EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A BOUNDARY VALUE PROBLEM ARISING FROM GRANULAR MATTER THEORY

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Abstract. We consider a system of PDEs of Monge-Kantorovich type that, in the isotropic case, describes the stationary configurations of two-layers models in granular matter theory with a general source and a general boundary data. We propose a new weak formulation which is consistent with the physical model and permits us to prove existence and uniqueness results.

1. Introduction

The model system usually considered for the description of the stationary configurations of sandpiles on a container is the Monge-Kantorovich type system of PDEs

$$\begin{cases} -\operatorname{div}(v\,Du) = f & \text{in } \Omega, \\ |Du| \leq 1, \ v \geq 0 & \text{in } \Omega, \\ (1-|Du|)v = 0 & \text{in } \Omega, \\ u \leq \phi & \text{on } \partial\Omega, \\ u = \phi & \text{on } \Gamma_f \end{cases} \tag{1}$$

(see, e.g., [2, 4, 6, 15]). The data of the problem are the flat surface of the container $\Omega \subseteq \mathbb{R}^2$, the profile of the rim ϕ, and the density of the source $f \geq 0$, whereas the set Γ_f is a subset of $\partial\Omega$, defined in terms of the other data, that will be specified below.

The dynamical behaviour of the granular matter is pictured by the pair (u, v), where u is the profile of the standing layer, whose slope has not to exceed a critical value ($|Du| \leq 1$) in order to prevent avalanches, while $v \geq 0$ is the thickness of the rolling layer. The condition $(1 - |Du|)v = 0$ corresponds to require that the matter runs down only in the region where the slope of the heaps is maximal.

The set Γ_f (which depends on the source f, the geometry of Ω, and on the boundary datum ϕ) is the part of the border where every admissible profile u touches the rim, in such a way the exceeding sand can fall down (see Definitions in Section 3). We underline that the set Γ_f is not an additional datum of the problem, but it is constructed in terms of the other data (see (9) for its precise definition).

The main contribution of our results to the theory concerns the uniqueness of the v–component for general boundary value problems, based on a new weak formulation of the continuity equation $-\operatorname{div}(v\,Du) = f$ in Ω.

Date: October 26, 2012.

2010 *Mathematics Subject Classification.* Primary 35A02, Secondary 35J25.

Key words and phrases. Boundary value problems, mass transfer theory.

The case of the open table problem, corresponding to $u = \phi = 0$ on $\partial\Omega$, is already completely understood (see e.g. [4, 5, 6, 17] and the references therein). Namely, if d_Ω denotes the distance function from the boundary of Ω, it is possible to construct a function $v_f \geq 0$, $v_f \in L^1(\Omega)$ such that the pair (d_Ω, v_f) is a solution to (1) (we underline that the continuity equation is understood in the sense of distributions). Moreover it turns out that v_f is the unique admissible v–component, and every profile u must coincides with d_Ω where the transport is active.

These results validate the model for the open table problem, since they depicted the sole physically acceptable situation: the mass transport density v has to be uniquely determined by the data of the problem, while the profile u could be different from the maximal one only where the mass transportation does not act.

Moreover the profile is unique (and maximal) if and only if the source f pours sand along the ridge of the maximal profile (i.e. on the closure of the set where d_Ω is not differentiable).

As far as we know, only the following two particular cases of non-homogeneous boundary conditions were considered in literature.

In [13], mostly devoted to a numerical point of view, the problem of the open table with walls (corresponding to $u = \phi = 0$ on a regular portion Γ of $\partial\Omega$, and $\phi = +\infty$ in $\partial\Omega \setminus \Gamma$) is considered. In order to take into account the fact that the sand can flow out from the table only through Γ, the weak formulation of the continuity equation proposed in [13] is the following:

$$\int_\Omega v\langle Du,\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall\psi \in C_c^\infty(\mathbb{R}^2 \setminus \overline{\Gamma}).$$

Under suitable regularity assumptions on the geometry of the sandpile, it is proved that there exists a function $v_f \geq 0$, $v_f \in L^1(\Omega)$ such that the pair (d_Γ, v_f) is a solution to (1), where d_Γ is the distance function from Γ.

A different approach to non-homogeneous boundary conditions was recently proposed in [12]. In that paper we considered only admissible boundary data, that is continuous functions ϕ on $\partial\Omega$ that coincide on the boundary with the related Lax–Hopf function u_ϕ. In the model, this corresponds to treating the so called tray table problem, where the boundary datum ϕ gives the height of the rim. The requirements are that the border of the rim is always reached ($u = \phi$ on $\partial\Omega$), and that the continuity equation is satisfied in the sense of distributions. The existence of a solution is obtained, in analogy with the open table problem, by exhibiting an explicit function $v_f \geq 0$, $v_f \in L^1(\Omega)$ such that the pair (u_ϕ, v_f) is a solution to (1). Moreover a necessary and sufficient condition for the uniqueness of the u-component can be obtained, with minor changes, as in the homogeneous case.

The main novelty in the analysis of the non-homogeneous case concerns the lack of uniqueness of the v–component. Namely, the boundary datum ϕ modifies the geometry of the directions along which the sand falls down. In particular it may happen that a family of transport rays passing across Ω covers a set of positive measure, so that it is possible to transport any additional mass along these rays, keeping the total flux unchanged.

In the present paper we shall deal with general boundary data, thus allowing the presence of walls as well as of exit points at different heights. The main goal will be to modify the weak formulation of the continuity equation in order to gain the uniqueness of the v–component, without loosing information concerning the u–component, then validating the model in a very general case.

Moreover we shall not require that the profiles have to reach the height of the rim at every point of $\partial\Omega$ where they a–priori could agree, compatibly with their gradient constraint (i.e. at every point where the maximal profile u_ϕ agrees with ϕ). It is perfectly clear that, during the evolution ending with the stationary state, the sandpile grows under the action of the source, so that it is not reasonable to require that $u = \phi$ in the part of the boundary not reached by those transport rays along which no sand is poured. For this reason we relax the boundary condition, requiring $u \leq \phi$ on $\partial\Omega$, and by selecting the region $\Gamma_f \subseteq \partial\Omega$ where $u = \phi$ in terms of f and of the geometry of the transport rays.

The region Γ_f also dictates the test functions in the weak formulation of the continuity equation. Namely, we require that a solution (u, v) to (1) has to satisfy

$$\int_\Omega v\langle Du,\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall\psi \in C_c^\infty\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right),$$

thus taking into account the fact that the sand cannot exit from $\partial\Omega \setminus \Gamma_f$.

We present all the results in a more general setting, which takes into account the possibility of homogeneous anisotropies (see also [8, 9, 10, 11]). More precisely, we shall consider the following system of PDEs in the open, bounded and connected set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary:

$$\begin{cases} -\operatorname{div}(v\, D\rho(Du)) = f & \text{in } \Omega, \\ \rho(Du) \leq 1,\ v \geq 0 & \text{in } \Omega, \\ (1 - \rho(Du))v = 0 & \text{in } \Omega, \\ u \leq \phi & \text{on } \partial\Omega, \\ u = \phi & \text{on } \Gamma_f, \end{cases} \tag{2}$$

in the unknowns $v \in L^1(\Omega)$, $u \in W^{1,\infty}(\Omega)$. (Here and in the following we understand that $u \in W^{1,\infty}(\Omega)$ denotes the Lipschitz extension to $\overline{\Omega}$ of u.) In this formulation:

- $\rho\colon \mathbb{R}^n \to [0, +\infty)$ is the gauge function of a compact convex set $K \subseteq \mathbb{R}^n$, of class C^1 and containing the origin in its interior;
- $f \in L^1(\Omega)$, $f \geq 0$;
- $\phi\colon \partial\Omega \to \mathbb{R}^+$ is a lower semicontinuous function, $\phi \not\equiv +\infty$.

The plan of the paper is the following. After recalling some notation an basic results, in Section 3 we give the precise formulation of the problem, showing that under the assumptions listed above we do not have, in general, neither existence nor uniqueness of solutions (see Examples 3.4 and 3.5). In order to overcome these obstructions we then introduce an additional geometric assumption (see (H5) below), which guarantees that the mass is transported straight to the boundary. This condition is automatically satisfied if $\phi = 0$, while in the general case may fail, possibly causing both concentration of mass transportation on sets of lower dimension (described by measure-type transport densities) or branching of transport paths (and thus multiplicity of transport densities).

In Section 4 we prove that (2) always admit solutions, showing that there exists a non-negative function $v_f \in L^1(\Omega)$ such that the pair (u_ϕ, v_f) is a solution (being u_ϕ the Lax-Hopf function defined in (6) below). The main ingredient for this step is a disintegration formula for the Lebesgue measure proved by S. Bianchini in [3]. In addition, we prove a preliminary but fundamental uniqueness result: if (u_ϕ, v) is a solution to (2), then $v = v_f$. In order to get this result, we show that we need to strengthen the geometric assumption (H5) (see Example 3.8 and assumption (H6) below).

Section 5 is devoted to the characterization of all solutions and, consequently, to the uniqueness result. More precisely, we show that there exists a minimal profile u_f such that every solution to (2) is of the form (u, v_f) with $u_f \le u \le u_\phi$. In particular, the v-component is unique, whereas the u-component is unique (and coincides with u_ϕ) if and only if the support of the source f covers the set of the endpoints of the transport rays. Finally, in Section 6 we briefly rewrite our results in the isotropic case (1), and we discuss their interpretation in terms of granular matter models.

2. Notation and preliminaries

General notation. The standard scalar product of $x, y \in \mathbb{R}^n$ will be denoted by $\langle x,\, y\rangle$, while $|x|$ will denote the Euclidean norm of x. Concerning the segment joining x with y, we set

$$[\![x, y]\!] := \{tx + (1-t)y;\ t \in [0,1]\}, \qquad]\!]x, y[\![:= [\![x, y]\!] \setminus \{x, y\}.$$

Given a set $A \subset \mathbb{R}^n$, its interior, its closure and its boundary will be denoted by $\operatorname{int} A$, $\overline{A}$ and ∂A respectively.

We shall denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ respectively the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure. Given a measure μ and a μ-measurable set F, the symbol $\mu\lfloor F$ will denote the restriction of μ to the set F.

If $g\colon \Omega \to \mathbb{R}$ is a measurable function, we shall denote by $\operatorname{spt} g$ the essential support of g, that is the complement in Ω of the union of all relatively open subsets $A \subset \Omega$ such that $g = 0$ a.e. in A. Notice that $\operatorname{spt} g$ is a relatively closed set in Ω, but need not to be closed as a subset of $\mathbb{R}^n$.

Convex geometry. Let us now fix the notation and the basic results concerning the convex set which plays the röle of gradient constraint for the u-component in (2). In the following we shall assume that

(3) $\qquad K$ is a compact, convex subset of $\mathbb{R}^n$ of class C^1, with $0 \in \operatorname{int} K$.

Let us denote by K^0 the polar set of K, that is

$$K^0 := \{p \in \mathbb{R}^n;\ \langle p,\, x\rangle \le 1\ \forall x \in K\}\,.$$

We recall that, if K satisfies (3), then K^0 is a compact, strictly convex subset of $\mathbb{R}^n$ containing the origin in its interior, and $K^{00} = (K^0)^0 = K$ (see, e.g., [19]).

The gauge function $\rho\colon \mathbb{R}^n \to \mathbb{R}$ of K is defined by

$$\rho(\xi) := \inf\{t \ge 0;\ \xi \in tK\} = \max\{\langle \xi,\, \eta\rangle,\ \eta \in K^0\}\,, \quad \xi \in \mathbb{R}^n\,.$$

It is straightforward to see that ρ is a positively 1-homogeneous convex function such that $K = \{\xi \in \mathbb{R}^n\colon\ \rho(\xi) \le 1\}$. The gauge function of the set K^0 will be denoted by ρ^0.

The properties of the gauge functions needed in the paper are collected in the following theorem.

Theorem 2.1. *Assume that $K \subseteq \mathbb{R}^n$ satisfies (3). Then the following hold:*
(i) ρ is continuously differentiable in $\mathbb{R}^n \setminus \{0\}$, and

$$\rho(\xi + \eta) \leq \rho(\xi) + \rho(\eta) \quad \forall\, \xi, \eta \in \mathbb{R}^n .$$

(ii) K^0 is strictly convex, and

$$\rho^0(\xi + \eta) \leq \rho^0(\xi) + \rho^0(\eta), \quad \forall\, \xi, \eta \in \mathbb{R}^n$$
$$\rho^0(\xi + \eta) = \rho^0(\xi) + \rho^0(\eta) \ \Leftrightarrow\ \exists \lambda \geq 0 \ :\ \xi = \lambda\, \eta \ \text{or}\ \eta = \lambda\, \xi.$$

(iii) For every $\xi \neq 0$, $D\rho(\xi)$ belongs to ∂K^0, and

$$\langle D\rho(\xi), \xi\rangle = \rho(\xi), \quad \langle p, \xi\rangle < \rho(\xi)\ \forall p \in K^0,\ p \neq D\rho(\xi) .$$

Proof. See [19], Section 1.7. □

In what follows we shall consider $\mathbb{R}^n$ endowed with the possibly asymmetric norm $\rho^0(x - y)$, $x, y \in \mathbb{R}^n$. By Theorem 2.1(ii), the unit ball K^0 of ρ^0 is strictly convex but, under the sole assumption (3), it need not be differentiable. Moreover, the Minkowski structure $(\mathbb{R}^n, \rho^0)$ is not a metric space in the usual sense, since ρ^0 need not be symmetric (for an introduction to non-symmetric metrics see [14]). Finally, since K^0 is compact and $0 \in \operatorname{int} K^0$, then the convex metric is equivalent to the Euclidean one, that is there exist c_1, $c_2 > 0$ such that $c_1|\xi| \leq \rho^0(\xi) \leq c_2|\xi|$ for every $\xi \in \mathbb{R}^n$.

Curves. In the following Ω will denote an open, bounded, connected subset of $\mathbb{R}^n$ with Lipschitz boundary. Let us denote by $\Gamma_{y,x}$ the family of absolutely continuous paths in $\overline{\Omega}$ connecting y to x:

$$\Gamma_{y,x} := \{\gamma \in AC([0,1], \overline{\Omega}),\ \gamma(0) = y,\ \gamma(1) = x\} .$$

For every absolutely continuous curve $\gamma\colon [0,1] \to \mathbb{R}^n$, let us denote by $L(\gamma)$ its length with respect to the convex metric associated to ρ^0, that is

$$L(\gamma) := \int_0^1 \rho^0(\gamma'(t))\, dt .$$

Since $\overline{\Omega}$ is a compact subset of $\mathbb{R}^n$, by a standard compactness argument we have that for every $x, y \in \overline{\Omega}$ there exists a (distance) minimizing curve $\tilde{\gamma} \in \Gamma_{y,x}$ such that $L(\tilde{\gamma}) \leq L(\gamma)$ for every $\gamma \in \Gamma_{y,x}$ (see e.g. [1, Thm. 4.3.2], [7, §14.1]).

The main motivation for introducing the convex metric associated to ρ^0 is the fact that the Sobolev functions with the gradient constrained to belong to K are the locally 1-Lipschitz functions with respect to ρ^0, as stated in the following result (see [16, Chap. 5]).

Lemma 2.2. *Assume that the set $K \subset \mathbb{R}^n$ satisfies (3). Let ρ^0 be the gauge function of K^0, let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let $u\colon \Omega \to \mathbb{R}$. Then the following properties are equivalent.*

(i) *u is a locally 1-Lipschitz function with respect to ρ^0, i.e.*

$$u(x_2) - u(x_1) \leq \rho^0(x_2 - x_1) \quad \text{for every } [\![x_1, x_2]\!] \subset \Omega. \tag{4}$$

(ii) *$u \in W^{1,\infty}(\Omega)$, and $Du(x) \in K$ for a.e. $x \in \Omega$.*

(iii) *$u(x) - u(y) \leq L(\gamma)$ for every $x, y \in \Omega$ and every $\gamma \in \Gamma_{y,x}$.*

3. Formulation of the problem

In this section we shall give the definition of solution (u, v) to the PDEs system

$$\begin{cases} -\operatorname{div}(v\, D\rho(Du)) = f & \text{in } \Omega, \\ \rho(Du) \leq 1,\ v \geq 0 & \text{in } \Omega, \\ (1-\rho(Du))v = 0 & \text{in } \Omega, \\ u \leq \phi & \text{on } \partial\Omega, \\ u = \phi & \text{on } \Gamma_f. \end{cases} \tag{5}$$

The basic assumptions are:

(H1) Ω is an open, bounded, connected subset of $\mathbb{R}^n$ with Lipschitz boundary;
(H2) ρ is the gauge function of a convex set $K \subseteq \mathbb{R}^n$ satisfying (3);
(H3) f belongs to $L^1_+(\Omega)$, the set of non-negative integrable functions in Ω;
(H4) $\phi\colon \partial\Omega \to (-\infty, +\infty]$ is a lower semicontinuous (l.s.c.) function, $\phi \not\equiv +\infty$.

As we shall see in Examples 3.4, 3.5 and 3.8, this set of assumptions is not enough in order to have existence and uniqueness of solutions. Two additional assumptions on the geometry of the problem will be introduced in the remaining part of this section.

The set $\Gamma_f \subseteq \partial\Omega$ where the u component is forced to agree with the boundary datum ϕ, is dictated by the data of the problem. Using the terminology of the Optimal Transport Theory, Γ_f corresponds to the set of initial points of those transport rays on which the transport is active. For a rigorous definition, some additional notation is in order.

Let $u_\phi\colon \overline{\Omega} \to \mathbb{R}$ be the Lax-Hopf function defined by

$$u_\phi(x) := \inf\left\{\phi(y) + L(\gamma) :\ y \in \partial\Omega,\ \gamma \in \Gamma_{y,x}\right\}, \quad x \in \overline{\Omega}. \tag{6}$$

It is clear that for every $x \in \Omega$ the infimum in (6) is attained, that is there exist $y \in \partial\Omega$ and a minimizing curve $\gamma \in \Gamma_{y,x}$ such that $u_\phi(x) = \phi(y) + L(\gamma)$.

Definition 3.1. For $x \in \overline{\Omega}$ we call *geodesic through* x any curve $\gamma \in \Gamma_{y,x}$, $y \in \partial\Omega$, satisfying $u_\phi(x) = \phi(y) + L(\gamma)$. Moreover, we say that a geodesic through x is (forward) *maximal* if its image is not a proper subset of the image of another geodesic through x.

We recall that u_ϕ is a Lipschitz function in $\overline{\Omega}$, $\rho(Du_\phi) = 1$ a.e. in Ω, and it is the maximal function in the space X defined by

$$X := \left\{u \in W^{1,\infty}(\Omega) :\ Du \in K \text{ a.e. in } \Omega,\ u \leq \phi \text{ on } \partial\Omega\right\}.$$

(Since Ω has a Lipschitz boundary, we understand that all functions in $W^{1,\infty}(\Omega)$ are extended to Lipschitz continuous functions in $\overline{\Omega}$.) We shall show that the Lax-Hopf function u_ϕ is always an admissible u-component in (5), regardless of the source f (see Theorem 4.1).

The function u_ϕ dictates the geometry of the transportation, that is the mass produced by the source f runs along the geodesics associated to u_ϕ, and falls down at their initial points.

For every $x \in \overline{\Omega}$ we denote by $\Pi(x)$ the full set of projections of x on $\partial\Omega$, i.e.

$$\Pi(x) := \{y \in \partial\Omega :\ \exists \text{ a geodesic } \gamma \in \Gamma_{y,x} \text{ through } x\}, \tag{7}$$

whereas $\Pi^M(x)$ will denote the set of maximal projections, i.e.

$$\Pi^M(x) := \{y \in \partial\Omega : \exists \text{ a maximal geodesic } \gamma \in \Gamma_{y,x} \text{ through } x\}.$$

Let us define the set of the initial points of maximal geodesics

$$\Gamma_\phi := \{y \in \partial\Omega\colon \exists \text{ a maximal geodesic } \gamma \in \Gamma_{y,x},\ x \in \Omega\} = \bigcup_{x\in\Omega} \Pi^M(x) \tag{8}$$

and the subset of Γ_ϕ of the initial points of those maximal geodesics where the source f is active

$$\Gamma_f := \{y \in \partial\Omega\colon \exists \text{ maximal geodesic } \gamma \in \Gamma_{y,x},\ x \in \operatorname{spt} f\} = \bigcup_{x\in\operatorname{spt} f} \Pi^M(x). \tag{9}$$

The set Γ_f turns out to be, in the sandpile problem, the actual portion of Γ_ϕ (depending on the source of matter f) where the sand falls down. The set $\partial\Omega \setminus \Gamma_\phi$ is the part of the boundary closed by the walls.

We are now in a position to fix the rigorous meaning of problem (5). The functional setting for the unknowns (u, v) in (5) is $X_f \times L^1_+(\Omega)$, where

$$X_f := \{u \in X : u = \phi \text{ on } \Gamma_f\}, \quad L^1_+(\Omega) := \{v \in L^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}.$$

Definition 3.2. A pair (u, v) is a solution to (5) if

(i) $(u, v) \in X_f \times L^1_+(\Omega)$;
(ii) $(1 - \rho(Du))v = 0$ a.e. in Ω;
(iii) for every $\psi \in C^\infty_c\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$

$$\int_\Omega v\langle D\rho(Du),\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx.$$

Remark 3.3. This notion of solution generalizes the one given in [13] for the table problem with walls. The weak formulation in (iii) corresponds to the continuity equation for the optimal transport problem subjected to the condition that the mass can flow away from Ω only through Γ_f.

The following two examples illustrate some points that should be taken into account in order to deal with existence and uniqueness results for (5). In order not to interrupt the main flow of the exposition, the details are postponed to Section 7.

Example 3.4. Under our general assumptions (H1)–(H4), problem (5) need not admit a solution in the sense of Definition 3.2. Namely, let $\Omega := \Omega_1 \cup \Omega_2 \subset \mathbb{R}^2$, where

$$\Omega_1 := \left\{(r\cos\theta,\, r\sin\theta) : 0 < \theta \leq \frac{\pi}{2},\ 0 < r < 1\right\},$$
$$\Omega_2 := \left\{x = (x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < 0,\ |x_2| < 1\right\}$$

(see Figure 1). Let $\phi\colon \partial\Omega \to \mathbb{R}$ be the function

$$\phi(x) = \begin{cases} 0, & x \in S := [-1, 0] \times \{-1\}, \\ +\infty, & \text{otherwise.} \end{cases}$$

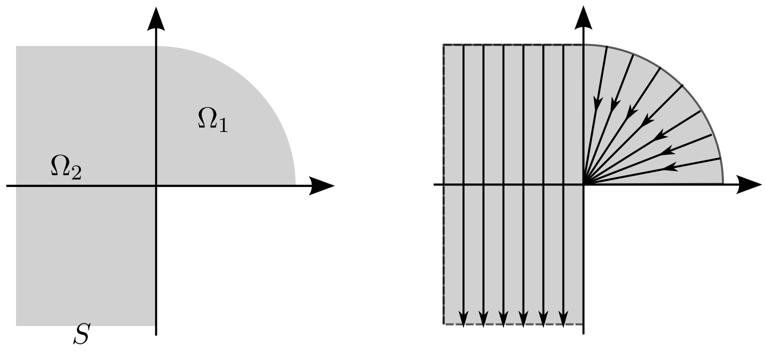


FIGURE 1. The set Ω in Example 3.4

Let us consider the isotropic case ($\rho(\xi) = |\xi|$) with a constant source $f \equiv 1$. The Lax-Hopf function is

$$u_\phi(x) = \begin{cases} 1 + |x|, & \text{if } x \in \Omega_1, \\ 1 + x_2, & \text{if } x \in \Omega_2, \end{cases}$$

so that $\Gamma_\phi = \Gamma_f = (-1, 0] \times \{-1\}$, and the geometry of the geodesics is the one depicted in Figure 1 right. In particular the mass collected in the region Ω_1 is transported to the origin, and from the origin there is a unique transport ray R going to Γ_ϕ. The concentration of the mass on the single ray R corresponds to the fact that the transport density v should be a measure with a singular part concentrated on R, which is not allowed in Definition 3.2.

Example 3.5. Even in the case of existence of solutions to (5), the uniqueness of the transport density v may fail if geodesics can bifurcate in the interior. Namely, let $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \mathbb{R}^2$, where

$$\begin{aligned} \Omega_1 &:= \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \ -1 < x_1 < 0,\ |x_2| < 1\right\}, \\ \Omega_2 &:= \left\{(r\cos\theta,\, r\sin\theta) : \ \frac{\pi}{4} < \theta \le \frac{\pi}{2},\ 0 < r < 1\right\}, \\ \Omega_3 &:= \left\{(r\cos\theta,\, r\sin\theta) : \ -\frac{\pi}{2} \le \theta < -\frac{\pi}{4},\ 0 < r < 1\right\} \end{aligned}$$

(see Figure 2). Let $\phi\colon \partial\Omega \to \mathbb{R}$ be the function defined by

$$\phi(x) = \begin{cases} 0, & x \in S_1 \cup S_3, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $S_1 := [-1, 0] \times \{-1\}$ and $S_3 := \{(\cos\theta, \sin\theta) : \ -\pi/2 < \theta \le -\pi/4\}$. Let us consider the isotropic case ($\rho(\xi) = |\xi|$) with a constant source $f \equiv 1$.

In this example the multiplicity of v-components depends on the fact that geodesics are not forward unique. Namely, given a point $y \in S_3$ and a point $z \in \Omega_2$, the curve $\gamma := [\![y, 0]\!] \cup [\![0, z]\!]$ is a geodesic. It is then clear that we have a lot of geodesics branching at 0, so that the mass collected in the region Ω_2 is transported to the origin, and from the origin it can be distributed in infinitely many ways to any point of S_3.

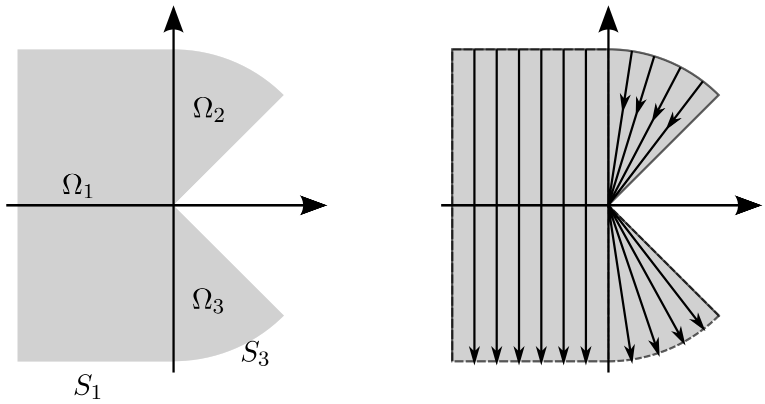


FIGURE 2. The set Ω in Example 3.5

In order to exclude the phenomena depicted in Examples 3.4 and 3.5, we propose the following additional geometric assumption:

(H5) For every $x \in \Omega$ and for every $y \in \Pi(x)$ the segment $]\!]y, x[\![$ is contained in Ω.

As a consequence of (H5), every geodesic through a point $x \in \Omega$ is maximal and its support is a segment. In particular, the geodesics cannot bifurcate (i.e., they are non-branching in the interior).

Under the assumption (H5), the Lax-Hopf function can be written as

$$u_\phi(x) := \min\left\{\phi(y) + \rho^0(x-y) : \ y \in \partial\Omega\right\}, \quad x \in \overline{\Omega}, \tag{10}$$

while

$$\Gamma_\phi = \{y \in \partial\Omega \colon \ \exists\, x \in \Omega \text{ s.t. }]\!]y, x[\![\subset \Omega \text{ and } u_\phi(x) = \phi(y) + \rho^0(x-y)\},$$
$$\Gamma_f = \{y \in \partial\Omega \colon \ \exists\, x \in \operatorname{spt} f \text{ s.t. }]\!]y, x[\![\subset \Omega \text{ and } u_\phi(x) = \phi(y) + \rho^0(x-y)\}.$$

For every $x \in \Omega$, let $\Delta(x)$ be the set of directions through x

$$\Delta(x) := \left\{\frac{x-y}{\rho^0(x-y)} : \ y \in \Pi(x)\right\}, \qquad x \in \Omega,$$

where $\Pi(x)$ is the set of projections defined in (7) that, under assumption (H5), can be written as

$$\Pi(x) = \{y \in \Gamma_\phi \colon \ u_\phi(x) = \phi(y) + \rho^0(x-y)\}.$$

Let $D \subset \Omega$ be the set of those points with multiple projections, that is

$$D := \{x \in \Omega : \ \Delta(x) \text{ is not a singleton}\},$$

and for every $x \in \Omega \setminus D$, let $p(x)$ and $d(x)$ denote the unique elements in $\Pi(x)$ and $\Delta(x)$ respectively, i.e.

$$\{p(x)\} = \Pi(x), \quad \{d(x)\} = \Delta(x), \qquad x \in \Omega \setminus D. \tag{11}$$

It can be easily checked that u_ϕ grows linearly along every segment joining $x \in \Omega$ to $y \in \Pi(x)$. Let us denote by $b(x)$ be the normal distance from the set D, defined by

$$b(x) := \begin{cases} \sup\{t \geq 0;\ u_\phi(x + sd(x))) = u_\phi(x) + s,\ \forall s \in [0, t]\}, & x \in \Omega \setminus D\,, \\ 0 & x \in D\,, \end{cases}$$

and let J be the set

$$J := \bigcup_{x\in\Omega} q(x), \quad q(x) := x + b(x)d(x),$$

where we understand that $q(x) = x$ if $x \in D$.

Definition 3.6 (Transport ray). We shall call *transport ray* through $x \in \Omega$ any segment $[\![p, q(x)]\!]$, $p \in \Pi(x)$. If $[\![p, q]\!]$ is a transport ray, the points p and q will be called respectively the *initial* and the *final point* of the ray.

It is clear from the definition that, if $x \in \Omega \setminus D$, then there is a unique transport ray $[\![p(x), q(x)]\!]$ through x. In this case there exists a unique number $a(x) \in (-\infty, 0)$ such that

$$p(x) = x + a(x)\, d(x) \qquad (x \in \Omega \setminus D).$$

Moreover, if assumption (H5) holds, according to Definition 3.1 the segment $[\![p(x), x]\!]$ is a maximal geodesic through x. On the other hand, if $x \in D$, any segment $[\![p, x]\!]$, with $p \in \Pi(x)$, is a transport ray through x.

The transport rays correspond to the segments where u_ϕ grows linearly with maximal slope, i.e.

$$u_\phi(x + t\, d(x)) = u_\phi(x) + t \qquad \forall x \in \Omega \setminus D,\ \forall t \in [a(x), b(x)]\,. \tag{12}$$

Let Σ be the set of those points where u_ϕ is not differentiable. The relationships between the singular sets related to the problem are the following (see [3] and [12, Prop. 6.4]).

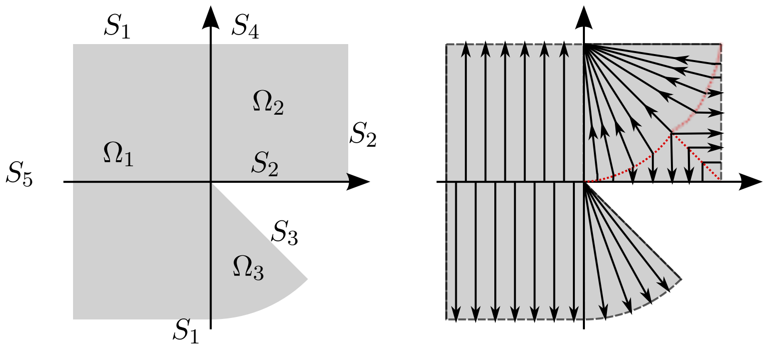
Proposition 3.7. *The sets Σ, D and J have zero Lebesgue measure. Moreover, $D \subset \Sigma$ and $D \subset J \subset \overline{D}$, with possibly strict inclusions. In addition, if u_ϕ is differentiable at $x \in \Omega$, then x has a unique projection and $\Delta(x) = \{D\rho(Du_\phi(x))\}$.*

Under the assumptions (H1)–(H5) we will be able to construct a mass transport density v_f such that the pair (u_ϕ, v_f) solves the system (5) (see Section 4). Unfortunately, these assumptions are not enough to ensure the uniqueness of the mass density v, due to the possibility of transporting a fictitious amount of mass along those transport rays $]\!]p, q[\![$ with both endpoints on $\partial\Omega$. The weak formulation of the continuity equation (Definition 3.2(iii)) prevents this possibility only for $q \in \partial\Omega \setminus \overline{\Gamma}_f$. The following example shows that a single $q \in \overline{\Gamma}_f$ may be the final point of a set of rays covering a region with positive Lebesgue measure.

Example 3.8. Let $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \mathbb{R}^2$, where

$$\Omega_1 := (-1, 0) \times (-1, 1), \qquad \Omega_2 := [0, 1) \times (0, 1),$$

$$\Omega_3 := \left\{(r\cos\theta, r\sin\theta) :\ r \in (0, 1),\ -\frac{\pi}{2} < \theta < -\frac{\pi}{4}\right\}\,,$$



and let $\phi\colon \partial\Omega \to \mathbb{R}$ be a function such that, in the isotropic case with source $f = 1$, the geometry of transport rays is the one depicted in Figure 3. In particular $\Gamma_\phi = \Gamma_f = S_1 \cup S_2$, where

$$S_1 := ((-1, 0] \times \{-1, 1\}) \cup \{(\cos\theta, \sin\theta) : -\frac{\pi}{2} < \theta < -\frac{\pi}{4}\}\,.$$
$$S_2 := ((0, 1) \times \{0\}) \cup (\{1\} \times (0, 1))$$

(see the details in Section 7). Notice that $(0, 0) \in \overline{\Gamma}_\phi$ is the final point of all the transport rays covering Ω_3. Hence, we can construct a $w \in L^1_+(\Omega_3)$ such that $\mathrm{div}(w\, Du_\phi) = 0$, which can be added, once prolonged to zero in $\Omega \setminus \Omega_3$, to any admissible v, loosing the uniqueness of mass transport density.

In order to exclude the behaviour described in Example 3.8 we need the following additional assumption:

(H6) $\left(\overline{J \cap \partial\Omega}\right) \cap \overline{\Gamma}_\phi = \emptyset$.

It is worth to remark that, as a consequence of (H5), one has $(J \cap \partial\Omega) \cap \Gamma_\phi = \emptyset$ (see Lemma 3.9 below), so that (H6) can be viewed as a mild additional assumption in order to separate the sets $J \cap \partial\Omega$ and Γ_ϕ.

Lemma 3.9. *Assume that (H1), (H2), (H4) and (H5) hold, and let $[\![p_1, q_1]\!]$, $[\![p_2, q_2]\!]$ be two distinct non-trivial transport rays. Then $q_1 \neq p_2$. In other words, $J \cap \Gamma_\phi = \emptyset$.*

Proof. Assume by contradiction that $q_1 = p_2$, and let $\gamma \in \Gamma_{p_1,q_2}$ be the curve whose support is $[\![p_1, p_2]\!] \cup [\![p_2, q_2]\!]$. By (12) we get

$$u_\phi(q_2) = u_\phi(p_2) + \rho^0(q_2 - p_2) = u_\phi(p_1) + \rho^0(p_2 - p_1) + \rho^0(q_2 - p_2) = u_\phi(p_1) + L(\gamma),$$

hence the curve γ is a geodesic, in contradiction with (H5). □

4. Existence of solutions

For the reader's convenience we collect here all the assumptions that have been introduced in the previous section.

(H1) Ω is an open, bounded, connected subset of $\mathbb{R}^n$ with Lipschitz boundary;
(H2) ρ is the gauge function of a convex set $K \subseteq \mathbb{R}^n$ satisfying (3);
(H3) f belongs to $L^1_+(\Omega)$;
(H4) $\phi\colon \partial\Omega \to (-\infty, +\infty]$ is a l.s.c. function, $\phi \not\equiv +\infty$;
(H5) For every $x \in \Omega$ and for every $y \in \Pi(x)$ the segment $]\!]y, x[\![$ is contained in Ω;

(H6) $\left(\overline{J \cap \partial\Omega}\right) \cap \overline{\Gamma}_\phi = \emptyset$.

This section will be devoted to the proof of the following result.

Theorem 4.1. *Let (H1)-(H6) hold. Then there exists a unique $v_f \in L^1_+(\Omega)$ such that the pair (u_ϕ, v_f) is a solution to the PDEs system (5).*

This theorem gives a partial uniqueness result for the transport density, since it states that the v-component associated to the profile $u = u_\phi$ is unique. We shall see in Section 5, Theorem 5.5, that, indeed, v_f is the only admissible v-component of every solution (u, v).

Since $u_\phi \in X_f$, and $\rho(Du_\phi) = 1$ a.e. in Ω, in order to prove the existence part of Theorem 4.1 it is enough to show that

$$\int_\Omega v_f \langle D\rho(Du_\phi),\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall \psi \in C^\infty_c\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right). \tag{13}$$

This can be done with a minor effort, since it turns out that we can choose as v-component the same function constructed in [3, 12] for the case of admissible boundary data, that is the one that satisfies

$$\int_\Omega v_f \langle D\rho(Du_\phi),\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall \psi \in C^\infty_c(\Omega). \tag{14}$$

In order to write the explicit form of the function v_f, we need the fundamental result concerning the disintegration of the Lebesgue measure along the transport rays due to S. Bianchini (see [3, Thm. 5.8]). The proof of this formula is mainly based on the fact that the divergence of the vector field d defined in (11) is a locally finite Radon measure in Ω. Moreover, decomposing $\operatorname{div} d$ into its absolutely continuous and singular part (w.r.t. the Lebesgue measure), $\operatorname{div} d = (\operatorname{div} d)_{ac}\mathcal{L}^n + (\operatorname{div} d)_s$, it turns out that $(\operatorname{div} d)_s$ is a positive measure in Ω (see [3], Section 5). These properties allow to describe the evolution of the Hausdorff measure of the $(n-1)$-dimensional sections of "cylinders" of transport rays (see Remark 4.3).

Theorem 4.2. *Let (H1), (H2) and (H4) hold. Then there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ of subsets of Ω with the following properties.*

(i) *$A_k \subset (\Omega \setminus D) \cap \{\langle x,\, e_{j_k}\rangle = z_k\}$ for some $j_k \in \{1, \dots, n\}$ and $z_k \in \mathbb{R}$.*
(ii) *A_k is measurable w.r.t. the $(n-1)$-dimensional Hausdorff measure.*
(iii) *The sets $T_k := \bigcup_{x\in A_k}]\!]p(x), q(x)[\![$, $k \in \mathbb{N}$, are pairwise disjoint, Lebesgue measurable subsets of Ω, and $\mathcal{L}^n(\Omega \setminus \bigcup_k T_k) = 0$.*
(iv) *For every $h \in L^1(\Omega)$, the function $t \mapsto h(x+td(x))\, \alpha(td_{j_k}(x), x)$, where $d_{j_k}(x) := \langle d(x),\, e_{j_k}\rangle$, belongs to $L^1(a(x), b(x))$ for every $x \in \bigcup_k A_k$. Moreover, the disintegration formula*

$$\int_\Omega h\, dx = \sum_k \int_{A_k} \left(\int_{a(x)}^{b(x)} h(x + td(x))\, \alpha(td_{j_k}(x), x)\, dt \right) d_{j_k}(x)\, d\mathcal{H}^{n-1}(x)$$

holds, and for $\mathcal{H}^{n-1}$-a.e. $x \in \bigcup_k A_k$ the function $\alpha(\cdot, x)$ is the solution of the linear ODE

$$\begin{cases} \dfrac{d}{dt}\alpha(td_{j_k}(x), x) = (\operatorname{div} d)_{ac}(x + td(x))\, \alpha(td_{j_k}(x), x), & t \in (a(x), b(x))\,, \\ \alpha(0, x) = 1\,. \end{cases} \tag{15}$$

Moreover, this solution is strictly positive.

We remark that the decomposition introduced in Theorem 4.2 is clearly not unique. In the following we shall always assume that such a decomposition has been fixed. Once the decomposition is given, we shall use the notation

$$\alpha(x+td(x)) \equiv \alpha(td_{j_k}(x),x)\, d_{j_k}(x), \qquad k\in\mathbb{N},\ x\in A_k,\ t\in(a(x),b(x)),$$

so that α is defined and strictly positive on the set

$$\Omega' := \left\{ x+td(x):\ x\in\bigcup_k A_k;\ t\in(a(x),b(x)) \right\} = \bigcup_k T_k \tag{16}$$

i.e., by Theorem 4.2(iii), almost everywhere on Ω. For every $x\in\Omega'$ the ODE (15) along transport rays can be written as

$$\frac{d}{dt}\alpha(x+td(x)) = (\operatorname{div} d)_{ac}(x+td(x))\,\alpha(x+td(x)),$$

and $\alpha=1$ on $\bigcup_k A_k$, while the disintegration formula for the Lebesgue measure given in Theorem 4.2(iv) simply becomes

$$\int_\Omega h\,dx = \sum_k \int_{A_k}\left(\int_{a(x)}^{b(x)} h(x+td(x))\alpha(x+td(x))\,dt\right)\, d\mathcal{H}^{n-1}(x)\,. \tag{17}$$

Remark 4.3. It can be of interest to understand the geometrical meaning of the function $\alpha(t,x)$ in Theorem 4.2(iv). For fixed $k\in\mathbb{N}$ and $t\in\mathbb{R}$, let us define the map

$$\varphi^t\colon A_k\to\mathbb{R}^n, \qquad x\mapsto\varphi^t(x) := x+t\,\frac{d(x)}{d_{j_k}(x)}\,,$$

and let $A_k^t := \varphi^t(A_k)$. Since A_k is contained in the hyperplane $\{x:\ \langle x,\, e_{j_k}\rangle = z_k\}$, we have that $A_k^t\subset\{x: \langle x,\, e_{j_k}\rangle = z_k+t\}$. Let us consider the measure μ on A_k such that the push-forward $(\varphi^t)_\sharp\mu$ of μ is $\mathcal{H}^{n-1}\lfloor A_k^t$, that is,

$$\mu\left((\varphi^t)^{-1}(F)\right) = \mathcal{H}^{n-1}(F) \qquad \text{for every } \mathcal{H}^{n-1}\text{-measurable set } F\subseteq A_k^t. \tag{18}$$

It turns out that the measure μ defined by (18) is absolutely continuous w.r.t. $\mathcal{H}^{n-1}\lfloor A_k$ and $\mu=\alpha(t,x)\mathcal{H}^{n-1}\lfloor A_k$ (see [3, Lemma 5.4]).

Let $C\subset\Omega$ be a "cylinder" of rays of the form

$$C=\{x+sd(x):\ x\in A_k,\ s\in(a(x),b(x))\}\ ,$$

and let χ_C denote its characteristic function. Then, for every $h\in L^1(C)$, using Fubini's theorem and the definition of μ we have that

$$\begin{aligned}
\int_C h\,d\mathcal{L}^n &= \int_{\mathbb{R}}\int_{A_k^t} h\,\chi_C\,d\mathcal{H}^{n-1}\,dt\\
&= \int_{\mathbb{R}}\int_{A_k^t} (h\,\chi_C)(\varphi^t(x))\alpha(t,x)\,d\mathcal{H}^{n-1}(x)\,dt\\
&= \int_{A_k}\int_{a(x)}^{b(x)} h(x+s\,d(x))\alpha(sd_{j_k}(x),x)d_{j_k}(x)\,ds\,d\mathcal{H}^{n-1}(x)\,.
\end{aligned}$$

Hence, roughly speaking, the disintegration formula (17) corresponds to covering Ω with cylinders of transport rays and applying Fubini's theorem on each cylinder.

We are now in a position to show that the problem of finding the solution v to the equation

$$\int_\Omega v\langle D\rho(Du_\phi),\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall \psi \in C_c^\infty\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right) \tag{19}$$

can be solved by the Method of Characteristics. We start recalling another result proved in [3, Sect. 7].

Theorem 4.4. *Let (H1)–(H4) hold. If $v \in L^1_+(\Omega)$ satisfies*

$$\int_\Omega v\langle D\rho(Du_\phi),\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,, \qquad \forall \psi \in C_c^\infty(\Omega), \tag{20}$$

then for a.e. $x \in \Omega$ the function v is locally absolutely continuous along the ray $t \mapsto x + td(x)$, $t \in (a(x), b(x))$, and satisfies

$$\frac{d}{dt}\left[v(x+td(x))\,\alpha(x+td(x))\right] = -f(x+td(x))\,\alpha(x+td(x))\,, \quad t \in (a(x), b(x)). \tag{21}$$

Moreover, if the final point $q(x) := x + b(x)d(x)$ belongs to Ω, then

$$\lim_{t\to b(x)^-} v(x+td(x))\,\alpha(x+td(x)) = 0. \tag{22}$$

Finally, the function $v_f \colon \Omega \to \mathbb{R}$ defined by

$$v_f(x) := \int_0^{b(x)} f(x+td(x))\,\frac{\alpha(x+td(x))}{\alpha(x)}\, dt, \qquad a.e.\ x \in \Omega\,, \tag{23}$$

belongs to $L^1_+(\Omega)$ and is a solution to (20).

Remark 4.5. The fact that v_f satisfies (21) along almost every ray can be easily proved observing that, for every $x \in \Omega'$,

$$v_f(x+td(x)) = \int_t^{b(x)} f(x+sd(x))\,\frac{\alpha(x+sd(x))}{\alpha(x+td(x))}\, ds,$$

so that

$$(v_f\,\alpha)(x+td(x)) = \int_t^{b(x)} (f\,\alpha)(x+sd(x))\, ds.$$

From this last equality we also see that v_f satisfies the terminal condition (22) for every $x \in \Omega'$, and not only for those points $x \in \Omega$ satisfying $q(x) \in \Omega$.

Remark 4.6. From the definition (23) of v_f and the fact that $\alpha(\cdot, x)$ is strictly positive in $(a(x), b(x))$, we deduce that the essential support of v_f coincides with the closure of the set

$$\left\{\bigcup \rrbracket p(x), x\llbracket :\ x \in \operatorname{spt} f\right\}.$$

In particular $v_f = 0$ along all the transport rays starting from the points of $\Gamma_\phi \setminus \Gamma_f$.

Remark 4.7. Notice that neither (H5) nor (H6) are needed in order to find a solution to (20). Namely, the weak formulation (20) corresponds to the continuity equation of the optimal mass transport problem in which the mass can freely flow away from Ω at the first time it touches the boundary. Hence, the phenomena depicted in Examples 3.4 and 3.5 cannot happen.

Since all functions $\psi \in C_c^\infty(\Omega)$ (extended to 0 in $\mathbb{R}^n \setminus \Omega$) are admissible in the weak formulation (19) of the transport equation, it is clear that every solution v to (19) satisfies (21) along almost every ray.

Using the disintegration formula for the Lebesgue measure it is not difficult to prove that the function v_f defined in (23) above is a solution to (13), provided that the mass can flow away at the initial points of the transport rays, i.e.

$$\left\{ p(x) :\ x \in \bigcup_k A_k,\ \rrbracket p(x), q(x) \llbracket \cap \operatorname{spt}(f) \neq \emptyset \right\} \subseteq \Gamma_f .$$

This condition is clearly satisfied if (H5) holds.

Theorem 4.8. *Assume that* (*H*1)*-*(*H*5) *hold. Then* (u_ϕ, v_f) *is a solution to* (14)*.*

Proof. From Theorem 4.4, the function v_f satisfies the ODE (21) along almost every ray $x + td(x)$, $t \in (a(x), b(x))$. Moreover, as already observed in Remark 4.5,

$$(v_f\alpha)(q(x)) := \lim_{t \to b(x)^-} (v_f\alpha)(x + td(x)) = 0, \qquad \mathcal{H}^{n-1}\text{-a.e. } x \in \bigcup_k A_k. \tag{24}$$

It is not restrictive to assume that (21) and (24) hold for every x belonging to the set Ω' defined in (16).

Since, by Proposition 3.7, $\Omega' \subseteq \Omega \setminus J \subseteq \Omega \setminus D$, we have that, for every $x \in \Omega'$, $d(x) = D\rho(Du_\phi(x))$ and, for $\psi \in C_c^\infty\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$,

$$\langle D\rho(Du_\phi))(x + td(x)),\, D\psi(x + td(x))\rangle = \langle d(x),\, D\psi(x + td(x))\rangle = \psi'(x + td(x))\,,$$

where the prime denotes differentiation w.r.t. t. Hence the disintegration formula (17) gives

$$\begin{aligned} &\int_\Omega v_f \langle D\rho(Du_\phi),\, D\psi\rangle\, dx \\ &= \sum_k \int_{A_k} \int_{a(x)}^{b(x)} (v_f\alpha)(x + td(x))\psi'(x + td(x))\, dt\, d\mathcal{H}^{n-1}(x)\,. \end{aligned} \tag{25}$$

An integration by parts in the inner integral of (25) and formula (21) lead to

$$\begin{aligned} &\int_{a(x)}^{b(x)} (v_f\alpha)(x + td(x))\psi'(x + td(x))\, dt \\ &= (v_f\alpha\psi)(q(x)) - (v_f\alpha\psi)(p(x)) + \int_{a(x)}^{b(x)} (f\alpha)(x + td(x))\psi(x + td(x))\, dt\,, \end{aligned} \tag{26}$$

where we use the convention

$$(v_f\alpha)(p(x)) := \lim_{t \to a(x)+} (v_f\alpha)(x + td(x)).$$

If $p(x)$ belongs to $\overline{\Gamma}_f$ then the test function ψ vanishes in a neighbourhood of $p(x)$. On the other hand, if $p(x) \notin \overline{\Gamma}_f$, then f vanishes on the ray $]\!]p(x), q(x)[\![$, so that, by (21), $v_f\alpha$ is constant along that ray. Since $(v_f\alpha)(q(x)) = 0$, then also $(v_f\alpha)(p(x)) = 0$.

Finally, using again the disintegration formula, the weak formulation (19), (24), and the boundary conditions $(v_f\alpha\psi)(p(x)) = (v_f\alpha)(q(x)) = 0$, we conclude that v_f is a solution to (19), i.e. the pair (u_ϕ, v_f) is a solution to (5). $\square$

In order to prove a uniqueness result we need to exploit assumption (H6).

Theorem 4.9. *Assume that* $(H1)$-$(H6)$ *hold. Then a function* $v \in L^1_+(\Omega)$ *satisfies* (19) *if and only if, for a.e.* $x \in \Omega \setminus D$*, the function* v *is locally absolutely continuous along the ray* $t \mapsto x + td(x)$*,* $t \in (a(x), b(x))$ *and*

$$\begin{cases} \dfrac{d}{dt}\left[v(x+td(x))\,\alpha(x+td(x))\right] = -f(x+td(x))\,\alpha(x+td(x)), \\ \lim\limits_{t\to b(x)^-} v(x+td(x))\,\alpha(x+td(x)) = 0. \end{cases} \tag{27}$$

As a consequence, the function v_f *defined in* (23) *is the unique solution of* (19).

Proof. By Theorem 4.2(iii) it is clear that two functions both satisfying (27) along almost every ray must coincide almost everywhere. Hence the uniqueness result will be achieved once we prove that every solution to (19) satisfies (27).

Let $v \in L^1_+(\Omega)$ be any solution to (19), and let $\psi \in C^\infty_c\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$. Notice that v is a solution to (20); by Theorem 4.4, it is not restrictive to assume that (21) holds for every x belonging to the set Ω' defined in (16), whereas (22) holds for every $x \in \Omega'$ such that $q(x) \in \Omega$.

Reasoning as in the proof of Theorem 4.8 (see (25) and (26) with v instead of v_f), and using the fact that v is a solution to (19), we obtain

$$\sum_k \int_{A_k} \left[(v\alpha\psi)(q(x)) - (v\alpha\psi)(p(x))\right] d\mathcal{H}^{n-1}(x) = 0\,.$$

By (22), if $q(x)$ belongs to Ω, then $(v\alpha)(q(x)) = 0$. Moreover, if $p(x)$ belongs to $\overline{\Gamma}_f$ then the test function ψ vanishes in a neighbourhood of $p(x)$. Hence the formula above is equivalent to

$$\sum_k \int_{B_k} (v\alpha\psi)(q(x))\, d\mathcal{H}^{n-1}(x) - \sum_k \int_{C_k} (v\alpha\psi)(p(x))\, d\mathcal{H}^{n-1}(x) = 0\,, \tag{28}$$

where

$$B_k := \{x \in A_k :\ q(x) \in \partial\Omega\}\,, \qquad C_k := \{x \in A_k :\ p(x) \notin \overline{\Gamma}_f\}\,.$$

Let us consider the following subsets of $\partial\Omega$:

$$\mathcal{B} := \left\{q(x) :\ x \in \bigcup_k B_k\right\} \subseteq J \cap \partial\Omega, \qquad \mathcal{C} := \left\{p(x) :\ x \in \bigcup_k C_k\right\} \subseteq \Gamma_\phi \setminus \overline{\Gamma}_f\,.$$

From the very definition of $\mathcal{C}$, (H6) and the inclusion $\Gamma_f \subseteq \Gamma_\phi$ we deduce that

$$\overline{\mathcal{B}} \cap \overline{\Gamma}_f = \emptyset, \qquad \overline{\mathcal{B}} \cap \overline{\mathcal{C}} = \emptyset. \tag{29}$$

Hence, choosing $\delta > 0$ such that $B_\delta(\mathcal{B}) \cap \mathcal{C} = B_\delta(\mathcal{B}) \cap \Gamma_f = \emptyset$, by (29) we can construct a function $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\psi > 0 \quad \text{on } \mathcal{B}, \qquad \psi = 0 \quad \text{on } [\mathbb{R}^n \setminus B_\delta(\mathcal{B})] \supset \Gamma_f \cup \mathcal{C}\,,$$

so that $\psi \in C_c^\infty\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$. Recalling that $v\alpha \geq 0$, and using ψ as a test function in (28) we conclude that $(v\alpha)(q(x)) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in B_k$, $k \in \mathbb{N}$, so that the boundary condition in (27) holds. □

Another easy consequence of both the representation formula for v_f and the disintegration (17) is the following stability result.

Theorem 4.10 (Stability)**.** *For every $f_1, f_2 \in L^1_+(\Omega)$ there holds*

$$\|v_{f_1} - v_{f_2}\|_{L^1(\Omega)} \leq \operatorname{diam}(\Omega)\, \|f_1 - f_2\|_{L^1(\Omega)}\,,$$

where $\operatorname{diam}(\Omega)$ *denotes the diameter of the set* Ω.

Proof. From the disintegration formula (17), the representation formula (23) and Remark 4.5 we have that

$$\begin{aligned}
\|v_{f_1} - v_{f_2}\|_{L^1(\Omega)} &= \sum_k \int_{A_k} \left(\int_{a(x)}^{b(x)} |(v_{f_1} - v_{f_2})(x + td(x))|\, \alpha(x + td(x))\, dt \right) d\mathcal{H}^{n-1}(x) \\
&\leq \sum_k \int_{A_k} \left[\int_{a(x)}^{b(x)} \left(\int_t^{b(x)} (|f_1 - f_2|\, \alpha)(x + sd(x)) ds \right) dt \right] d\mathcal{H}^{n-1}(x) \\
&\leq \operatorname{diam}(\Omega) \sum_k \int_{A_k} \left(\int_{a(x)}^{b(x)} (|f_1 - f_2|\, \alpha)(x + sd(x)) ds \right) d\mathcal{H}^{n-1}(x) \\
&= \operatorname{diam}(\Omega)\, \|f_1 - f_2\|_{L^1(\Omega)}\,,
\end{aligned}$$

completing the proof. □

5. Uniqueness of the solutions

Up to now we have proved that there exists a unique $v = v_f$ such that the pair (u_ϕ, v_f) solves (5). In this section we shall prove that, actually, v_f is the unique admissible v–component for (5).

For what concerns the u component, we start by proving that the Monge–Kantorovich system (5) is the Euler-Lagrange condition for the minimum problem with gradient constraint

$$\min\left\{ -\int_\Omega f u \, dx :\ u \in X_f \right\}. \tag{30}$$

It is clear that, since f is non-negative and u_ϕ is the maximal element in X_f, then u_ϕ is a solution to (30). Moreover every u which minimizes (30) has to agree with u_ϕ on $\operatorname{spt} f$.

Theorem 5.1. *Assume that* $(H1)$*–*$(H5)$ *hold. The minimum problem* (30) *and the system of PDEs* (5) *are equivalent in the following sense.*

(i) *$u \in X_f$ is a solution to* (30) *if and only if there exists $v \in L^1_+(\Omega)$ such that (u, v) is a solution to* (5).

(ii) *Let* $u \in X_f$ *be a solution to* (30). *Then* (u, v) *is a solution to* (5) *if and only if* (u_ϕ, v) *is a solution to* (5).

Proof. The proof is similar to the one of Theorem 5.3 in [12]; for the reader's convenience we sketch here the main steps.

In what follows we will freely use that fact that, by a density argument, the difference $u - w$ of two functions $u, w \in X_f$ can be used as test function in the weak formulation (19) of the transport equation.

Let us denote by I_K the indicator function of the set K, that is

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K\,, \\ +\infty & \text{if } x \in \mathbb{R}^n \setminus K\,, \end{cases}$$

so that

$$F(u) := \int_\Omega [I_K(Du) - fu]\, dx = -\int_\Omega fu\, dx, \qquad \forall u \in X_f\,.$$

Since the gauge function ρ is differentiable in $\mathbb{R}^n \setminus \{0\}$, the subgradient of I_K can be explicitly computed, obtaining

$$\partial I_K(\xi) = \begin{cases} \{\alpha D\rho(\xi) \colon \alpha \geq 0\} & \text{if } \xi \in \partial K, \\ \emptyset & \text{if } \xi \notin K, \\ \{0\} & \text{if } \xi \in \operatorname{int} K \end{cases}$$

(see e.g. [18, Sect. 23]).

Hence, if (u, v) is a solution to (5), by condition (ii) in Definition 3.2, we have that $v(x) D\rho(Du(x)) \in \partial I_K(Du(x))$ for a.e. $x \in \Omega$, so that, for every $w \in X_f$

$$F(w) - F(u) \geq \int_\Omega v \langle D\rho(Du),\, Dw - Du \rangle\, dx - \int_\Omega f\,(w - u)\, dx = 0\,,$$

where the last equality follows from the fact that $w - u$ can be taken as test function in (19). This proves that u is a solution to (30).

Assume now that $u \in X_f$ is a minimizer for F, so that $f(u - u_\phi) = 0$ a.e. in Ω, due to the maximality of u_ϕ in X_f and the fact that $f \geq 0$. Again we can choose $u - u_\phi$ as test function in the transport equation (19) solved by u_ϕ, getting

$$0 = \int_\Omega v_f \langle D\rho(Du_\phi),\, Du - Du_\phi \rangle\, dx = -\int_\Omega v_f (1 - \langle D\rho(Du_\phi),\, Du \rangle)\, dx\,.$$

On the other hand, by Theorem 2.1(ii) and the fact that $Du \in K$ a.e. in Ω, we have

$$1 - \langle D\rho(Du_\phi(x)),\, Du(x) \rangle \geq 0, \ \text{a.e. } x \in \Omega\,,$$
$$1 - \langle D\rho(Du_\phi(x)),\, Du(x) \rangle = 0 \iff D\rho(Du_\phi(x)) = D\rho(Du(x)),$$

so that $v_f D\rho(Du_\phi) = v_f D\rho(Du)$ and $v_f(1 - \rho(Du)) = 0$ a.e. in Ω, that is (u, v_f) is a solution of (5). This concludes the proof of (i).

Let us prove (ii). The previous computation shows that if (u_ϕ, v) is a solution of (5), and $u \in X_f$ is a solution of the minimum problem (30), then also (u, v) is a solution of (5). Finally, let (u, v) be a solution to (5). Upon observing that $v\,\rho(Du) = v$ a.e. in Ω, and choosing $u - u_\phi$ as test function in the weak formulation of the transport equation

$$-\operatorname{div}(v\, D\rho(Du)) = f, \qquad \text{in } \Omega\,,$$

we conclude that (u_ϕ, v) is a solution to (5). □

As a consequence of the previous results we obtain the following uniqueness result for the v component.

Corollary 5.2. *Assume that* $(H1)$*–*$(H6)$ *hold. If* (u, v) *is a solution to* (5)*, then* $v = v_f$ *in* Ω*, and* $u = u_\phi$ *on* $\operatorname{spt} f$*.*

Proof. Let (u, v) be a solution to (5). By Theorem 5.1(i) we have that u is a solution to (30) hence, by Theorem 5.1(ii), (u_ϕ, v) is a solution to (5). The conclusion now follows from Theorem 4.9. □

We now introduce another element of X_f, which will play the röle of the minimal admissible profile. This minimal profile u_f depends on the source f and is defined by

$$u_f(x) := \sup\{u_\phi(z) - L(\gamma) : \; z \in \operatorname{spt}(f), \; \gamma \in \Gamma_{x,z}\}\,, \tag{31}$$

with the convention $u_f \equiv -\infty$ if $f \equiv 0$. (We recall that, by definition, $\operatorname{spt}(f)$ is a relatively closed set in Ω.)

Proposition 5.3. *Assume that* $(H1)$*–*$(H4)$ *hold. If* $f \not\equiv 0$ *then the function* u_f *defined in* (31) *belongs to* X_f *and* $u_f = u_\phi$ *on* $\operatorname{spt} f$*. Moreover, every function* $u \in X_f$ *such that* $u = u_f$ *in* $\operatorname{spt}(f)$ *satisfies* $u_f \le u \le u_\phi$ *in* $\overline{\Omega}$*.*

Proof. By the very definition of u_f, we have that u_f satisfies (4) in Ω, hence, by Lemma 2.2, $u_f \in W^{1,\infty}(\Omega)$ and $Du_f \in K$ a.e. in Ω. Moreover, for every $z \in \operatorname{spt}(f)$ and $y \in \partial\Omega$ we have that

$$\phi(y) \ge u_\phi(y) \ge u_\phi(z) - L(\gamma), \qquad \forall \gamma \in \Gamma_{y,z},$$

so that $\phi(y) \ge u_f(y)$. In order to prove that $u_f = \phi$ on Γ_f, let $y \in \Gamma_f$ and let $z \in \overline{\operatorname{spt}(f)}$ be such that there exists a maximal geodesics $\gamma \in \Gamma_{y,z}$, i.e. $u_\phi(z) = \phi(y) + L(\gamma)$. Then

$$\phi(y) \ge u_f(y) \ge u_\phi(z) - L(\gamma) = \phi(y).$$

It remains to prove that, if $u \in X_f$ coincides with u_f on $\operatorname{spt}(f)$, then $u \ge u_f$. (The inequality $u \le u_\phi$ is trivially satisfied by the maximality of u_ϕ in X_f.) Namely, for every $x \in \Omega$ there exist $z \in \overline{\operatorname{spt}(f)}$ and $\gamma \in \Gamma_{x,z}$ such that $u_f(x) = u_\phi(z) - L(\gamma)$, hence by Lemma 2.2(iii)

$$u_f(x) + L(\gamma) = u_\phi(z) = u(z) \le u(x) + L(\gamma)$$

i.e. $u_f(x) \le u(x)$. □

Remark 5.4. Since $u_f = u_\phi$ in $\operatorname{spt}(f)$, exploiting the explicit representation formula (23) of v_f, we can infer that $u_f = u_\phi$ in $\operatorname{spt}(v_f)$ (see also Remark 4.6).

The following result is the analogous of Theorem 7.2 in [12].

Theorem 5.5 (Uniqueness)**.** *Assume that* $(H1)$*–*$(H6)$ *hold. Then a function* $u \in X_f$ *is a solution to* (30) *if and only if* $u_f \le u \le u_\phi$*. Moreover, the function* u_f *coincides with* u_ϕ *in* Ω *(and hence* u_ϕ *is the unique solution to* (30)*) if and only if* $J \subseteq \overline{\operatorname{spt}(f)}$*.*

Proof. Let us first consider the trivial case $f \equiv 0$. In this case $X_f = X$, $u_f \equiv -\infty$ and every function $u \in X$ is a solution to the minimum problem (30).

Let now assume that $f \not\equiv 0$. The first assertion follows from Proposition 5.3 and from the fact that $u \in X_f$ is a solution to (30) if and only if $u = u_\phi$ on $\mathrm{spt}(f)$.

Let us prove the uniqueness result.

Let $J \subseteq \overline{\mathrm{spt}(f)}$. From Proposition 5.3 we have that $u_f = u_\phi$ on $\overline{\mathrm{spt}(f)}$, hence on J. Let $x \in \Omega \setminus J$ be given, and let $q(x) \in J$ be the endpoint of the ray through x. We have

$$u_\phi(x) = u_\phi(q(x)) - \rho^0(q(x) - x) = u_f(q(x)) - \rho^0(q(x) - x) \le u_f(x) \le u_\phi(x)\,,$$

and hence $u_f(x) = u_\phi(x)$.

Assume now that $u_\phi = u_f$ in Ω, and assume, by contradiction, that there exists a point $x_0 \in J$, $x_0 \notin \overline{\mathrm{spt}(f)}$. By definition, there exist $z \in \overline{\mathrm{spt}(f)}$ and a curve $\gamma \in \Gamma_{x_0,z}$ such that

$$u_f(x_0) = u_\phi(z) - L(\gamma). \tag{32}$$

Moreover, by (10), if $y_0 \in \Pi(x_0)$ one has

$$u_\phi(x_0) = u_\phi(y_0) + \rho^0(x_0 - y_0). \tag{33}$$

Since, by assumption, $u_\phi(x_0) = u_f(x_0)$, (32) and (33) yield

$$\rho^0(z - y_0) \le \rho^0(x_0 - y_0) + L(\gamma) = u_\phi(z) - u_\phi(y_0) \le \rho^0(z - y_0),$$

i.e., $\rho^0(x_0 - y_0) + L(\gamma) = \rho^0(z - y_0)$. From Theorem 2.1(ii) it follows that $[\![y_0, x_0]\!] \cup \gamma = [\![y_0, z]\!]$, that is $L(\gamma) = \rho^0(z - x_0)$ and $x_0 \in [\![y_0, z]\!]$. Finally, since $z \neq x_0$, by (32) and the definition of final point we have that $x_0 \notin J$, a contradiction. $\square$

For the reader's convenience, we summarize here the results we have obtained.

Corollary 5.6. *Assume that (H1)–(H6) hold. Then*

(i) *If (u, v) is a solution to (5), then $v = v_f$.*
(ii) *(u, v_f) is a solution to (5) if and only if $u \in X$ and $u_f \le u \le u_\phi$ in Ω.*
(iii) *(u_ϕ, v_f) is the unique solution to (5) if and only if $J \subseteq \overline{\mathrm{spt}\, f}$.*

6. The isotropic case: application to sandpiles

In this section we shall focus our attention on the isotropic case $\rho(\xi) = |\xi|$, $\xi \in \mathbb{R}^n$, ($n = 2$ in the model problem) translating the results of the previous sections in terms of description of the equilibrium configurations for sandpiles on a flat table with a vertical rim, and comparing these results with the ones known in literature.

In what follows we always assume that (H1), (H3), (H4), (H5) and (H6) hold. Furthermore, when considering the sandpile model, one could prefer not to allow profiles with negative height. If this is the case, one has to assume that

(a) The boundary datum ϕ (corresponding to the height of the rim) is non-negative;
(b) The space X_f contains only non-negative functions;
(c) The minimal function u_f, defined in (31), is replaced by $\max\{u_f, 0\}$.

If the sand is poured by a vertical source f on the table occupying the region Ω, the admissible equilibrium configurations can be depicted in terms of the pair (u, v), the profile of the standing layer and the thickness of the rolling layer, which are solutions of the PDEs system (1), that is

(i′) $(u, v) \in X_f \times L^1_+(\Omega)$;
(ii′) $(1 - |Du|)v = 0$ a.e. in Ω;
(iii′) for every $\psi \in C^\infty_c\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$

$$\int_\Omega v\langle Du,\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx\,,$$

where

$$\Gamma_f = \{y \in \partial\Omega\colon\ \exists x \in \operatorname{spt} f \text{ s.t. } \rrbracket y, x\llbracket \subset \Omega \text{ and } u_\phi(x) = \phi(y) + |x - y|\}$$

is defined in term of the Lax-Hopf function

$$u_\phi(x) = \min\{\phi(y) + |x - y|,\ y \in \partial\Omega\}.$$

Following (10), we have that

$$u_\phi(x) = \min\{\phi(y) + |x - y|,\ y \in \Gamma_\phi\}, \qquad x \in \Omega,$$

where

$$\Gamma_\phi = \{y \in \partial\Omega\colon\ \exists x \in \Omega\ \text{ s.t. } \rrbracket y, x\llbracket \subset \Omega \text{ and } u_\phi(x) = \phi(y) + |x - y|\}.$$

Hence in the sandpiles model, Γ_ϕ is the set of the initial points of the transport rays where the matter is allowed to run down, that is where the v–component could be non-zero.

On the other hand, Γ_f is the set of the initial points of the transport rays along which the transport is active. It is the effective border of the container, where the standing layer fills the gap with the rim of height ϕ (since $u = u_f = u_\phi = \phi$ on Γ_f for every admissible profile, see Theorem 5.5) and the exceeding sand falls down.

Finally, $\partial\Omega \setminus \Gamma_\phi$ is the part of the boundary closed by walls that the sand cannot overcome, no matter what the source is.

The results obtained in the previous sections, interpreted in the light of the sandpile model, are the following.

Theorem 6.1. *Assume that* $(H1)$*,* $(H3)$*–*$(H6)$ *hold. Then:*

(i) *The thickness of the rolling layer is uniquely determined by the data of the problem.*
(ii) *The admissible profiles of the standing layer are wedged between the minimal profile* u_f *and the maximal profile* u_ϕ*. In particular, every admissible profile has to agree with the maximal one in the region where the transport is active.*
(iii) *There exists a unique admissible configuration if and only if the source pours sand on the whole ridge of the maximal profile.*

These results extend the known results for the open table problem without walls (see [4, 5, 6, 17]) and with walls (see [13]). Moreover, we have clarified the issue of the lack of uniqueness of the rolling layer we have faced in a less general setting in [12].

7. Examples

In this section we detail some computation that were skipped while presenting Examples 3.4, 3.5 and 3.8 in Section 3.

In all examples the metric is isotropic, i.e. $\rho(\xi) = |\xi|$. Moreover, we assumed that $f \equiv 1$, so that the set Γ_f (defined in (9)) of initial points of maximal geodesics intersecting the support of f coincides with the set Γ_ϕ (defined in (8)) of all initial points of maximal geodesics. In the first two examples only assumptions (H1)–(H4) hold, so that maximal geodesics need not be segments. In any case, by Proposition 5.3 we have that $u_f = u_\phi$, so that the Lax–Hopf function u_ϕ is the unique candidate for the u-component of the solution.

Concerning the v-component, by Theorem 4.4 we have that, along almost every transport ray, the ODE (21) is satisfied. Moreover, if the final point q of a transport ray does not belong to the closure of the set of initial points, then $v\alpha$ must vanish on q (see the end of the proof of Theorem 4.9).

Details on Example 3.4. We have already observed that every solution (u, v) of (5) must satisfy $u = u_\phi$ in Ω, whereas $v(x) = 1 - x_2$ for every $x = (x_1, x_2) \in \Omega_2$, so that, for every test function $\psi \in C_c^\infty(\mathbb{R}^2 \setminus \overline{\Gamma}_f)$,

$$\int_{\Omega_2} (v\langle Du_\phi,\, D\psi\rangle - \psi)\, dx = 0.$$

On the other hand, v must be locally absolutely continuous along any ray $(0,1) \ni r \mapsto (r\cos\theta, r\sin\theta)$, $0 < \theta < \pi/2$, and

$$\frac{d}{dr}\left[r\, v(r\cos\theta, r\sin\theta)\right] = r, \qquad -r \in (0,1),\ 0 < \theta < \frac{\pi}{2}$$

(see (21)). Then, on Ω_1, we have that

$$v(r\cos\theta, r\sin\theta) = \frac{1 - r^2}{2r} + \frac{c(\theta)}{r} \qquad r \in (0,1),\ 0 < \theta < \frac{\pi}{2}$$

for some integrable function $c \geq 0$. Now, an explicit computation in polar coordinates gives

$$\int_{\Omega_1} (v\langle Du_\phi,\, D\psi\rangle - \psi)\, dx = \int_0^{\pi/2} \left[c(\theta)\psi(\cos\theta, \sin\theta) - \left(\frac{1}{2} + c(\theta)\right)\psi(0)\right] d\theta$$

and this last integral clearly cannot vanish for every choice of the test function ψ.

Details on Example 3.5. The Lax-Hopf function can be easily computed:

$$u_\phi(x) = \begin{cases} 1 + x_2, & \text{if } x \in \Omega_1, \\ 1 + |x|, & \text{if } x \in \Omega_2, \\ 1 - |x|, & \text{if } x \in \Omega_3. \end{cases}$$

It is clear that $\Gamma_\phi = \Gamma_f = (S_1 \cup S_3) \setminus \{(-1,-1), (2^{-1/2}, -2^{-1/2})\}$.

We claim that every pair $(u_\phi, v_f + w)$, with

$$v_f(x) := \begin{cases} 1 + x_2, & \text{if } x \in \Omega_1, \\ (1 - |x|^2)/(2|x|), & \text{if } x \in \Omega_2, \\ |x|/2, & \text{if } x \in \Omega_3, \end{cases}$$

and

$$w(r\cos\theta, r\sin\theta) := \begin{cases} c(\theta)/r & \text{on } \Omega_3, \\ 0 & \text{on } \Omega \setminus \Omega_3, \end{cases} \tag{34}$$

with

$$c \in L^1_+(-\pi/2, -\pi/4), \qquad \int_{-\pi/2}^{-\pi/4} c(\theta)\, d\theta = \frac{\pi}{8}\,,$$

is a solution to (5) in the sense of Definition 3.2. Notice that, as usual, the function v_f is obtained solving the ODE (21), while w is a solution to $-\operatorname{div}(wDu_\phi) = 0$ in Ω, with $\int_{\Omega_3} w = \mathcal{L}^2(\Omega_2)$. Namely, given $\psi \in C_c^\infty(\mathbb{R}^2 \setminus \overline{\Gamma}_f)$ and denoting by

$$I_j := \int_{\Omega_j} [(v_f + w)\langle Du_\phi,\, D\psi\rangle - \psi]\, dx, \qquad j = 1, 2, 3,$$

passing in polar coordinates we easily get

$$I_1 = 0, \qquad I_2 = -\frac{\pi}{8}\,\psi(0), \qquad I_3 = \psi(0) \int_{-\pi/2}^{-\pi/4} c(\theta)\, d\theta = \frac{\pi}{8}\,\psi(0),$$

hence $I_1 + I_2 + I_3 = 0$ and the claim is proved.

Details on Example 3.8. The boundary datum corresponding to the geometry of transport rays depicted in Figure 3 is the continuous function $\phi\colon \partial\Omega \to \mathbb{R}$ satisfying

$$\phi(x) = \begin{cases} 0 & x \in S_1 := ((-1, 0] \times \{-1, 1\}) \cup \{(\cos\theta, \sin\theta) : -\frac{\pi}{2} < \theta < -\frac{\pi}{4}\} \\ 1 & x \in S_2 := ((0, 1) \times \{0\}) \cup (\{1\} \times (0, 1)) \\ 1 - \sqrt{x_1^2 + x_2^2} & x \in S_3 := \{(x_1, -x_1),\ x_1 \in (0, \frac{1}{\sqrt{2}})\} \\ x_1 & x \in S_4 := (0, 1) \times \{1\} \\ 1 - |x_2| & x \in S_5 := \{-1\} \times [-1, 1]\,. \end{cases}$$

The Lax–Hopf function can be computed as follows:

$$u_\phi(x) = \begin{cases} 1 - |x_2| & x \in \Omega_1 \\ 1 - \sqrt{x_1^2 + x_2^2} & x \in \Omega_3 \\ 1 + x_2 & x \in \Omega_2' := \{x_1 \in (0, 1),\ 0 < x_2 < \frac{x_1^2}{4}\} \\ 2 - x_1 & x \in \Omega_2'' := \{x_2 \in (0, 1),\ 1 - \frac{(1 - x_2^2)}{4} < x_1 < 1\} \\ \sqrt{x_1^2 + (1 - x_2)^2} & x \in \Omega_2 \setminus (\Omega_2' \cup \Omega_2'')\,. \end{cases}$$

Hence $\Gamma_\phi = \Gamma_f = S_1 \cup S_2$. Since (H1)–(H5) are satisfied, by Theorem 4.8 the pair (u_ϕ, v_f) is a solution to (5).

On the other hand $(0,0) \in \overline{\Gamma}_f$ is the endpoint of all the transport rays covering Ω_3, so that (H6) is not satisfied. Hence, if $w\colon \Omega \to \mathbb{R}$ is a function of the form (34), with $c \in L^1_+(-\pi/2, -\pi/4)$, then for every test function $\psi \in C_c^\infty\left(\mathbb{R}^n \setminus \overline{\Gamma}_f\right)$ one has

$$\int_\Omega w\langle Du_\phi, D\psi\rangle dx = \int_{\Omega_3} w\langle Du_\phi, D\psi\rangle dx = 0,$$

so that the pair $(u_\phi, v_f + w)$ is a solution to (5).

References

[1] L. Ambrosio and P. Tilli, Topics on analysis in metric spaces, Oxford Lecture Series in Mathematics and its Applications, vol. 25, Oxford University Press, Oxford, 2004. MR 2039660 (2004k:28001)

[2] G. Aronsson, L. C. Evans, and Y. Wu, *Fast/slow diffusion and growing sandpiles*, J. Differential Equations **131** (1996), no. 2, 304–335. MR MR1419017 (97i:35068)

[3] S. Bianchini, *On the Euler-Lagrange equation for a variational problem*, Discrete Contin. Dyn. Syst. **17** (2007), no. 3, 449–480 (electronic). MR 2276421 (2007k:49048)

[4] P. Cannarsa and P. Cardaliaguet, *Representation of equilibrium solutions to the table problem for growing sandpiles*, J. Eur. Math. Soc. (JEMS) **6** (2004), 435–464.

[5] P. Cannarsa, P. Cardaliaguet, G. Crasta, and E. Giorgieri, *A Boundary Value Problem for a PDE Model in Mass Transfer Theory: Representation of Solutions and Applications*, Calc. Var. Partial Differential Equations **24** (2005), 431–457.

[6] P. Cannarsa, P. Cardaliaguet, and C. Sinestrari, *On a differential model for growing sandpiles with non-regular sources*, Comm. Partial Differential Equations **34** (2009), no. 7-9, 656–675. MR MR2560296

[7] L. Cesari, Optimization—theory and applications, Applications of Mathematics (New York), vol. 17, Springer-Verlag, New York, 1983.

[8] G. Crasta and A. Malusa, *The distance function from the boundary in a Minkowski space*, Trans. Amer. Math. Soc. **359** (2007), 5725–5759.

[9] G. Crasta and A. Malusa, *On a system of partial differential equations of Monge-Kantorovich type*, J. Differential Equations **235** (2007), 484–509.

[10] G. Crasta and A. Malusa, *A sharp uniqueness result for a class of variational problems solved by a distance function*, J. Differential Equations **243** (2007), 427–447.

[11] G. Crasta and A. Malusa, *A variational approach to the macroscopic electrodynamics of anisotropic hard superconductors*, Arch. Rational Mech. Anal. **192** (2009), 87–115.

[12] G. Crasta and A. Malusa, *A nonhomogeneous boundary value problem in mass transfer theory*, Calc. Var. Partial Differential Equations **44** (2012), 61–80.

[13] G. Crasta and S. Finzi Vita, *An existence result for the sandpile problem on flat tables with walls*, Netw. Heterog. Media **3** (2008), no. 4, 815–830. MR 2448942 (2010b:35219)

[14] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, english ed., Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR MR2307192 (2007k:53049)

[15] K.P. Hadeler and C. Kuttler, *Dynamical models for granular matter*, Granular Matter **2** (1999), 9–18.

[16] P.L. Lions, Generalized solutions of Hamilton-Jacobi equations, Pitman, Boston, 1982.

[17] L. Prigozhin, *Variational model of sandpile growth*, European J. Appl. Math. **7** (1996), 225–235.

[18] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton, NJ, 1970.

[19] R. Schneider, Convex bodies: the Brunn–Minkowski theory, Cambridge Univ. Press, Cambridge, 1993.

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