

METRIC CYCLES, CURVES AND SOLENOIDS

VLADIMIR GEORGIEV AND EUGENE STEPANOV

ABSTRACT. We prove that every one-dimensional real Ambrosio-Kirchheim current with zero boundary (i.e. a cycle) in a lot of reasonable spaces (including all finite-dimensional normed spaces) can be represented by a Lipschitz curve parameterized over the real line through a suitable limit of Cesàro means of this curve over a subsequence of symmetric bounded intervals (viewed as currents). It is further shown that in such spaces, if a cycle is indecomposable, i.e. does not contain “nontrivial” subcycles, then it can be represented again by a Lipschitz curve parameterized over the real line through a limit of Cesàro means of this curve over every sequence of symmetric bounded intervals, that is, in other words, such a cycle is a solenoid.

1. INTRODUCTION

The paper is aimed at studying the notion of so-called *solenoids* or solenoidal vector charges, i.e. one-dimensional currents (in the classical sense of Whitney in a Euclidean spaces or in the sense of metric currents of Ambrosio-Kirchheim in a generic metric space) with zero boundary given by a limit of Cesàro means of some Lipschitz curve parameterized over the real line (viewed as currents as well). This notion is well related to the subject of optimal transportation, though in a quite indirect way. Namely, one of the topics close to the optimal transportation is weak KAM theory and the existence of Mather’s minimal measures. The strong analogy between Mather’s theory of minimal measures in Lagrangian dynamics and the optimal mass transportation theory has been shown by L.C. Evans and further explored by many researchers. In the study of Mather’s minimal measures recently the result on decomposition of one-dimensional cycles (currents without boundary) into “elementary” cycles represented by Lipschitz curves has proved to be very helpful [2, 4]. The “elementary” cycles used in such a decomposition are exactly solenoids. For normal currents in a Euclidean space such a result has been first proven in [14]. In [11] an analogous decomposition result has been proven for Ambrosio-Kirchheim metric currents in arbitrary metric spaces. Namely, it has been shown that if T is a metric current in a complete metric space E with, say, a compact support (we just mention this case for simplicity), and has $\partial T = 0$ (i.e. is a cycle), then there is a finite positive measure η over the space $\text{Lip}_1(\mathbb{R}; E)$ of 1-Lipschitz curves in E parameterized over the real line such that for η -a.e. θ there is a limit

$$S_\theta = \lim_{t \rightarrow +\infty} \frac{1}{2t} \llbracket \theta \llbracket [-t, t] \rrbracket \rrbracket$$

The work of the first author was supported in part by the project 2008BLM8BB “Analisi nello spazio delle fasi per E.D.P.” of the Italian Ministry of Research. The work of the second author was financed by GNAMPA, by RFBR grant #11-01-00825, by the project 2008K7Z249 “Trasporto ottimo di massa, disuguaglianze geometriche e funzionali e applicazioni” of the Italian Ministry of Research.

in the weak sense of metric currents, where $[\![\sigma]\!]$ stands for the current associated with the curve σ , while

$$T = \int_{\text{Lip}_1(\mathbb{R}; E)} S_\theta d\eta(\theta),$$

$$\mathbb{M}(T) = \int_{\text{Lip}_1(\mathbb{R}; E)} \mathbb{M}(S_\theta) d\eta(\theta) = \eta(\text{Lip}_1(\mathbb{R}; E)),$$

$\mathbb{M}(T)$ standing for the mass of T , so that in particular, S_θ is a metric current of unit mass for η -a.e. $\theta \in \text{Lip}_1(\mathbb{R}; E)$, and, further, we may assume $\theta(\mathbb{R}) \subset \text{supp } S_\theta \subset \text{supp } T$ for η -a.e. θ . The currents S_θ associated to curves θ are usually called, following [14], elementary solenoids. A closely related object is that of an asymptotic cycle introduced by S. Schwartzman in [12] and further studied in [13]. The latter appears quite natural as well in the study of the problem of representation of homology classes of manifolds (see [7, 9, 8, 6]). Last but not least one has to mention the applications of a decomposition of every cycle in solenoids in the problems of approximation of a given vector field by some vector field with better properties (i.e. harmonic approximation), see [5] and references therein.

What has been mentioned above well justifies the interest in a more profound study of solenoids, i.e. cycles T satisfying

$$T = \lim_{t \rightarrow +\infty} \frac{1}{2t} [\![\theta|_{[-t, t]}\!]$$

in the weak sense of currents for some curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$. Even a weaker notion seems to be interesting, namely, we say that a current T is represented by some curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$, if the above limit exists just for some increasing subsequence of t (it is worth making the following comparison: if the notion of a solenoid is directly related to that of a Schwartzman asymptotic cycle, then the set of cycles represented by the same curve corresponds in a similar way to the notion a Schwartzman balanced cluster [7]). In this paper we show that in a lot of reasonable spaces including all finite-dimensional normed ones every cycle (i.e. a real one-dimensional metric current without boundary), up to normalization by its mass, is represented by some curve. We further show that if a cycle T represented by a curve is indecomposable, i.e. intuitively, does not contain subcycles different from λT , $\lambda \in [0, 1]$, then it is a solenoid. The results we provide seem in fact to be interesting already in classical settings (E being a Euclidean space or a smooth connected Riemannian manifold, the cycles being intended in the sense of classical Whitney currents, i.e. elements of some space dual to a space of smooth differential forms), but the use of the theory of Ambrosio-Kirchheim currents gives us the possibility to obtain some generalization to more general, and not even finite-dimensional, metric spaces. However, we are not interested in providing the most general results here and limit ourselves mainly to illustrating the described phenomena. Thus when possible we provide the results for the settings where they are easier to obtain (say, in finite-dimensional normed spaces rather than in general metric spaces) and provide only those generalizations to metric space settings which are almost “free of charge” by the use of the theory of Ambrosio-Kirchheim currents. Such generalizations, when they seem to be too technical, are located in the appendix.

2. NOTATION AND PRELIMINARIES

The metric spaces are always in the sequel assumed to be complete. A metric space will be called connected by rectifiable arcs, if every couple of points in this space can be connected by a Lipschitz curve of finite length. The parametric length of a Lipschitz curve $\theta: [a, b] \rightarrow E$ will be denoted by $\ell(\theta)$. If d is the distance of the

metric space E , we let

$$d_*(x, y) := \inf\{\ell(\theta) : \theta: [a, b] \rightarrow E \text{ Lipschitz curve, } \theta(a) = x, \theta(b) = y\}.$$

For a compact set $K \subset E$, we let $\text{diam}_* K$ to stand for the diameter of K with respect to the distance d_* . It is worth recalling that $d_* \geq d$. The metric space is called *quasiconvex*, if $d_*(x, y) \leq Cd(x, y)$ for some $C > 0$ and for all $\{x, y\} \subset E$.

For a set $D \subset E$ of a metric space E with distance d we employ the usual notation $D^c := E \setminus D$, $\text{dist}(x, D) := \inf\{d(x, y) : y \in D\}$ whenever $x \in E$, and $(D)_\varepsilon := \{x \in E : \text{dist}(x, D) < \varepsilon\}$.

The notation $\text{Lip}(X, Y)$ (resp. $\text{Lip}_k(X, Y)$ and $\text{Lip}_b(X, Y)$) for metric spaces X and Y stands for the set of all Lipschitz maps (resp. all Lipschitz maps with Lipschitz constant k , the set of bounded Lipschitz maps) $f: X \rightarrow Y$. We omit the reference to Y in case $Y = \mathbb{R}$ and write just $\text{Lip}(X)$, $\text{Lip}_k(X)$, $\text{Lip}_b(X)$ respectively.

For metric currents we use the same notation as in [10] (i.e. almost identical to that of [1], with the only exception of the notation for the mass measure). In particular, $D^k(E) = \text{Lip}_b(E) \times (\text{Lip}(E))^k$ stands for the space of metric k -forms, its elements (i.e. k -forms) being denoted by $f d\pi$, where $f \in \text{Lip}_b(E)$, $\pi \in (\text{Lip}(E))^k$, $\mathcal{M}_k(E)$ stands for the space of k -dimensional metric currents, $\mathbb{M}(T)$ stands for the mass of a current T , and μ_T stands for the mass measure associated to this current. The one-dimensional current associated to a Lipschitz curve $\theta: [a, b] \rightarrow E$ will be denoted by $\llbracket \theta \rrbracket$, namely,

$$\llbracket \theta \rrbracket(f d\pi) := \int_a^b f(\theta(t)) d\pi(\theta(t))$$

for every $f d\pi \in D^1(E)$. Recall that $\mathbb{M}(\llbracket \theta \rrbracket) \leq \ell(\theta)$. The weak topology in $\mathcal{M}_k(E)$ is defined by the family of seminorms $\{T \mapsto |T(\omega)| : \omega \in D^k(E)\}$. We write $S \leq T$ for currents $\{S, T\} \subset \mathcal{M}_k(E)$, when $\mathbb{M}(S) + \mathbb{M}(T - S) = \mathbb{M}(T)$. $T \in \mathcal{M}_k(E)$ is called a *cycle*, if $\partial T = 0$, and if T and S are cycles and $T \leq S$, we say that T is a *subcycle* of S .

The current $T \in \mathcal{M}_k(E)$ will be called *tight* if so is the measure μ_T , and, similarly, the sequence $\{T_j\} \subset \mathcal{M}_k(E)$ will be called *tight* if so is the sequence of measures $\{\mu_{T_j}\}$. All the measures considered in the sequel, unless explicitly stated otherwise, are assumed to be finite positive Borel measures over a metric space E where they are defined. Recall that it is consistent with Zermelo-Fraenkel set theory to assume that all such measures are tight (and hence, in particular, all the metric currents are tight); and it is always true when E is a Polish (i.e. complete separable metric) space. The narrow topology on measures is defined by duality with the space $C_b(E)$ of continuous bounded functions.

3. CYCLES AND CURVES

Before introducing the concept of a solenoid we study the following weaker notion.

Definition 3.1. *We say that a cycle $T \in \mathcal{M}_1(E)$ is represented by a curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$, if there is a sequence $s_k \nearrow +\infty$ with*

$$T = \lim_{k \rightarrow +\infty} \frac{1}{2s_k} \llbracket \theta \llcorner [-s_k, s_k] \rrbracket.$$

in the weak sense of currents.

Note that this definition makes sense since

$$\partial T = \lim_{k \rightarrow +\infty} \frac{1}{2s_k} \partial \llbracket \theta \llcorner [-s_k, s_k] \rrbracket = \lim_{k \rightarrow +\infty} \frac{1}{2s_k} (-\delta_{-s_k} + \delta_{s_k}) = 0.$$

We also remark that the same curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$ may represent different cycles (corresponding to different diverging sequences $\{s_k\}$). Finally, it is clear that for a cycle $T \in \mathcal{M}_1(E)$ represented by some curve one has $\mathbb{M}(T) \leq 1$. We will show that in

a lot of reasonable metric spaces (including, of course, all finite-dimensional normed spaces) every cycle with bounded support, normalized by its mass, is represented by some curve. In fact, the main idea, as well as the main difficulty, is in the proof of this result for finite-dimensional normed spaces which we provide in this section. Since this section is mainly propaedeutic, we are not interested in obtaining the most general representation result, and will be satisfied with the finite-dimensional one. Some easy generalizations to infinite-dimensional spaces are given in section A.

Theorem 3.2. *For every cycle $T \in \mathcal{M}_1(E)$ with bounded support in a finite-dimensional normed space E one has that $T/\mathbb{M}(T)$ is represented by some curve.*

Proof. Combine Corollary 3.8, Lemma 3.9 and Lemma 3.6 below. \square

The rest of the section contains the technical assertions used in the proof of Theorem 3.2. The complete metric space E here and in the sequel is supposed to be generic (i.e. not necessarily finite-dimensional normed) unless otherwise explicitly stated. We will first make the following useful observations.

Remark 3.3. If $T \in \mathcal{M}_1(E)$, $\mathbb{M}(T) = 1$, is represented by some curve, then since

$$\mathbb{M}(T) \leq \liminf_{k \rightarrow +\infty} \frac{1}{2s_k} \mathbb{M}(\llbracket \theta_{\perp}[-s_k, s_k] \rrbracket) \leq \limsup_{k \rightarrow +\infty} \frac{1}{2s_k} \mathbb{M}(\llbracket \theta_{\perp}[-s_k, s_k] \rrbracket) \leq 1 = \mathbb{M}(T),$$

we have that

$$\mu_T = \lim_{k \rightarrow +\infty} \frac{1}{2s_k} \mu_{\llbracket \theta_{\perp}[-s_k, s_k] \rrbracket}$$

in the narrow sense of measures, so that in particular the sequence of measures $\{\frac{1}{2s_k} \mu_{\llbracket \theta_{\perp}[-s_k, s_k] \rrbracket}\}$ is uniformly tight.

Remark 3.4. If $T \in \mathcal{M}_1(E)$, $\mathbb{M}(T) = 1$, is represented by some curve, and $B \subset E$ is a Borel set, then denoting $T_j := \frac{1}{2s_j} \llbracket \theta_{\perp}[-s_j, s_j] \rrbracket$, one has for each subsequence of j such that $\{\mu_{T_j}(B)\}$ is convergent, the relationship

$$(3.1) \quad \lim_j \mu_{T_j}(B) = \lim_j \frac{1}{2s_j} \mathcal{L}^1(\theta^{-1}(B) \cap [-s_j, s_j]).$$

In fact,

$$\begin{aligned} \mu_{T_j}(B) &\leq \frac{1}{2s_j} \mathcal{L}^1(\theta^{-1}(B) \cap [-s_j, s_j]), \\ \mu_{T_j}(B^c) &= \mathbb{M}(T_j) - \mu_{T_j}(B) \leq \frac{1}{2s_j} \mathcal{L}^1(\theta^{-1}(B^c) \cap [-s_j, s_j]) \\ &= 1 - \frac{1}{2s_j} \mathcal{L}^1(\theta^{-1}(B) \cap [-s_j, s_j]), \end{aligned}$$

which means

$$\left| \mu_{T_j}(B) - \frac{1}{2s_j} \mathcal{L}^1(\theta^{-1}(B) \cap [-s_j, s_j]) \right| \leq 1 - \mathbb{M}(T_j),$$

and minding that $\lim_j \mathbb{M}(T_j) = 1$ we get (3.1).

Remark 3.5. If

$$T = \lim_{k \rightarrow +\infty} \frac{1}{2s_k} \llbracket \theta_{\perp}[-s_k, s_k] \rrbracket.$$

in the weak sense of currents for some $\theta \in \text{Lip}_1(\mathbb{R}; E)$ and some sequence $s_k \nearrow +\infty$, then

$$T = \lim_{k \rightarrow +\infty} \frac{1}{2t_k} \llbracket \theta_{\perp}[-t_k, t_k] \rrbracket$$

again in the weak sense of currents, whenever $t_k/s_k \rightarrow 1$. In fact, assuming without loss of generality that $t_k > s_k$, we get

$$\begin{aligned} \mathbb{M} \left(\frac{1}{2t_k} \llbracket \theta_{\perp}[-t_k, t_k] \rrbracket - \frac{1}{2s_k} \llbracket \theta_{\perp}[-s_k, s_k] \rrbracket \right) &\leq \mathbb{M} \left(\frac{1}{2t_k} \llbracket \theta_{\perp}[-t_k, -s_k] \cup [s_k, t_k] \rrbracket \right) \\ &\quad + \left| \frac{1}{2t_k} - \frac{1}{2s_k} \right| \mathbb{M} (\llbracket \theta_{\perp}[-s_k, s_k] \rrbracket) \\ &\leq \frac{1}{2t_k} 2|t_k - s_k| + \left| \frac{1}{2t_k} - \frac{1}{2s_k} \right| 2s_k \rightarrow 0 \end{aligned}$$

as $k \rightarrow +\infty$, which shows the claim.

The following statement holds true.

Lemma 3.6. *Let E be a quasiconvex metric space, and $\{T_j\} \subset \mathcal{M}_1(E)$ be a tight sequence of cycles represented by some curves, $\lim_j T_j = T$ in the weak sense of currents. Then T is represented by some curve.*

Proof. Since

$$T_j = \lim_{k \rightarrow +\infty} \frac{1}{2s_k^j} \llbracket \theta_j \llcorner [-s_k^j, s_k^j] \rrbracket$$

for some $\theta_j: \mathbb{R} \rightarrow E$ with $\text{Lip } \theta_j \leq 1$, we define the curve $\theta: \mathbb{R} \rightarrow E$ inductively as follows.

Let $K \subset E$ be such a compact set that $\mu_{T_j}(K^c) \leq 1/4$. Note that for every $\varepsilon > 0$ and for each $j \in \mathbb{N}$ the set $\theta_j^{-1}((K)_\varepsilon) \subset \mathbb{R}$ is unbounded both from above and from below. In fact, suppose the contrary, i.e. that the latter set is bounded, say, from above, i.e. $\theta_j(t) \notin (K)_\varepsilon$ for all $t > \bar{t}_\varepsilon^j$. Then

$$\begin{aligned} \mu_{T_j}(K) &\leq \mu_{T_j}((K)_\varepsilon) \leq \liminf_{k \rightarrow +\infty} \frac{1}{2s_k^j} \mu_{\llbracket \theta_j \llcorner [-s_k^j, s_k^j] \rrbracket}((K)_\varepsilon) \\ &\leq \liminf_{k \rightarrow +\infty} \frac{1}{2s_k^j} \mathcal{L}^1(\{t \in [-s_k^j, s_k^j] : \theta_j(t) \in (K)_\varepsilon\}) \\ &\leq \liminf_{k \rightarrow +\infty} \frac{1}{2s_k^j} \mathcal{L}^1(\{t \in [-s_k^j, \bar{t}_\varepsilon^j] : \theta_j(t) \in (K)_\varepsilon\}) \\ &\leq \lim_{k \rightarrow +\infty} \frac{|\bar{t}_\varepsilon^j + s_k^j|}{2s_k^j} = \frac{1}{2}, \end{aligned}$$

which contradicts the choice of K (the latter provides $\mu_{T_j}(K) > 3/4$).

Let $K_\nu^j \subset E$ be such a compact set that

$$\frac{1}{2s_k^j} \mu_{\llbracket \theta_j \llcorner [-s_k^j, s_k^j] \rrbracket}((K_\nu^j)^c) + \frac{1}{2s_k^j} \mu_{\llbracket \theta(\theta_j \llcorner [-s_k^j, s_k^j]) \rrbracket}((K_\nu^j)^c) \leq 1/\nu,$$

and $K_\nu \subset E$ be such a compact set that $\mu_{T_j}(K_\nu^c) \leq 1/\nu$, and define

$$X := \bigcup_{j,\nu} K_\nu^j \cup \bigcup_{\nu} K_\nu.$$

Let d_w stand for the distance over $\{T_j\}$ provided by lemma A.1 from [11], and denote by $\|\cdot\|_0$ the Kantorovich-Rubinstein norm metrizing the narrow topology on positive finite Borel measures over \bar{X} (see [3][theorem 8.3.2]).

Set $\varepsilon := 1$. Let $r_1 := s_1^1$ and θ coincide with θ_1 over $[-r_1, r_1]$. For each $j \in \mathbb{N}$ we choose

$$\alpha_j^- > \text{diam}_*((K)_\varepsilon \cup \{\theta_j(-r_j)\})$$

such that $\theta_j(-r_j - \alpha_j^-) \in (K)_\varepsilon$ and an

$$\alpha_j^+ > \text{diam}_*((K)_\varepsilon \cup \{\theta_j(r_j)\})$$

such that $\theta_j(r_j + \alpha_j^+) \in (K)_\varepsilon$ (note that in the case (ii) we may simply take $\alpha_j^- = \alpha_j^+ > \text{diam}_* K$). Let now $r_{j+1} := s_k^{j+1}$ with k such that

$$s_k^{j+1} > 2^{2r_j + \alpha_j^- + \alpha_j^+} \quad \text{and} \quad d_w \left(T_{j+1}, \frac{1}{2s_k^{j+1}} \llbracket \theta_{j\llcorner}[-s_k^{j+1}, s_k^{j+1}] \rrbracket \right) \leq \frac{1}{j},$$

$$\left\| \mu_{T_{j+1}} - \frac{1}{2s_k^{j+1}} \mu_{\llbracket \theta_{j\llcorner}[-s_k^{j+1}, s_k^{j+1}] \rrbracket} \right\|_0 \leq \frac{1}{j},$$

and let θ

- (a) coincide with θ_{j+1} over $[-r_{j+1}, -r_j - \alpha_j^-] \cup [r_j + \alpha_j^+, r_{j+1}]$, while
- (b) $\theta_{j\llcorner}[-r_j - \alpha_j^-, -r_j]$ be an arbitrary curve with Lipschitz constant bounded by 1 connecting $\theta_{j+1}(-r_j - \alpha_j^-)$ with $\theta_j(-r_j)$ (such a curve exists since $d_*(\theta_{j+1}(-r_j - \alpha_j^-), \theta_j(-r_j)) < \alpha_j^-$ by the choice of α_j^-),
- (c) and, analogously, $\theta_{j\llcorner}[r_j, r_j + \alpha_j^+]$ be an arbitrary curve with Lipschitz constant bounded by 1 connecting $\theta_j(r_j)$ with $\theta_{j+1}(r_j + \alpha_j^+)$.

Clearly, with this construction

$$(3.2) \quad T = \lim_{j \rightarrow +\infty} T_j = \lim_{j \rightarrow +\infty} \frac{1}{2r_j} \llbracket \theta_{j\llcorner}[-r_j, r_j] \rrbracket$$

in distance d_w . But the sequence $\{\mu_{T_j}\}$ is precompact in the narrow topology of measures, hence in the norm $\|\cdot\|_0$, hence so is the sequence $\{\frac{1}{2r_j} \mu_{\llbracket \theta_{j\llcorner}[-r_j, r_j] \rrbracket}\}$, and therefore, the latter is uniformly tight by the Prokhorov theorem for nonnegative measures (theorem 8.6.4 from [3]). Thus, by lemma A.1 from [11], the convergence in (3.2) is also in the weak topology of currents. But

$$\begin{aligned} \mathbb{M} \left(\frac{1}{2r_{j+1}} (\llbracket \theta_{j\llcorner}[-r_{j+1}, r_{j+1}] \rrbracket - \llbracket \theta_{j\llcorner}[-r_{j+1}, r_{j+1}] \rrbracket) \right) \\ = \frac{1}{2r_{j+1}} \mathbb{M} \left((\llbracket \theta_{j\llcorner}[-r_j - \alpha_j^-, r_j + \alpha_j^+] \rrbracket - \llbracket \theta_{j\llcorner}[-r_j - \alpha_j^-, r_j + \alpha_j^+] \rrbracket) \right) \\ \leq 2\mathcal{L}^1([-r_j - \alpha_j^-, r_j + \alpha_j^+]) / 2r_{j+1} \\ = (2r_j + \alpha_j^- + \alpha_j^+) / r_{j+1} \rightarrow 0 \end{aligned}$$

as $j \rightarrow +\infty$, and hence

$$T = \lim_{j \rightarrow +\infty} \frac{1}{2r_j} \llbracket \theta_{j\llcorner}[-r_j, r_j] \rrbracket,$$

in the weak sense of currents, which shows the thesis. \square

The following auxiliary statements have been used in the proof of the above Theorem 3.2.

Lemma 3.7. *Let $T \in \mathcal{M}_1(E)$, where E is a metric space connected by rectifiable arcs, be of the form*

$$T = \sum_{i=1}^m \alpha_i \llbracket \theta_i \rrbracket,$$

where $\mathbb{M}(T) = \sum_{i=1}^m \alpha_i \ell(\theta_i) = 1$, $\alpha_i > 0$, $i = 1, \dots, m$, and each θ_i is a simple closed curve, so that, in particular, $\partial T = 0$. Then T is represented by some curve.

Proof. Let each θ_i , $i = 1, \dots, m$, be parameterized by arclength over $[0, \ell(\theta_k)]$. Choose an $x_0 \in E$ and let σ_i stand for the Lipschitz curves parameterized by arclength and connecting x_0 to $\theta_i(0)$, and $\tilde{\sigma}_i$ stand for the same curves covered in the opposite direction (i.e. $\tilde{\sigma}_i(t) := \sigma_i(\ell(\sigma_i) - t)$, $t \in [0, \ell(\sigma_i)]$), and, finally, set

$$d := \sum_{i=1}^m 2\ell(\sigma_k).$$

Define now inductively

$$t_0 := 0, \quad t_{k+1} := (2^k + 1)t_k + d.$$

Obviously $t_k \rightarrow +\infty$ (since $t_k \geq kd$) and $t_{k+1}/t_k = d/t_k + (2^k + 1) \rightarrow \infty$ as $k \rightarrow \infty$. Let θ_i^k stand for a Lipschitz curve defined as the composition of the curves σ_i with θ_i covered n_i^k times and then with $\tilde{\sigma}_i$, where $n_i^k := \lfloor \alpha_i 2^k t_k \rfloor$. Clearly,

$$\ell(\theta_i^k) = 2\ell(\sigma_i) + n_i^k \ell(\theta_i),$$

while $\llbracket \theta_i^k \rrbracket = n_i^k \llbracket \theta_i \rrbracket$. We then define θ over each interval $[t_k, t_{k+1}]$ as the composition of the curves θ_i^k , $i = 1, \dots, m$, and parameterized with velocity not greater than one, which is possible because

$$\begin{aligned} \ell(\theta_{\perp}[t_k, t_{k+1}]) &= \sum_{i=1}^m \ell(\theta_i^k) = \sum_{i=1}^m (2\ell(\sigma_i) + n_i^k \ell(\theta_i)) \\ &= d + \sum_{i=1}^m n_i^k \ell(\theta_i) \leq d + 2^k t_k \sum_{i=1}^m \alpha_i \ell(\theta_i) \\ &= d + 2^k t_k = t_{k+1} - t_k. \end{aligned}$$

Now, bearing in mind that

$$\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket = \sum_{i=1}^m n_i^k \llbracket \theta_i \rrbracket,$$

we have

$$\begin{aligned} \mathbb{M}\left(T - \frac{\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket}{2^k t_k}\right) &= \mathbb{M}\left(\sum_{i=1}^m \left(\alpha_i - \frac{n_i^k}{2^k t_k}\right) \llbracket \theta_i \rrbracket\right) \\ &= \frac{1}{2^k t_k} \mathbb{M}\left(\sum_{i=1}^m (\alpha_i 2^k t_k - \lfloor \alpha_i 2^k t_k \rfloor) \llbracket \theta_i \rrbracket\right) \\ &\leq \frac{1}{2^k t_k} \sum_{i=1}^m \ell(\theta_i) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. But then

$$\begin{aligned} \mathbb{M}\left(\frac{\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket}{t_{k+1}} - \frac{\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket}{2^k t_k}\right) &= \frac{t_{k+1} - 2^k t_k}{t_{k+1}} \mathbb{M}\left(\frac{\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket}{2^k t_k}\right) \\ &= \frac{t_k + d}{t_{k+1}} \mathbb{M}\left(\frac{\llbracket \theta_{\perp}[t_k, t_{k+1}] \rrbracket}{2^k t_k}\right) \\ &= \frac{t_k + d}{t_{k+1}} (\mathbb{M}(T) + o(1)) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Finally,

$$\mathbb{M}\left(\frac{\llbracket \theta_{\perp}[0, t_k] \rrbracket}{t_{k+1}}\right) \leq \frac{t_k}{t_{k+1}} \rightarrow 0,$$

so that we have

$$\mathbb{M}\left(T - \frac{\llbracket \theta_{\perp}[0, t_{k+1}] \rrbracket}{t_{k+1}}\right) \rightarrow 0$$

as $k \rightarrow \infty$. Extending the definition of θ to the whole \mathbb{R} defining $\theta(t)$ for $t < 0$ in a symmetric way, we will have

$$\mathbb{M}\left(T - \frac{\llbracket \theta_{\perp}[-t_{k+1}, t_{k+1}] \rrbracket}{2t_{k+1}}\right) \rightarrow 0,$$

which shows the statement. \square

Corollary 3.8. *For every polyhedral current $T \in \mathcal{M}_1(E)$ in a finite-dimensional normed space E having $\partial T = 0$ one has that $T/\mathbb{M}(T)$ is represented by some curve.*

Proof. Represent $T/\mathbb{M}(T)$ as in the statement of the Lemma 3.7 and apply the latter. \square

Lemma 3.9. *Let E be a finite-dimensional normed space. Then for every cycle $T \in \mathcal{M}_1(E)$ with bounded support there is a sequence of polyhedral currents $T_\nu \in \mathcal{M}_1(E)$ with $\partial T_\nu = 0$, uniformly bounded supports and such $T_\nu \rightarrow T$ weakly in the sense of currents, while $\mathbb{M}(T_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$.*

Proof. We denote by $\mathcal{F}(T)$ the flat norm of T defined by

$$\mathcal{F}(T) := \inf\{\mathbb{M}(A) + \mathbb{M}(B) : A \in \mathcal{M}_k(E), B \in \mathcal{M}_{k+1}(E), A + \partial B = T\}.$$

By lemma C.1 from [10] there are polyhedral currents $T'_\nu \in \mathcal{M}_1(E)$ with uniformly bounded supports and such that $\mathcal{F}(T'_\nu - T) \rightarrow 0$, while $\mathbb{M}(T'_\nu) \rightarrow \mathbb{M}(T)$ as $\nu \rightarrow \infty$. In particular,

$$T = T'_\nu + A_\nu + \partial B_\nu$$

with $A_\nu \in \mathcal{M}_1(E)$, $B_\nu \in \mathcal{M}_2(E)$ and $\mathbb{M}(A_\nu) \rightarrow 0$, $\mathbb{M}(B_\nu) \rightarrow 0$ as $\nu \rightarrow \infty$. Since

$$\partial A_\nu = -\partial T'_\nu$$

is polyhedral, by Lemma 3.10 below we may choose $A'_\nu \in \mathcal{M}_1(E)$ polyhedral with $\partial A'_\nu = \partial A_\nu$ and $\mathbb{M}(A'_\nu) \leq \mathbb{M}(A_\nu)$. Define now

$$T_\nu := T'_\nu + A'_\nu.$$

One has that T_ν are polyhedral and

$$\partial T_\nu = \partial T'_\nu + \partial A'_\nu = \partial T'_\nu + \partial A_\nu = \partial T = 0.$$

We have also

$$T = T_\nu + A_\nu - A'_\nu + \partial B_\nu,$$

and since $\mathbb{M}(A_\nu - A'_\nu) \leq 2\mathbb{M}(A_\nu) \rightarrow 0$, we conclude $\mathcal{F}(T_\nu - T) \rightarrow 0$. It remains to observe that

$$\begin{aligned} |\mathbb{M}(T) - \mathbb{M}(T_\nu)| &\leq |\mathbb{M}(T) - \mathbb{M}(T'_\nu)| + |\mathbb{M}(T'_\nu) - \mathbb{M}(T_\nu)| \\ &\leq |\mathbb{M}(T) - \mathbb{M}(T'_\nu)| + \mathbb{M}(A'_\nu) \leq |\mathbb{M}(T) - \mathbb{M}(T'_\nu)| + \mathbb{M}(A_\nu) \rightarrow 0 \end{aligned}$$

as $\nu \rightarrow \infty$. \square

Lemma 3.10. *Let E be a finite-dimensional normed space, and let $A \in \mathcal{M}_1(E)$. Then there exists an $A' \in \mathcal{M}_1(E)$ such that $\partial A' = \partial A$ and $\mathbb{M}(A') \leq \mathbb{M}(A)$. Moreover, if ∂A is a finite sum of signed Dirac masses, then one may choose A' polyhedral.*

Proof. We refer to theorem A.1 from [10] for the existence of an $A' \in \mathcal{M}_1(E)$ which provides the minimum of the mass functional $T \mapsto \mathbb{M}(T)$ among all $T \in \mathcal{M}_1(E)$ satisfying $\partial T = \partial A$, so that, in other words, $\mathbb{M}(A')$ be the classical Monge-Kantorovich cost of optimal transportation the measure $(\partial A)^+$ to $(\partial A)^-$ (where φ^\pm stand for the positive and negative parts of a finite signed Borel measure φ respectively). The same theorem applied in the case when ∂A is a finite sum of signed Dirac masses, shows that one can choose A' so that $\mu_{A'}$ be concentrated over geodesics (i.e. segments) connecting these masses, that is, in this case one may choose A' polyhedral. \square

4. INDECOMPOSABLE CYCLES

In this section we will be interested in the following notion.

Definition 4.1. *We say that a cycle $T \in \mathcal{M}_1(E)$ is indecomposable, if $S \leq T$ with $\partial S = 0$ implies $S = \lambda T$ for some $\lambda \in [0, 1]$.*

We will study now the representation of indecomposable cycles by curves. The following observation makes sometimes calculations (or just writing them) a bit easier.

Remark 4.2. If $T \in \mathcal{M}_1(E)$ is an indecomposable cycle represented by a curve and $\mathbb{M}(T) = 1$, then

$$T = \lim_{s_j \rightarrow +\infty} \frac{1}{2s_j} \llbracket \theta_{\mathbb{L}}[-s_j, s_j] \rrbracket,$$

implies

$$T = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\mathbb{L}}[0, s_j] \rrbracket = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\mathbb{L}}[-s_j, 0] \rrbracket.$$

In fact, denoting

$$\begin{aligned} T_1 &:= \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\mathbb{L}}[0, s_j] \rrbracket, \\ T_2 &:= \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\mathbb{L}}[-s_j, 0] \rrbracket, \end{aligned}$$

we have $T = (T_1 + T_2)/2$, while by construction $\mathbb{M}(T_i) \leq 1$, $i = 1, 2$, which implies $\mathbb{M}(T_1) = \mathbb{M}(T_2) = 1$, because $\mathbb{M}(T) = 1$. Thus $T_1/2 \leq T$ which means $T_1/2 = \lambda T$ since T is indecomposable, and hence

$$\frac{1}{2} = \mathbb{M}(T_1/2) = \lambda \mathbb{M}(T),$$

so that $\lambda = 1/2$, i.e. $T_1 = T$, and analogously we get $T_2 = T$.

The following auxiliary result will be used in our construction.

Proposition 4.3. *Let $T \in \mathcal{M}_1(E)$ be an indecomposable cycle represented by a curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$, $\mathbb{M}(T) = 1$, and*

$$T = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\mathbb{L}}[0, s_j] \rrbracket,$$

for some $\{s_j\} \subset \mathbb{R}$ and $\theta \in \text{Lip}_1(\mathbb{R}; E)$. Let $\{r_j\} \subset \mathbb{R}$ be such that $r_j \leq s_{j+1}$ and $r_j = \alpha s_{j+1} + o(s_{j+1})$ as $j \rightarrow +\infty$ for some $\alpha > 0$. Then

$$T = \lim_{r_j \rightarrow +\infty} \frac{1}{r_j} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket.$$

Proof. In view of Lemma 4.4 we may suppose that for a subsequence of r_j (not relabeled)

$$\lim_{r_j \rightarrow +\infty} \frac{1}{r_j} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket := T'.$$

We have then

$$(4.1) \quad \lim_{j \rightarrow +\infty} \frac{1}{s_{j+1}} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket = \lim_{j \rightarrow +\infty} \frac{r_j}{s_{j+1}} \cdot \frac{1}{r_j} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket = \alpha T',$$

so that

$$(4.2) \quad \lim_{j \rightarrow +\infty} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\mathbb{L}}[0, s_j] \rrbracket - \frac{1}{s_{j+1}} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket \right) = T - \alpha T'.$$

On the other hand,

$$\begin{aligned} \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\mathbb{L}}[0, s_{j+1}] \rrbracket - \frac{1}{s_{j+1}} \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket \right) &= \frac{1}{s_{j+1}} \mathbb{M} (\llbracket \theta_{\mathbb{L}}[0, s_{j+1}] \rrbracket - \llbracket \theta_{\mathbb{L}}[0, r_j] \rrbracket) \\ &= \frac{1}{s_{j+1}} \mathbb{M} (\llbracket \theta_{\mathbb{L}}[r_j, s_{j+1}] \rrbracket) \\ &\leq \frac{s_{j+1} - r_j}{s_{j+1}} \rightarrow 1 - \alpha \end{aligned}$$

as $j \rightarrow +\infty$. Thus $\mathbb{M}(T - \alpha T') \leq 1 - \alpha$, which gives

$$\mathbb{M}(\alpha T') + \mathbb{M}(T - \alpha T') \leq \alpha + (1 - \alpha) = 1 = \mathbb{M}(T),$$

that is, $\alpha T' \leq T$. This implies $\alpha T' = \lambda T$ for some $\lambda \in [0, 1]$, since T is indecomposable.

We show now $\mathbb{M}(T') = 1$. In fact, one has $\mathbb{M}(T') \leq 1$. But

$$\begin{aligned} \frac{1}{s_{j+1}} (\mathbb{M}(\llbracket \theta_{\perp}[0, r_j] \rrbracket) + \mathbb{M}(\llbracket \theta_{\perp}[r_j, s_{j+1}] \rrbracket)) &\leq \frac{1}{s_{j+1}} (\ell(\theta_{\perp}[0, r_j]) + \ell(\theta_{\perp}[r_j, s_{j+1}])) \\ &= \frac{1}{s_{j+1}} \ell(\theta_{\perp}[0, s_{j+1}]), \end{aligned}$$

and hence keeping in mind

$$1 = \liminf_j \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[0, s_{j+1}] \rrbracket \right) \leq \liminf_j \frac{1}{s_{j+1}} \ell(\theta_{\perp}[0, s_{j+1}]) \leq 1,$$

we get

$$(4.3) \quad \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[0, r_j] \rrbracket \right) + \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[r_j, s_{j+1}] \rrbracket \right) \leq 1 + o(1)$$

as $j \rightarrow \infty$. We have then

$$(4.4) \quad \lim_j \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[0, r_j] \rrbracket \right) = \mathbb{M}(\alpha T').$$

In fact, recalling (4.1) and supposing that for some sequence of j (not relabeled) one has

$$\lim_j \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[0, r_j] \rrbracket \right) < \mathbb{M}(\alpha T'),$$

we get passing to the limit in (4.3) while taking into account (4.1) and (4.2), that

$$1 = \mathbb{M}(T) \leq \mathbb{M}(\alpha T') + \mathbb{M}(T - \alpha T') < 1,$$

which is a contradiction showing (4.4). But from (4.4) we get

$$\lim_j \frac{1}{r_j} \mathbb{M}(\llbracket \theta_{\perp}[0, r_j] \rrbracket) = \mathbb{M}(T').$$

as $r_j \rightarrow +\infty$. Thus

$$\begin{aligned} \mathbb{M}(\llbracket \theta_{\perp}[0, s_{j+1}] \rrbracket) &\leq \mathbb{M}(\llbracket \theta_{\perp}[0, r_j] \rrbracket) + (s_{j+1} - r_j) \\ &= \mathbb{M}(T')r_j + (s_{j+1} - r_j) + o(r_j) \\ &= \mathbb{M}(T')r_j + (s_{j+1} - r_j) + o(s_{j+1}), \end{aligned}$$

which gives

$$1 = \mathbb{M}(T) \leq \mathbb{M} \left(\frac{1}{s_{j+1}} \llbracket \theta_{\perp}[0, s_{j+1}] \rrbracket \right) = 1 + (\mathbb{M}(T') - 1) \frac{r_j}{s_{j+1}} + o(1),$$

and passing to a limit as $j \rightarrow +\infty$, we get $\mathbb{M}(T') - 1 \geq 0$, i.e. $\mathbb{M}(T') = 1$.

It suffices now to notice that

$$\alpha \mathbb{M}(T') = \alpha = \lambda \mathbb{M}(T) = \lambda,$$

and therefore $T' = T$ as claimed. \square

The following lemma has been used in the above proof.

Lemma 4.4. *Under the conditions of Proposition 4.3 the sequence of measures $\{\mu_{R_j}\}$ corresponding to currents*

$$R_j := \frac{1}{r_j} \llbracket \theta_{\perp}[0, r_j] \rrbracket,$$

is uniformly tight (so that the sequence $\{R_j\}$ is compact in the weak topology of currents).

Proof. Note that for $T_j := \llbracket \theta_{\perp}[0, s_{j+1}] \rrbracket / s_{j+1}$ one has $\lim_j T_j \rightharpoonup T$ in the weak sense of currents and $\lim_j \mu_{T_j} = \mu_T$ in the narrow sense of measures (by Remark 3.3 with $[0, s_{j+1}]$ instead of $[-s_j, s_j]$), and hence the sequence $\{\mu_{T_j}\}$ is uniformly tight. Therefore, for every $\varepsilon > 0$ there is a $\tilde{K} \subset E$ compact such that $\mu_{T_j}(\tilde{K}^c) \leq \varepsilon$. Note also that

$$\lim_j \mu_{T_j}(\tilde{K}^c) = \lim_j \frac{1}{s_{j+1}} \mathcal{L}^1(\theta^{-1}(\tilde{K}^c) \cap [0, s_{j+1}])$$

whenever the first limit exists (by Remark 3.4 with $[0, s_{j+1}]$ instead of $[-s_j, s_j]$). We have then

$$\begin{aligned} \mu_{R_j}(\tilde{K}^c) &\leq \frac{1}{r_j} \mathcal{L}^1(\theta_j^{-1}(\tilde{K}^c) \cap [0, r_j]) \\ &\leq \frac{s_{j+1}}{r_j} \cdot \frac{1}{s_{j+1}} \mathcal{L}^1(\theta_j^{-1}(\tilde{K}^c) \cap [0, s_{j+1}]), \end{aligned}$$

hence for some $k \in \mathbb{N}$ one has

$$\mu_{R_j}(\tilde{K}^c) \leq 2 \frac{s_{j+1}}{r_j} \mu_{T_j}(\tilde{K}^c) \leq C\varepsilon$$

for all $j \geq k$ and for some $C > 0$ independent of j and ε . Letting \hat{K} to be such a compact set that

$$\mu_{R_j}(\hat{K}^c) \leq C\varepsilon, \quad j = 1, \dots, k-1,$$

then for $K := \hat{K} \cup \tilde{K}$ one has $\mu_{R_j}(K^c) \leq C\varepsilon$ so that the sequence $\{R_j\}$ is compact in the weak topology of currents by compactness theorem 5.2 from [1]. \square

5. SOLENOIDS

Let us introduce now another definition.

Definition 5.1. *A cycle $T \in \mathcal{M}_1(E)$ is called solenoid, if there exists a curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$ such that*

$$T = \lim_{s \rightarrow +\infty} \frac{1}{2s} \llbracket \theta_{\perp}[-s, s] \rrbracket,$$

the limit being taken over all $s \in \mathbb{R}$. The set of all solenoids $T \in \mathcal{M}_1(E)$ will be denoted by $\text{Sol}(E)$.

Clearly, being a solenoid is stronger than just being representable by a curve.

The following result holds true.

Theorem 5.2. *Let E be a quasiconvex metric space, and $T \in \mathcal{M}_1(E)$ be an indecomposable cycle represented by a curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$, $\mathbb{M}(T) = 1$. Then $T \in \text{Sol}(E)$.*

Proof. Let

$$T = \lim_j S'_j, \quad \text{where } S'_j := \frac{1}{s'_j} \llbracket \theta_{\perp}[0, s'_j] \rrbracket,$$

the limit being, as usual, in the weak sense of currents, for some sequence $s'_j \nearrow +\infty$. We will assume $\lim_j s'_j / s'_{j+1} = 0$ since otherwise there is nothing to prove. In fact, if $\lim_j s'_j / s'_{j+1} = \alpha > 0$ for some subsequence of $\{s'_j\}$, then for every increasing sequence $\{r_j\}$, up to passing to a further subsequence of $\{s'_j\}$ (not relabeled) we may consider any subsequence of $\{r_j\}$ (again not relabeled) such that $r_j \in [s'_j, s'_{j+1}]$, and hence $\liminf_j r_j / s'_{j+1} \geq \alpha > 0$, which by Proposition 4.3 means that

$$T = \lim_{j \rightarrow +\infty} \frac{1}{r_j} \llbracket \theta_{\perp}[0, r_j] \rrbracket$$

for every subsequence of $\{r_j\}$, and hence, for the whole sequence $\{r_j\}$.

The proof will be achieved in two steps.

Step 1. Consider a compact set K such that $\mu_T(K^c) \leq 1/8$. Fix an arbitrary $\varepsilon > 0$. We have then that for every $j \in \mathbb{N}$ there is a point $s_j \in [3s'_{j+1}/4, s'_{j+1}]$ such that $\theta(s_j) \in (K)_\varepsilon$. In fact, otherwise

$$\begin{aligned} \mu_{s'_{j+1}}((K)_\varepsilon) &= \frac{1}{s'_{j+1}} \mu_{[\theta_\perp[0, s'_{j+1}]]}((K)_\varepsilon) \\ &\leq \frac{1}{s'_{j+1}} \mathcal{L}^1(\{t \in [0, s'_{j+1}] : \theta(t) \in (K)_\varepsilon\}) \\ &= \frac{1}{s'_{j+1}} \mathcal{L}^1(\{t \in [0, 3s'_{j+1}/4] : \theta(t) \in (K)_\varepsilon\}) \leq \frac{3}{4}, \end{aligned}$$

thus, passing to a limit as $j \rightarrow \infty$, we have

$$\mu_T(K) \leq \mu_T((K)_\varepsilon) \leq \liminf_j \mu_{s'_{j+1}}((K)_\varepsilon) \leq \frac{3}{4},$$

which, recalling that $\mathbb{M}(T) = 1$, contradicts the assumption $\mu_T(K^c) \leq 1/8$.

Choosing a subsequence of $\{s_j\}$ (not relabeled) such that s_j/s'_{j+1} converge as $j \rightarrow \infty$ to some $\alpha > 0$, we have by Proposition 4.3 that

$$T = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} [\theta_\perp[0, s_j]].$$

and $\theta(s_j) \in (K)_\varepsilon$ for all $j \in \mathbb{N}$. Without loss of generality we also will assume $\theta(0) \in (K)_\varepsilon$. Note that by construction one still has $\lim_j s_j/s_{j+1} = 0$.

Step 2. Fix a $d > \text{diam}_*(K)_\varepsilon$. Define a new curve $\sigma: [0, +\infty) \rightarrow E$ as follows. Let σ coincide with θ over $[0, s_1]$. For every $j \in \mathbb{N}$ we let $\sigma'_j \in \text{Lip}_1([0, d'_j]; E)$ to be an arcwise parameterized curve connecting $\sigma(s_j)$ with $\sigma(0) = \theta(0)$, with $d'_j := d_*(\sigma(s_j), \sigma(0))$, set

$$\begin{aligned} d''_j &:= d_*(\sigma(s_j), \theta(s_{j+1})) \\ n_j &:= \left\lfloor \frac{s_{j+1} - d''_j}{s_j + d'_j} \right\rfloor \\ \delta''_j &:= s_{j+1} - n_j(s_j + d'_j), \end{aligned}$$

and $\sigma''_j \in \text{Lip}_1([0, \delta''_j]; E)$ to be a curve connecting $\sigma(s_j)$ with $\theta(s_{j+1})$ with velocity bounded by one (which is possible since $\delta''_j \geq d''_j$). Observe that clearly $d'_j \leq d$ and $d''_j \leq d$, while

$$\delta''_j \leq s_j + d'_j + d''_j \leq s_j + 2d.$$

Define now σ over $[s_j, s_{j+1}]$ as a composition of σ'_j with $\theta_\perp[0, s_j]$ repeated n_j times and then connected by σ''_j to $\sigma(s_{j+1}) := \theta(s_{j+1})$. This inductively defines σ over $[0, +\infty)$. Extending σ to $(-\infty, 0)$ symmetrically σ over the whole \mathbb{R} .

Consider now an arbitrary increasing sequence $\{r_i\}$, $\lim_i r_i = +\infty$. For every $i \in \mathbb{N}$ we find a $j = j(i)$ such that $r_i \in [s_{j(i+1)-1}, s_{j(i+1)}]$ and calculate

$$\lim_{r_i \rightarrow +\infty} \frac{1}{r_i} [\sigma_\perp[0, r_i]].$$

Clearly $j(i) \rightarrow +\infty$ when $i \rightarrow +\infty$. For the sake of brevity of notation we write $j := j(i+1) - 1$. We consider several cases.

CASE 1. $\lim_i r_i/s_j = +\infty$. Then, if $r_i \in [s_{j+1} - \delta''_j, s_{j+1}]$ for a subsequence of i , one has

$$\lim_{s_{j+1} \rightarrow +\infty} \frac{1}{s_{j+1}} [\sigma_\perp[0, s_{j+1}]] = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} [\theta_\perp[0, s_j]] = T,$$

because

$$\frac{1}{r_i} [\sigma_\perp[0, r_i]] = \frac{n_i}{r_i} [\theta_\perp[0, s_j]] + \Delta_i,$$

where

$$\Delta_i = \frac{1}{r_i} (\llbracket \sigma_{\perp}[0, s_j] \rrbracket + n_j \llbracket \sigma'_j \rrbracket + \llbracket \sigma''_j \llbracket 0, r_i - s_{j+1} + \delta''_j \rrbracket \rrbracket),$$

so that

$$\mathbb{M}(\Delta_i) \leq \frac{s_j}{r_i} + \frac{n_j d'_j}{r_i} + \frac{d''_j}{r_i} \leq \frac{s_j}{r_i} + \frac{n_j d}{r_i} + \frac{d}{r_i} \rightarrow 0$$

as $j \rightarrow \infty$, while $\lim_j s_j n_j / s_{j+1} = 1$.

Otherwise, setting

$$\rho_i := \left\lfloor \frac{r_i}{s_j + d'_j} \right\rfloor,$$

we get

$$\frac{1}{r_i} \llbracket \sigma_{\perp}[0, r_i] \rrbracket = \frac{\rho_j}{r_i} \llbracket \theta_{\perp}[0, s_j] \rrbracket + R_i$$

with

$$\mathbb{M}(R_i) \leq \frac{s_j}{r_i} + \frac{\rho_i d'_j}{r_i} + \frac{s_j + d'_j}{r_i} \leq \frac{2s_j}{r_i} + \frac{\rho_i d}{r_i} + \frac{s_j + d}{r_i} \rightarrow 0$$

as $i \rightarrow \infty$, and keeping in mind that $\lim_i s_j \rho_i / r_i = 1$, we get that the above limit is T .

CASE 2. $\lim_i r_i / s_j =: \alpha > 1$. Consider first for simplicity the case $r_i = n s_j$ for some $n \in \mathbb{N}$. Then

$$\frac{1}{r_i} \llbracket \sigma_{\perp}[0, r_i] \rrbracket = \frac{(n-1)}{n s_j} \llbracket \theta_{\perp}[0, s_j] \rrbracket + \frac{1}{s_j} \llbracket \sigma_{\perp}[0, s_j] \rrbracket + R'_i,$$

where

$$\mathbb{M}(R'_i) \leq \frac{n d'_j}{r_i} \leq \frac{n d}{r_i} \rightarrow 0$$

as $i \rightarrow \infty$. Thus, keeping in mind that

$$\lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \sigma_{\perp}[0, s_j] \rrbracket = \lim_{s_j \rightarrow +\infty} \frac{1}{s_j} \llbracket \theta_{\perp}[0, s_j] \rrbracket = T,$$

we have

$$\lim_i \frac{1}{r_i} \llbracket \sigma_{\perp}[0, r_i] \rrbracket = \left(1 - \frac{1}{n}\right) \lim_j \llbracket \theta_{\perp}[0, s_j] \rrbracket + \frac{1}{n} \lim_j \llbracket \sigma_{\perp}[0, s_j] \rrbracket = T.$$

The case of a generic $\alpha > 1$ follows by applying Proposition 4.3 (with σ instead of θ and $n s_j$ instead of s_j). \square

Theorem 5.2 implies that if E is the space where every tight cycle with bounded support is represented by a curve (a fairly general collection of such spaces including of course, all finite-dimensional vector spaces, are provided by Theorem 3.2, Proposition A.1 and Corollary A.2), then every indecomposable tight cycle in $\mathcal{M}_1(E)$ with bounded support is, up to normalization by its mass, a solenoid.

Note that Theorem 5.2 gives only sufficient conditions for a cycle to be a solenoid. In fact, solenoids are not necessarily indecomposable as the following example shows.

Example 5.3. Let $\{z_1, z_2, z_3\} \subset E := \mathbb{R}^2$ be vertices of an equilateral triangle with side 3, and let $T_i \in \mathcal{M}_1(\mathbb{R}^2)$ be a cycle represented by the oriented unit circumference S_i with center z_i , having density $1/6\pi$, $i = 1, \dots, 3$ (for definiteness, assume that the orientation of all the three cycles are, say, clockwise). In this way, defined $T := T_1 + T_2 + T_3$ we get $\mathbb{M}(T) = 1$. We show now that T is a solenoid (although, clearly it is not indecomposable: in fact, $T_i \leq T$ and $T_i \neq \lambda T$ for all $\lambda \in [0, 1]$). We further abuse slightly the notation and let S_i stand both for the cycle (as a metric current), i.e. the oriented circumference (so that $T_i = S_i/6\pi$), and its support, i.e. the circumference itself, the exact meaning being always clear from the context.

We have to define a curve $\theta \in \text{Lip}_1(\mathbb{R}; E)$ such that the limit

$$(5.1) \quad \lim_{s_m \rightarrow +\infty} \frac{1}{2s_m} \llbracket \theta_{\perp}[-s_m, s_m] \rrbracket = T,$$

for all sequences $s_m \nearrow +\infty$. For this purpose denote by $b_i b_j$ the segment of length 1 on the side $z_i z_j$ of the triangle $\Delta z_1 z_2 z_3$ such that its endpoints b_i and b_j are the intersections of S_i and S_j with the segment $z_i z_j$ respectively.

For the intervals $\Delta_k := [k^2, (k+1)^2]$ consider the partition

$$\Delta_k = [k^2, k^2 + 1] \cup \varepsilon_{k1} \cup \gamma_k^{12} \cup \varepsilon_{k2} \cup \gamma_k^{23} \cup \varepsilon_{k3} \cup [(k+1)^2 - 1, (k+1)^2],$$

where $\varepsilon_{k1}, \varepsilon_{k2}, \varepsilon_{k3}$ are three open intervals of equal lengths

$$|\varepsilon_{k1}| = |\varepsilon_{k2}| = |\varepsilon_{k3}|$$

and γ_{k1}, γ_{k2} are closed intervals of length 1. Then we have

$$(5.2) \quad |\varepsilon_{kj}| = \frac{(k+1)^2 - k^2}{3} - 4 = \frac{2k}{3} + O(1)$$

as $k \rightarrow +\infty$.

Then we consider the curve $\theta \in \text{Lip}_1([0, +\infty); E)$ such that

- (i) $\theta_{\perp} \varepsilon_{kj}$ covers the circumference S_j in the clockwise direction starting from b_j with the unit velocity for the time $2\pi n_{kj}$, where $n_{kj} := \lfloor |\varepsilon_{kj}| / 2\pi \rfloor$, and then staying in the point b_j for the rest of the time $|\varepsilon_{kj}| - 2\pi n_{kj}$,
- (ii) $\theta_{\perp} \gamma_k^{12}$ covers $b_1 b_2$ starting from b_1 and ending at b_2 with unit velocity,
- (iii) $\theta_{\perp} \gamma_k^{23}$ covers $b_2 b_3$ starting from b_2 and ending at b_3 with unit velocity,
- (iv) $\theta_{\perp} [k^2, k^2 + 1]$ and $\theta_{\perp} [(k+1)^2 - 1, (k+1)^2]$ cover $b_3 b_1$ starting from b_3 and ending at b_1 with constant velocity $1/2$.

Of course this can be done only for $k \gg 1$, i.e. for $k \geq k_0$ for some $k_0 \in \mathbb{N}$, and we define $\theta_{\perp} \Delta_k$ for all $k \leq k_0$ in an arbitrary way so as to have $\theta \in \text{Lip}_1([0, +\infty); E)$. Finally, let $\theta_{\perp}(-\infty, 0]$ be defined symmetrically. This gives a $\theta \in \text{Lip}_1(\mathbb{R}; E)$.

To verify (5.1) we first show it for the case $s_m := m^2 \in \mathbb{N}$. One has

$$\llbracket \theta_{\perp}[0, m^2] \rrbracket = \llbracket \theta_{\perp}[0, k_0^2] \rrbracket + \sum_{j=1}^3 \sum_{k=k_0}^m \llbracket \theta_{\perp} \varepsilon_{kj} \rrbracket + R_m,$$

where $\mathbb{M}(R_m) \leq 5(m - k_0)$, and therefore,

$$\begin{aligned} \lim_m \frac{1}{m^2} \llbracket \theta_{\perp}[0, m^2] \rrbracket &= \sum_{j=1}^3 \lim_m \frac{1}{m^2} \sum_{k=k_0}^m \llbracket \theta_{\perp} \varepsilon_{kj} \rrbracket \\ &= \sum_{j=1}^3 \lim_m \frac{1}{m^2} \sum_{k=k_0}^m n_{kj} S_j = \sum_{j=1}^3 S_j \left(\lim_m \frac{1}{m^2} \sum_{k=k_0}^m n_{kj} \right) \\ &= \sum_{j=1}^3 S_j \left(\lim_m \frac{1}{m^2} \sum_{k=k_0}^m \frac{2k}{3} \cdot \frac{1}{2\pi} \right) = \frac{1}{6\pi} \sum_{j=1}^3 S_j = T, \end{aligned}$$

where the limits of currents are intended in the weak sense. Since $\theta_{\perp}(-\infty, 0]$ is defined symmetrically, then one has

$$\lim_m \frac{1}{2m^2} \llbracket \theta_{\perp}[-m^2, m^2] \rrbracket = \lim_m \frac{1}{m^2} \llbracket \theta_{\perp}[0, m^2] \rrbracket = T,$$

which is exactly (5.1) for the particular case being verified.

Take now an arbitrary sequence $s_m \nearrow +\infty$ and let k_m be such that $s_m \in [k_m^2, (k_m + 1)^2]$. Then

$$\lim_m \frac{1}{2s_m} \llbracket \theta_{\perp}[-s_m, s_m] \rrbracket = \lim_m \frac{1}{2s_m} \llbracket \theta_{\perp}[-k_m^2, k_m^2] \rrbracket,$$

because $\mathbb{M}(\llbracket \theta_{\perp}[-s_m, k_m^2] \rrbracket) = \mathbb{M}(\llbracket \theta_{\perp}[k_m^2, s_m] \rrbracket) \leq (k_m + 1)^2 - k_m^2 = o(k_m^2) = o(s_m)$ as $m \rightarrow \infty$. But

$$\lim_m \frac{1}{2s_m} \llbracket \theta_{\perp}[-k_m^2, k_m^2] \rrbracket = \lim_m \frac{1}{2k_m^2} \llbracket \theta_{\perp}[-k_m^2, k_m^2] \rrbracket = T,$$

which concludes the proof of (5.1).

APPENDIX A. SOME EASY GENERALIZATIONS TO INFINITE-DIMENSIONAL SPACES

We provide here a couple of easy, though quite technically looking results generalizing Theorem 3.2 to not necessarily finite-dimensional spaces. We are not really interested in obtaining the most general results, since the main idea is well illustrated by Theorem 3.2, and thus we limit ourselves here to provide some fairly general results that are almost “free of charge” by using the theory of metric currents.

Proposition A.1. *Let E be a Banach space with metric approximation property. For every tight cycle $T \in \mathcal{M}_1(E)$ with bounded support one has that $T/\mathbb{M}(T)$ is represented by some curve.*

Proof. When E is a finite-dimensional normed space, the result is given by Theorem 3.2. Suppose that E is a Banach space with metric approximation property. Let $\{T_n\}$ be a sequence of currents over finite-dimensional subspaces E_n of E be provided by lemma A.6 from [11] (minding the remark A.7 from the same paper), namely, $T_n \rightharpoonup T$ in the weak sense of currents, $\mathbb{M}(T_n) \rightarrow \mathbb{M}(T)$ as $n \rightarrow \infty$, and $\partial T_n = 0$. Thus each T_n (as a cycle over a finite-dimensional normed space) is represented by some curve in $\text{Lip}_1(\mathbb{R}; E_n)$, hence in $\text{Lip}_1(\mathbb{R}; E)$, and again applying Lemma 3.6 we get the result. \square

Of course the above Proposition A.1 provides a rather wide class of spaces where every cycle is (up to a normalization by its mass) represented by some curve, which includes all finite-dimensional normed spaces, Hilbert spaces and a lot of reasonable Banach spaces (for instance, Lebesgue spaces). However, if we are interested in generic metric spaces without any linear structure, the following corollary might be helpful. To formulate it, we recall that a closed subset $X \subset Y$ of a metric space Y is called a 1-Lipschitz neighborhood retract, if there is an open $U \subset Y$ satisfying $X \subset U$ and a map $p \in \text{Lip}_1(\bar{U}, X)$ such that $p(x) = x$ for all $x \in X$.

Corollary A.2. *Let X be a quasiconvex metric space isometrically embedded in Banach space Y with metric approximation property as a 1-Lipschitz neighborhood retract of the latter. Then for every tight cycle $T \in \mathcal{M}_1(X)$ with bounded support one has that $T/\mathbb{M}(T)$ is represented by some curve.*

Proof. Assume without loss of generality $\mathbb{M}(T) = 1$. Let $j: X \rightarrow Y$ be an isometric embedding. Then $j_{\#}T$ is represented by some curve in Y by Proposition A.1, and thus $j_{\#}T$ is represented by a curve in $j(X)$ by Lemma A.3 (applied with $\bar{V} := j(X)$). In other words,

$$j_{\#}T = \lim_k \frac{1}{2s_k} \llbracket \theta'_{\perp}[-s_k, s_k] \rrbracket$$

for some $\theta' \in \text{Lip}_1(\mathbb{R}; j(X))$ and some sequence $s_k \nearrow +\infty$. Therefore,

$$T = \lim_k \frac{1}{2s_k} \llbracket \theta_{\perp}[-s_k, s_k] \rrbracket,$$

where $\theta(t) := j^{-1}(\theta'(t))$, which concludes the proof. \square

The following lemma has been used in the proof of Corollary A.2.

Lemma A.3. *Let Y be a metric space and $\bar{V} \subset Y$ be a closed set, while \bar{V} is quasiconvex, and there is an open $U \subset Y$, $\bar{V} \subset U$ and a 1-Lipschitz map $p: \bar{U} \rightarrow \bar{V}$ satisfying $p(v) = v$ for all $v \in \bar{V}$. If T is represented by some curve in Y with $\mathbb{M}(T) = 1$ and μ_T concentrated over \bar{V} , then T is represented by some curve in \bar{V} . Moreover, if*

$$T = \lim_k \frac{1}{2s_k} \llbracket \theta' \llcorner [-s_k, s_k] \rrbracket,$$

for some $\theta' \in \text{Lip}_1(\mathbb{R}; Y)$ and some sequence $s_k \nearrow +\infty$, then

$$T = \lim_k \frac{1}{2s_k} \llbracket \theta \llcorner [-s_k, s_k] \rrbracket = T$$

for some $\theta \in \text{Lip}_1(\mathbb{R}; \bar{V})$ and the same sequence $\{s_k\}$.

Proof. Let

$$\Delta := (\theta')^{-1}(\bar{U}^c) = \sqcup_i (\alpha_i, \beta_i),$$

where $\theta'(\alpha_i) \in \bar{U}$ and $\theta'(\beta_i) \in \bar{U}$. We denote then $\bar{\theta}'(t) := p(\theta'(t))$ whenever $\theta'(t) \notin \Delta$ and define $\sigma \in \text{Lip}(\mathbb{R}; \bar{V})$ by setting $\sigma(s) := \bar{\theta}'(s)$ whenever $s \notin \Delta$, while over each interval (α_i, β_i) we let σ be a curve θ_i in \bar{V} connecting $\bar{\theta}'(\alpha_i)$ with $\bar{\theta}'(\beta_i)$ parameterized over $[\alpha_i, \beta_i]$ and having length

$$\ell(\theta_i) \leq C d_Y(\bar{\theta}'(\alpha_i), \bar{\theta}'(\beta_i)) \leq C |\beta_i - \alpha_i|,$$

where d_Y stands for the distance in Y . Let now

$$f(s) := \begin{cases} |\dot{\sigma}|(s) = |\dot{\theta}_i|(s), & s \in (\alpha_i, \beta_i), \\ 1, & \text{otherwise.} \end{cases}$$

$$t(s) := \int_0^s f(\tau) d\tau, \quad \theta(t) := \sigma(s(t)).$$

It is immediate to calculate then

$$|\dot{\theta}|(t) := \begin{cases} 1, & s \in (\alpha_i, \beta_i), \\ |\dot{\theta}|(s(t)), & \text{otherwise,} \end{cases}$$

so that $\theta \in \text{Lip}_1(\mathbb{R}; \bar{V})$.

Denoting

$$S_k := \frac{1}{2s_k} \llbracket \theta' \llcorner [-s_k, s_k] \setminus \Delta \rrbracket, T_k := \frac{1}{2t_k} \llbracket \theta' \llcorner [-s_k, s_k] \setminus \Delta \rrbracket,$$

$$T'_k := \frac{1}{2s_k} \llbracket \theta' \llcorner [-s_k, s_k] \rrbracket,$$

so that $T_k \in M_1(\bar{U})$ and $S_k \in M_1(\bar{U})$ we have

$$\mathbb{M}(S_k - T'_k) \leq \frac{\mathcal{L}^1([-s_k, s_k] \cap \Delta)}{2s_k} \rightarrow 0$$

as $k \rightarrow \infty$, because by Remark 3.4 one has

$$\begin{aligned} \limsup_k \frac{\mathcal{L}^1([-s_k, s_k] \cap \Delta)}{2s_k} &= \limsup_k \mu_{\frac{1}{2s_k} \llbracket \theta' \llcorner [-s_k, s_k] \rrbracket}(\bar{U}^c) \\ &\leq \limsup_k \mu_{\frac{1}{2s_k} \llbracket \theta' \llcorner [-s_k, s_k] \rrbracket}(U^c) \\ &\leq \mu_T(U^c) = 0. \end{aligned}$$

Further, letting $t_k := t(s_k)$ and minding that

$$(1.1) \quad s_k \leq t_k \leq s_k + C \mathcal{L}^1([-s_k, s_k] \cap \Delta),$$

and thus $\lim_k t_k/s_k = 1$, which gives

$$\mathbb{M}(S_k - T_k) = \left(1 - \lim_k \frac{t_k}{s_k}\right) \mathbb{M}(S_k) = 0$$

(because $\mathbb{M}(S_k) \leq 1$). Thus $\lim_k S_k = \lim_k T_k = \lim_k T'_k = T$ in the weak sense of currents.

We estimate now

$$\begin{aligned} \mathbb{M}\left(p_{\#}S_k - \frac{1}{2s_k}[\sigma_{\perp}[-s_k, s_k]]\right) &\leq \frac{1}{2s_k}\mathbb{M}([\sigma_{\perp}[-s_k, s_k] \cap \Delta]) \\ &\leq C\frac{1}{2s_k}\mathcal{L}^1([-s_k, s_k] \cap \Delta) \rightarrow 0, \end{aligned}$$

but then

$$(1.2) \quad \begin{aligned} \mathbb{M}\left(\frac{1}{2t_k}[\theta_{\perp}[t(-s_k), t_k]] - p_{\#}T_k\right) &= \mathbb{M}\left(\frac{1}{2t_k}[\sigma_{\perp}[-s_k, s_k]] - p_{\#}T_k\right) \\ &= \frac{s_k}{t_k}\mathbb{M}\left(\frac{1}{2s_k}[\sigma_{\perp}[-s_k, s_k]] - p_{\#}S_k\right) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Since also

$$-s_k - C\mathcal{L}^1([-s_k, s_k] \cap \Delta) \leq t(-s_k) \leq -s_k,$$

combining this estimate with (1.1), we get

$$|t(-s_k) + t_k| \leq C\mathcal{L}^1([-s_k, s_k] \cap \Delta),$$

and thus, denoting by I_k the interval with endpoints $t(-s_k)$ and $-t_k$, we get

$$\begin{aligned} \mathbb{M}\left(\frac{1}{2t_k}[\theta_{\perp}[-t_k, t_k]] - \frac{1}{2t_k}[\theta_{\perp}[t(-s_k), t_k]]\right) &= \frac{1}{2t_k}\mathbb{M}([\theta_{\perp}I_k]) \\ &\leq \frac{s_k}{t_k}\frac{1}{2s_k}\mathcal{L}^1(I_k) \leq \frac{s_k}{t_k}\frac{|t(-s_k) + t_k|}{2s_k} \\ &\leq \frac{s_k}{t_k}\frac{C\mathcal{L}^1([-s_k, s_k] \cap \Delta)}{2s_k} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Combined with (1.2), this gives

$$\mathbb{M}\left(\frac{1}{2t_k}[\theta_{\perp}[-t_k, t_k]] - p_{\#}T_k\right) \rightarrow 0$$

as $k \rightarrow \infty$. Now, since in the weak sense of currents we have $\lim_k p_{\#}T_k = p_{\#}\lim_k T_k = p_{\#}T = T$ (because p is identity over \bar{V}), then

$$\lim_k \frac{1}{2t_k}[\theta_{\perp}[-t_k, t_k]] = T$$

in the weak sense of currents. Finally, since $\lim_k t_k/s_k = 1$, then from Remark 3.5 one has

$$\lim_k \frac{1}{2s_k}[\theta_{\perp}[-s_k, s_k]] = T$$

in the weak sense of currents, which completes the proof. \square

REFERENCES

- [1] L. Ambrosio and B. Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.
- [2] V. Bangert. Minimal measures and minimizing closed normal one-currents. *Geom. Funct. Anal.*, 9(3):413–427, 1999.
- [3] V. Bogachev. *Measure Theory, vol. I and II*. Springer-Verlag, 2006.
- [4] L. De Pascale, M.S. Gelli, and L. Granieri. Minimal measures, one-dimensional currents and the Monge-Kantorovich problem. *Calc. Var. Partial Differential Equations*, 27(1):1–23, 2006.
- [5] M. B. Dubashinskiĭ. On uniform approximation by harmonic and almost harmonic vector fields. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 389(Issledovaniya po Lineinym Operatoram i Teorii Funktsii. 38):58–84, 285, 2011. in Russian.
- [6] V. Muñoz and R. Pérez Marco. Ergodic solenoidal homology. II. Density of ergodic solenoids. *Aust. J. Math. Anal. Appl.*, 6(1):Art. 11, 8, 2009.
- [7] V. Muñoz and R. Pérez Marco. Schwartzman cycles and ergodic solenoids. <http://arxiv.org/abs/0910.2837>, 2009.

- [8] V. Muñoz and R. Pérez Marco. Ergodic solenoidal homology: realization theorem. *Comm. Math. Phys.*, 302(3):737–753, 2011.
- [9] V. Muñoz and R. Pérez Marco. Ergodic solenoids and generalized currents. *Rev. Mat. Complut.*, 24(2):493–525, 2011.
- [10] E. Paolini and E. Stepanov. Decomposition of acyclic normal currents in a metric space. *J. Funct. Anal.*, 263(11):3358–3390, 2012.
- [11] E. Paolini and E. Stepanov. Structure of metric cycles and normal one-dimensional currents. 2012. <http://cvgmt.sns.it/paper/1891/>.
- [12] S. Schwartzman. Asymptotic cycles. *Ann. of Math. (2)*, 66:270–284, 1957.
- [13] S. Schwartzman. Asymptotic cycles on non-compact spaces. *Bull. London Math. Soc.*, 29(3):350–352, 1997.
- [14] S.K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids and the structure of normal one-dimensional currents. *St. Petersburg Math. J.*, 5(4):841–867, 1994.

DIPARTIMENTO DI MATEMATICA “L. TONELLI”, UNIVERSITÀ DI PISA, LARGO B. PONTECORVO 5, 56127 PISA, ITALY.

E-mail address: `georgiev@mail.dm.unipi.it`

ST.PETERSBURG BRANCH OF THE STEKLOV MATHEMATICAL INSTITUTE OF THE RUSSIAN ACADEMY OF SCIENCES, FONTANKA 27, 191023 ST.PETERSBURG, RUSSIA AND DEPARTMENT OF MATHEMATICAL PHYSICS, FACULTY OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIJ PR. 28, OLD PETERHOF, 198504 ST.PETERSBURG, RUSSIA

E-mail address: `stepanov.eugene@gmail.com`