Large solutions for the elliptic 1-Laplacian with absorption

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Abstract

In this paper we give a general condition on the absorption term of the 1-Laplace elliptic equation for the existence of suitable large solutions. This condition can be considered as the correspondent Keller-Osserman condition for the p-Laplacian, in the case $p = 1$. We also provide conditions that guarantee uniqueness for solutions to such problems.

Keywords: Large solutions, 1-Laplacian, Keller-Osserman Condition, Maximal solutions
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1. Introduction

The study of both existence and uniqueness of large solutions for nonlinear partial differential equations with absorption goes back to the pioneering papers by Keller ([25]) and Osserman ([32]). Starting from there a huge literature has been devoted to the study of such a problems in particular for their connection to several branches of mathematics as Differential Geometry, Probability, and Control Theory (see for instance [31, 19, 27, 26]).

From the mathematical point of view the main idea is that, roughly, the existence of solutions (to a certain PDE) that blow up on the boundary of a domain is strictly related to the absorption role played by suitable lower order terms. Far to provide a complete list of references we refer to [18, 34, 33, 22, 17, 39, 21, 29, 30, 36] and references therein for a review on the subject. To be a bit more precise, let Ω be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary. The existence of a large solution for problem

$$\begin{cases} \Delta_p u = u^q & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

can be proved provided $q > p - 1$ (see for instance [18]).

For a general increasing and continuous nonlinearity f, with $f(0) = 0$, problem

$$\begin{cases} \Delta_p u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a solution if and only f satisfies the so-called Keller-Osserman condition;

$$\int_1^\infty \frac{1}{F(s)^{\frac{1}{p}}}\, ds < +\infty\,, \tag{1.3}$$

with $F(s) = \int_0^s f(t)\, dt$.

In this paper we deal with both existence and uniqueness of solutions to the 1-Laplace problem with absorption

$$\begin{cases} \Delta_1 u := \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = f(u) & \text{in } \Omega \\ \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

under different conditions on both f and Ω. The study of problems involving 1-Laplace type operators arises, for instance, in the study of image restoration as well as in some optimal design problems in the theory of torsion (see for instance [24, 40, 8] and references therein for a review on the main applications). A systematic mathematical study of this type of problems began with the works [5, 6] and a comprehensive treatment of the 1-Laplace diffusive term can be found in the monograph [8].

We will call our solutions *Large Solutions* in order to be consistent with the existing literature, though, as we will see, the boundary condition $u = +\infty$ should be understood in a very weak sense (see also Remark 3.2 later). In fact, as we will see, depending on the assumptions (to be specified later) on f and Ω, very different situations occur: large solutions turn out to be, in many cases, globally bounded and they can be regarded as *maximal solutions* for the equation in (1.4). Anyway, depending, respectively, on the boundary regularity of Ω and on the behavior of f, then the value $+\infty$ can be attained at some points (Remark 4.1) of $\partial\Omega$ or maximality can be lost (Remark 4.3). We want to stress that this range of phenomena is intrinsic to the 1-Laplacian operator. In fact, as first observed in Osher and Sethian's celebrated paper [38] (see also the monograph [37]), the 1-Laplacian operator is closely related to the mean curvature operator in the following way: consider the surface given by the level set $\{u(x) = k\}$; then its unit normal is formally given by $\mathbf{n}(x) = \frac{Du}{|Du|}$. Therefore, the mean curvature of the surface at the point x is formally given by

$$\mathbf{H}(x) = \operatorname{div}(\mathbf{n})(x) = \operatorname{div}\left(\frac{Du}{|Du|}\right)(x);$$

i.e. the 1-Laplacian operator. This relationship clearly shows that the behavior at the boundary $\partial\Omega$ of the solutions to (1.4) might depend on the mean curvature of the boundary itself, and in particular, as we will see, on its boundedness.

Though both Neumann and Dirichlet 1-Laplace type problems with absorption terms have been considered (see for instance [16] and references therein) we want to stress that, as a by-product of our arguments, existence and uniqueness for suitable nonhomogeneous Dirichlet boundary value problems will also be obtained (see Remark 4.4), some of these, to our knowledge, being missed in the existing literature.

Let us describe the structure of the paper. After a preliminary section where we recall some basic definitions and results, and we define the notation we shall use throughout the paper, in Section 3 we define a notion of solution to (1.4) for the case of Ω having a Lipschitz boundary and under very general assumptions on the nonlinearity f. We prove that, if it exists, this type of solution is maximal among all distributional solutions and, as a consequence, we obtain uniqueness for such solutions.

In Section 4 we give the existence result for solutions to problem (1.4). The formal idea is the following one: after the change of variables $v = f(u)$, then, using the homogeneity of the 1-Laplacian operator, (1.4) formally transforms into

$$\begin{cases} \Delta_1 v = v & \text{in } \Omega \\ \\ v = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

We show that this procedure is correct in case of the domain satisfying a uniform interior ball condition or being a convex body.

Existence and uniqueness of solutions to (1.5) can be immediately derived from the recent work [36], where the main issue is the study of both existence and uniqueness of large solutions for parabolic problems without lower order absorption terms whose model is

$$\begin{cases} u_t = \Delta_p u & \text{in } Q_T, \\ u = u_0 & \text{on } \{0\} \times \Omega, \\ u = +\infty & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.6}$$

where $u_0 \in L^1_{loc}(\Omega)$ is a nonnegative function and Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary and $1 \le p < 2$. In this case, the naïve idea is that, clearly, the term u_t plays itself an absorption role of growth of order1 and this allows to avoid, for instance, the interior blow up of the approximating solutions, at least, for small p.

The results about (1.5) allow us to show that (1.4) has a unique solution in the case of f being an increasing function defined on $\mathbb{R}$ with f^{-1} locally Lipschitz continuous in $]0, +\infty[$. This assumption takes the place of the Keller-Osserman condition for higher growth operators. As a matter of fact, the 1-Laplacian is singular enough to play an absorption role by itself, thanks to its homogeneity property. So that, though, as we said, solutions can attain the values $+\infty$ on the boundary, essentially no assumptions are needed on the behavior of $f(s)$ at infinity.

Finally, in Section 5 we prove a stability result. Solutions to (1.4), in the case of Ω being of class C^2 and under natural Keller-Osserman type conditions on f (that will be specified later), can be obtained through a stability procedure by taking the limit as $p \to 1^+$ in (1.2).

Summarizing, the different hypotheses on f we will use are the following:

(H_1). f is increasing.

(H_2). f is increasing, with f^{-1} increasing and locally Lipschitz continuous in $]0, +\infty[$.

(H_3). f is continuous and increasing, $f(0) = 0$ and it verifies $f(s) \ge cs^{\overline{q}}$ for some $c, \overline{q} > 0$.

We sum up here the results obtained. The least assumption on the bounded set Ω is to have a Lipschitz boundary.

- (H_1) gives uniqueness of large solutions (Corollary 3.2). It is moreover a sharp condition (Remark 4.3).
- (H_2) is a sufficient condition for existence of solutions in case Ω satisfies a uniform interior ball condition or it is a convex body (Theorem 4.5). Therefore, (H_2) can be considered as the corresponding Keller-Osserman condition for $p = 1$.
- If Ω is a C^2 domain, then (H_3) is enough to prove that solutions to (1.2) (which exist for small p's since (H_3) implies (1.3)) converge to the solution to (1.4) as $p \to 1^+$.

2. Preliminaires and notations

In this section we collect the main notation and some useful results we will use in our analysis.

2.1. Functions of bounded variations and some generalizations

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total mass in Ω is called a *function of bounded variation.* The class of such functions will be denoted by $BV(\Omega)$ and $|Du|$ will denote the total variation of the measure Du.

A measurable set $E \subset \mathbb{R}^N$ is said to be of finite perimeter in Ω if $\chi_E \in BV(\Omega)$. In this case, the perimeter of E in Ω is defined as $Per(E,\Omega) := |D\chi_E|(\Omega)$. We shall use the notation $Per(E) := Per(E,\mathbb{R}^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is a countably $(N-1)$-rectifiable set with finite $\mathcal{H}^{N-1}$ measure, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Moreover, the outer unit normal $\nu^E(x)$ exists at $\mathcal{H}^{N-1}$ almost all points x of $\partial^* E$. It holds that the measure $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$.

For further information and properties concerning functions of bounded variation and sets of finite perimeter we refer to [4], [20] or [43].

We consider the following truncature functions. For $k > 0$, let $T_k(s) := \max(\min(-k,s),k)$. Given any function u and $a,b \in \mathbb{R}$ we shall use the notation $[u \geq a] = \{x \in \mathbb{R}^N : u(x) \geq a\}$, and similarly for the sets $[u > a]$, $[u \leq a]$, $[u < a]$, etc., while the symbol $u \llcorner E$ will indicate the restriction of the function u to the measurable set $E \subset \mathbb{R}^N$.

Given a real function $f(s)$, we define its positive part as $f^+(s) = \max(0, f(s))$. For our purposes, we need to consider the function spaces

$$TBV(\Omega) := \left\{u \in L^1(\Omega) \ : \ T_k(u) \in BV(\Omega), \ \ \forall\, k > 0\right\},$$

$$TBV_{loc}(\Omega) := \left\{u \in L^1_{loc}(\Omega) \ : \ T_k(u) \in BV(\Omega), \ \ \forall\, k > 0\right\},$$

and to give a sense to the Radon-Nikodym derivative (with respect to the Lebesgue measure) ∇u of Du for a function $u \in TBV_{loc}(\Omega)$.

Lemma 2.1. *[7, Lemma 1] For every $u \in TBV_{loc}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{[|u|<k]} \qquad \mathcal{L}^N - \text{a.e.}, \ \ \forall\, k > 0. \tag{2.1}$$

Thanks to this result we define ∇u for a function $u \in TBV_{loc}(\Omega)$ as the unique function v which satisfies (2.1). Obviously, if $w \in W^{1,1}_{\rm loc}(\Omega)$, then the generalized gradient turns out to coincide with the classical distributional one. This notation will be used throughout in the sequel.

We denote by $\mathcal{P}$ the set of nondecreasing Lipschitz continuous functions $S : [0,+\infty[\to \mathbb{R}$ satisfying $S'(s) = 0$ for $|s|$ large enough. We recall the following result.

Lemma 2.2. *[7, Lemma 2] If $u \in TBV(\Omega)$, then $S(u) \in BV(\Omega)$ for every $S \in \mathcal{P}$. Moreover, $\nabla S(u) = S'(u)\nabla u$* $\mathcal{L}^N$-a.e.

2.2. A generalized Green's formula

We shall need several results from [10] (see also [8]). Let

$$X(\Omega) = \left\{ z \in L^\infty(\Omega; \mathbb{R}^N) \ : \ \operatorname{div}(z) \in L^1(\Omega) \right\}$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z, Dw), \varphi \rangle := -\int_\Omega w\, \varphi \operatorname{div}(z)\, dx - \int_\Omega w\, z \cdot \nabla \varphi\, dx. \tag{2.2}$$

In [10] it is proved that (z, Dw) is a Radon measure in Ω verifying

$$(z, Dw)(\Omega) = \int_\Omega z \cdot \nabla w\, dx \qquad \forall\ w \in W^{1,1}(\Omega) \cap L^\infty(\Omega).$$

Moreover, for all $w \in BV(\Omega) \cap L^\infty(\Omega)$, (z, Dw) is absolutely continuous with respect to the total variation of Dw and it holds,

$$(z, Dw)(B) = \int_\Omega \theta(z, Dw)|Dw| \le \|z\|_\infty |Dw|(B), \tag{2.3}$$

for any Borel set $B \subseteq \Omega$, where $\theta(z, Dw)$ is the Radon-Nikodym derivative of (z, Dw) with respect to $|Dw|$.

We will use the following result which is derived again from the work in [10].

Lemma 2.3. *Let $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in X(\Omega)$. Then,*

(a) Coarea formula [10, Proposition 2.7.(ii)]:

$$(z, Dw)(B) = \int_{-\infty}^{\infty} (z, D\chi_{[w>\lambda]})(B)\, d\lambda\,, \quad \text{for any Borel set } B \subset \Omega.$$

(b) Let α be an increasing function. Then, a slight modification of [10, Proposition 2.8] shows that if $\alpha(w) \in BV(\Omega) \cap L^\infty(\Omega)$, then

$$\theta(z, D(\alpha(w)))(x) = \theta(z, Dw)(x)\,, \quad |Dw| - \text{ a.e. in } \Omega.$$

In [10], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$, denoted by $[z, \nu]$, is defined. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ is established

$$\int_\Omega w \operatorname{div}(z)\ dx + (z, Dw)(\Omega) = \int_{\partial\Omega} [z, \nu] w\ d\mathcal{H}^{N-1}. \tag{2.4}$$

3. Definition and uniqueness of large solutions

Throughout this section, f is considered to be an increasing function and Ω is a bounded set in $\mathbb{R}^N$ with Lipschitz boundary. We give the following definition of distributional solution to the equation in (1.4) which is the natural extension to the classical one (see [5, 8, 36] and references therein).

Definition 3.1. *We say that $u \in TBV(\Omega)$ is a distributional solution of*

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = f(u)$$

if there exists $z \in X(\Omega)$, with $\|z\|_\infty \leq 1$, such that $f(u) = \operatorname{div} z$ in $\mathcal{D}'(\Omega)$ and $(z, DT_k(u)) = |DT_k(u)|$ as measures, for any $k > 0$.

Remark 3.1. *Observe that if $u \in TBV(\Omega)$ is a distributional solution of* (1.4) *in the sense of Definition 3.1, then $u \llcorner_{\Omega'}$ is a distributional solution of* (1.4) *for all $\Omega' \subset \Omega$ with Lipschitz boundary. In fact it suffices to take $z' := z \llcorner_{\Omega'}$ since, by the fact that $(z, DT_k(u)) = |DT_k(u)|$ as measures and* (2.3)*, then $(z, DT_k(u))(B) = |DT_k(u)|(B)$ for any Borel set $B \subset \Omega$; in particular for $B = \Omega'$. Then, again by* (2.3)*, $(z', DT_k(u) \llcorner_{\Omega'}) = |DT_k(u) \llcorner_{\Omega'}|$.*

Here is our definition of *large solution* for problem (1.4).

Definition 3.2. *We say that $u \in TBV(\Omega)$ is a (large) solution to* (1.4) *if there exists $z \in X(\Omega)$, with $\|z\|_\infty \leq 1$, such that*

$$f(u) = \operatorname{div} z\,, \quad \text{in } \mathcal{D}'(\Omega)\,, \tag{3.1}$$

$$(z, DT_k(u)) = |DT_k(u)|\,, \quad \text{as measures for any } k > 0 \text{ and} \tag{3.2}$$

$$[z, \nu] = 1\,, \quad \text{for a.e. } x \in \partial\Omega \tag{3.3}$$

Remark 3.2. *Let us make some comments about Definition 3.2. First of all, we note that in case that $v \in BV(\Omega) \cap L^\infty(\Omega)$, then, by* (2.3)*, condition* (3.2) *is equivalent to*

$$(z, Dv) = |Dv| \quad \text{as measures.} \tag{3.4}$$

Secondly, observe that a large solution is nothing but a distributional solution verifying the boundary condition (3.3)*. Observe that, since $\|z\|_\infty \leq 1$, then* (3.3) *forces the vector field z to be parallel to the outward unit exterior normal to the boundary and to have its biggest possible magnitude at the boundary. This is the mild way in which condition "$u = +\infty$" must be understood. Also observe that solutions to Dirichet problems involving the 1-Laplacian as the diffusion term do not verify, in general, the boundary condition in a classical trace sense* *(see e.g. [6], [35]). Usually, if the Dirichlet constraint is "$u = g$ at $\partial\Omega$", with $g \in L^1(\partial\Omega)$, then this condition transforms into*

$$[z,\nu] \in \text{sign}\,(T_k(g) - T_k(u))\,, \quad \mathcal{H}^{N-1} \text{ at } \partial\Omega$$

for any $k > 0$, where sign *is the multivalued sign function.*

We finally note that, for the parabolic case without absorption studied in [36], the condition at the boundary for a large solution is the same as (3.3)*. Moreover, this condition produces solutions to be maximal as the following result shows.*

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. If u is a large solution of* (1.4) *and $\overline{u}$ is a distributional solution of* (1.4)*, then $u \geq \overline{u}$ for a.e. $x \in \Omega$*

Proof. By definition, there exist $z, \overline{z} \in X(\Omega)$ such that

$$f(u) = \text{div} z\,, \tag{3.5}$$

$$f(\overline{u}) = \text{div}\overline{z}. \tag{3.6}$$

We multiply (3.5) by $-(T_k(\overline{u})-T_k(u))^+$, and integrate by parts in Ω. We obtain

$$-\int_\Omega (T_k(\overline{u}) - T_k(u))^+ f(u) \overset{(3.3)}{=} \int_\Omega (z, D(T_k(\overline{u}) - T_k(u))^+) - \int_{\partial\Omega} (T_k(\overline{u}) - T_k(u))^+ \, d\mathcal{H}^{N-1}\,.$$

Similarly,

$$\int_\Omega (T_k(\overline{u}) - T_k(u))^+ f(\overline{u}) = -\int_\Omega (\overline{z}, D(T_k(\overline{u}) - T_k(u))^+) + \int_{\partial\Omega} [\overline{z},\nu](T_k(\overline{u}) - T_k(u))^+ \, d\mathcal{H}^{N-1}\,.$$

Adding both equalities we get,

$$\int_\Omega (T_k(\overline{u}) - T_k(u))^+ (f(\overline{u}) - f(u)) = -\int_\Omega (\overline{z} - z, D(T_k(\overline{u}) - T_k(u))^+) - \int_{\partial\Omega} (1 - [z,\nu])(T_k(\overline{u}) - T_k(u))^+ \, d\mathcal{H}^{N-1}.$$

Finally, since in view of estimate (2.3),

$$(\overline{z} - z, D(T_k(\overline{u}) - T_k(u))^+)(B) \geq 0,$$

for any borel set $B \subseteq \Omega$ and using that $\|z\|_\infty, \|\overline{z}\|_\infty \leq 1$, we get

$$\int_\Omega (T_k(\overline{u}) - T_k(u))^+ (f(\overline{u}) - f(u))\, dx \leq 0.$$

Therefore, letting $k \to \infty$ and since f is increasing, we obtain the desired result.

Note that, in particular, if we apply Theorem 3.1 to two large solutions we get the following result:

Corollary 3.2. *If f is an increasing function, there exists at most one large solution of* (1.4)*.*

4. Existence result with a general absorption term

In this section we address to the analysis of (1.4) under different regularity conditions on the domain Ω.

In order to do that, we first address to the case of $f(s) = s$; i.e: problem (1.5). The existence of solutions to (1.5) (and uniqueness in the case of Ω having a smooth boundary, see Section 4.1) will follow in a quite standard way from some recent tools available in the literature. On the other hand, if Ω is a convex body, then the solutions can be explicitly constructed even if the domain does not satisfy any further regularity condition (i.e. a uniform interior ball condition).

This permits to obtain existence of solutions to (1.4) under very general conditions on the absorption term f. Our analysis shows that the sufficient condition on f in (1.4) for obtaining a solution is that f is increasing in $\mathbb{R}$ and f^{-1} is increasing and Lipschitz in the domain of the solution to (1.5).

We begin with the following definition and example of existence of large solutions for a very specific class of domains:

Definition 4.1. *We say that a bounded convex set E of class $C^{1,1}$ is calibrable if there exists a vector field $\xi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that $\|\xi\|_\infty \le 1$, $(\xi, D\chi_E) = |D\chi_E|$ as measures, and*

$$-\mathrm{div}\xi = \lambda_E \chi_E \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

for some constant λ_E. In this case [2, page 6], $\lambda_E = \frac{Per(E)}{|E|}$ and integrating by parts in E it is easily seen that $[\xi, \nu^E] = -1$, $\mathcal{H}^{N-1} - a.e$ in ∂E.

As it is proved in [2, Theorem 9], a bounded and convex set E is calibrable if and only if the following condition holds:

$$(N-1)\|\mathbf{H}_E\|_\infty \le \lambda_E = \frac{Per(E)}{|E|},$$

where $\mathbf{H}_E$ denotes the ($\mathcal{H}^{N-1}$-a.e. defined) mean curvature of ∂E. In particular, if $E = B_R(0)$, for some $R > 0$, then E is calibrable.

Example 4.1. *If Ω is a calibrable set, then $v = \frac{Per(\Omega)}{|\Omega|}$ is the large solution to* (1.5). *It suffices to take the restriction to Ω of the vector field in the definition of calibrability; i.e.: $z := -\xi \llcorner_\Omega$, since*

$$(z, Dv)(\Omega) \overset{(2.4)}{=} -\int_\Omega \left(\frac{Per(\Omega)}{|\Omega|}\right)^2 dx$$
$$+ \int_{\partial\Omega} -[\xi, \nu^\Omega] \frac{Per(\Omega)}{|\Omega|} \, d\mathcal{H}^{N-1} = 0 = |Dv|(\Omega).$$

We next follow with the proof of the existence of a large solutions to (1.5) when the domain Ω is smooth enough.

4.1. The case of the domain verifying a uniform interior ball condition

Let Ω satisfying a uniform interior ball condition: i.e. there exists $s_\Omega > 0$ such that for every $x \in \Omega$ with $dist(x, \partial\Omega) < s_\Omega$, there is $z_x \in \partial\Omega$ such that $|x - z_x| = dist(x, \partial\Omega)$ and $B(x_0, s_\Omega) \subset \Omega$ with $x_0 := z_x + s_\Omega \frac{x-z_x}{|x-z_x|}$. In the same way, one can define the uniform exterior ball condition by replacing Ω with $\mathbb{R}^N \setminus \Omega$. As is proved in [3, Corollary 3.14] a domain with compact boundary is of class $C^{1,1}$ if and only if it satisfies both a uniform interior ball condition and an exterior one. This result is implicitly used in Section 5. From now on, s_Ω will denote the radius of the uniform interior ball condition corresponding to Ω. In [36] an operator $\mathcal{A}$ associated to the elliptic problem

$$\begin{cases} -\text{div}\left(\dfrac{Dv}{|Dv|}\right) = w & \text{in } \Omega \\ \\ v = +\infty & \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

is defined. More precisely, we have the following:

Definition 4.2. *We say that $(v, w) \in \mathcal{A}$ iff $v, w \in L^1(\Omega)$, $v \in TBV(\Omega)$ and there exists $z \in X(\Omega)$ with $\|z\|_\infty \le 1$, $w = -\text{div} z$ in $\mathcal{D}'(\Omega)$ such that*

$$[z, \nu] = 1\,, \qquad \mathcal{H}^{N-1} - \text{a.e in } \partial\Omega$$

and

$$(z, DS(v))(\Omega) = |DS(v)|(\Omega)\,, \quad \text{for all } S \in \mathcal{P}. \tag{4.2}$$

The following result holds true ([36, Theorem 5.2.]):

Theorem 4.1. *The operator $\mathcal{A}$ is m-completely accretive in $L^1(\Omega)$ with dense domain.*

Theorem 4.2. *Let Ω satisfy a uniform interior ball condition. Then, there exists a unique large solution $v \in BV(\Omega) \cap L^\infty(\Omega)$ to (1.5).*

Proof. By the definition of m-accretivity (we do not enter into the details, see for instance [15],[13]) and as a consequence of Theorem 4.1, we get that for any $w \in L^1(\Omega)$, there exists a unique solution of

$$\begin{cases} -\text{div}\left(\dfrac{Dv}{|Dv|}\right) = w - v & \text{in } \Omega \\ \\ v = +\infty & \text{on } \partial\Omega\,, \end{cases} \tag{4.3}$$

in the sense that $(v, w - v) \in \mathcal{A}$. We just take now $w = 0$ and we get the result by observing that, by [36, Remark 5.3], $v \in BV(\Omega) \cap L^\infty(\Omega)$.

4.2. The case of a convex domain

Let us consider Ω being a nontrivial convex body in $\mathbb{R}^N$; i.e. a compact convex subset of $\mathbb{R}^N$ with a nonempty interior.

We recall the approach and several results given in [1] which we gather together in the next theorem:

Theorem 4.3 ([1], Proposition 2.4 and Remark 2.3). *Consider the problem*

$$(P)_\lambda := \min_{F\subseteq\Omega} Per(F) - \lambda|F|$$

Then, there is a unique convex set $K \subseteq \Omega$ of class $C^{1,1}$ (the Cheeger set, which is moreover calibrable, see [1, 14] for details) which is a solution of $(P)_{\lambda_K}$ with $\lambda_D := \frac{Per(D)}{|D|}$ for any $D \subseteq \Omega$. For any $\lambda > \lambda_K$ there is a unique minimizer Ω_λ of $(P)_\lambda$, which is moreover convex, and the function $\lambda \to \Omega_\lambda$ is increasing and continuous and $\Omega_\lambda \to \Omega$ as $\lambda \to \infty$.

Let K be the Cheeger set contained in Ω defined in the previous result. For each $\lambda \in (0, +\infty)$ let Ω_λ be the minimizer of problem $(P)_\lambda$. We take $\Omega_\lambda = \emptyset$ for any $\lambda < \lambda_K$. Using the monotonicity of Ω_λ and the fact that $|\Omega \setminus \cup\{\Omega_\lambda : \lambda > 0\}| = 0$ we may define the *variational mean curvature* as

$$H_\Omega(x) := \begin{cases} -\inf\{\lambda : x \in \Omega_\lambda\} & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.4}$$

In [11] (see also [23, Theorem 2.3]) it is established that, if Ω is a set of finite perimeter, then $\|H_\Omega\|_1 = Per(\Omega)$ and

$$\int_{\Omega_\lambda} H_\Omega(x)\, dx = -Per(\Omega_\lambda)\,.$$

Thanks to this result, in [2, Lemma 7] the authors are able to construct a vector field $\xi_\Omega \in X(\mathbb{R}^N)$ with $\|\xi_\Omega\|_\infty \le 1$ such that $\mathrm{div}\xi_\Omega = -H_\Omega$ in $L^1(\mathbb{R}^N)$, and

$$(\xi_\Omega, D\chi_{\Omega_\lambda})(\mathbb{R}^N) = Per(\Omega_\lambda)\,, \quad \text{for any } \lambda > 0.$$

We next follow the construction of the solution to the Cauchy problem for the Total Variation Flow build up in [2, Theorem 17] and [1, Remark 2.5] in order to obtain a maximal solution to (1.5).

Theorem 4.4. *Let Ω be a non-trivial convex body and let H_Ω be the variational mean curvature given by* (4.4)*. Then, $v(x) := -H_\Omega(x)$ is a large solution to* (1.5)*. Moreover, if $\|H_\Omega\|_\infty < +\infty$, then, Ω is of class $C^{1,1}$.*

Proof. First of all, we have that $v \in TBV(\Omega)$ since, for $\lambda \le k$,

$$Per([T_k(v) \ge \lambda], \Omega) = Per([v \ge \lambda], \Omega) = Per(\Omega_\lambda),$$

which is finite. Then, by the coarea formula ([4, Theorem 3.40]),

$$|DT_k(v)|(\Omega) = \int_0^k Per([T_k(v) \geq \lambda], \Omega)\ d\lambda < +\infty.$$

Let ξ_Ω be the vector field obtained above and let us take $z := -\xi_\Omega$. Obviously $[v \leq \lambda] = \Omega_\lambda \subseteq \Omega$ and $z \in X(\Omega)$. Moreover, the same computations as in [2, Theorem 17] show that

$$[z, \nu^\Omega] = 1\,, \quad \mathcal{H}^{N-1}\text{−a.e on } \partial\Omega\,, \text{and} \tag{4.5}$$

$$[z, \nu^{\Omega_\lambda}] = 1\,, \quad \mathcal{H}^{N-1}\text{−a.e on } \partial\Omega_\lambda. \tag{4.6}$$

Finally, by Lemma 2.3[a],

$$(z, DT_k(v))(\Omega) = \int_0^\infty (z, D\chi_{[T_k(v)\geq\lambda]})(\Omega)\, d\lambda$$

$$= -\int_0^k \int_{(\partial^*[v\geq\lambda])\cap\Omega} [z, \nu^{[v\geq\lambda]}] d\mathcal{H}^{N-1}\, d\lambda$$

$$= \int_0^k \int_{(\partial^*[v\geq\lambda])\cap\Omega} [z, \nu^{[v\leq\lambda]}] d\mathcal{H}^{N-1}\, d\lambda$$

$$= \int_0^k \int_{(\partial^*[v\geq\lambda])\cap\Omega} [z, \nu^{\Omega_\lambda}] d\mathcal{H}^{N-1}\, d\lambda$$

$$\stackrel{(4.6)}{=} \int_0^k Per([v \geq \lambda], \Omega) d\lambda = |DT_k(v)|(\Omega)\,.$$

Together with (2.3), this proves that v is a large solution to (1.5). By Corollary 3.2, it is its unique solution. Finally, suppose by contradiction that $\|H_\Omega\|_\infty < C$. Then, $\Omega_\lambda = \Omega$ for all $\lambda \geq C$. Since Ω_λ is a solution to (P_λ), we proceed as in [2, Proposition 2.7] to show that the mean curvature of Ω is bounded which, together with its convexity, proves that Ω is $C^{1,1}$.

Remark 4.1. *Let us emphasize the following qualitative property of large solutions. Observe that, as a corollary of Theorem 4.4 in case Ω is a nontrivial convex body which is not $C^{1,1}$, then the large solution to* (1.5) *is not bounded. Heuristically, this means that the large solution takes the values $+\infty$ at those points in the boundary of the convex body which do not have a finite mean curvature (e.g. at the "corners").*

4.3. Existence of solutions: the general case

We next show that the existence of solutions to (1.5) permits us to show existence of a large solution of (1.4) in the case of Ω being either a domain satisfying a uniform interior ball condition or a convex body.

Theorem 4.5. *Let f be an everywhere defined increasing function such that f^{-1} is locally Lipschitz continuous in $]0,+\infty[$. Then, there exists a large solution of* (1.4)*.*

Proof. Let $v \in TBV(\Omega)$ be a large solution to (1.5) obtained in Theorems 4.2 and 4.4 under different conditions on Ω. Then, there exists $z \in X(\Omega)$ such that

$$\mathrm{div} z = v\,, \quad \text{in } \mathcal{D}'(\Omega)\,,$$

$$(z, DT_k(v)) = |DT_k(v)|\,, \quad \text{as measures for any } k > 0 \text{ and}$$
$$[z,\nu] = 1\,, \quad \text{for a.e. } x \in \partial\Omega\,.$$

Now, we take a ball B_R, sufficiently large such that $\Omega \subset B_R$ and we let $v_R(x) := \frac{N}{R}$ to be the unique large solution to (1.5) in B_R as seen in Example 4.1. Then, by Remark 3.1, v_R is a distributional solution to (1.5) in Ω. By Theorem 3.1, then $v(x) \geq v_R(x) = \frac{N}{R}$, a.e. $x \in \Omega$.

So that, if we take $u := f^{-1}(v)$, by chain's rule in BV ([4, Theorem 4.4]), $u \in TBV(\Omega)$. First of all,

$$\mathrm{div} z = f(u)\,, \quad \text{in } \mathcal{D}'(\Omega).$$

Secondly, since $T_k(u) = f^{-1}(T_{f(k)}(v))$, by Lemma 2.3(b)

$$(z, DT_k(u)) = \theta(z, DT_k(u))|DT_k(u)| = \theta(z, DT_{f(k)}(v))|DT_k(u)| \overset{(3.2)}{=} |DT_k(u)|\,,$$

as measures, which, coupled with $[z,\nu] = 1$ shows that u is a large solution of (1.4).

Remark 4.2. *Observe that the previous theorem shows that there exist solutions to* (1.4) *with nonlinearities which do not verify condition (1.3) for any $p \geq 1$. One can take for instance $f(s) = log(1+s)$.*

Remark 4.3. *Let $C \in \mathbb{R}$ be a constant and Ω be $C^{1,1}$. We consider now the case of $f \equiv C$. A trivial integration by parts shows that for* $\mathrm{div} z = v$ *and $[z,\nu] = 1$ to hold, then, necessarily $C = \frac{Per(\Omega)}{|\Omega|}$. Then, Ω must be a calibrable set and $f = \frac{Per(\Omega)}{|\Omega|}$. In this case, large solutions exist. In fact, $v = \tilde{C}$ for any constant $\tilde{C}$ is a large solution to* (1.4) *in the sense of Definition 3.2. However, since f is not increasing, Theorem 3.1 does not hold. In this sense, we can say that there is not a maximal solution to* (1.4)*, consistent with the case of the $p-$Laplacian for $p > 1$. With the same analysis, it can be proved that if f is constant in a nontrivial interval around $\frac{Per(\Omega)}{|\Omega|}$, then, large solutions are not unique. Therefore, the strict monotonicity is also a necessary condition for obtaining uniqueness of large solutions.*

Remark 4.4. *Let $g : \partial\Omega \to \mathbb{R}$. With the additional hypothesis (with respect to those in Theorem 4.5) on f that $f \circ g \in L^1(\partial\Omega)$ and a similar proof of Theorems 4.5 and 3.1, one can easily obtain existence and uniqueness of solutions to problem*

$$\begin{cases} \Delta_1 u = f(u) & \text{in } \Omega \\ \\ u = g & \text{on } \partial\Omega\,, \end{cases} \tag{4.7}$$

with $g \in L^1(\partial\Omega)$ from the unique solutions (see [6]) to problem

$$\begin{cases} \Delta_1 v = v & \text{in } \Omega \\ \\ v = f(g) & \text{on } \partial\Omega\,, \end{cases}$$

in the sense that $u \in BV(\Omega)$ and there exists $z \in X(\Omega)$ such that

$$f(u) = \mathrm{div} z\,, \quad \text{in } \mathcal{D}'(\Omega)\,,$$

$$(z, Du) = |Du|\,, \quad \text{as measures and}$$
$$[z,\nu] \in \mathrm{sign}(T_k(g) - T_k(u))\,, \quad \mathcal{H}^{N-1} \text{ a.e. } x \in \partial\Omega,$$

for any $k > 0$. To our knowledge, problem (4.7) *has not been studied in the literature in this generality.*

5. Existence of solutions obtained as limit for $p \to 1^+$

Here we want to show a stability type approach to the existence of a large solution for the 1-Laplace equation with absorption. The result has a proper independent interest as it highlights the direct connection with standard large solutions associated to p-Laplace type problems with absorption. To perform the analysis we have to restrict our assumptions on both the domain and the nonlinearity f. Let Ω be a bounded domain of class C^2.

Concerning f we assume (H3), that is $f(0) = 0$, f is continuous and increasing and there exists $\overline{q} > 0$ and $c > 0$ such that

$$f(s) \geq c s^{\overline{q}} \tag{5.1}$$

for any $s \in [0, \infty)$.

We want to take the limit when $p \to 1^+$ in problem

$$\begin{cases} \Delta_p u_p = f(u_p) & \text{in } \Omega, \\ u_p = +\infty & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

to obtain a large solution of

$$\begin{cases} \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = f(u) & \text{in } \Omega \\ \\ u = +\infty & \text{on } \partial\Omega\,. \end{cases} \tag{5.3}$$

As $\overline{q}$ is fixed, without loss of generality we can always think about $p < 1+\overline{q}$ in order for the Keller-Osserman condition to be satisfied for *any* p near 1.

The following stability result will be proved along this section and it is the main result in it.

Theorem 5.1. *Let f verify (H3). Then, there is a sequence of solutions to* (5.2), $\{u_p\}_p \subset W^{1,p}_{loc}(\Omega)$ *such that u_p converges in $L^1_{loc}(\Omega)$, as $p \to 1^+$, to the unique solution to* (5.3).

Remark 5.1. *Observe that we work on very general conditions on the nonlinearity f which do not guarantee uniqueness of solutions to* (5.2) *(for instance we do not assume condition (5.5) below). Anyway, thanks to Corollary 3.2 uniqueness is achieved in the limit as p goes to one.*

We first recall the notion of weak solution to (5.2).

Definition 5.1. *A function $u_p \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution to Problem (5.2) if $T_k(u_p) = k$ on $\partial\Omega$, for any $k > 0$, and*

$$\int_\omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla v + \int_\omega f(u_p)v = 0 \tag{5.4}$$

for any $v \in W^{1,p}_0(\omega)$ and $\omega \subset\subset \Omega$.

Existence and uniqueness of nonnegative solutions u_p to this problem are studied in [18] (see also [34]). In particular, existence is obtained (see [34, Theorem 3.3]) under the condition on f to be an increasing and continuous function with $f(0) = 0$ and satisfying (1.3). Under more restrictions on f (see [34, Corollary 4.5]) it is shown that the solution satisfies

$$\lim_{x\to\partial\Omega} \frac{u_p(x)}{\Psi_p^{-1}(dist(x,\partial\Omega))} = 1 \tag{5.5}$$

uniformly, where

$$\Psi_p(t) := \int_t^{+\infty} \frac{1}{(p'F(s))^{\frac{1}{p}}}\, ds\,.$$

Here $p' = \frac{p}{p-1}$ is the conjugate exponent of p. In [34] it is proved that (5.5) yields uniqueness. Note that, if we proceed in a formal way from (5.5) and we let $p \to 1^+$ we obtain that the limit solution when $p \to 1^+$ must be a bounded function. However, this argument is purely formal. Moreover, we want to show that this is in fact the case for any nonlinearity f verifying (H3) (which does not imply, in general, the condition on f needed for (5.5) to be true (see condition (A3) in [34]).

In order to perform our stability argument, we will need some careful local a priori estimates on the solutions u_p to (5.2). Since we need these estimates to be nondegenerate as p approaches 1, we have to explicit all the constants appearing in the calculations in order to control them. For the sake of simplicity, throughout this section C will indicate any positive constant (that may change his value from line to line) that could depend on N, Ω, but not on p; if needed we will also use symbols as, for instance, $C_{N,|\Omega|}$ in order to stress the dependence of the constant on N and $|\Omega|$.

As we already mentioned, without loss of generality, we can suppose p small enough. Note that, if $p < \frac{3}{2}$, then we can apply a *controlled Sobolev inequality* that reads as follows: let $v \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega |\nabla v|^p \geq \left(\frac{2N-3}{3N-2}\right)^2 \left(\int_\Omega |v|^{p^*}\right)^{\frac{p}{p^*}}, \tag{5.6}$$

for any $p \in (1, \frac{3}{2})$ with p^* being the Sobolev conjugate of p (see for instance [9]).

5.1. Basic a priori estimates

In order to obtain local estimates we need to construct suitable cut-off functions. To do this we will use a technical lemma whose proof can be found in [28, Lemma 1.1].

Lemma 5.2. *Let f be an increasing function such that $f(0) = 0$, and satisfying (5.1), and let K be a positive constant. Then there exists a smooth function $\varphi : [0,1] \mapsto [0,1]$, with $\varphi(0) = \varphi'(0) = 0$, $\varphi(1) = 1$, such that*

$$t^p \frac{\varphi'(\sigma)^p}{\varphi(\sigma)^{p-1}} \leq \frac{1}{K} t f(t) \varphi(\sigma) + 1,$$

for any $\sigma \in [0,1]$, $t \geq 0$.

Remark 5.2. *Observe that, a priori, φ can depend on p. In fact, Lemma 5.2 is nothing but a generalization of Young's Inequality. In the model case in which $f(s) = cs^{\overline{q}}$, with $\overline{q} > p-1$, then $\varphi(\sigma) = \sigma^{\frac{p(\overline{q}+1)}{\overline{q}-p+1}}$.*

For $0 < r < R \leq 1$, we will consider cut-off functions ξ on balls $B_r \subset B_R \subset\subset \Omega$, that is, smooth functions in $C_0^1(B_R)$, such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on B_r. Observe that, if we set $\eta := \varphi(\xi)$, where φ is given by Lemma 5.2, then η is a cut-off function for B_r as well in B_R.

Local energy estimate. Here we prove a local estimate for ∇u_p in $(L^p_{loc}(\Omega))^N$.

Theorem 5.3. *Let u_p be a weak solutions to (5.2). Then*

$$\int_{B_r} |\nabla u_p|^p \leq C_{r,R}\,. \tag{5.7}$$

Proof. Let ξ be a cut-off function for B_r in B_R such that

$$|\nabla \xi| \leq \frac{C}{R-r},$$

φ be as in Lemma 5.2, and consider the cut-off function $\eta := \varphi(\xi)$. We can take $v = u_p\eta$ as test in (5.4) and we obtain

$$\int_{B_R} |\nabla u_p|^p \eta + \int_{B_R} f(u_p)u_p\eta = \int_{B_R} |\nabla u_p|^{p-2}\nabla u_p \nabla\xi\varphi'(\xi)u_p.$$

Now, also using Young's inequality, we have

$$\begin{aligned}
&\left|\int_{B_R} |\nabla u_p|^{p-2}\nabla u_p\nabla\xi\varphi'(\xi)u_p\right| \\
&\le \int_{B_R} |\nabla u_p|^{p-1}|\nabla\xi|\varphi'(\xi)u_p \le \frac{C}{R-r}\int_{B_R} |\nabla u_p|^{p-1}\varphi(\xi)^{\frac{p-1}{p}}\frac{\varphi'(\xi)}{\varphi(\xi)^{\frac{p-1}{p}}}u_p \\
&\le \frac{C\varepsilon}{p'(R-r)}\int_{B_R} |\nabla u_p|^p\eta + \frac{C}{\varepsilon^{p-1}p(R-r)}\int_{B_R}\frac{\varphi'(\xi)^p}{\varphi(\xi)^{p-1}}u_p^p \\
&\le \frac{1}{2}\int_{B_R} |\nabla u_p|^p\eta + \frac{C}{(R-r)}\int_{B_R}\frac{\varphi'(\xi)^p}{\varphi(\xi)^{p-1}}u_p^p\,,
\end{aligned}$$

where, in the last inequality, we also choose $\varepsilon = \frac{(R-r)}{2C}$.

Therefore, we have

$$\frac{1}{2}\int_{B_R} |\nabla u_p|^p\eta + \int_{B_R} f(u_p)u_p\eta \le \frac{C}{(R-r)}\int_{B_R}\frac{\varphi'(\xi)^p}{\varphi(\xi)^{p-1}}u_p^p\,. \tag{5.8}$$

Now, we apply Lemma 5.2 with $t = u_p$ and $K = \frac{R-r}{C}$, in order to obtain

$$\frac{C}{(R-r)}\int_{B_R}\frac{\varphi'(\xi)^p}{\varphi(\xi)^{p-1}}u_p^p \le \int_{B_R} f(u_p)u_p\eta + \frac{C}{R-r}|B_R|,$$

which, together with (5.8), yields the desired result since $\eta \equiv 1$ on B_r.

We will also need the following *global BV* bound on the truncations of u_p.

Lemma 5.4. *Let u_p be a weak solution of (5.2). Then*

$$\int_\Omega |\nabla T_k(u_p)|\,dx \le C(kf(k))^{\frac{1}{p}}\,. \tag{5.9}$$

Proof. Fix $k \in [0, +\infty[$ and let $\Omega_{p,k} := \{x \in \Omega : u_p(x) \le k\}$. First we prove that $\Omega_{p,k} \subset\subset \Omega$. In fact, suppose by contradiction that this is not the case. So that, the exist $x \in \partial\Omega \cap \overline{\Omega_{p,k}}$ and a sequence $\{x_n\} \subset \Omega_{p,k}$, with x_n that converges to x. Now, since $u_p(x_n) \le k$ we deduce that

$$\Psi_p(u_p(x_n)) \ge \Psi_p(k),$$

that is

$$\frac{\Psi_p(u_p(x_n))}{dist(x_n, \partial\Omega)} \ge \frac{\Psi_p(k)}{dist(x_n, \partial\Omega)},$$

where the right hand side of the previous inequality diverges as x_n approaches x. This is a contradiction since

$$\lim_{x_n \to x} \frac{\Psi_p(u_p(x_n))}{dist(x_n, \partial\Omega)} = 1,$$

as proved in [34, Theorem 4.4].

Now we are allowed to take $T_k(u_p) - k$ as test function in (5.2). Therefore,

$$\int_\Omega |\nabla T_k(u_p)|^p \, dx = \int_{\Omega_{p,k}} |\nabla T_k(u_p)|^p \, dx$$

$$= \int_{\Omega_{p,k}} f(u_p)(k - T_k(u_p)) \, dx \le f(k)k|\Omega|.$$

Finally, by Hölder's inequality,

$$\int_\Omega |\nabla T_k(u_p)| \, dx \le \left(\int_\Omega |\nabla T_k(u_p)|^p \, dx \right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \le C(kf(k))^{\frac{1}{p}} .$$

Local boundedness of the solutions. First, we prove an upper bound for solutions to (5.2) in case the domain is a ball:

Lemma 5.5. *Let $\Omega = B_R$. Then, there is a weak solution to* (5.2)*, $u_p \in W^{1,p}_{loc}(B_R) \cap L^\infty_{loc}(B_R)$ such that*

$$u_p \le \Psi_p^{-1} \left(\frac{1}{p'N} \left(R - \left(\frac{|x|^p}{R} \right)^{\frac{1}{p-1}} \right) \right). \tag{5.10}$$

Proof. The existence and the stated regularity of the weak solution follows from [34, Theorem 3.3]. However, we need to recall how this weak solution is obtained (see [34] for the details). We let $u_{p,n} \in W^{1,p}(\Omega)$ to be the unique solution to

$$\begin{cases} \Delta_p u_{p,n} = f(u_{p,n}) & \text{in } \Omega \\ u_{p,n} = n & \text{on } \partial\Omega \end{cases} \tag{5.11}$$

Then, since the sequence $\{u_{p,n}\}$ is increasing with respect to n, it converges pointwise to a function u_p. This function u_p is a weak solution to (5.2) with the stated regularity.

For (5.10), we closely follow the proof of [30, Lemma 4.1]. We let

$$\tilde{F}(r) := \frac{f \circ \Psi_p^{-1}(r)}{(p'F \circ \Psi_p^{-1}(r))^{\frac{1}{p'}}}.$$

Then, $w_p := \Psi_p(u_p)$ is a distributional solution to

$$\begin{cases} \Delta_p w_p = \tilde{F}(w_p)[|\nabla w_p|^p - 1] & \text{in } B_R \\ \\ w_p = 0 & \text{on } \partial B_R \end{cases}. \tag{5.12}$$

We consider now w_p^0 to be the solution to

$$\begin{cases} \Delta_p w_p^0 = -\dfrac{N^{2-p}}{R} & \text{in } B_R \\ \\ w_p^0 = 0 & \text{on } \partial B_R\,. \end{cases} \tag{5.13}$$

Then, w_p^0 is explicitly given by (see e.g. [42])

$$w_p^0 = \frac{1}{p'N}\left(R - \left(\frac{|x|^p}{R}\right)^{\frac{1}{p-1}}\right).$$

We next show that w_p^0 is a subsolution to (5.12).

In the case $f(s) = cs^q$, a direct computation shows that

$$\tilde{F}(r) = \frac{(p-1)(q+1)}{(q+1-p)r}.$$

Therefore,

$$\begin{aligned} &\Delta_p w_p^0 + \tilde{F}(w_p^0)(1 - |\nabla w_0|^p) \\ &= -\frac{N^{2-p}}{R} + \frac{(q+1)p^{\frac{1}{p'}}N}{(q+1-p)\left(R - \left(\frac{|x|^p}{R}\right)^{\frac{1}{p-1}}\right)}\left(1 - \frac{1}{N^p}\left(\frac{|x|}{R}\right)^{p'}\right) \\ &\geq -\frac{N^{2-p}}{R} + \frac{N}{\left(R - \left(\frac{|x|^p}{R}\right)^{\frac{1}{p-1}}\right)}\left(1 - \frac{1}{N^p}\left(\frac{|x|}{R}\right)^{p'}\right) \\ &= \frac{N}{R}\left(\frac{\left(1 - \frac{1}{N^p}\left(\frac{|x|}{R}\right)^{p'}\right)}{\left(1 - \left(\frac{|x|}{R}\right)^{p'}\right)} - N^{1-p}\right) \geq 0. \end{aligned}$$

From here, we can proceed exactly as in [30, Lemma 4.1] to conclude that $w_p \geq w_p^0$ in B_R. Then,

$$u_p = \Psi_p^{-1}(w_p) \leq \Psi_p^{-1}(w_p^0)\,. \tag{5.14}$$

In the case of a general f satisfying (5.1) we argue by comparison. We consider the approximating solutions to problem (5.2), that is the weak solutions $u_{p,n}$ to problem (5.11) that exist and are unique (see for instance [34, Theorem A]). Observe that thanks to (5.1) we have that $u_{p,n}$ is a subsolution to problem

$$\begin{cases} -\Delta_p v_{p,n} + c v_{p,n}^{\overline{q}} = 0 & \text{in } \Omega, \\ v_{p,n} = n & \text{on } \partial\Omega, \end{cases}$$

so that, by standard comparison, $u_{p,n} \leq v_{p,n}$, and the bound (5.14) is obtained as n goes to ∞ as u_p is the a.e. limit of $u_{p,n}$.

Local BV estimate and local estimate on the vector field. We are in the position to give the essential estimates in order to pass to the limit in the approximating problems (5.2). We prove the following:

Theorem 5.6. *Let $0 < r < R$, then it holds,*

$$\int_{B_r} |\nabla u_p| \leq \tilde{C}_{r,R}, \tag{5.15}$$

and for any $1 < q < p'$,

$$\int_{B_r} |\nabla u_p|^{(p-1)q} \leq \tilde{C}_{r,R,q}\,. \tag{5.16}$$

Proof. Using Hölder's inequality and (5.7), we have

$$\int_{B_r} |\nabla u_p| \leq \left(\int_{B_r} |\nabla u_p|^p \right)^{\frac{1}{p}} |B_r|^{\frac{1}{p'}} \leq \tilde{C}_{r,R}.$$

Furthermore, in the same way, we have, for any $q < p'$,

$$\int_{B_r} |\nabla u_p|^{(p-1)q} \leq \left(\int_{B_r} |\nabla u_p|^p \right)^{\frac{q}{p'}} |B_r|^{\frac{p'-q}{p'}} \leq \tilde{C}_{r,R,q}.$$

Observe that, by virtue of Theorem 5.3, the constants $\tilde{C}_{r,R}$ and $\tilde{C}_{r,R,q}$, that in principle do depend on p, are uniformly controlled, and so they can be chosen to be independent of this parameter as p approaches 1.

5.2. Passage to the limit

Observe that, from both the L^∞_{loc} bound on u_p and (5.15) we deduce that, in particular, the sequence $\{u_p\}$ is locally bounded in $W^{1,1}(\Omega)$, so that we can find a subsequence, not relabeled, such that u_p converges in $L^1_{loc}(\Omega)$ and a.e. in Ω to a function $u \in L^1_{loc}(\Omega)$. As a first step, we need to explicit the L^∞ bound on u_p in order to obtain a global bound on the limit u .

Global L^∞ bound on u. Let s_Ω denote the radius given by the uniform interior ball condition. Then, we can cover Ω with interior balls with radius bigger than or equal to s_Ω. By Lemma 5.5, we have that

$$u_p \leq \Psi_p^{-1}\left(\frac{1}{p'N}\left(s_\Omega - \left(\frac{|x-x_0|^p}{s_\Omega}\right)^{\frac{1}{p-1}}\right)\right) \quad \text{for any } x \in B_{s_\Omega}(x_0) \subset \Omega.$$

Let us consider, without loss of generality, the case $f(s) = s^q$. Reasoning as before, the result for a general f satisfying (5.1) will easily follow by comparison.

In this case we have,

$$\Psi_p^{-1}(s) = \left(\left(\frac{(q+1)}{p'} \right)^{\frac{1}{p}} \frac{p}{(q+1-p)s} \right)^{\frac{p}{q+1-p}} .$$

Then,

$$u_p \leq \left(\frac{(q+1)^{\frac{1}{p}} p p'^{\frac{1}{p'}} N}{(q+1-p) \left(s_\Omega - \left(\frac{|x-x_0|^p}{s_\Omega} \right)^{\frac{1}{p-1}} \right)} \right)^{\frac{p}{q+1-p}} \overset{p \to 1^+}{\to} \left(\frac{(q+1)N}{q\ s_\Omega} \right)^{\frac{1}{q}} ,$$

that is,

$$u \leq \left(\frac{(q+1)N}{q\ s_\Omega} \right)^{\frac{1}{q}} , \quad \text{a.e } x \in \Omega.$$

In the general case we have,

$$f(u) \leq \frac{(q+1)N}{q\ s_\Omega} , \quad \text{a.e } x \in \Omega. \tag{5.17}$$

Convergence of the term $|\nabla u_p|^{p-2}\nabla u_p$. Observe that by (5.15) and (5.9) $u \in TBV_{loc}(\Omega)$. As, by what we have just showed, $u \in L^\infty(\Omega)$ we deduce that $u \in BV(\Omega)$.

Furthermore, let $\omega \subset\subset \Omega$. We have that $|\nabla u_p|^{p-2}\nabla u_p$ is weakly relatively compact in $L^1(\omega;\mathbb{R}^N)$. This is an easy consequence of (5.16). In particular, we may assume that there exists $z_\omega \in L^1(\omega, \mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_\omega , \quad \text{as } p \to 1^+ \quad \text{weakly in } L^1(\omega, \mathbb{R}^N).$$

Following the proof of [5, Lemma 1], we can prove that $\|z_\omega\|_\infty \leq 1$. Moreover, by a diagonal argument we can find $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$ and a subsequence (not relabeled) such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z , \quad \text{as } p \to 1^+ \quad \text{weakly in } L^1_{loc}(\Omega, \mathbb{R}^N).$$

On the other hand, taking $\varphi \in C_0^\infty(\omega)$ in (5.4) and letting $p \to 1^+$ we obtain that

$$\text{div} z = f(u) \quad \text{in } \mathcal{D}'(\omega).$$

Finally, using (5.17), we deduce that

$$\text{div} z = f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

We now use the lower semicontinuity in $L^1(\Omega)$ (see for instance [10]) of the energy functional defined by

$$\mathcal{F}_k(v) := \begin{cases} \displaystyle\int_\Omega |Dv| + \int_{\partial\Omega} |k - v|\, d\mathcal{H}^{N-1} & \text{if } v \in BV(\Omega) \\ \qquad\qquad +\infty & \text{otherwise} \end{cases}$$

As we already pointed out, Lemma 5.4 allows us to deduce the relative strong compactness in $L^1(\Omega)$ of $T_k(u_p)$, so that, reasoning as in Lemma 5.4 and using Young's inequality, we have that

$$\int_\Omega |DT_k(u)| + \int_{\partial\Omega} |k - T_k(u)|\, d\mathcal{H}^{N-1} \le \liminf_{p\to 1^+} \int_\Omega |DT_k(u_p)|$$

$$\le \liminf_{p\to 1^+} \frac{1}{p} \int_\Omega |DT_k(u_p)|^p + \lim_{p\to 1^+} \frac{|\Omega|}{p'}$$

$$= \liminf_{p\to 1^+} \frac{1}{p} \int_\Omega f(u_p)(k - T_k(u_p))\, dx = \int_\Omega f(u)(k - T_k(u))\, dx$$

$$= \int_\Omega \mathrm{div} z(k - T_k(u))\, dx = \int_\Omega -(z, DT_k(u)) + \int_{\partial\Omega} [z,\nu](k - T_k(u))\, d\mathcal{H}^{N-1}$$

$$\le \int_\Omega |DT_k(u)| + \int_{\partial\Omega} |k - T_k(u)|\, d\mathcal{H}^{N-1}.$$

From here, since the measures $|DT_k(u)|$ are supported in Ω, letting $k \ge f^{-1}(\frac{(q+1)N}{qs_\Omega})$ we obtain that

$$(z, Du)(\Omega) = |Du|(\Omega) \ \text{ and}$$

$$[z,\nu] = 1\,, \qquad \mathcal{H}^{N-1} - \text{a.e. in } \partial\Omega.$$

Therefore, by (2.3) and Remark 3.2, u is a large solution of (5.3) in the sense of Definition 3.2.

Remark 5.3. *Observe that, a straightforward modification of our arguments in Section 4 allows us to easily extend the result of existence and uniqueness of large solutions to a larger class of nonhomogeneous problems of the type*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + f(u) = g(x) & \text{in } \Omega \\[2ex] \qquad\qquad u = +\infty & \text{on } \partial\Omega\,, \end{cases}$$

with $g \in L^1(\Omega)$ (cfr. with (4.3)). On the other hand, in order to deal with the stability result contained in the last section we have to restrict the class of data g to $L^m(\Omega)$, with $m > N$ in order to get locally bounded approximating solutions.

We would like to conclude by showing that the upper bounds obtained in (5.17) and (5.19) are or can be taken to be the optimal ones in some particular cases.

Remark 5.4. *Let us consider, as a model, the case of $f(s) = u^q$ for some $q > \frac{1}{N-1}$. With a small modification of our argument we can show that, in this* *case, we can find an optimal upper bound for u. In fact, if we take $\tilde{w}_p^0$ to be the solution to*

$$\begin{cases} \Delta_p \tilde{w}_p^0 = -\dfrac{N^{2-p}}{R}\left(\dfrac{q+1}{q+1-p}\right)^{\frac{1}{p'}} & \text{in } B_R \\ \\ \tilde{w}_p^0 = 0 & \text{on } \partial B_R \end{cases} \tag{5.18}$$

and we follow the proof of Lemma 5.5, using $q > \frac{1}{N-1}$, we get that for p sufficiently close to 1,

$$u_p \leq \Psi^{-1}(\tilde{w}_p^0) \overset{p\to 1^+}{\to} \left(\frac{N}{R}\right)^{\frac{1}{q}}.$$

Therefore, in this case the procedure to bound the approximate solutions ends up with the exact constant as it can be deduced by Remark 4.1 together with the existence result given by Theorem 4.5. As a second example, the bound is optimal also for the exponential case $f(s) = e^s$ (whose solution exists and it unique as, in view of Remark 5.3, we can subtract and add the constant 1 in order to satisfy our assumptions on f). In fact, in this case, $\Psi_p^{-1}(r) = \log\left(\frac{(p-1)p^{p-1}}{r^p}\right)$ and $\tilde{F}(r) = \frac{p-1}{r}$. Then, a direct computation shows that

$$u_p(r) \leq \log\left(\frac{p^{2p-1}N^p}{(p-1)^{p-1}\left(s_\Omega - \frac{|x|^p}{s_\Omega}^{\frac{1}{p-1}}\right)^p}\right) \overset{p\to 1^+}{\longrightarrow} \log\left(\frac{N}{s_\Omega}\right) \tag{5.19}$$

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