# The Bernstein Theorem in Higher Dimensions

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*Dedicated to the memory of Guido Stampacchia*

**Abstract.** – *In this work we have reconsidered the famous paper of Bombieri, De Giorgi and Giusti [4] and, thanks to the software Mathematica® we made it possible for anybody to control the difficult computations.*

## 1. – Introduction

On April 27th, 1978, Guido Stampacchia died in Paris. His coffin arrived to Pisa for the Adieu Ceremonial, in the Courtyard of Scuola Normale.

That day I was there, and I could not meet Ennio De Giorgi. The apparent absence of him seemed impossible to me, considering the lasting and close friendship between Ennio and Guido.

Many years later, Sergio Spagnolo told me that Ennio remained, the whole duration of the Ceremonial, hidden in a corner of the building. The modesty did not permit Ennio to show around his sorrow.

Guido had been a putative father for Ennio; he protected a younger and precious man, as long as he could. In March 1961, De Giorgi was invited to Paris for giving two talks. Stampacchia wanted to join him in the trip. There exists a beautiful picture of the two friends, on the Rive Gauche with Nôtre Dame behind them. A French mathematician told me that Stampacchia in Paris seemed to be the Assistant Professor of De Giorgi. He was wrong. Stampacchia had never been an academic servant. He could be said a devotee of Mathematics.

In 1955 Guido pushed Ennio into studying the XIX Hilbert's Problem [5]. And, in 1968 Guido pushed Enrico Bombieri to work with De Giorgi. The cooperation promoted by Stampacchia, produced two very important mathematical results.

This paper, written by Umberto Massari, Michele Miranda and myself, is offered to the "Unione Matematica Italiana", to be published in an issue of the Bollettino UMI dedicated to Guido Stampacchia. It wants to be considered a gift for the young mathematicians, and could be easier to read, than the original one of Enrico Bombieri, Ennio De Giorgi and Enrico Giusti [4].

We have used the software Mathematica® [19], following the PhD thesis of Danilo Benarros [2] (see also [3] and [16]), a Brazilian from Manhaus, who spent four years in Trento for his mathematical work.

## 2. – The regularity of minimal boundaries and extension of Bernstein's Theorem

In 1961, Ennio De Giorgi proved the regularity of minimal boundaries at points where the tangent hyperplane exists. The details of the proof were published in two booklets [6, 7] of the "Scuola Normale Superiore" Mathematical Seminar. These booklets were later inserted in [8] (see also [10]).

At the same time, Wendell H. Fleming proved the full regularity of minimal boundaries in $\mathbb{R}^3$. The paper appeared in "Rendiconti del Circolo Matematico di Palermo" [11] and contained a new proof of the classical Bernstein's Theorem [17].

De Giorgi's conjecture of full regularity of minimal boundaries and Fleming's conjecture of the Bernstein's Theorem, started sharing one destiny in all dimensions.

In August 1962, De Giorgi and Fleming met for the first time in Genova, guested by professor Jorés Cecconi. Right after, they met again in Stockholm, at the International Mathematical Union Congress. At the farewell, Fleming invited De Giorgi to spend a semester in the USA.

In February 1964, De Giorgi sailed to New York and remained in the States until June. Fleming met him at the sea–port, where De Giorgi offered him a mathematical gift: the improvement of Fleming's Theorem, which became

"If $\mathbb{R}^n$ has no singular minimal boundaries, then the Bernstein's Theorem is true for $n$–variable functions".

As a Corollary, De Giorgi got the first extension of Bernstein's Theorem to 3–variable functions [9].

In 1965, a student of Herbert Federer, Frederic J. Almgren Jr, came to Pisa to show his extension to 4–variables [1].

In 1967, a student of Shiin S. Chern, James H. Simons proved a strong result for analytic cones [18]. Assuming that the cone has zero first fundamental form, and the second fundamental form non negative, at all points except the vertex, Simons proved that the cone is a hyperplane, until dimension 7. The stop was justified by Simons with the cone

$$C = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 - |y|^2 = 0\}.$$

## 3. – Simons' cone is a singular minimal boundary and generalised solutions exist in 8–variables

This section has to be considered as an introduction of the results obtained by Enrico Bombieri, Ennio De Giorgi and Enrico Giusti in [4].

An easy proof for the minimal property of the Simons' cone was presented by Umberto Massari and Mario Miranda in [12].

Considering the fact that the cone can be written as

$$C = \left\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^4 - |y|^4 = 0\right\}$$

they studied the behavior of the function

$$\varphi(u, v) = \frac{u^4 - v^4}{4}, \qquad u = |x|, \quad v = |y|,$$

with respect to the minimal surface operator. Then they computed the form

$$\mathcal{E}(\varphi) = \left(\varphi_{uu} + \varphi_{vv} + 3\left(\frac{\varphi_u}{u} + \frac{\varphi_v}{v}\right)\right) + \\ + \left(\varphi_v^2\varphi_{uu} + \varphi_u^2\varphi_{vv} - 2\varphi_u\varphi_v\varphi_{uv} + 3(\varphi_u^2 + \varphi_v^2)\left(\frac{\varphi_u}{u} + \frac{\varphi_v}{v}\right)\right).$$

i.e.

$$\mathcal{E}(\varphi) = (1 + |\nabla\varphi|^2)\Delta\varphi + \langle H\varphi\nabla\varphi, \nabla\varphi\rangle.$$

They got

$$\mathcal{E}\left(\frac{u^4 - v^4}{4}\right) = 3(u^2 - v^2)(2 + (u^2 + v^2)(u^2 - v^2)^2),$$

and also

$$(u^2 - v^2)\mathcal{E}\left(\frac{u^4 - v^4}{4}\right) \geq 0, \quad \forall u, \forall v;$$

which means that $\frac{u^4-v^4}{4}$ is a sub-solution when it is positive and a super-solution when it is negative.

Therefore, for any $k > 0$, there holds

$$\mathcal{E}\left(k\frac{u^4 - v^4}{4}\right) = 3k(u^2 - v^2)(2 + k^2(u^2 + v^2)(u^2 - v^2)^2)$$

and

$$\lim_{k\to+\infty} \text{graph}\left(k\frac{u^4 - v^4}{4}\right) = C \times \mathbb{R},$$

where $C$ is the Simons' cone. Due to the differential properties of the function $k\frac{u^4-v^4}{4}$, one obtains that the cylinder $C \times \mathbb{R}$ is a minimal boundary in $\mathbb{R}^9$, and also $C$ is a minimal boundary in $\mathbb{R}^8$.

In the second part of this section, we will use the function $\frac{u^4-v^4}{4}$ to prove the existence of a generalised solution $S$, with

$$S(u, v) > 0, \quad \text{if } u > v; \quad S(u, v) < 0, \qquad \text{if } u < v.$$

Let us consider, for any $\varrho \geq 1$, the ball

$$B_\varrho = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 + |y|^2 \leq \varrho^2\},$$

and the solution $S_\varrho$ of the minimal surface equation in $B_\varrho$, with boundary values $\frac{u^4-v^4}{4}$ at $u^2 + v^2 = \varrho^2$ (see [13])

The $S_\varrho$ are increasing when positive and decreasing when negative, i.e. positive in $\{u > v\} \cap \{u^2 + v^2 < \varrho^2\}$ and negative in $\{u < v\} \cap \{u^2 + v^2 < \varrho^2\}$. Therefore, there exists a function $S$

$$S = \lim_{\varrho\to+\infty} S_\varrho$$

with values at all $(u, v) \in [-\infty, +\infty]$, and $S$ is a generalised minimal solution in $\mathbb{R}^8$ (see [15] and [14]).

## 4. – A classical non–trivial solution in 8–variables

We will follow, from now on, the Bombieri, De Giorgi and Giusti [4] indications and our existence of generalised minimal solutions.

Bombieri, De Giorgi and Giusti chose the function

$$\varphi(u,v) = \frac{u^2-v^2}{2}\sqrt{\frac{u^2+v^2}{2}}$$

instead of our $\frac{u^4-v^4}{4}$, with the advantage of a homogeneous function of degree 3 instead of 4.

Their second decision was to consider the function

$$\phi(u,v) = \frac{u^2-v^2}{2}\left(1+\sqrt{\frac{u^2+v^2}{2}}\left(1+A\left|\frac{u^2-v^2}{u^2+v^2}\right|^a\right)\right)$$

with $a = \frac{31}{96}$ and $A > 1$ to be fixed.

Both $\varphi(u,v)$ and $\phi(u,v)$ are continuous functions, and

$$\phi(u,v) \geq \varphi(u,v) \geq 0, \qquad \text{for } u \geq v;$$

and

$$\phi(u,v) \leq \varphi(u,v) \leq 0, \qquad \text{for } u \leq v.$$

At this point, Bombieri, De Giorgi and Giusti had to face the comparison of $\mathcal{E}(\phi)$ with 0 in the area $\{(u,v) : u > v\}$.

They had to work hard to get something interesting, but not the full success.

We took advantage of the work made by Danilo Benarros [2] and the Mathematica® book [19], to get the following inequality:

$$\mathcal{E}(\phi) \leq 0, \qquad \text{for } \frac{u^2-v^2}{2} \geq \frac{96}{\sqrt{31}} := \gamma.$$

We only need to study the case $u > v$; let us define

$$z = \frac{u^2-v^2}{2}, \quad r = \frac{u^2+v^2}{2}, \quad s = \sqrt{r}, \quad t = \frac{z}{r}$$

so that, since $0 < t < 1$, we can write

$$\phi(u,v) = \Gamma(z,r) = z\left(1+\sqrt{r}\left(1+A\left(\frac{z}{r}\right)^a\right)\right) = z(1+s(1+At^a)).$$

We obtain in this way that

$$\begin{aligned}\mathcal{E}(\phi) &= (1+\phi_u^2+\phi_v^2)\left(\phi_{uu}+\phi_{vv}+\frac{3\phi_u}{u}+\frac{3\phi_v}{v}\right) - (\phi_u^2\phi_{uu}+2\phi_u\phi_v\phi_{uv}+\phi_v^2\phi_{vv}) \\ &= \underbrace{\phi_{uu}+\phi_{vv}+\frac{3\phi_u}{u}+\frac{3\phi_v}{v}}_{=\mathcal{D}\phi} + \\ &\quad + \underbrace{\phi_v^2\phi_{uu}+\phi_u^2\phi_{vv}-2\phi_u\phi_v\phi_{uv}+3(\phi_u^2+\phi_v^2)\left(\frac{\phi_u}{u}+\frac{\phi_v}{v}\right)}_{=\mathcal{E}_0\phi}.\end{aligned}$$

We also have that

$$\phi_u = (\Gamma_r + \Gamma_z)u, \quad \phi_v = (\Gamma_r - \Gamma_z)v, \quad \phi_{uv} = (\Gamma_{rr} - \Gamma_{zz})uv,$$

$$\phi_{uu} = \Gamma_r + \Gamma_z + (\Gamma_{rr} + \Gamma_{zz} + 2\Gamma_{rz})u^2, \quad \phi_{vv} = \Gamma_r - \Gamma_z + (\Gamma_{rr} + \Gamma_{zz} - 2\Gamma_{rz})v^2.$$

From these formulae, taking into account that $0 < t < 1$, we get

$$\phi_u = s\sqrt{1+t}(g+f), \quad \phi_v = s\sqrt{1-t}(g-f), \quad \phi_{uv} = \sqrt{1-t^2}(G-F),$$

$$\phi_{uu} = g + f + (1+t)(G+F+2H), \quad \phi_{vv} = g - f + (1-t)(G+F-2H),$$

where we have defined

$$f = \Gamma_z, \quad g = \Gamma_r, \quad F = r\Gamma_{zz}, \quad G = r\Gamma_{rr}, \quad H = r\Gamma_{rz}.$$

In conclusion, we obtain the following formulae

$$\mathcal{D}\phi = 2(F + G + 2tH + 4g) \tag{1}$$

$$\begin{aligned} \mathcal{E}_0\phi &= 2r\Big((g^2 - f^2)(g + tf) + 6g(f^2 + g^2 + 2tfg) \\ &\quad + 2(1-t^2)(f^2G + g^2F - 2fgH)\Big). \end{aligned} \tag{2}$$

A direct computation gives

$$f = 1 + \sqrt{r}\left(1 + (1+a)A\left(\frac{z}{r}\right)^a\right) = 1 + s(1 + (1+a)At^a);$$

$$g = \frac{z}{2\sqrt{r}} + \sqrt{r}A\left(\frac{z}{r}\right)^{a+1}\left(\frac{1}{2} - a\right) = \frac{st}{2} + sAt^{a+1}\left(\frac{1}{2} - a\right);$$

$$F = a(a+1)A\sqrt{r}\left(\frac{z}{r}\right)^{-1+a} = a(a+1)Ast^{-1+a};$$

$$G = -\frac{z}{4\sqrt{r}} - A\left(\frac{1}{4} - a^2\right)\sqrt{r}\left(\frac{z}{r}\right)^{a+1} = -\frac{st}{4} - A\left(\frac{1}{4} - a^2\right)st^{a+1};$$

$$H = \sqrt{r} + A\left(\frac{1}{2} - a\right)(1+a)\sqrt{r}\left(\frac{z}{r}\right)^a = s + A\left(\frac{1}{2} - a\right)(1+a)st^a.$$

The next step is to substitute the previous values in formulae (1) and (2); this computation is rather long and complicated, but it can be performed with the aid of some computer's software, such as Mathematica® [19] (the command to get the result can be found in Appendix A). There holds:

$$\mathcal{D}\phi = \frac{11}{2}st + 2a(a+1)Ast^{-1+a} + A\left(\frac{11}{2} - 10a - 2a^2\right)st^{a+1}.$$

Since the three terms are positive for $A > 1$, we have that

$$\begin{aligned} \mathcal{D}\phi &\leq Ast^{-1+a}\left(\frac{11}{2} + 2a^2 + 2a + \frac{11}{2} - 10a - 2a^2\right) \\ &\leq 9Ast^{-1+a}. \end{aligned}$$

Regarding $\mathcal{E}_0$, we have that $\mathcal{E}_0\phi/2r$ is a polynomial of degree three in the $s$–variable

$$\frac{\mathcal{E}_0\phi}{2r} = -t + c_1 s + c_2 s^2 + c_3 s^3.$$

We estimate now the coefficient $c_1$, $c_2$ and $c_3$. For $c_1$, it can be written as $c_1 = c_{11}+c_{12}+c_{13}$ where

$$\begin{aligned} c_{11} &= -t + \frac{t^3}{2} < 0 \\ c_{12} &= At^{1+a}(-1-8a+2a^2) \\ c_{13} &= At^{3+a}(\frac{1}{2} - 2a^2) \geq 0. \end{aligned}$$

We also have that,

$$\begin{aligned} c_{12} + c_{13} &\leq At^{1+a}(-1-8a+2a^2) + At^{1+a}(\frac{1}{2} - 2a^2) \\ &= At^{1+a}\left(-\frac{1}{2} - 8a\right) \leq 0 \end{aligned}$$

and then $c_1 \leq 0$. For the coefficient $c_2$ we have $c_2 = c_{21} + c_{22} + c_{23} + c_{24} + c_{25}$ with

$$\begin{aligned} c_{21} &= 3aAt^{1+a}(-3+2a) < 0 \\ c_{22} &= 9aA^2t^{1+2a}(-1-a) < 0 \\ c_{23} &= \frac{21}{4}t^3 > 0 \\ c_{24} &= 3At^{3+a}\left(\frac{7}{2} - 5a - 2a^2\right) > 0 \\ c_{25} &= 3A^2t^{3+2a}\left(\frac{7}{4} - 5a + 3a^2\right) > 0. \end{aligned}$$

First of all, we have that

$$\begin{aligned} c_{21} + c_{23} &\leq 3aAt^{1+a}(-3+2a) + \frac{21}{4}t^{1+a} \\ &= 3t^{1+a}\left(-3aA + 2a^2A + \frac{7}{4}\right) \\ &\leq 3t^{1+a}\left(-\frac{19}{36}A + \frac{7}{4}\right) \leq 0 \end{aligned}$$

if $A \geq \frac{63}{19}$. We also have that

$$\begin{aligned} c_{22} + c_{25} &\leq -9aA^2t^{1+2a}(1+a) + 3A^2t^{1+2a}\left(\frac{7}{4} - 5a + 3a^2\right) \\ &= -3A^2t^{1+2a}\left(8a - \frac{7}{4}\right). \end{aligned}$$

Since

$$c_{24} \leq 3At^{1+2a}\left(\frac{7}{2} - 5a - 2a^2\right),$$

we then obtain that

$$\begin{aligned} c_{22} + c_{24} + c_{25} &\leq -3At^{1+2a}\left(A\left(8a - \frac{7}{4}\right) - \frac{7}{2} + 5a + 2a^2\right) \\ &\leq -3At^{1+2a}\left(\frac{A}{4} - \frac{17}{8}\right) < 0 \end{aligned}$$

if $A > \frac{17}{2}$. At the end we have that also $c_2 \leq 0$.

It remains to consider the last coefficient; we have that $c_3 = c_{31} + c_{32} + c_{33} + c_{34} + c_{35} + c_{36} + c_{37}$ where

$$\begin{aligned} c_{31} &= \frac{3}{2} a A t^{1+a}(-1+3a) < 0 \\ c_{32} &= -3a(1+a)A^2 t^{1+2a} < 0 \\ c_{33} &= -3aA^3 t^{1+3a}\left(\frac{1}{2} + \frac{5}{2}a + 2a^2\right) < 0 \\ c_{34} &= \frac{45}{8} t^3 > 0 \\ c_{35} &= \frac{3}{2} A t^{3+a}\left(\frac{45}{4} - 11a - 3a^2\right) > 0 \\ c_{36} &= 3A^2 t^{3+2a}\left(\frac{45}{8} - 11a + a^2\right) > 0 \\ c_{37} &= 3A^3 t^{3+3a}\left(\frac{15}{8} - \frac{11}{2}a + \frac{5}{2}a^2 + 2a^3\right) > 0. \end{aligned}$$

We have the following estimate

$$\begin{aligned} c_{32} + c_{34} + c_{35} &\leq 3At^{1+2a}\left(-a(1+a)A + \frac{15}{2} - \frac{11}{2}a - \frac{3}{2}a^2\right) \\ &\leq 3At^{1+2a}\left(-\frac{5}{16}A + \frac{193}{32}\right) < 0 \end{aligned}$$

if $A \geq \frac{193}{10}$. Moreover,

$$\begin{aligned} c_{33} + c_{37} &\leq -3aA^3 t^{1+3a}\left(\frac{1}{2} + \frac{5}{2}a + 2a^2\right) \\ &\quad +3A^3 t^{1+3a}\left(\frac{15}{8} - \frac{11}{2}a + \frac{5}{2}a^2 + 2a^3\right) \\ &= -3A^3 t^{1+3a}\left(6a - \frac{15}{8}\right) \leq 0. \end{aligned}$$

So our final choice of $A$ is $A > \frac{193}{10}$, and so $A = 20$ will do. With this choice of the parameters $a$ and $A$ we have obtained that

$$\mathcal{E}_0\phi \leq c_{31},$$

and then, for the minimal surface operator applied to $\phi$ there holds

$$\mathcal{E}\phi = \mathcal{D}\phi + \mathcal{E}_0\phi \leq sAt^{-1+a}\left(9 - (1-3a)3a(rt)^2\right).$$

We then have $\mathcal{E}\phi \leq 0$ only under the condition $rt \geq \gamma$.

The third decision of Bombieri, De Giorgi and Giusti was to introduce in their calculi the function

$$K(\sigma) = \int_0^\sigma \exp\left(B\int_\tau^{+\infty} \frac{1}{w^{1-a} + w^{1+a}} dw\right) d\tau, \qquad \sigma > 0,$$

and $B > 0$ to be fixed.

The first and second derivatives of $K$ are

$$K'(\sigma) = \exp\left(B\int_{\sigma}^{+\infty}\frac{1}{w^{1-a}+w^{1+a}}dw\right) > 1$$

$$K''(\sigma) = -\frac{B}{\sigma^{1-a}+\sigma^{1+a}}K'(\sigma).$$

Therefore

$$\mathcal{E}(K(\phi)) = K'(\phi)\left(\mathcal{D}\phi + K'(\phi)^2\mathcal{E}_0\phi - B\frac{\phi_u^2+\phi_v^2}{\phi^{1-a}+\phi^{1+a}}K'(\phi)\right).$$

So the inequality

$$\mathcal{E}K(\phi) \le 0, \qquad \text{for } \frac{u^2-v^2}{2} \ge \gamma$$

remains true.

And we only have to prove, for $\frac{u^2-v^2}{2} \le \gamma$,

$$\mathcal{D}(\phi) - B\frac{\phi_u^2}{\phi^{1-a}+\phi^{1+a}} \le 0.$$

Since

$$\phi_u \ge u\left(1+\sqrt{\frac{u^2+v^2}{2}}(1+At^a)\right) = uQ,$$

it is sufficient to prove

$$9Ast^{-1+a} \le \frac{BrQ^2}{(rt)^{1-a}Q^{1-a}+(rt)^{1+a}Q^{1+a}},$$

i.e.

$$9Ast^{-a}\left(\frac{1}{Q^{1+a}}+\frac{(rt)^{2a}}{Q^{1+a}}\right) \le B.$$

And, for $s \le 1$, thanks to $Q \ge 1$, we get

$$9Ast^{-a}\left(\frac{1}{Q^{1+a}}+\frac{(rt)^{2a}}{Q^{1+a}}\right) \le 9A(1+\gamma^{2a}).$$

For $s \ge 1$, thanks to $Q \ge s$, we get

$$9Ast^{-a}\left(\frac{1}{Q^{1+a}}+\frac{(rt)^{2a}}{Q^{1+a}}\right) \le 9A(1+\gamma^{2a}).$$

Finally, choosing $B \ge 9A(1+\gamma^{2a})$ we get the solution of our problem.

## A. – The Mathematica®'s commands

In this section we present the command (together with the output) that can be used with Mathematica (version 6) in order to obtain the result of the present paper.

```
In[1] := f[s_]:=1 + s * (1 + (1 + a) * A * t^a)
In[2] := g[s_]:=(s * t)/2 + s * A * t^(1 + a) * (1/2 - a)
In[3] := F[s_]:=a * (a + 1) * A * s * t^(-1 + a)
In[4] := G[s_]:= - (s * t)/4 - A * (1/4 - a^2) * s * t^(1 + a)
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$\boldsymbol{In[5] := H[s\_]:= s/2 + A*(1/2-a)*(1+a)*s*t\char`^a}$

$\boldsymbol{In[6] := D[s\_]:= 2*(F[s]+G[s]+2*t*H[s]+4*g[s])}$

$\boldsymbol{In[7] := Simplify[D[s]]}$

$Out[7] = -\frac{s\left(-11t^2-4a(1+a)At^a+\left(-11+20a+4a^2\right)At^{2+a}\right)}{2t}$

$\boldsymbol{In[8] := E0[s\_]:=}$

$\boldsymbol{2*r*((g[s]\char`^2-f[s]\char`^2)*(g[s]+t*f[s])}$
$\boldsymbol{+6*g[s]*(f[s]\char`^2+g[s]\char`^2+2*t*f[s]*g[s])}$
$\boldsymbol{+2*(1-t\char`^2)*(f[s]\char`^2*G[s]+g[s]\char`^2*F[s]-2*f[s]*g[s]*H[s]))}$

$\boldsymbol{In[9] := Simplify[E0[s]]}$

$$Out[9] = \tfrac{1}{4}rt\left(-8-4s\left(2-t^2-2\left(-1-8a+2a^2\right)At^a+\left(-1+4a^2\right)At^{2+a}\right)\right.$$
$$+6s^2\left(7t^2+4a(-3+2a)At^a-12a(1+a)A^2t^{2a}-2\left(-7+10a+4a^2\right)At^{2+a}\right.$$
$$\left.+\left(7-20a+12a^2\right)A^2t^{2+2a}\right)$$
$$+3s^3\left(15t^2+4a(-1+3a)At^a-8a(1+a)A^2t^{2a}\right.$$
$$-4a\left(1+5a+4a^2\right)A^3t^{3a}-\left(-45+44a+12a^2\right)At^{2+a}$$
$$\left.\left.+\left(45-88a+8a^2\right)A^2t^{2+2a}+\left(15-44a+20a^2+16a^3\right)A^3t^{2+3a}\right)\right)$$

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