REGULARITY RESULTS FOR VERY DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. We consider a family of elliptic equations introduced in the context of traffic congestion. They have the form $\nabla\cdot(\nabla\mathcal{F}(\nabla u))=f$, where $\mathcal{F}$ is a convex function which vanishes inside some convex set and is elliptic outside. Under some natural assumptions on $\mathcal{F}$ and f, we prove that the function $\nabla\mathcal{F}(\nabla u)$ is continuous in any dimension, extending a previous result valid only in dimension 2 [14].

RÉSUMÉ. Dans ce papier, nous considérons une famille d'équations elliptiques introduites dans le contexte d'un problème de transport congestionné. Ces équations sont de la forme $\nabla\cdot(\nabla\mathcal{F}(\nabla u))=f$, où $\mathcal{F}$ est une fonction convexe qui vaut zéro sur un ensemble convexe et est uniformément elliptique au dehors de cet ensemble. Sous des conditions naturelles sur $\mathcal{F}$ et f, on démontre que la fonction $\nabla\mathcal{F}(\nabla u)$ est continue en toutes dimensions, ce qui étend un résultat précèdent en dimension 2 [14].

1. INTRODUCTION

Given a bounded open subset Ω of $\mathbb{R}^n$, a convex function $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}$, and an integrable function $f:\Omega\to\mathbb{R}$, we consider a function $u:\Omega\to\mathbb{R}$ which locally minimizes the functional

$$\int_\Omega \mathcal{F}(\nabla u)+fu. \tag{1.1}$$

When $\nabla^2\mathcal{F}$ is uniformly elliptic, namely there exist $\lambda,\Lambda>0$ such that

$$\lambda\,\mathrm{Id}\le\nabla^2\mathcal{F}\le\Lambda\,\mathrm{Id},$$

the regularity results of u in terms of $\mathcal{F}$ and f are well known.

If $\mathcal{F}$ degenerates at only one point, then several results are still available. For instance, in the case of the p-Laplace equation with zero right hand side, that is when $\mathcal{F}(v)=|v|^p$ and $f=0$, the $C^{1,\alpha}$ regularity of u has been proved by Uraltseva [19], Uhlenbeck [18], and Evans [10] for $p\ge2$, and by Lewis [13] and Tolksdorff [17] for $p>1$ (see also [7, 20]). Notice that in this case the equation is uniformly elliptic outside the origin.

More in general, one can consider functions whose degeneracy set is a convex set: for example, for $p>1$ one may consider

$$\mathcal{F}(v)=\frac{1}{p}(|v|-1)_+^p\qquad\forall\,v\in\mathbb{R}^n, \tag{1.2}$$

so that the degeneracy set is the entire unit ball. There are many Lipschitz results on u in this context [11, 9, 2], and in general no more regularity than L^∞ can be expected on ∇u. Indeed, when $\mathcal{F}$ is given by (1.2) and f is identically 0, every 1-Lipschitz function solves the equation. However, as proved in [14] in dimension 2, something more can be said about the regularity of $\nabla\mathcal{F}(\nabla u)$, since either it vanishes or we are in the region where the equation is more elliptic.

Key words and phrases. Degenerate elliptic PDEs, traffic congestion, continuity of the gradient.

The problem of minimizing the energy (1.1) with the particular choice of $\mathcal{F}$ given in (1.2) arises in the context of traffic congestion. Indeed, it is equivalent to the problem

$$\min\left\{\int_\Omega |\sigma| + \frac{1}{p'}|\sigma|^{p'} : \sigma \in L^{p'}(\Omega),\ \nabla\cdot\sigma = f,\ \sigma\cdot\nu_{\partial\Omega} = 0\right\}, \tag{1.3}$$

where p' satisfies $1/p+1/p'=1$, and σ represents the traffic flow. The particular choice of $\mathcal{F}$, or equivalently of its convex conjugate $\mathcal{F}^*$ which appears in (1.3) as an integrand, satisfies two demands: $\mathcal{F}^*$ has more than linear growth at infinity (so to avoid "congestion") and satisfies $\liminf_{w\to 0}|\nabla\mathcal{F}^*(w)|>0$ (which means that moving in an empty street has a nonzero cost). As shown in [3], the unique optimal minimizer $\bar\sigma$ turns out to be exactly $\nabla\mathcal{F}(\nabla u)$, where $\mathcal{F}$ is defined by (1.2).

In this paper we prove that, if $\mathcal{F}$ vanishes on some convex set E and is elliptic outside such a set, then $\mathcal{H}(\nabla u)$ is continuous for any continuous function $\mathcal{H}:\mathbb{R}^n\to\mathbb{R}$ which vanishes on E. In particular, by applying this result with $\mathcal{H}=\partial_i\mathcal{F}$ $(i=1,\dots,n)$ where $\mathcal{F}$ is as in (1.2), our continuity result implies that $\bar\sigma=\nabla\mathcal{F}(\nabla u)$ (the minimizer of (1.3)) is continuous in the interior of Ω. This result is important for the following reason: as shown in [5] (see also [3]), one can build a measure on the space of possible paths starting from $\bar\sigma$, and this optimal traffic distribution satisfies a Wardrop equilibrium principle: no traveler wants to change his path, provided all the others keep the same strategy. In other words, every path which is followed by somebody is a geodesics with respect to the metric $g(\bar\sigma(x))\,\mathrm{Id}$ (where $g(t)=1+t^{p-1}$ is the so-called "congestion function"), which is defined in terms of the traffic distribution itself. Hence, our continuity result shows that the metric is continuous (so, in particular, well defined at every point), which allows to set and study the geodesic problem in the usual sense.

Since we want to allow any bounded convex set as degeneracy set for $\mathcal{F}$, before stating the result we introduce the notion of norm associated to a convex set, which is used throughout the paper to identify the nondegenerate region. Given a bounded closed convex set $E\subseteq\mathbb{R}^n$ such that 0 belongs to $\mathrm{Int}(E)$ (the interior of E), and denoting by tE the dilation of E by a factor t with respect to the origin, we define $|\cdot|_E$ as

$$|e|_E := \inf\{t>0 : e\in tE\}. \tag{1.4}$$

Notice that $|\cdot|_E$ is a convex positively 1-homogeneous function. However $|\cdot|_E$ is not symmetric unless E is symmetric with respect to the origin.

The main result of the paper proves that, in the context introduced before, $\nabla\mathcal{F}(\nabla u)$ is continuous.

Theorem 1.1. *Let n be a positive integer, Ω a bounded open subset of $\mathbb{R}^n$, $f\in L^q(\Omega)$ for some $q>n$. Let E be a bounded, convex set with $0\in\mathrm{Int}(E)$. Let $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}$ be a convex nonnegative function such that $\mathcal{F}\in C^2(\mathbb{R}^n\setminus\overline{E})$. Let us assume that for every $\delta>0$ there exist $\lambda_\delta,\Lambda_\delta>0$ such that*

$$\lambda_\delta I \le \nabla^2\mathcal{F}(x)\le \Lambda_\delta I \qquad \text{for a.e. } x \text{ such that } 1+\delta\le |x|_E\le 1/\delta. \tag{1.5}$$

Let $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional

$$\int_\Omega \mathcal{F}(\nabla u)+fu.$$

Then, for any continuous function $\mathcal{H}:\mathbb{R}^n\to\mathbb{R}$ such that $\mathcal{H}=0$ on E, we have

$$\mathcal{H}(\nabla u)\in C^0(\Omega). \tag{1.6}$$

More precisely, for every open set $\Omega'\Subset\Omega$ there exists a modulus of continuity $\omega:[0,\infty)\to[0,\infty)$ for $\mathcal{H}(\nabla u)$ on Ω', which depends only on the modulus of continuity of $\mathcal{H}$, on the modulus of continuity *of $\nabla^2\mathcal{F}$, on the functions $\delta \to \lambda_\delta, \delta \to \Lambda_\delta$, and on $\|\nabla u\|_\infty$ in a neighborhood of Ω', such that*

$$\omega(0) = 0 \qquad \text{and} \qquad \big|\mathcal{H}(\nabla u(x)) - \mathcal{H}(\nabla u(y))\big| \le \omega(|x-y|) \text{ for any } x, y \in \Omega'. \tag{1.7}$$

In particular, if $\mathcal{F} \in C^1(\mathbb{R}^n)$ then $\nabla\mathcal{F}(\nabla u) \in C^0(\Omega)$.

Remark 1.2. In the hypothesis of Theorem 1.1 the Lipschitz regularity of u is always satisfied under mild assumptions on $\mathcal{F}$. For instance, if $\mathcal{F}$ is uniformly elliptic outside a fixed ball, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. In [2] many other cases are studied. For example, the Lipschitz regularity of u holds true for our model case $(|x|-1)_+^p$ for every $p > 1$.

Remark 1.3. The regularity result of Theorem 1.1 is optimal without any further conditions about the degeneracy of $\mathcal{F}$ near E. More precisely, there exist functions $\mathcal{F}$ satisfying our assumptions and $\mathcal{H}$ Lipschitz such that $\mathcal{H}(\nabla u)$ is not Hölder continuous for any exponent. Indeed, let us consider the minimizer of the functional (1.1) with $f = n$. The minimizer can be explicitly computed from the Euler equation and turns out to be $\mathcal{F}^*$, where $\mathcal{F}^*$ is the convex conjugate of $\mathcal{F}$. We consider a radial function $\mathcal{F}$. Let ω be a modulus of strict convexity for $\mathcal{F}$ outside E, i.e.,

$$\big(\nabla\mathcal{F}(x) - \nabla\mathcal{F}(y)\big)\cdot(x-y) \ge \omega(|x-y|)|x-y| \qquad \forall\, x, y \in \mathbb{R}^n \setminus B_1,\ x = ty,\ t > 0. \tag{1.8}$$

Then the function ω^{-1} is a modulus of continuity of $\nabla\mathcal{F}^*$. Hence it suffices to choose $\mathcal{F}$ so that ω^{-1} is not Hölder continuous.

For simplicity, we construct an explicit example in dimension 1, although it can be easily generalized to any dimension considering a radial function $\mathcal{F}$.

Let

$$G(t) := \begin{cases} e^{-1/(|t|-1)^2} & \text{if } |t| > 1, \\ 0 & \text{if } |t| \le 1, \end{cases}$$

and let $F \in C^\infty(\mathbb{R})$ be a convex function which coincides with G in a $(-1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$. Then the function $u : \mathbb{R} \to \mathbb{R}$ defined as

$$u(x) := \int_0^{|x|} [F']^{-1}(s)\, ds$$

solves the Euler-Lagrange equation $\big(F'(u'(x))\big)' = 1$ (note that the function $F' : \mathbb{R} \setminus [-1,1] \to \mathbb{R} \setminus \{0\}$ is invertible, so u is well defined), and it is easy to check that, given $\mathcal{H}(x) := (|x|-1)_+$, the function $\mathcal{H}(u') = \big([F']^{-1} - 1\big)_+$ is not Hölder continuous at 0.

Theorem 1.1 has been proved in dimension 2 with $E = B_1(0)$ by Santambrogio and Vespri in [14]. Their proof is based on a method by Di Benedetto and Vespri [8], which is very specific to the two dimensional case: using the equation they prove that either the oscillation of the solution is reduced by a constant factor when passing from a ball $B_r(0)$ to a smaller ball $B_{\varepsilon r}(0)$, or the Dirichlet energy in the annulus $B_r(0) \setminus B_{\varepsilon r}(0)$ is at least a certain value, which is scale invariant in dimension 2. Since the Dirichlet energy is assumed to be finite in the whole domain, this proves a decay for the oscillation.

In this paper we generalize the result to dimension n and with a general convex set of degeneracy, using a different method and following some ideas of a paper by Wang [20] in the case of the p-laplacian. We divide regions where the gradient is degenerate from nondegeneracy regions. The rough idea is the following: if no partial derivative of u is close to $|\nabla u|$ in a set of positive measure inside a ball, then $|\nabla u|$ is smaller (by a universal factor) in a smaller ball. If u has a nondegenerate partial derivative in a set of large measure, then its slope in the center of the ball is nondegenerate and the ellipticity of the equation provides regularity of u.

Theorem 1.1 is obtained from the following result through an approximation argument, which allows us to deal with smooth functions.

Theorem 1.4. *Let E be a bounded, strictly convex set with $0 \in \mathrm{Int}(E)$. Let $f \in C^0(B_2(0))$ and let $q > n$. Let $\mathcal{F} \in C^\infty(\mathbb{R}^n)$ be a convex function, fix $\delta > 0$, and assume that there exist constants $\lambda, \Lambda > 0$ such that*

$$\lambda I \leq \nabla^2 \mathcal{F}(x) \leq \Lambda I \qquad \text{for every } x \text{ such that } 1 + \frac{\delta}{2} \leq |x|_E. \tag{1.9}$$

Let $u \in C^2(B_2(0))$ be a solution of

$$\nabla \cdot (\nabla \mathcal{F}(\nabla u)) = f \qquad \text{in } B_2(0). \tag{1.10}$$

satisfying $\|\nabla u\|_{L^\infty(B_2(0))} \leq M$.

Then there exist $C > 0$ and $\alpha \in (0,1)$, depending only on the modulus of continuity of $\nabla^2 \mathcal{F}$, and on $E, \delta, M, q, \|f\|_{L^q(B_2(0))}$, λ, and Λ, such that

$$\|(|\nabla u|_E - (1+\delta))_+\|_{C^{0,\alpha}(B_1(0))} \leq C. \tag{1.11}$$

The paper is structured as follows: in Section 2 we prove a compactness result for a class of elliptic equations which are nondegenerate only in a small neighborhood of the origin. Then, in Section 3, we provide a way of separating degeneracy points from nondegeneracy points, and in Section 4 we prove $C^{1,\alpha}$ regularity of u at any point where the equation is nondegenerate. Finally, Section 5 is devoted to the proof of Theorems 1.4 and 1.1.

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2. Compactness result for a degenerate equation

In this section we prove a regularity result for a class of degenerate fully nonlinear elliptic equations. The argument follows the lines of [15, Corollary 3.3], although there are some main differences: First, in [15, Corollary 3.3] regularity is proved in the class of fully nonlinear equations with a degeneracy depending on the hessian of the solution, whereas in our case the degeneracy is in the gradient. Moreover only right hand sides in L^∞ are considered there, while in our context we are allowed to take them in L^n. Allowing f to be in L^n introduce several additional difficulties, in particular in the proof of Lemma 2.4. In addition, we would like to notice that the proofs of Lemmas 2.3 and 2.4 do not seem to easily adapt to the case $f \in L^n$ if in addition we allow a degeneracy in the hessian as in [15] (more precisely, in this latter case neither (2.9) nor (2.19) would allow to deduce that the equation is uniformly elliptic at the contact points).

We also notice that, with respect to [15], we prove a slightly weaker statement which is however enough for our purposes: instead of showing the the L^∞ norm of u decays geometrically, we only prove that its oscillation decays. The reason for this is just that the proof of this latter result is slightly simpler. However, by using the whole argument in the proof of [15, Theorem 1.1] one could replace $\operatorname{osc} u$ with $\|u\|_\infty$ in the statements of Proposition 2.2 and Theorem 2.1.

We keep the notation as similar as possible to the one of [15]. We assume for simplicity that $u \in C^2$ and f continuous, but these regularity assumptions are not needed (though verified for our application) and the same proof could be carried out in the context of viscosity solutions (as done in [15]).

Let $\mathcal{S} \subseteq \mathbb{R}^{n\times n}$ be the space of symmetric matrices in $\mathbb{R}^n$, $F : B_1(0) \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ be a measurable function, and consider the fully nonlinear equation

$$F(x, u(x), \nabla u(x), \nabla^2 u(x)) = f(x). \tag{2.1}$$

Let $\delta > 0$. We consider the following assumptions on F.

(H1) F is elliptic, namely for every $x \in B_1(0)$, $z \in \mathbb{R}$, $v \in \mathbb{R}^n$, $M, N \in \mathcal{S}$ with $N \geq 0$

$$F(x, z, v, M + N) \geq F(x, z, v, M).$$

(H2) F is uniformly elliptic in a neighborhood of $\nabla u = 0$ with ellipticity constants $0 < \lambda \leq \Lambda$: namely, for every $x \in B_1(0)$, $z \in \mathbb{R}$, $v \in B_\delta(0)$, $M, N \in \mathcal{S}$ with $N \geq 0$

$$\Lambda\|N\| \geq F(x, z, v, M + N) - F(x, z, v, M) \geq \lambda\|N\|.$$

(H3) Small planes are solutions of (2.1), namely for every $x \in B_1(0)$, $z \in \mathbb{R}$, $v \in B_\delta(0)$,

$$F(x, z, v, 0) = 0.$$

Given $M \in \mathcal{S}$, let M^+ and M^- denote its positive and negative part, respectively, so that $M = M^+ - M^-$ and $M^+, M^- \geq 0$. Applying (H2) twice and using (H3), we have

$$\Lambda\|M^+\| - \lambda\|M^-\| \geq F(x, z, p, M) \geq \lambda\|M^+\| - \Lambda\|M^-\| \tag{2.2}$$

for every $x \in B_1(0)$, $z \in \mathbb{R}$, $v \in B_\delta(0)$, $M \in \mathcal{S}$.

In this section we will call *universal* any positive constant which depends only on n, λ, Λ.

Theorem 2.1. *Let $\delta > 0$, $F : B_1(0) \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ a measurable function which satisfies (H1), (H2), and (H3), $f \in C^0(B_1(0))$, and assume that $u \in C^2(B_1(0))$ solves* (2.1). *Then there exist universal constants $\nu, \varepsilon, \kappa, \rho \in (0, 1)$ such that if $\delta' > 0$ and $k \in \mathbb{N}$ satisfy*

$$\operatorname{osc}_{B_1(0)} u \leq \delta' \leq \rho^{-k}\kappa\delta, \qquad \|f\|_{L^n(B_1(0))} \leq \varepsilon\delta', \tag{2.3}$$

then

$$\operatorname{osc}_{B_{\rho^s}(0)} u \leq (1 - \nu)^s\delta' \qquad \forall\, s = 0, ..., k + 1. \tag{2.4}$$

As we will show at the end of this section, Theorem 2.1 follows by an analogous result at scale 1 (stated in the following proposition) and a scaling argument.

Proposition 2.2. *Let $\delta > 0$, $F : B_1(0) \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ a measurable function which satisfies (H1), (H2), and (H3), $f \in C^0(B_1(0))$, and assume that $u \in C^2(B_1(0))$ solves* (2.1).

Then there exist universal constants $\nu, \varepsilon, \kappa, \rho \in (0, 1)$ such that if δ' satisfies

$$\operatorname{osc}_{B_1(0)} u \leq \delta' \leq \kappa\delta, \qquad \|f\|_{L^n(B_1(0))} \leq \varepsilon\delta', \tag{2.5}$$

then

$$\operatorname{osc}_{B_\rho(0)} u \leq (1 - \nu)\delta'.$$

Before proving this result, we state and prove three basic lemmas. The first lemma gives an estimate on the contact set of a family of paraboloids with fixed opening in terms of the measure of the set of vertices. The proof is a simple variant of the one of [15, Lemma 2.1].

Lemma 2.3. *Let $\delta > 0$, F, λ, Λ, f, and u be as in Proposition 2.2. Fix $a \in (0, \delta/2)$, let $K \subseteq B_1(0)$ be a compact set, and define $A \subseteq \overline{B_1(0)}$ to be the set of contact point of paraboloids with vertices in K and opening $-a$, namely the set of points $x \in \overline{B_1(0)}$ such that there exists $y \in K$ which satisfies*

$$\inf_{z \in B_1(0)} \left\{ \frac{a}{2}|y - z|^2 + u(z) \right\} = \frac{a}{2}|y - x|^2 + u(x). \tag{2.6}$$

Assume that $A \subset B_1(0)$.

Then there exists a universal constant $c_0 > 0$*, such that*

$$c_0|K| \leq |A| + \int_A \frac{|f(x)|^n}{a^n}\, dx. \tag{2.7}$$

Proof. Since by assumption $A \subset B_1(0)$, for every $x \in A$, given $y \in K$ which satisfies (2.6), we have that

$$\nabla u(x) = -a(x-y). \tag{2.8}$$

Let $T : A \to K$ be the map which associates to every contact point x the vertex of the paraboloid, namely

$$T(x) := \frac{\nabla u(x)}{a} + x.$$

Notice that $T \in C^1(\overline{A})$ and $K = T(A)$. From (2.8) we have that, at each contact point $x \in A$,

$$|\nabla u(x)| = a|x-y| \leq 2a \leq \delta,$$

hence from (H2) the equation is uniformly elliptic at x. Moreover we have that $-a\,\mathrm{Id} \leq \nabla^2 u(x)$, so it follows by (2.2) that

$$-a\,\mathrm{Id} \leq \nabla^2 u(x) \leq \frac{\Lambda a + |f(x)|}{\lambda}\,\mathrm{Id} \qquad \forall\, x \in A. \tag{2.9}$$

In addition, from the change of variable formula we have that

$$|K| = |T(A)| \leq \int_A \det \nabla T(x)\, dx = \int_A \det\left(\frac{\nabla^2 u(x)}{a} + \mathrm{Id}\right) dx \tag{2.10}$$

Since each eigenvalue of the matrix $\nabla u(x)/a + \mathrm{Id}$ lies in the interval $[0, (1+\Lambda/\lambda) + |f(x)|/(\lambda a)]$ (see (2.9), we get

$$\det\left(\frac{\nabla^2 u(x)}{a} + I\right) \leq C_0\left[1 + \frac{|f(x)|^n}{a^n}\right]$$

for some universal constant C_0. Hence, it follows from (2.10) that

$$|K| \leq C_0|A| + C_0\int_A \frac{|f(x)|^n}{a^n}\, dx,$$

which proves (2.7) with $c_0 = 1/C_0$. □

Before stating the next lemma we introduce some notation.

Given u as before, for every $b > 0$ we define A_b be the set of $x \in B_1(0)$ such that $u(x) \leq b$ and the function u can be touched from below at x with a paraboloid of opening $-b$, namely there exists $y \in \overline{B_1(0)}$ such that

$$\inf_{z \in B_1(0)} \left\{\frac{b}{2}|y-z|^2 + u(z)\right\} = \frac{b}{2}|y-x|^2 + u(x). \tag{2.11}$$

In addition, given $g \in L^1(B_1(0))$, we denote by $M[g]$ the maximal function associated to g, namely

$$M[g](x) := \sup\left\{ \fint_{B_r(z)} g(y)\, dy : B_r(z) \subseteq B_1(0), x \in B_r(z)\right\}.$$

A fundamental property of maximal functions is a weak-L^1 estimate (see for instance [16]): there exists a constant C_n depending only on the dimension such that

$$|\{x : M[g](x) > t\}| \le \frac{C_n \|g\|_{L^1(B_1(0))}}{t} \qquad \forall\, t > 0,\ \forall\, g \in L^1(B_1(0)). \tag{2.12}$$

Given f as before, for every $b > 0$ we denote by M_b the set

$$M_b := \{x \in B_1(0) : M[|f|^n](x) \le b^n\}.$$

Lemma 2.4. *Let $\delta > 0$, F, λ, Λ, f and u be as in Proposition 2.2. Let $a > 0$, $\overline{B_{4r}(x_0)} \subset B_1(0)$.*
Then there exist universal constants $\tilde{C} \ge 2$ and $\tilde{c}, \mu > 0$, such that if $a \le \delta/\tilde{C}$, and

$$B_r(x_0) \cap A_a \cap M_{\mu a} \ne \emptyset$$

then

$$|B_{r/8}(x_0) \cap A_{\tilde{C}a}| \ge \tilde{c}|B_r(x_0)|. \tag{2.13}$$

Proof. Let $x_1 \in B_r(x_0) \cap A_a \cap M_{\mu a}$ and $y_1 \in B_1(0)$ be the vertex of the paraboloid which satisfies (2.11) with x_1. Let $P_{y_1}(x)$ be the tangent paraboloid, namely

$$P_{y_1}(x) = u(x_1) + \frac{a}{2}|x_1 - y_1|^2 - \frac{a}{2}|x - y_1|^2.$$

• *Step 1: There exist universal constants $C_0, C_1 > 0$ such that if $a \le \delta/C_0$, then there is $z \in B_{r/16}(x_0)$ such that*

$$u(z) \le P_{y_1}(z) + C_1 a r^2. \tag{2.14}$$

Let $\alpha > 0$ be a large universal constant which we choose later, and define $\varphi : \mathbb{R}^n \to \mathbb{R}$ as

$$\varphi(x) := \begin{cases} \alpha^{-1}(32^\alpha - 1) & \text{if } |x| < 32^{-1} \\ \alpha^{-1}(|x|^{-\alpha} - 1) & \text{if } 32^{-1} \le |x| \le 1 \\ 0 & \text{if } 1 < |x|. \end{cases} \tag{2.15}$$

Given $x_3 \in B_r(x_1) \cap B_{r/32}(x_0)$ we consider the function $\psi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\psi(x) := P_{y_1}(x) + a r^2 \varphi\left(\frac{x - x_3}{r}\right) \qquad \forall\, x \in \mathbb{R}^n.$$

We slide the function ψ from below until it touches the function u. Let x_4 be the contact point. Since the function φ is radial and decreasing in the radial direction, from

$$-a r^2 \varphi\left(\frac{x_4 - x_3}{r}\right) \le u(x_4) - \psi(x_4) \le \min_{x \in B_1(0)}\{u(x) - \psi(x)\} \le -a r^2 \varphi\left(\frac{x_1 - x_3}{r}\right) \tag{2.16}$$

we deduce that $|x_4 - x_3| \le |x_1 - x_3| \le r$. In particular since $|x_4 - x_0| \le |x_4 - x_3| + |x_3 - x_0| \le 2r$ and $\overline{B_{2r}(x_0)} \subset B_1(0)$ (by assumption), the contact point is inside $B_1(0)$. We now distinguish two cases:

- *Case 1: There exists $x_3 \in B_r(x_1) \cap B_{r/32}(x_0)$ such that the contact point x_4 lies inside $B_{r/32}(x_3)$.*

In this case we have $|x_4 - x_0| \le |x_4 - x_3| + |x_3 - x_0| \le r/16$. In addition, the last two inequalities in (2.16) give that $u(x_4) - \psi(x_4) \le 0$. Hence

$$u(x_4) \le \psi(x_4) = P_{y_1}(x_4) + a r^2 \varphi\left(\frac{x_4 - x_3}{r}\right) \le P_{y_1}(x_4) + a r^2 \|\varphi\|_{L^\infty(\mathbb{R}^n)},$$

which proves that $z = x_4$ satisfies (2.14) with $C_1 := \|\varphi\|_{L^\infty(\mathbb{R}^n)}$ (without any restriction on a).

- *Case 2: For every $x_3 \in B_r(x_1) \cap B_{r/32}(x_0)$ the contact point x_4 satisfies* $1/32 < |x_4 - x_3| < 1$.
At the contact point we have that

$$\nabla u(x_4) = \nabla\psi(x_4) = -a(x_4 - y_1) + ar\nabla\varphi\left(\frac{x_4 - x_3}{r}\right). \tag{2.17}$$

Hence, if we choose C_0 such that $C_0 \geq 2 + \|\varphi\|_{L^\infty(\mathbb{R}^n)}$ we get

$$|\nabla u(x_4)| \leq a|x_4 - y_1| + ar\|\varphi\|_{L^\infty(\mathbb{R}^n)} < a(2 + \|\varphi\|_{L^\infty(\mathbb{R}^n)}) \leq C_0 a \leq \delta,$$

which shows that the equation (2.1) is uniformly elliptic at x_4 thanks to our assumptions on F.
Computing the second derivatives of ψ at x_4 we get

$$\begin{aligned}\nabla^2\psi(x_4) &= -aI + a\nabla^2\varphi\left(\frac{x_4 - x_3}{r}\right)\\ &= a\left(-I - \left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha} I + (2+\alpha)\frac{(x_4 - x_3)\otimes(x_4 - x_3)}{r^2}\left(\frac{r}{|x_4 - x_3|}\right)^{4+\alpha}\right),\end{aligned}$$

hence from (H1) and (2.2) applied with $M = \nabla^2\psi(x_4)$ we obtain (since ψ touches u from below at x_4, we have $\nabla^2 u(x_4) \geq \nabla^2\psi(x_4)$)

$$\begin{aligned}f(x_4) &= F(x_4, u(x_4), \nabla u(x_4), \nabla^2 u(x_4))\\ &\geq F(x_4, u(x_4), \nabla u(x_4), \nabla^2\psi(x_4))\\ &\geq a\left(-\Lambda - \Lambda\left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha} + (2+\alpha)\lambda\left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha}\right)\\ &= a\left(-\Lambda + ((2+\alpha)\lambda - \Lambda)\left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha}\right).\end{aligned}$$

Choosing α big enough so that $(2+\alpha)\lambda - \Lambda \geq \Lambda + 1$, and using that $|x_4 - x_3| \leq r$, we obtain

$$\frac{f(x_4)}{a} \geq -\Lambda + (\Lambda + 1)\left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha} \geq 1. \tag{2.18}$$

In addition,

$$\nabla^2 u(x_4) \geq \nabla^2\psi(x_4) = -aI + a\nabla^2\varphi\left(\frac{x_4 - x_3}{r}\right) \geq a\left(-1 - \left(\frac{r}{|x_4 - x_3|}\right)^{2+\alpha}\right) I \geq -(1 + 32^{2+\alpha})a\,\mathrm{Id},$$

so by applying the second inequality in (2.2) to $M = \nabla^2 u(x_4)$, we get

$$\lambda\|\nabla^2 u(x_4)^+\| \leq F(x_4, u(x_4), \nabla u(x_4), \nabla^2 u(x_4)) + \Lambda\|\nabla^2 u(x_4)^-\| \leq |f(x_4)| + \Lambda(1 + 32^{2+\alpha})a,$$

that is

$$\frac{\nabla^2 u(x_4)}{a} \leq C_2\left(1 + \frac{|f(x_4)|}{a}\right)\mathrm{Id}, \tag{2.19}$$

for some $C_2 > 0$ universal.

Let us consider K the set of contact points x_4 as x_3 varies in $B_{r/32}(x_0)$ (as we observed before, $K \subseteq B_{2r}(x_0)$), and let $T : K \to \mathbb{R}^n$ be the map which associates to every contact point x_4 the corresponding x_3, which is given by (see (2.17))

$$T(x) = x - r(\nabla\varphi)^{-1}\left(\frac{\nabla u(x) + a(x - y_1)}{ar}\right)$$

(note that $\nabla\varphi$ is an invertible function in the annulus $1/32 < |x| < 1$ and $(\nabla\varphi)^{-1}$ can be explicitly computed). Since $T(K) = B_r(x_1) \cap B_{r/32}(x_0)$, we deduce that there exists a constant c_n, depending only on the dimension, such that $c_n r^n \le |B_r(x_1) \cap B_{r/32}(x_0)| = |T(K)|$. Therefore, from the area formula,

$$c_n r^n \le \int_K |\det \nabla T(x)|\, dx \tag{2.20}$$

We now observe that

$$\nabla T(x) = \mathrm{Id} - \left(\nabla^2\varphi \circ (\nabla\varphi)^{-1}\left(\frac{\nabla^2 u(x) + a(x-y_1)}{ar}\right)\right)^{-1} \frac{\nabla u(x) + aI}{a},$$

so from (2.19) and (2.18) we get

$$\begin{aligned}\|\nabla T(x)\| &\le 1 + \|(\nabla^2\varphi)^{-1}\|_{L^\infty(B_1\setminus B_{1/32})}\left(1 + C_2 + C_2\frac{|f(x)|}{a}\right)\\ &\le \left(1 + \|(\nabla^2\varphi)^{-1}\|_{L^\infty(B_1\setminus B_{1/32})}(1+2C_2)\right)\frac{|f(x)|}{a}.\end{aligned}$$

Hence, combining this bound with (2.20) we get

$$c_n r^n \le C_3 \int_K \frac{|f(x)|^n}{a^n}\, dx \le C_3 \int_{B_{2r}(x_0)} \frac{|f(x)|^n}{a^n}\, dx,$$

where $C_3 > 0$ is universal. Since $B_{2r}(x_0) \subseteq B_{3r}(x_1)$ and $B_{3r}(x_1) \subset B_1(0)$ (note $B_{3r}(x_1)$ is included in $B_{4r}(x_0)$, which is contained inside $B_1(0)$ by assumption), we conclude

$$c_n r^n \le C_3 \int_{B_{3r}(x_1)} \frac{|f(x)|^n}{a^n}\, dx \le C_3 M[|f|^n](x_1)\frac{|B_{3r}(x_1)|}{a^n}. \tag{2.21}$$

Recalling that by assumption $M(|f|^n)(x_1) \le \mu^n a^n$, choosing μ small enough so that $\mu^n < c_n/(C_3|B_3(0)|2^n)$, we obtain

$$C_3 M(|f|^n)(x_1)\frac{|B_1(0)|2^n r^n}{a^n} \le C_3\mu^n|B_1(0)|2^n r^n < c_n r^n,$$

which contradicts (2.21).

• *Step 2: Conclusion of the proof.* From now on, we assume that $a \le \delta/C_0$, so that the conclusion of Step 1 holds.

Let $C_4 > 0$ be a universal constant which will be fixed later, and for every $y \in B_{r/64}(z)$ we consider the paraboloid

$$Q_y(x) := P_{y_1}(x) - C_4\frac{a}{2}|x-y|^2.$$

It can be easily seen that for every y the function $Q_y(x)$ is a paraboloid with opening $-(C_4+1)a$ and vertex

$$\frac{y_1 + C_4 y}{1 + C_4}. \tag{2.22}$$

Let slide Q_y from below until it touches the graph of u. We claim that the contact point $\bar{x}$ lies inside $B_{r/16}(z) \subset B_{r/8}(x_0)$.

Indeed if $|\bar{x} - z| \ge r/16$ we have that

$$|\bar{x} - y| \ge |\bar{x} - z| - |z - y| \ge \frac{r}{16} - \frac{r}{64} \ge \frac{r}{32},$$

so, thanks to (2.14),

$$\min_{x\in B_1(0)}\left\{u(x)-P_{y_1}(x)+C_4\frac{a}{2}|x-y|^2\right\}\le u(z)-P_{y_1}(z)+C_4\frac{a}{2}|z-y|^2\le C_1ar^2+C_4\frac{a}{2}\left(\frac{r}{64}\right)^2. \tag{2.23}$$

On the other hand, since $u\ge P_{y_1}$ we have

$$u(\bar x)-P_{y_1}(\bar x)+C_4\frac{a}{2}|\bar x-y|^2\ge C_4\frac{a}{2}\left(\frac{r}{32}\right)^2,$$

which contradicts (2.23) if we choose C_4 sufficiently large. This proves in particular that

$$\bar x\in B_{r/16}(z)\subset B_{r/8}(x_0). \tag{2.24}$$

We now show that the contact points satisfy $u(\bar x)\le C_4a$. Indeed, since by assumption $P_{y_1}(x_1)=u(x_1)\le a$ and all points lie inside $B_1(0)$, we have

$$P_{y_1}(\bar x)=u(x_1)+\frac{a}{2}|x_1-y_1|^2-\frac{a}{2}|\bar x-y_1|^2\le a+4a=5a,$$

so from (2.23) we obtain

$$u(\bar x)\le P_{y_1}(\bar x)-C_4\frac{a}{2}|\bar x-y|^2+C_1ar^2+C_4\frac{a}{2}\left(\frac{r}{64}\right)^2\le 5a+C_1ar^2+C_4\frac{a}{2}\left(\frac{r}{64}\right)^2,$$

which is less than C_4a provided that C_4 is chosen sufficiently large.

We now observe that, as y varies in $B_{r/64}(z)$, the set of vertices of the paraboloids is a ball around $\frac{y_1+C_4z}{1+C_4}$ of radius $\frac{C_4r}{64(1+C_4)}$ (see (2.22)). Hence, recalling (2.24) and that $u\le C_4a$ at the contact points, it follows from Lemma 2.3 that

$$c\left(\frac{C_4r}{64(1+C_4)}\right)^n|B_1(0)|\le|B_{r/8}(x_0)\cap A_{C_4a}|+\int_{B_{r/8}(x_0)}\frac{|f(x)|^n}{a^n}\,dx.$$

Since the last integral can be estimated with

$$\int_{B_{2r}(x_1)}\frac{|f(x)|^n}{a^n}\,dx\le M[|f|^n](x_1)\frac{|B_{2r}(x_1)|}{a^n}\le\mu^nr^n|B_2(0)|,$$

we conclude that (2.13) holds with $\tilde C:=\max\{C_0,C_4\}$, provided μ is sufficiently small. □

The following measure covering lemma is proved by Savin in [15, Lemma 2.3] in a slightly different version.

Lemma 2.5. *Let $\sigma,r_0\in(0,1)$, and let D_0,D_1 be two closed sets satisfying*

$$\emptyset\ne D_0\subseteq D_1\subseteq\overline{B_{r_0}(0)}.$$

Assume that whenever $x\in B_{r_0}(0)$ and $r>0$ satisfy

$$B_{4r}(x)\subseteq B_1(0),\qquad B_{r/8}(x)\subseteq B_{r_0}(0),\qquad \overline{B_r(x)}\cap D_0\ne\emptyset$$

then

$$|B_{r/8}(x)\cap D_1|\ge\sigma|B_r(x)|.$$

Then, if $r_0>0$ is sufficiently small we get

$$|B_{r_0}(0)\setminus D_1|\le(1-\sigma)|B_{r_0}(0)\setminus D_0|. \tag{2.25}$$

Although the proof is a minor variant of the argument of Savin in [15, Lemma 2.3], we give the argument for completeness. As we will see from the proof, a possible choice for r_0 is $1/13$.

Proof. Given $x_0 \in B_{r_0}(0) \setminus D_0$, set $\bar r := \mathrm{dist}(x_0, D_0) \le 2r_0$, and define

$$x_1 := x_0 - \frac{\bar r}{7}\frac{x_0}{|x_0|}, \qquad r := \frac{8}{7}\bar r.$$

Then it is easy to check that

$$B_{r/8}(x_1) \subset B_{r/4}(x_0) \cap B_{r_0}(0), \qquad \overline{B_r(x_1)} \cap D_0 = \emptyset.$$

In addition, since $r \le 3r_0$ and $|x_1| < r_0$,

$$B_{4r}(x_1) \subset B_{13r_0}(0) \subseteq B_1(0) \qquad \text{provided } r_0 \le 1/13.$$

Hence, using our assumptions we get

$$|B_{r/4}(x_0) \cap B_{r_0}(0) \cap D_1| \ge |B_{r/8}(x_1) \cap D_1| \ge \sigma|B_r(x_1)| = \sigma|B_r(x_0)| \ge \sigma|B_{r_0}(0) \cap B_r(x_0)|.$$

Now, for every $x \in B_{r_0}(0) \setminus D_0$ we consider the ball centered at x and radius $r := \mathrm{dist}(x, D_0)$, and we apply Vitali covering's Lemma to this family to extract a subfamily $\{B_{r_i}(x_i)\}$ such that the balls $B_{r_i/3}(x_i)$ (and so in particular also the balls $B_{r_i/4}(x_i)$) are disjoint. Hence

$$\sigma|B_{r_0}(0) \setminus D_0| \le \sigma \sum_i |(B_{r_i}(x_i) \cap B_{r_0}) \setminus D_0| \le \sum_i |B_{r_i/4}(x_i) \cap B_{r_0}(0) \cap (D_1 \setminus D_0)| \le |B_{r_0} \cap (D_1 \setminus D_0)|,$$

from which the result follows easily. □

Proof of Proposition 2.2. Let c_0 be the constant from Lemma 2.3, and $\tilde C$, $\tilde c$, μ the constants given by Lemma 2.4. Also, we fix $r_0 > 0$ sufficiently small so that Lemma 2.5 applies, and we define $r_1 := r_0/8$.

Let $\nu < 1/2$ and N be universal constants (to be chosen later) satisfying $N\nu \ll 1$, set $a := N\nu\delta'$, $m := \inf_{B_1(0)} u$ and assume by contradiction that there exists $x_0 \in B_{r_0/2}(0)$ such that

$$u(x_0) - m < \nu\delta', \tag{2.26}$$

and in addition

$$\sup_{B_{r_1}(0)} u - m > \delta'/2. \tag{2.27}$$

(Note that if either (2.26) or (2.27) fails, then $\mathrm{osc}_{B_{r_1}(0)} u \le (1-\nu)\delta'$, so the statement is true with $\rho = r_1$).

We define the sets A_a as before but replacing u with the nonnegative function $u - m$, that is A_a is the set of points where $u - m$ is bounded by a and can be touched from below with a paraboloid of opening $-a$.

- *Step 1: The following holds:*

$$|B_{r_0}(0) \cap A_a| \ge \frac{c_0|B_{r_1}(0)|}{2}, \qquad |M_{\mu a}| > |B_1| - \frac{c_0|B_{r_1}(0)|}{2}. \tag{2.28}$$

To prove this, for every $y \in B_{r_1}(0)$ we consider the paraboloid

$$P_y(x) := \frac{a}{2}\left((r_0 - r_1)^2 - |x-y|^2\right).$$

We observe that

$$P_y \le 0 \qquad \text{for } |x| \ge r_0$$

(because $|x-y| \geq |x|-|y| \geq r_0 - r_1$), while $|x-y| \leq |x|+|y| \leq r_0/2 + r_1$ for $x \in B_{r_0/2}(0)$, which implies (recall that $a = N\nu\delta'$)

$$P_y(x) \geq \frac{a}{2}\left((r_0-r_1)^2 - \left(\frac{r_0}{2}+r_1\right)^2\right) > \nu\delta' \geq u(x_0) - m \qquad \forall\, x \in B_{r_0/2}(0) \tag{2.29}$$

provided N is sufficiently large. Moreover $P_y(x) \leq a$ for every $x, y \in B_1(0)$.

Hence, let us slide the paraboloids P_y from below until they touch the function $u - m$. Let A be the contact set as y varies inside $B_{r_1}(0)$. By what said before it follows that the contact points are contained inside $B_{r_0}(0)$. In addition, thanks to (2.26) and (2.29), at any contact point x we have

$$\begin{aligned} 0 &> u(x_0) - m - \nu\delta' \geq \min_{z\in B_1(0)} \{u(z) - m - P_y(z)\} \\ &= u(x) - m - P_y(x) \geq u(x) - m - a, \end{aligned}$$

which proves that $A \subset B_{r_0}(0) \cap A_a$. From Lemma 2.3 applied to $K = B_{r_1}(0)$ we obtain

$$|B_{r_0}(0)\cap A_a| \geq |A| \geq c_0|B_{r_1}(0)| - \int_A \frac{|f(x)|^n}{a^n}\,dx \geq c_0|B_{r_1}(0)| - \int_{B_1(0)} \frac{|f(x)|^n}{a^n}\,dx \geq c_0|B_{r_1}(0)| - \frac{\varepsilon^n}{N^n\nu^n},$$

while the maximal estimate (2.12) gives

$$|B_1(0)\setminus M_{\mu a}| \leq \frac{C_n\|f\|^n_{L^n(B_1(0))}}{(\mu a)^n} \leq \frac{C_n\varepsilon^n}{\mu^n N^n \nu^n},$$

hence (2.28) is satisfied provided ε is sufficiently small.

- *Step 2: There exists a constant $\bar C > 0$, depending only on the dimension, such that*

$$|B_{r_0}(0)\setminus A_{\tilde C^k a}| \leq \bar C(1-\tilde c)^k \qquad \text{provided } \tilde C^{k+1}a \leq \delta. \tag{2.30}$$

From (2.28) it follows that

$$B_{r_0}(0)\cap A_a \cap M_{\mu a} \neq \emptyset.$$

Since the sets A_a and M_a are increasing with respect to k, this implies that

$$B_{r_0}(0)\cap A_{\tilde C^k a} \cap M_{\mu \tilde C^k a} \neq \emptyset \qquad \forall\, k \in \mathbb{N}, \tag{2.31}$$

where $\tilde C \geq 2$ is as in Lemma 2.4.

Now, for every $k \in \mathbb{N}$ such that $\tilde C^{k+1}a \leq \delta$ we apply Lemma 2.5 to the closed sets

$$D_0 := \overline{B_{r_0}(0)} \cap A_{\tilde C^k a} \cap M_{\mu\tilde C^k a}, \qquad D_1 := \overline{B_{r_0}(0)} \cap A_{\tilde C^{k+1} a}.$$

Since D_0 is nonempty (see (2.31)), Lemma 2.4 applied with $\tilde C^k a$ instead of a proves that assumption of Lemma 2.5 are satisfied with $\sigma = \tilde c > 0$. Therefore

$$\begin{aligned} |B_{r_0}(0)\setminus A_{\tilde C^{k+1}a}| &\leq (1-\tilde c)|B_{r_0}(0)\setminus (A_{\tilde C^k a}\cap M_{\mu\tilde C^k a})| \\ &\leq (1-\tilde c)\left(|B_{r_0}(0)\setminus A_{\tilde C^k a}| + |B_{r_0}(0)\setminus M_{\mu\tilde C^k a}|\right). \end{aligned} \tag{2.32}$$

Applying (2.32) inductively for every positive integer k such that $\tilde{C}^{k+1}a \le \delta$ and using the maximal estimate (2.12), we obtain

$$\begin{aligned} |B_{r_0}(0) \setminus A_{\tilde{C}^k a}| &\le (1-\tilde{c})^k |B_{r_0}(0) \setminus A_a| + \sum_{i=1}^{k} (1-\tilde{c})^i |B_{r_0}(0) \setminus M_{\tilde{C}^{k-i}a}| \\ &\le (1-\tilde{c})^k |B_{r_0}(0)| + \sum_{i=1}^{k} (1-\tilde{c})^i \frac{C_n \|f\|^n_{L^n(B_1(0))}}{\mu^n \tilde{C}^{n(k-i)} a^n}, \end{aligned}$$

so by (2.5) we get (recall that $a = N\nu\delta'$)

$$\begin{aligned} |B_{r_0}(0) \setminus A_{\tilde{C}^k a}| &\le (1-\tilde{c})^k \left[|B_{r_0}(0)| + \frac{C_n \varepsilon^n}{\mu^n N^n \nu^n} \sum_{i=1}^{k} \frac{1}{((1-\tilde{c})\tilde{C}^n)^{k-i}} \right] \\ &\le (1-\tilde{c})^k \left[|B_{r_0}(0)| + \frac{C_n \varepsilon^n}{\mu^n N^n \nu^n} \sum_{i=0}^{\infty} \frac{1}{((1-\tilde{c})\tilde{C}^n)^{i}} \right]. \end{aligned} \tag{2.33}$$

Assuming without loss of generality that $\tilde{c} \le 1/2$, $\tilde{C} \ge 3$, and $\varepsilon \le \mu N \nu / C_n^{-1/n}$ we have

$$|B_{r_0}(0) \setminus A_{\tilde{C}^k a}| \le (1-\tilde{c})^k \left[|B_{r_0}(0)| + \sum_{i=0}^{\infty} \left(\frac{2}{3^n} \right)^i \right],$$

which proves (2.30).

- *Step 3: Let $E := \{x \in B_{r_0}(0) : u(x) - m \ge \delta'/4\}$. Then*

$$|E| \ge \frac{c_0 |B_{r_1}(0)|}{2}. \tag{2.34}$$

For every $y \in B_{r_1}(0)$ we consider the paraboloid

$$Q_y(x) := \frac{\delta'}{(r_0 - r_1)^2} |x-y|^2 + \frac{\delta'}{4},$$

and we slide it from above (in Step 1 we slided paraboloids from below) until it touches the graph of $u - m$ inside $B_1(0)$. It is easy to check that, since $|x-y| \ge |x| - |y|$, we have

$$Q_y(x) > \delta' \ge u(x) - m \qquad \text{for } |x| \ge r_0$$

(recall that $y \in B_{r_1}(0)$ and $u - m \le \delta'$ inside $B_1(0)$), while by (2.27)

$$\sup_{B_{r_1}(0)} Q_y \le \delta'/2 < \sup_{B_{r_1}(0)} u - m \tag{2.35}$$

(recall that $r_0 = 8r_1$), so the contact point lies inside $B_{r_0}(0)$. If we denote by A' the contact set as y varies inside $B_{r_1}(0)$ applying Lemma 2.3 "from above" (namely to the function $-u(x) + m$ touched from below by the paraboloids $-Q_y(x)$) with $a = 2\delta'/(r_0 - r_1)^2$ (notice that $\delta' \le \kappa\delta$, so $a \le \delta/2$ if κ is sufficiently small) we obtain

$$|A'| \ge c_0 |B_{r_1}(0)| - \int_{A'} \frac{|f(x)|^n}{a^n}\, dx \ge c_0 |B_{r_1}(0)| - \frac{\varepsilon^n}{N^n \nu^n}. \tag{2.36}$$

Moreover, it follows by (2.35) tha$u - m \ge \delta'/4$ at every contact point. This implies that the contact set A' is contained in E, so the desired estimate follows by (2.36).

• *Step 4: Conclusion.* Let $k_0 \in \mathbb{N}$ be the largest number such that $\tilde{C}^{k_0+1} a \leq \delta'/4$. Since $\delta' \leq \delta$, by Step 2 we get

$$|B_{r_0}(0) \setminus A_{\tilde{C}^{k_0} a}| \leq \bar{C}(1-\tilde{c})^{k_0}.$$

On the other hand, since

$$E \subset \Big\{ x \in B_{r_0}(0) : u(x) - m > \tilde{C}^{k_0} a \Big\} \subset B_{r_0}(0) \setminus A_{\tilde{C}^{k_0} a},$$

it follows by Step 3 that

$$\frac{c_0 |B_{r_1}(0)|}{2} \leq \bar{C}(1-\tilde{c})^{k_0}.$$

Since $k_0 \sim |\log_{\tilde{C}}(N\nu)|$ (recall that $a = N\nu\delta'$), we get a contradiction by first fixing N large enough (so that all the previous arguments apply) and then choosing ν sufficiently small. □

Proof of Theorem 2.1. Let $\nu, \varepsilon, \kappa, \rho \in (0,1)$ be the constants of Proposition 2.2. Without loss of generality we assume that $\nu, \rho \leq 1/2$. We prove (2.4) by induction on s. For $s = 0$ the result is true by assumption. We prove the result for $s+1$ given the one for s. Let $\widetilde{F} : B_1(0) \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ be

$$\widetilde{F}(x, z, p, M) := \rho^s F(x, \rho^s z, p, \rho^{-s} M),$$

and consider the function

$$v(x) := \rho^{-s} u(\rho^s x) \qquad \forall\, x \in B_1(0).$$

Then $\widetilde{F}$ satisfies the same assumptions (H1), (H2), and (H3) which are satisfied by F with the same ellipticity constants λ and Λ, and v solves the fully nonlinear equation

$$\widetilde{F}(x, v(x), \nabla v(x), \nabla^2 v(x)) = \rho^s f(\rho^s x).$$

By inductive hypothesis

$$\|v\|_{L^\infty(B_{\rho^s}(0))} = \rho^{-s} \|u\|_{L^\infty(B_{\rho^s}(0))} \leq \rho^{-s}(1-\nu)^s \delta' \leq \rho^{-s}\delta' \leq \rho^{k-s}\kappa\delta \leq \kappa\delta. \tag{2.37}$$

Also, by (2.3),

$$\|\rho^s f(\rho^s x)\|_{L^n(B_1(0))} = \|f\|_{L^n(B_{\rho^s}(0))} \leq \|f\|_{L^n(B_1(0))} \leq \varepsilon\delta' \leq \varepsilon \rho^{-s}(1-\nu)^s \delta'.$$

Hence, we apply Proposition 2.2 to v with $\rho^{-s}(1-\nu)^s\delta'$ instead of δ', to obtain

$$\rho^{-s} \|u\|_{L^\infty(B_{\rho^{s+1}}(0))} = \|v\|_{L^\infty(B_\rho(0))} \leq \rho^{-s}(1-\nu)^{s+1}\delta',$$

which proves the inductive step.

□

3. Separation between degenerancy and nondegeneracy

First, we introduce some notation regarding the norm induced by a convex set E (see (1.4)).

We denote by E^* the ball in the dual norm

$$E^* := \{ e^* \in \mathbb{R}^n : e^* \cdot e \leq 1 \quad \forall\, e \in E \}. \tag{3.1}$$

It can be easily seen that with this definition

$$|e|_E = \sup\{ e^* \cdot e : e^* \in E^* \} \qquad \forall\, e \in \mathbb{R}^n.$$

We denote by d_E (and d_{E^*}, respectively) the smallest radius such that $E \subseteq B_{d_E}(0)$, ($E^* \subseteq B_{d_{E^*}}(0)$, respectively). Notice that

$$d_E = \max\{ |e| : |e|_E = 1 \} \tag{3.2}$$

Similarly, we denote by $\widetilde{d}_E$ the biggest radius such that $B_{\widetilde{d}_E}(0) \subseteq E$. It satisfies

$$|e|_E \leq |e|/\widetilde{d}_E \qquad \forall\, e \in \mathbb{R}^n. \tag{3.3}$$

Moreover, if E is strictly convex, then we can define map $\ell : \partial E^* \to \partial E$, where $\ell_{e^*} := \ell(e^*)$ is the unique element of ∂E such that $|\ell_{e^*}|_E = e^* \cdot \ell_{e^*}$ (in other terms, $\{x \cdot e^* = 1\}$ is a supporting hyperplane for E at ℓ_{e^*}). In addition, again by the strict convexity of E, ℓ is continuous in the following sense: for every $\varepsilon_0 > 0$ there exists $\eta(\varepsilon_0) > 0$ such that

$$e \in \overline{E}, \quad e^* \in \partial E^*, \quad 1 - \eta(\varepsilon_0) \leq e^* \cdot e \leq 1 \qquad \Rightarrow \quad |e - \ell_{e^*}| \leq \varepsilon_0. \tag{3.4}$$

In the following lemma we prove that, at every scale, if none of the partial derivatives of u is close to the L^∞ norm of $|\nabla u|_E$ in a set of large measure, then $|\nabla u|_E$ decays by a fixed amount on a smaller ball. As we will see in the next section, if this case does not occur, then the equation is nondegenerate and we can prove that u is $C^{1,\alpha}$ there.

As we will see below, a key observation being the proof of the next result is the fact that the function $v_{e^*}(x) := (\partial_{e^*} u(x) - (1+\delta))_+$ solves

$$\partial_i[\partial_{ij}\mathcal{F}(\nabla u(x))\partial_j v_{e^*}(x)] \geq \partial_{e^*} f(x), \tag{3.5}$$

and the equation might be assumed to be uniformly elliptic, since the values of the coefficients $\partial_{ij}\mathcal{F}(\nabla u(x))$ are not relevant when $|\nabla u(x)| \leq 1+\delta$ (since at that points $v_{e^*} = 0$).

Lemma 3.1. *Fix $\eta > 0$, and let δ, $\mathcal{F}$, E, λ, Λ, M, f, and u be as in Theorem 1.4. For every $i \in \mathbb{N}$ set*

$$d_i := \sup\{(|\nabla u(x)|_E - (1+\delta))_+ : x \in B_{2^{-i}(0)}\},$$

and assume that there exists $k \in \mathbb{N}$ such that

$$\sup_{e^* \in \partial E^*} |\{x \in B_{2^{-2i-1}}(0) : (\partial_{e^*} u(x) - (1+\delta))_+ \geq (1-\eta)d_{2i}\}| \leq (1-\eta)|B_{2^{-2i-1}}(0)| \qquad \forall\, i = 0, ..., k. \tag{3.6}$$

Then there exists $\alpha \in (0,1)$ and $C_0 > 0$, depending only on η, M, q, $\|f\|_{L^q(B_1(0))}$, d_{E^*},$\widetilde{d}_E$, δ, λ, and Λ, such that

$$d_{2i} \leq C_0 2^{-2i\alpha} \qquad \forall\, i = 0, ..., k+1. \tag{3.7}$$

Proof. Given $e^* \in \partial E^*$, we differentiate (1.10) in the direction of e^* to obtain

$$\partial_i[\partial_{ij}\mathcal{F}(\nabla u(x))\partial_j(\partial_{e^*} u(x))] = \partial_{e^*} f(x).$$

Since the function $t \mapsto (t-(1+\delta))_+$ is convex, it follows that the function $v_{e^*}(x) := (\partial_{e^*} u(x) - (1+\delta))_+$ is a subsolution of the above equation, that is (3.5) holds.

Note that, since $v_{e^*}(x)$ is constant where $|\nabla u|_E \leq 1+\delta$ and $\mathcal{F}$ is uniformly elliptic on the set $\{|\nabla u|_E \geq 1+\delta/2\}$ (see (1.9)), we can change the coefficients outside this region to ensure that the equation is uniformly elliptic everywhere, with constants λ and Λ. Applying [12, Theorem 8.18] to the function $d_{2i} - v_{e^*}(x)$ (which is a nonnegative supersolution inside $B_{2^{-2i}}(0)$), we obtain that there exists a constant $c_0 := c_0(n, \lambda, \Lambda) > 0$ such that

$$\inf\bigl\{d_{2i} - v_{e^*}(x) : x \in B_{2^{-2i-2}}(0)\bigr\} \geq c_0 2^{2in} \int_{B_{2^{-2i-1}}(0)} (d_{2i} - v_{e^*}(x))\, dx \; - 2^{-2i(1-n/q)} \|f e^*\|_{L^q(B_{2^{-2i}}(0))}$$

We estimate the integral in the right hand side considering only the set

$$\bigl\{x \in B_{2^{-2i-1}}(0) : v_{e^*}(x) \leq (1-\eta)d_{2i}\bigr\}.$$

There, the integrand is greater than ηd_{2i} and the measure of the set is greater than $\eta|B_{2^{-2i-1}}(0)|$ (by (3.6)), hence

$$\begin{aligned}\inf\big\{d_{2i} - v_{e^*}(x) : x \in B_{2^{-2i-2}}(0)\big\} &\geq c_0 2^{2in}\eta^2 d_{2i}|B_{2^{-2i-1}}(0)| - 2^{-2i(1-n/q)}\|fe^*\|_{L^q(B_1(0))} \\ &\geq c_0\eta^2 d_{2i}|B_{1/2}(0)| - 2^{-2i(1-n/q))}\|f\|_{L^q(B_1(0))}d_{E^*}.\end{aligned} \tag{3.8}$$

We now distinguish two cases, depending whether

$$\frac{c_0|B_{1/2}(0)|\eta^2}{2} d_{2i} \geq d_{E^*}2^{-2i(1-n/q))}\|f\|_{L^q(B_1)} \tag{3.9}$$

holds or not.

- *Case 1:* (3.9) *holds.* In this case we obtain from (3.8) that

$$v_{e^*}(x) \leq \left(1 - \frac{c_0|B_{1/2}(0)|\eta^2}{2}\right) d_{2i} \qquad \forall\, x \in B_{2^{-2i-2}}(0).$$

Since $e^* \in \partial E^*$ is arbitrary and

$$\sup_{e^*\in\partial E^*} v_{e^*}(x) = \left(\sup_{e^*\in\partial E^*} \partial_{e^*}u(x) - (1+\delta)\right)_+ = (|\nabla u(x)|_E - (1+\delta))_+ \qquad \forall\, x \in B_1(0),$$

we get

$$(|\nabla u(x)|_E - (1+\delta))_+ \leq \left(1 - \frac{c_0|B_{1/2}(0)|\eta^2}{2}\right) d_{2i} \qquad \forall\, x \in B_{2^{-2i-2}}(0),$$

that is

$$d_{2(i+1)} \leq \left(1 - \frac{c_0|B_{1/2}(0)|\eta^2}{2}\right) d_{2i}. \tag{3.10}$$

- *Case 2:* (3.9) *fails.* In this case we get

$$d_{2(i+1)} \leq d_{2i} \leq C' 2^{-2i(1-n/q)}. \tag{3.11}$$

for some constant C' depending only on η, n, λ, Λ, d_{E^*}, and $\|f\|_{L^q(B_1(0))}$.

Let us choose $\alpha \in (0,1)$ such that

$$\alpha \leq 1 - n/q, \qquad 1 - \frac{c_0|B_{1/2}(0)|\eta^2}{2} \leq 2^{-2\alpha},$$

and $C_0 := \max\{M/\widetilde{d}_E, 4C'\}$ (recall that M is an upper bound for $|\nabla u|$ inside $B_2(0)$). We prove the result by induction over i.

Since $|\nabla u(x)|_E \leq |\nabla u(x)|/\widetilde{d}_E \leq M/\widetilde{d}_E$ (see (3.3)), we have that $d_0 \leq M/\widetilde{d}_E$, so the statement is true for $i = 0$.

Assuming the result for i, if (3.9) holds, then from (3.10) and the inductive hypothesis we obtain

$$d_{2(i+1)} \leq \left(1 - \frac{c_0|B_{1/2}(0)|\eta^2}{2}\right) d_{2i} \leq 2^{-2\alpha} \cdot C_0 2^{-2i\alpha},$$

while if (3.9) fails then (3.11) gives

$$d_{2(i+1)} \leq C' 2^{-2i(1-n/q))} \leq C' 2^{-2i\alpha} \leq 4C' \cdot 2^{-2(i+1)\alpha} \leq C_0 2^{-2(i+1)\alpha}.$$

This proves the inductive step on $d_{2(i+1)}$, and concludes the proof. □

4. Regularity at nondegenerate points

In the following lemma we prove that in a neighborhood of a nondegenerate point the function u is close to a linear function with a nondegenerate slope. In Proposition 4.3 we prove that this implies $C^{1,\alpha}$ regularity of u at the nondegenerate point. The proof is based on an approximation argument with solutions of a smooth elliptic operator, which is stated in Lemma 4.2 and whose proof is based on the compactness result of Section 2.

We recall that E^* denotes the dual of a convex set E, and $|\cdot|_E$ the norm associated to E (see (3.1) and (1.4)).

Lemma 4.1. *Let $\delta, \eta, \zeta > 0$, and let E be a strictly convex set.*

Let $u : B_1(0) \to \mathbb{R}$ with $u(0) = 0$ and $|\nabla u(x)|_E \leq \zeta + \delta + 1$ for every $x \in B_1(0)$. Let us assume that there exists $e^* \in \partial E^*$ such that

$$\big|\big\{x \in B_1 : (\partial_{e^*}u(x) - (1+\delta))_+ \geq (1-\eta)\zeta\big\}\big| \geq (1-\eta)|B_1(0)|. \tag{4.1}$$

Then for every $\varepsilon > 0$ there exists η depending only on E and n, and constants $A \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that

$$|u(x) - A \cdot x - b| \leq \varepsilon(\zeta + \delta + 1) \qquad \forall\, x \in B_1(0). \tag{4.2}$$

In addition $|A|_E = \zeta + \delta + 1$ and $|b| \leq C(\zeta + \delta + 1)$, where C depends only on E.

Proof. First of all, by standard Sobolev inequalities, there exists a constant C_0 such that for every $u \in W^{1,2n}(B_1(0))$

$$\left|u(x) - \fint_{B_1(0)} u(y)\,dy\right| \leq C_0 \left(\fint_{B_1(0)} |\nabla u(y)|^{2n}\,dy\right)^{1/(2n)} \qquad \forall\, x \in B_1(0). \tag{4.3}$$

Recalling that $\ell : \partial E^* \to \partial E$ denotes the duality map, we apply (4.3) to the function $u(x) - (\zeta + \delta + 1)\ell_{e^*} \cdot x$. Thus, setting m to be the average of $u(x)/(\zeta + \delta + 1)$ inside $B_1(0)$, we obtain

$$\big|u(x) - (\zeta+\delta+1)\ell_{e^*} \cdot x - m(\zeta+\delta+1)\big|^{2n} \leq C_0^{2n} \fint_{B_1(0)} |\nabla u(y) - (\zeta+\delta+1)\ell_{e^*}|^{2n}\,dy \tag{4.4}$$

for every $x \in B_1(0)$. We estimate the integral in (4.4) by splitting it into two sets.

Let $\varepsilon_0 > 0$ be a constant that we choose later. Since by assumption $|\nabla u(x)|_E \leq \zeta + \delta + 1$ for every $x \in B_1(0)$, and in addition

$$\big\{x \in B_1 : \partial_{e^*}u(x) \geq (1-\eta)\zeta + \delta + 1\big\} \subseteq \big\{x \in B_1 : e^* \cdot \nabla u(x) \geq (1-\eta)\big(\zeta + \delta + 1\big)\big\},$$

we apply (3.4) with $e = \nabla u(x)/(\zeta + \delta + 1)$ to deduce that

$$\frac{1}{|B_1(0)|} \int_{\{(\partial_{e^*}u - (1+\delta))_+ \geq (1-\eta)d\}} |\nabla u(y) - (\zeta+\delta+1)\ell_{e^*}|^{2n}\,dy \leq (\zeta+\delta+1)^{2n}\varepsilon_0^{2n},$$

provided $\eta \leq \eta(\varepsilon_0)$.

On the other hand, since the complement has measure less than $\eta|B_1(0)|$, we simply estimate the integrand there with $C_E(\zeta + \delta + 1)^{2n}$, where C_E is a constant depending only on E.

Hence, by choosing first ε_0 so that $C_0^{2n}\varepsilon_0^{2n} \leq \varepsilon^{2n}/2$, and then $\eta \leq \eta(\varepsilon_0)$ sufficiently small so that so that $C_0^{2n}C_E^{2n}\eta \leq \varepsilon^{2n}/2$, from (4.4) we easily obtain (4.2). □

Lemma 4.2. *Let $\delta > 0$, and let $a_{ij} \in C^0(\mathbb{R}^n)$ be bounded coefficients uniformly elliptic in $B_\delta(0)$, namely there exist $\lambda, \Lambda > 0$ such that*

$$\lambda I \leq a_{ij}(v) \leq \Lambda I \qquad \forall\, v \in B_\delta(0).$$

Then, for every $\tau > 0$ there exist $\sigma(\tau) > 0$, $\mu(\tau) > 0$, which depend only on τ and on the modulus of continuity of a_{ij}, such that the following holds: For every $\theta \le \sigma(\tau)$, $f \in C^0(B_1(0))$ such that $\|f\|_{L^n(B_1(0))} \le \mu(\tau)$, and $w \in C^2(B_1(0))$ such that $\|w\|_{L^\infty(B_1(0))} \le 1$ and

$$a_{ij}(\theta\nabla w)\partial_{ij} w = f \qquad \text{in } B_1(0),$$

there exists $v : B_1(0) \to \mathbb{R}$ such that

$$a_{ij}(0)\partial_{ij} v = 0 \qquad \text{in } B_1(0) \tag{4.5}$$

and

$$\|v - w\|_{L^\infty(B_{1/2}(0))} \le \tau.$$

Proof. By contradiction, there exists $\tau > 0$ and sequences $\theta_m \to 0$, $\mu_m \to 0$ and functions $w_m, f_m : B_1(0) \to \mathbb{R}$ such that $\|w_m\|_{L^\infty(B_1(0))} \le 1$, $\|f_m\|_{L^n(B_1(0))} \le \mu_m$,

$$a_{ij}(\theta_m\nabla w_m)\partial_{ij} w_m = f_m \qquad \text{in } B_1(0), \tag{4.6}$$

but for every function $v : B_1(0) \to \mathbb{R}$ satisfying (4.5) we have that

$$\|v - w_m\|_{L^\infty(B_{1/2}(0))} \ge \tau \qquad \forall\, m \in \mathbb{N}. \tag{4.7}$$

We prove that up to subsequence (not relabeled)

$$w_m \to w_\infty \qquad \text{locally uniformly in } B_1(0) \tag{4.8}$$

and that w_∞ satisfies (4.5), which contradicts (4.7).

Consider $\Omega \Subset B_1(0)$, let $d_\Omega = \operatorname{dist}(\Omega, \mathbb{R}^n \setminus B_1(0))$, and for every $m \in \mathbb{N}$ and $x_0 \in \Omega$ we consider the function

$$u_m(x) := \frac{\theta_m}{d_\Omega}\big(w_m(x_0 + d_\Omega x) - w_m(x_0)\big) \qquad \forall\, x \in B_1(0),$$

which solves

$$a_{ij}(\nabla u_m(x))\partial_{ij} u_m(x) = \theta_m d_\Omega f_m(d_\Omega x) \qquad \forall\, x \in B_1(0).$$

We apply Theorem 2.1 to $F(x, z, p, M) = a_{ij}(p)M_{ij}$ (which satisfies all the assumptions) and let $\nu, \varepsilon, \kappa, \rho > 0$ be the constants introduced in that theorem. Thus, if $\delta' > 0$ and $k \in \mathbb{N}$ satisfy

$$\operatorname{osc}_{B_1(0)} u_m \le \delta' \le \rho^{-k}\kappa\delta, \qquad \|\theta_m d_\Omega f_m(d_\Omega x)\|_{L^n(B_1(0))} \le \varepsilon\delta' \tag{4.9}$$

then

$$\operatorname{osc}_{B_{\rho^s}(0)} u_m \le (1-\nu)^s\delta' \qquad \forall\, s = 0, ..., k+1.$$

We want to apply it with $\delta' = \theta_m$. Hence, define k_m to be the biggest positive integer such that $\theta_m \le 2^{-k_m}\kappa\delta$. Since

$$\|f_m\|_{L^n(B_1(0))} \le \varepsilon$$

for m sufficiently large, we get

$$\|\theta_m d_\Omega f_m(d_\Omega x)\|_{L^n(B_1(0))} = \|\theta_m f_m(x)\|_{L^n(B_{d_\Omega}(0))} \le \theta_m\|f_m(x)\|_{L^n(B_1(0))} \le \varepsilon\theta_m.$$

Hence (4.9) is satisfied, and we get

$$\operatorname{osc}_{B_{\rho^s}(0)} u_m \le (1-\nu)^s\theta_m \qquad \forall\, s = 0, ..., k_m + 1$$

which can be rewritten in terms of w_m as

$$\operatorname{osc}_{B_{\rho^s}(0)}(w_m(x_0 + d_\Omega x)) \le (1-\nu)^s d_\Omega \qquad \forall\, s = 0, ..., k_m + 1. \tag{4.10}$$

Let $\alpha := -\log_\rho(1-\nu)$. From (4.10) we obtain that, for every m large enough, w_m is α-Hölder on points at distance at least $\rho^{-k_m} d_\Omega$, namely there exists C independent on m such that for every m large enough

$$|w_m(x) - w_m(y)| \le C|x-y|^\alpha \qquad \forall\, x, y \in \Omega : |x-y| \ge 2^{-k_m} d_\Omega. \tag{4.11}$$

Since $k_m \to \infty$ as $m \to \infty$, it can be easily seen, with the same proof as the one of Ascoli-Arzela theorem, that the family $\{w_m\}_{m\in\mathbb{N}}$ of functions satisfying $\|w_m\|_{L^\infty(B_1(0))} \le 1$ and (4.11) is relatively compact with respect to the uniform convergence in Ω. Letting Ω vary in a countable family of open sets compactly supported in $B_1(0)$ which cover $B_1(0)$, with a diagonal argument we obtain (4.8).

We claim that w_∞ solves (4.5) in the viscosity sense. Indeed, assume by contradiction that w_∞ is not a supersolution of (4.5) in the viscosity sense. Then there exists a function $\varphi \in C^2(B_1(0))$ and a point $x_0 \in B_1(0)$ such that $\varphi(x_0) = w_\infty(x_0)$, $\varphi(x) < w_\infty(x)$ for every $x \in B_1(0) \setminus \{x_0\}$, and $a_{ij}(0)\partial_{ij}\varphi(x_0) > 0$. Since φ is C^2, there exists $r > 0$ such that

$$a_{ij}(0)\partial_{ij}\varphi(x) > 0 \qquad \forall\, x \in B_r(x_0). \tag{4.12}$$

Since φ touches w_∞ strictly at x_0 and $w_m \to w_\infty$ uniformly, for every $m \in \mathbb{N}$ large enough there exist $c_m \in \mathbb{R}$ and $x_m \in \overline{B_r(x_0)}$ such that $c_m + \varphi(x_m) = w_m(x_m)$, and $c_m + \varphi(x) \le w_m(x)$ for every $x \in B_r(x_0)$. In addition, $c_m \to 0$ and $x_m \to x_0$ as $m \to \infty$.

Let $h := \inf_{\partial B_{r/2}(x_0)}(w_\infty - \varphi)/2 > 0$. Since c_m converge to 0 and w_m converge to w_∞, for every m large enough $h \le \inf_{\partial B_{r/2}(x_0)}(w_m + c_m - \varphi)$. Let $(w_m + c_m - \varphi - h)^-$ be the negative part of the function $w_m + c_m - \varphi - h$, and let Γ_m be the convex envelope of $(w_m + c_m - \varphi - h)^-$ in $B_r(x_0)$.

Since the function $w_m + c_m - \varphi - h$ is of class C^2, it is a classical fact that Γ_m is of class $C^{1,1}$ inside $B_r(x_0)$ (see for instance [6]).

For every m let E_m be the contact set between $w_m + c_m - \varphi - h$ and Γ_m in $B_{r/2}(x_0)$, namely

$$E_m := \{x \in B_{r/2}(x_0) : w_m(x) + c_m - \varphi(x) - h = \Gamma_m(x)\}.$$

Recalling (4.12), we see that the function $w_m + c_m - \varphi - h$ solves

$$a_{ij}(\theta_m \nabla w_m)\partial_{ij}(w_m + c_m - \varphi - h) = f_m - a_{ij}(\theta_m\nabla w_m)\partial_{ij}\varphi < f_m - [a_{ij}(\theta_m\nabla w_m) - a_{ij}(0)]\partial_{ij}\varphi \tag{4.13}$$

in $B_r(x_0)$. In addition, since Γ_m is convex, has oscillation h and vanishes on $\partial B_r(x_0)$, it is easy to see that

$$|\nabla \Gamma_m(x)| \le \frac{2h}{r} \qquad \forall x \in B_{r/2}(0). \tag{4.14}$$

Since at the contact points the gradient of $w_m - \varphi$ coincides with the gradient of Γ_m, it follows that, for every $x \in E_m$,

$$a_{ij}(\theta_m\nabla w_m) - a_{ij}(0) = a_{ij}(\theta_m(\nabla\varphi + \nabla\Gamma_m)) - a_{ij}(0).$$

Hence the equation (4.13) is uniformly elliptic at the contact points for m large enough and in addition the term $a_{ij}(\theta_m\nabla w_m) - a_{ij}(0)$ converges uniformly to 0 on E_m as $m \to \infty$.

Hence, applying the Alexandroff-Bakelman-Pucci estimate [4, Theorem 3.2] we obtain

$$\begin{aligned} h - c_m &\le \sup_{B_{r/2}(x_0)} (w_m + c_m - \varphi - h)^- \le Cr \left\|(f_m + (a_{ij}(\theta_m\nabla w_m) - a_{ij}(0))\partial_{ij}\varphi)^+\right\|_{L^n(E_m)} \\ &\le Cr\left(\|f_m\|_{L^n(B_1(0))} + \|a_{ij}(\theta_m\nabla w_m) - a_{ij}(0)\|_{L^n(E_m)}\|\varphi\|_{C^2(B_1(0))}\right), \end{aligned} \tag{4.15}$$

where $C > 0$ depends only on n, λ and Λ, and letting $m \to \infty$ we get

$$h \le Cr \liminf_{m\to+\infty}\left[\|f_m\|_{L^n(B_1(0))} + \|a_{ij}(\theta_m\nabla w_m) - a_{ij}(0)\|_{L^n(E_m)}\|\varphi\|_{C^2(B_1(0))}\right] = 0,$$

a contradiction. A symmetric argument proves also that w_∞ is a subsolution of (4.5).

Therefore w_∞ solves (4.5) in the viscosity sense, and being (4.5) a uniformly elliptic equation with constant coefficients, w_∞ is actually a classical solution. This fact and (4.8) contradict (4.7). □

We prove an improvement of flatness result when the gradient is nondegenerate. In the following proposition the assumption $f \in L^q(B_1(0))$ for some $q > n$ plays a crucial role, and this is the optimal assumption one can make. Indeed, even for the Laplace equation $\Delta u = f$, the $C^{1,\alpha}$ regularity of the solution u is false for $f \in L^n$ (since $W^{2,n}$ does not embed into $C^{1,\alpha}$).

Proposition 4.3. *Let δ, $\mathcal{F}$, E, λ, Λ, f, u, and M be as in Theorem 1.4. There exist $\delta_0, \mu_0 > 0$, depending only on the modulus of continuity of $\nabla^2\mathcal{F}$, and on δ, λ, and Λ, such that the following holds:*

If $\|f\|_{L^q(B_2(0))} \le \delta_0\mu_0$ and for any $x \in B_{1/2}(0)$ there exist $A_x \in \mathbb{R}^n$ and $b_x \in \mathbb{R}$ such that $1+\delta \le |A_x|_E \le M$ and $|u(y) - A_x \cdot y - b_x| \le \delta_0$ for every $y \in B_1(0)$, then

$$|u(y) - u(x) - A\cdot(y-x)| \le C|y-x|^{1+\alpha} \qquad \forall\, y \in B_1(0) \tag{4.16}$$

with $\alpha := 1 - n/q$, C depends only on δ, n, λ, and Λ, and $A \in \mathbb{R}^n$ satisfies

$$|A - A_0| \le \frac{\widetilde{d}_E}{4}\delta. \tag{4.17}$$

In particular $u \in C^{1,\alpha}(B_{1/4}(0))$ (with bounds depending only on the modulus of continuity of $\nabla^2\mathcal{F}$, on δ, n, λ, and Λ), and $|\nabla u|_E \ge 1 + \delta/2$ inside $B_{1/4}(0)$.

Proof. We prove (4.16) for $x = 0$. Up to a vertical translation, we can assume without loss of generality that $u(0) = 0$. It suffices to show that there exists $r \in (0,1)$ such that, for every $k \in \mathbb{N}\cup\{0\}$, there is a linear function $L_k(y) = A_k \cdot y + b_k$ satisfying

$$|u(y) - L_k(y)| \le \delta_0 r^{k(\alpha+1)} \qquad \forall\, y \in B_{r^k}(0),$$

$$|A_k - A_{k+1}| \le C'\delta_0 r^{k\alpha} \qquad |b_k - b_{k+1}| \le C'\delta_0 r^{k(\alpha+1)}. \tag{4.18}$$

For $k = 0$ the result is true by assumption.

Now we prove the result for $k+1$ assuming it for $0, \dots, k$. Let us consider the rescaled function

$$w(y) := \frac{u(r^k y) - L_k(r^k y)}{\delta_0 r^{k(\alpha+1)}} \qquad \forall\, y \in B_1(0). \tag{4.19}$$

Observe that, by the inductive hypothesis, $|w| \le 1$ inside $B_1(0)$ and w solves the equation

$$\partial_{ij}\mathcal{F}(A_k + \delta_0 r^{k\alpha}\nabla w(y))\partial_{ij} w(y) = \frac{r^{k(1-\alpha)}}{\delta_0} f(r^k y) \qquad \text{in } B_1(0).$$

Recalling that $\alpha = 1 - n/q$, by a change of variable and Hölder inequality we get

$$\|r^k f(r^k y)\|_{L^n(B_1(0))} = \|f\|_{L^n(B_{r^k}(0))} \le |B_1(0)|^{1/q} r^{k\alpha}\|f\|_{L^q(B_{r^k}(0))} \le |B_1(0)|^{1/q} r^{k\alpha}\|f\|_{L^q(B_2(0))}. \tag{4.20}$$

Since $\|f\|_{L^q(B_2(0))} \le \delta_0\mu_0$, we get

$$\frac{r^{k(1-\alpha)}}{\delta_0}\|f(r^k y)\|_{L^n(B_1(0))} \le |B_1(0)|^{1/q}\mu_0. \tag{4.21}$$

Recalling (3.3) and (4.18), by the inductive assumption we get

$$\widetilde{d}_E \sum_{i=0}^{k-1} |A_i - A_{i+1}|_E \le \sum_{i=0}^{k-1} |A_i - A_{i+1}| \le C'\delta_0 \sum_{i=0}^{k-1} r^{i\alpha} \le C'\delta_0 \sum_{i=0}^{\infty} r^{i\alpha} \le \frac{\widetilde{d}_E}{4}\delta,$$

provided we choose δ_0 small enough. Hence $A_k \notin E$, and more precisely

$$1 + \frac{3}{4}\delta \le |A_0|_E - \sum_{i=0}^{k-1} |A_i - A_{i+1}|_E \le |A_k|_E \le |A_0|_E + \sum_{i=0}^{k-1} |A_i - A_{i+1}|_E \le M + \frac{\delta}{4}. \tag{4.22}$$

Define $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ as $a_{ij}(v) := \partial_{ij}\mathcal{F}(A_k + v)$. Then by (3.3) and (4.22) we have

$$B_{\widetilde{d}_E\delta/4}(A_k) \subseteq \left\{ |y - A_k|_E \le \frac{\delta}{4} \right\} \subseteq \left\{ |y|_E \ge 1 + \frac{\delta}{2} \right\},$$

so by assumption (1.9) on $\mathcal{F}$ we get

$$\lambda I \le \nabla^2 \mathcal{F}(v) \le \Lambda I \qquad \text{for any } v \in B_{\widetilde{d}_E\delta/4}(A_k),$$

which implies that the coefficients a_{ij} are uniformly elliptic inside $B_{\widetilde{d}_E\delta/4}(0)$ with constants λ, Λ.

Let σ and μ be the functions provided by Lemma 4.2. If δ_0 is small enough so that $\delta_0 r^{k\alpha} \le \sigma(r^{1+\alpha}/2)$, and μ_0 is small enough so that $|B_1(0)|^{1/q}\mu_0 \le \mu(r^{1+\alpha}/2)$, Lemma 4.2 applied to w implies the existence of a function $v : B_1(0) \to \mathbb{R}$ such that

$$\partial_{ij}\mathcal{F}(A_k)\partial_{ij} v = 0 \qquad \text{in } B_1(0)$$

and

$$|v(y) - w(y)| \le \frac{r^{1+\alpha}}{2} \qquad \forall\, y \in B_{1/2}(0). \tag{4.23}$$

In particular, since $|v(y)| \le |v(y) - w(y)| + |w(y)| \le 3/2$ in $B_{1/2}(0)$, and v solves a uniformly elliptic equation with constant coefficients, there exist $C' > 0$ (depending only on n, λ, Λ) and a linear function $L(y) = A \cdot y + b$, such that

$$|v(y) - A \cdot y - b| \le C'|y|^2 \qquad \forall\, y \in B_{1/4}(0).$$

In particular, if $C' r^{1-\alpha} \le 1/2$ and $r \le 1/4$, we get

$$|v(y) - A \cdot y - b| \le C' r^2 \le \frac{r^{1+\alpha}}{2} \qquad \forall\, y \in B_r(0). \tag{4.24}$$

Hence, first we choose $0 < r < 1/4$ such that

$$C' r^{1-\alpha} \le \frac{1}{2},$$

then fix δ_0 such that

$$\delta_0 r^{k\alpha} \le \sigma(r^{1+\alpha}/2) \qquad \text{and} \qquad C'\delta_0 \sum_{i=0}^{\infty} r^{i\alpha} \le \frac{\widetilde{d}_E}{4}\delta,$$

and finally take μ_0 such that

$$|B_1(0)|^{1/q}\mu_0 \le \mu(r^{1+\alpha}/2).$$

Then from (4.23) and (4.24) we get

$$|w(y) - A \cdot y - B| \le |w(y) - v(y)| + |v(y) - A \cdot y - B| \le r^{1+\alpha} \qquad \forall\, y \in B_r(0),$$

which can be rewritten in terms of u as (see (4.19))

$$|u(y) - L_{k+1}(y)| \le \delta_0 r^{(k+1)(\alpha+1)} \qquad \forall\, y \in B_{r^{k+1}}(0),$$

where

$$L_{k+1}(y) := L_k(y) - \delta_0 r^{k(\alpha+1)} L\left(\frac{y}{r^k}\right).$$

It is easy to check that (4.18) holds for some C' large enough independent of δ_0 and r, and this concludes the proof of the inductive step.

Also, it follows from (4.18) and the definition of δ_0 that

$$|A_k - A_0| \le \sum_{i=0}^{k-1} |A_i - A_{i+1}| \le \frac{\widetilde{d}_E}{4}\delta, \tag{4.25}$$

which proves (4.17) in the limit.

Finally, the fact that (4.16) implies that $u \in C^{1,\alpha}(B_{1/4}(0))$ is standard (see for instance [6, Lemma 3.1]). □

5. Proof of Theorems 1.4 and 1.1

Proof of Theorem 1.4. For any $x_0 \in B_1(0)$ and $r \in (0,1)$, we have

$$\int_{B_1(0)} |rf(x_0+rx)|^p\,dx = \int_{B_r(x_0)} r^{p-n}|f(x)|^p\,dx \le r^{p-n}\|f\|^p_{L^p(B_2(0))}.$$

Let μ_0 and δ_0 be as in Proposition 4.3. Fix $r < 1/2$ small enough such that $r^{1-n/p}\|f\|_{L^p(B_1(0))} \le \delta_0\mu_0$, so that

$$\|rf(x_0+rx)\|_{L^p(B_1(0))} \le \delta_0\mu_0. \tag{5.1}$$

Consider now the function $w : B_1(0) \to \mathbb{R}$ given by

$$w(x) := \frac{1}{r}u(x_0+rx) \qquad \forall\, x \in B_1(0),$$

which by (1.10) solves

$$\partial_i[\partial_{ij}\mathcal{F}(\nabla w(x))\partial_j w(x))] = rf(x_0+rx). \tag{5.2}$$

Our goal is to show that the quantity

$$\sup_{x\in B_{2^{-i}}(0)} \big\{|(|\nabla w(x)|_E - 1-\delta)_+ - (|\nabla w(0)|_E - 1 - \delta)_+|\big\} \qquad \forall\, i \in \mathbb{N}, \tag{5.3}$$

decays geometrically.

For every $i \in \mathbb{N}$ set

$$d_i := \sup_{x\in B_{2^{-i}}(0)} (|\nabla w(x)|_E - (1+\delta))_+,$$

and let k be the smallest value of $i \in \mathbb{N}$ such that

$$\sup_{e^*\in\partial E^*} \big|\big\{x \in B_{2^{-2i-1}(0)} : (\partial_{e^*}w(x) - (1+\delta))_+ \ge (1-\eta)d_{2i}\big\}\big| \ge (1-\eta)|B_{2^{-2i-1}}(0)| \tag{5.4}$$

($k = \infty$ if there is no such i). By Lemma 3.1 there exists a constant $C_0 > 0$ and $\alpha_0 \in (0,1)$ such that

$$d_{2i} \le C_0 2^{-2i\alpha_0} \qquad \forall\, i = 0, \dots, k. \tag{5.5}$$

If $k = \infty$, then there is nothing to prove. Assume then that k is finite.

For every $k+1 \le i \le 2k$ we estimate d_{2i} with d_{2k}, and from (5.5) applied to d_{2k} we obtain

$$d_{2i} \le d_{2k} \le C_0 2^{-2k\alpha_0} \le C_0 2^{-i\alpha_0}. \tag{5.6}$$

We now scale the function w in order to preserve its gradient:

$$v(x) := 2^{2k+1}(w(2^{-2k-1}x) - w(0)) \qquad \forall\, x \in B_1(0).$$

Since $\nabla v(x) = \nabla w(2^{-2k-1}x)$, from (5.4) we obtain that there exists $e^* \in \partial E^*$ such that

$$\left|\left\{x \in B_1(0) : (\partial_{e^*} v(x) - (1+\delta))_+ \geq (1-\eta) d_{2k}\right\}\right| \geq (1-\eta)|B_1(0)|. \tag{5.7}$$

Moreover, we have that $|\nabla v(x)|_E \leq d_{2k} + \delta + 1 \leq M/\widetilde{d}_E$ for every $x \in B_1(0)$ (recall (3.3)). Hence, from Lemma 4.1 applied to v with $\varepsilon = \delta_0 \widetilde{d}_E/M$ (with δ_0 as in Proposition 4.3) and $\zeta = d_{2k}$, there exist $A \in \mathbb{R}^n$ with $|A|_E = d_{2k+1} + \delta + 1$ and $b \in \mathbb{R}$ such that

$$|v(x) - A \cdot x - b| \leq \varepsilon(d_{2k+1} + \delta + 1) \leq \varepsilon M/\widetilde{d}_E = \delta_0 \qquad \forall\, x \in B_1(0). \tag{5.8}$$

From (5.2), (5.1), and (5.8), the hypothesis of Proposition 4.3 are satisfied, so there exists a constant C_1, depending only on δ, n, λ, and Λ, such that

$$|\nabla v(x) - \nabla v(0)| \leq C_1 |x|^{\alpha_1} \qquad \forall\, x \in B_{1/4}(0),$$

where $\alpha_1 := 1 - n/q$. Since the function $x \to (|x| - 1 - \delta)_+$ is 1-Lipschitz, we get

$$|(|\nabla w(x)| - 1 - \delta)_+ - (|\nabla w(0)| - 1 - \delta)_+| \leq |\nabla w(x) - \nabla w(0)| = |\nabla v(2^{2k+1}x) - \nabla v(0)|,$$

for every $x \in B_{2^{-2k-2}}(0)$. In particular, for any $i \geq 2k+1$ and $x \in B_{2^{-2i}}(0)$ we have

$$|(|\nabla w(x)| - 1 - \delta)_+ - (|\nabla w(0)| - 1 - \delta)_+| \leq C_1 2^{(2k+1)\alpha_1} |x|^{\alpha_1} \leq C_1 2^{(2k+1-2i)\alpha_1} \leq C_1 2^{-i\alpha_1}. \tag{5.9}$$

Setting $\bar{C} := 2\max\{C_0, C_1\}$ and $\bar{\alpha} := \min\{\alpha_0, \alpha_1\}/2$, from (5.5), (5.6), and (5.9), we obtain that for every $i \in \mathbb{N}$

$$\sup_{x \in B_{2^{-2i}(0)}} \left\{|(|\nabla w(x)|_E - 1 - \delta)_+ - (|\nabla w(0)|_E - 1 - \delta)_+|\right\} \leq \bar{C} 2^{-2i\bar{\alpha}},$$

namely

$$\sup_{x \in B_{2^{-2i}r}(x_0)} \left\{|(|\nabla u(x)|_E - 1 - \delta)_+ - (|\nabla u(x_0)|_E - 1 - \delta)_+|\right\} \leq \bar{C} 2^{-2i\bar{\alpha}},$$

from which (1.11) follows easily. □

Proof of Theorem 1.1. Let $\Omega' \Subset \Omega'' \Subset \Omega''' \Subset \Omega$ and set $M := \|\nabla u\|_{L^\infty(\Omega''')}$ (M is finite because u is locally Lipschitz inside Ω). Recall that $\mathcal{F}$ is C^2 outside $\overline{E}$, so in particular it is C^2 for $|v| > d_E$ (recall (3.3)).

We now want to find a functional $\mathcal{G} \in C^2(\mathbb{R}^n \setminus \overline{E})$ which coincides with $\mathcal{F}$ inside $B_M(0)$ (so that $\mathcal{F}(\nabla u) = \mathcal{G}(\nabla u)$ inside Ω''') but $\mathcal{G}$ is quadratic at infinity. We follow a construction used in [1].

Let $M' = \sup\{\mathcal{F}(v) : v \in B_{M+2d_E}(0)\}$. Let $\psi : [0,\infty) \to \mathbb{R}$ be a C^∞ function such that $\psi(t) = t$ in $[0, M'+1]$, and $\psi(t) = M'+2$ in $[M'+2, \infty]$. Since $\mathcal{F}$ is coercive, the function $\psi(\mathcal{F}(v))$ is constant outside a some ball. Hence

$$N := \sup_{|v| > M + d_E} |\nabla^2[\psi \circ \mathcal{F}](v)|$$

is finite. Let $\phi \in C^2(\mathbb{R}^n)$ be a convex function such that $\phi(x) = 0$ for every $x \in B_{M+d_E}(0)$, $\nabla^2\phi(x) \leq (2N+1)\,\mathrm{Id}$ for every $x \in \mathbb{R}^n$ and $\nabla^2\phi(x) \geq (N+1)\,\mathrm{Id}$ for every $x \in \mathbb{R}^n \setminus B_{M+2d_E}(0)$. Define

$$\mathcal{G}(v) := \psi(\mathcal{F}(v)) + \phi(v) \qquad \forall v \in \mathbb{R}^n. \tag{5.10}$$

Computing the Hessian of $\mathcal{G}$, we obtain that $\mathcal{G}$ is convex, that $\nabla^2\mathcal{G}(v) \leq (3N+1)\,\mathrm{Id}$ for every $|v| > M + d_E$ and that $\mathrm{Id} \leq \nabla^2\mathcal{G}(v)$ for every $|v| > M + 2d_E$. Since $\mathcal{G} = \mathcal{F}$ inside $B_{2d_E+M}(0)$ and u solves the Euler-Lagrange equation (1.10) in the sense of distributions, u solves also the Euler-Lagrange equation for $\mathcal{G}$, and so by convexity of $\mathcal{G}$ it is a minimizer for the functional

$$\int_{\Omega'''} \mathcal{G}(\nabla u) + fu.$$

By (1.5) we have that for every $\delta > 0$ small there exist $\lambda'_\delta, \Lambda'_\delta > 0$, depending only on $\lambda_{\delta/4}, \Lambda_{\delta/4}, N$, such that

$$\lambda'_\delta \,\mathrm{Id} \le \nabla^2 \mathcal{G}(v) \le \Lambda'_\delta \,\mathrm{Id} \qquad \text{for a.e. } v \text{ such that } 1 + \frac{\delta}{4} \le |v|_E. \tag{5.11}$$

Let ρ_ε be a standard mollification kernel whose support is contained in $B_\varepsilon(0)$ and let

$$\mathcal{G}_\varepsilon(x) := \rho_\varepsilon * \mathcal{G}(x) + \varepsilon|x|^2, \qquad f_\varepsilon(x) := \rho_\varepsilon * f(x),$$

$$u_\varepsilon := \operatorname{argmin}\left\{ \int_{\Omega'''} \mathcal{G}_\varepsilon(\nabla u) + f_\varepsilon u : u \in W^{1,2}(\Omega''') \right\}.$$

Note that $u_\varepsilon \in C^\infty(\Omega''')$ thanks to the regularity of $\mathcal{G}_\varepsilon$ and f_ε, and thanks to the uniform convexity of $\mathcal{G}_\varepsilon$. From (5.11), for every δ small there exist $\lambda''_\delta, \Lambda''_\delta > 0$, depending only on $\lambda'_\delta, \Lambda'_\delta, N$, such that, for $\varepsilon \le \delta/(4\widetilde{d}_E)$,

$$\lambda''_\delta \,\mathrm{Id} \le \nabla^2 \mathcal{G}_\varepsilon(v) \le \Lambda''_\delta \,\mathrm{Id} \qquad \text{for a.e. } v \text{ such that } 1 + \frac{\delta}{2} \le |v|_E. \tag{5.12}$$

Differentiating the Euler equation solved by u_ε with respect to ∂_e for any $e \in S^{n-1}$ we obtain that

$$\partial_i[\partial_{ij}\mathcal{G}_\varepsilon(\nabla u_\varepsilon(x))\partial_j(\partial_e u_\varepsilon(x))] = \partial_e f_\varepsilon(x). \tag{5.13}$$

Hence the function $v_\varepsilon(x) := (|\nabla u_\varepsilon(x)| - (1 + d_E))_+$ is a subsolution of the equation

$$\partial_i[\partial_{ij}\mathcal{G}_\varepsilon(\nabla u_\varepsilon(x))\partial_j v] \le \partial_e f.$$

As we already observed in the proof of Lemma 3.1, this equation is uniformly elliptic because the values of $\partial_{ij}\mathcal{G}_\varepsilon(\nabla u_\varepsilon(x))$ are not important when $|\nabla u_\varepsilon(x)| \le 1 + d_E$. Hence, we can apply [12, Theorem 8.17] to obtain

$$\|(|\nabla u_\varepsilon(x)| - (1+d_E))_+\|_{L^\infty(\Omega'')} \le C'(1 + \|(|\nabla u_\varepsilon(x)| - (1+d_E))_+\|_{L^2(\Omega''')}) \le C'(1 + \|\nabla u_\varepsilon(x)\|_{L^2(\Omega''')}) \tag{5.14}$$

for some constant C' depending only on $n, \lambda_{\delta_0}, \Lambda_{\delta_0}, \Omega'', \Omega'''$ (for some δ_0 small).

Since the function G_ε has quadratic growth at infinity, we get

$$\|\nabla u_\varepsilon(x)\|_{L^2(\Omega'')} \le C\biggl(1 + \int_{\Omega'''} G_\varepsilon(\nabla u_\varepsilon(x))\,dx\biggr). \tag{5.15}$$

From the boundedness of energies of u_ε, (5.14), and (5.15), it follows that the functions u_ε are M'-Lipschitz for ε small.

Let E_δ be a strictly convex set such that $E \subseteq E_\delta \subseteq (1 + \delta/2)E$. Since

$$\left\{|v|_{E_\delta} > 1 + \frac{\delta}{2}\right\} = \left\{v \notin \left(1 + \frac{\delta}{2}\right) E_\delta\right\} \subseteq \left\{v \notin \left(1 + \frac{\delta}{2}\right) E\right\},$$

from (5.12) it follows that $\lambda''_\delta I \le \nabla^2 \mathcal{G}_\varepsilon(x) \le \Lambda''_\delta$ for a.e. x such that $1 + \frac{\delta}{2} \le |x|_{E_\delta}$. Applying Theorem 1.4 to u_ε and E_δ, by a covering argument we deduce that there exists a constant D_δ (independent of ε) such that

$$|(|\nabla u_\varepsilon(x)|_{E_\delta} - 1 - \delta)_+ - (|\nabla u_\varepsilon(y)|_{E_\delta} - 1 - \delta)_+| \le D_\delta |x - y|^\alpha \qquad \forall\, x, y \in \Omega'. \tag{5.16}$$

Without loss of generality, up to adding a constant to u_ε we can assume that $u_\varepsilon(0) = 0$. Hence, since $|\nabla u_\varepsilon| \le M$, we obtain that, up to adding a constant a subsequence,

$$u_\varepsilon \to u_0 \qquad \text{uniformly in } \Omega'$$

and

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 \qquad \text{weakly}^* \text{ in } L^\infty(\Omega') \tag{5.17}$$

for some Lipschitz function u_0. We claim that $\nabla u_0 = \nabla u$ outside E and that

$$(|\nabla u_\varepsilon(x)|_{E_\delta} - 1 - \delta)_+ \to (|\nabla u(x)|_{E_\delta} - 1 - \delta)_+ \qquad \text{strongly in } L^p(\Omega') \tag{5.18}$$

for every $p < \infty$.

Indeed, from the convergence of the energies on a sequence of local minimizers, and thanks to the uniform convergence of $\mathcal{G}_\varepsilon$ to $\mathcal{G}$ on $B_{M'}(0)$, we have that

$$\int_{\Omega'} \mathcal{G}(\nabla u(x))\,dx = \lim_{\varepsilon\to 0}\int_{\Omega'} \mathcal{G}_\varepsilon(\nabla u_\varepsilon(x))\,dx = \lim_{\varepsilon\to 0}\int_{\Omega'} \mathcal{G}(\nabla u_\varepsilon(x))\,dx = \int_{\Omega'} \mathcal{G}(\nabla u_0(x))\,dx, \tag{5.19}$$

Since $\mathcal{G}$ is strictly convex outside E, it follows by standard results in the calculus of variations that $\nabla u_0 = \nabla u$ outside E and (5.18) holds (a possible way to show these facts, is to consider the Young measure ν_x generated by ∇u_ε, and show that $\nu_x = \delta_{\nabla u(x)}$ for a.e. x such that $\nabla u(x) \notin E$).

Hence, thanks to (5.18), we can take the limit as $\varepsilon \to 0$ in (5.16) to obtain $(|\nabla u|_{E_\delta} - 1 - \delta)_+ \in C^{0,\alpha}(\Omega')$. In particular, the set

$$A_\delta := \big\{x \in \Omega' : |\nabla u(x)|_{E_\delta} > 1 + \delta\big\}$$

is open. Moreover, from the choice of E_δ, it follows easily that

$$F_\delta := \big\{x \in \Omega' : |\nabla u(x)|_{E} > 1 + 2\delta\big\} \subset A_\delta \tag{5.20}$$

Since every partial derivative of u solves (5.13) (with $\varepsilon = 0$) which is uniformly elliptic inside A_δ, from De Giorgi regularity theorem it follows that $\nabla u \in C^{0,\alpha'}(F_\delta)$, with $C^{0,\alpha'}$ norm bounded by a constant which depends only on α, M, δ, λ_δ, Λ_δ, A_δ, and f. By the arbitrariness of δ, we deduce that ∇u is continuous inside the open set $\{|\nabla u|_E > 1\}$ with a universal modulus of continuity.

We also note that, since the functions $(|v|_{E_\delta} - 1 - \delta)_+$ converge uniformly to $(|v|_E - 1)_+$ on $B_{M'}(0)$, we get that $(|\nabla u|_{E_\delta} - 1 - \delta)_+$ converge uniformly to $(|\nabla u|_E - 1)_+$, so also $(|\nabla u|_E - 1)_+$ is continuous with a universal modulus of continuity.

Combining this fact with the continuity of ∇u inside $\{|\nabla u|_E > 1\}$ and the fact that $\mathcal{H}$ is continuous and vanishes on E, it is easy to check that $\mathcal{H}(\nabla u)$ is continuous (again with a universal modulus of continuity) everywhere inside Ω'.

□

References

[1] A.Abbondandolo, A.Figalli: *High action orbits for Tonelli Lagrangians and superlinear Hamiltonians on compact configuration spaces.* J. Differential Equations **234** (2007), no. 2, 626–653.

[2] L.Brasco: *Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation.* Nonlinear Anal. **74** (2011), 516–531.

[3] L.Brasco, G.Carlier, F.Santambrogio: *Congested traffic dynamics, weak flows and very degenerate elliptic equations.* J. Math. Pures Appl. (9) **93** (2010), no. 6, 652–671.

[4] L.Caffarelli, X.Cabré: *Fully nonlinear elliptic equations.* American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[5] G.Carlier, C.Jimenez, F.Santambrogio: *Optimal transportation with traffic congestion and Wardrop equilibria.* SIAM J. Control Optim. **47** (2008), 1330–1350.

[6] G.De Philippis, A.Figalli: *Optimal regularity of the convex envelope.* Preprint (2012).

[7] E.DiBenedetto: *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal. **7** (1983), 827–850.

[8] E.DiBenedetto, V.Vespri: *On the singular equation $\beta(u)_t = \Delta u$.* Arch. Rational Mech. Anal. **132** (1995), 247–309.

[9] L.Esposito, G.Mingione, C.Trombetti: *On the Lipschitz regularity for certain elliptic problems.* Forum Math. **18** (2006), 263–292.

[10] L.C.Evans: *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.* J. Differential Equations **45** (1982), 356–373.

[11] I.Fonseca, N.Fusco, P.Marcellini: *An existence result for a nonconvex variational problem via regularity.* ESAIM Control Optim. Calc. Var. **7** (2002), 69–95.

[12] D.Gilbarg, N.S.Trudinger: *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin (2001).

[13] J.L.Lewis: *Regularity of the derivatives of solutions to certain degenerate elliptic equations.* Indiana Univ. Math. J. **32** (1983), 849–858.

[14] F.Santambrogio, V.Vespri: *Continuity in two dimensions for a very degenerate elliptic equation.* Nonlinear Anal. **73** (2010), 3832–3841.

[15] O.Savin: *Small perturbation solutions for elliptic equations.* Comm. Partial Differential Equations **32** (2007), 557–578.

[16] E.M.Stein: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[17] P.Tolksdorff: *Regularity for a more general class of quasilinear elliptic equations.* J. Differential Equations **51** (1984), 126–150.

[18] K.Uhlenbeck:*Regularity for a class of non-linear elliptic systems.* Acta Math. **138**(1977), 219–240.

[19] N.N.Uraltseva: *Degenerate quasilinear elliptic systems.* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 184–222.

[20] L.Wang: *Compactness methods for certain degenerate elliptic equations.* J. Differential Equations **107** (1994), 341–350.

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