutive choice, that of the functional space (see also [10], [11], [15])

can consider the volume as given by the value over the third-rank, skew-symmetric contravariant tensor $a_1 \wedge a_2 \wedge a_3 \in \Lambda_3\left(\mathbb{R}^3\right)$ of a third-rank, skew-symmetric covariant tensor $\omega \in \Lambda^3\left(\mathbb{R}^3\right)$. In terms of the dual basis to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the tensor ω writes $\omega = \sqrt{\det g}\, dx^1 \wedge dx^2 \wedge dx^3$, with g the metric.

An analogous remark holds for the oriented area determined by two non-collinear vectors a_1 and a_2. In this case we can select an element of $\Lambda^2\left(\mathbb{R}^2\right)$ and check its value over $a_1 \wedge a_2$.

These two remarks put just in evidence the role of the wedge product $\wedge$ in computing volumes and areas.

2.4. **3-vectors and strain measures.** Select linearly independent vectors a_1, a_2, and a_3 at a point x in $\mathcal{B}$ and consider maps of the type

$$a_1 \wedge a_2 \wedge a_3 \longmapsto Fa_1 \wedge a_2 \wedge a_3,$$

$$a_1 \wedge a_2 \wedge a_3 \longmapsto Fa_1 \wedge Fa_2 \wedge a_3,$$

$$a_1 \wedge a_2 \wedge a_3 \longmapsto Fa_1 \wedge Fa_2 \wedge Fa_3.$$

$M(F)$ is the third-rank, skew-symmetric contravariant tensor given by

$$\begin{gathered} M(F) := a_1 \wedge a_2 \wedge a_3 + \\ + Fa_1 \wedge a_2 \wedge a_3 + a_1 \wedge Fa_2 \wedge a_3 + a_1 \wedge a_2 \wedge Fa_3 + \\ + Fa_1 \wedge Fa_2 \wedge a_3 + Fa_1 \wedge a_2 \wedge Fa_3 + a_1 \wedge Fa_2 \wedge Fa_3 + \\ + Fa_1 \wedge Fa_2 \wedge Fa_3 = \\ = (a_1, Fa_1) \wedge (a_2, Fa_2) \wedge (a_3, Fa_3) \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3). \end{gathered}$$

Once bases in $\mathcal{E}^3$ and $\tilde{\mathcal{E}}^3$ are selected, the components of $M(F)$ are the number 1 and all components of F (in the terms where F acts on just one vector) and $\mathrm{cof} F$ (where F is applied to two vectors), and $\det F$ (the last term). All ingredients appearing in the strain measures are then in $M(F)$.

(1) In fact, the local volume change, computed relatively to the initial volume, is given by $\det F - 1$.
(2) According to the Nanson formula, the variation of area, once again computed relatively to the initial area, is given by $|(\mathrm{cof} F)| - 1$, where n is the normal orienting the parallelogram delimited by two of the three vectors a_1, a_2, a_3. And the normal is considered as a covector.
(3) The deformation tensor measures the elongation of lines and the variation of angles. Two versions of it are common: the first one is the difference of metrics $E := \frac{1}{2}(C - g)$, with $C := F^* \tilde{g} F$, the second one is a relative difference of metrics, namely $\hat{E} := \frac{1}{2}\left(\hat{C} - \hat{I}\right)$, with $\hat{I}$ the second-rank identity with components δ^A_B, and $\hat{C} := g^{-1} C = F^T F$.

In constructing the strain measures, we have in mind that $F := Du(x)$. However, we can construct $M(F)$ even in case of incompatible strain, that is when the map $x \longmapsto F(x)$ is such that $\mathrm{curl} F \neq 0$, but always $\det F > 0$. Independently on whether the strain is compatible, $M(F)$ belong to $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. However, not all elements of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ are of the type $M(F)$. In other words, not all $M \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ are generated by only one linear operator.

Two constants, say ζ and a , and two independent linear operators, e.g. H and A, determine, in fact, a generic element M of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. With respect to the bases in $\mathbb{R}^3$ and $\tilde{\mathbb{R}}^3$, namely $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$, every $3-$vector $M \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ has the form

$$\begin{aligned} M &= \zeta \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \sum_{i,J}^{3} (-1)^{J-1} H^{iJ} \mathbf{e}_{\bar{J}} \wedge \tilde{\mathbf{e}}_i + \\ &+ \sum_{i,J}^{3} (-1)^{i-1} A^{iJ} \mathbf{e}_J \wedge \tilde{\mathbf{e}}_{\bar{\imath}} + a \tilde{\mathbf{e}}_1 \wedge \tilde{\mathbf{e}}_2 \wedge \tilde{\mathbf{e}}_3, \end{aligned}$$

where $\bar{J}$ is the complementary multi-index to J with respect to $(1,2,3)$ and $\bar{\imath}$ has an analogous relation with i (for example, if $J = 1$, then $\bar{J} = (2,3)$ and $\mathbf{e}_{\bar{J}} = \mathbf{e}_2 \wedge \mathbf{e}_3$, and the same holds for the index i and its pertinent $\bar{\imath}$)[3]. We put in evidence the algebraic signs to allow a direct identification of the coefficients in the special case $M = M(F)$.

For the sake of conciseness, we write $M = (\zeta, H, A, a)$ or $M = (\zeta, \mathsf{M})$, with M indicating the three-plet (H, A, a). The latter notation suggests also the orthogonal decomposition $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) = \mathbb{R}^3 \times \mathcal{V}$. The generic element of $\mathcal{V}$ is then of the type

$$\mathsf{M} = \sum_{i,J}^{3} (-1)^{J-1} H^{iJ} e_{\bar{J}} \wedge \tilde{e}_i + \sum_{i,J}^{3} (-1)^{i-1} A^{iJ} e_J \wedge \tilde{e}_{\bar{\imath}} + a \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3.$$

M coincides with $M(F)$ when $\zeta = 1$, $H = Fg^{-1}$, $A = \tilde{g}\ \mathrm{cof} F$ when $\mathrm{cof} F$ is defined by $(\det F)\left(F^{-1}\right)^*$ or $A = \mathrm{cof} F g^{-1}$ when we consider $\mathrm{cof} F$ as given by $(\det F)\left(F^{-1}\right)^T$, and $a = \det F$. In short, we shall write $\mathsf{M}(F) = (F, \mathrm{cof} F, \det F)$ when $M = M(F)$. Remind that a special case of $M(F)$ is when $M = M(Du)$.

Our remarks do not indicate an alternative way to define strain measures, because the map $F \longmapsto M(F)$ is one-to-one. They just pave the way to the possibility of using methods of the geometric measure theory to deduce at least qualitative information on the problem that we discuss here.

A number of sets appear in the subsequent developments. They are all subsets of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. We summarize here the relevant notations:

$$\mathcal{V} := \{\mathsf{M} = (F, A, a)\},$$

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[3] The construction leading to $M(F)$ can be naturally extended. To this aim, consider a vector basis $e_1, ..., e_n$ in $\mathbb{R}^n$ and another basis $\bar{e}_1, ..., \bar{e}_N$ in $\mathbb{R}^N$. For any linear map $G : \mathbb{R}^n \to \mathbb{R}^N$, $M(G)$ is defind by

$$M(G) := \Lambda_n (id \times G)(e_1 \wedge ... \wedge e_n) = (e_1, G(e_1)) \wedge ... \wedge (e_n, G(e_n))$$

which is analogous to

$$M(G) = \sum_{k=0}^{\min(n,N)} M_{(k)}(G),$$

where, by indicating by α a multi-index in $(1, 2, ..., n)$ with length $|\alpha|$ and by $\bar{\alpha}$ its complement always in $(1, 2, ..., n)$, β another multi-index, with $sign(\alpha, \bar{\alpha})$ the sign of the permutation from $(1, 2, ..., n)$ into $(\alpha_1, ..., \alpha_k, \bar{\alpha}_1, ..., \bar{\alpha}_{n-k})$, the component $M_{(k)}(G)$ is given by

$$M_{(k)}(G) = \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} sign(\alpha, \bar{\alpha})\, M_{\bar{\alpha}}^{\beta}(\mathbf{G})\, e_\alpha \wedge \bar{e}_\beta.$$

$\mathbf{G}$ indicates the matrix associated with G, so that $M_{\bar{\alpha}}^{\beta}(\mathbf{G})$ is the determinant of the submatrix of $\mathbf{G}$ made of the rows and the columns indexed by β and $\bar{\alpha}$ respectively. By definition $M_0^0(\mathbf{G}) := 1$, and $M(G)$ is a simple $n-$vector in $\Lambda_n\left(\mathbb{R}^n \times \mathbb{R}^N\right)$, a linear space with dual indicated by $\Lambda^n\left(\mathbb{R}^n \times \mathbb{R}^N\right)$.

$$\mathcal{V}_{+} := \{\mathsf{M} = (F, A, a) \mid a > 0\},$$

$$\mathcal{V}_{+,F} := \{\mathsf{M}(F) = (F, \mathrm{cof} F, \det F) \mid \det F > 0\},$$

$$\Sigma_1 := \{1\} \times \mathcal{V}, \quad \Sigma_{1,+} := \{1\} \times \mathcal{V}_{+},$$

$$\Sigma_{1,+,F} := \{1\} \times \mathcal{V}_{+,F}.$$

3. Contact actions and 1-forms

What has been done in previous section with F can be repeated for the first Piola-Kirchhoff stress. The construction is dual to the previous one. Reasons for it emerge once we think of the way we represent commonly contact actions in a continuous body.

Consider an oriented plane cutting ideally the current shape $u(\mathcal{B})$ in two pieces $u(\mathcal{B})^{+}$ and $u(\mathcal{B})^{-}$, where the algebraic signs distinguish between the positive and negative orientation of the normal $\bar{n}$ to the plane. According to Cauchy's view, at a generic point y of the section, the contact interaction between $u(\mathcal{B})^{+}$ and $u(\mathcal{B})^{-}$ is locally described *only* by a force t, the **tension**, which depends on $y := u(x)$ and $\bar{n}$. Given a velocity field $y \longmapsto v(y)$, which can be considered even *virtual*, the tension t is **defined** at y by the power $\mathsf{t}(y, \bar{n}) \cdot v(y)$ developed along v. Hence, since $v(y)$ is a vector, to define $\mathsf{t}(y, \bar{n}) \cdot v(y)$ without resorting to the *additional* structure of scalar product, $\mathsf{t}(y, \bar{n})$ must be considered a covector, an element of the cotangent space to $u(\mathcal{B})$ at y, in short $T^{*}_{u(x)}u(\mathcal{B})$, the space of linear maps over $T_{u(x)}u(\mathcal{B})$, coinciding with $\Lambda^{1}(\tilde{\mathbb{R}}^{3})$. This way, $\mathsf{t}(y, \bar{n}) \cdot v(y)$ is the value taken over $v(y)$ by the covector $\mathsf{t}(y, \bar{n})$. It appears in the external power of actions over a generic subset of $u(\mathcal{B})$ with non-vanishing volume, together with the covector density of the body forces. Imposing the invariance of such a power under isometry-based changes in observers in $\tilde{\mathcal{E}}^{3}$ implies the validity of the integral balances of forces and couples. The assumption that the body actions are bounded implies also the validity of the action-reaction principle, namely $\mathsf{t}(y, n_a) = -\mathsf{t}(y, -n_a)$. Continuity of t with respect to y, the integral balances, and the standard tetrahedron argument assure the existence of a second-rank tensor σ depending *only* on y, **Cauchy stress tensor**, such that

$$\mathsf{t}(y, n_a(y)) = \sigma(y)\, n_a(y).$$

Previous assumptions have been variously weakened in the available literature (see, e.g., results and comments in [19], [20], [21], [22]). However, more than analyzing the weakest conditions assuring Cauchy theorem, our attention is focused here on some aspects of the geometry in the tetrahedron argument used in the standard proof.

If we consider the normal as a covector, Cauchy stress tensor appears as a linear operator mapping $T^{*}_{u(x)}u(\mathcal{B})$ onto itself, namely

$$\sigma(u(x)) \in \mathrm{Hom}\left(T^{*}_{u(x)}u(\mathcal{B}), T^{*}_{u(x)}u(\mathcal{B})\right) \simeq \mathrm{Hom}\left(\Lambda^{1}(\tilde{\mathbb{R}}^{3}), \Lambda^{1}(\tilde{\mathbb{R}}^{3})\right) \simeq \tilde{\mathbb{R}}^{3*} \otimes \tilde{\mathbb{R}}^{3}.$$

The cut in $u(\mathcal{B})$ can be considered as the image of another ideal cut in the reference place $\mathcal{B}$, oriented at x – the point connected with $y := u(x)$ by the deformation u – by a normal n. By pulling back to the reference place the second component of σ, we construct the **first Piola-Kirchhoff stress,** P, a linear operator depending only on x, which maps n onto t, attached at y. Formally, we write the standard relation

$$P := (\det F)\,\sigma F^{-*} \in \mathrm{Hom}\left(T_x^*\mathcal{B}, T_{u(x)}^* u\,(\mathcal{B})\right).$$

Consider at $x \in \mathcal{B}$ three linearly independent covectors $\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$, which are a basis of $\mathbb{R}^{3*}$. By using $P : \mathbb{R}^3 \longrightarrow \tilde{\mathbb{R}}^3$, we have maps

$$\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3 \longmapsto \mathsf{c}_1 \wedge \mathsf{c}_2 \wedge P\mathsf{c}_3,$$

$$\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3 \longmapsto \mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3,$$

$$\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3 \longmapsto P\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3.$$

Then we can define a 3−covector $\omega\,(P)$ by

$$\begin{gathered}\omega\,(P) := \mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3 + \\ +\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge P\mathsf{c}_3 + \mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge \mathsf{c}_3 + P\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3 + \\ +\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3 + P\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge P\mathsf{c}_3 + P\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge \mathsf{c}_3 + \\ P\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3 = \\ = (\mathsf{c}_1, P\mathsf{c}_1) \wedge (\mathsf{c}_2, P\mathsf{c}_2) \wedge (\mathsf{c}_3, P\mathsf{c}_3) \in \Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3).\end{gathered}$$

The addenda determining $\omega\,(P)$ furnish information on the local tension (the terms of the type $\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge P\mathsf{c}_3$), the 'averaged' stress over surfaces (the terms of the type $\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3$), and a pressure associated with changes in volume ($P\mathsf{c}_1 \wedge P\mathsf{c}_2 \wedge P\mathsf{c}_3$ is just proportional to $\mathsf{c}_1 \wedge \mathsf{c}_2 \wedge \mathsf{c}_3$).

$\omega\,(P) \in \Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ is dual to $M\,(F) \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ as P is dual to F.

However, not all the elements of $\Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ are of the type $\omega\,(P)$.

Let dx^1, dx^2, dx^3 and dy^1, dy^2, dy^3 be the dual bases of $\mathbb{R}^{3*}$ and $\tilde{\mathbb{R}}^{3*}$, respectively. Any $\omega \in \Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ has the form

$$\begin{aligned}\omega \quad : \quad &= \beta dx^1 \wedge dx^2 \wedge dx^3 + \sum_{i,J=1}^{3} (-1)^{J-1} r_{iJ} dx^{\bar{J}} \wedge dy^i + \\ &+ \sum_{i,J=1}^{3} (-1)^{i-1} s_{iJ} dx^J \wedge dy^{\bar{\imath}} + \varsigma dy^1 \wedge dy^2 \wedge dy^3,\end{aligned}$$

where $\bar{J}$ and $\bar{\imath}$ have the same meaning of those in the formula that defines M. β and δ are scalars. r and s are linear operators.

Even in the case of 3−covectors, for the sake of conciseness we write $\omega = (\beta, r, s, \varsigma)$ or $\omega = (\beta, \varpi)$, with ϖ indicating the three-plet (r, s, δ). The latter notation suggests also the orthogonal decomposition $\Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) = \mathbb{R}^3 \times \mathcal{V}^*$. The generic element of $\mathcal{V}^*$ is then of the type

$$\varpi = \sum_{i,J=1}^{3} (-1)^{J-1} r_{iJ} dx^{\bar{J}} \wedge dy^i + \sum_{i,J=1}^{3} (-1)^{i-1} s_{iJ} dx^J \wedge dy^{\bar{\imath}} + \varsigma dy^1 \wedge dy^2 \wedge dy^3.$$

For $\omega \in \Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ and $M \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$, the product $\omega \cdot M$ (the natural duality pairing between ω and M, namely $\omega \cdot M = \omega\,(M)$) is defined by

$$\omega \cdot M := \beta\zeta + \sum_{i,J=1}^{3} r_{iJ} H^{iJ} + \sum_{i,J=1}^{3} s_{iJ} A^{iJ} + \varsigma a.$$

4. Energy

We call **Cauchy's bodies** (or **materials**) those with mechanical behavior represented by a scheme in which their morphology (a word indicating collectively the macroscopic and minute aspects of their geometry) is just described by the place that they occupy in space, and the inner contact actions are associated only with the crowding and the shearing of the material elements, so they are described by covectors over the space of velocities.

The denomination is selected to remind that such a scheme is not the sole possible option: refined descriptions of the body morphology to account for minute material changes imply enrichments in the representation of the inner actions (see discussions in [14]).[4]

In common treatises, bodies are called **simple** when they are Cauchy's and the constitutive structures depend just on the deformation gradient F besides x, in the case of inhomogeneous materials. We have already mentioned that in finite strain (isothermal) elasticity, the elastic energy density is assumed to be polyconvex function of F, that is a convex function of the convex hull containing triplets $(F, \mathrm{cof} F, \det F)$ (see detailed discussions in [4]).

We can rewrite the condition on the energy by considering $\Sigma_{1,+}$, the convex hull of the set $\{M(F)\}$, as the *state space* of a simple body. The choice could appear questionable for $\Sigma_{1,+}$ includes incompatible strains. Precisely, $\Sigma_{1,+}$ contains $3-$vectors not necessarily associated with a linear operator F and even when we take an element of the type $M(F)$, it is not necessarily $M(Du)$, i.e. the strain is not always compatible with a deformation. Also, when we take a polyconvex elastic energy and try to determine its minimizers, the minimizing sequences of F, $\mathrm{cof} F$, and $\det F$ are independent with each other. However, if along the minimizing process we have strain compatibility, it is possible to prove that the compatibility is preserved in the limit.

For any piecewise-C^1 curve $\gamma : [-1,1] \longrightarrow \Sigma_{1,+}$, define a functional $\mathsf{w}(\omega, \gamma)$ by

$$\mathsf{w}(\omega,\gamma) := \int_{-1}^{1} \omega(t) \cdot \dot{\gamma}(t) \; dt,$$

where $\dot{\gamma}(t)$ is tangent to γ at the point $\gamma(t)$ and $\omega(t)$ is dual to $\dot{\gamma}(t)$.

Since

$$\omega = \sum_{k=0}^{3} \omega_{(k)}, \quad \gamma(t) = \sum_{k=0}^{3} \gamma_{(k)},$$

the product $\omega(t) \cdot \dot{\gamma}(t) = \sum_{k=0}^{3} \omega_{(k)}(\gamma(t)) \cdot \dot{\gamma}_{(k)}(t)$ involves

(1): the power density $\omega_{(1)} \cdot \dot{\gamma}_{(1)}$ determined by line strain,

---

[4] When we account for the minute geometry of the material microstructure through additional descriptors (e.g. a direction or a second-rank tensor at every point, as in the case of nematic liquid crystals, with the inclusion in the two choice of different details about the arrangement of the matter), contact actions associated with inhomogeneous microstructural changes and related self-actions arise (microstresses) and they are additional to those due just to the crowding and the shearing of the material elements, due to the macroscopic deformation.

(2): a power associated with volume changes, and given by $(\omega_{(3)} \cdot \dot{\gamma}_{(3)}(t)) = (-p\dot{\gamma}_{(3)})(t)$, with p the scalar appearing in $\omega_{(3)} = -pdy^1 \wedge dy^2 \wedge dy^3$ with the meaning of a pressure[5], and

(3): terms given by the multiplication of the components of $\omega_{(2)}$ with the ones of $\dot{\gamma}_{(2)}(t)$, and representing the power over coordinate planes in a local frame, and for every local frame.

For these reasons we call $\mathsf{w}(\omega, \gamma)$ **extended virtual internal** (or **inner**) **work**, the extension being referred to the possibility of incompatible strain.

The space $\Sigma_{1,+}$ is pathwise connected. The requirements that (i) w is non-negative along closed paths and (ii) any path in $\Sigma_{1,+}$ is physically realizable are tantamount to impose that w vanishes along all closed paths, a requirement of conservativeness, indeed. These conditions are, in fact, equivalent to the existence of a function $e : \Sigma_{1,+} \longrightarrow \mathbb{R}$ such that

$$\omega(M) = de(M), \quad \forall M \in \Sigma_{1,+}. \tag{4.1}$$

In this setting we consider then e as the **stored energy density**.

Physically, a more realistic choice could be to define the energy over $\Sigma_{1,+,F}$, rather than the whole $\Sigma_{1,+}$. However, $\Sigma_{1,+,F}$ is the image of the set of 3×3 matrices with real entries and positive determinant, obtained through the map $F \longmapsto M(F) = (1, F, \mathrm{cof}F, \det F)$, and it is not convex. Hence, if we want to define a convex function over it (as it would be the energy density when, for M of the type $M(F)$, we consider it as a polyconvex function of F), we must consider the convex hull of $\Sigma_{1,+,F}$, which is $\Sigma_{1,+}$.

For piecewise-C^1 paths γ in $\Sigma_{1,+,F}$, the extended virtual internal power density at a given t reads

$$\begin{aligned} \omega(t) \cdot \dot{\gamma}(t) &\equiv \omega \cdot \dot{M}(F) = \\ &= \omega \cdot \frac{dM}{dF}(F)\dot{F} = \omega \frac{dM}{dF}(F) \cdot \dot{F}. \end{aligned}$$

In the last product in the previous expression, namely $\omega\frac{dM}{dF}(F)$, the fifth-rank tensor $\frac{dM}{dF}(F)$, which is skew-symmetric in the first three indices, acts – it is applied from the *right* – as an element of the space

$$\mathrm{Hom}\left(\Lambda^3(\mathcal{B} \times \tilde{\mathbb{R}}^3), \mathrm{Hom}\left(T_x^*\mathcal{B}, T_y^*\mathcal{B}_a\right)\right).$$

At $x \in \mathcal{B}$, it maps linearly $\Lambda^3(\mathcal{B} \times \tilde{\mathbb{R}}^3)$ onto $\mathrm{Hom}\left(T_x^*\mathcal{B}, T_y^*\mathcal{B}_a\right)$, which includes the **first Piola-Kirchhoff stress** P. Hence, we write

$$P = \omega\frac{dM(F)}{dF},$$

and we can identify the product $\omega\frac{dM(F)}{dF} \cdot \dot{F}$ with the standard density of internal power of simple bodies, namely $P \cdot \dot{F}$. In components we have

$$P_i^A = \omega_{\alpha\beta\gamma}\frac{d(M(F))^{\gamma\beta\alpha}}{dF_A^i}.$$

The result is then summarized below.

[5] In our previous notations, $-p = \varsigma$.

Proposition 1. *The first Piola-Kirchhoff stress admits a multiplicative decomposition into a third-rank skew-symmetric tensor (the value at x of a form $x \longmapsto \omega(x)$ over the reference place $\mathcal{B}$) and a fifth-rank tensor which is skew-symmetric in the first three indices. In the elastic setting, the first tensor, namely ω, is the sole part related with the energy by the relation (4.1), while the second factor, namely $\frac{dM(F)}{dF}$, is just a geometric projector depending only on the specific deformation along which the first Piola-Kirchhoff stress is evaluated.*

In particular, we have

$$\begin{aligned} P(F) &= \omega(M(F)) \frac{dM}{dF}(F) = \\ &= \omega_{(1)}(M(F)) I + \omega_{(2)}(M(F)) \frac{d\mathrm{cof} F}{dF}(F) + \\ &\quad + \omega_{(3)}(M(F)) \frac{d\det F}{dF}(F), \end{aligned}$$

where I is now the identity in $\mathrm{Hom}(\mathbb{R}^{3*}, \tilde{\mathbb{R}}^{3*})$. The derivative $\partial \det F / \partial F$ equals $\mathrm{cof} F$. Moreover, by identifying covariant with contravariant components (that is $\mathbb{R}^3$ with its dual), for the component $\frac{\partial(\mathrm{cof}F)_{iA}}{\partial F_{jB}}$ we get

$$\frac{\partial(\mathrm{cof}F)_{iA}}{\partial F_{jB}} = \mathsf{e}_{ijm} F_{mC} \mathsf{e}_{CAB},$$

with e Ricci's permutation index.

For the composition map $\tilde{e}(F) := e(M(F))$, with $F \in M_{3\times 3}$, we then have

$$P = \partial_F \tilde{e}(F)$$

which is

$$P(F) = \frac{de(M(F))}{dM(F)} \frac{dM(F)}{dF}.$$

As a consequence of the construct presented so far, we then have

- **(1):** a variational characterization of the stress form ω,
- **(2):** a natural framework for evaluating ground states (equilibrated configurations) with stress form ω, constrained to take values in some convex admissibility set.

Item 2 refers to circumstances that we find in discussing the occurrence of the phase transition from elastic to plastic or brittle behavior.

5. Some elements of convex analysis

We find expedient to recall certain notions of convex analysis. For further details the reader can refer to standard treatises like, e.g., [12] and [18].

We refer to a couple of finite-dimensional normed vector spaces $\mathcal{V}$ and $\mathcal{V}^*$, which are dual one another.

Given a convex and lower semicontinuous (l.s.c.) function $f : \mathcal{V} \to (-\infty, +\infty]$, we call *conjugate* the map $f^* : \mathcal{V}^* \to (-\infty, +\infty]$, defined by

$$f^*(\varpi) = \sup_{\mathsf{M}\in\mathcal{V}} (\varpi \cdot \mathsf{M} - f(\mathsf{M})), \quad \forall \varpi \in \mathcal{V}^*.$$

Again, f^* is convex and l.s.c.. The conjugation operation is involutive: f^{**} coincides with f. An element ϖ of the dual space $\mathcal{V}^*$ is called a *subgradient* of f at $\mathsf{M} \in \mathcal{V}$

if and only if $f(\mathsf{Z}) \geq f(\mathsf{M}) + \varpi \cdot (\mathsf{Z} - \mathsf{M})$ for any $\mathsf{Z} \in \mathcal{V}$. The element ϖ of $\mathcal{V}^*$ satisfying the previous inequality is not necessarily unique, when it exists.

The set of all subgradients of f at M is a subset of $\mathcal{V}^*$. It is called *subdifferential* of f at M and is denoted by $\partial f(\mathsf{M})$. Since the conjugation operation is involutive, we get $\varpi \in \partial f(\mathsf{M})$ if and only if $\mathsf{M} \in \partial f^*(\varpi)$. When f is differentiable at M, the subgradient of f at M contains only the differential $df(\mathsf{M})$, so that $\partial f(\mathsf{M}) = \{df(\mathsf{M})\}$.

Fenchel inequality, namely

$$f(\mathsf{M}) + f^*(\varpi) \geq \varpi \cdot \mathsf{M}, \quad \forall (\mathsf{M}, \varpi) \in \mathcal{V} \times \mathcal{V}^*,$$

with the equality sign holding if and only if $\varpi \in \partial f(\mathsf{M})$ (or, equivalently, $\mathsf{M} \in \partial f^*(\varpi)$), is the standard way to express what is usually called *convex duality.* Another way can be followed by introducing the *Lagrangian* of f, precisely the function $\mathcal{L} : \mathcal{V} \times \mathcal{V}^* \longrightarrow \bar{\mathbb{R}}$ defined by

$$\mathcal{L}(\mathsf{M}, \varpi) := f(\mathsf{M}) + f^*(\varpi) - \varpi \cdot \mathsf{M}, \quad (\mathsf{M}, \varpi) \in \mathcal{V} \times \mathcal{V}^*.$$

Fenchel inequality can be then expressed by affirming that $\mathcal{L}(\mathsf{M}, \varpi) \geq 0$ and the pair (M, ϖ) is such that $\mathcal{L}(\mathsf{M}, \varpi) = 0$ *if and only if* $\varpi \in \partial f(\mathsf{M})$ (or, equivalently, $\mathsf{M} \in \partial f^*(\varpi)$).

For $K \subset \mathcal{V}^*$, denote by $\mathbb{I}_K$ the *indicator function* $\mathbb{I}_K : \mathcal{V}^* \longrightarrow \{0, +\infty\}$ defined by

$$\mathbb{I}_K(\varpi) = \begin{cases} 0 & \text{if } \varpi \in K, \\ +\infty & \text{if } \varpi \notin K. \end{cases}$$

$\mathbb{I}_K(\varpi)$ is a convex function if and only if K is a convex set and is also l.s.c. if and only if K is closed.

Let $f : \mathcal{V} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c., and K be a bounded convex set in $\mathcal{V}^*$, containing the origin in its interior. Define

$$w^*(\varpi) := f^*(\varpi) + \mathbb{I}_K(\varpi), \quad \varpi \in \mathcal{V}^*.$$

$w^*(\varpi) = f^*(\varpi)$ if $\varpi \in K$ and moreover, $w^*(\varpi) = f^*(\varpi)$ implies $\varpi \in K$, provided $f^*(\varpi) < \infty$.

Proposition 2. *$\partial w^*(\varpi) = \emptyset$ if $\varpi \notin K$. $\partial w^*(\varpi) = \partial f^*(\varpi)$, if $\varpi \in \mathrm{int}K$. For $\varpi \in \partial K$, $\partial w^*(\varpi) \supset \partial f^*(\varpi)$.*

Proof. The first two claims are trivial because w^* and f^* agree in a neighborhood of ϖ. Moreover, if $\varpi \in K$ and $\mathsf{M} \in \partial f^*(\varpi)$, then $f^*(\eta) \geq f^*(\varpi) + (\eta - \varpi) \cdot \mathsf{M}$. Consequently, $w^*(\eta) \geq f^*(\eta) \geq w^*(\varpi) + (\eta - \varpi) \cdot \mathsf{M}$, i.e. $\mathsf{M} \in \partial w^*(\varpi)$. □ □

Consider the conjugate function of w^*, namely $w : \mathcal{V} \longrightarrow \bar{\mathbb{R}}$, given by

$$w(\mathsf{M}) := w^{**}(\mathsf{M}) = \sup_{\varpi \in \mathcal{V}^*} (\varpi \cdot \mathsf{M} - w^*(\varpi)), \quad \mathsf{M} \in \mathcal{V}.$$

Since $w^* \geq f^*$, we trivially have $w = w^{**} \leq f^{**} = f$.

Proposition 3. *Assume that K is compact. Then the following statements hold true.*

(1) *$w(\mathsf{M}) = f(\mathsf{M})$ if and only if $K \cap \partial f(\mathsf{M}) \neq \emptyset$.*
(2) *There exist real constants c_1 and c_2, with $c_1 > 0$, such that*

$$w(\mathsf{M}) \leq c_1 |\mathsf{M}| + c_2, \quad \forall \mathsf{M} \in \mathcal{V}.$$

(3) *If K contains 0 in its interior, there exist real constants $c_3 > 0$ and $c_4 \geq 0$, such that*

$$w(\mathsf{M}) \geq c_3 |\mathsf{M}| - c_4, \quad \forall \mathsf{M} \in \mathcal{V}.$$

(4) $\partial w(\mathsf{M}) \subset K$ *and* $\mathrm{int}K \cap \partial f(\mathsf{M}) = \mathrm{int}K \cap \partial w(\mathsf{M})$.

(5) *If f is superlinear at infinity, that is $f(\mathsf{M})/|\mathsf{M}| \longrightarrow \infty$ as $|\mathsf{M}| \longrightarrow \infty$, then $\partial w(\lambda \mathsf{M}) \in \partial K$ for λ a number large enough.*

Proof. 1. Assume that $\hat{\varpi} \in K \cap \partial f(\mathsf{M})$, i.e. $\hat{\varpi} \in K$ and $f(\mathsf{M}) + f^*(\hat{\varpi}) = \langle \hat{\varpi}, \mathsf{M} \rangle$. Since $w^*(\varpi) \geq f^*(\varpi)$ for any ϖ, we get $w(\mathsf{M}) = w^{**}(\mathsf{M}) \leq f^{**}(\mathsf{M}) = f(\mathsf{M})$ for any M. On the other hand,

$$w(\mathsf{M}) = \sup_{\varpi \in K} (\varpi \cdot \mathsf{M} - f^*(\varpi)) \geq \hat{\varpi} \cdot \mathsf{M} - f^*(\hat{\varpi}) = f(\mathsf{M})$$

by Fenchel inequality. Conversely, assume that $w(\mathsf{M}) = f(\mathsf{M})$. Since K is compact, there exists $\hat{\varpi} \in K$ such that $w(\mathsf{M}) = \hat{\varpi} \cdot \mathsf{M} - f^*(\hat{\varpi})$. If $w(\mathsf{M}) = f(\mathsf{M})$, then $f(\mathsf{M}) + f^*(\hat{\varpi}) = \hat{\varpi} \cdot \mathsf{M}$, hence $\hat{\varpi} \in \partial f(\mathsf{M})$ by Fenchel theorem.

2. Since K is compact,

$$w(\mathsf{M}) = \sup_{\varpi \in K} (\varpi \cdot \mathsf{M} - f^*(\varpi)) \leq c_1 |\mathsf{M}| + c_2,$$

where $c_1 := \sup_{\varpi \in K} |\varpi|$ and $c_2 := \max(0, \sup_{\varpi \in K}(-f^*(\varpi)))$.

3. By assumption, the ball $B(0, c_3)$, centered at 0 and with radius c_3, is contained in K, namely $B(0, c_3) \subset K$, for some $c_3 > 0$. Therefore, $c_3 \frac{\varpi}{|\varpi|} \in K$, for any $\varpi \in K$, so that

$$w(\mathsf{M}) = \sup_{\varpi \in K} (\varpi \cdot \mathsf{M} - f^*(\varpi)) \geq c_3 |\mathsf{M}| - c_4,$$

with $c_4 := \sup_{\varpi \in K} f^*(\varpi)$.

4. Since $\xi \in \partial w(\mathsf{M})$ if and only if $\mathsf{M} \in \partial w^*(\xi)$, the result follows from Proposition 2.

5. By contradiction, if $\varpi \in \mathrm{int}K \cap \partial w(\lambda \mathsf{M})$ for some λ, it follows that $\varpi \in \partial f(\lambda \mathsf{M})$ as a consequence of the previous item. If λ is large enough, the last conclusion is a contradiction because f is superlinear at infinity. □

Let $w : \mathcal{V} \longrightarrow \mathbb{R}$ be convex and l.s.c. For $t > 0$ and $\mathsf{M} \in \mathcal{V}$, define

$$W_t(\mathsf{M}) := t w\left(\frac{\mathsf{M}}{t}\right).$$

Evidently, $W_1(\mathsf{M}) = w(\mathsf{M})$. The function $W(t, \mathsf{M}) := W_t(\mathsf{M})$ is a convex, l.s.c. function on $\mathbb{R}^+ \times \mathcal{V}$, which is also positively homogeneous of degree 1, i.e.

$$W(\lambda t, \lambda \mathsf{M}) = \lambda W(t, \mathsf{M}), \qquad \forall \lambda \in \mathbb{R}^+, \ (t, \mathsf{M}) \in \mathbb{R}^+ \times \mathcal{V}. \tag{5.1}$$

In particular, fixing t, the function W_t is convex and l.s.c. on $\mathcal{V}$ and, for $t > 0$, $\partial W_t(\mathsf{M}) = \partial w(\mathsf{M}/t)$. The so-called **recession function** of W_t, namely

$$W_0(\mathsf{M}) := \lim_{t \to 0^+} W_t(\mathsf{M}),$$

is then well defined, convex, and such that $\mathrm{dom} W_0(\mathsf{M}) = \mathcal{V}$, if w has linear growth at infinity. Moreover, if both W_1 and W_0 are of class C^1, from $W_t(\mathsf{M}) \longrightarrow W_0(\mathsf{M})$ we get $dW_t(\mathsf{M}) \longrightarrow dW_0(\mathsf{M})$ for any $\mathsf{M} \in \mathcal{V}$.

Proposition 4. *If W and W_0 are differentiable, then*

$$\partial W_0(\mathsf{M}) = \{dW_0(\mathsf{M})\} \subset \partial K, \qquad \forall \mathsf{M} \in \mathcal{V}.$$

Proof. $dW_t(\mathsf{M}) = dw(\mathsf{M}/t) \in K$ and we get the inclusion $\partial W_t(\mathsf{M}) \subset K$ for $t > 0$. Since $W_t(\mathsf{M}) \longrightarrow W_0(\mathsf{M})$ pointwise, and W_t and W_0 are convex and of class C^1, we have $dW_t(\mathsf{M}) \longrightarrow dW_0(\mathsf{M})$. Therefore, $dW_0(\mathsf{M}) \in K$ since $dW_t(\mathsf{M}) \in K$ and K is closed. If $dW_t(\mathsf{M}) \in \mathrm{int}(K)$, then, since in the interior of K we have $dW_t(\mathsf{M}) = dw(\mathsf{M}/t) = df(\mathsf{M}/t)$, we would have $df(\mathsf{M}/t) \in \mathrm{int}(K)$, which is impossible for t small enough, thanks to item 5 of Proposition 3. By evaluating then the limit as $t \longrightarrow 0^+$, we then get $dW_0(\mathsf{M}) \notin \mathrm{int}(K)$, so it belongs to the boundary of K. □

6. Deformations with constrained stress

Stress constraints appear in standard descriptions of elastic-perfectly-plastic or elastic-brittle behavior. They are described by **admissibility criteria** for the stress to be sustained point by point by the material. There is a multitude of such criteria in the current literature. Each criterion refers to certain aspects of the material behavior. In general, they are expressed by prescribing that the stress – σ or P, depending on circumstances – should belong to a certain convex set to assure elastic behavior of the material. By exploiting the link between stress and strain, assured by the energy in absence of viscous parts of the stress, the admissibility criterion can be expressed even in terms of strain.

In small strain regime (namely when $|Du| << 1$) and linear elastic behavior below the stress barrier, convexity of the energy and the analogous property for the stress admissibility region allow a natural and synthetic description of the related mechanics in terms of convex duality (see, e.g., [1], [2], [3], [7], [13], [16], [23], [24]).

The setting does not translate entirely in large strain regime, due to non-convexity of the energy density with respect to F. However, having in mind polyconvexity and considering the energy density e as a convex function of $M \in \Sigma_{1,+}$ or $\mathsf{M} \in \mathcal{V}_+$, we can use once again methods of convex analysis.

To this aim, *in contrast with common treatments*, as anticipated in the introduction, *we assign the admissibility criterion as a bound on the stress form* $x \longmapsto \omega(x)$.

Once we have determined a deformation related with an admissible ω by the energy, we recover the relevant (admissible in this sense) stress by multiplying from the right ω by $\frac{dM(F)}{dF}$.

We shall presume that $\omega(x)$ belongs to a convex subset K of $\Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. It is not immediate to recover by the action of $\frac{dM(F)}{dF}$ a relevant convex admissibility set for P because $\frac{dM(F)}{dF}$ depends on a specific deformation F, which should be even incompatible, while K is prescribed disregarding the deformation.

A constraint on ω rather than P implies the possibility of constraining stresses associated with independent strains of lines, surfaces, and volume that could occur in a neighborhood of the point considered in case of locally incompatible deformations.

Two steps are in general necessary for determining existence of minimizers of energies by direct methods in calculus of variations: (i) the extension of the class of competitors to some topological space in such a way that energy bounded level sets are compact, and (ii) a companion re-definition of the functional under scrutiny over such an extended class as a lower semicontinuous function – this is the so-called relaxed energy. Once the first step has been completed, the need of enlarging the functional setting to non-smooth functions appears often. It is hard to compute explicitly the *relaxed energy*, which is in general unknown, even in non-linear elasticity of simple bodies. Often, heuristic choices are helpful.

For the problem tackled here, subsets of the class of Cartesian currents, introduced and discussed in [10] and [11], are appropriate spaces of competitors. Within their setting we analyze below the existence of minimizers for the energy with constrained stresses in case of Dirichlet boundary conditions. To go into details, some functional notions need to be called upon.

First we remind that a $k-$**current** T on $\mathbb{R}^n$, $k \leq n$, is a linear continuous operator on the space of $k-$forms ω on $\mathbb{R}^n$ with C_c^∞ coefficients. As pointed out previously, T has the meaning of an inner power in the present setting.

The so-called **boundary** of T, denoted by ∂T, is a $(k-1)-$current on $\mathbb{R}^n$ defined by duality by the exterior differential operator d on $(k-1)-$forms, namely $\partial T(\eta) := T(d\eta)$ for any $(k-1)-$form η on $\mathbb{R}^n$. The **total variation** of a current T is then defined as the nonnegative measure

$$\|T\|(f) := \sup\left\{T(\eta) \ \mid\ |\eta(z)| \leq f(z),\ \eta \in C_c^0(\mathbb{R}^n)\right\},$$

and the **mass** of T is the value of the total variation over 1, denoted by $\mathbf{M}(T)$, namely

$$\mathbf{M}(T) := \|T\|(1).$$

We commonly say that *a sequence of $k-$currents $\{T_i\}$ weakly converges to a current* T, and we write $T_i \rightharpoonup T$, *if* $T_i(\omega) \longrightarrow T(\omega)$ *for every $k-$form* ω. Observe that

$$\mathbf{M}(T) \leq \liminf_{i\to\infty} \mathbf{M}(T_i),$$

if $T_i \rightharpoonup T$.

Let $\mathcal{H}^k$ be the $k-$dimensional Hausdorff measure over $\mathbb{R}^n$. A $k-$dimensional $\mathcal{H}^k-$rectifiable subset[6] $\mathcal{T}$ of $\mathbb{R}^n$, a unit $3-$vector field $\overrightarrow{T}(x)$, defined $\mathcal{H}^k$ a.e. on $\mathcal{T}$ and a non-negative integer-valued $\mathcal{H}^k-$summable map θ on $\mathcal{T}$ identify a current T by

$$T(\omega) := \int_{\mathcal{T}} \omega(z) \cdot \overrightarrow{T}(z)\ \theta(z)\,\mathcal{H}^k(dz),$$

commonly called a $k-$**dimensional integer rectifiable current**. The notation $T = \tau(\mathcal{T}, \theta, \overrightarrow{T})$ is the classical one to indicate them.

The total variation $\|T\|$ of T is then given by $\|T\| := \theta\ \mathcal{H}^k$, i.e.,

$$\|T\|(f) = \int_{\mathcal{T}} f(z)\,\theta(z)\,\mathcal{H}^k(dz) \quad \forall f \in C_c^\infty(\mathbb{R}^n),$$

so that the mass $\mathbf{M}$ of T is

$$\mathbf{M}(T) = \int_{\mathcal{T}} \theta(z)\,d\mathcal{H}^k(z).$$

Closure and compactness of integer rectifiable current classes are assured by the fundamental Federer-Fleming theorem [8].

Theorem 1 (Federer-Fleming)**.** *Let $\{T_i\}$ be a sequence of $k-$dimensional integer rectifiable currents such that* $\sup_i\{\mathbf{M}(T_i) + \mathbf{M}(\partial T_i)\} < \infty$. *Then there exists a $k-$dimensional integer rectifiable current T and a subsequence $\{T_{i_k}\}$ of $\{T_i\}$ such that $T_{i_k} \rightharpoonup T$. Moreover, ∂T is also a $(k-1)$-dimensional integer rectifiable current.*

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[6] That is a set $\mathcal{T} = \cup_{k=1}^\infty \mathcal{T}_k + \mathcal{T}_0$, where every $\mathcal{T}_k$, $k = 1, 2, ...$, is a Borel set of a $k-$dimensional smooth manifold and the $k-$dimensional Hausdorff measure of $\mathcal{T}_0$ is zero, namely $\mathcal{H}^3(\mathcal{T}_0) = 0$.

For a $3-$dimensional integer rectifiable current $T = \tau(\mathcal{T}, \theta, \overrightarrow{T})$ in $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, the unit $3-$vector $\overrightarrow{T} \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ writes

$$\overrightarrow{T}(x,y) = (\zeta, F, A, a)(x,y)$$

with the constraint

$$\zeta^2 + |F|^2 + |A|^2 + a^2 = 1.$$

Consider two bounded, connected, open sets, $\mathcal{B} \subset \mathbb{R}^3$ and $\tilde{\mathcal{B}} \subset \tilde{\mathbb{R}}^3$, with $\mathcal{L}^3(\partial\mathcal{B}) = \mathcal{L}^3\left(\partial\tilde{\mathcal{B}}\right) = 0$ ($\mathcal{L}^3$ is Lebesgue volume measure). In the Cartesian product $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ indicate also by $\pi : \mathbb{R}^3 \times \tilde{\mathbb{R}}^3 \to \mathbb{R}^3$ and $\tilde{\pi} : \mathbb{R}^3 \times \tilde{\mathbb{R}}^3 \to \tilde{\mathbb{R}}^3$ the orthogonal projections onto $\mathbb{R}^3$ and $\tilde{\mathbb{R}}^3$, respectively.

Definition 1. *For $\mathcal{T}$ a $3-$dimensional $\mathcal{H}^3-$rectifiable subset of $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, a 3D integer rectifiable current $T := \tau(\mathcal{T}, \theta, \overrightarrow{T})$ is said to be a* ***weak diffeomorphism*** *in the class* $\text{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$ *if $\overrightarrow{T} = (\zeta, F, A, a)$ is characterized by*

(i): $\zeta \geq 0$ and $a \geq 0$, $\mathcal{H}^3$ a.e.,
(ii):

$$\int_{\mathcal{T}\cap\pi^{-1}(x)} \zeta(z)\,\theta(z)\,d\mathcal{H}^k(z) = 1 \quad for\ \mathcal{L}^3\ a.e.\ x \in \mathcal{B},$$

$$\int_{\mathcal{T}\cap\tilde{\pi}^{-1}(y)} a(z)\,\theta(z)\,d\mathcal{H}^k(z) = 1 \quad for \mathcal{L}^3 a.e. y \in \tilde{\mathcal{B}},$$

(iii): $\text{spt}(T) \subset \overline{\mathcal{B}} \times \overline{\tilde{\mathcal{B}}}$ and $\text{spt}(\partial T) \subset \partial\overline{\mathcal{B}} \times \partial\overline{\tilde{\mathcal{B}}}$.

The last requirement, namely $\text{spt}(\partial T) \subset \partial\overline{\mathcal{B}} \times \partial\overline{\tilde{\mathcal{B}}}$, is tantamount to impose that $\partial T = 0$ in $\mathcal{B} \times \tilde{\mathcal{B}}$. Such a condition excludes maps describing deformations with nucleation of voids or discontinuities like the ones associated with the formation of fractures (see also related discussions in [9]).

We can take $\tilde{\mathcal{B}}$ to be coincident with the actual placement of the body, obtained through a deformation of $\mathcal{B}$. As a consequence, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ can be viewed as the projections of the graph $\mathcal{G}_u$ of the deformation u connecting them one another. $\mathcal{G}_u$ is a subset of the Cartesian product $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$.

If u is a C^1 orientation preserving diffeomorphism that maps $\partial\mathcal{B}$ into $\partial\tilde{\mathcal{B}}$, integration over the graph $\mathcal{G}_u$ is a $k-$dimensional integer rectifiable current $G_u = \tau(\mathcal{G}_u, \theta, \overrightarrow{G}_u)$ in $\text{dif}(\bar{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$. In particular, we have

$$\overrightarrow{G}_u := \frac{M(Du(x))}{|M(Du(x))|},$$

and

$$\pi_{\#}\left(\mathcal{H}^3\right) = |M(Du(x))|\,dx,$$

where $\pi_{\#}$ is the counterpart in terms of measures of π. Then, we write

$$G_u(\omega) = \int_{\mathcal{B}} \omega(x, u(x)) \cdot M(Du(x))\ dx$$

for every smooth $3-$form with compact support in $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$.

The remark allows us to interpret the current associated with a bijective orientation preserving differentiable map as an internal work in the extended sense *suggested by the natural expression of the mechanics of simple bodies in terms of forms that we have developed here.*

In going back to the class of weak diffeomorphisms, however, we notice that it includes degenerate diffeomorphisms. Consider, in fact, a point $x_0 \in \partial\mathcal{B}$. The integration over the 3−dimensional surface $(\mathcal{B} \times \{x_0\}) \cup \left(\{x_0\} \times \tilde{\mathcal{B}}\right)$ belongs to $\mathrm{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$.

We take the elastic energy density as a function defined over $\mathcal{V}$ by extension from $\mathcal{V}_+$, rather than on $\Sigma_{1,+}$. We do not consider, then, the component 1 of $M \in \Sigma_{1,+}$.

The basic ingredients of our analysis are then

- **(1):** the space $\mathcal{V}$ defined at the end of Section 2 and its dual $\mathcal{V}^*$,
- **(2):** the energy density $e : \mathcal{V} \longrightarrow \bar{\mathbb{R}}$, a convex l.s.c. function over $\mathcal{V}$, and
- **(3):** the stress constraint, expressed by prescribing a convex compact set $K \subset \mathcal{V}^*$ with $0 \in \mathrm{int}K$.

The energy density e is then a function of F, $\mathrm{cof}F$, and $\det F$ alone.

We then define $w^*(\varpi) := e^*(\varpi) + \mathbb{I}_K(\varpi)$, with $\varpi \in \mathcal{V}^*$ and compute $w : \mathcal{V} \longrightarrow \bar{\mathbb{R}}$, given by $w = w^{**}$. By construction, w is convex and l.s.c. Hence, by Proposition 3 it follows that w – we can call it **stored energy density** – grows linearly at infinity:

$$c_3 |\mathsf{M}| - c_4 \leq w(\mathsf{M}) \leq c_1 |\mathsf{M}| + c_2, \tag{6.1}$$

for any $\mathsf{M} \in \mathcal{V}$. It is then possible to have *strain concentration (localization).*

Since the unit tangent vector $\overrightarrow{T} = (\zeta, F, A, a)$ that orients the integration domain of a generic weak diffeomorphism (the concept, in fact, is not strictly associated with a deformation) may possibly have $\zeta = 0$ and/or $a = 0$, we need to extend the stored energy density to the whole $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. To this aim we use the extension $W : \bar{\mathbb{R}}^+ \times \mathcal{V} \longrightarrow \bar{\mathbb{R}}$ of $w : \mathcal{V} \longrightarrow \bar{\mathbb{R}}$, as defined in previous sections. The space $\bar{\mathbb{R}}^+ \times \mathcal{V}$, we remind, is defined by

$$\bar{\mathbb{R}}^+ \times \mathcal{V} := \left\{ M \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \mid M = (\zeta, \mathsf{M}), \zeta \geq 0 \right\}.$$

And we put, as above,

$$W(\zeta, \mathsf{M}) := \begin{cases} W_\zeta(\mathsf{M}) := \zeta w\left(\frac{\mathsf{M}}{\zeta}\right), & \text{if } \zeta > 0, \\ W_0(\mathsf{M}) & \text{if } \zeta = 0, \end{cases} \tag{6.2}$$

with $W_0(\mathsf{M})$ the recession function of $w(\mathsf{M})$. We call $W(\zeta, \mathsf{M})$ **extended stored energy density including the constraint**. It is by definition l.s.c. and convex. Moreover, since $w(\mathsf{M}) \geq c_3 |\mathsf{M}| - c_4$, for $\zeta > 0$ we have

$$W(\zeta, \mathsf{M}) = \zeta w\left(\frac{\mathsf{M}}{\zeta}\right) \geq c_3 \zeta \left|\frac{\mathsf{M}}{\zeta}\right| - c_4 \zeta = c_3 |\mathsf{M}| - c_4 \zeta$$

and

$$W(0, \mathsf{M}) = W_0(\mathsf{M}) = \lim_{\zeta \longrightarrow 0} \zeta w\left(\frac{\mathsf{M}}{\zeta}\right) \geq c_3 |\mathsf{M}|.$$

Hence, we have

$$W(\zeta, \mathsf{M}) \geq c_3 |\mathsf{M}| - c_4 \zeta \text{ for any } \zeta \geq 0. \tag{6.3}$$

Then, for $T = \tau(\mathcal{T}, \theta, \overrightarrow{T}) \in \operatorname{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$, the **extended global energy including the constraint** is given by

$$\mathcal{E}(T) := \int_{\mathcal{T}} W(\overrightarrow{T}(z)) d\|T\|(z) = \int_{\mathcal{T}} W(\overrightarrow{T}(z))\, \theta(z)\, d\mathcal{H}^3(z).$$

From the 1−homogeneity of W, it follows that, for a smooth diffeomorphism $u : \mathcal{B} \longrightarrow \tilde{\mathcal{B}}$, we get

$$\mathcal{E}(G_u) = \int_{\mathcal{B}} w(M(Du))\ dx,$$

which is the stored energy we start with – compare Section 7.

Theorem 2. *Let* Γ *be a two-dimensional current of finite mass and zero boundary on* $\partial\mathcal{B}\times\partial\tilde{\mathcal{B}}$. *Assume that the class*

$$\mathcal{A} := \left\{ T \in \operatorname{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}}) \mid \partial T = \Gamma \right\}$$

is not empty. In this class the functional $\mathcal{E}(T)$, *the extended global energy including the constraint, attains a minimum.*

Proof. Let $T := \tau(\mathcal{T}, \theta, \overrightarrow{T})$ belong to $\operatorname{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$. Set

$$\mathcal{T}_0 = \left\{ z \in \mathcal{T} \mid \overrightarrow{T}(z) = (0, \mathsf{M}(z)) \right\}$$

and

$$\mathcal{T}_+ := \mathcal{T} \backslash \mathcal{T}_0.$$

From (6.3), since ζ ranges from 0 to 1, as $|\overrightarrow{T}| = 1$, we get

$$W(\overrightarrow{T}(z)) \geq \begin{cases} c_3 \left|\overrightarrow{T}(z)\right| - (c_3 + c_4), & \text{if } z \in \mathcal{T}_+, \\ c_3 \left|\overrightarrow{T}(z)\right| & \text{if } z \in \mathcal{T}_0. \end{cases}$$

Hence, taking into account that $\left|\overrightarrow{T}(z)\right| = 1$ and (ii) of Definition 1,

$$c_3 \mathbf{M}(T \llcorner \mathcal{T}_0) = c_3 \int_{\mathcal{T}_0} \theta(z)\ d\mathcal{H}^3(z) \leq \mathcal{E}(T \llcorner \mathcal{T}_0),$$

with $T \llcorner \mathcal{T}_0$ the restriction of T to $\mathcal{T}_0$, and

$$c_3 \mathbf{M}(T \llcorner \mathcal{T}_+) = c_3 \int_{\mathcal{T}_+} \theta(z)\ d\mathcal{H}^3(z) \leq \int_{\mathcal{T}_+} \theta(z)\, W(\overrightarrow{T}(z))\ d\mathcal{H}^3(z) +$$

$$+ (c_3 + c_4) \int_{\mathcal{T}_+} \zeta(z)\, \theta(z)\ dz = \mathcal{E}(T \llcorner \mathcal{T}_0) + (c_3 + c_4)\, \mathcal{L}^3(\mathcal{B}),$$

where $\mathcal{L}^3$ is the $3D$ Lebesgue measure. Eventually, we get

$$c_3 \mathbf{M}(T) \leq \mathcal{E}(T) + (c_3 + c_4)\, \mathcal{L}^3(\mathcal{B}). \tag{6.4}$$

Define $\lambda := \inf_{T \in \mathcal{A}} \mathcal{E}(T)$. Let $\{T_k\} \subset \mathcal{A}$ a minimizing sequence. Then $\mathbf{M}(\partial T_k) = \mathbf{M}(\Gamma) < \infty$ for every k, and, by (6.4), it follows that

$$c_3 \mathbf{M}(T_k) \leq \mathcal{E}(T_k) + (c_3 + c_4)\, \mathcal{L}^3(\mathcal{B}).$$

Hence the elements T_k of the sequence have equibounded masses, so the Federer-Fleming closure-compactness theorem applies. Hence, there exists a $3D$ integer rectifiable current $T = \tau(\mathcal{T}, \theta, \overrightarrow{T})$, with $\overrightarrow{T} = (\zeta, F, A, a)$, and a subsequence $\{T_{k_n}\}$

of $\{T_k\}$ such that $T_{k_n} \rightharpoonup T$. Since the extended stored energy density $W(\zeta, \mathsf{M})$ is convex and l.s.c., the map $T \longmapsto \mathcal{E}(T)$ is lower semicontinuous,

$$\mathcal{E}(T) \leq \liminf_{k\to\infty} \mathcal{E}(T_k) = \lambda,$$

hence $\mathcal{E}(T) = \lambda$. Moreover, since weak convergence implies $\zeta \geq 0$, $a \geq 0$, and the validity of the equalities in Definition 1 for T, we can conclude that $T \in \mathcal{A}$. □

The previous theorem assures the existence of minimizers for a boundary value problem where the datum Γ is expressed by the boundary of a weak diffeomorphism, say ϑ, namely $\Gamma = \partial\vartheta$ (strong anchoring condition in the terminology of [11]), i.e. it is assigned in terms of extended internal work. Minimizers are themselves weak diffeomorphisms.

The result is then reconciled with the standard view on continuum mechanics of simple bodies (see [11] about it). It also assures us that minimizers are weak diffeomeorphisms with zero boundary in $\mathcal{B} \times \mathcal{B}_a$ as long as the stress constraint is satisfied. Hence, even critical states are described by weak diffeomorphisms. As such, they do not display fractures or holes such as the ones due to cavitation.

7. Stress form associated with a deformation

We discuss here the characteristic features of the stress associated with the minimizers emerging in Theorem 2. To this aim, it is expedient to remind some steps of the procedure followed so far.

For any $\zeta \geq 0$, let $W_\zeta : \mathcal{V} \longrightarrow \bar{\mathbb{R}}$ be given by $W_\zeta(\mathsf{M}) := W(\overrightarrow{T})$ if $\overrightarrow{T} = (\zeta, \mathsf{M})$. Let us assume that W_ζ is differentiable for any $\zeta \geq 0$. In this case, we have seen that

$$\partial W_\zeta(\mathsf{M}) = \zeta \partial w \left(\frac{\mathsf{M}}{\zeta}\right) \subset K, \qquad \partial W_0(\mathsf{M}) \subset \partial K.$$

Let $T = \tau(\mathcal{T}, \theta, \overrightarrow{T}) \in \mathrm{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$ be a weak diffeomorphism such that

$$\mathcal{E}(T) := \int_{\mathcal{T}} W(\overrightarrow{T}(z)) \ \|T\|(dz) < \infty,$$

where $\|T\| := \theta\, \mathcal{H}^3$ denotes the mass measure of T, with density θ. With every unit $3-$vector $\overrightarrow{T} \in \mathbb{R}^+ \times \mathcal{V}$, we associate *the corresponding stress form* taking values on $\mathcal{V}^*$, namely

$$\varpi(\overrightarrow{T}) := dW_\zeta(\mathsf{M}), \quad \text{if } \overrightarrow{T} = (\zeta, \mathsf{M}).$$

Therefore, since $T = \tau(\mathcal{T}, \theta, \overrightarrow{T}) \in \mathrm{dif}(\overline{\mathcal{B}}, \overline{\tilde{\mathcal{B}}})$ contains (in the generalized sense described so far) the elements necessary to construct standard strain measures, the product $\overrightarrow{T}(z) \ d\|T\|(z)$ can be considered as a **generalized strain measure,** with a corresponding **stress measure,** $S(dz)$, given by

$$S(dz) := \varpi(\overrightarrow{T}(z)) \ d\|T\|(z).$$

S is a vector-valued measure with values in $\mathcal{V}^*$, with bounded coefficients with respect to $\|T\|$.

Once again, the traditional view is recovered when we remind that for every diffeomorphism $T = (\mathcal{T}, \theta, \overrightarrow{T})$, we can uniquely associate with it a $BV(\mathbb{R}, \tilde{\mathbb{R}}^3)$ map u with some peculiar properties (see [10] and [11] for further details). Specifically, with $Du = Du^{(a)} + Du^{(s)}$, the decomposition of the gradient measure Du into absolutely continuous and singular components,

$$\left|M\left(Du^{(a)}\right)\right| := \sqrt{1 + \left|\mathsf{M}\left(Du^{(a)}\right)\right|^2},$$

with

$$\mathsf{M}\left(Du^{(a)}\right) := \left(Du^{(a)}, \mathrm{cof}Du^{(a)}, \det Du^{(a)}\right),$$

is summable. Moreover, by denoting by G_u the integration over the graph of u over $\mathcal{B}_+$, which is the set of Lebesgue points of u and $Du^{(a)}$, we have

$$T \llcorner \mathcal{B}_+ \times \hat{\mathbb{R}}^3 = G_u := (\mathcal{T}_+, 1, \overrightarrow{\mathcal{G}}_u),$$

where

$$\mathcal{T}_+ = \{z = (x, u(x)) \mid x \in \mathcal{B}_+\} \qquad \mathcal{H}^3 - a.e.,$$

$$\overrightarrow{\mathcal{G}}_u = \overrightarrow{T}(x, u(x)) = \frac{1}{\left|M\left(Du^{(a)}(x)\right)\right|}\left(1, \mathsf{M}\left(Du^{(a)}(x)\right)\right).$$

Moreover, with $\pi : \mathcal{B} \times \tilde{\mathbb{R}}^3 \longrightarrow \mathcal{B}$, the projection onto the reference configuration and $\pi_\#$ its counterpart in terms of measures, we have also

$$\pi_\# \|T\| \llcorner \mathcal{T}_+ = \left|M\left(Du^{(a)}(x)\right)\right| dx$$

and

$$\begin{aligned} \pi_\#(S \llcorner \mathcal{T}_+) &= \pi_\#\left(\varpi(\overrightarrow{T})\|T\| \llcorner \mathcal{T}_+\right) = \varpi(x, u(x))\left|M\left(Du^{(a)}\right)(x)\right| \ dx = \\ &= dW_1(T_\flat) \ dx = dw\left(\mathsf{M}\left(Du^{(a)}(x)\right)\right) dx. \end{aligned}$$

The following items summarize essential aspects of the analysis developed so far.

***(i)*:** The restriction $u : \mathcal{B}_+ \longrightarrow \tilde{\mathbb{R}}^3$ represents the part of the deformation that is classical: it is not associated with strain localization.

***(ii)*:** The global stored energy on $\mathcal{B}_+$, $\mathcal{E}(T)$, is given by

$$\int_{\mathcal{T}_+} W(\overrightarrow{T}(z)) \ \|T\|(dz) = \int_{\mathcal{B}_+} w\left(M\left(Du^{(a)}(x)\right)\right) \ dx. \tag{7.1}$$

***(iii)*:** The projection of the stress measure over the reference configuration is absolutely continuou with respect to $\mathcal{L}^3$ with density given by $\frac{\partial w}{\partial M}\left(M\left(Du^{(a)}(x)\right)\right)$. T*he first Piola-Kirchhoff stress is well defined just on* $\mathcal{B}_+$ and is given by the map

$$F \longmapsto \frac{\partial w}{\partial M}\left(M\left(Du^{(a)}(x)\right)\right)\frac{dM}{dF}\left(Du^{(a)}(x)\right).$$

***(iv)*:** The complementary set $\mathcal{B}\backslash\mathcal{B}_+$ has zero measure. Denote by $\hat{\pi} : \mathbb{R}^3 \times \hat{\mathbb{R}}^3 \longrightarrow \hat{\mathbb{R}}^3$ the orthogonal projection onto the second factor. Then $\hat{\pi}(\mathcal{T})$ is the current place, while $\hat{\pi}(\mathcal{T}\backslash\mathcal{T}_+)$ is the image of $\mathcal{B}\backslash\mathcal{B}_+$ determined by the deformation. To get the total energy, we need to add to the part over $\mathcal{T}_+$, the one corresponding to the elastic behavior, the 'spurious' contributions pertaining to $\mathcal{T}\backslash\mathcal{T}_+$ where phenomena as swelling or prodroms of plastic slips occur. Strains correspond to both states, as we shall discuss in the next two sections. Moreover, the current

$$T^S = T \llcorner (\mathcal{B}\backslash\mathcal{B}_+) \times \hat{\mathbb{R}}^3 = (\mathcal{T}\backslash\mathcal{T}_+, \theta, \|T\|)$$

contributes to weak balance equations.

8. Weak balances involving the stress form

Once again, let us identify the set $\tilde{\mathcal{B}}$ in $\tilde{\mathbb{R}}^3$, involved in the definition of weak diffeomorphism, with the current shape of the body under scrutiny. Also, when a weak diffeomorphism $T := \tau(\mathcal{T}, \theta, \overrightarrow{T})$ appears in the following lines, it is presumed to be a member of $\mathcal{A}$, the class appearing in the existence theorem.

Our extended stored energy functional is a parametric integral. Its first variation can be computed as in [8]. For reader's convenience, we repeat it briefly.

For every $h \in C_c^\infty\left(\mathcal{B}, \mathbb{R}^3\right)$, $k \in C_c^\infty(\tilde{\mathcal{B}}, \tilde{\mathbb{R}}^3)$, and $\varepsilon \in \mathbb{R}$, set

$$\rho(z) := (h(x), k(y)),$$

with $z = (x, y)$, $x \in \mathcal{B}$, $y \in \tilde{\mathcal{B}}$, and define the map

$$\phi_\varepsilon : \mathcal{B} \times \tilde{\mathcal{B}} \longrightarrow \mathbb{R}^3 \times \tilde{\mathbb{R}}^3$$

by

$$\phi_\varepsilon(z) := z + \varepsilon \rho(z).$$

For $|\varepsilon|$ small enough, the two components of ϕ_ε, namely $x \longmapsto x + \varepsilon h(x)$ and $y \longmapsto y + \varepsilon k(y)$, are orientation preserving diffeomorphisms of both the reference and current places $\mathcal{B}$ and $\tilde{\mathcal{B}}$, respectively. Hence, ϕ_ε is univalent on $\mathcal{T}$. Moreover, the current $\phi_{\varepsilon\#}T$, the push forward of T along ϕ_ε, belongs to $\mathcal{A}$.

Since the integrand W in (7.1) is positively homogeneous of degree 1, the area formula yields

$$\mathcal{E}\left(\phi_{\varepsilon\#}T\right) = \int W(\overrightarrow{\phi_{\varepsilon\#}T})\, d\left\|\phi_{\varepsilon\#}T\right\| = \int_{\mathcal{T}} W(\xi_\varepsilon(z))\, d\|T\|(z),$$

where $\xi_\varepsilon(z) := \Lambda_3(D\phi_\varepsilon(z))\overrightarrow{T}(z)$, i.e.

$$\xi_\varepsilon(z) := D\phi_\varepsilon(z) v_1 \wedge D\phi_\varepsilon(z) v_2 \wedge D\phi_\varepsilon(z) v_3,$$

if we express $\overrightarrow{T}(z)$ as $v_1 \wedge v_2 \wedge v_3$. Moreover, since $D\phi_\varepsilon(z) := \mathrm{I} + \varepsilon D\rho(z)$, I the identity in $M_{6\times 6}$, we compute

$$\begin{aligned} \xi_\varepsilon(z) &= (\mathrm{I} + \varepsilon D\rho(z)) v_1 \wedge (\mathrm{I} + \varepsilon D\rho(z)) v_2 \wedge (\mathrm{I} + \varepsilon D\rho(z)) v_3 = \\ &= \overrightarrow{T}(z) + \varepsilon L(D\rho(z)) + O\left(\varepsilon^2\right), \end{aligned}$$

where, for every $R \in M_{6\times 6}$, $L(R)$ denotes the linear operator

$$L(R) : \Lambda_3(\mathbb{R}^3, \tilde{\mathbb{R}}^3) \longrightarrow \Lambda_3(\mathbb{R}^3, \tilde{\mathbb{R}}^3)$$

with values

$$L(R)\,\mathrm{t} := Rv_1 \wedge v_2 \wedge v_3 + v_1 \wedge Rv_2 \wedge v_3 + v_1 \wedge v_2 \wedge Rv_3 \tag{8.1}$$

if $\mathrm{t} = v_1 \wedge v_2 \wedge v_3 \in \Lambda_3(\mathbb{R}^3, \tilde{\mathbb{R}}^3)$. Hence, we conclude that the map $\varepsilon \longmapsto \xi_\varepsilon$ is differentiable at $\varepsilon = 0$ and

$$\frac{\partial \xi_\varepsilon(z)}{\partial \varepsilon}\Big|_{\varepsilon=0} = L(D\rho(z))\overrightarrow{T}(z).$$

As a special case, consider now variations only in the actual shape of the simple body under scrutiny, which is tantamount to set $h(x) = 0$ so that $\phi_\varepsilon(z) := (x, y + \varepsilon k(y))$. The components of the six-dimensional vector $(\mathrm{I} + \varepsilon D(0, k(y)))v$, with v any vector of the three-plet v_1, v_2, v_3, in the reference place $\mathcal{B}$, are independent of ε.

Take a basis e_1, e_2, e_3 in $\mathbb{R}^3$. The component of $\xi_\varepsilon(z)$ proportional to the element $e_1 \wedge e_2 \wedge e_3$ is also independent of ε, and the $e_1 \wedge e_2 \wedge e_3$ component of $L(D\rho(z))\overrightarrow{T}(z)$ is zero.

Proposition 5. *Assume that the extended stored energy density $W(t, \mathsf{M}) : \mathbb{R}^+ \times \mathcal{V} \longrightarrow \bar{\mathbb{R}}$ is differentiable in M for all $t \geq 0$. Then*

$$\delta\mathcal{E}(T,(0,k)) = \int_{\mathcal{T}} (0, \varpi(\overrightarrow{T}(z)))L(D(0,k(y)))\overrightarrow{T}(x,y)\ \ d\|T\|(x,y).$$

To be more explicit, take once again the three-plet e_1, e_2, e_3 as basis in $\mathbb{R}^3$, and write $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ for the one in $\tilde{\mathbb{R}}^3$. Consider also two 3−multi-indices α and β such that the sum of their lengths is 3, namely $|\alpha| + |\beta| = 3$. As a matter of notation, 0 will denote the empty multi-index, while $\bar{0}$ will be used for the full multi-index, namely $\bar{0} = \{1,2,3\}$, so that $e_{\bar{0}} \wedge \tilde{e}_0 = e_1 \wedge e_2 \wedge e_3$ and $e_0 \wedge \tilde{e}_{\bar{0}} = \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3$. This way $\{e_\alpha \wedge \tilde{e}_\beta\}_{|\alpha|+|\beta|=3}$ is a basis for $\Lambda_3(\mathbb{R}^3, \tilde{\mathbb{R}}^3)$, which can be perhaps more explicitly written as $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$.

Denote by $W_{,e_\alpha \wedge \tilde{e}_\beta}(\mathrm{t})$ the derivative of W at t in the 'direction' $e_\alpha \wedge \tilde{e}_\beta$. The stress form at t has components listed in the 1×19 matrix

$$\left[W_{,e_\alpha \wedge \tilde{e}_\beta}(\mathrm{t})\right], \quad |\alpha| + |\beta| = 3, \quad |\beta| > 0.$$

Moreover, $L(D(0,k))\overrightarrow{T}$ has components listed in the 20×1 matrix

$$\left[L(D\rho)^{\alpha\beta}\right], \quad |\alpha| + |\beta| = 3.$$

By using (8.1), we can compute that $L(D(0,k))^{\bar{0}0} = 0$, so that

$$\delta\mathcal{E}(T,(0,k)) = \int_{\mathcal{T}} \sum_{\substack{|\alpha|+|\beta|=3 \\ |\alpha|>0}} W_{,e_\alpha \wedge \tilde{e}_\beta}(\overrightarrow{T})(L(D(0,k))\overrightarrow{T})^{\alpha\beta}\ d\|T\|(x,y).$$

Similarly, we may consider variations only in the reference place by setting $\rho(z) = (h(x), 0)$. In this case, the components of the six-dimensional vector $(\mathrm{I} + \varepsilon D(h(x), y))v$, referred to the actual place, are independent of ε. Consequently, the component of $L(D(h,0))\overrightarrow{T}$ with respect to $\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3$ vanishes. By assuming that the extended stored energy W be differentiable on $\mathbb{R}^+ \times \mathcal{V}$, we conclude that

$$\delta\mathcal{E}(T,(h,0)) = \int_{\mathcal{T}} \left\langle dW(\overrightarrow{T}), L(D(h,0))\overrightarrow{T}\right\rangle\ d\|T\|(x,y),$$

or, more explicitly, by taking into account that $(L(D(h,0))\overrightarrow{T})^{\bar{0}0} = 0$,

$$\delta\mathcal{E}(T,(h,0)) = \int_{\mathcal{T}} \sum_{\substack{|\alpha|+|\beta|=3 \\ |\beta|>0}} W_{,e_\alpha \wedge \tilde{e}_\beta}(\overrightarrow{T})(L(D(h,0))\overrightarrow{T})^{\alpha\beta}\ d\|T\|(x,y).$$

In evaluating both variations $\delta\mathcal{E}(T,(0,k))$ and $\delta\mathcal{E}(T,(h,0))$, it is useful to remind that the partial derivatives of W are homogeneous functions of degree zero and the map $\overrightarrow{T} \longmapsto L(D\rho(z))\overrightarrow{T}$ is linear.

Assume that the set $\mathcal{T}$ in $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, called upon every time we have constructed a weak diffeomorphism $T = (\mathcal{T}, 1, \bar{T})$, can be parametrized by using another set, say $\mathcal{S}$, in $\mathbb{R}^3$ and a Lipschitz map $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ such that $\psi(\mathcal{S}) = \mathcal{T}$, so $\psi(s) = (x,y) \in \mathcal{T}$, for any $s \in \mathcal{S}$, and $\psi|_{\mathcal{S}}$ is injective. Moreover, assume that we have over $\mathcal{S}$ a $k-$dimensional integer rectifiable current $S = \tau(\mathcal{S}, 1, \overrightarrow{S})$ such that T be the push-forward of S along ψ, namely $T := \psi_{\#}S$. In this case, by setting $\xi(s) := \Lambda_3 (D\psi(s)) \overrightarrow{S}$, the variations $\delta_{(0,k)}\mathcal{E}(T)$ and $\delta_{(h,0)}\mathcal{E}(T)$ rewrite as

$$\delta\mathcal{E}(T,(0,k)) = \int_{\mathcal{S}} \sum_{\substack{|\alpha|+|\beta|=3 \\ |\alpha|>0}} W_{,e_\alpha\wedge\tilde{e}_\beta}(\xi)(L(D(0,k))\,\xi)^{\alpha\beta}\ d\|S\|(s),$$

$$\delta\mathcal{E}(T,(h,0)) = \int_{\mathcal{S}} \sum_{\substack{|\alpha|+|\beta|=3 \\ |\beta|>0}} W_{,e_\alpha\wedge\tilde{e}_\beta}(\xi)(L(D(h,0))\,\xi)^{\alpha\beta}(s)\ \ d\|S\|(s).$$

The relevant balance equations emerge by equalizing to zero the first variations above. The vertical variation $\delta\mathcal{E}(T,(0,k))$ determines the weak form of the balance of forces in terms of the Cauchy stess, while the horizontal variation $\delta\mathcal{E}(T,(h,0))$ generates the balance of configurational action free of purely dissipative driving forces (the setting considered here excludes them, in fact).

9. Examples

Example 1 (smooth diffeomorphisms). *Let $u : \mathcal{B} \longrightarrow \tilde{\mathbb{R}}^3$ be a smooth diffeomorphism. Take $T = G_u$, with G_u the integration current over u. Then, by taking into account (7.1),*

$$\mathcal{E}(T) = \int_{\mathcal{B}} w(\mathsf{M}(Du(x)))\ dx = \int_{\mathcal{B}} \tilde{w}(Du(x))\ dx$$

once we write $\tilde{w}(F) := w(F, \mathrm{cof}F, \det F)$. As a consequence,

$$\delta\mathcal{E}(T,(0,k)) = \int_{\mathcal{B}} \tilde{w}_{,F^i_A}(Du(x))\, D_A u^j(x)\, D_j k^i(u(x))\ dx,$$

with D the derivative with respect to y. Capital indices refer to coordinates in $\mathcal{B}$ while the lower-case ones are related with coordinates in $u(\mathcal{B})$. The equality $\delta\mathcal{E}(T,(0,k)) = 0$, for any $k \in C^1_c(u(\mathcal{B}), \tilde{\mathbb{R}}^3)$ is a weak form of the standard balance of forces (bulk actions are not considered from the beginning of our analysis) written in terms of Kirchhoff stress, which is Cauchy stress when multiplied by $\det Du$. Moreover, we get

$$\delta\mathcal{E}(T,(h,0)) = \int_{\mathcal{B}} (\tilde{w}(Du(x))\,\delta_{AB} - D_A u^i(x))\tilde{w}_{,F^i_B}(Du(x))\, D_A h^B(x)\ dx.$$

The equality $\delta\mathcal{E}(T,(h,0)) = 0$ for any $h \in C^1_c(\mathcal{B}, \mathbb{R}^3)$ is the balance of configurational actions in absence of body forces in conservative setting.[7]

The subsequent examples show that balance equations may involve stresses in the ‘plasticized’ region.

Example 2 (blow up of a point in a ball). *We imagine here that the reference place $\mathcal{B}$ coincides with the ball B_2 of radius 2. We consider a smooth diffeomorphism $u : \mathcal{B} \longrightarrow \tilde{\mathbb{R}}^3$, and we indicate by $\mathcal{B}_a$ the current shape $u(\mathcal{B})$, as usual. As*

[7] See [11], pp. 264-268, for further analytical details.

parametrization set for the graph we take also the three-dimensional ball B_2 *and introduce a map* $\psi : B_2 \longrightarrow \mathcal{B} \times \tilde{\mathbb{R}}^3$ *defined by*

$$\psi\left(s\right) := \begin{cases} \left(0, u\left(2s\right)\right) & if\ \ |s| < \frac{1}{2}, \\ \left(\frac{2|s|-1}{|s|}s, u(\frac{s}{|s|})\right) & if\ \frac{1}{2} \le |s| < 1, \\ \left(s, u\left(s\right)\right) & if\ 1 \le |s| < 2. \end{cases}$$

By construction, ψ *is injective and Lipschitz. By indicating by* $[[B_2]]$ *the integration over* B_2*, we find that* $T := \psi_{\#}\left[[B_2]\right]$ *is a* $3-$*dimensional integer rectifiable current on* $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$*, with multiplicity* 1*, which is, in fact, a weak diffeomorphism which coincides with the graph ofu at* $\partial\mathcal{B} \times \partial\mathcal{B}_a$*. For this reason, we write*

$$\psi_{\#}\left[[B_2]\right] = \tau\left(\mathcal{T}, 1, \overrightarrow{T}\right).$$

Moreover, $\psi_{\#}\left[\left[B_2 \backslash B_{1\backslash 2}\right]\right]$*, with* $B_{1\backslash 2}$ *the ball of radius* $\frac{1}{2}$ *is the integration over the graph of the map* $\mathfrak{u} : \mathcal{B}\backslash\left\{0\right\} \longrightarrow \tilde{\mathbb{R}}^3$ *defined by*

$$\mathfrak{u}\left(x\right) := \begin{cases} u(\frac{x}{|x|}) & if\ \ |x| < 1, \\ u(x) & if\ 1 \le |x| < 2, \end{cases}$$

while $\psi_{\#}\left[\left[B_{1\backslash 2}\right]\right]$ *is the integration over the* $3-$*dimensional set* $\{0\} \times u\left(B_1\right)$*, a subset of* $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ *with tangent vector* ξ *at* $z = (0, y)$ *given by*

$$\xi\left(z\right) = \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3,$$

for all $y \in \mathcal{B}_a$*. Moreover, by setting* $R = D\left(0, k\left(y\right)\right)$ *we get*

$$R\tilde{e}_j = \sum_{i=1}^{3} D_j k^i\left(y\right)\tilde{e}_i,$$

while, if $R = D\left(h\left(x\right), 0\right)$*, we find* $R\tilde{e}_i = 0$*, so that*

$$\begin{aligned} L\left(D\left(0, k\left(y\right)\right)\right)\xi &= R\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 + \tilde{e}_1 \wedge R\tilde{e}_2 \wedge \tilde{e}_3 + +\tilde{e}_1 \wedge \tilde{e}_2 \wedge R\tilde{e}_3 \\ &= \operatorname{div} k\left(y\right)\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3, \end{aligned}$$

and

$$L\left(D\left(h\left(x\right), 0\right)\right)\xi = 0.$$

As a consequence, we find

$$\mathcal{E}\left(T\right) = \int_{B_2} w\left(D\mathfrak{u}\left(x\right)\right)\ dx + \frac{4}{3}\pi W\left(\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3\right)$$

where $\frac{4}{3}\pi$ *is the volume of the ball with unit radius, and*

$$\delta\mathcal{E}\left(T, (0, k)\right) = \delta\mathcal{E}\left(G_{\mathfrak{u}}, (0, k)\right) + W_{,\tilde{e}_1\wedge\tilde{e}_2\wedge\tilde{e}_3}\left(\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3\right)\int_{\mathcal{B}_a} \operatorname{div} k\left(y\right)\ dy,$$

and

$$\delta\mathcal{E}\left(T, (h, 0)\right) = \delta\mathcal{E}\left(G_{\mathfrak{u}}, (h, 0)\right),$$

for every $k \in C_c^1(\mathcal{B}_a, \tilde{\mathbb{R}}^3)$ *and every* $h \in C_c^1(\mathcal{B}, \mathbb{R}^3)$*.*

The analysis of the blow-up of a point in a disc in a two-dimensional space and the one of a line in a cylinder wrapped around it in three-dimensional setting require the same analyses developed in the lines above.

Example 3 (**two-dimensional slip**). *In 2D ambient space, for* $B_1 :=]-1,1[\subset \mathbb{R}$, $B_2 :=]-2,2[\subset \mathbb{R}$, *take as a reference place* $\mathcal{B}$ *the rectangle* $B_1 \times B_2$. *A point in* $B_1 \times B_2$ *is then* $x := (x_1, x_2)$. *To parametrize the graph, define the map*

$$\psi : B_1 \times B_2 \longrightarrow (B_1 \times B_2) \times \tilde{\mathbb{R}}^2$$

by

$$\psi(s_1, s_2) := \begin{cases} (s_1, s_2+1; s_1, s_2+1) & \text{if } -2 \le s_2 < -1, \\ \left(s_1, 0; s_1 + \frac{\beta(s_2+1)}{2}, 0\right) & \text{if } -1 \le s_2 < 1, \\ (s_1, s_2-1; s_1+\beta, s_2-1) & \text{if } 1 \le s_2 < 2, \end{cases}$$

with $\beta > 0$. *The map* ψ *is Lipschitz and injective onto its image. The integration over the image, namely* $T := \psi_{\#}[[B_1 \times B_2]]$ *is a* $2-$*dimensional integer rectifiable current on* $\mathbb{R}^2 \times \tilde{\mathbb{R}}^2$ *with multiplicity 1, which is, in fact, a weak diffeomorphism. Then, we write*

$$\psi_{\#}[[B_1 \times B_2]] = \tau(\mathcal{T}, 1, \overrightarrow{T}).$$

Moreover, $\psi_{\#}[[B_1 \times (B_2 \backslash B_1)]]$ *is the integration over the graph of the map* $\mathfrak{u} : B_1 \times (B_2 \backslash \{0\}) \longrightarrow \tilde{\mathbb{R}}^2$ *defined by*

$$\mathfrak{u}(x_1, x_2) := \begin{cases} (x_1+\beta, x_2) & \text{if } x_2 > 0, \\ (x_1, x_2) & \text{if } x_2 < 0. \end{cases}$$

The current shape $\mathcal{B}_a$ *is here once again* $\mathfrak{u}(\mathcal{B})$. *Specifically,* $\psi_{\#}[[B_1 \times B_2]]$ *is the integration over the* $2-$*dimensional parallelogram* $\mathcal{S}$ *in* $\mathbb{R}^4$ *with vertices* $(-1,0,-1,0)$, $(1,0,1,0)$, $(-1+\beta,0,-1+\beta,0)$, $(1+\beta,0,1+\beta,0)$. *The relevant tangent plane is generated by the vectors* $(1,0,1,0)$ *and* $(0,0,\beta/2,0)$, *so that*

$$\xi(s) = (e_1 + \tilde{e}_1) \wedge \frac{\beta}{2}\tilde{e}_1 = \frac{\beta}{2} e_1 \wedge \tilde{e}_1,$$

with e_1 *in the basis over* $B_1 \times B_2$ *and* $\tilde{e}_1$, $\tilde{e}_2$ *the vectors of the basis in* $\tilde{\mathbb{R}}^2$. *Moreover,*

$$D(0, k(y))\, e_1 = 0, \quad D(0, k(y))\, \tilde{e}_1 = \sum_{i=1}^{2} D_1 k^i(y)\, \tilde{e}_i,$$

$$D(h(x), 0)\, e_1 = \sum_{i=1}^{2} D_1 h^i(x)\, e_i, \quad D(h(x), 0)\, \tilde{e}_1 = 0.$$

Therefore, we find

$$L(D(0, k(y)))(e_1 \wedge \tilde{e}_1) = \sum_{i=1}^{2} D_1 k^i(y)\, e_1 \wedge \tilde{e}_i,$$

$$L(D(h(x), 0))(e_1 \wedge \tilde{e}_1) = \sum_{i=1}^{2} D_1 h^i(x)\, e_i \wedge \tilde{e}_1,$$

so that

$$\mathcal{E}(T) = \int_{B_1 \times B_2} w(D\mathfrak{u}(x_1, x_2))\ dx_1 dx_2 + W(e_1 \wedge \tilde{e}_1)\, \mathcal{H}^3(\mathcal{S}),$$

with

$$\delta\mathcal{E}(T, (0,k)) = \delta_{(0,k)}\mathcal{E}(G_{\mathfrak{u}}) + \sum_{i=1}^{2} W_{,e_1 \wedge \tilde{e}_i}(e_1 \wedge \tilde{e}_1) \int_{-1}^{1} dx_1 \int_{x_1}^{x_1+\beta} D_1 k^i(y_1, 0)\ dy_1,$$

and

$$\delta\mathcal{E}(T, (h,0)) = \delta_{(h,0)}\mathcal{E}(G_{\mathfrak{u}}) + \beta \sum_{i=1}^{2} W_{,e_i \wedge \tilde{e}_1}(e_1 \wedge \tilde{e}_1) \left(h^i(1,0) - h^i(-1,0)\right),$$

for every $k \in C_c^1(\mathcal{B}_a, \tilde{\mathbb{R}}^2)$ *and every* $h \in C_c^1(\mathcal{B}, \mathbb{R}^2)$*, where the vertical and horizontal variations of* $\mathcal{E}(G_{\mathfrak{u}})$ *are respectively indicated by* $\delta_{(0,k)}$ *and* $\delta_{(h,0)}$*.*

Example 4 (three-dimensional slip). *Once again take* $B_1 := \left]-1,1\right[\subset \mathbb{R}$ *and* $B_2 := \left]-2,2\right[\subset \mathbb{R}$*. We identify the reference place* $\mathcal{B}$ *with the product* $B_1 \times B_1 \times B_2$*. A point* x *in* $\mathcal{B}$ *will be then individualized by the triple* (x_1, x_2, x_3) *while the generic one, namely* y*, in the current place will be specified by the triple* (y_1, y_2, y_3)*. Define the map*

$$\psi : B_1 \times B_1 \times B_2 \longrightarrow (B_1 \times B_1 \times B_2) \times \tilde{\mathbb{R}}^3$$

by

$$\psi(s_1, s_2, s_3) := \begin{cases} (s_1, s_2, s_3 + 1; s_1, s_2, s_3 + 1) & if \ -2 \le s_3 < -1, \\ \left(s_1, s_2, 0; s_1 + \frac{\beta(s_3+1)}{2}, s_2, 0\right) & if \ -1 \le s_3 < 1, \\ (s_1, s_2, s_3 - 1; s_1 + \beta, s_2, s_3 - 1) & if \ 1 \le s_3 < 2, \end{cases}$$

with $\beta > 0$*. The map* ψ *is injective and Lipschitz onto its image. The integration over the image, namely* $T := \psi_{\#}[[B_1 \times B_1 \times B_2]]$ *is a* $3-$*dimensional integer rectifiable current on* $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ *with multiplicity 1, which is, once again, a weak diffeomorphism. Then, we write*

$$\psi_{\#}[[B_1 \times B_1 \times B_2]] = \tau(\mathcal{T}, 1, \overrightarrow{T}),$$

Moreover,

$$\psi_{\#}[[B_1 \times B_1 \times (B_2 \backslash B_1)]]$$

is the integration over the graph of the map $\mathfrak{u} : B_1 \times B_1 \times (B_2 \backslash \{0\}) \longrightarrow \tilde{\mathbb{R}}^3$ *defined by*

$$\mathfrak{u}(x_1, x_2, x_3) := \begin{cases} (x_1 + \beta, x_2, x_3) & if \ x_3 > 0, \\ (x_1, x_2, x_3) & if \ x_3 < 0. \end{cases}$$

The current shape $\mathcal{B}_a$ *is once again* $\mathfrak{u}(\mathcal{B})$*. Specifically,* $\psi_{\#}[[B_1 \times B_1 \times B_2]]$ *is the integration over the* $3-$*dimensional prism* $\mathcal{S}$ *in* $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ *with vertices*

$$(-1,-1,0,-1,-1,0), (-1,1,0,-1,1,0), (1,-1,0,1,-1,0),$$
$$(1,1,0,1,1,0), (-1,-1,0,-1+\beta,-1,0), (-1,1,0,-1+\beta,1,0),$$
$$(1,-1+\beta,0,1+\beta,-1,0), (1,1,0,1+\beta,1,0).$$

The relevant tangent plane is generated by the vectors

$$(1,0,0,1,0,0), (0,1,0,0,1,0), (0,0,0,\beta/2,0,0),$$

so that

$$\begin{aligned} \xi(s) &= (e_1 + \tilde{e}_1) \wedge (e_2 + \tilde{e}_2) \wedge \frac{\beta}{2}\tilde{e}_1 = \\ &= \frac{\beta}{2} e_1 \wedge e_2 \wedge \tilde{e}_1 - \frac{\beta}{2} e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_2, \end{aligned}$$

with e_1*,* e_2 *in the basis over* $B_1 \times B_1 \times B_2$ *and* $\tilde{e}_1$*,* $\tilde{e}_2$ *in the basis in* $\tilde{\mathbb{R}}^3$*. Moreover, for every* $j = 1, 2, 3$*, we calculate*

$$D(0, k(y)) e_j = 0, \quad D(0, k(y)) \tilde{e}_j = \sum_{i=1}^{3} D_j k^i(y) \tilde{e}_i,$$

$$D(h(x), 0) e_j = \sum_{i=1}^{3} D_j h^i(x) e_i, \quad D(h(x), 0) \tilde{e}_j = 0.$$

Therefore, we find

$$L\left(D\left(0,k\right)\right)\left(e_1\wedge e_2\wedge\tilde{e}_1\right)=\sum_{i=1}^{3}D_1k^i\left(y\right)e_1\wedge e_2\wedge\tilde{e}_i,$$

$$L\left(D\left(0,k\right)\right)\left(e_1\wedge\tilde{e}_1\wedge\tilde{e}_2\right)=\sum_{i<j}\left(D_1k^iD_2k^j-D_1k^jD_2k^i\right)\left(e_1\wedge\tilde{e}_i\wedge\tilde{e}_j\right),$$

$$L\left(D\left(h,0\right)\right)\left(e_1\wedge e_2\wedge\tilde{e}_1\right)=\sum_{i<j}\left(D_1h^iD_2h^j-D_1h^jD_2h^i\right)\left(e_i\wedge e_j\wedge\tilde{e}_1\right)$$

$$L\left(D\left(h,0\right)\right)\left(e_1\wedge\tilde{e}_1\wedge\tilde{e}_2\right)=\sum_{i=1}^{3}D_1h^i\left(x\right)e_j\wedge\tilde{e}_1\wedge\tilde{e}_2,$$

for every $k\in C_c^1(\mathcal{B}_a,\tilde{\mathbb{R}}^3)$ *and every* $h\in C_c^1(\mathcal{B},\mathbb{R}^3)$*, so that the energy* $\mathcal{E}\left(T\right)$ *and its variations can be explicitly evaluated. In Example 3 the* $3-$*vector* ξ *orienting* $\psi_{\#}\left[\left[B_1\times B_2\right]\right]$ *includes just line slips. In contrast, here,* ξ *orients* $\psi_{\#}\left[\left[B_1\times B_1\times B_2\right]\right]$ *and includes both line and surface slips, the latter ones associated with* $\left(e_1\wedge\tilde{e}_1\wedge\tilde{e}_2\right)$*.*

In all previous examples a smooth diffeomorphism can be superimposed to the ones considered. The description has analogies with what is described by the multiplicative decomposition of the deformation gradient in elasto-plasticity.

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