Higher differentiability of minimizers of variational integrals with Sobolev coefficients

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Abstract

In this paper we consider integral functionals of the form

$$\mathfrak{F}(v,\Omega)=\int_\Omega F(x,Dv(x))\,\mathrm{d}x$$

with convex integrand satisfying p growth conditions with respect to the gradient variable.
As a novel feature, the dependence of the integrand on the x-variable is allowed to be through a Sobolev function. We prove local higher differentiability results for local minimizers of the functional $\mathfrak{F}$, establishing uniform higher differentiability estimates for solutions to a class of auxiliary problems, constructed adding singular higher order perturbations to the integrand. Furthermore, we prove a dimension free higher integrability result for the gradient of local minimizers, by the use of a weighted version of the Gagliardo-Nirenberg interpolation inequality.

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1 Introduction and Statement of Results

We prove higher differentiability results for minimizers of convex variational integrals of the form

$$\mathfrak{F}(v,O)=\int_O F(x,Dv(x))\,\mathrm{d}x \tag{1.1}$$

with convex integrand F satisfying p growth conditions with respect to the gradient variable. The functionals $\mathfrak{F}$ are defined for Sobolev maps $v\in \mathrm{W}^{1,r}(\Omega,\mathbb{R}^N)$, $r>1$, and open subsets O of a fixed bounded and open subset Ω of $\mathbb{R}^n$. Our main concern is the multi–dimensional vectorial case n, $N\geq 2$, but our results remain new also in the scalar case $N=1$.
There exists a wide literature concerning the regularity of minimizers of the functional $\mathfrak{F}(v,O)$ in case the integrand F is assumed to satisfy the following

assumptions

$$\begin{cases} c_1|\xi|^p \le F(x,\xi) \le c_2(\mu^2+|\xi|^2)^{\frac{p}{2}} \\ \\ \nu(\mu^2+|\xi|^2)^{\frac{p-2}{2}}|\eta|^2 \le \langle D_{\xi\xi}F(x,\xi)\eta,\eta\rangle \\ \\ |F(x_1,\xi)-F(x_2,\xi)| \le \omega(|x_1-x_2|)(\mu^2+|\xi|^2)^{\frac{p-2}{2}} \end{cases} \tag{1.2}$$

for some positive constants c_1, c_2, ν, a parameter $\mu \ge 0$, for every $\xi, \eta \in \mathbb{R}^{N\times n}$ and where the dependence on the x-variable is Hölder continuous with some exponent α, i.e.

$$\omega(\rho) = \min\{\rho^\alpha, 1\} \qquad\qquad (\alpha, 1]. \tag{1.3}$$

For an exhaustive treatment of the regularity of F-minimizers under the assumptions at (1.2), also in case F is quasiconvex with respect to the gradient variable, we refer the interested reader to [21, 22] and the references therein. In the last few years, the study of the regularity has been successfully carried out under weaker assumptions on the function $\omega(\rho)$, which, roughly speaking, measures the continuity of the integrand F with respect to the x-variable. In particular, in [18] (see also [13, 14]), a partial $C^{0,\alpha}$ regularity result has been established relaxing the assumption $(1.2)_3$ in a continuity assumption of the type

$$\lim_{\rho\to 0}\omega(\rho) = 0.$$

Very recently, the result of [18] has been extended in [4] to functionals that have discontinuous dependence on the x-variable, through a VMO coefficient.

Our aim here is to study the regularity of the gradient of the F- minimizers of the functional $\mathfrak{F}(v, O)$, relaxing both the assumptions at $(1.2)_1$ and $(1.2)_3$. More precisely, we will deal with degenerate functionals and we will replace the Hölder continuous dependence of the integrand with respect to the x-variable at $(1.2)_3$ with a suitable Sobolev regularity assumption. A simple model case of the functionals we have in mind is

$$\mathfrak{I}(v, O) = \int_O a(x) f(Dv)\, \mathrm{d}x$$

where $a(x)$ lies in a suitable Sobolev class and that can be unbounded. Since our goal is to obtain the regularity at level of the gradient of the minimizer (not of the function itself as in the above quoted papers), with respect to the hypotheses of [4], we have to assume a slightly stronger regularity for the integrand F with respect to x, still dealing with the case of discontinuous coefficients.
In order to state the results precisely, we shall briefly introduce and discuss our hypotheses.
Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be an integrand satisfying for an exponent $p \ge 2$, a function g such that $g^{\frac{p}{p-1}+1}(x) \in \mathrm{W}^{1,n}(\Omega)$ and a constant $L > 0$, the following set of hypotheses:

$$\xi \mapsto F(x,\xi) \text{ is a strictly convex } C^2 \text{ function for a.e. } x \in \Omega \tag{H1}$$

$$\frac{1}{g(x)}|\xi|^p \le F(x,\xi) \le g(x)|\xi|^p \tag{H2}$$

One can easily see that the convexity assumption (H1) and the growth condition (H2) imply

$$|D_\xi F(x,\xi)| \le c(p)g(x)|\xi|^{p-1} \tag{H3}$$

Concerning the dependence on the x-variable we shall assume that

$$x \mapsto F(x,\xi) \text{ is weakly differentiable for every } \xi \in \mathbb{R}^{\mathrm{N}\times\mathrm{n}} \tag{H4}$$

and

$$|D_x D_\xi F(x,\xi)| \le L|Dg(x)||\xi|^{p-1}\,. \tag{H5}$$

The convexity assumption on the integrand F can be expressed as the following degenerate ellipticity condition on the matrix $D_{\xi\xi}F$

$$\langle D_{\xi\xi}F(x,\xi)\eta,\eta\rangle \ge \frac{1}{g(x)}|\xi|^{p-2}|\eta|^2 \tag{H6}$$

for all $\xi \in \mathbb{R}^{N\times n}$, $\eta \in \mathbb{R}^{N\times n}$.
Let us give the definition of local minimizer:

Definition 1.1. A mapping $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local F–minimizer if $F(Du) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\mathrm{supp}\varphi} F(x,Du)\,\mathrm{d}x \le \int_{\mathrm{supp}\varphi} F(x,Du+D\varphi)\,\mathrm{d}x$$

for any $O \Subset \Omega$ and any $\varphi \in \mathrm{C}^\infty_0(O,\mathbb{R}^N)$.

By virtue of our assumptions on the integrand F, a local F–minimizer minimizer u solves the corresponding Euler Lagrange system

$$\int_\Omega \langle D_\xi F(x,Du), D\varphi\rangle\, dx = 0$$

for every $\varphi \in \mathrm{C}^\infty_0(O,\mathbb{R}^N)$.
Remark that assumptions (H3) and (H6) implies the following

$$|Du|^p + |D_\xi F(x,Du)|^{\frac{p}{p-1}} \le \widetilde{g}(x)\langle D_\xi F(x,Du), Du\rangle \tag{1.4}$$

with $\widetilde{g} = g(1+g^{\frac{p}{p-1}})$. The assumption (H1) implies that we are dealing with the genuine anisotropic case, since the ratio between the eigenvalues can be unbounded and the matrix $D_{\xi\xi}F$ satisfies a degenerate ellipticity condition. The function $g^{\frac{p}{p-1}+1}$ appearing in the right hand side of inequality (1.4), that measures the degree of degeneracy of our problem, is assumed to belong to the Sobolev class $\mathrm{W}^{1,n}(\Omega)$. It is well known that $\mathrm{W}^{1,n}(\Omega) \subsetneq VMO$, i.e. the following condition

$$\lim_{r\to 0}\sup_{\rho\le r} \fint_{B_\rho(x_0)} |g^{p'+1}(x) - (g^{p'+1})_{\rho,x_0}| = 0,$$

holds for every ball $B_r(x_0) \Subset \Omega$, (see [9]), where we denoted by $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of p. We also have, through the classical Moser Trudinger inequality, that $g^{p'+1}$ is exponentially integrable, i.e.

$$\int_\Omega \exp(\lambda g^{\frac{(p'+1)n}{n-1}})\,\mathrm{d}x < +\infty$$

for some constant $\lambda > 0$, depending on the $\mathrm{W}^{1,n}$- norm of $g^{p'+1}$. Hence, combining Hölder's inequality in Orlicz-Sobolev spaces with the requirement that $F(Du) \in L^1_{\mathrm{loc}}(\Omega)$, we obtain that

$$\int_{\Omega'} |Du|^p \log^{-\frac{n-1}{(p'+1)n}}(e + |Du|)\,\mathrm{d}x < \infty \tag{1.5}$$

i.e. the gradient of a local minimizer belongs to the Orlicz-Zygmund class $\mathrm{L}^p \log^{-\frac{n-1}{(p'+1)n}} L_{\mathrm{loc}}(\Omega; \mathbb{R}^{N\times n})$.

Regularity results for scalar minimizers of functionals, as well as for solutions of partial differential equations, with exponentially integrable degeneracy can be found in [5, 6, 7, 25, 26, 19] where higher integrability results have been obtained in the scale of Orlicz-Zygmund classes by means of Gehring–type inequalities (see Theorem 2.4 in Section 2). With the use of Young's inequality in Orlicz spaces, one can easily check that there exist two positive constants, both depending on the norm of the function $g^{p'+1}$ in the exponential class $\mathrm{Exp}_{\frac{n}{n-1}}(\Omega)$, such that

$$\int_{\Omega} |Du|^r\,\mathrm{d}x - c_1 \le \int_{\Omega} F(x, Du(x))\,\mathrm{d}x \le \int_{\Omega} |Du|^q\,\mathrm{d}x - c_1 \tag{1.6}$$

with $r < p < q$. Therefore assumption (H2), together with the exponential integrability of $g^{p'+1}$, entails an integral version of the well-known (p, q) growth conditions, introduced in the celebrated papers by Marcellini (see in particular [29, 30, 31]) and that have since attracted much attention.

The regularity of minimizers of functionals satisfying (p, q) growth conditions has been widely investigated, both in the scalar and in the vectorial setting (see for example [1, 2, 3, 11, 12, 15, 16, 27, 28, 32]).

In particular, the higher differentiability of the gradient is usually deduced by means of difference quotient methods. A different approach has been introduced in [8] and it is based on establishing higher differentiability estimates for solutions to a class of auxiliary problems, constructed adding singular higher order perturbations to the integrand.

The main idea here is to treat the regularity of minimizers of degenerate functionals with the tools needed to deal with functionals satisfying (p, q) growth condition.

In fact, with arguments similar to those in [8], we show that the Sobolev regularity of the coefficient is a sufficient condition to establish a higher differentiability result for the gradient of the minimizers. More precisely, we have the following

Theorem 1.2. *Let $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be an integrand satisfying the assumptions (H1)–(H6) for an exponent $2 \le p < n$ and a function g such that $g^{p'+1} \in \mathrm{W}^{1,n}(\Omega)$. If $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local F–minimizer, then*

$$V_p(Du) \in \mathrm{W}^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$$

where s is any exponent such that $s < 2$ and where $V_p(Du)$ is defined as

$$V_p(Du) := |Du|^{\frac{p-2}{2}} Du.$$

Furthermore, there exists a radius $R_0 = R_0(n, N, L, p)$ such that whenever $B_{2R} \subset B_{R_0} \Subset \Omega$ we have the Caccioppoli type inequality

$$\int_{B_R} \frac{1}{g} |D(V(Du))|^2 \, \mathrm{d}x \leq \frac{c}{R^6} \left(\int_{\Omega} (|g|^2 + |D(g^2)|)^n \, \mathrm{d}x \right)^{\frac{6}{n}} \left(\int_{B_{2R}} F(x, Du) \, \mathrm{d}x \right)$$

for a constant $c = c(n, N, L, p)$.

As far as we know, no higher differentiability results are available for minimizers of functionals that depend on the x-variable through a Sobolev function. Nevertheless, we'd like to mention that in [24] the authors deal with functionals depend on the x-variable through a coefficient that belongs to a fractional Sobolev space but satisfies also a Hölder's condition with arbitrarily small exponent.
Moreover, in [10], the higher differentiability of solutions to a non degenerate Beltrami equation with a Sobolev coefficient is achieved in the two dimensional setting.
A continuity result for solutions of linear elliptic equations with Sobolev coefficients has been established in [33].
Our results here don't cover the case $p = 2 = n$ which requires different tools and that will be treated in a forthcoming paper.

We remark that as a consequence of the Sobolev imbedding, Theorem 1.3 yields that the gradient of a F-mimimizer belongs to the space $\mathrm{L}^{\frac{n\tilde{p}}{n-2}}$, for every $\tilde{p} < p$. Hence, in case $p > n - 2$, the gradient of a F-mimimizer belongs to $\mathrm{L}^{\tilde{p}+2}$, for every $\tilde{p} < p$. Here, in case $F(x, \xi) = \tilde{F}(x, |\xi|)$ combining a suitable weighted version of the Gagliardo Nirenberg interpolation inequality (see Lemma 2.3 below) with a local boundedness result for F-mimimizers, we show that this higher integrability persists for F-minimizers without any restriction on the growth exponent p. More precisely, we have the following

Theorem 1.3. *Let $F \colon \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ satisfy the conditions (H1)– (H6) for an exponent $2 \leq p < n$ and a function g such that $g^{p'+1} \in \mathrm{W}^{1,n}(\Omega)$. Suppose in addition that $F(x, \xi) = \tilde{F}(x, |\xi|)$. If $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local F–minimizer, then*

$$Du \in \mathrm{L}^{\tilde{p}+2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N \times n})$$

for every $\tilde{p} < p$. Furthermore, there exists a radius $\bar{R} = \bar{R}(n, N, L, p)$ such that whenever $B_{2R} \subset B_{\bar{R}} \Subset \Omega$ we have the following inequality

$$\int_{B_{\frac{R}{2}}} \frac{1}{g} |Du|^{p+2} \, \mathrm{d}x \leq c ||u||^2_{L^\infty(B_R)} \left(\int_{\Omega} (|g|^2 + |D(g^2)|)^n \, \mathrm{d}x \right)^{\frac{6}{n}} \left(\int_{B_{2R}} F(x, Du) \, \mathrm{d}x \right)$$

for a constant $c = c(n, N, L, p, R)$.

The higher integrability exponent of the gradient of a F-minimizer is dimension–free since we are going to prove that the minimizers are locally bounded. Dimension–free higher integrability exponents have been established in [8] for a priori bounded minimizer of autonomous integrals, satisfying nonstandard growth conditions and in [23] for the second gradient of the F-minimizer, but under much more severe growth conditions (i.e. (H2), (H6) with $p = 2$ and Lipschitz continuous dependence with respect to x.).

The plan of the paper is the following. We have collected standard preliminary material in Section 2, which at the same time serves as our reference for notation. The proofs of the higher differentiability result stated in Theorem 1.2 and of the higher integrability result stated in Theorem 1.3 are presented in Sections 3 and 4, respectively.

2 Preliminaries

For matrices ξ, $\eta \in \mathbb{R}^{N\times n}$ we write $\langle \xi, \eta\rangle := \mathrm{trace}(\xi^T\eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi,\xi\rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. When $a\in\mathbb{R}^N$ and $b\in\mathbb{R}^n$ we write $a\otimes b\in\mathbb{R}^{N\times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column. Observe that $|a\otimes b| = |a||b|$, where $|a|$, $|b|$ denote the usual euclidean norms of a in $\mathbb{R}^N$, b in $\mathbb{R}^n$, respectively.
When $F\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is sufficiently differentiable we write

$$D_\xi F(x,\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(x,\xi{+}t\eta) \quad \text{and} \quad D_{\xi\xi}F(x,\xi)[\eta,\eta] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} F(x,\xi{+}t\eta)$$

for ξ, $\eta\in\mathbb{R}^{N\times n}$.
We recall the definition of the auxiliary function V_p as

$$V_p(\xi) = V(\xi) := |\xi|^{\frac{p-2}{2}}\xi.$$

For later reference, we note that, for a C^2 map w and for $p\ge 2$, a routine calculation yields

$$\left|D\Big[V(Dw)\Big]\right|^2 \le \frac{p^2}{4}|Dw|^{p-2}|D^2w|^2 \tag{2.1}$$

Now, we state an iteration lemma, which is very well-known, in a version suitable for our purposes.

Lemma 2.1. *Let $\Phi\colon [\frac{R}{2}, R]\to\mathbb{R}$ be a bounded nonnegative function on the interval $[\frac{R}{2}, R]$ where $R>0$. Assume that for all $\frac{R}{2}\le r<s\le R$ we have*

$$\Phi(r)\le \vartheta\Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\alpha} + \frac{D}{(s-r)^\beta}$$

where $\vartheta\in(0,1)$, A, B, C, $D\ge 0$ and $0<\alpha<\beta$ are constants. Then there exists a constant $c = c(\vartheta,\beta)$ such that

$$\Phi\left(\frac{R}{2}\right)\le c\left(A + \frac{B}{R^2} + \frac{C}{R^\alpha} + \frac{D}{R^\beta}\right)$$

See for instance [22], pp. 191–192, for a proof that can easily be adapted to cover the above statement too.
Let P be an increasing function from $P(0)=0$ to $\lim_{t\to\infty} P(t) = \infty$ and continuously differentiable on $(0,\infty)$. The Orlicz class generated by the function $P(t)$ consists of the functions h for which there exists a constant $\lambda = \lambda(h)>0$ such that

$$P\left(\frac{|h|}{\lambda}\right)\in L^1(\Omega).$$

In particular, the Orlicz-Zygmund classes $L^s \log^\alpha L$, $1 < s < \infty$, $\alpha \in \mathbb{R}$, are Orlicz classes generated by a function $P(t) \simeq t^s \log^\alpha(e+t)$ as $t \to \infty$.
For $\alpha > 0$, the dual Orlicz space to $L \log^\alpha L(\Omega)$ is the space $\mathrm{Exp}_{\frac{1}{\alpha}}(\Omega)$, generated by a function $Q(t) \simeq \exp(t^{\frac{1}{\alpha}})-1$, as $t \to \infty$. For $\alpha > 0$, the following elementary inequalities hold true

$$s^{p_1} \le s^p \log^{-\alpha}(e+s) \le s^p \le s^p \log^\alpha(e+s) \le s^{p_2} \qquad \forall s \ge 1 \tag{2.2}$$

where $p_1 < p < p_2$. The following Sobolev imbedding Theorem in the Orlicz-Sobolev setting can be found in [17]

Theorem 2.2. *Let $h \in W_0^{1,1}(\Omega)$ be a function such that $|Dh| \in L^n \log^{-\sigma} L(\Omega)$, some $0 \le \sigma < 1$. Then*

$$h \in EXP_{\frac{n}{n+\sigma-1}}(\Omega)$$

Obviously, Theorem 2.2 in case $\sigma = 0$ gives back the Moser-Trudinger embedding Theorem.
Now, we give a weighted version of the classical Gagliardo Nirenberg interpolation inequality

Lemma 2.3. *Let $p > 1$. For $\eta \in C_c^1(\Omega)$ with $\eta \ge 0$, $g \in \mathrm{W}_0^{1,n}$ with $g \ge 1$ and $u \in \mathrm{W}^{1,1} \cap L^\infty$ such that*

$$\frac{|D(V(Du)|}{\sqrt{g}} \in L^2(\Omega)\,,$$

we have

$$\begin{aligned} &\int_\Omega \eta^2 \frac{1}{g} |Du|^{p+2}\,\mathrm{d}x \le c(p+1)^2 \int_\Omega \eta^2 \frac{1}{g} |u|^2 |D(V(Du))|^2\,\mathrm{d}x \\ + \quad & c\int_\Omega \eta^2 |u|^2 \frac{|Dg|^2}{g^3} |Du|^p\,\mathrm{d}x + c\int_\Omega |u|^2 \frac{1}{g} |\nabla\eta|^2 |Du|^p\,\mathrm{d}x \end{aligned} \tag{2.3}$$

for an absolute positive constant c.

Proof. Integration by parts yields

$$\begin{aligned} &\int_\Omega \eta^2 \frac{1}{g} |Du|^{p+2}\,\mathrm{d}x = \\ = \quad & \int_\Omega \left\langle \eta^2 \frac{1}{g} |Du|^p Du, Du \right\rangle \mathrm{d}x = -\int_\Omega D\left[\eta^2 \frac{1}{g} Du |Du|^p\right] \cdot u\,\mathrm{d}x \\ \le \quad & (p+1)\int_\Omega \eta^2 \frac{1}{g} |u| |Du|^p |D^2 u|\,\mathrm{d}x + \int_\Omega \eta^2 |u| |Du|^{p+1} \frac{|Dg|}{g^2}\,\mathrm{d}x \\ + \quad & 2\int_\Omega \eta |u| |\nabla\eta| \frac{1}{g} |Du|^{p+1}\,\mathrm{d}x = I_1 + I_2 + I_3 \end{aligned} \tag{2.4}$$

We estimate I_1 by using the Young's inequality as follows

$$I_1 \le \frac{1}{16}\int_\Omega \eta^2 \frac{1}{g} |Du|^{p+2}\,\mathrm{d}x + c(p+1)^2 \int_\Omega \eta^2 \frac{1}{g} |u|^2 |Du|^{p-2} |D^2 u|^2\,\mathrm{d}x \tag{2.5}$$

Similarly, we have

$$I_2 \le \frac{1}{16}\int_\Omega \eta^2 \frac{1}{g} |Du|^{p+2}\,\mathrm{d}x + c\int_\Omega \eta^2 |u|^2 \frac{|Dg|^2}{g^3} |Du|^p\,\mathrm{d}x \tag{2.6}$$

and

$$I_3 \leq \frac{1}{16}\int_\Omega \eta^2\frac{1}{g}|Du|^{p+2}\,\mathrm{d}x + c\int_\Omega |u|^2\frac{1}{g}|\nabla\eta|^2|Du|^p\,\mathrm{d}x \tag{2.7}$$

Hence, inserting (2.5), (2.6) and (2.7) in (2.4), we get

$$\begin{aligned}
\int_\Omega \eta^2\frac{1}{g}|Du|^{p+2}\,\mathrm{d}x &\leq \frac{3}{16}\int_\Omega \eta^2\frac{1}{g}|Du|^{p+2}\,\mathrm{d}x \\
+ \quad & c(p+1)^2\int_\Omega \eta^2\frac{1}{g}|u|^2|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
+ \quad & c\int_\Omega \eta^2|u|^2\frac{|Dg|^2}{g^3}|Du|^p\,\mathrm{d}x + c\int_\Omega |u|^2\frac{1}{g}|\nabla\eta|^2|Du|^p\,\mathrm{d}x
\end{aligned}$$

Reabsorbing the first integral in the right hand side by the left hand side in previous estimate, we conclude with

$$\begin{aligned}
\int_\Omega \eta^2\frac{1}{g}|Du|^{p+2}\,\mathrm{d}x &\leq c(p+1)^2\int_\Omega \eta^2\frac{1}{g}|u|^2|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
+ \quad c\int_\Omega \eta^2|u|^2\frac{|Dg|^2}{g^3}|Du|^p\,\mathrm{d}x &+ c\int_\Omega |u|^2\frac{1}{g}|\nabla\eta|^2|Du|^p\,\mathrm{d}x
\end{aligned} \tag{2.8}$$

i.e. the thesis.

□

We conclude this section with an higher integrability result for the gradient of a local F–minimizer in the scale of Orlicz-Zygmund classes.

Theorem 2.4. *Let $F\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfy the conditions (H1)– (H6) for a function g such that $g^{\frac{p}{p-1}+1}\in \mathrm{W}^{1,n}$, and an exponent $2\leq p<n$. If $u\in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local F–minimizer, then*

$$|Du|\in \mathrm{L}^p_{\mathrm{loc}}\log^\alpha L(\Omega),$$

for every $\alpha>0$.

The proof can be easily obtained through the same arguments of [5, 19, 25].

3 Proof of Theorem 1.2

Our aim is to show that $V(Du)\in \mathrm{W}^{1,\tilde p}_{\mathrm{loc}}(\Omega)$, $\forall \tilde p<2$. Before proceeding with the proof, we need to carry out an approximation procedure, which is essentially based on the arguments contained in [8]. Here we give a weighted version, which takes into account the degeneracy of the integrand.
Fix a subdomain with a smooth boundary $\Omega'\Subset\Omega$ and take $k\in\mathbb{N}$, so large that we have the continuous embedding $\mathrm{W}^{k,2}(\Omega')\hookrightarrow \mathrm{C}^2(\overline{\Omega'})$. For a smooth kernel $\phi\in \mathrm{C}^\infty_c(B_1(0))$ with $\phi\geq 0$ and $\int_{B_1(0)}\phi=1$, we consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put $\tilde u_\varepsilon := \phi_\varepsilon * u$ on Ω' for each positive $\varepsilon<\operatorname{dist}(\Omega',\partial\Omega)$. By (1.5) and (2.2) we have that $Du\in L^{\tilde p}$ for every $\tilde p<p$and hence

$$\tilde u_\varepsilon \to u \text{ as } \varepsilon\searrow 0 \text{ strongly in } \mathrm{W}^{1,\tilde p}(\Omega')\quad \forall \tilde p<p. \tag{3.1}$$

Moreover we record that, for a suitable function $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ with $\tilde{\varepsilon} \searrow 0$ as $\varepsilon \searrow 0$, also

$$\tilde{\varepsilon} \int_{\Omega'} |D^k \tilde{u}_\varepsilon|^2 \to 0 \text{ as } \varepsilon \searrow 0. \tag{3.2}$$

For small $\varepsilon > 0$, we let $u_\varepsilon \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$ denote a minimizer to the functional

$$v \mapsto \int_{\Omega'} \Big(F(x, Dv) + \frac{\tilde{\varepsilon}}{2} |D^k v|^2 \Big)$$

on the Sobolev class $\mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$. The existence of u_ε is easily established by the direct method. Next two Lemmas are suitable versions for our purposes of Lemma 8 and Lemma 9 in [8]. We give it here for the sake of completeness.

Lemma 3.1. *For each* $\varphi \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_0(\Omega')$,

$$0 = \int_{\Omega'} \Big(\langle D_\xi F(x, Du_\varepsilon), D\varphi \rangle + \tilde{\varepsilon} \langle D^k u_\varepsilon, D^k \varphi \rangle \Big). \tag{3.3}$$

Furthermore, $u_\varepsilon \in \mathrm{W}^{2k,2}_{\mathrm{loc}}(\Omega')$.

Proof. The minimality of u_ε yields the weak form of the Euler–Lagrange system (3.3) by straight forward means since by our choice of k we have that Du_ε and $D\varphi \in \mathrm{L}^\infty(\Omega')$. The additional regularity of u_ε then follows from standard elliptic regularity theory if we notice that (3.3) can be rewritten as

$$\Delta^k u_\varepsilon = \frac{(-1)^k}{\tilde{\varepsilon}} \Big(\mathrm{div}\ D_\xi F(x, Du_\varepsilon) \Big) \tag{3.4}$$

where the composition of k Laplacians on the left–hand side acts row–wise, and as usual is understood in the distributional sense on Ω'. Since by our choice of k, $u_\varepsilon \in \mathrm{C}^2(\overline{\Omega'})$, the right–hand side of (3.4) belongs in particular to $\mathrm{L}^2(\Omega')$ from which we deduce $u_\varepsilon \in \mathrm{W}^{2k,2}_{\mathrm{loc}}(\Omega')$. □

Lemma 3.2. *As* $\varepsilon \searrow 0$, *we have that*

$$\int_{\Omega'} |Du_\varepsilon - Du|^{\tilde{p}} \, \mathrm{d}x \to 0, \qquad \forall \tilde{p} < p$$

and

$$\int_{\Omega'} F(x, Du_\varepsilon) \, \mathrm{d}x \to \int_{\Omega'} F(x, Du) \, \mathrm{d}x.$$

Proof. By the minimality of u_ε, we have that there exists $\varepsilon_0 > 0$ such that

$$\int_{\Omega'} \Big(F(x, Du_\varepsilon) + \frac{\tilde{\varepsilon}}{2} |D^k u_\varepsilon|^2 \Big) \le \int_{\Omega'} \Big(F(x, D\tilde{u}_\varepsilon) + \frac{\tilde{\varepsilon}}{2} |D^k \tilde{u}_\varepsilon|^2 \Big) \tag{3.5}$$

for all $0 < \varepsilon \le \varepsilon_0$. Now, by Theorem 2.4, the duality between the spaces $L \log^\alpha L$ and $EXP_{\frac{1}{\alpha}}$ and standard properties of mollifiers we obtain that

$$\lim_{\varepsilon \searrow 0} \int_{\Omega'} F(x, D\tilde{u}_\varepsilon) \le \int_{\Omega'} F(x, Du) \tag{3.6}$$

Fatou's Lemma allows us to conclude that

$$\int_{\Omega'} F(x, D\tilde{u}_\varepsilon) \to \int_{\Omega'} F(x, Du) \quad \text{as} \quad \varepsilon \searrow 0. \tag{3.7}$$

In view of (3.2), (3.6) and (3.7), inequality (3.5) implies

$$\limsup_{\varepsilon \searrow 0} \int_{\Omega'} \Big(F(x, Du_\varepsilon) + \frac{\tilde{\varepsilon}}{2} |D^k u_\varepsilon|^2 \Big) \leq \int_{\Omega'} F(x, Du). \tag{3.8}$$

By the left inequality in the assumption (H2) we have that $F(x,\xi) \geq \frac{1}{g(x)} |\xi|^p$ for all $\xi \in \mathbb{R}^{N \times n}$. Therefore estimate (3.8) and Hölder's inequality imply

$$\begin{aligned}
\int_{\Omega'} |Du_\varepsilon|^{\tilde{p}} &\leq \Big(\int_{\Omega'} g^{\frac{\tilde{p}}{p-\tilde{p}}} \Big)^{\frac{p-\tilde{p}}{p}} \Big(\int_{\Omega'} \frac{1}{g} |Du_\varepsilon|^p \Big)^{\frac{\tilde{p}}{p}} \\
&\leq c(n,p) \Big(\int_{\Omega'} \exp(g^{\frac{(p'+1)n}{n-1}}) \Big)^{\frac{p-\tilde{p}}{p}} \Big(\int_{\Omega'} \frac{1}{g} |Du_\varepsilon|^p \Big)^{\frac{\tilde{p}}{p}} \\
&\leq c(n,p) \Big(\int_{\Omega'} \exp(g^{\frac{(p'+1)n}{n-1}}) \Big)^{\frac{p-\tilde{p}}{p}} \Big(\int_{\Omega'} F(x, Du) \Big)^{\frac{\tilde{p}}{p}}
\end{aligned} \tag{3.9}$$

Hence by the assumption on g and Theorem 2.2, the family (Du_ε) is bounded in $\mathrm{L}^{\tilde{p}}(\Omega')$, $\forall \tilde{p} < p$, and since $u_\varepsilon = \tilde{u}_\varepsilon$ in the sense of trace on $\partial\Omega'$ a standard lower semicontinuity result together with the minimality of u allow us to conclude that

$$\liminf_{\varepsilon \searrow 0} \int_{\Omega'} F(x, Du_\varepsilon) \geq \int_{\Omega'} F(x, Du).$$

By virtue of (3.8), this implies that

$$\int_{\Omega'} \frac{\tilde{\varepsilon}}{2} |D^k u_\varepsilon|^2 \to 0 \tag{3.10}$$

and

$$\int_{\Omega'} F(x, Du_\varepsilon) \to \int_{\Omega'} F(x, Du)\,,$$

as $\varepsilon \searrow 0$. In order to conclude the proof, it is sufficient to note that, by the assumption (H6), standard calculations imply that

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{g(x)} \Big(|D\tilde{u}_\varepsilon|^2 + |Du_\varepsilon|^2 \Big)^{\frac{p-2}{2}} |D\tilde{u}_\varepsilon - Du_\varepsilon|^2 \\
\leq\ & c \int_{\Omega'} \Big(F(x, D\tilde{u}_\varepsilon) - F(x, Du_\varepsilon) - \langle D_\xi F(x, Du_\varepsilon), D\tilde{u}_\varepsilon - Du_\varepsilon \rangle \Big).
\end{aligned}$$

Here we have by Lemma 3.1,

$$\int_{\Omega'} \langle D_\xi F(x, Du_\varepsilon), D\tilde{u}_\varepsilon - Du_\varepsilon \rangle = -\tilde{\varepsilon} \int_{\Omega'} \langle D^k u_\varepsilon, D^k \tilde{u}_\varepsilon - D^k u_\varepsilon \rangle. \tag{3.11}$$

Hence using with Hölder's inequality, (3.2) and (3.10), it follows that

$$\int_{\Omega'} \frac{1}{g(x)} \Big(|D\tilde{u}_\varepsilon|^2 + |Du_\varepsilon|^2 \Big)^{\frac{p-2}{2}} |D\tilde{u}_\varepsilon - Du_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \searrow 0.$$

Because $p \geq 2$, we conclude by well-known means that

$$\int_{\Omega'} \frac{1}{g(x)} |D\tilde{u}_\varepsilon - Du_\varepsilon|^p \to 0,$$

and therefore also

$$\int_{\Omega'} |Du_\varepsilon - Du|^{\tilde{p}} \to 0, \qquad\qquad \forall \tilde{p} < p,$$

by a simple use of Hölder's inequality, as in (3.9). This concludes the proof. $\square$

We are now ready to embark on the core of the proof of Theorem 1.2.

Proof. Fix $B_{2R} = B_{2R}(x_0) \subset \Omega'$, radii $R \leq r < s \leq 2R \leq 2$ and a smooth cut-off function ρ satisfying $1_{B_r} \leq \rho \leq 1_{B_s}$ and $|D^i \rho| \leq \left(\frac{2}{s-r}\right)^i$ for each $i \in \mathbb{N}$. According to Lemma 3.1, we can test the Euler–Lagrange system (3.3) with $\varphi = \rho^{2k} D_j^2 u_\varepsilon$ for each direction $1 \leq j \leq n$:

$$\begin{aligned} 0 \quad &= \quad \int_{\Omega'} \Big\langle D_\xi F(x, Du_\varepsilon), D_j^2 Du_\varepsilon \Big\rangle \rho^{2k} + \int_{\Omega'} \Big\langle D_\xi F(x, Du_\varepsilon), D_j^2 u_\varepsilon \otimes D\Big(\rho^{2k}\Big) \Big\rangle \\ &\quad\ +\tilde{\varepsilon} \int_{\Omega'} \Big\langle D^k u_\varepsilon, D^k \Big(D_j^2 u_\varepsilon \rho^{2k}\Big) \Big\rangle \\ &=: \quad I + II + III. \end{aligned} \tag{3.12}$$

Integration by parts yields

$$\begin{aligned} I \quad &= \quad -\int_{\Omega'} \Big(\rho^{2k} \Big\langle D_j \Big(D_\xi F(x, Du_\varepsilon)\Big), D_j Du_\varepsilon \Big\rangle + 2k\rho^{2k-1} D_j \rho \Big\langle D_\xi F(x, Du_\varepsilon), D_j Du_\varepsilon \Big\rangle\Big) \\ &= \quad -\int_{\Omega'} \Big(\rho^{2k} D_{\xi\xi} F(x, Du_\varepsilon) \Big[D_j Du_\varepsilon, D_j Du_\varepsilon\Big] + \rho^{2k} \langle D_{x_j} D_\xi F(x, Du_\varepsilon), D_j Du_\varepsilon \rangle\Big) \\ &\quad\ -\int_{\Omega'} \Big(2k \Big\langle \rho^{2k-1} D_j \rho D_\xi F(x, Du_\varepsilon), D_j Du_\varepsilon \Big\rangle\Big) \\ &\leq \quad -\int_{\Omega'} \rho^{2k} \frac{1}{g(x)} |Du_\varepsilon|^{p-2} |D_j Du_\varepsilon|^2 + c(L) \int_{\Omega'} \rho^{2k} |D_{x_j} g| |Du_\varepsilon|^{p-1} |D_j Du_\varepsilon| \\ &\quad\ +c(p,k) \int_{\Omega'} \rho^{2k-1} |D_j \rho| g(x) |Du_\varepsilon|^{p-1} |D_j Du_\varepsilon|, \end{aligned}$$

where we used (H6), (H3) and (H5). Hence, using Young's inequality in the last two integrals, we obtain

$$\begin{aligned} I \quad &\leq \quad -\int_{\Omega'} \frac{\rho^{2k}}{g(x)} |Du_\varepsilon|^{p-2} |D_j Du_\varepsilon|^2 + \frac{1}{2} \int_{\Omega'} \frac{\rho^{2k}}{g(x)} |Du_\varepsilon|^{p-2} |D_j Du_\varepsilon|^2 \\ &\quad + \quad c(p,k) \int_{\Omega'} \rho^{2(k-1)} |D_j \rho|^2 g^3(x) |Du_\varepsilon|^p \\ &\quad + \quad c(p,L) \int_{\Omega'} \rho^{2k} g(x) |Dg|^2 |Du_\varepsilon|^p. \end{aligned} \tag{3.13}$$

By virtue of (H3) and Cauchy–Schwarz' inequality, we get

$$II \quad \leq \quad c(p,k) \int_{\Omega'} g(x) |Du_\varepsilon|^{p-1} \rho^{2k-1} |D\rho| |D_j^2 u_\varepsilon| \leq c(p,k) \int_{\Omega'} g^3(x) |Du_\varepsilon|^p \rho^{2(k-1)} |D\rho|^2$$

$$+ \quad \frac{1}{4}\int_{\Omega'} \frac{\rho^{2k}}{g(x)} |Du_\varepsilon|^{p-2}|D_jDu_\varepsilon|^2. \tag{3.14}$$

In order to estimate III, we argue as in [8] writing

$$III = \tilde{\varepsilon}\int_{\Omega'} \Big\langle D^k u_\varepsilon, D_j D^k\Big(\rho^{2k}D_ju_\varepsilon\Big) - D^k\Big(D_j\big(\rho^{2k}\big)D_ju_\varepsilon\Big)\Big\rangle$$

and integrating the first term by parts,

$$\begin{aligned} III &= -\tilde{\varepsilon}\int_{\Omega'}\Big(\Big\langle D_jD^ku_\varepsilon, D^k\Big(\rho^{2k}D_ju_\varepsilon\Big)\Big\rangle - \tilde{\varepsilon}\int_{\Omega'}\Big\langle D^ku_\varepsilon, D^k\Big(D_j\big(\rho^{2k}\big)D_ju_\varepsilon\Big)\Big\rangle\Big) \\ &=: III_1 + III_2. \end{aligned}$$

We estimate these terms by use of Cauchy-Schwarz' inequality, Leibniz' product formula and the assumptions on $D^i\rho$ (simplifying also by use of $s - r \le 1$):

$$\begin{aligned} III_1 &\le -\tilde{\varepsilon}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 + \frac{c_k\tilde{\varepsilon}}{(s-r)^k}\int_{\Omega'}\rho^k|D_jD^ku_\varepsilon|\sum_{i=0}^{k-1}|D^iD_ju_\varepsilon| \\ &\le -\frac{2\tilde{\varepsilon}}{3}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 + \frac{c_k\tilde{\varepsilon}}{(s-r)^{2k}}\int_{B_{2R}}\Big(\sum_{i=0}^{k-1}|D^iD_ju_\varepsilon|\Big)^2 \\ &\le -\frac{2\tilde{\varepsilon}}{3}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 + \frac{c_k\tilde{\varepsilon}}{(s-r)^{2k}}\int_{B_{2R}}\sum_{i=0}^{k-1}|D^iD_ju_\varepsilon|^2 \end{aligned}$$

for a (new) constant c_k. Likewise,

$$III_2 \le \frac{\tilde{\varepsilon}}{3}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 + \frac{c_k\tilde{\varepsilon}}{(s-r)^{2k+2}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}|D^iD_ju_\varepsilon|^2 + |D^ku_\varepsilon|^2\right),$$

where we remark that the increased power of the factor $(s-r)$ is due to the presence of an additional D_j-derivative on ρ^{2k} in III_2. Collecting the above bounds and adjusting the constant c_k we arrive at

$$III \le -\frac{\tilde{\varepsilon}}{3}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 + \frac{c_k\tilde{\varepsilon}}{(s-r)^{2k+2}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}|D^iD_ju_\varepsilon|^2 + |D^ku_\varepsilon|^2\right). \tag{3.15}$$

Inserting the bounds (3.13), (3.14), (3.15) in (3.12) and using the properties of ρ we get for each $1 \le j \le n$:

$$\begin{aligned} \frac{1}{4}\int_{\Omega'}\frac{\rho^{2k}}{g(x)}|Du_\varepsilon|^{p-2}|D_jDu_\varepsilon|^2 \quad &+ \quad \frac{\tilde{\varepsilon}}{3}\int_{\Omega'}\rho^{2k}|D_jD^ku_\varepsilon|^2 \\ &\le \frac{c(p,k)}{(s-r)^2}\int_{B_s\setminus B_r} g^3(x)|Du_\varepsilon|^p \\ &+ c(p,L)\int_{\Omega'}\rho^{2k}|Dg|^2 g(x)|Du_\varepsilon|^p \\ &\quad + \frac{c\tilde{\varepsilon}}{(s-r)^{2k+2}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}|D_jD^iu_\varepsilon|^2 + |D^ku_\varepsilon|^2\right). \end{aligned}$$

Adding up these inequalities over $j \in \{1, \dots, n\}$ and adjusting the constants we arrive at

$$
\begin{aligned}
& \int_{\Omega'} \frac{\rho^{2k}}{g(x)} |Du_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 + \frac{4\tilde{\varepsilon}}{3} \int_{\Omega'} \rho^{2k} |D^{k+1} u_\varepsilon|^2 \\
\leq \; & \frac{c(n,p,k)}{(s-r)^2} \int_{B_s \setminus B_r} g^3 |Du_\varepsilon|^p + c(n,L,p) \int_{\Omega'} \rho^{2k} g(x) |Dg|^2 |Du_\varepsilon|^p \\
+ \; & \frac{A(\varepsilon)}{(s-r)^{2k+2}}, \qquad (3.16)
\end{aligned}
$$

where $A(\varepsilon)$ is independent of r, s and where by virtue of (3.10) in Lemma 3.2, through the Gagliardo Nirenberg interpolation inequality,

$$
A(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \searrow 0.
$$

Omitting the second term on the left–hand side, the above inequality simplifies to

$$
\begin{aligned}
& \int_{\Omega'} \frac{\rho^{2k}}{g(x)} |Du_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 \leq \frac{c(n,p,k)}{(s-r)^2} \int_{B_s \setminus B_r} g^3 |Du_\varepsilon|^p \\
& + c(n,p,L) \int_{\Omega'} \rho^{2k} g |Dg|^2 |Du_\varepsilon|^p + \frac{A(\varepsilon)}{(s-r)^{2k+2}}. \qquad (3.17)
\end{aligned}
$$

Now, elementary calculations imply that

$$
\begin{aligned}
& \left| D\left(\rho^k \frac{1}{\sqrt{g}} V(Du_\varepsilon) \right) \right|^2 \\
\leq \; & c \left[\rho^{2k} \frac{1}{g} |Du_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 + \rho^{2k} \frac{1}{g^3} |Dg|^2 |V(Du_\varepsilon)|^2 + k^2 \rho^{2k-2} \frac{1}{g} |D\rho|^2 |V(Du_\varepsilon)|^2 \right],
\end{aligned}
$$

where c is a constant depending on p but independent of ε . Integrating previous estimate over the set Ω' and using (3.17) we get

$$
\begin{aligned}
& \int_{\Omega'} \left| D\left(\rho^k \frac{1}{\sqrt{g}} V(Du_\varepsilon) \right) \right|^2 \\
\leq \; & \frac{c(n,p,k)}{(s-r)^2} \int_{B_s \setminus B_r} g^3 |Du_\varepsilon|^p + c(n,p,L) \int_{\Omega'} \rho^{2k} g |Dg|^2 |Du_\varepsilon|^p \\
+ \; & \frac{c(p,k)}{(s-r)^2} \int_{B_s \setminus B_r} \frac{1}{g} |Du_\varepsilon|^p + c(p) \int_{\Omega'} \rho^{2k} \frac{1}{g^3} |Dg|^2 |Du_\varepsilon|^p \\
& + \frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}}, \qquad (3.18)
\end{aligned}
$$

where we set $\tilde{A}(\varepsilon) = cA(\varepsilon)$. Since $g \geq 1$, inequality (3.18) simplifies to

$$
\begin{aligned}
& \int_{\Omega'} \left| D\left(\rho^k \frac{1}{\sqrt{g}} V(Du_\varepsilon) \right) \right|^2 \\
\leq \; & \frac{c_1(n,p,k)}{(s-r)^2} \int_{B_s \setminus B_r} g^3 |Du_\varepsilon|^p + c_2(n,p,L) \int_{\Omega'} \rho^{2k} g |Dg|^2 |Du_\varepsilon|^p \\
& + \frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}} . \qquad (3.19)
\end{aligned}
$$

Sobolev imbedding Theorem and the definition of $V(Du)$ yield

$$\begin{aligned}
\left(\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{pn}{n-2}}\right)^{\frac{n-2}{n}} &= \left(\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|V(Du_\varepsilon)|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\le C_S^2\int_{\Omega'}\left|D\left(\rho^k\frac{1}{\sqrt{g}}V(Du_\varepsilon)\right)\right|^2 \qquad (3.20)
\end{aligned}$$

for a constant $C_S = C_S(n,N)$. Combining (3.19) and (3.20), we obtain

$$\begin{aligned}
&\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{pn}{n-2}}\\
\le\ & \frac{c_1(n,p,k,N)}{(s-r)^{\frac{2n}{n-2}}}\left(\int_{B_s\setminus B_r}g^3|Du_\varepsilon|^p\right)^{\frac{n}{n-2}} + c_2(n,p,L,N)\left(\int_{\Omega'}\rho^{2k}g|Dg|^2|Du_\varepsilon|^p\right)^{\frac{n}{n-2}}\\
+\ & \left(\frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}}\right)^{\frac{n}{n-2}}\\
=:\ & I + II + \left(\frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}}\right)^{\frac{n}{n-2}}. \qquad (3.21)
\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}
II &\le c_2(n,p,L,N)\left(\int_{\Omega'}g^{\frac{n}{2}}|Dg|^n\right)^{\frac{2}{n-2}}\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{np}{n-2}}\\
&\le c_2(n,p,L,N)\left(\int_{\Omega'}g^{n}|Dg|^n\right)^{\frac{2}{n-2}}\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{np}{n-2}} \qquad (3.22)
\end{aligned}$$

where we used again that $g \ge 1$. Now, since the assumption on g implies $g^2 \in \mathrm{W}^{1,n}(\Omega)$, the absolute continuity of the integral allows us to choose Ω' such that

$$\left(\int_{\Omega'}g^n|Dg|^n\right)^{\frac{2}{n-2}} < \frac{1}{4c_2(n,p,L,N)} \qquad (3.23)$$

in order to have that

$$II \le \frac{1}{4}\left(\int_{\Omega'}\rho^{\frac{2kn}{n-2}}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{np}{n-2}}\right)^{\frac{n-2}{n}}. \qquad (3.24)$$

For the estimation of I, we use Hölder's and Young's inequalities as follows

$$\begin{aligned}
I &\le \frac{c_1}{(s-r)^{\frac{2n}{n-2}}}\left(\int_{B_s\setminus B_r}g^{4n}\right)^{\frac{1}{n-2}}\left(\int_{B_s\setminus B_r}\frac{1}{g}|Du_\varepsilon|^p\right)^{\frac{n}{2(n-2)}}\\
&\cdot\left(\int_{B_s\setminus B_r}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{np}{n-2}}\right)^{\frac{1}{2}}\\
&\le \int_{B_s\setminus B_r}\frac{1}{g^{\frac{n}{n-2}}}|Du_\varepsilon|^{\frac{np}{n-2}}\\
&+ \frac{c_1}{(s-r)^{\frac{4n}{n-2}}}\left(\int_{B_s\setminus B_r}g^{4n}\right)^{\frac{2}{n-2}}\left(\int_{B_s\setminus B_r}\frac{1}{g}|Du_\varepsilon|^p\right)^{\frac{n}{n-2}} \qquad (3.25)
\end{aligned}$$

for a new constant $c_1 = c_1(n,p,k,N)$, independent of ε. Inserting (3.25) and (3.24) in (3.21) we get

$$\begin{aligned}\int_{\Omega'} \rho^{\frac{2kn}{n-2}} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{pn}{n-2}} &\leq \frac{1}{4}\int_{\Omega'} \rho^{\frac{2kn}{n-2}} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{np}{n-2}} + \int_{B_s\setminus B_r} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{np}{n-2}} \\ &+ \frac{c_1}{(s-r)^{\frac{4n}{n-2}}} \left(\int_{B_s\setminus B_r} g^{4n}\right)^{\frac{2}{n-2}} \left(\int_{B_s\setminus B_r} \frac{1}{g}|Du_\varepsilon|^p\right)^{\frac{n}{n-2}} \\ &+ \left(\frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}}\right)^{\frac{n}{n-2}} \end{aligned} \tag{3.26}$$

Reabsorbing the first integral in the right hand side by the left hand side and using that ρ equals 1 on B_r, we get

$$\begin{aligned}\int_{B_r} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{pn}{n-2}} &\leq \int_{\Omega'} \rho^{\frac{2kn}{n-2}} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{pn}{n-2}} \leq \frac{4}{3}\int_{B_s\setminus B_r} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{np}{n-2}} \\ &+ \frac{c}{(s-r)^{\frac{4n}{n-2}}} \left(\int_{B_s\setminus B_r} g^{4n}\right)^{\frac{2}{n-2}} \left(\int_{B_s\setminus B_r} \frac{1}{g}|Du_\varepsilon|^p\right)^{\frac{n}{n-2}} \\ &+ \left(\frac{\tilde{A}(\varepsilon)}{(s-r)^{2k+2}}\right)^{\frac{n}{n-2}} \end{aligned} \tag{3.27}$$

where c denotes a constant independent of ε. We use the hole filling trick of Widman to obtain

$$\begin{aligned} &\int_{B_R} \frac{1}{g^{\frac{n}{n-2}}} |Du_\varepsilon|^{\frac{pn}{n-2}} \\ \leq\ & \frac{c}{R^{\frac{4n}{n-2}}} \left(\int_{B_{2R}} g^{4n}\right)^{\frac{2}{n-2}} \left(\int_{B_{2R}} \frac{1}{g}|Du_\varepsilon|^p\right)^{\frac{n}{n-2}} + \left(\frac{\tilde{A}(\varepsilon)}{R^{2k+2}}\right)^{\frac{n}{n-2}} \\ \leq\ & \frac{c}{R^{\frac{4n}{n-2}}} \left(\int_{B_{2R}} g^{4n}\right)^{\frac{2}{n-2}} \left(\int_{B_{2R}} F(x, Du_\varepsilon)\,\mathrm{d}x\right)^{\frac{n}{n-2}} + \left(\frac{\tilde{A}(\varepsilon)}{R^{2k+2}}\right)^{\frac{n}{n-2}} \end{aligned}$$

by the use of assumption (H1). Since, by Lemma 3.2, we have that the integral

$$\int_{B_{2R}} F(x, Du_\varepsilon)\,\mathrm{d}x$$

is bounded independently of ε, it follows that the sequence (Du_ε) is bounded in $\mathrm{L}^{\frac{n\tilde{p}}{n-2}}(B_R, \mathbb{R}^{N\times n})$, for all $\tilde{p} < p$. So, by the arbitrariness of the ball $B_{2R}(x_0) \subset \Omega'$ and a simple covering argument, we conclude that (Du_ε) is bounded in $\mathrm{L}^{\frac{n\tilde{p}}{n-2}}_{\mathrm{loc}}(\Omega', \mathbb{R}^{N\times n})$. Therefore by passing to the limit as $\varepsilon \searrow 0$ we get

$$\int_{B_R} \frac{1}{g^{\frac{n}{n-2}}} |Du|^{\frac{pn}{n-2}} \leq \frac{c}{R^{\frac{4n}{n-2}}} \left(\int_{B_{2R}} g^{4n}\right)^{\frac{2}{n-2}} \left(\int_{B_{2R}} F(x, Du)\,\mathrm{d}x\right)^{\frac{n}{n-2}} \tag{3.28}$$

and hence, since by the assumption on g we have $g^2 \in \mathrm{W}^{1,n}(\Omega)$, we get

$$\left(\int_{B_R} \frac{1}{g^{\frac{n}{n-2}}} |Du|^{\frac{pn}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{c}{R^4} \left(\int_\Omega (|g|^2 + |D(g^2)|)^n\right)^{\frac{4}{n}} \int_{B_{2R}} F(x, Du)\,\mathrm{d}x \tag{3.29}$$

In view of (3.17) and (3.29), it then also follows that $(V(Du_\varepsilon))$ is bounded in $W^{1,s}_{\mathrm{loc}}(\Omega', \mathbb{R}^{N\times n})$, for every $s < 2$ and passing to limit as $\varepsilon \searrow 0$ we obtain the following estimate

$$\int_{B_R} \frac{1}{g} |D(V(Du))|^2 \,\mathrm{d}x \le \frac{c}{R^6} \left(\int_\Omega (|g|^2 + |D(g^2)|)^n \,\mathrm{d}x \right)^{\frac{6}{n}} \int_{B_{2R}} F(x, Du) \,\mathrm{d}x \,. \tag{3.30}$$

This concludes the proof. □

As a simple consequence of previous Theorem, we have

Corollary 3.3. *Let $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy the assumptions (H1)– (H6) for a function g such that $g^{p'+1} \in W^{1,n}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and an exponent $2 < p \le n$. If $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local F–minimizer, then*

$$V_p(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$$

Furthermore, there exists a radius $R_0 = R_0(n, N, L, p)$ such that whenever $B_{2R} \subset B_{R_0} \subset \Omega$ we have the Caccioppoli type inequality

$$\int_{B_R} |D(V(Du))|^2 \,\mathrm{d}x \le \frac{c}{R^6} \left(\int_\Omega (|g|^2 + |D(g^2)|)^n \,\mathrm{d}x \right)^{\frac{6}{n}} \left(\int_{B_{2R}} F(x, Du) \,\mathrm{d}x \right)$$

for a constant $c = c(n, N, L, p, ||g||_\infty)$.

4 Proof of Theorem 1.3

The proof of the higher integrability result stated in Theorem 1.3 will be achieved establishing first that F- minimizers are locally bounded in Ω and then using the weighted interpolation result of Lemma 2.3. The local boundedness of minimizers, which could be of interest by itself, is contained in the following

Lemma 4.1. *Let $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy the conditions (H1)– (H6) for an exponent $2 \le p < n$ and a function g such that $g^{p'+1} \in W^{1,n}_0(\Omega)$. Suppose in addition that $F(x, \xi) = \tilde{F}(x, |\xi|)$. If $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local F–minimizer, then there exists $R_1 = R_1(n, N, p) > 0$ such that for every ball $B_R \subset B_{R_1} \Subset \Omega$*

$$\sup_{B_{\frac{R}{2}}} |u| \le \frac{c(p, n, N)}{R^{\frac{n-p}{p_1}}} \left(\int_\Omega (|g|^2 + |D(g^2)|)^n \right)^{\frac{n-p}{np_1}} \left(\int_{B_R} |u|^{\frac{np_2}{n-p}} \,\mathrm{d}x \right)^{\frac{n-p}{np_2}} \tag{4.1}$$

for an exponent $p_2 < p$.

Proof. The proof, which is rather technical, will be given in two steps. In the first one, we establish uniform estimates for the minimizers of suitable approximating problems, while in the second one we conclude showing that these estimates are preserved in passing to limit.
Step 1. The a priori estimate
Fix a subdomain with a smooth boundary $\Omega' \Subset \Omega$ and for a smooth kernel $\phi \in C^\infty_c(B_1(0))$ with $\phi \ge 0$ and $\int_{B_1(0)} \phi = 1$, we consider the corresponding family of

mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put $\tilde{u}_\varepsilon := \phi_\varepsilon * u$ on Ω' for each positive $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$. Let us denote by v_ε the unique minimizer of the Dirichlet problem

$$\min_{v \in \tilde{u}_\varepsilon + W_0^{1,p}(\Omega')} \int_{\Omega'} F_\varepsilon(x, Dv)\,\mathrm{d}x \tag{4.2}$$

where we set

$$F_\varepsilon(x,\xi) = \frac{F(x,\xi)}{1+\varepsilon g} + \frac{\varepsilon g}{1+\varepsilon g}|\xi|^p\,.$$

It is well known (see for example [5, 25, 26]) that $F_\varepsilon(x,\xi)$ satisfies the following bounds (uniform with respect to ε)

$$\frac{1}{g(x)}|\xi|^p \le F_\varepsilon(x,\xi) \le g(x)|\xi|^p \tag{A1}$$

$$|D_x D_\xi F_\varepsilon(x,\xi)| \le c(p,L)|Dg(x)||\xi|^{p-1}\,, \tag{A2}$$

and

$$\langle D_{\xi\xi}F_\varepsilon(x,\xi)\eta,\eta\rangle \ge \frac{1}{g(x)}|\xi|^{p-2}|\eta|^2\,. \tag{A3}$$

One can easily check that

$$|D_\xi F_\varepsilon(x,\xi)| \le c(p)g(x)|\xi|^{p-1} \tag{A4}$$

for positive constants independent of ε. Moreover, $F_\varepsilon(x,\xi)$ satisfies the following bounds (uniform with respect to x)

$$\frac{\varepsilon}{1+\varepsilon}|\xi|^p \le F_\varepsilon(x,\xi) \le \frac{1+\varepsilon}{\varepsilon}|\xi|^p\,. \tag{A5}$$

Fix $B_{2R} = B_{2R}(x_0) \subset \Omega'$, radii $R \le r < s \le 2R \le 2$ and a smooth cut-off function η satisfying $1_{B_r} \le \eta \le 1_{B_s}$ and $|D\eta| \le \frac{2}{s-r}$. Let us consider the function $\varphi = \eta^p|v_\varepsilon|^\sigma v_\varepsilon$ where σ is a positive exponent. Since by Theorem 2.3 in [28], v_ε is bounded in Ω', we are allowed to plug φ as test function in the Euler-Lagrange equation associated to the functional defined in (4.2). We get

$$\begin{aligned}
0 &= \int_{\Omega'} D_\xi F_\varepsilon(x,Dv_\varepsilon)D\varphi\,\mathrm{d}x \\
&= \int_{\Omega'} \eta^p|v_\varepsilon|^\sigma\langle D_\xi F_\varepsilon(x,Dv_\varepsilon),Dv_\varepsilon\rangle\,\mathrm{d}x \\
&+ \sigma\int_{\Omega'} \eta^p\langle D_\xi F_\varepsilon(x,Dv_\varepsilon),Dv_\varepsilon\rangle|v_\varepsilon|^\sigma\,\mathrm{d}x \\
&+ p\int_{\Omega} \eta^{p-1}|v_\varepsilon|^\sigma\langle D_\xi F_\varepsilon(x,Dv_\varepsilon),v_\varepsilon\otimes\nabla\eta\rangle\,\mathrm{d}x \\
&:= I + II + III
\end{aligned} \tag{4.3}$$

Then we estimate I and II thanks to the ellipticity assumption (A3):

$$I + II \ge (\sigma+1)\int_{\Omega'} \eta^p\frac{1}{g(x)}|v_\varepsilon|^\sigma|Dv_\varepsilon|^p\,\mathrm{d}x \tag{4.4}$$

We can estimate $|III|$ by using the growth condition on the first derivatives of F_ε expressed by the inequality (A4) and Young's inequality as follows

$$\begin{aligned} |III| &\leq c(p)\int_{\Omega'} \eta^{p-1} g(x)|v_\varepsilon|^{\sigma+1}|Dv_\varepsilon|^{p-1}|\nabla\eta| \,\mathrm{d}x \\ &\leq c(p)\int_{\Omega'} \eta^{p-1} g(x)|v_\varepsilon|^{\sigma+1}|Dv_\varepsilon|^{p-1}|\nabla\eta| \,\mathrm{d}x \\ &\leq \frac{1}{2}(\sigma+1)\int_{\Omega'} \eta^p \frac{1}{g(x)}|v_\varepsilon|^{\sigma}|Dv_\varepsilon|^p \,\mathrm{d}x \\ &+ \frac{c(p)}{(\sigma+1)^{p-1}}\int_{\Omega'} g^{2p-1}(x)|\nabla\eta|^p|v_\varepsilon|^{\sigma+p} \,\mathrm{d}x \end{aligned} \tag{4.5}$$

Inserting estimates (4.4) and (4.5) in (4.3) and reabsorbing we have that

$$\begin{aligned} &(\sigma+1)\int_{\Omega'} \eta^p \frac{1}{g(x)}|v_\varepsilon|^{\sigma}|Dv_\varepsilon|^p \,\mathrm{d}x \\ \leq\ & \frac{c(p)}{(\sigma+1)^{p-1}}\int_{\Omega'} g^{2p-1}(x)|\nabla\eta|^p|v_\varepsilon|^{\sigma+p} \,\mathrm{d}x\,. \end{aligned} \tag{4.6}$$

We can write previous inequality as follows

$$\begin{aligned} &\int_{\Omega'} \eta^p \frac{1}{g(x)}|v_\varepsilon|^{\sigma}|Dv_\varepsilon|^p \,\mathrm{d}x \\ \leq\ & \frac{c(p)}{(\sigma+1)^p(s-r)^p}\int_{B_s\setminus B_r} g^{2p-1}(x)|v_\varepsilon|^{\sigma+p} \,\mathrm{d}x\,, \end{aligned} \tag{4.7}$$

by the use of the properties of η. Now, we calculate

$$\begin{aligned} &\left|D\left(\eta\frac{1}{g^{\frac{1}{p}}(x)}|v_\varepsilon|^{\frac{\sigma}{p}}v_\varepsilon\right)\right|^p \\ \leq\ & c(p)\eta^p\frac{|Dg|^p}{g^{p+1}}|v_\varepsilon|^{\sigma+p} + c(p)\left(\frac{\sigma+p}{p}\right)^p\eta^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma}|Dv_\varepsilon|^p \\ +\ & c(p)|\nabla\eta|^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma+p} \end{aligned}$$

Integrating previous inequality over Ω' and using (4.7), we get

$$\begin{aligned} &\int_{\Omega'}\left|D\left(\eta\frac{1}{g^{\frac{1}{p}}(x)}|v_\varepsilon|^{\frac{\sigma}{p}}v_\varepsilon\right)\right|^p \mathrm{d}x \\ \leq\ & \frac{c(p)}{(s-r)^p}\frac{(\sigma+p)^p}{(\sigma+1)^p}\int_{B_s\setminus B_r} g^{2p-1}(x)|v_\varepsilon|^{\sigma+p}\,\mathrm{d}x \\ +\ & c(p)\int_{\Omega'}\eta^p\frac{|Dg|^p}{g^{p+1}}|v_\varepsilon|^{\sigma+p}\,\mathrm{d}x + c(p)\int_{\Omega'}|\nabla\eta|^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma+p}\,\mathrm{d}x \end{aligned} \tag{4.8}$$

Sobolev imbedding Theorem yields

$$\int_{\Omega'}\left(\eta^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}}$$

$$
\begin{aligned}
&\le \frac{c(p,n,N)}{(s-r)^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\left(\int_{B_s\setminus B_r} g^{2p-1}(x)|v_\varepsilon|^{\sigma+p}\,dx\right)^{\frac{n}{n-p}}\\
&+ c(p,n,N)\left(\int_{\Omega'}\eta^p\frac{|Dg|^p}{g^{p+1}}|v_\varepsilon|^{\sigma+p}\,dx\right)^{\frac{n}{n-p}}\\
&:= A+B \qquad (4.9)
\end{aligned}
$$

where we used that, since $g(x)\ge 1$, one has $\frac{1}{g}\le g^{2p-1}$ and that $\frac{\sigma+p}{\sigma+1}>1$. We estimate B using Hölder's inequality as follows

$$
B\le c(p,n,N)\left(\int_{\Omega'}\left(\eta^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}}\mathrm{d}x\right)\left(\int_{B_R}\left(\frac{|Dg|^p}{g^p}\right)^{\frac{n}{p}}\mathrm{d}x\right)^{\frac{p}{n-p}}
$$

We choose $R<R_1=R_1(n,p,N)$ such that

$$
\left(\int_{B_{R_1}}\frac{|Dg|^n}{g^n}\,\mathrm{d}x\right)^{\frac{p}{n-p}}<\frac{1}{2c(n,p,N)}
$$

and hence

$$
B\le\frac{1}{2}\int_{\Omega'}\left(\eta^p\frac{1}{g(x)}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}}\mathrm{d}x \qquad (4.10)
$$

In order to estimate A, we interpolate as follows

$$
\frac{1}{\sigma+p}=\frac{\vartheta}{\frac{n(\sigma+p)}{n-p}}+\frac{1-\vartheta}{\frac{np_1}{n-p}},
$$

for a fixed $p_1<p$. We find

$$
\vartheta=\frac{n}{n-p}-\frac{p(\sigma+p)}{(\sigma+p-p_1)(n-p)}
$$

that is positive and strictly less than 1 for $\sigma>\frac{p^2-n(p-p_1)}{n-p}$. Hence we can use Hölder's and Young's inequalities as follows

$$
\begin{aligned}
A &\le \frac{c(p,n,N)}{(s-r)^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\left(\int_{B_s\setminus B_r} g^{2p-1}(x)|v_\varepsilon|^{\sigma+p}\,dx\right)^{\frac{n}{n-p}}\\
&\le \frac{c(p,n,N)}{(s-r)^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\left(\int_{B_s\setminus B_r}\left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}}dx\right)^{\vartheta}\\
&\qquad\cdot\left(\int_{B_s\setminus B_r} g^{\chi}(x)|v_\varepsilon|^{\frac{np_1}{n-p}}\,dx\right)^{\frac{(1-\vartheta)(\sigma+p)}{p_1}}\\
&\le \frac{1}{8}\int_{B_s\setminus B_r}\left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}}dx\\
&+ \left[\frac{c(p,n,N)}{(s-r)^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{1}{1-\vartheta}}\left(\int_{B_s\setminus B_r} g^{\chi}(x)|v_\varepsilon|^{\frac{np_1}{n-p}}\right)^{\frac{\sigma+p}{p_1}} \qquad (4.11)
\end{aligned}
$$

where in order to simplify the notation we set $\chi = (2p+1-\vartheta)\frac{np_1}{(1-\vartheta)(n-p)(\sigma+p)}$. Inserting estimates (4.11) and (4.10) in (4.9), we get

$$
\begin{aligned}
&\int_{\Omega'} \left(\eta^p \frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} \\
\leq\; & \frac{1}{2}\int_{\Omega'} \left(\eta^p \frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} dx + \frac{1}{8}\int_{B_s\setminus B_r} \left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} dx \\
+\; & \left[\frac{c(p,n,N)}{(s-r)^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{1}{1-\vartheta}} \left(\int_{B_s\setminus B_r} g^{\chi}(x)|v_\varepsilon|^{\frac{np_1}{n-p}}\right)^{\frac{\sigma+p}{p_1}}
\end{aligned}
\tag{4.12}
$$

Reabsorbing the first integral in the right hand side by the left hand side and using the hole filling trick of Widman, we get

$$
\begin{aligned}
&\int_{B_{\frac{R}{2}}} \left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} \mathrm{d}x \\
\leq\; & \left[\frac{c(p,n,N)}{R^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{1}{1-\vartheta}} \left(\int_{B_R} g^{\chi}(x)|v_\varepsilon|^{\frac{np_1}{n-p}}\right)^{\frac{\sigma+p}{p_1}}
\end{aligned}
\tag{4.13}
$$

Recalling that $1-\vartheta = \frac{pp_1}{(\sigma+p-p_1)(n-p)}$ we obtain

$$
\begin{aligned}
&\left(\int_{B_{\frac{R}{2}}} \left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{n(\sigma+p)}} \\
\leq\; & \left[\frac{c(p,n)}{R^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{\sigma+p-p_1}{\sigma+p}\frac{(n-p)^2}{npp_1}} \left(\int_{B_R} g^{\chi}(x)|v_\varepsilon|^{\frac{np_1}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np_1}}
\end{aligned}
$$

Hölder's inequality yields

$$
\begin{aligned}
\left(\int_{B_{\frac{R}{2}}} \left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{n(\sigma+p)}} &\leq \left[\frac{c(p,n)}{R^{\frac{np}{n-p}}}\left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{\sigma+p-p_1}{\sigma+p}\frac{(n-p)^2}{npp_1}} \\
&\cdot \left(\int_{B_R} g^{\chi\frac{p_2}{p_2-p_1}}(x)\right)^{\frac{(n-p)(p_2-p_1)}{np_1p_2}} \left(\int_{B_R} |v_\varepsilon|^{\frac{np_2}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np_2}}
\end{aligned}
\tag{4.14}
$$

where $p_1 < p_2 < p$.

Step 2. Conclusion
Since the functional F_ε satisfies (A1)–(A4), we are legitimate to use estimate (3.29) for v_ε

$$
\begin{aligned}
&\int_{B_R} \frac{1}{g^{\frac{n}{n-2}}}|Dv_\varepsilon|^{\frac{pn}{n-2}} \mathrm{d}x \leq \frac{C}{R^4}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\right)^{\frac{4}{n}} \int_{B_{2R}} F_\varepsilon(x, Dv_\varepsilon)\,\mathrm{d}x \\
\leq\; & \frac{C}{R^4}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\right)^{\frac{4}{n}} \int_{B_{2R}} F_\varepsilon(x, D\tilde{u}_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C}{R^4}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\right)^{\frac{4}{n}} \int_{B_{2R}} g|D\tilde{u}_\varepsilon|^p \,\mathrm{d}x \\
&\leq \frac{C}{R^4}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\right)^{\frac{5}{n}} \left(\int_{B_{2R}} |D\tilde{u}_\varepsilon|^{\frac{2np}{2n-1}} \,\mathrm{d}x\right)^{\frac{2n-1}{2n}} \qquad (4.15)
\end{aligned}
$$

by the minimality of v_ε and the growth condition (A1). Since $\frac{2np}{2n-1} < \frac{np}{n-2}$, by Theorem 1.1 we have that $Du \in \mathrm{L}^{\frac{2np}{2n-1}}$ and hence $\tilde{u}_\varepsilon$ strongly converges to u as $\varepsilon \to 0$ in $W^{1,\frac{2np}{2n-1}}$. Therefore there exists a positive constant C independent of ε such that

$$
\int_{B_R} \frac{1}{g^{\frac{n}{n-2}}} |Dv_\varepsilon|^{\frac{pn}{n-2}} \,\mathrm{d}x \leq C \qquad (4.16)
$$

and then the sequence Dv_ε is bounded in $\mathrm{L}^{\frac{n\tilde{p}}{n-2}}(B_R, \mathbb{R}^{N\times n})$, for all $\tilde{p} < p$ and a standard diagonal argument give a subsequence $v_j = (v_\varepsilon)_j$ weakly converging to a function v, whose gradient Dv belongs to $\mathrm{L}^{\frac{n\tilde{p}}{n-2}}(B_R, \mathbb{R}^{N\times n})$, for all $\tilde{p} < p$. Moreover $v_j = (v_\varepsilon)_j$ strongly converges to v in L^γ, for every $\gamma < \frac{np}{n-2-p}$ (if $p < n-2$) or γ is any number if $p \geq n-2$. Our next aim is to show that the limit function v coincides with the minimizer u of the functional $\mathfrak{F}(u, B_R)$. The lower semicontinuity of the integral and the definition of F_ε imply

$$
\begin{aligned}
\int_{B_R} F(x, Dv)\,\mathrm{d}x &\leq \lim_{\varepsilon\to 0}\int_{B_R} F(x, Dv_\varepsilon)\,\mathrm{d}x \leq \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon)(1+\varepsilon g)\,\mathrm{d}x \\
&\leq \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon) + \lim_{\varepsilon\to 0}\varepsilon\int_{B_R} gF_\varepsilon(x, Dv_\varepsilon)\,\mathrm{d}x \\
&\leq \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon) + \lim_{\varepsilon\to 0}\varepsilon\int_{B_R} g^2|Dv_\varepsilon|^p\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon)\,\mathrm{d}x \\
&+ \lim_{\varepsilon\to 0}\varepsilon\left(\int_\Omega (|g|^2+|D(g^2)|)^n\right)^{\frac{3}{2n}}\left(\int_{B_R}\frac{1}{g^{\frac{n}{n-2}}}|Dv_\varepsilon|^{\frac{pn}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\
&= \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon)\,\mathrm{d}x \qquad (4.17)
\end{aligned}
$$

where we used (4.16). Estimate (4.17), the minimality of v_ε, Jensen's inequality and Dominated Convergence Theorem yield

$$
\begin{aligned}
\int_{B_R} F(x, Dv)\,\mathrm{d}x &\leq \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, Dv_\varepsilon)\,\mathrm{d}x \leq \lim_{\varepsilon\to 0}\int_{B_R} F_\varepsilon(x, D\tilde{u}_\varepsilon)\,\mathrm{d}x \\
&\leq \int_{B_R} F(x, Du)\,\mathrm{d}x \qquad (4.18)
\end{aligned}
$$

Since v and u coincide on ∂B_R in the sense of traces, by the minimimality of u, we have

$$
\int_{B_R} F(x, Du)\,\mathrm{d}x \leq \int_{B_R} F(x, Dv)\,\mathrm{d}x
$$

and the strict convexity of F implies that v and u coincide a.e. in B_R. Taking the limit as $\varepsilon \to 0$ in (4.14), by the use of Fatou's Lemma and the fact that

$v_j = (v_\varepsilon)_j$ strongly converges to u in L^γ, for every $\gamma < \frac{np}{n-p}$, we get

$$\left(\int_{B_{\frac{R}{2}}} \left(\frac{1}{g}|u|^{\sigma+p}\right)^{\frac{n}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{n(\sigma+p)}} \leq \liminf_{\varepsilon\to 0} \left(\int_{B_{\frac{R}{2}}} \left(\frac{1}{g}|v_\varepsilon|^{\sigma+p}\right)^{\frac{n}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{n(\sigma+p)}}$$

$$\leq \left[\frac{c(p,n,N)}{R^{\frac{np}{n-p}}} \left(\frac{\sigma+p}{\sigma+1}\right)^{\frac{np}{n-p}}\right]^{\frac{\sigma+p-p_1}{\sigma+p}\frac{(n-p)^2}{npp_1}}$$

$$\cdot \left(\int_{B_R} g^{\frac{\chi p_2}{p_2-p_1}}(x)\right)^{\frac{(n-p)(p_2-p_1)}{np_1p_2}} \left(\int_{B_R} |u|^{\frac{np_2}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np_2}} \tag{4.19}$$

Letting $\sigma \to \infty$ in (4.19), taking into account that $\chi \to 2n$ and using the assumption $g^2 \in \mathrm{W}^{1,n}(\Omega)$, we conclude that

$$\begin{aligned}\sup_{B_{\frac{R}{2}}} |u| &\leq \frac{c(p,n,N)}{R^{\frac{n-p}{p_1}}} \left(\int_{B_R} g^{\frac{2np_2}{p_2-p_1}}(x)\right)^{\frac{(n-p)(p_2-p_1)}{np_1p_2}} \left(\int_{B_R} |u|^{\frac{np_2}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np_2}} \\ &\leq \frac{c(p,n,N)}{R^{\frac{n-p}{p_1}}} \left(\int_{\Omega} (|g|^2 + |D(g^2)|)^n\right)^{\frac{n-p}{np_1}} \left(\int_{B_R} |u|^{\frac{np_2}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np_2}} \end{aligned}\tag{4.20}$$

□

Now, combining Lemmas 4.1 and 2.3, we are ready to give the following

Proof of Theorem 1.3. Let $B_R \Subset \Omega$ with $2R < \bar{R} = \min\{R_0, R_1\}$, where R_0, R_1 are determined in Theorem 1.2 and Lemma 4.1 respectively and let $\eta \in C^1_c(\Omega)$ be a cut off function between $B_{\frac{R}{2}}$ and B_R. By the higher differentiability result of Theorem 1.3 and Lemma 4.1, we are legitimate to apply Lemma 2.3, thus obtaining

$$\begin{aligned}&\int_\Omega \eta^2 \frac{1}{g}|Du|^{p+2}\,\mathrm{d}x \leq c(p)\int_\Omega \eta^2\frac{1}{g}|u|^2|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\ &+ \ c\int_\Omega \eta^2|u|^2\frac{|Dg|^2}{g^3}|Du|^p\,\mathrm{d}x + c\int_\Omega |u|^2\frac{1}{g}|\nabla\eta|^2|Du|^p\,\mathrm{d}x\end{aligned}\tag{4.21}$$

Previous inequality obviously implies that

$$\begin{aligned}&\int_\Omega \eta^2\frac{1}{g}|Du|^{p+2}\,\mathrm{d}x \leq c(p)||u||^2_{L^\infty(B_R)}\int_\Omega \eta^2\frac{1}{g}|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\ + \ & c||u||^2_{L^\infty(B_R)}\int_\Omega \eta^2\frac{|Dg|^2}{g^3}|Du|^p\,\mathrm{d}x + c||u||^2_{L^\infty(B_R)}\int_\Omega |\nabla\eta|^2|Du|^p\,\mathrm{d}x \\ \leq \ & c(p)||u||^2_{L^\infty(B_R)}\int_\Omega \eta^2\frac{1}{g}|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\ + \ & c||u||^2_{L^\infty(B_R)}\left(\int_\Omega \eta^2\frac{|Dg|^n}{g^n}\,\mathrm{d}x\right)^{\frac{2}{n}}\left(\int_\Omega \eta^2\frac{1}{g^{\frac{n}{n-2}}}|Du|^{\frac{pn}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \\ + \ & c||u||^2_{L^\infty(B_R)}\int_\Omega |\nabla\eta|^2\frac{1}{g}|Du|^p\,\mathrm{d}x\end{aligned}$$

Estimates (3.29) and (3.30) of Theorem 1.2 yield

$$\begin{aligned}
& \int_{B_{\frac{R}{2}}} \frac{1}{g}|Du|^{p+2}\,\mathrm{d}x \\
\leq\ & \frac{c||u||^2_{L^\infty(B_R)}}{R^6}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\,\mathrm{d}x\right)^{\frac{6}{n}}\left(\int_{B_{2R}}\frac{1}{g}|Du|^p\right) \\
+\ & \frac{c||u||^2_{L^\infty(B_R)}}{R^4}\left(\int_\Omega (|g|^2+|D(g^2)|)^n\,\mathrm{d}x\right)^{\frac{6}{n}}\left(\int_{B_{2R}}\frac{1}{g}|Du|^p\right) \\
+\ & \frac{c||u||^2_{L^\infty(B_R)}}{R^2}\int_{B_R}\frac{1}{g}|Du|^p\,\mathrm{d}x
\end{aligned}$$

i.e. the conclusion. □

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