# THE 1-HARMONIC FLOW WITH VALUES INTO A HYPER-OCTANT OF THE $N$-SPHERE

L. GIACOMELLI, J. M. MAZÓN AND S. MOLL

ABSTRACT. We prove the existence of solutions to the 1-harmonic flow –i.e., the formal gradient flow of the total variation of a vector field with respect to the $L^2$-distance– from a domain of $\mathbb{R}^m$ into a hyper-octant of the $N$-dimensional unit sphere, $\mathbb{S}_+^{N-1}$, under homogeneous Neumann boundary conditions. In particular, we characterize the lower-order term appearing in the Euler-Lagrange formulation in terms of the "geodesic representative" of a BV-director field on its jump set. Such characterization relies on a lower semi-continuity argument which leads to a nontrivial and non-convex minimization problem: to find a shortest path between two points on $\mathbb{S}_+^{N-1}$ with respect to a metric which penalizes the closeness to their geodesic midpoint.

## CONTENTS



*Key words and phrases.* Harmonic flows, total variation flow, nonlinear parabolic systems, lower semi-continuity and relaxation, non-convex variational problems, geodesics, Riemannian manifolds with boundary, image processing.

2010 *Mathematics Subject Classification.* 35K92, 35K55, 35K67, 49J45, 49Q20, 53C22, 53C44, 58E20, 68U10.

## 1. Introduction

Throughout the paper, $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$. For a smooth map $\mathbf{u} : \Omega \to \mathbb{S}^{N-1}$ and $1 \leq p < \infty$, the $p$-energy of $\mathbf{u}$ is given by

$$E_p(\mathbf{u}) = \int_\Omega |D\mathbf{u}|^p \, \mathrm{d}x.$$

A critical point $\mathbf{u} \in C^1(\Omega; \mathbb{S}^{N-1})$ of the $p$-energy, a *$p$-harmonic map*, formally satisfies the Euler-Lagrange equation

$$-\operatorname{div}\left(|D\mathbf{u}|^{p-2} D\mathbf{u}\right) = |D\mathbf{u}|^p \mathbf{u}. \tag{1.1}$$

The term $|D\mathbf{u}|^p$ plays the role of a Lagrange multiplier corresponding to the pointwise constraint $|\mathbf{u}| = 1$.

One well-known method to obtain (distributional) solutions to (1.1), the so-called *heat-flow method* introduced by J. Eells and J.H. Sampson in [22] for $p = 2$ in the general framework of Riemannian manifolds, consists in looking at long time limits of solutions to

$$\mathbf{u}_t = \operatorname{div}(|D\mathbf{u}|^{p-2} D\mathbf{u}) + \mathbf{u}|D\mathbf{u}|^p, \quad |\mathbf{u}| = 1. \tag{1.2}$$

Equation (1.2) is also a prototype for often quite complicated reaction-diffusion systems for the evolution of director fields which arise in various contexts –multi-grain problems [39], theory of liquid crystals [37], ferromagnetism [20], and image processing [41]. For $p > 1$, equation (1.2) with various boundary conditions has been widely studied over the last decades; referenced discussions of the cases $p = 2$ and $p \in (1, \infty)$ may be found e.g. in [10, 11, 15, 43] and [17, 38, 40], respectively.

Here we are interested in the case $p = 1$, for which (1.2) formally reads as

$$\mathbf{u}_t = \operatorname{div}\left(\frac{D\mathbf{u}}{|D\mathbf{u}|}\right) + \mathbf{u}|D\mathbf{u}|, \quad \mathbf{u} \in \mathbb{S}^{N-1}. \tag{1.3}$$

More precisely, we focus on the homogeneous Neumann problem for (1.3) when the target space is a compact subset $\mathbb{A}$ of $\mathbb{S}^{N-1}$; i.e.,

$$\begin{cases} \mathbf{u}_t = \operatorname{div}\left(\dfrac{D\mathbf{u}}{|D\mathbf{u}|}\right) + \mathbf{u}|D\mathbf{u}|, \quad \mathbf{u} \in \mathbb{A} \subseteq \mathbb{S}^{N-1} & \text{in } Q_T = (0,T)\times\Omega \\ \dfrac{D\mathbf{u}}{|D\mathbf{u}|}\nu = 0 & \text{on } S_T = (0,T)\times\partial\Omega \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot), \quad \mathbf{u}_0 \in \mathbb{A} & \text{in } \Omega\,, \end{cases} \tag{1.4}$$

(here $\nu$ denotes the outward unit normal to $\partial\Omega$). Problem (1.4) was proposed as a tool to denoise either two-dimensional image gradients and optical flows, in which case $N = 2$ and $\mathbb{A} = \mathbb{S}^1$ [44], or color images by smoothing the chromaticity data while preserving the contrast, in which case $N = 3$ and $\mathbb{A}$ is an octant of the sphere [45].

While the scalar and unconstrained version of (1.3), i.e. the so-called total variation flow, is by now well understood since the pioneering paper [6] (see the monograph [7] and the references therein or [12] for an up–to–date reference list), an existence theory for (1.3) is still open in general. Special cases considered so far dealt with piece-wise constant data [33, 35, 36], initial data with "small" energy [32], and rotationally symmetric solutions [34, 18, 30]. We refer to [28] for a detailed discussion of previous attempts to obtain a solution to (1.4) given in the earlier works [9, 25].

In dealing with (1.3), the most delicate issue is of course the interpretation of the bounded matrix $\mathbf{Z}$ which represents $\frac{D\mathbf{u}}{|D\mathbf{u}|}$ and of the measure $\boldsymbol{\mu}$ which represents $\mathbf{u}|D\mathbf{u}|$, the latter being the product between a measure and a possibly discontinuous function. Very recently, an interpretation of (1.3) has been proposed by the authors in [28]: in summary,

$$\mathbf{u}_t(t) - \operatorname{div}\mathbf{Z}(t) \in \mathbf{u}_g\,|D\mathbf{u}|(t),\ \mathbf{u}(t) \in \mathbb{A} \quad \text{for a.e. } t \in [0,T] \tag{1.5}$$

in the sense of distributions, where $\mathbf{Z}(t)$ is a bounded matrix that represents $\frac{D\mathbf{u}(t)}{|D\mathbf{u}(t)|}$ (the precise meaning is given in Proposition 3.5) and $\mathbf{u}_g|D\mathbf{u}|(t)$ denotes a set of vector-valued measures which are oriented as $\mathbf{u}(t)^*$ (the precise representative of $\mathbf{u}(t)$) and have total variation density $|D\mathbf{u}(t)|$. For $N = 2$, this interpretation has led to the existence and uniqueness of a solution to (1.4) when $\mathbb{A}$ is a semicircle [28, Theorems 4.1 and 5.1] together with the existence of a solution when $\mathbb{A} = \mathbb{S}^1$ and $\mathbf{u}_0 \in BV(\Omega;\mathbb{S}^1)$ has no jumps of an "angle" larger than $\pi$.

The aim of this paper is to prove an existence result, according to the same interpretation, for an arbitrary dimension of the target sphere. We consider (1.4) in the first hyper-octant of the $N$-sphere:

$$\mathbb{A} = \mathbb{S}^{N-1}_+ := \{(x_1, \ldots, x_N) \in \mathbb{S}^{N-1} \ : \ x_i \geq 0 \text{ for } \ i = 1, \ldots, N\}$$

(a natural assumption in the context of image processing, see above). Note that in this case, for every pair $\mathbf{u}_-, \mathbf{u}_+ \in \mathbb{S}^{N-1}_+$ there exists a unique *geodesic midpoint*, $\mathbf{u}_g = (\mathbf{u}_+ + \mathbf{u}_-)/|\mathbf{u}_+ + \mathbf{u}_-|$ (see Definition 3.1). Hence we may define the *geodesic representative of* $\mathbf{u} \in BV(\Omega; \mathbb{S}^{N-1}_+)$, $\mathbf{u}_g := \mathbf{u}^*/|\mathbf{u}^*|$ (see Definition 3.2 and Remark 3.3) and the set of measures in (1.5) reduces to the singleton $\mathbf{u}(t)_g|D\mathbf{u}(t)|$.

The complete definition of solution and the statement of the main result are given in Definition 3.4 and Theorem 3.6, respectively. We obtain a solution as limit of a sequence of solutions to the following approximating problems (see Proposition 3.7 and Lemma 3.8):

$$\begin{cases} \mathbf{u}^\varepsilon_t = \operatorname{div} \mathbf{Z}^\varepsilon + \boldsymbol{\mu}^\varepsilon, \quad \mathbf{u}^\varepsilon \in \mathbb{S}^{N-1}_+ & \text{in } \Omega_T \\ [\mathbf{Z}^\varepsilon, \nu] = 0 & \text{on } S_T \\ \mathbf{u}^\varepsilon(0, \cdot) = \mathbf{u}^\varepsilon_0(\cdot) & \text{in } \ \Omega, \end{cases}$$

where

$$\mathbf{Z}^\varepsilon = \varepsilon^\alpha \nabla \mathbf{u}^\varepsilon + \frac{\nabla \mathbf{u}^\varepsilon}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}, \quad \boldsymbol{\mu}^\varepsilon = \varepsilon^\alpha \mathbf{u}^\varepsilon |\nabla \mathbf{u}^\varepsilon|^2 + \mathbf{u}^\varepsilon \frac{|\nabla \mathbf{u}^\varepsilon|^2}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}, \tag{1.6}$$

and the initial data suitably converge to a given $\mathbf{u}_0 \in BV(\Omega; \mathbb{S}^{N-1}_+)$ (see Lemma 3.9). The strategy we follow is completely different from that in [28], where the special structure of $\mathbb{S}^1$ was heavily used. Its core, neglecting any technicality and concentrating on the crucial issues, may be summarized as follows (see also [29] for a slightly more detailed discussion). By fairly standard compactness arguments, we obtain convergence of $\mathbf{u}^\varepsilon$, $\mathbf{Z}^\varepsilon$, and $\boldsymbol{\mu}^\varepsilon$ to $\mathbf{u}$, $\mathbf{Z}$, and $\boldsymbol{\mu}$, respectively (see Step 1 in the proof of Theorem 3.6). The functions $\mathbf{u}$ and $\mathbf{Z}$ can be seen to satisfy, for a.e. $t \in [0, T]$,

$$\mathbf{u}_t(t) - \operatorname{div} \mathbf{Z}(t) = \boldsymbol{\mu}(t) \quad \text{in } \ \mathcal{M}(\Omega; \mathbb{R}^N).$$

Then we show, by a relatively soft argument which however requires quite a few preliminaries, that

$$\boldsymbol{\mu} = *(*(\mathbf{Z} \wedge \mathbf{u}) \wedge D\mathbf{u}) \quad \text{and} \quad |\boldsymbol{\mu}(t)| \leq |D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0, T] \tag{1.7}$$

(see Step 2 in the proof of Theorem 3.6). Hence, in order to identify $\boldsymbol{\mu}$ it suffices to show that

$$\mathbf{u}(t)_g \cdot \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \geq 1 \quad \text{for a.e. } \ t \in [0, T], \tag{1.8}$$

where $\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}$ denotes the Radon-Nikodým derivative of $\boldsymbol{\mu}(t)$ with respect to $|D\mathbf{u}(t)|$. Indeed, (1.7) and simple vectorial identities then imply that

$$\boldsymbol{\mu}(t) = \mathbf{u}(t)_g|D\mathbf{u}(t)| \quad \text{for a.e. } t \in [0,T]$$

(see Step 6 in the proof of Theorem 3.6). In view of (1.6), the lower bound (1.8) for the diffuse part of $\boldsymbol{\mu}$ follows from (a suitable modification of, see §2.6) a relaxation result in [2], applied to each of the components of

$$\mathcal{F}(\mathbf{v}) := \int_\Omega \mathbf{v}(x)|\nabla \mathbf{v}(x)|\,\mathrm{d}x$$

(see Step 4 in the proof of Theorem 3.6). On the other hand, the same argument would lead to a sub-optimal lower bound on $\boldsymbol{\mu}(t)$ over the jump set of $\mathbf{u}(t)$ (see Remark 3.10). Moreover, the results in [2] can not be directly applied to $\mathbf{u}(t)_g \cdot \boldsymbol{\mu}^\varepsilon(t)$ since $\mathbf{u}(t)_g$ is a discontinuous function (though a very special one). For these reasons, we revisit the blow-up argument in [26] and the dimensional reduction argument in [27] to conclude that

$$\mathbf{u}(t)_g \cdot \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \quad \geq \quad \frac{1}{|\mathbf{u}(t)_+(x) - \mathbf{u}(t)_-(x)|} \inf_{\boldsymbol{\gamma} \in \tilde{\Gamma}_N} \int_0^1 \mathbf{u}(t)_g(x) \cdot \boldsymbol{\gamma}(s)|\boldsymbol{\gamma}'(s)|\,\mathrm{d}s \tag{1.9}$$

for a.e. $t$ and $\mathcal{H}^{m-1}$-a.e. $x \in J_{\mathbf{u}(t)}$, where

$$\tilde{\Gamma}_N := \left\{\boldsymbol{\gamma} \in W^{1,1}((0,1);\mathbb{S}^{N-1}_+) : \ \boldsymbol{\gamma}(0) = \mathbf{u}(t)_-(x),\ \boldsymbol{\gamma}(1) = \mathbf{u}(t)_+(x)\right\} \tag{1.10}$$

(see Step 5 in the proof of Theorem 3.6). The minimization problem which appears on the right-hand side of (1.9) is crucial in our argument. In Section 4 we argue that

$$\begin{aligned} &\min_{\boldsymbol{\gamma}\in\Gamma} \int_0^1 \mathbf{u}_g \cdot \boldsymbol{\gamma}(s)|\boldsymbol{\gamma}'(s)|\,\mathrm{d}s = |\mathbf{u}_+ - \mathbf{u}_-|, \quad \text{where} \\ &\Gamma = \left\{\boldsymbol{\gamma} \in W^{1,1}((0,1);\mathbb{S}^{N-1}_+) : \ \boldsymbol{\gamma}(0) = \mathbf{u}_-,\ \boldsymbol{\gamma}(1) = \mathbf{u}_+\right\} \end{aligned} \tag{1.11}$$

(see Theorem 4.1). Together with (1.9), (1.11) yields the lower bound (1.8) on the jump set of $\mathbf{u}(t)$, too.

The minimization problem in (1.11) is equivalent to finding –and characterizing the length of– shortest paths between $\mathbf{u}_-$ and $\mathbf{u}_+$ in a Riemannian manifold with boundary whose metric penalizes the closeness to $\mathbf{u}_g$. In addition, the metric may degenerate at a point of the manifold: for instance, if $N = 3$, $\mathbf{u}_- = (0,0,1)$, and $\mathbf{u}_+ = (0,1,0)$, then $\mathbf{u}_g \cdot (1,0,0) = 0$. In these respects, the minimization problem has a geometrical interest of its own.

It turns out that the minimum in (1.11) is achieved by the standard geodesic on $\mathbb{S}^{N-1}_+$ connecting $\mathbf{u}_-$ and $\mathbf{u}_+$, see Lemma 4.2. Nevertheless, the analysis of (1.11) is highly nontrivial for two reasons. Firstly, one has to characterize the length of candidate shortest paths which may in principle intersect, and/or de-touch from, the boundary of the manifold. Secondly, the functional in (1.11) is genuinely non-convex: Indeed, besides the aforementioned standard geodesic, it always possesses a second smooth critical point, which we show not to be a shortest path. In addition, in the extreme cases in which $\mathbf{u}_+$ and $\mathbf{u}_-$ are two distinct "vertices" of $\mathbb{S}^{N-1}_+$, the functional in (1.11) possesses a second shortest path which is not a critical point: it follows the boundary of $\mathbb{S}^{N-1}_+$ and passes through the point of degeneracy. For instance, if $N = 3$, $\mathbf{u}_- = (0,0,1)$, and $\mathbf{u}_+ = (0,1,0)$, then $\mathbf{u}_g = (0,1,1)/\sqrt{2}$ and the curve

$$\boldsymbol{\gamma}(s) = \begin{cases} (\sin(\pi s), 0, \cos(\pi s)) & \text{if } \ s \in [0,1/2] \\ (\sin(\pi s), -\cos(\pi s), 0) & \text{if } \ s \in (1/2,1] \end{cases}$$

is such that

$$\int_0^1 \mathbf{u}_g \cdot \boldsymbol{\gamma}(s)|\boldsymbol{\gamma}'(s)|\,\mathrm{d}s = 2\int_0^{1/2} \frac{\cos(\pi s)}{\sqrt{2}}\pi\,\mathrm{d}s = \sqrt{2} = |\mathbf{u}_+ - \mathbf{u}_-|.$$

Finally, we note that if the paths in $\Gamma$ are allowed to take values in a set $\mathbb{A}$ which contains $\mathbb{S}^{N-1}_+$, then in general the standard geodesic is not a minimizer and (1.11) does not hold; an example is given in Remark 4.4.

The paper is organized as follows. In Section 2 we collect the definitions and results which we need concerning multi-vector fields, functions of bounded variations, a generalized Green's formula, tensor fields, and lower semi-continuity of integral functionals. In Section 3 we introduce the concept of solution and we prove the existence of solution to (1.4). Section 4 is devoted to the minimization problem in (1.11).

## 2. Preliminaries

In this section we introduce some notation and some preliminary results that we need in the sequel.

*General notations.* Throughout this paper $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure and $\mathcal{L}^m$ the $m$-dimensional Lebesgue measure. We denote by $\mathcal{M}(\Omega;\mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued finite Radon measures on $\Omega$ (see [5, Def. 1.40]). We recall that $\mathcal{M}(\Omega;\mathbb{R}^N)$ is the dual space of $C_0(\Omega;\mathbb{R}^N)$. Throughout, the subscript $_0$ denotes spaces of compactly supported functions. We denote by $\mathcal{D}(\Omega;\mathbb{R}^N) := C_0^\infty(\Omega;\mathbb{R}^N)$. When $N = 1$ we often do not specify the target space (e.g., $\mathcal{M}(\Omega) = \mathcal{M}(\Omega;\mathbb{R})$). Finally, if $\mathbb{A} \subset \mathbb{R}^N$ is compact and $\Upsilon(\Omega;\mathbb{R}^N)$ is a space of functions, we sometimes use the notation $\Upsilon(\Omega;\mathbb{A}) := \big\{\mathbf{u} \in \Upsilon(\Omega;\mathbb{R}^N) : \ \mathbf{u}(x) \in \mathbb{A} \ \text{ for } \mathcal{L}^m\text{-a.e. } x \in \Omega\big\}$.

2.1. **Multi-vectors.** Here we recall some definitions and basic properties about multi-vectors that we need in our analysis. We refer to e.g. [24, Chapter 1] and [19, Chapter 1] for details.

The spaces $\Lambda_0(\mathbb{R}^N)$ and $\Lambda_1(\mathbb{R}^N)$ coincide with $\mathbb{R}$ and $\mathbb{R}^N$, respectively. For $2 \le k \le \mathbb{N}$, the $k$-th exterior power of $\mathbb{R}^N$, denoted by $\Lambda_k(\mathbb{R}^N)$, is a set spanned by elements of the form

$$\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k, \quad \mathbf{u}_i \in \mathbb{R}^N, \ i = 1, \dots, k$$

(elements of this form are called "generators") and subject to the following rules:

$$\left.\begin{array}{c} (a\mathbf{v} + b\mathbf{w}) \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k = a(\mathbf{v} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k) + b(\mathbf{w} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k), \\ \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \text{ changes sign if two entries are transposed,} \\ \text{for any basis } \{\mathbf{e}_1, \dots \mathbf{e}_n\} \text{ of } \mathbb{R}^N, \\ \{\mathbf{e}_\alpha := \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} \ : \alpha \in I(k,N)\} \text{ is a basis for } \Lambda_k(\mathbb{R}^N), \end{array}\right\} \tag{2.1}$$

where we use the standard notation for ordered multi-indexes:

$$I(k,N) = \{\alpha = (\alpha_1, \dots, \alpha_k) \ : \ \alpha_i \ \text{ integers}, \ 1 \le \alpha_1 < \cdots < \alpha_k \le N\}.$$

The elements of $\Lambda_k(\mathbb{R}^N)$ are called multi-vectors (or $k$-vectors), and $\Lambda_k(\mathbb{R}^N)$ is a vector space of dimension $\left(\begin{smallmatrix} N \\ k \end{smallmatrix}\right)$. We will use the well-known equality [19, Formula 1.68]:

$$|\mathbf{a}|^2|\mathbf{b}|^2 = (\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\wedge\mathbf{b})^2 \quad \text{for all } \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^N. \tag{2.2}$$

Given $k, p \in \{0, \dots, N\}$ with $k + p \le N$, there exists a unique bilinear map $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \to \boldsymbol{\lambda} \wedge \boldsymbol{\mu}$ from $\Lambda_k(\mathbb{R}^N) \times \Lambda_p(\mathbb{R}^N)$ to $\Lambda_{k+p}(\mathbb{R}^N)$, whose effect on generators is

$$(\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k) \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p.$$

Such map satisfies

$$\boldsymbol{\lambda} \wedge \boldsymbol{\mu} = (-1)^{-kp}(\boldsymbol{\mu} \wedge \boldsymbol{\lambda}) \quad \text{for } \ \boldsymbol{\lambda} \in \Lambda_k(\mathbb{R}^N), \ \boldsymbol{\mu} \in \Lambda_p(\mathbb{R}^N). \tag{2.3}$$

The *Hodge-star operator* is an isomorphism from $\Lambda_k(\mathbb{R}^N)$ to $\Lambda_{N-k}(\mathbb{R}^N)$, defined on the basis as

$$*(\mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k}) := \mathbf{e}_{\alpha_{k+1}} \wedge \cdots \wedge \mathbf{e}_{\alpha_N}, \tag{2.4}$$

where $\{\alpha_1, \ldots, \alpha_N\}$ has positive signature. In particular, in what follows we will systematically identify $\Lambda_{N-1}(\mathbb{R}^N)$ with $\mathbb{R}^N$. We will use the following well known formulas:

$$*(*\boldsymbol{\lambda}) = (-1)^{k(N-k)}\boldsymbol{\lambda} \quad \text{for all} \ \ \boldsymbol{\lambda} \in \Lambda_k(\mathbb{R}^N) \tag{2.5}$$

(see e,g, [19, (1.64)]) and

$$\mathbf{a} \wedge *(\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) * \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) * \mathbf{c}, \quad \text{for all} \ \ \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^N \tag{2.6}$$

(see e.g. [19, Table 1.2]). It follows from (2.3), (2.6), and (2.5) that

$$|\mathbf{b}|^2\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} - *(*(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{b}) \quad \text{for all} \ \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^N. \tag{2.7}$$

Introducing the norm

$$|\boldsymbol{\lambda}|_k = \left( \sum_{\alpha \in I(k,N)} |\lambda_\alpha|^2 \right)^{\frac{1}{2}}, \quad \text{where} \quad \boldsymbol{\lambda} = \sum_{\alpha \in I(k,N)} \lambda_\alpha \mathbf{e}_\alpha \tag{2.8}$$

and using (2.4), it is immediate to see that

$$| * \boldsymbol{\lambda}|_{N-k} = |\boldsymbol{\lambda}|_k \quad \text{for any } \boldsymbol{\lambda} \in \Lambda_k(\mathbb{R}^N). \tag{2.9}$$

Finally, we recall that, given $\boldsymbol{\lambda} \in \Lambda_k(\mathbb{R}^N)$ and $\boldsymbol{\eta} \in \Lambda_p(\mathbb{R}^N)$ such that one of them is a generator, then

$$|\boldsymbol{\lambda} \wedge \boldsymbol{\eta}|_{k+p} \leq |\boldsymbol{\lambda}|_k |\boldsymbol{\eta}|_p \tag{2.10}$$

(see [24, pag. 32]).

2.2. **Vector valued functions.** Let $(X, \|\cdot\|)$ a Banach space with dual $X'$ and let $U \subset \mathbb{R}^d$ be a bounded open set endowed with the Lebesgue measure $\mathcal{L}^d$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $X$ and $X'$. A function $u : U \to X$ is called *simple* if there exist $x_1, \ldots, x_n \in X$ and $U_1, \ldots, U_n$ $\mathcal{L}^m$-measurable subsets of $U$ such that $u = \sum_{i=1}^n x_i \chi_{U_i}$. The function $u$ is called *strongly measurable* if there exists a sequence of simple functions $\{u_n\}$ such that $\|u_n(x) - u(x)\| \to 0$ as $n \to +\infty$ for almost all $x \in U$. If $1 \leq p < \infty$, then $L^p(U; X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : U \to X$ with

$$\|u\|_p := \left( \int_U \|u(x)\|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Endowed with this norm, $L^p(U; X)$ is is a Banach space. For $p = \infty$, the symbol $L^\infty(U; X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : U \to X$ such that

$$\|u\|_\infty := \ \mathrm{esssup}\{\|u(x)\| : \ x \in U\} < \infty.$$

If $U = (0,T)$, we write $L^p(0,T;X) = L^p((0,T);X)$. For $1 \le p < \infty$, $L^{p'}(0,T;X')$ ($\frac{1}{p}+\frac{1}{p'}=1$) is isometric to a subspace of $(L^p(0,T;X))'$, with equality if and only if $X'$ has the Radon-Nikodým property (see for instance [21]).

We consider the vector space $\mathcal{D}(U;X) := C_0^\infty(U;X)$, endowed with the topology for which a sequence $\varphi_n \to 0$ as $n \to +\infty$ if there exists $K \subset U$ compact such that $\operatorname{supp}(\varphi_n) \subset K$ for any $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to 0$ uniformly on $K$ as $n \to +\infty$ for all multi-index $\alpha$. We denote by $\mathcal{D}'(U;X)$ the space of distributions on $U$ with values in $X$, that is, the set of all linear continuous maps $T : \mathcal{D}(U;X) \to \mathbb{R}$. As is well known, $L^p(U;X) \subset \mathcal{D}'(U;X)$ through the standard continuous injection. Given $T \in \mathcal{D}'(U;X)$, the distributional derivative of $T$ is defined by

$$\langle D_i T, \varphi\rangle := -\langle T, \partial_i \varphi\rangle \quad \text{for any } \varphi \in \mathcal{D}(U;X) \text{ and any } i \in \{1,\dots,d\}. \tag{2.11}$$

*General notations for matrices.* If $\mathbf{A} = (a_i^\ell)$ is an $N \times m$ matrix, we write $a^\ell = (a_1^\ell, \dots a_m^\ell)$ for $1 \le \ell \le N$ and $\mathbf{a}_i = (a_i^1, \dots, a_i^N)$ for $1 \le i \le m$. If $\mathbf{B} = (b_i^\ell)$ is also an $N \times m$ matrix, we let

$$\mathbf{A} : \mathbf{B} = \sum_{\ell=1}^N \sum_{i=1}^m a_i^\ell b_i^\ell \quad \text{and} \quad |\mathbf{A}| = (\mathbf{A}:\mathbf{A})^{\frac12} = \left(\sum_{\ell=1}^N \sum_{i=1}^m (a_i^\ell)^2\right)^{\frac12}.$$

Given $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathbb{R}^{N\times m}$ and $\mathbf{b} \in \mathbb{R}^N$, we let

$$\begin{aligned} \mathbf{A} \wedge \mathbf{b} &:= (\mathbf{a}_1 \wedge \mathbf{b}, \dots, \mathbf{a}_m \wedge \mathbf{b}), \\ *(\mathbf{A} \wedge \mathbf{b}) &:= (*(\mathbf{a}_1 \wedge \mathbf{b}), \dots, *(\mathbf{a}_m \wedge \mathbf{b})). \end{aligned}$$

2.3. **Functions of bounded variation.** A vector-field $\mathbf{u} \in L^1(\Omega;\mathbb{R}^N)$ has bounded variation, $\mathbf{u} \in BV(\Omega;\mathbb{R}^N)$, if there is an $N \times m$ matrix $D\mathbf{u}$, whose components $D_i u^\ell$ are finite Radon measures, such that

$$\sum_{\ell=1}^N \int_\Omega u^\ell \operatorname{div} \varphi^\ell \, \mathrm{d}x = -\sum_{\ell=1}^N \sum_{i=1}^m \int_\Omega \varphi_i^\ell \, \mathrm{d}D_i u^\ell \quad \text{for all} \quad \varphi \in \left(C_0^1(\Omega;\mathbb{R}^N)\right)^m.$$

Its variation measure $|D\mathbf{u}|$ is a finite Radon measure defined on open sets $U \subseteq \Omega$ by

$$|D\mathbf{u}|(U) = \sup\left\{ \sum_{\ell=1}^N \int_U u^\ell \operatorname{div}\varphi^\ell \, \mathrm{d}x : \ \varphi \in \left(C_0^1(U;\mathbb{R}^N)\right)^m, \ \|\varphi\|_\infty \le 1 \right\}.$$

The matrix-valued Radon measure $D\mathbf{u}$ is decomposed into three mutually orthogonal measures (see [5, 23, 46]):

$$D\mathbf{u} = \nabla \mathbf{u} \mathcal{L}^m + D^c \mathbf{u} + D^j \mathbf{u},$$

where $\nabla\mathbf{u}$ denotes the Radon–Nikodým derivative of $D\mathbf{u}$ with respect to $\mathcal{L}^m$. The *Cantor part* $D^c\mathbf{u}$ is supported on the set of *Lebesgue points of* $\mathbf{u}$, $\Omega \setminus S_{\mathbf{u}}$, i.e. those points $x \in \Omega$ for which there exists $\tilde{\mathbf{u}}(x) \in \mathbb{R}^N$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^m(B_\rho(x))} \int_{B_\rho(x)} |\mathbf{u}(y) - \tilde{\mathbf{u}}(x)| \, \mathrm{d}y = 0.$$

The *jump part* $D^j\mathbf{u}$ is supported on the set of *approximate jump points of* $\mathbf{u}$, $J_{\mathbf{u}}$, i.e.those points $x \in \Omega$ for which there exist $\mathbf{u}_+(x) \neq \mathbf{u}_-(x) \in \mathbb{R}^N$ and $\nu_{\mathbf{u}}(x) \in \mathbb{S}^{m-1}$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^m(B^\pm_\rho(x, \nu_{\mathbf{u}}(x)))} \int_{B^\pm_\rho(x,\nu_{\mathbf{u}}(x))} |\mathbf{u}(y) - \mathbf{u}_\pm(x)| \, \mathrm{d}y \;\; = \;\; 0,$$

where

$$B^\pm_\rho(x, \nu_{\mathbf{u}}(x)) = \{y \in B_\rho(x) \ : \ \langle y - x, \nu_{\mathbf{u}}(x)\rangle \gtrless 0\}.$$

The jump set $J_{\mathbf{u}}$ is a Borel subset of $S_{\mathbf{u}}$ that satisfies $\mathcal{H}^{m-1}(S_{\mathbf{u}} \backslash J_{\mathbf{u}}) = 0$. The *precise representative* $\mathbf{u}^* : \Omega\backslash(S_{\mathbf{u}}\backslash J_{\mathbf{u}}) \to \mathbb{R}^N$ of $\mathbf{u}$ is defined to be equal to $\tilde{\mathbf{u}}$ on $\Omega\backslash S_{\mathbf{u}}$ and equal to $\frac{\mathbf{u}_- + \mathbf{u}_+}{2}$ on $J_{\mathbf{u}}$. In what follows, we identify $\mathbf{u} = \tilde{\mathbf{u}} = \mathbf{u}^*$ on $\Omega \setminus S_{\mathbf{u}}$.

2.4. **A generalized Green's formula.** Let

$$X_{\mathcal{M}}(\Omega) = \{z \in L^\infty(\Omega; \mathbb{R}^m) \ : \ \operatorname{div} z \in \mathcal{M}(\Omega)\}$$

and

$$\mathcal{M}_{\mathcal{H}}(\Omega; \mathbb{R}^N) := \left\{\boldsymbol{\mu} \in \mathcal{M}(\Omega; \mathbb{R}^N) : \ |\boldsymbol{\mu}|(B) = 0 \ \text{ for any Borel set } B \subset \Omega\text{: } \mathcal{H}^{m-1}(B) = 0\right\}.$$

In [8, Theorem 1.2] (see also [7, 16]), the weak trace on $\partial\Omega$ of the normal component of $z \in X_{\mathcal{M}}(\Omega)$ is defined. Namely, it is proved that there exists a linear operator $[\cdot, \nu] : X_{\mathcal{M}}(\Omega) \to L^\infty(\partial\Omega)$ such that $\| \, [z, \nu] \, \|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$ for all $z \in X_{\mathcal{M}}(\Omega)$ and $[z, \nu]$ coincides with the pointwise trace of the normal component if $z$ is smooth:

$$[z, \nu](x) = z(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \ \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^m).$$

It follows from [16, Proposition 3.1] or [3, Proposition 3.4] that

$$\operatorname{div} z \in \mathcal{M}_{\mathcal{H}}(\Omega) \qquad \text{for all } z \in X_{\mathcal{M}}(\Omega). \tag{2.12}$$

Therefore, given $z \in X_{\mathcal{M}}(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$, the functional $(z, Du) \in \mathcal{D}'(\Omega)$ given by

$$\langle (z, Du), \varphi\rangle := -\int_\Omega u^* \, \varphi \, \mathrm{d}(\operatorname{div} z) - \int_\Omega u \, z \nabla\varphi \, \mathrm{d}x \tag{2.13}$$

is well defined, and the following holds (in [14], see Lemma 5.1, Theorem 5.3, and the discussion after Lemma 5.4):

**Lemma 2.1.** *Let $z \in X_{\mathcal{M}}(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the functional $(z, Du) \in \mathcal{D}'(\Omega)$ defined by* (2.13) *is a Radon measure which is absolutely continuous with respect to $|Du|$. Furthermore,*

$$\int_\Omega u^* \,\mathrm{d}(\operatorname{div} z) + (z, Du)(\Omega) = \int_{\partial\Omega} [z, \nu] u \,\mathrm{d}\mathcal{H}^{m-1}$$

*and*

$$\operatorname{div}(zu) = u^* \operatorname{div} z + (z, Du) \quad \text{as measures.}$$

We will use the vector-valued version of Lemma 2.1. To this aim, we introduce the space

$$\mathbf{X}^N_{\mathcal{M}}(\Omega) = \left\{ \mathbf{Z} = (z^1, \dots, z^N)^T : z^\ell \in X_{\mathcal{M}}(\Omega) \ \text{ for } \ell = 1, \dots, N \right\}.$$

Given $\mathbf{Z} \in \mathbf{X}^N_{\mathcal{M}}(\Omega)$ and $\mathbf{u} \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, we use the following notation:

$$\begin{aligned} \operatorname{div} \mathbf{Z} &:= (\operatorname{div} z^1, \dots, \operatorname{div} z^N), \\ [\mathbf{Z}, \nu] &:= ([z^1, \nu], \dots, [z^N, \nu]), \\ \mathbf{Z} : D\mathbf{u} &:= \sum_{\ell=1}^N (z^\ell, Du^\ell). \end{aligned}$$

Then, as an immediate consequence of (2.12) and Lemma 2.1, the following holds:

**Corollary 2.2.** *Let $\mathbf{Z} \in \mathbf{X}^N_{\mathcal{M}}(\Omega)$. Then $\operatorname{div} \mathbf{Z} \in \mathcal{M}_{\mathcal{H}}(\Omega; \mathbb{R}^N)$. Furthermore, for any $\mathbf{u} \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, $\mathbf{Z} : D\mathbf{u}$ is a Radon measure which is absolutely continuous with respect to $|D\mathbf{u}|$,*

$$\int_\Omega \mathbf{u}^* \cdot \,\mathrm{d}(\operatorname{div} \mathbf{Z}) + (\mathbf{Z} : D\mathbf{u})(\Omega) = \int_{\partial\Omega} [\mathbf{Z}, \nu] \cdot \mathbf{u} \,\mathrm{d}\mathcal{H}^{m-1} \tag{2.14}$$

*and*

$$\operatorname{div}(\mathbf{Z}^T \mathbf{u}) = \mathbf{u}^* \cdot \operatorname{div} \mathbf{Z} + \mathbf{Z} : D\mathbf{u} \quad \text{as measures.} \tag{2.15}$$

2.5. **Multi-vector fields.** Let $U \subset \mathbb{R}^d$. A multi-vector distribution in $U$ is a linear continuous map $\boldsymbol{\lambda} \in \mathcal{D}'(U; \Lambda_k(\mathbb{R}^N))$ (see §2.2). It may be expressed in terms of the basis (2.1) as

$$\boldsymbol{\lambda} = \sum_{\alpha \in I(k,N)} \lambda_\alpha \mathbf{e}_\alpha, \quad \text{with } \lambda_\alpha \in \mathcal{D}'(U; \mathbb{R}^N) \text{ for any } \alpha \in I(k, N).$$

Then, according to (2.11),

$$D_i\boldsymbol{\lambda} = \sum_{\alpha \in I(k;N)} D_i\lambda_\alpha \mathbf{e}_\alpha \quad \text{for any } i \in \{1,\dots,d\}. \tag{2.16}$$

From (2.16), the following two identities are easily seen to hold for $k, p \in \mathbb{N}$ and $i \in \{1,\dots,d\}$:

$$D_i(\boldsymbol{\lambda} \wedge \boldsymbol{\eta}) = D_i\boldsymbol{\lambda} \wedge \boldsymbol{\eta} + \boldsymbol{\lambda} \wedge D_i\boldsymbol{\eta} \tag{2.17}$$

for any $\boldsymbol{\lambda} \in L^2(U;\Lambda_k(\mathbb{R}^N))$ such that $D_i\boldsymbol{\lambda} \in L^2(U;\Lambda_k(\mathbb{R}^N))$ and any $\boldsymbol{\eta} \in L^2(U;\Lambda_p(\mathbb{R}^N))$ such that $D_i\boldsymbol{\eta} \in L^2(U;\Lambda_p(\mathbb{R}^N))$;

$$*(D_i\boldsymbol{\lambda}) = D_i(*\boldsymbol{\lambda}) \qquad \text{for any } \boldsymbol{\lambda} \in \mathcal{D}'(U;\Lambda_k(\mathbb{R}^N)). \tag{2.18}$$

For any $k \in \mathbb{N}$, $(\Lambda_k(\mathbb{R}^N))^m$ is a Banach space. We use the norm

$$\|\mathcal{A}\| := \left(\sum_{i=1}^m |\mathcal{A}_i|_k^2\right)^{\frac{1}{2}}, \quad \text{for } \mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_m)$$

with $|\cdot|_k$ given by (2.8).

We will now state and prove the analogue of Corollary 2.2 for a multi-vector field $\mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_m) \in L^\infty(\Omega;(\Lambda_k(\mathbb{R}^N))^m)$. We define

$$\operatorname{div}\mathcal{A} := \sum_{i=1}^m D_i(\mathcal{A}_i). \tag{2.19}$$

It will suffice to our purposes that $\operatorname{div}\mathcal{A}$ is square-integrable. Hence, we introduce the space

$$\mathbf{X}_2(\Omega;\Lambda_{N-2}(\mathbb{R}^N)) := \left\{\mathcal{A} \in L^\infty(\Omega;(\Lambda_{N-2}(\mathbb{R}^N))^m) \ : \ \operatorname{div}\mathcal{A} \in L^2(\Omega;\Lambda_{N-2}(\mathbb{R}^N))\right\}.$$

The following holds:

**Lemma 2.3.** *Let $\mathcal{A} \in \mathbf{X}_2(\Omega;\Lambda_{N-2}(\mathbb{R}^N))$ and $\mathbf{u} \in BV(\Omega;\mathbb{R}^N) \cap L^2(\Omega;\mathbb{R}^N)$. Then the functional $*(\mathcal{A} \wedge D\mathbf{u}) : \mathcal{D}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ defined by*

$$\langle *(\mathcal{A} \wedge D\mathbf{u}), \Phi\rangle := -\int_\Omega *(\operatorname{div}\mathcal{A} \wedge \mathbf{u}) \cdot \Phi \,\mathrm{d}x - \sum_{i=1}^m \int_\Omega *(\mathcal{A}_i \wedge \mathbf{u}) \cdot \partial_i\Phi \,\mathrm{d}x \tag{2.20}$$

*is an $\mathbb{R}^N$-valued Radon measure on $\Omega$, absolutely continuous with respect to $|D\mathbf{u}|$, with*

$$|*(\mathcal{A} \wedge D\mathbf{u})|(B) \le \|\mathcal{A}\|_\infty |D\mathbf{u}|(B) \quad \text{for any Borel set } B \subseteq \Omega. \tag{2.21}$$

*Furthermore*

$$\operatorname{div}(*(\mathcal{A}\wedge\mathbf{u})) = *(\mathcal{A}\wedge D\mathbf{u}) + *(\operatorname{div}\mathcal{A}\wedge\mathbf{u})\mathcal{L}^m \quad \textit{as measures.} \tag{2.22}$$

*Proof.* Since $\Omega$ has compact Lipschitz boundary, it follows from [5, Theorem 3.21, Remark 3.22, and Corollary 3.80] that the sequence $\mathbf{u}_n := (T\mathbf{u})\star\rho_n \in C^\infty(\overline{\Omega})$ (here $T$ denotes an extension operator) is such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $BV(\Omega;\mathbb{R}^N)$, $\int_\Omega |\nabla\mathbf{u}_n|\,\mathrm{d}x \to |D\mathbf{u}|(\Omega)$, and $\mathbf{u}_n \to \mathbf{u}^*$ $\mathcal{H}^{m-1}$-a.e. in $\overline{\Omega}$. Furthermore, by construction and since $\mathbf{u}\in L^2(\Omega;\mathbb{R}^N)$, $\mathbf{u}_n\to\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^N)$. Then

$$\langle *(\mathcal{A}\wedge D\mathbf{u}),\Phi\rangle \overset{(2.20)}{=} -\lim_{n\to\infty}\left(\int_\Omega *(\operatorname{div}\mathcal{A}\wedge\mathbf{u}_n)\cdot\Phi\,\mathrm{d}x + \sum_{i=1}^m\int_\Omega *(\mathcal{A}_i\wedge\mathbf{u}_n)\cdot\partial_i\Phi\,\mathrm{d}x\right).$$

Integrating by parts and using (2.18), we obtain

$$\begin{aligned}\langle *(\mathcal{A}\wedge D\mathbf{u}),\Phi\rangle &= -\lim_{n\to\infty}\left(\int_\Omega *(\operatorname{div}\mathcal{A}\wedge\mathbf{u}_n)\cdot\Phi\,\mathrm{d}x - \sum_{i=1}^m\int_\Omega *\left(\partial_i(\mathcal{A}_i\wedge\mathbf{u}_n)\right)\cdot\Phi\,\mathrm{d}x\right)\\ &\overset{(2.17)}{=} \lim_{n\to\infty}\sum_{i=1}^m\int_\Omega *(\mathcal{A}_i\wedge\partial_i\mathbf{u}_n)\cdot\Phi\,\mathrm{d}x.\end{aligned}$$

Therefore, applying Hölder and Cauchy-Schwarz inequalities,

$$\begin{aligned}|\langle *(\mathcal{A}\wedge D\mathbf{u}),\Phi\rangle| &\overset{(2.9)}{\le} \|\Phi\|_\infty \lim_{n\to\infty}\sum_{i=1}^m\int_\Omega |\mathcal{A}_i\wedge\partial_i\mathbf{u}_n|_{N-1}\,\mathrm{d}x\\ &\overset{(2.10)}{\le} \|\Phi\|_\infty \lim_{n\to\infty}\int_\Omega\sum_{i=1}^m |\partial_i\mathbf{u}_n||\mathcal{A}_i|_{N-2}\,\mathrm{d}x\\ &\le \|\Phi\|_\infty \lim_{n\to\infty}\int_\Omega |\nabla\mathbf{u}_n|\left(\sum_{i=1}^m|\mathcal{A}_i|^2_{N-2}\right)^{\frac12}\mathrm{d}x\\ &\le \|\Phi\|_\infty\|\mathcal{A}\|_\infty \lim_{n\to\infty}\int_\Omega|\nabla\mathbf{u}_n|\,\mathrm{d}x\\ &= \|\Phi\|_\infty\|\mathcal{A}\|_\infty|D\mathbf{u}|(\Omega).\end{aligned}$$

The arbitrariness of $\Phi$ completes the proof of (2.21). It follows from (2.20) and (2.19) that $\operatorname{div}(*(\mathcal{A}\wedge\mathbf{u}))$ is also a $\mathbb{R}^N$-valued Radon measure in $\Omega$, and (2.22) follows from (2.19). □

2.6. **Lower semi-continuity of integral functionals over $W^{1,1}(\Omega;\mathbb{S}^{N-1}_+)$.** Let $f:\Omega\times\mathbb{S}^{N-1}_+\to\mathbb{R}_+$ and consider the energy functional defined in $L^1(\Omega;\mathbb{S}^{N-1}_+)$ by

$$\mathcal{F}_f(\mathbf{v}):=\begin{cases}\displaystyle\int_\Omega f(x,\mathbf{v}(x))|\nabla\mathbf{v}(x)|\,\mathrm{d}x & \text{if } \mathbf{v}\in W^{1,1}(\Omega;\mathbb{S}^{N-1}_+)\\ +\infty & \text{otherwise}\end{cases}$$

The purpose of this section is to restate, to the extent we need in the present setting, a few lower semi-continuity results obtained in [27, 2] (see also [31] for related results when the target space is a general manifold). We consider the following hypotheses for $f$:

(H1) $f$ is continuous and non-negative;

(H2) (uniform boundedness) a positive constant $C_1$ exists such that

$$|f(x,\mathbf{s})|\le C_1\quad\text{for all }\ (x,\mathbf{s})\in\Omega\times\mathbb{S}^{N-1}_+;$$

(H3) for every compact set $U\subset\Omega$ there exist a continuous function $\omega$, with $\omega(0)=0$, such that

$$|f(x,\mathbf{s})-f(x',\mathbf{s}')|=\omega(|x-x'|+|\mathbf{s}-\mathbf{s}'|)\quad\text{for all }\ (x,\mathbf{s}),(x',\mathbf{s}')\in U\times\mathbb{S}^{N-1}_+.$$

For $\varsigma\in\mathbb{R}^m$ such that $|\varsigma|=1$, we define $Q_\varsigma:=R_\varsigma\left[-\frac12,\frac12\right]^m$, where $R_\varsigma$ denotes a rotation such that $R_\varsigma e_m=\varsigma$. Given $\mathbf{a},\mathbf{b}\in\mathbb{S}^{N-1}_+$, we set

$$K_f(x,\mathbf{a},\mathbf{b},\varsigma):=\inf\left\{\int_{Q_\varsigma}f(x,\mathbf{v}(y))|\nabla\mathbf{v}(y)|\,\mathrm{d}y:\ \mathbf{v}\in\mathcal{P}(\mathbf{a},\mathbf{b},\varsigma)\right\},\tag{2.23}$$

where

$$\mathcal{P}(\mathbf{a},\mathbf{b},\varsigma):=\left\{\mathbf{v}\in W^{1,1}(Q_\varsigma;\mathbb{S}^{N-1}_+):\ \mathbf{v}(x)=\mathbf{a}\ \text{ if } x\cdot\varsigma=-\frac12,\ \mathbf{v}(x)=\mathbf{b}\ \text{ if } x\cdot\varsigma=\frac12\right\}.\tag{2.24}$$

The following holds:

**Lemma 2.4.** *Assume (H1). Then*

$$\begin{aligned}&K_f(x,\mathbf{a},\mathbf{b},\varsigma)\\ &=\ \inf\left\{\int_0^1 f(x,\boldsymbol{\gamma}(t))|\dot{\boldsymbol{\gamma}}(t)|\,\mathrm{d}t:\ \boldsymbol{\gamma}\in W^{1,1}((0,1);\mathbb{S}^{N-1}_+)\ ,\ \boldsymbol{\gamma}(0)=\mathbf{a}\,,\ \boldsymbol{\gamma}(1)=\mathbf{b}\right\}.\end{aligned}\tag{2.25}$$

The proof of Lemma 2.4 is identical to that of [27, Proposition 2.6], where the same result has been proved (under more general assumptions on the energy density) when the target space is $\mathbb{R}^N$ rather that $\mathbb{S}^{N-1}_+$: therefore we omit it.

In order to obtain a lower bound on the lower semi-continuous envelope of $\mathcal{F}_f$, in particular of its jump part, one needs an approximation Lemma which relates a generic sequence in $W^{1,1}(Q_\varsigma;\mathbb{S}^{N-1}_+)$, converging to a step function, to a non-generic one in $\mathcal{P}(\mathbf{a},\mathbf{b},\varsigma)$:

**Lemma 2.5.** *Assume (H1) and (H2).*

*Let $\mathbf{a},\mathbf{b}\in\mathbb{S}^{N-1}_+$ and let $\mathbf{v}_n\in W^{1,1}(Q_\varsigma;\mathbb{S}^{N-1}_+)$ such that $\mathbf{v}_n\to\mathbf{u}_0$ in $L^1(Q_\varsigma;\mathbb{S}^{N-1}_+)$, where*

$$\mathbf{u}_0(x):=\begin{cases}\mathbf{b} & \text{if } \langle x,\varsigma\rangle\geq 0\\ \mathbf{a} & \text{if } \langle x,\varsigma\rangle<0\end{cases}$$

*Then a sequence $\mathbf{w}_n\in\mathcal{P}(\mathbf{a},\mathbf{b},\varsigma)$ exists such that $\mathbf{w}_n\to\mathbf{u}_0$ in $L^1(Q_\varsigma;\mathbb{S}^{N-1}_+)$ and*

$$\liminf_{n\to\infty}\int_{Q_\varsigma} f(x,\mathbf{v}_n)|\nabla\mathbf{v}_n|\,\mathrm{d}x\geq\limsup_{n\to\infty}\int_{Q_\varsigma} f(x,\mathbf{w}_n)|\nabla\mathbf{w}_n|\,\mathrm{d}x.$$

Lemma 2.5 may be proved following line by line that of Lemma 5.2 in [2], where the same result was proved (under more general assumptions on the energy density) when the target space is $\mathbb{S}^{N-1}$, and therefore we omit it. We just mention that the proof may in fact be simplified in the present setting, by using the standard projection onto $\mathbb{S}^{N-1}_+$ (see estimate (3.23) and Lemma 3.9 below for a related approximation result).

Let $\mathcal{G}_f$ be the functional defined in $BV(\Omega;\mathbb{S}^{N-1}_+)$ by

$$\mathcal{G}_f(\mathbf{v}):=\int_\Omega f(x,\mathbf{v})|\nabla\mathbf{v}|\,\mathrm{d}x+\int_{J(\mathbf{v})}K_f(x,\mathbf{v}_-,\mathbf{v}_+,\nu_{\mathbf{v}})\,\mathrm{d}\mathcal{H}^{m-1}+\int_\Omega f(x,\mathbf{v})\,\mathrm{d}|D^c\mathbf{v}|$$

(and $+\infty$ elsewhere). Under an additional coercivity assumption on $f$, and when the target space is $\mathbb{S}^{N-1}$, in [2, Proposition 5.1] it is proved that $\mathcal{G}_f$ coincides with the lower semi-continuous envelope of $\mathcal{F}_f$ with respect to the $L^1$-convergence. Of course, coercivity is crucial for the upper bound, in that it guarantees that *any* sequence along which $\mathcal{G}_f$ is bounded has a convergent subsequence. However, it may be dropped when only a lower bound is needed, provided it is a-priori known that a sequence has good convergence properties. Indeed, the following holds:

**Proposition 2.6.** *Let $f$ satisfy (H1)-(H3) and let $\mathbf{v}_n\in W^{1,1}(\Omega;\mathbb{S}^{N-1}_+)$ such that $\mathbf{v}_n\rightharpoonup\mathbf{v}\in BV(\Omega;\mathbb{S}^{N-1}_+)$ and $\mathbf{v}_n\to\mathbf{v}$ in $L^1(\Omega;\mathbb{S}^{N-1}_+)$. Then*

$$\mathcal{G}_f(\mathbf{v})\leq\liminf_{n\to\infty}\mathcal{F}_f(\mathbf{v}_n).$$

Given Lemma 2.5, the proof follows line by line that of [2, Proposition 5.1], and the difference between the target spaces ($\mathbb{S}^{N-1}$ versus $\mathbb{S}^{N-1}_+$) is harmless: therefore we omit it.

## 3. Existence of solutions

In this section we introduce the notion of solutions to (1.4) and we prove their existence.

As is mentioned in Section 2.3, on its jump set $J_{\mathbf{u}}$ a function $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ has a jump discontinuity between two distinct values, $\mathbf{u}_+$ and $\mathbf{u}_-$, and the value of the precise representative of $\mathbf{u}$ is given by $(\mathbf{u}_+ + \mathbf{u}_-)/2$. Note that $(\mathbf{u}_+ + \mathbf{u}_-)/2$ is the midpoint of the segment which connects $\mathbf{u}_+$ and $\mathbf{u}_-$. In this sense, $(\mathbf{u}_+ + \mathbf{u}_-)/2$ has natural counterparts in $\mathbb{S}^{N-1}$ endowed with the standard geodesic distance $d_g$ on $\mathbb{S}^{N-1}$, the *geodesic midpoints*:

**Definition 3.1.** Let $\mathbb{A}$ be a geodesically convex subset of $\mathbb{S}^{N-1}$ and let $\mathbf{u}_-, \mathbf{u}_+ \in \mathbb{A}$. A point $\mathbf{u}_g \in \mathbb{A}$ is called a *geodesic midpoint* on $\mathbb{A}$ between $\mathbf{u}_-$ and $\mathbf{u}_+$ if:

(i) $\mathbf{u}_g$ belongs to a greatest circle of $\mathbb{S}^{N-1}$ passing through $\mathbf{u}_-$ and $\mathbf{u}_+$;
(ii) $d_g(\mathbf{u}_g, \mathbf{u}_-) = d_g(\mathbf{u}_g, \mathbf{u}_+)$.

In particular, when $\mathbb{A} = \mathbb{S}^{N-1}_+$, geodesic midpoints are uniquely determined:

$$\mathbf{u}_g = \frac{\mathbf{u}_- + \mathbf{u}_+}{|\mathbf{u}_- + \mathbf{u}_+|} \quad \text{for all } \mathbf{u}_-, \mathbf{u}_+ \in \mathbb{S}^{N-1}_+.$$

Thus we can introduce the notion of *geodesic representative of* $\mathbf{u} \in BV(\Omega; \mathbb{S}^{N-1}_+)$:

**Definition 3.2.** Let $\mathbf{u} \in BV(\Omega; \mathbb{S}^{N-1}_+)$. The *geodesic representative* $\mathbf{u}_g : \Omega \backslash (S_{\mathbf{u}} \backslash J_{\mathbf{u}}) \to \mathbb{S}^{N-1}_+$ of $\mathbf{u}$ is defined by

$$\mathbf{u}_g = \begin{cases} \mathbf{u}^* & \text{on } \Omega \backslash S_{\mathbf{u}} \\ \mathbf{u}^*/|\mathbf{u}^*| & \text{on } J_{\mathbf{u}}. \end{cases}$$

Note that $\mathbf{u}_g \in BV(\Omega; \mathbb{S}^{N-1}_+)$ since $\mathbf{u}_+$ and $\mathbf{u}_-$ are $\mathcal{H}^{m-1}$-measurable on $J_{\mathbf{u}}$ (see [5, Prop. 3.69]). Hence the following Radon measures are well defined:

$$|\mathbf{u}^*||D\mathbf{u}| := |\nabla \mathbf{u}|\mathcal{L}^m + |D^c\mathbf{u}| + |\mathbf{u}^*||\mathbf{u}_+ - \mathbf{u}_-|\, \mathcal{H}^{m-1} \llcorner_{J_{\mathbf{u}}}, \tag{3.1}$$

$$\mathbf{u}_g|D\mathbf{u}| := \mathbf{u}\,(|\nabla \mathbf{u}|\mathcal{L}^m + |D^c\mathbf{u}|) + \mathbf{u}_g|\mathbf{u}_+ - \mathbf{u}_-|\, \mathcal{H}^{m-1} \llcorner_{J_{\mathbf{u}}}. \tag{3.2}$$

Moreover, $\mathbf{u}_g|D\mathbf{u}| \in \mathcal{M}_{\mathcal{H}}(\Omega; \mathbb{R}^N)$ (see §2.4).

**Remark 3.3.** As shown in the proof of Lemma 3.9, the projections onto $\mathbb{S}^{N-1}_+$ of the mollifications of $\mathbf{u}$ point-wise converge to $\mathbf{u}_g$ in $\Omega$. In this sense, the geodesic representative $\mathbf{u}_g$ is a natural representative for BV-vector fields with values into $\mathbb{S}^{N-1}_+$.

We are now ready to introduce the concept of solution for (1.4).

**Definition 3.4.** Let $\mathbb{A} = \mathbb{S}_+^{N-1}$, $T > 0$, and $\mathbf{u}_0 \in BV(\Omega; \mathbb{S}_+^{N-1})$. A function

$$\mathbf{u} \in L^\infty(0,T; BV(\Omega;\mathbb{R}^N)) \cap C(0,T;L^1(\Omega;\mathbb{R}^N)), \quad \mathbf{u}_t \in L^2(0,T;L^2(\Omega;\mathbb{R}^N))$$

is a solution to (1.4) in $Q_T$ if $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u} \in \mathbb{S}_+^{N-1}$ a.e. in $Q_T$, and there exists a matrix-valued function $\mathbf{Z} \in L^\infty(Q_T, \mathbb{R}^{N\times m})$, with $\|\mathbf{Z}\|_\infty \leq 1$ and $\mathbf{Z}(t) \in \mathbf{X}_\mathcal{M}^N(\Omega)$ for almost all $t \in (0,T)$, such that

$$\mathbf{u}_t(t) - \operatorname{div} \mathbf{Z}(t) = \mathbf{u}(t)_g\, |D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0,T], \tag{3.3}$$

$$\mathbf{u}_t(t) \wedge \mathbf{u}(t) = \operatorname{div}(\mathbf{Z}(t) \wedge \mathbf{u}(t)) \quad \text{in } L^2(\Omega; \Lambda_2(\mathbb{R}^N)) \quad \text{for a.e. } t \in [0,T], \tag{3.4}$$

$$\mathbf{Z}^T \mathbf{u} = 0 \quad \text{a.e. in } Q_T, \tag{3.5}$$

and

$$[\mathbf{Z}(t), \nu] = 0 \quad \mathcal{H}^{m-1}\text{-a.e. on } \partial\Omega \text{ for a.e. } t \in [0,T]. \tag{3.6}$$

The next observation clarifies the concept of solution in Definition 3.4.

**Proposition 3.5.** *Let* $\mathbf{u}$ *be a solution of* (1.4) *in the sense of Definition 3.4. Then*

$$\mathbf{Z}(t) : D\mathbf{u}(t) = |\mathbf{u}(t)^*|\, |D\mathbf{u}(t)| \quad \textit{as measures for a.e. } t \in (0,T). \tag{3.7}$$

*Proof.* We take any $\varphi \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned}
\int_\Omega \varphi \, \mathrm{d}(\mathbf{Z}(t) : D\mathbf{u}(t)) &\overset{(2.15)}{=} -\int_\Omega \varphi \mathbf{u}(t)^* \cdot \mathrm{d}(\operatorname{div} \mathbf{Z}(t)) - \int_\Omega (\mathbf{Z}(t)^T \mathbf{u}(t)) \cdot \nabla\varphi \, \mathrm{d}x \\
&\overset{(3.5),(3.3)}{=} \int_\Omega \varphi \mathbf{u}^*(t) \cdot \mathrm{d}\left(\mathbf{u}_t(t) + \mathbf{u}(t)_g |D\mathbf{u}(t)|\right) \\
&= \int_\Omega \varphi \mathbf{u}(t)^* \cdot \mathrm{d}\left(\mathbf{u}(t)_g |D\mathbf{u}(t)|\right),
\end{aligned}$$

where in the last line we have used the facts that $|\mathbf{u}(t)| = 1$, $\mathbf{u}_t \in L^2(Q_T; \mathbb{R}^N)$ and the fact that $\mathbf{u}(t)_g |D\mathbf{u}(t)| \in \mathcal{M}_\mathcal{H}(\Omega; \mathbb{R}^N)$ a.e. $t \in (0,T)$. Finally, by (3.1) we get

$$\begin{aligned}
\int_\Omega \varphi \, \mathrm{d}(\mathbf{Z}(t) : D\mathbf{u}(t)) &= \int_\Omega \varphi \, \mathrm{d}(|\nabla \mathbf{u}(t)| \mathcal{L}^m + |D^c(\mathbf{u}(t))|) + \int_{J_{\mathbf{u}(t)}} \varphi |\mathbf{u}(t)^*| |\mathbf{u}(t)_+ - \mathbf{u}(t)_-| \, \mathrm{d}\mathcal{H}^{m-1} \\
&= \int_\Omega \varphi \, \mathrm{d}(|\mathbf{u}(t)^*|\, |D\mathbf{u}(t)|).
\end{aligned}$$

□

Our main result is the following existence theorem.

**Theorem 3.6.** *For any $T > 0$ and any $\mathbf{u}_0 \in BV(\Omega; \mathbb{S}^{N-1}_+)$ there exists a solution $\mathbf{u}$ to (1.4) in the sense of Definition 3.4.*

To prove Theorem 3.6 we need to recall or establish several results. The first one follows as a particular case from [9, Thm. 4.1, (4.24) and (4.25)] (with $\lambda = \mathbf{g} = 0$ and $p = 2$).

**Proposition 3.7.** *Let $\varepsilon > 0$, $T > 0$ and $\alpha > 0$. If $\mathbf{u}_0^\varepsilon \in W^{1,2}(\Omega; \mathbb{S}^{N-1})$, then there exists*

$$\mathbf{u}^\varepsilon \in L^\infty(0,T; W^{1,2}(\Omega;\mathbb{R}^N)) \cap W^{1,2}(0,T; L^2(\Omega;\mathbb{R}^N))$$

*such that $\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}_0^\varepsilon$,*

$$|\mathbf{u}^\varepsilon| = 1 \quad \text{a.e. in } Q_T, \tag{3.8}$$

*and $\mathbf{u}^\varepsilon$ is a weak solution to*

$$\begin{cases} \mathbf{u}_t^\varepsilon = \operatorname{div} \mathbf{Z}^\varepsilon + \boldsymbol{\mu}^\varepsilon & \text{in } Q_T \\ [\mathbf{Z}^\varepsilon, \nu] = 0 & \text{in } S_T, \end{cases} \tag{3.9}$$

*where*

$$\mathbf{Z}^\varepsilon = \varepsilon^\alpha \nabla \mathbf{u}^\varepsilon + \frac{\nabla \mathbf{u}^\varepsilon}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}} \quad \text{and} \quad \boldsymbol{\mu}^\varepsilon = \varepsilon^\alpha \mathbf{u}^\varepsilon |\nabla \mathbf{u}^\varepsilon|^2 + \mathbf{u}^\varepsilon \frac{|\nabla \mathbf{u}^\varepsilon|^2}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}, \tag{3.10}$$

*in the sense that*

$$\int_0^T \int_\Omega (\mathbf{u}_t^\varepsilon \cdot \mathbf{v} + \mathbf{Z}^\varepsilon : \nabla \mathbf{v} - \boldsymbol{\mu}^\varepsilon \cdot \mathbf{v})\, \mathrm{d}x\, \mathrm{d}t = 0 \quad \textit{for all } \mathbf{v} \in C^1(\overline{Q}_T; \mathbb{R}^N). \tag{3.11}$$

*Furthermore, the following holds:*

$$(\mathbf{Z}^\varepsilon)^T \mathbf{u}^\varepsilon = 0 \quad \text{a.e. in } Q_T\,, \tag{3.12}$$

$$\mathbf{u}_t^\varepsilon \cdot \mathbf{u}^\varepsilon = 0 \quad \text{a.e. in } Q_T\,, \tag{3.13}$$

$$\mathbf{u}_t^\varepsilon \wedge \mathbf{u}^\varepsilon = \operatorname{div}(\mathbf{Z}^\varepsilon \wedge \mathbf{u}^\varepsilon), \tag{3.14}$$

$$J_\alpha^\varepsilon(\mathbf{u}^\varepsilon(t)) + \int_0^t \int_\Omega |\mathbf{u}_t^\varepsilon|^2\, \mathrm{d}x\, \mathrm{d}s \le J_\alpha^\varepsilon(\mathbf{u}_0) \quad \text{for a.e. } t \in [0,T], \tag{3.15}$$

*where the energy functional $J_\alpha^\varepsilon$ is defined as*

$$J_\alpha^\varepsilon(\mathbf{v}) := \varepsilon^\alpha \int_\Omega |\nabla \mathbf{v}(x)|^2\, \mathrm{d}x + \int_\Omega \sqrt{|\nabla \mathbf{v}(x)|^2 + \varepsilon^2}\, \mathrm{d}x, \quad \mathbf{v} \in W^{1,2}(\Omega;\mathbb{R}^N),$$

*and a positive $\varepsilon$-independent constant $C$ exists such that*

$$\|\operatorname{div} \mathbf{Z}^\varepsilon\|_{L^2(0,T;L^1(\Omega;\mathbb{R}^N))} \le C, \tag{3.16}$$

$$\|\operatorname{div}(\mathbf{Z}^\varepsilon \wedge \mathbf{u}^\varepsilon)\|_{L^2(0,T;L^2(\Omega;\Lambda_2(\mathbb{R}^N)))} \le C, \tag{3.17}$$

$$\varepsilon^{\frac{\alpha}{2}} \|\nabla \mathbf{u}^\varepsilon(t)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{N\times m}))} \le C. \tag{3.18}$$

We next show that if $\mathbf{u}_0^\varepsilon$ takes values in the first hyper-octant, then also $\mathbf{u}^\varepsilon$ does:

**Lemma 3.8.** *If $\mathbf{u}_0^\varepsilon \in W^{1,2}(\Omega;\mathbb{S}^{N-1}_+)$, then the weak solution to Problem* (3.9) *given by Proposition 3.7 verifies $\mathbf{u}^\varepsilon \in \mathbb{S}^{N-1}_+$ a.e. in $Q_T$.*

*Proof.* Let $(s)^- = \max\{0,-s\}$ and let $(\mathbf{u}^\varepsilon)^- = ((u^{\varepsilon,1})^-,\dots,(u^{\varepsilon,N})^-)$. Pick a sequence of smooth functions $\mathbf{v}_n$ such that $\mathbf{v}_n \to (\mathbf{u}^\varepsilon)^-$ in $L^2(0,T;W^{1,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$ as $n \to +\infty$. Choosing $\mathbf{v} = \mathbf{v}_n$ in (3.11) and passing to the limit as $n \to +\infty$, we obtain on one hand

$$\int_0^T \int_\Omega (\mathbf{u}^\varepsilon)^- \cdot \mathbf{u}_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \left(\varepsilon^\alpha + \frac{1}{\sqrt{\varepsilon^2 + |\nabla \mathbf{u}^\varepsilon|^2}}\right) |\nabla (\mathbf{u}^\varepsilon)^-|^2 (1 - |(\mathbf{u}^\varepsilon)^-|^2)\,\mathrm{d}x\,\mathrm{d}t \ge 0.$$

On the other hand, since $\mathbf{u}^\varepsilon \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^N))$,

$$0 \le \int_0^T \int_\Omega (\mathbf{u}^\varepsilon)^- \cdot \mathbf{u}_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left(|(\mathbf{u}_0)^-|^2 - |(\mathbf{u}^\varepsilon(T))^-|^2\right)\,\mathrm{d}x = -\int_\Omega |(\mathbf{u}^\varepsilon(T))^-|^2\,\mathrm{d}x,$$

hence the negative part of each component remains 0 for all times. $\square$

Provided $\alpha$ is large enough, any function in $BV(\Omega;\mathbb{S}^{N-1}_+)$ can be approximated in $W^{1,2}(\Omega;\mathbb{S}^{N-1}_+)$ in such a way that the initial energy is controlled.

**Lemma 3.9.** *Given $\mathbf{u}_0 \in BV(\Omega;\mathbb{S}^{N-1}_+)$ and $\alpha > m$, there exist $\mathbf{u}_0^\varepsilon \in W^{1,2}(\Omega;\mathbb{S}^{N-1}_+)$ such that*

$$\mathbf{u}_0^\varepsilon \to \mathbf{u}_0 \quad \textit{in} \quad L^p(\Omega;\mathbb{R}^N) \textit{ for all } \; p<\infty \qquad \textit{as} \;\; \varepsilon \to 0 \tag{3.19}$$

$$\mathbf{u}_0^\varepsilon \to (\mathbf{u}_0)_g \quad \mathcal{H}^{m-1}\textit{-a.e. in} \;\; \Omega \qquad \textit{as} \;\; \varepsilon \to 0$$

$$J_\alpha^\varepsilon(\mathbf{u}_0^\varepsilon) \to L < +\infty \qquad \text{as } \varepsilon \to 0. \tag{3.20}$$

*Proof.* We will construct $\mathbf{u}_0^\varepsilon$ as the projection onto $\mathbb{S}^{N-1}_+$ of the convolution of a suitable extension $T\mathbf{u}_0$ of $\mathbf{u}_0$ with a standard mollifier. In order to do this, we proceed as in [5, Proposition 3.21], to which we refer for further details (see also [13, Theorem 9.7]).

Since $\overline{\Omega}$ is compact, there exists a finite collection $\{R_i\}_{i\in I}$ of open rectangles, whose union $B$ contains $\overline{\Omega}$, which satisfies the following property: for any $i \in I$, either

(a) $R_i \subset \Omega$

or

(b) $\partial\Omega \cap R_i$ is the graph of a Lipschitz function defined on one face $L_i$ of $R_i$ and the closure of $\partial\Omega \cap R_i$ intersects neither $\overline{L_i}$ nor the closure of the face opposite to $L_i$.

Let $\Omega_i = \Omega \cap R_i$. In case (b), up to a translation, a rotation, and an homothety, we have $R_i = L_i \times (-1, 1)$ with $\Omega_i$ on the upper side of $R_i$ (i.e., $\Omega_i = \{x = (y, z) : \ z > \phi_i(y)\}$). A vertical deformation $\varphi : R_i \to R_i$ exists such that $\varphi(\Omega_i) = R_i^+ = L_i \times (0, 1)$ and both $\varphi$ and its inverse are Lipschitz. Given $u \in BV(\Omega)$, the operator $T_i : R_i \to \mathbb{R}$ is defined as the identity in case (a) and as

$$T_i(u) = T_i'(u \circ \varphi^{-1}) \circ \varphi, \quad \text{where } \ T_i'(u)(y, z) = u(y, |z|)$$

in case (b). Note that, since $|\mathbf{u}_0| = 1$ a.e. in $\Omega$, $\varphi$ and its inverse are Lipschitz, and $T_i'$ does not change the value of $u$, we have that

$$U_i := \{x \in R_i : \ |(T_i(u_0^1), \ldots, T_i(u_0^N))| \neq 1\} \quad \text{has zero measure.}$$

Let $\{\eta_i\}_{i\in I}$ be a partition of unity relative to $\{R_i\}_{i\in I}$, i.e. $supp(\eta_i) \subset R_i$, $0 \leq \eta_i \leq 1$ for any $i \in I$ and there exists $r > 0$ such that $\sum_{i\in I} \eta_i \equiv 1$ in a neighbourhood of $\overline{\Omega}$ containing $\Omega \oplus B_r$. We now define

$$T\mathbf{u}_0 : B = \bigcup_{i\in I} R_i \to \mathbb{R}^N, \quad T\mathbf{u}_0 := \left( \sum_{i\in I} T_i(u_0^1)\eta_i, \ldots, \sum_{i\in I} T_i(u_0^N)\eta_i \right).$$

It is readily checked that $T \in BV(\Omega \oplus B_r; \mathbb{R}^N)$. Let now $k > 0$ be the cardinality of $I$ and $U = \cup_{i\in I} U_i$ (a set of zero measure). We observe that

$$|T\mathbf{u}_0(x)| \geq \frac{1}{k} \quad \text{for all } \ x \in \Omega \oplus B_r \setminus U. \tag{3.21}$$

Indeed, for each $x \in (\Omega \oplus B_r) \setminus U$ there exists $i(x) \in I$ such that $\eta_{i(x)}(x) \geq \frac{1}{k}$: since each component of $\mathbf{u}_0$ is non-negative and $x \notin U_{i(x)}$,

$$|T\mathbf{u}_0(x)|^2 \geq \frac{1}{k^2} \left( (T_{i(x)}(u_0^1))^2 + \ldots + (T_{i(x)}(u_0^N))^2 \right) = \frac{1}{k^2}.$$

Given $\varepsilon < r$, let $\rho_\varepsilon(x) := \varepsilon^{-m} \rho(\frac{x}{\varepsilon})$ be a standard mollifier. As is well known (see e.g. [5, Remark 3.22]) $T\mathbf{u}_0 \star \rho_\varepsilon$ converges to $T\mathbf{u}_0$ strictly in $BV(\Omega; \mathbb{R}^N)$ and strongly in $L^1(\Omega; \mathbb{R}^N)$. Since $\|T\mathbf{u}_0 \star \rho_\varepsilon\|_\infty \leq 1$, the last convergence upgrades to

$$T\mathbf{u}_0 \star \rho_\varepsilon \to T\mathbf{u}_0 \quad \text{in } \ L^p(\Omega; \mathbb{R}^N) \quad \text{for all } \ 1 \leq p < \infty. \tag{3.22}$$

By (3.21) and since $(T(\mathbf{u}_0))^\ell \geq 0$ for $\ell = 1, \ldots, N$, a direct computation shows that

$$|T\mathbf{u}_0 \star \rho_\varepsilon(x)| \geq \frac{1}{k\sqrt{N}} \quad \text{for all } \ x \in \Omega. \tag{3.23}$$

In addition, it follows from [5, Corollary 3.80] that $T\mathbf{u}_0 \star \rho_\varepsilon \to (T\mathbf{u}_0)^* = \mathbf{u}_0^*$ pointwise in $\Omega \setminus (S_{\mathbf{u}_0} \setminus J_{\mathbf{u}_0})$. Together with (3.23), this implies that

$$\mathbf{u}_0^\varepsilon := \frac{T\mathbf{u}_0 \star \rho_\varepsilon}{|T\mathbf{u}_0 \star \rho_\varepsilon|} \to (\mathbf{u}_0)_g \quad \mathcal{H}^{m-1}\text{-a.e. in } \Omega. \tag{3.24}$$

Furthermore, (3.23) and (3.22) easily imply that $\mathbf{u}_0^\varepsilon \to \mathbf{u}_0$ in $L^p(\Omega;\mathbb{R}^N)$ for all $1 \le p < \infty$. Finally, applying the chain rule and (3.23), [5, Prop. 3.2], and [5, Thm. 2.2(b)] (in this order), we see that

$$\int_\Omega |\nabla \mathbf{u}_0^\varepsilon| \,\mathrm{d}x \le C \int_\Omega |\nabla (T\mathbf{u}_0 \star \rho_\varepsilon)| \,\mathrm{d}x = C \int_\Omega |(DT\mathbf{u}_0) \star \rho_\varepsilon| \,\mathrm{d}x \le C|DT\mathbf{u}_0|(\Omega \oplus B_\varepsilon). \tag{3.25}$$

Similarly

$$\int_\Omega |\nabla \mathbf{u}_0^\varepsilon|^2 \,\mathrm{d}x \le C \int_\Omega |(DT\mathbf{u}_0) \star \rho_\varepsilon|^2 \,\mathrm{d}x \le C\|(DT\mathbf{u}_0) \star \rho_\varepsilon\|_\infty \int_\Omega |(DT\mathbf{u}_0) \star \rho_\varepsilon| \,\mathrm{d}x,$$

and using the definition of $\rho_\varepsilon$ we conclude that

$$\varepsilon^\alpha \int_\Omega |\nabla \mathbf{u}_0^\varepsilon|^2 \,\mathrm{d}x \le C\varepsilon^{\alpha - m} \left(|DT\mathbf{u}_0|(\Omega \oplus B_\varepsilon)\right)^2. \tag{3.26}$$

Inequalities (3.25) and (3.26), together with (3.24), complete the proof. ☐

We are now ready to prove Theorem 3.6.

*Proof of Theorem 3.6.* We proceed along various steps. In the first step, we use the previous lemmas, together with standard compactness arguments, to identify a triplet $(\mathbf{u}, \mathbf{Z}, \boldsymbol{\mu})$. In the second step we identify $\boldsymbol{\mu}$ in terms of $\mathbf{u}$ and $\mathbf{Z}$, which automatically yields an upper bound on $|\boldsymbol{\mu}|$. In the third step, collecting the information of the previous two steps, we note that $\mathbf{u}$ satisfies all the properties in Definition 3.4 except for

$$\boldsymbol{\mu}(t) = \mathbf{u}(t)_g |D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0,T], \tag{3.27}$$

to which the rest of the proof is devoted. In the fourth step we use the lower semi-continuity results in §2.6 to prove a lower bound on $\boldsymbol{\mu}(t)$ over the diffuse support of $|D\mathbf{u}(t)|$. In the fifth step we revise the blow-up argument given in [26, 27] to obtain a lower bound on $\boldsymbol{\mu}(t)$ over $J_{\mathbf{u}(t)}$. Finally, in the sixth step we complete the proof.

**Step 1: Passage to the limit.** Let $\mathbf{u}_0^\varepsilon$ and $\mathbf{u}^\varepsilon$ as given by Lemma 3.9 and Proposition 3.7, respectively. By Lemma 3.8, $\mathbf{u}^\varepsilon \in \mathbb{S}_+^{N-1}$ a.e. in $Q_T$. By (3.8), (3.20), and (3.15), a positive constant $C$ (independent of $\varepsilon$) exists such that

$$\sup_{t\in(0,T)} \|\mathbf{u}^\varepsilon\|_{W^{1,1}(\Omega)} \leq C, \tag{3.28}$$

$$\|\mathbf{u}_t^\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))} \leq C. \tag{3.29}$$

We recall that $BV(\Omega;\mathbb{R}^N)$ is compactly embedded in $L^1(\Omega;\mathbb{R}^N)$ ([5, Theorem 3.23]). Hence the Aubin-Simon compactness criterion [42, Corollary 8.4], together with (3.28) and (3.29), implies that

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{in } C(0,T;L^1(\Omega;\mathbb{R}^N)) \text{ and a.e. in } Q_T \tag{3.30}$$

for a subsequence. By the lower semi-continuity of the total variation [5, Remark 3.5], (3.30) and (3.15) imply that

$$\mathbf{u} \in L^\infty(0,T;BV(\Omega;\mathbb{R}^N)). \tag{3.31}$$

From (3.30) and (3.19) we have

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{3.32}$$

and, using also (3.8),

$$|\mathbf{u}| = 1 \quad \text{a.e. in } Q_T. \tag{3.33}$$

By a standard interpolation argument, the boundedness of $\mathbf{u}^\varepsilon$ in $L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^N))$ and (3.30) imply that

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{in } L^p(0,T;L^q(\Omega;\mathbb{R}^N)) \text{ for all } p,q \in [1,\infty) \text{ and a.e. in } Q_T. \tag{3.34}$$

Moreover, it follows from (3.29) that

$$\mathbf{u}_t^\varepsilon \rightharpoonup \mathbf{u}_t \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^N)). \tag{3.35}$$

By (3.15), (3.20), and (3.18), a subsequence exists such that

$$\varepsilon^\alpha \nabla\mathbf{u}^\varepsilon \rightharpoonup 0 \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{N\times m})), \tag{3.36}$$

$$\frac{\nabla\mathbf{u}^\varepsilon}{\sqrt{|\nabla\mathbf{u}^\varepsilon|^2+\varepsilon^2}} \overset{*}{\rightharpoonup} \mathbf{Z} \qquad \text{in } L^\infty(Q_T;\mathbb{R}^{N\times m}). \tag{3.37}$$

Recalling the definition (3.10) of $\mathbf{Z}^\varepsilon$, by (3.36) and (3.37) we obtain that

$$\mathbf{Z}^\varepsilon \rightharpoonup \mathbf{Z} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{N\times m})), \tag{3.38}$$

and from (3.37) we also obtain that

$$\|\mathbf{Z}\|_{L^\infty(Q_T)} \leq 1. \tag{3.39}$$

Since $\{\boldsymbol{\mu}^\varepsilon\}$ is bounded in $L^\infty(0,T;L^1(\Omega;\mathbb{R}^N))$, and

$$L^\infty(0,T;L^1(\Omega;\mathbb{R}^N)) \subset L^\infty(0,T;\mathcal{M}(\Omega;\mathbb{R}^N)) \subset \left(L^1(0,T;C_0(\Omega;\mathbb{R}^N))\right)'$$

(see §2.2), we have

$$\boldsymbol{\mu}^\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{\mu} \quad \text{in } \left(L^1(0,T;C_0(\Omega;\mathbb{R}^N))\right)'. \tag{3.40}$$

Analogously, by (3.16),

$$\operatorname{div}\mathbf{Z}^\varepsilon \overset{*}{\rightharpoonup} \operatorname{div}\mathbf{Z} \quad \text{in } \left(L^2(0,T;C_0(\Omega;\mathbb{R}^N))\right)'. \tag{3.41}$$

Passing to the limit as $\varepsilon \to 0$ in $(3.9)_1$ (using (3.35), (3.41), and (3.40)) we obtain

$$\mathbf{u}_t - \operatorname{div}\mathbf{Z} = \boldsymbol{\mu} \quad \text{in } \left(L^2(0,T;C_0(\Omega;\mathbb{R}^N))\right)'. \tag{3.42}$$

Passing to the limit as $\varepsilon \to 0$ in (3.12) (using (3.38) and (3.34)), in (3.13) (using (3.35) and (3.34)), and in (3.14) (using (3.35), (3.38) and (3.34)), we get that

$$\mathbf{Z}^T\mathbf{u} = 0 \qquad \text{a.e. in } Q_T, \tag{3.43}$$

$$\mathbf{u}_t \cdot \mathbf{u} = 0 \qquad \text{a.e. in } Q_T, \tag{3.44}$$

$$\mathbf{u}_t(t) \wedge \mathbf{u}(t) = \operatorname{div}(\mathbf{Z}(t) \wedge \mathbf{u}(t)) \qquad \text{in } L^2(\Omega;\Lambda_2(\mathbb{R}^N)) \text{ for a.e. } t \in [0,T]. \tag{3.45}$$

**Step 2: The intermediate identification of $\boldsymbol{\mu}$ and its upper bound.** We claim that

$$\boldsymbol{\mu} = *(*(\mathbf{Z} \wedge \mathbf{u}) \wedge D\mathbf{u}) \in L^\infty(0,T;\mathcal{M}(\Omega;\mathbb{R}^N)) \tag{3.46}$$

with

$$|\boldsymbol{\mu}(t)| \le |D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0,T]. \tag{3.47}$$

Let

$$\mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_m) := *(\mathbf{Z} \wedge \mathbf{u}) \in L^\infty(Q_T;(\Lambda_{N-2}(\mathbb{R}^N))^m). \tag{3.48}$$

We have

$$*(\mathbf{u}_t \wedge \mathbf{u}) \overset{(3.45)}{=} *(\operatorname{div}(\mathbf{Z} \wedge \mathbf{u})) \overset{(2.18)}{=} \operatorname{div}\left(*(\mathbf{Z} \wedge \mathbf{u})\right) = \operatorname{div}\mathcal{A}, \tag{3.49}$$

hence $\mathcal{A}(t) \in \mathbf{X}_2(\Omega;\Lambda_{N-2}(\mathbb{R}^N))$ for a.e. $t$. Therefore, by Lemma 2.3, $*(\mathcal{A}(t) \wedge D\mathbf{u}(t)) \in \mathcal{M}(\Omega;\mathbb{R}^N)$ for almost every $t$ with

$$\begin{aligned} |*(\mathcal{A}(t) \wedge D\mathbf{u}(t))| &\overset{(2.21)}{\le} \|\mathcal{A}(t)\|_\infty |D\mathbf{u}(t)| \overset{(2.9)}{=} \|\mathbf{Z}(t) \wedge \mathbf{u}(t)\|_\infty |D\mathbf{u}(t)| \\ &\overset{(2.10),(3.33),(3.39)}{\le} |D\mathbf{u}(t)| \end{aligned} \tag{3.50}$$

and in addition

$$*(\mathcal{A}(t) \wedge D\mathbf{u}(t)) \overset{(2.22)}{=} -*\left(\operatorname{div}\mathcal{A}(t) \wedge \mathbf{u}(t)\right)\mathcal{L}^m + \operatorname{div}(*(\mathcal{A}(t) \wedge \mathbf{u}(t))). \tag{3.51}$$

It follows from (3.50) and (3.31) that

$$*(\mathcal{A}\wedge D\mathbf{u})\in L^\infty(Q_T;\mathcal{M}(\Omega;\mathbb{R}^N)). \tag{3.52}$$

Using (3.51), we see that

$$\begin{aligned}\mathbf{u}_t &\overset{(3.33)}{=} |\mathbf{u}|^2\mathbf{u}_t \overset{(2.7)}{=} (\mathbf{u}_t\cdot\mathbf{u})\mathbf{u} - *(*(\mathbf{u}_t\wedge\mathbf{u})\wedge\mathbf{u})\\ &\overset{(3.44),(3.49)}{=} -*(\operatorname{div}\mathcal{A}\wedge\mathbf{u})\\ &\overset{(3.51)}{=} *(\mathcal{A}\wedge D\mathbf{u}) - \operatorname{div}((*(\mathcal{A}\wedge\mathbf{u}))).\end{aligned} \tag{3.53}$$

On the other hand,

$$\begin{aligned}-*(\mathcal{A}\wedge\mathbf{u}) &\overset{(3.48)}{=} -(*(*(\mathbf{Z}\wedge\mathbf{u})\wedge\mathbf{u}))\\ &= -(*(*(\mathbf{z}_1\wedge\mathbf{u})\wedge\mathbf{u}),\ldots,*(*(\mathbf{z}_m\wedge\mathbf{u})\wedge\mathbf{u}))\\ &\overset{(2.7)}{=} \big(|\mathbf{u}|^2\mathbf{z}_1-(\mathbf{u}\cdot\mathbf{z}_1)\mathbf{u},\ldots,|\mathbf{u}|^2\mathbf{z}_m-(\mathbf{u}\cdot\mathbf{z}_m)\mathbf{u}\big)\\ &= \mathbf{Z}-(\mathbf{Z}^T\mathbf{u})\mathbf{u} \overset{(3.43)}{=} \mathbf{Z}.\end{aligned} \tag{3.54}$$

Combining (3.53) and (3.54) we obtain

$$\mathbf{u}_t = *(\mathcal{A}\wedge D\mathbf{u}) + \operatorname{div}\mathbf{Z},$$

which together with (3.42), (3.48), and (3.52), implies (3.46). Finally, (3.47) follows immediately from (3.46) and (3.50).

**Step 3: Intermediate summary.** It follows from (3.42), (3.35), and (3.46) that $\operatorname{div}\mathbf{Z}\in L^2(0,T;\mathcal{M}(\Omega;\mathbb{R}^N))$. Hence (3.42) upgrades to

$$\mathbf{u}_t(t)-\operatorname{div}\mathbf{Z}(t)=\boldsymbol{\mu}(t)\quad\text{as measures for a.e. } t\in[0,T], \tag{3.55}$$

In particular,

$$\mathbf{Z}(t)\in\mathbf{X}^N_{\mathcal{M}}(\Omega)\qquad\text{for a.e. } t\in[0,T]. \tag{3.56}$$

Thus the weak trace $[\mathbf{Z}(t),\nu]$ on $\partial\Omega$ of the normal component of $\mathbf{Z}(t)$ is well defined, and for all smooth $\mathbf{w}$ we have

$$\begin{aligned}\int_0^T\int_{\partial\Omega}[\mathbf{Z},\nu]\cdot\mathbf{w}\,\mathrm{d}\mathcal{H}^{m-1}\,\mathrm{d}t &\overset{(2.14)}{=} \int_0^T\left(\int_\Omega \mathbf{w}\cdot\,\mathrm{d}(\operatorname{div}\mathbf{Z})+\mathbf{Z}:\nabla\mathbf{w}\,\mathrm{d}x\right)\mathrm{d}t\\ &\overset{(3.38),(3.41)}{=} \lim_{\varepsilon\to0}\left(\int_0^T\int_\Omega(\mathbf{w}\cdot\operatorname{div}\mathbf{Z}^\varepsilon+\mathbf{Z}^\varepsilon:\nabla\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\right)\overset{(3.9)_2}{=}0.\end{aligned}$$

Hence

$$[\mathbf{Z}(t),\nu]=0\quad\mathcal{H}^{m-1}\text{- a.e. on }\partial\Omega\text{ for a.e. } t\in[0,T]. \tag{3.57}$$

Collecting (3.31), (3.30), (3.35), (3.32), (3.33), (3.39), (3.56), (3.43), (3.45), and (3.57), we see that all the properties of $\mathbf{u}$ stated in Definition 3.4 are satisfied except for (3.3). In view of (3.55), in order to prove (3.3) it remains to show (3.27).

**Step 4: The lower bound on $\boldsymbol{\mu}$ over the diffuse support of $|D\mathbf{u}|$.** In view of (3.47), $\boldsymbol{\mu}(t)$ can be decomposed as

$$\boldsymbol{\mu}(t) = \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\left(|\nabla \mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|\right) + \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}|\mathbf{u}(t)_+ - \mathbf{u}(t)_-|\mathcal{H}^{m-1}\llcorner J_{\mathbf{u}(t)}, \quad (3.58)$$

where $\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \in (L^1(\Omega; |D\mathbf{u}(t)|))^N$ denotes the Radon-Nikodým derivative of $\boldsymbol{\mu}(t)$ with respect to $|D\mathbf{u}(t)|$. We claim that

$$\mathbf{u}(t)\cdot\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \geq 1 \quad (|\nabla\mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \ \Omega. \quad (3.59)$$

We first notice that

$$\mu^{\varepsilon,\ell} \overset{(3.10)}{\geq} u^{\varepsilon,\ell}(\sqrt{\varepsilon^2 + |\nabla\mathbf{u}^\varepsilon|^2} - \varepsilon) \geq u^{\varepsilon,\ell}(|\nabla \mathbf{u}^\varepsilon| - \varepsilon), \quad \ell = 1, \dots N. \quad (3.60)$$

For any $\varphi \in C(\overline{\Omega}; [0,\infty))$, $0 \leq \psi \in L^1((0,T))$, and $\ell \in \{1,\dots,N\}$, we have

$$\begin{aligned}\int_0^T \psi(t)\left(\int_\Omega \varphi \,\mathrm{d}\mu^\ell(t)\right)\mathrm{d}t &\overset{(3.40)}{=} \lim_{\varepsilon\to 0}\int_0^T \psi(t)\left(\int_\Omega \varphi\mu^{\varepsilon,\ell}(t)\,\mathrm{d}x\right)\mathrm{d}t \\ &\overset{(3.60)}{\geq} \liminf_{\varepsilon\to 0}\int_0^T \psi(t)\left(\int_\Omega \varphi u^{\varepsilon,\ell}(t)|\nabla\mathbf{u}^\varepsilon(t)|\,\mathrm{d}x\right)\mathrm{d}t.\end{aligned}$$

We claim that for a.e. $t \in (0,T)$

$$\mathbf{u}^\varepsilon(t) \rightharpoonup \mathbf{u}(t) \quad \text{in } BV(\Omega;\mathbb{R}^N) \text{ as } \varepsilon \to 0. \quad (3.61)$$

Indeed, in view of (3.28), for a.e. $t$ we have $\|\mathbf{u}^\varepsilon(t)\|_{W^{1,1}(\Omega)} < \infty$. Take any of such $t$ and assume by contradiction that (3.61) does not hold, i.e. that $\mathbf{u}^\varepsilon(t) \not\rightharpoonup \mathbf{u}(t)$ for a subsequence. By (3.28), a further subsequence would exist such that $\mathbf{u}^\varepsilon(t) \rightharpoonup \tilde{\mathbf{u}}$ for some $\tilde{\mathbf{u}} \in BV(\Omega;\mathbb{R}^N)$. On the other hand, because of (3.30), $\mathbf{u}^\varepsilon(t) \to \mathbf{u}(t)$ in $L^1(\Omega;\mathbb{R}^N)$: hence $\tilde{\mathbf{u}} = \mathbf{u}(t)$, a contradiction.

In view of (3.61) and of (3.30), we may apply Proposition 2.6 to the right-hand side of (3.60) with $f = f_{\varphi,\ell} : \Omega \times \mathbb{R}^N \to [0,\infty)$ defined by $f_{\varphi,\ell}(x,\mathbf{s}) := \varphi(x) s^\ell |\xi|$. This implies that

$$\begin{aligned}
&\int_0^T \psi(t) \left( \int_\Omega \varphi \, \mathrm{d}\mu^\ell(t) \right) \mathrm{d}t \\
&\quad \geq \int_0^T \psi(t) \left( \int_\Omega \varphi u^\ell(t) \left( |\nabla \mathbf{u}(t)| \, \mathrm{d}x + \mathrm{d}|D^c \mathbf{u}(t)| \right) + \int_{J_{\mathbf{u}(t)}} \varphi K_t^\ell \, \mathrm{d}\mathcal{H}^{m-1} \right) \mathrm{d}t,
\end{aligned}$$

where

$$K_t^\ell = \inf \left\{ \int_0^1 \gamma^\ell(\tau) |\dot{\boldsymbol{\gamma}}(\tau)| \, \mathrm{d}\tau : \boldsymbol{\gamma} \in W^{1,1}((0,1); \mathbb{S}_+^{N-1}) \ , \ \boldsymbol{\gamma}(0) = \mathbf{u}(t)_- \, , \boldsymbol{\gamma}(1) = \mathbf{u}(t)_+ \right\}. \tag{3.62}$$

By the arbitrariness of $\psi$ we conclude that

$$\int_\Omega \varphi \, \mathrm{d}\mu^\ell(t) \geq \int_\Omega \varphi u^\ell(t) \left( |\nabla \mathbf{u}(t)| \, \mathrm{d}x + \mathrm{d}|D^c \mathbf{u}(t)| \right) + \int_{J_{\mathbf{u}(t)}} \varphi K_t^\ell \, \mathrm{d}\mathcal{H}^{m-1} \quad \text{for all } \varphi \in C(\overline{\Omega}) \tag{3.63}$$

for a.e. $t \in [0,T]$ and for all $\ell \in \{1,\ldots,N\}$. Recalling (3.58), (3.63) yields

$$\frac{\mu^\ell(t)}{|D\mathbf{u}(t)|} \ \geq \ u^\ell(t) \quad (|\nabla \mathbf{u}(t)| \mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \ \Omega$$

for a.e. $t \in [0,T]$ and all $\ell = 1,\ldots N$. The inequality (3.59) now follows at once recalling that $|\mathbf{u}(t)| = 1$ a.e. in $\Omega$.

**Remark 3.10.** On the jump set $J_{\mathbf{u}(t)}$, the above argument would yield

$$|\mathbf{u}_+(t) - \mathbf{u}_-(t)| \mathbf{u}(t)_g \cdot \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \geq \mathbf{u}(t)_g \cdot (K_t^1, \ldots, K_t^N) \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}}(t).$$

Unfortunately, by (3.62) and obvious properties of the infimum,

$$\begin{aligned}
&\mathbf{u}(t)_g \cdot (K_t^1, \ldots, K_t^N) \\
&\quad \leq \ \inf \left\{ \int_0^1 \mathbf{u}(t)_g \cdot \boldsymbol{\gamma}(\tau) |\dot{\boldsymbol{\gamma}}(\tau)| \, \mathrm{d}\tau : \boldsymbol{\gamma} \in W^{1,1}((0,1); \mathbb{S}_+^{N-1}), \boldsymbol{\gamma}(0) = \mathbf{u}(t)_-, \boldsymbol{\gamma}(1) = \mathbf{u}(t)_+ \right\},
\end{aligned} \tag{3.64}$$

whilst, as we shall see, it is the right-hand side of (3.64) which yields the sharp lower bound on the jump part (cf. (3.70)-(3.73) below). On the other hand, we can not use the results in Proposition 2.6 directly on $\mathbf{u}^* \cdot \boldsymbol{\mu}^\varepsilon$, since $\mathbf{u}^*$ is a discontinuous function (though a very special one). This motivates the discussion that follows.

**Step 5: The lower bound on $\boldsymbol{\mu}$ over $J_{\mathbf{u}(t)}$.** We claim that

$$\mathbf{u}(t)_g \cdot \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \geq 1 \quad \mathcal{H}^{m-1}\text{-a.e. on } \; J_{\mathbf{u}(t)} \; . \tag{3.65}$$

It follows from (3.8) and (3.28) that for a.e. $t \in [0, T]$ there exists a subsequence $\varepsilon_k$ such that

$$\mathbf{u}^{\varepsilon_k}(t)|\nabla \mathbf{u}^{\varepsilon_k}(t)| \stackrel{*}{\rightharpoonup} \tilde{\boldsymbol{\mu}}(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N). \tag{3.66}$$

Then (3.60) and the fact that $u^{\varepsilon,\ell} \geq 0$ imply that

$$\mu^\ell(t) \geq \tilde{\mu}^\ell(t) \geq 0 \quad \text{as measures for a.e. } t \in [0, T], \quad \ell \in \{1, \ldots, N\}. \tag{3.67}$$

Hereafter, we argue for a fixed $t$ and we do not specify dependence on $t$ for notational convenience. Using Radon-Nikodým's Theorem [5, Theorem 1.28] we decompose $\tilde{\boldsymbol{\mu}}$ into four mutually orthogonal measures:

$$\tilde{\boldsymbol{\mu}} \;\; = \;\; \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}|\nabla \mathbf{u}|\mathcal{L}^N + \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}|D^c\mathbf{u}| + \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}|\mathbf{u}_+ - \mathbf{u}_-|\mathcal{H}^{m-1} \llcorner J_{\mathbf{u}} + (\tilde{\boldsymbol{\mu}})^o \, ,$$

with $(\tilde{\boldsymbol{\mu}})^o \perp |D\mathbf{u}|$. It follows from (3.67) and (3.58) that

$$\mathbf{u}_g \cdot \frac{\boldsymbol{\mu}}{|D\mathbf{u}|} \geq \mathbf{u}_g \cdot \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|} \quad \mathcal{H}^{m-1}\text{-a.e. on } \; J_{\mathbf{u}}.$$

Therefore, (3.65) is proved once we have shown that

$$\mathbf{u}_g \cdot \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|} \geq 1 \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}}. \tag{3.68}$$

To prove (3.68) we apply the same blow-up argument as in [26, Section 3].

From the Besicovitch Differentation Theorem [5, Theorem 2.22], for $\mathcal{H}^{m-1}$-a.e. $x_0 \in J_{\mathbf{u}}$ we have

$$\frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}(x_0) = \lim_{\delta \to 0} \frac{\tilde{\boldsymbol{\mu}}(x_0 + \delta Q_{\nu_{\mathbf{u}(x_0)}})}{|\mathbf{u}_+ - \mathbf{u}_-|\mathcal{H}^{m-1}(J_{\mathbf{u}} \cap (x_0 + \delta Q_{\nu_{\mathbf{u}(x_0)}}))},$$

where $Q_\varsigma$ is defined in §2.6. On the other hand, by [26, Lemma 2.6], for $\mathcal{H}^{m-1}$ a.e. $x_0 \in J_{\mathbf{u}}$ we also have

$$\lim_{\delta \to 0} \frac{1}{\delta^{m-1}} \int_{(x_0 + \delta Q_{\nu_{\mathbf{u}(x_0)}}) \cap J_{\mathbf{u}}} |\mathbf{u}_+(x) - \mathbf{u}_-(x)| \, d\mathcal{H}^{m-1} = |\mathbf{u}_+(x_0) - \mathbf{u}_-(x_0)|.$$

Therefore, letting for notational convenience

$$M = |\mathbf{u}_+(x_0) - \mathbf{u}_-(x_0)| \, ,$$

we obtain that

$$M \frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}(x_0) = \lim_{\delta \to 0} \frac{1}{\delta^{m-1}} \int_{x_0 + \delta Q_{\nu_{\mathbf{u}(x_0)}}} d\tilde{\boldsymbol{\mu}}.$$

Then, for any $\ell \in \{1, \ldots, N\}$, since the function $\chi_{x_0+\delta Q_{\nu_{\mathbf{u}(x_0)}}}$ is upper semi-continuous with compact support in $\Omega$ if $\delta$ is sufficiently small, we have

$$\begin{aligned} M\frac{\tilde{\mu}^\ell}{|D\mathbf{u}|}(x_0) &= \lim_{\delta\to 0}\frac{1}{\delta^{m-1}}\int_{x_0+\delta Q_{\nu_{\mathbf{u}(x_0)}}} \mathrm{d}\tilde{\mu}^\ell \\ &\overset{(3.66)}{\geq} \limsup_{\delta\to 0}\limsup_{k\to\infty}\frac{1}{\delta^{m-1}}\int_{x_0+\delta Q_{\nu_{\mathbf{u}(x_0)}}} u^{\varepsilon_k,\ell}|\nabla\mathbf{u}^{\varepsilon_k}|\,\mathrm{d}x \\ &= \limsup_{\delta\to 0}\limsup_{k\to\infty}\int_{Q_{\nu_{\mathbf{u}(x_0)}}} v^\ell_{\delta,k}(y)|\nabla\mathbf{v}_{\delta,k}(y)|\,\mathrm{d}y, \end{aligned} \tag{3.69}$$

where

$$\mathbf{v}_{\delta,k}(y) := \mathbf{u}^{\varepsilon_k}(x_0+\delta y).$$

We now observe that $\mathbf{v}_{\delta,k} \in W^{1,1}(Q_{\nu_{\mathbf{u}(x_0)}};\mathbb{R}^N)$ and (see [26, formula (3.2)])

$$\lim_{\delta\to 0}\lim_{k\to\infty}\|\mathbf{v}_{\delta,k}-\mathbf{w}_0\|_{L^1(Q_{\nu_{\mathbf{u}(x_0)}};\mathbb{R}^N)} = 0,$$

where

$$\mathbf{w}_0(y) := \begin{cases} \mathbf{u}_+(x_0) & \text{if } \ y\cdot\nu_{\mathbf{u}(x_0)} > 0 \\ \\ \mathbf{u}_-(x_0) & \text{if } \ y\cdot\nu_{\mathbf{u}(x_0)} < 0. \end{cases}$$

Then, by a diagonalization argument we may extract a subsequence $\mathbf{v}_k$ converging to $\mathbf{w}_0$ in $L^1(Q_{\nu_{\mathbf{u}(x_0)}};\mathbb{R}^N)$. It follows from (3.69) that

$$M\frac{\tilde{\mu}^\ell}{|D\mathbf{u}|}(x_0) \geq \lim_{k\to\infty}\int_{Q_{\nu_{\mathbf{u}(x_0)}}} v^\ell_k(y)|\nabla\mathbf{v}_k(y)|\,\mathrm{d}y.$$

Since $(u^\ell)^* \geq 0$ for all $\ell \in \{1, \ldots, N\}$, this implies that

$$M\left(\mathbf{u}_g\cdot\frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}\right)(x_0) \geq \lim_{k\to\infty}\int_{Q_{\nu_{\mathbf{u}(x_0)}}} \mathbf{u}_g(x_0)\cdot\mathbf{v}_k(y)|\nabla\mathbf{v}_k(y)|\,\mathrm{d}y.$$

The function $f(x,\mathbf{s}) = f(\mathbf{s}) = \mathbf{u}_g(x_0)\cdot\mathbf{s}$ is continuous, non-negative and bounded. Then, applying Lemma 2.5 we obtain a new sequence

$$\mathbf{w}_k \in \mathcal{P}(\mathbf{u}_+(x_0),\mathbf{u}_-(x_0),\nu_{\mathbf{u}(x_0)})$$

(with $\mathcal{P}$ given by (2.24)) converging to $\mathbf{w}_0$ in $L^1(Q_{\nu_{\mathbf{u}(x_0)}};\mathbb{R}^N)$ and such that

$$M\left(\mathbf{u}_g\cdot\frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}\right)(x_0) \ \geq \ \limsup_{k\to\infty}\int_{Q_{\nu_{\mathbf{u}(x_0)}}} \mathbf{u}_g(x_0)\cdot\mathbf{w}_k(y)|\nabla\mathbf{w}_k(y)|\,\mathrm{d}y$$

We may now apply Lemma 2.4. It follows from (2.23) and (2.25) that

$$M\left(\mathbf{u}_g\cdot\frac{\tilde{\boldsymbol{\mu}}}{|D\mathbf{u}|}\right)(x_0) \quad\geq \inf_{\boldsymbol{\gamma}\in\tilde{\Gamma}_N(\mathbf{u}_+(x_0),\mathbf{u}_-(x_0))} J_N[\mathbf{u}_+(x_0),\mathbf{u}_-(x_0)](\boldsymbol{\gamma}), \tag{3.70}$$

where

$$J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma}) := \int_0^1 \mathbf{v}_g\cdot\boldsymbol{\gamma}(t)|\dot{\boldsymbol{\gamma}}(t)|\,\mathrm{d}t, \quad \mathbf{v}_g := \frac{\mathbf{v}_0+\mathbf{v}_1}{|\mathbf{v}_0+\mathbf{v}_1|} \tag{3.71}$$

and

$$\tilde{\Gamma}_N(\mathbf{v}_0,\mathbf{v}_1) := \left\{\boldsymbol{\gamma}\in W^{1,1}((0,1);\mathbb{S}^{N-1}_+):\ \boldsymbol{\gamma}(0)=\mathbf{v}_0,\ \boldsymbol{\gamma}(1)=\mathbf{v}_1\right\}. \tag{3.72}$$

In view of (3.70), (3.68) and therefore (3.65) follows from

$$\inf_{\boldsymbol{\gamma}\in\tilde{\Gamma}_N(\mathbf{u}_+(x_0),\mathbf{u}_-(x_0))} J_N[\mathbf{u}_+(x_0),\mathbf{u}_-(x_0)](\boldsymbol{\gamma}) \geq M = |\mathbf{u}_+(x_0)-\mathbf{u}_-(x_0)|. \tag{3.73}$$

This last inequality will be proved in Theorem 4.1, to which the next Section is devoted.

**Step 6: Conclusion.** Recalling (3.58), the upper bound on $|\boldsymbol{\mu}|$ given by (3.47) immediately implies that

$$\left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\right| \leq 1 \quad |D\mathbf{u}(t)|\text{-a.e. in } \Omega \tag{3.74}$$

for a.e. $t\in[0,T]$. In particular, recalling (3.33),

$$\begin{aligned}\mathbf{u}(t)\cdot\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} &\leq 1 \quad (|\nabla\mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \Omega,\\ \mathbf{u}(t)_g\cdot\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} &\leq 1 \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}(t)}\end{aligned}$$

for a.e. $t\in[0,T]$. Combining these inequalities with the lower bounds in (3.59) and (3.65), we obtain

$$\mathbf{u}(t)\cdot\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} = 1 \quad (|\nabla\mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \Omega, \tag{3.75}$$

$$\mathbf{u}(t)_g\cdot\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} = 1 \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}(t)} \tag{3.76}$$

for a.e. $t\in[0,T]$. We are now ready to complete the proof. By (2.2), we have

$$\left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\wedge\mathbf{u}(t)\right|^2 = \left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\right|^2 - \left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\cdot\mathbf{u}(t)\right|^2 \quad (|\nabla\mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \Omega \tag{3.77}$$

and

$$\left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\wedge\mathbf{u}(t)_g\right|^2 = \left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\right|^2|\mathbf{u}(t)_g|^2 - \left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\cdot\mathbf{u}(t)_g\right|^2 \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}(t)}. \tag{3.78}$$

Now, from (3.74),(3.75) and (3.77), we get

$$\left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\wedge \mathbf{u}(t)\right|^2 = 0 \quad (|\nabla \mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \Omega,$$

and from (3.74),(3.76) and (3.78), we get

$$\left|\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|}\wedge \mathbf{u}(t)_g\right|^2 = 0 \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}(t)}.$$

Hence the wedge products on the left-hand side are zero.

Therefore, applying (2.7) and using once more the equalities in (3.75) and (3.76), we conclude that

$$\begin{aligned}
\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} &= |\mathbf{u}(t)|^2\,\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} = \mathbf{u}(t) \quad (|\nabla \mathbf{u}(t)|\mathcal{L}^m + |D^c(\mathbf{u}(t))|)\text{-a.e. in } \Omega,\\
\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} &= |\mathbf{u}(t)_g|^2\,\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} = \mathbf{u}(t)_g \quad \mathcal{H}^{m-1}\text{-a.e. on } J_{\mathbf{u}(t)}.
\end{aligned}$$

Plugging these expressions into (3.58) we obtain (3.27), and the proof is complete. $\square$

## 4. A non-convex variational problem

In this section we study the minimization of a non-convex functional and as a result we set the inequality (3.73). We will prove the following:

**Theorem 4.1.** *Let $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{S}^{N-1}$ and let $J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma})$ and $\tilde{\Gamma}_N(\mathbf{v}_0,\mathbf{v}_1)$ be given by* (3.71) *and* (3.72)*. If $\mathbf{v}_0\cdot\mathbf{v}_1 \geq 0$, then*

$$\min_{\boldsymbol{\gamma}\in\tilde{\Gamma}_N(\mathbf{v}_0,\mathbf{v}_1)} J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma}) = |\mathbf{v}_1 - \mathbf{v}_0|.$$

Of course, it suffices to consider $\mathbf{v}_0 \neq \mathbf{v}_1$. Up to a rotation, we may assume without loss of generality that

$$\mathbf{v}_g = \frac{\mathbf{v}_0+\mathbf{v}_1}{|\mathbf{v}_0+\mathbf{v}_1|} = \mathbf{e}_N \quad \text{and} \quad \mathbf{v}_0,\mathbf{v}_1 \in \operatorname{span}\{\mathbf{e}_{N-1},\mathbf{e}_N\}.$$

Since $\mathbf{v}_g$ is the geodesic midpoint and $\mathbf{v}_0\cdot\mathbf{v}_1 \geq 0$, there exists $\theta_0 \in (0,\pi/4]$ such that

$$\mathbf{v}_0 = (0,\ldots,0,\sin\theta_0,\cos\theta_0) \quad \text{and} \quad \mathbf{v}_1 = (0,\ldots,0,-\sin\theta_0,\cos\theta_0).$$

Then

$$|\mathbf{v}_1-\mathbf{v}_0| = 2\sin\theta_0. \tag{4.1}$$

A curve which attains the equality in (4.3) is easily obtained: it is just the geodesic with respect to the standard metric of $\mathbb{S}^{N-1}$.

**Lemma 4.2.** *Let* $\boldsymbol{\gamma}_{min}(t) = (0,\ldots,0,\sin((1-2t)\theta_0),\cos((1-2t)\theta_0)$. *Then* $J(\boldsymbol{\gamma}_{min}) = 2\sin\theta_0$.

After the above-mentioned rotation, $\mathbb{S}^{N-1}_+$ is transformed into a geodesic simplex $\mathcal{T}$ in $\mathbb{S}^{N-1}$. We consider a larger set of curves: let $\mathcal{P}_N(\mathbf{v}_0,\mathbf{v}_1)$ be given by

$$\mathcal{P}_N(\mathbf{v}_0,\mathbf{v}_1) = \{\mathbf{v}\in\mathbb{S}^{N-1} : \mathbf{v}\cdot\mathbf{v}_0 \ge 0, \quad \mathbf{v}\cdot\mathbf{v}_1 \ge 0\}$$

and let

$$\Gamma_N(\mathbf{v}_0,\mathbf{v}_1) = \left\{\boldsymbol{\gamma}\in W^{1,1}((0,1);\mathcal{P}_N(\mathbf{v}_0,\mathbf{v}_1)) : \ \boldsymbol{\gamma}(0)=\mathbf{v}_0, \quad \boldsymbol{\gamma}(1)=\mathbf{v}_1\right\}.$$

Then,

$$J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma}) = \int_0^1 \gamma^N(t)|\dot{\boldsymbol{\gamma}}(t)|\,\mathrm{d}t\,, \quad \text{for } \boldsymbol{\gamma}=(\gamma^1,\ldots,\gamma^N)\in\Gamma_N(\mathbf{v}_0,\mathbf{v}_1)\,.$$

Hence, recalling (4.1) and Lemma 4.2, it suffices to prove that

$$\inf_{\boldsymbol{\gamma}\in\Gamma_N(\mathbf{v}_0,\mathbf{v}_1)} J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma}) \ge 2\sin\theta_0. \tag{4.2}$$

We now show that the problem in $\mathbb{S}^{N-1}$ may be reduced to the same problem in $\mathbb{S}^2$. Let

$$\bar{\mathbf{v}}_i = (0,(-1)^i\sin\theta_0,\cos\theta_0), \quad i=0,1$$

denote the projection of $\mathbf{v}_i$ onto the three-dimensional subspace $\mathrm{span}\{\mathbf{e}_{N-2},\mathbf{e}_{N-1},\mathbf{e}_N\}$.

**Lemma 4.3.** *Let* $N\ge 4$. *Then*

$$\inf_{\boldsymbol{\gamma}\in\Gamma_N(\mathbf{v}_0,\mathbf{v}_1)} J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma}) \ge \inf_{\boldsymbol{\gamma}\in\Gamma_3(\bar{\mathbf{v}}_0,\bar{\mathbf{v}}_1)} J_3[\bar{\mathbf{v}}_0,\bar{\mathbf{v}}_1](\boldsymbol{\gamma}).$$

*Proof.* For any $\boldsymbol{\gamma}\in\Gamma_N(\mathbf{v}_0,\mathbf{v}_1)$, consider the curve

$$\tilde{\boldsymbol{\gamma}} = \left(0,\ldots,0,\sqrt{(\gamma^1)^2+\cdots+(\gamma^{N-2})^2},\gamma^{N-1},\gamma^N\right).$$

Clearly $\tilde{\boldsymbol{\gamma}}\in W^{1,1}((0,1),\mathbb{S}^{N-1})$. Since $\mathbf{v}_0$ and $\mathbf{v}_1$ belong to $\mathrm{span}\{\mathbf{e}_{N-1},\mathbf{e}_N\}$ and the projections of $\tilde{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ onto $\mathrm{span}\{\mathbf{e}_{N-1},\mathbf{e}_N\}$ coincide, $\tilde{\boldsymbol{\gamma}}\in W^{1,1}((0,1);\mathcal{P}_N(\mathbf{v}_0,\mathbf{v}_1))$ and the end-point conditions are satisfied. Therefore $\tilde{\boldsymbol{\gamma}}\in\Gamma_N(\mathbf{v}_0,\mathbf{v}_1)$. In addition, letting

$$\delta = (\gamma^1,\ldots,\gamma^{N-2}),$$

we may apply the chain rule given in [4, Corollary 3.2]: since $\delta \in W^{1,1}((0,1);\mathbb{R}^{N-2})$ and $f(x) = |x|$ is a Lipschitz function with $f(0) = 0$, then $|\delta| = f \circ \delta \in W^{1,1}((0,1);\mathbb{R})$, for almost every $t \in (0,1)$ the restriction of the function $f$ to the affine space

$$T_t^\delta := \{y \in \mathbb{R}^{N-2} \ : \ y = \delta(t) + \eta\dot{\delta}(t) \ \text{ for some } \ \eta \in \mathbb{R}\},$$

is differentiable at $\delta(t)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}|\delta| = \nabla(f|_{T_t^\delta})(\delta(t)) \cdot \dot{\delta}(t) \quad \text{for a.e.} \quad t \in (0,1).$$

Since the Lipschitz constant of $f$ is 1, we get that $\left|\frac{\mathrm{d}}{\mathrm{d}t}|\delta|\right| \le \left|\frac{\mathrm{d}}{\mathrm{d}t}\delta\right|$. Hence $\left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{\gamma}}\right| \le \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\gamma}\right|$, which implies that $J_N[\mathbf{v}_0,\mathbf{v}_1](\tilde{\boldsymbol{\gamma}}) \le J_N[\mathbf{v}_0,\mathbf{v}_1](\boldsymbol{\gamma})$ since $\tilde{\gamma}^N = \gamma^N$. Arguing as above, we also see that

$$\bar{\boldsymbol{\gamma}} = \left(\sqrt{(\gamma^1)^2 + \cdots + (\gamma^{N-2})^2}, \gamma^{N-1}, \gamma^N\right)$$

belongs to $\Gamma_3(\bar{\mathbf{v}}_0,\bar{\mathbf{v}}_1)$. Since $J_N[\mathbf{v}_0,\mathbf{v}_1](\tilde{\boldsymbol{\gamma}}) = J_3[\bar{\mathbf{v}}_0,\bar{\mathbf{v}}_1](\bar{\boldsymbol{\gamma}})$, the proof is complete. □

We hereafter let

$$\mathbf{v}_i := \bar{\mathbf{v}}_i, \ J := J_3[\mathbf{v}_0,\mathbf{v}_1], \ \mathcal{P} := \mathcal{P}_3(\mathbf{v}_0,\mathbf{v}_1), \ \Gamma := \Gamma_3(\mathbf{v}_0,\mathbf{v}_1).$$

Because of (4.2) and of Lemma 4.3, it suffices to prove that

$$\inf_{\boldsymbol{\gamma}\in\Gamma} J(\boldsymbol{\gamma}) \ge 2\sin\theta_0. \tag{4.3}$$

Proving (4.3) is far from trivial, both since the functional is genuinely non-convex (see Lemmas 4.9 and 4.10) and since the curves are constrained to an octant of the sphere. However, it is exactly for this reason that the lower bound holds:

**Remark 4.4.** In the extremal case $\theta_0 = \frac{\pi}{4}$, there are exactly two paths $\boldsymbol{\gamma}$ such that $J(\boldsymbol{\gamma}) = 2\sin\theta_0$: the one given in Lemma 4.2 and the one which coincides with $\partial\mathcal{P}$ (see the Introduction or Lemma 4.14 with $\varphi_0 = 0$ and $\varphi_1 = \pi/2$). If the constraint is removed, then the lower bound (4.3) does not hold any more: for instance, the curve

$$\boldsymbol{\gamma}(t) := \begin{cases} (0,\sin\theta,\cos\theta)\,, & \theta = \theta_0 + 3t\left(\frac{\pi}{2} - \theta_0\right) \in \left(\theta_0, \frac{\pi}{2}\right) & \text{if } \ 0 \le t \le \frac{1}{3} \\ (\sin\varphi,\cos\varphi,0)\,, & \varphi = 3\pi\left(t - \frac{1}{3}\right) \in (0,\pi) & \text{if } \ \frac{1}{3} < t \le \frac{2}{3} \\ (0,-\sin\theta,\cos\theta)\,, & \theta = \frac{\pi}{2} + 3\left(t - \frac{2}{3}\right)\left(\theta_0 - \frac{\pi}{2}\right) \in \left(\theta_0, \frac{\pi}{2}\right) & \text{if } \ \frac{2}{3} < t \le 1 \end{cases}$$

is such that

$$J(\boldsymbol{\gamma}) = 2\int_0^{1/3} \cos\theta\,|\dot{\theta}|\,\mathrm{d}t = 2\int_{\theta_0}^{\pi/2} \cos\theta\,\mathrm{d}\theta = 2(1 - \sin\theta_0),$$

hence $J(\boldsymbol{\gamma}) = 2(1-\sin\theta_0) < 2\sin\theta_0$ if $\theta_0 > \frac{\pi}{6}$.

We will often use spherical coordinates centered at $(0,0,1)$:

$$X(\varphi,\theta) := (\sin\varphi\sin\theta, \cos\varphi\sin\theta, \cos\theta). \tag{4.4}$$

In this case $\mathbf{v}_0 = X(0,\theta_0)$, $\mathbf{v}_1 = X(\pi,\theta_0)$, the functional reads as

$$J(\boldsymbol{\gamma}) = \int_0^1 \cos\theta(t)\sqrt{(\dot\theta(t))^2 + (\dot\varphi(t))^2\sin^2\theta(t)}\,\mathrm{d}t, \quad \text{where } \ \boldsymbol{\gamma}(t) = X(\varphi(t),\theta(t)), \tag{4.5}$$

and the constraint $\boldsymbol{\gamma}(t) \in \mathcal{P}$ is equivalent to

$$\theta(t) \in [0,\pi/2], \quad \theta(t) \le \arctan\left(\frac{1}{\tan\theta_0|\cos\varphi(t)|}\right) =: \theta^*(\varphi(t)). \tag{4.6}$$

It is convenient to cut-off from $\mathcal{P}$ a neighborhood of $z=0$: in this way, the new constraint has a smooth boundary and the density of $J$ does not degenerate. Thus, let $\theta^*_\varepsilon \in C^\infty(\mathbb{R})$ be such that:

$$\begin{aligned} &\theta^*_\varepsilon \text{ is } \pi\text{-periodic, even w.r.t. } \pi/2, \text{ increasing in } (0,\pi/2),\\ &\theta^*_\varepsilon(\varphi) = \theta^*(\varphi) \text{ if } |\tfrac{\pi}{2}-\varphi| \ge \varepsilon,\ \theta^*_\varepsilon < \pi/2, \text{ and } \left|(\theta^*_\varepsilon)'\right| \le C, \end{aligned} \tag{4.7}$$

for some $\varepsilon$-independent positive constant $C$. Note that here and after primes denote differentiation with respect to $\varphi$, and that the latter property of $\theta^*_\varepsilon$ may be fulfilled since $\theta^*$ is Lipschitz-continuous. Let now

$$\begin{aligned} \mathcal{P}_\varepsilon &:= \{X(\varphi,\theta):\ \varphi \in [0,2\pi],\ 0 \le \theta \le \theta^*_\varepsilon(\varphi)\},\\ \Gamma_\varepsilon &:= \left\{\boldsymbol{\gamma} \in W^{1,1}((0,1);\mathcal{P}_\varepsilon):\ \boldsymbol{\gamma}(0) = \mathbf{v}_0, \quad \boldsymbol{\gamma}(1) = \mathbf{v}_1\right\}. \end{aligned}$$

In what follows, $\omega(\varepsilon)$ denotes a generic positive universal function which goes to zero as $\varepsilon \to 0$. The next Lemma shows that we may equivalently work on $\mathcal{P}_\varepsilon$:

**Lemma 4.5.** *Assume that*

$$\inf_{\boldsymbol{\gamma}\in\Gamma_\varepsilon} J(\boldsymbol{\gamma}) \ge 2\sin\theta_0 - \omega(\varepsilon). \tag{4.8}$$

*Then* (4.3)*, and therefore Theorem 4.1, hold true.*

*Proof.* Given $\boldsymbol{\gamma} \in \Gamma$, we replace the parts of $\boldsymbol{\gamma}$ which enter into $\mathcal{P}\setminus\mathcal{P}_\varepsilon$ by arcs of $\partial\mathcal{P}_\varepsilon$. More precisely, let

$$I_\varepsilon = \{t \in (0,1) : \boldsymbol{\gamma}(t) \in \mathcal{P}\setminus\mathcal{P}_\varepsilon\}.$$

Since the spherical coordinates (4.4) are a bijection away from the north-pole $(0,0,1)$, in $I_\varepsilon$ we may define $\varphi(t)$ and $\theta(t)$ through $\boldsymbol{\gamma}(t) =: X(\varphi(t),\theta(t))$. Then, we let

$$\boldsymbol{\gamma}_\varepsilon(t) = \begin{cases} \boldsymbol{\gamma}(t) & \text{if } \ t \notin I_\varepsilon \\ (\varphi(t), \theta^*_\varepsilon(\varphi(t))) & \text{if } t \in I_\varepsilon. \end{cases}$$

It follows from (4.7) that

$$\left|\frac{\pi}{2} - \theta^*_\varepsilon(\varphi(t))\right| = \omega(\varepsilon) \tag{4.9}$$

We may now estimate:

$$J(\boldsymbol{\gamma}) - J(\boldsymbol{\gamma}_\varepsilon) \overset{(4.5)}{\geq} -\int_{I_\varepsilon} \cos\theta^*_\varepsilon(\varphi)|\dot\varphi|\sqrt{(\theta^{*\prime}_\varepsilon)^2 + \sin^2\theta^*_\varepsilon(\varphi)}\,\mathrm{d}t \overset{(4.7),(4.9)}{\geq} -\omega(\varepsilon)\int_0^1 |\dot\varphi|\,\mathrm{d}t.$$

Therefore

$$J(\boldsymbol{\gamma}) \overset{(4.8)}{\geq} 2\sin\theta_0 - \omega(\varepsilon)\left(1 + \int_0^1 |\dot\varphi|\,\mathrm{d}t\right).$$

Passing to the limit as $\varepsilon \to 0$, the arbitrariness of $\boldsymbol{\gamma} \in \Gamma$ yields (4.3). □

The rest of the section will be concerned with the proof of (4.8). Let

$$\Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) := \left\{\boldsymbol{\gamma} \in W^{1,1}((0,1);\mathcal{P}_\varepsilon) : \ \boldsymbol{\gamma}(0) = \mathbf{w}_0, \quad \boldsymbol{\gamma}(1) = \mathbf{w}_1\right\} \quad \text{for } \mathbf{w}_0,\mathbf{w}_1 \in \mathcal{P}_\varepsilon.$$

First of all, we note that the following holds:

**Lemma 4.6.** *For any $\mathbf{w}_0,\mathbf{w}_1 \in \mathcal{P}_\varepsilon$ there exists a minimizer $\boldsymbol{\gamma}$ of $J$ in $\Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1)$. Furthermore $\boldsymbol{\gamma} \in W^{1,\infty}((0,1);\mathbb{R}^3)$, $\gamma^3|\dot{\boldsymbol{\gamma}}| = J(\boldsymbol{\gamma})$ a.e. in $[0,1]$, and $\boldsymbol{\gamma}$ is also a minimizer of*

$$E(\boldsymbol{\gamma}) := \int_0^1 (\gamma^3(t))^2|\dot{\boldsymbol{\gamma}}(t)|^2\,\mathrm{d}t$$

*among all $\boldsymbol{\gamma} \in \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) \cap H^1((0,1);\mathbb{R}^3)$.*

Though we could appeal to general results on geodesics for Riemannian manifolds with boundary (see [1] and the references therein), we prefer to give a self-contained proof.

*Proof.* We preliminarily observe that

$$\begin{aligned}&\text{for all } \boldsymbol{\gamma} \in \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) \text{ there exists } \tilde{\boldsymbol{\gamma}} \in \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) \cap W^{1,\infty}((0,1);\mathbb{R}^3)\\ &\text{such that } \tilde\gamma^3(t)|\dot{\tilde{\boldsymbol{\gamma}}}(t)| = L := J(\boldsymbol{\gamma}) \text{ for a.e. } t \in [0,1].\end{aligned} \tag{4.10}$$

To see this, let

$$s(t) = \frac{1}{L}\int_0^t \gamma^3(\tau)|\dot{\boldsymbol{\gamma}}(\tau)|\,\mathrm{d}\tau. \tag{4.11}$$

Obviously $s \in W^{1,1}([0,1];[0,1])$, $s$ in non-decreasing, and $s(t_1) = s(t_2)$ if and only if $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t_1)$ in $[t_1, t_2]$. Therefore, for any $\sigma \in [0,1]$ either there exists a unique $t(\sigma)$ such that $s(t(\sigma)) = \sigma$, or there exists an interval $I_\sigma$ such that $s(t) = \sigma$ for all $t \in I_\sigma$, and in this case we let e.g. $t(\sigma) = \inf I_\sigma$, so that again $s(t(\sigma)) = \sigma$. Let now $\tilde{\boldsymbol{\gamma}}(\sigma) := \boldsymbol{\gamma}(t(\sigma))$. By construction,

$$\boldsymbol{\gamma}(t) = \tilde{\boldsymbol{\gamma}}(s(t)) \quad \text{for all } t \in [0,1]. \tag{4.12}$$

Note that $\tilde{\boldsymbol{\gamma}} \in W^{1,\infty}((0,1);\mathcal{P}_\varepsilon)$. Indeed,

$$\begin{aligned}
|\tilde{\boldsymbol{\gamma}}(\sigma_1) - \tilde{\boldsymbol{\gamma}}(\sigma_2)| \quad &= \quad |\boldsymbol{\gamma}(t(\sigma_1)) - \boldsymbol{\gamma}(t(\sigma_2))| \leq \int_{t(\sigma_1)}^{t(\sigma_2)} |\dot{\boldsymbol{\gamma}}(\tau)|\, \mathrm{d}\tau \\
&\overset{(4.11)}{\leq} \quad \frac{L}{\inf\limits_{\tau \in [t(\sigma_1), t(\sigma_2)]} \gamma^3(\tau)} |s(t(\sigma_1)) - s(t(\sigma_2))| \\
&\overset{(4.7)}{\leq} \quad \frac{L}{\omega(\varepsilon)} |\sigma_1 - \sigma_2|.
\end{aligned} \tag{4.13}$$

Hence, it follows from (4.12) and the chain rule formula given in [5, Theorem 3.101] that

$$\dot{\boldsymbol{\gamma}}(t) = \frac{\mathrm{d}\tilde{\boldsymbol{\gamma}}}{\mathrm{d}s}(s(t))\dot{s}(t) \quad \text{in} \quad L^1((0,1)). \tag{4.14}$$

Therefore

$$L = \int_0^1 \gamma^3(t)|\dot{\boldsymbol{\gamma}}(t)|\, \mathrm{d}t \overset{(4.14),(4.12)}{=} \int_0^1 \tilde{\gamma}^3(s(t)) \left|\frac{\mathrm{d}\tilde{\boldsymbol{\gamma}}}{\mathrm{d}s}(s(t))\right| \dot{s}(t)\, \mathrm{d}t = \int_0^1 \tilde{\gamma}^3(s) \left|\frac{\mathrm{d}\tilde{\boldsymbol{\gamma}}}{\mathrm{d}s}(s)\right| \mathrm{d}s. \tag{4.15}$$

On the other hand, given $s \in [0,1]$ and $\varepsilon > 0$, let $s_1, s_2 \in [0,1]$ with $|s_i - s| < \varepsilon$. Then

$$\tilde{\gamma}^3(s)|\tilde{\boldsymbol{\gamma}}(s_1) - \tilde{\boldsymbol{\gamma}}(s_2)| \overset{(4.13)}{\leq} \frac{\tilde{\gamma}^3(s)}{\inf\limits_{\tau \in [t(s_1), t(s_2)]} \gamma^3(\tau)} L|s_2 - s_1|. \tag{4.16}$$

If $\tau \in [t(s_1), t(s_2)]$ then, by the monotonicity of $s$ and since $s(\tau(s)) = s$, $s(\tau) \in [s_1, s_2]$. Hence

$$\inf_{\tau \in [t(s_1), t(s_2)]} \gamma^3(\tau) \overset{(4.12)}{=} \inf_{\tau \in [t(s_1), t(s_2)]} \tilde{\gamma}^3(s(\tau)) \geq \inf_{s \in [s_1, s_2]} \tilde{\gamma}^3(s). \tag{4.17}$$

Combining (4.16) and (4.17) and passing to the limit as $\varepsilon \to 0$ we obtain

$$\tilde{\gamma}^3(s) \left|\frac{\mathrm{d}\tilde{\boldsymbol{\gamma}}}{\mathrm{d}s}(s)\right| \leq L \quad \text{for a.e. } s \in [0,1],$$

which together with (4.15) concludes the proof of the claim (4.10).

We consider the functional $E$ defined on $G_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) := \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1) \cap H^1((0,1);\mathbb{R}^3)$. By the Cauchy-Schwarz inequality

$$(J(\boldsymbol{\gamma}))^2 \le E(\boldsymbol{\gamma}) \quad \text{for all } \boldsymbol{\gamma} \in G_\varepsilon(\mathbf{w}_0,\mathbf{w}_1). \tag{4.18}$$

Hence $\inf E(\boldsymbol{\gamma}) \ge \inf (J(\boldsymbol{\gamma}))^2$. On the other hand, let $\boldsymbol{\gamma}_n$ be a minimizing sequence for $J$, and let $\tilde{\boldsymbol{\gamma}}_n$ as given by (4.10): then $E(\tilde{\boldsymbol{\gamma}}_n) = (J(\boldsymbol{\gamma}_n))^2$, which means that $\inf E \le \inf J^2$. Therefore

$$\inf_{\boldsymbol{\gamma}\in G_\varepsilon(\mathbf{w}_0,\mathbf{w}_1)} E(\boldsymbol{\gamma}) = \inf_{\boldsymbol{\gamma}\in \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1)} (J(\gamma))^2.$$

The inf on the left-hand side is attained. Indeed, let $\boldsymbol{\gamma}_n$ be a minimizing sequence. By the coercivity of $E$ ensured by the definition of $\mathcal{P}_\varepsilon$, a subsequence (not relabeled) exists such that $\boldsymbol{\gamma}_n \to \boldsymbol{\gamma}$ weakly in $H^1((0,1);\mathcal{P}_\varepsilon)$ and in $C([0,1];\mathcal{P}_\varepsilon)$. Therefore $E(\boldsymbol{\gamma}) \le \liminf\limits_{n\to+\infty} E(\boldsymbol{\gamma}_n)$.

Let then $\boldsymbol{\gamma}_0$ be a minimizer of $E$, and let $\tilde{\boldsymbol{\gamma}}_0$ as given by (4.10). Then

$$E(\boldsymbol{\gamma}_0) \overset{(4.18)}{\ge} (J(\boldsymbol{\gamma}_0))^2 = (J(\tilde{\boldsymbol{\gamma}}_0))^2 = E(\tilde{\boldsymbol{\gamma}}_0),$$

i.e. $\tilde{\boldsymbol{\gamma}}_0$ is also a minimizer of $E$, and

$$(J(\boldsymbol{\gamma}))^2 = (J(\tilde{\boldsymbol{\gamma}}))^2 = E(\tilde{\boldsymbol{\gamma}}) \ge E(\tilde{\boldsymbol{\gamma}}_0) = (J(\tilde{\boldsymbol{\gamma}}_0))^2 \quad \text{for all } \boldsymbol{\gamma} \in \Gamma_\varepsilon(\mathbf{w}_0,\mathbf{w}_1),$$

hence $\tilde{\boldsymbol{\gamma}}_0$ (or $\boldsymbol{\gamma}_0$) is a minimizer of $J$. Therefore $J$ has a minimizer, too. □

The rest of the section is concerned with estimating the length of a minimizer of $J$ in $\Gamma_\varepsilon$ as given by Lemma 4.6, a *shortest path* in what follows. Our first observation concerns those shortest paths which pass through the north pole:

**Lemma 4.7.** *If a shortest path* $\boldsymbol{\gamma}$ *passes through* $(0,0,1)$*, then* $J(\boldsymbol{\gamma}) \ge 2\sin\theta_0$.

*Proof.* Let $t_0$ and $t_1$ be the first, respectively the last, time in which $\boldsymbol{\gamma} = (0,0,1)$. Then, using the spherical coordinates (4.4),

$$\begin{aligned} J(\boldsymbol{\gamma}) &\ge \int_0^{t_0} \cos\theta\sqrt{(\dot\theta)^2 + (\dot\varphi)^2\sin^2\theta}\,\mathrm{d}t + \int_{t_1}^1 \cos\theta\sqrt{(\dot\theta)^2 + (\dot\varphi)^2\sin^2\theta}\,\mathrm{d}t \\ &\ge \int_0^{t_0} \cos\theta|\dot\theta|\,\mathrm{d}t + \int_{t_1}^1 \cos\theta|\dot\theta|\,\mathrm{d}t = \int_0^{t_0} \left|\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta\right|\,\mathrm{d}t + \int_{t_1}^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta\right|\,\mathrm{d}t, \end{aligned}$$

and the Lemma follows since $\theta(t_0) = \theta(t_1) = 0$ and $\theta(0) = \theta(1) = \theta_0$. □

We may therefore restrict our attention to shortest paths not passing through the north pole. There, the spherical coordinates (4.4) are a diffeomorphism. In fact, we may also restrict our attention to those paths for which $\varphi$ is non-decreasing and which are symmetric with respect to $\varphi = \pi/2$. In what follows, we shall call them *symmetric* shortest paths.

**Lemma 4.8.** *Let $\boldsymbol{\gamma} = X(\varphi, \theta)$ be a shortest path not passing through* $(0,0,1)$*. Then $\varphi \in [0,\pi]$ and $\varphi$ is non-decreasing. Moreover, there exists a shortest path $\tilde{\boldsymbol{\gamma}} = X(\tilde{\varphi}, \tilde{\theta})$ not passing through* $(0,0,1)$ *such that $\tilde{\varphi}$ is symmetric with respect to $\pi/2$:*

$$\{(\tilde{\varphi}(t), \tilde{\theta}(t)) : \ t \in [0,1]\} = \{(\pi - \tilde{\varphi}(t), \tilde{\theta}(t)) : \ t \in [0,1]\}.$$

*Proof.* Without loss of generality, $\varphi(0) = 0$ and $\varphi(1) = (2k+1)\pi$ with $k \geq 0$. It is straightforward to see that $\max\{\varphi, 0\}$ and $\min\{\varphi, \pi\}$ both decrease the value of $J$, hence $k = 0$. Analogously, if $t_0 < t_1 < t_2$ are such that $\varphi(t_1) < \varphi(t_2) = \varphi(t_0)$, then replacing $\varphi$ with $\varphi(t_0)$ in $(t_0, t_2)$ decreases the value of $J$. Therefore $\varphi$ is non-decreasing along a shortest path.

In order to construct $\tilde{\boldsymbol{\gamma}}$, we claim that

$$J_1 \ := \int_0^{t_*} \cos\theta \sqrt{(\dot{\theta})^2 + (\dot{\varphi})^2 \sin^2\theta}\, \mathrm{d}t = \int_{t_*}^1 \cos\theta \sqrt{(\dot{\theta})^2 + (\dot{\varphi})^2 \sin^2\theta}\, \mathrm{d}t =: J_2 \tag{4.19}$$

for any $t^* \in (0,1)$ such that $\varphi(t_*) = \pi/2$. Suppose by contradiction that (4.19) does not hold. Then, without loss of generality, we can suppose $J_1 < J_2$. We define $\tilde{\boldsymbol{\gamma}}(t) = X(\tilde{\varphi}(t), \tilde{\theta}(t))$, where

$$\tilde{\theta}(t) = \begin{cases} \theta(t) & \text{if } \ t \leq t_* \\ \theta\left(\frac{t_*(1-t)}{1-t_*}\right) & \text{if } \ t > t_* \end{cases} \quad \text{and} \quad \tilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } \ t \leq t_* \\ \pi - \varphi\left(\frac{t_*(1-t)}{1-t_*}\right) & \text{if } \ t > t_*. \end{cases} \tag{4.20}$$

Then, by letting $\hat{t} = \frac{t_*(1-t)}{1-t_*}$ and using the 1-homogeneity of the integrands with respect to $t$, we see that

$$\begin{aligned} J(\tilde{\boldsymbol{\gamma}}) &= J_1 + \frac{t_*}{1-t_*} \int_{t_*}^1 \cos\theta(\hat{t}) \sqrt{(\dot{\theta}(\hat{t}))^2 + (\dot{\varphi}(\hat{t}))^2 \sin^2\theta(\hat{t})}\, \mathrm{d}t \\ &= J_1 + \int_0^{t_*} \cos\theta(\hat{t}) \sqrt{(\dot{\theta}(\hat{t}))^2 + (\dot{\varphi}(\hat{t}))^2 \sin^2\theta(\hat{t})}\, d\hat{t} = 2J_1 \\ &< J_1 + J_2 = J(\boldsymbol{\gamma}), \end{aligned} \tag{4.21}$$

a contradiction since $\boldsymbol{\gamma}$ is a shortest path. Therefore (4.19) holds. Then, defining $\tilde{\boldsymbol{\gamma}}(t) = X(\tilde{\varphi}(t), \tilde{\theta}(t))$ as in (4.20), it follows from (4.21) that $J(\tilde{\boldsymbol{\gamma}}) = 2J_1 = J_1 + J_2 = J(\boldsymbol{\gamma})$, hence $\tilde{\boldsymbol{\gamma}}$ is also a shortest path. ☐

We now characterize arcs of shortest paths contained in $\mathring{\mathcal{P}}_\varepsilon$.

**Lemma 4.9.** *Let $\boldsymbol{\gamma}$ be a shortest path not passing through $(0,0,1)$ and let $(t_0,t_1)$ be an interval in which $\boldsymbol{\gamma}|_{(t_0,t_1)} \subset \mathring{\mathcal{P}}_\varepsilon$. Then*

$$\frac{\cos\theta(t)\sin^2\theta(t)\dot\varphi(t)}{\sqrt{(\dot\theta(t))^2+(\dot\varphi(t))^2\sin^2\theta(t)}} = K \quad \textit{for all} \;\; t\in(t_0,t_1) \tag{4.22}$$

*for some $K\in[0,1/2]$. If $K=0$, then $\varphi$ is constant. If $K>0$, then $\varphi$ is strictly increasing, the function*

$$(\varphi(t_0),\varphi(t_1)) =: I \ni \varphi \mapsto \theta(t(\varphi)) \tag{4.23}$$

*is a smooth solution of*

$$\theta''\,\sin\theta\cos\theta = \big((\theta')^2(\cos^2\theta+\cos(2\theta)) + \cos(2\theta)\sin^2\theta\big) \tag{4.24}$$

*with*

$$\sin^2\theta(\cos^2\theta\sin^2\theta - K^2) = K^2(\theta')^2 \tag{4.25}$$

*and*

$$J(\boldsymbol{\gamma}\chi_{(t_0,t_1)}) = \int_{\varphi(t_0)}^{\varphi(t_1)} \cos\theta\sqrt{(\theta')^2+\sin^2\theta}\,d\varphi. \tag{4.26}$$

*Proof.* Up to a linear re-parametrization, $\boldsymbol{\gamma}$ is also a minimizer of $J$ in $\Gamma_\varepsilon(\boldsymbol{\gamma}(t_0),\boldsymbol{\gamma}(t_1))$. Hence, by Lemma 4.6, it is also a minimizer of $E$ in $\Gamma_\varepsilon(\boldsymbol{\gamma}(t_0),\boldsymbol{\gamma}(t_1))\cap H^1((0,1);\mathbb{R}^3)$. Since it does not touch the north-pole, $\boldsymbol{\gamma} = X(\varphi,\theta)$ with $\varphi$ and $\theta$ Lipschitz, and

$$E(\boldsymbol{\gamma}\chi_{(t_0,t_1)}) = \int_{t_0}^{t_1} \cos^2\theta(\dot\theta^2+\sin^2\theta\dot\varphi^2)\,\mathrm{d}t.$$

Taking the first variation with respect to $\varphi$, we obtain $\sin^2\theta\cos^2\theta\dot\varphi = H$, and (4.22) follows recalling that $\gamma^3|\dot{\boldsymbol{\gamma}}|$ is constant. Since $\sin\theta>0$ ($\boldsymbol{\gamma}$ does not cross the north pole), $\cos\theta>0$ ($\boldsymbol{\gamma}\in P_\varepsilon$), and $\varphi$ is non-decreasing (by Lemma 4.8), we see that $K\ge 0$. If $K=0$ then $\varphi$ is constant. If $K>0$ then $\dot\varphi>0$ in $(t_0,t_1)$ and we may use $\varphi$ as independent variable: letting $\theta$ as in (4.23), we have $\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\dot\theta}{\dot\varphi} \in L^\infty((t_0,t_1))$ (because of 4.22). Then (4.26) follows at once from (4.5) and the definition of $K$ may be rewritten as

$$\frac{\cos\theta\sin^2\theta}{\sqrt{(\theta')^2+\sin^2\theta}} = K, \tag{4.27}$$

which is equivalent to (4.25). From (4.25) one sees immediately that $K\le 1/2$. Differentiating (4.27) we obtain (4.24) in the sense of distributions, and a bootstrap argument starting from $\theta\in W^{1,\infty}((t_0,t_1))$ yields smoothness. □

If $\boldsymbol{\gamma} = X(\varphi, \theta(\varphi)) : (t_0, t_1) \to \mathbb{S}^2$ is a curve which does not pass through $(0, 0, 1)$ and such that $\varphi \in I := (\varphi(t_0), \varphi(t_1))$ is strictly increasing, following (4.26) we hereafter write (with a slight abuse of notation)

$$J(\boldsymbol{\gamma}\chi_{(t_0,t_1)}) = J_I(\theta) := \int_I \cos\theta \sqrt{(\theta')^2 + \sin^2\theta}\, \mathrm{d}\varphi, \quad J(\theta) := J_{(0,\pi)}(\theta).$$

In view of Lemma 4.9, it is convenient to state a few properties of the solutions to (4.24), some of which are visualized in Figure 1.

**Lemma 4.10.** *Let $\theta$ be any solution of* (4.24) *such that $\theta \in (0, \pi/2)$ at some point of its domain. Then:*

(a) *$\theta$ is globally defined, periodic, and $\theta \in (0, \pi/2)$;*
(b) *within a period, $\theta$ has a unique local (and therefore global) maximum, $\theta_M \geq \pi/4$, and a unique local (and therefore global) minimum, $\theta_m = \pi/2 - \theta_M$, it is symmetric with respect to its maximum (minimum) point;*
(c) *the period $P$ is larger than $\pi$;*
(d) *the length of each interval in which $\theta \leq \frac{\pi}{4}$ is at least $\frac{\pi}{\sqrt{2}}$;*
(e) *$\theta'$ has a unique local (and therefore global) maximum and a unique local (and therefore global) minimum.*

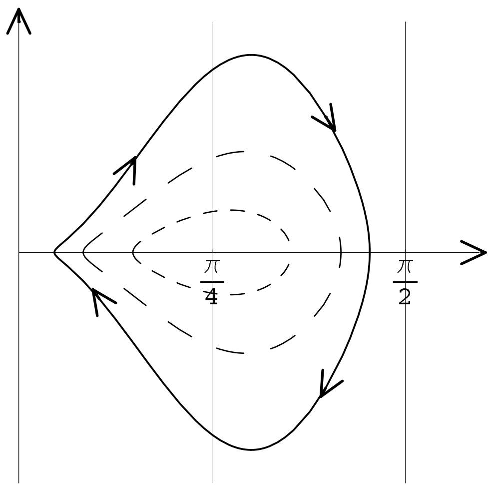


FIGURE 1. The phase plane $(\theta, \theta')$

*Proof.* (a) and (b) easily follow from (4.25), rewritten as

$$(\theta')^2 = \frac{1}{K^2}\sin^2\theta(\sin^2\theta\cos^2\theta - K^2) =: f_K(\theta), \quad K \in [0, 1/2], \tag{4.28}$$

and plotted in the phase space (see Figure 1). We just observe explicitly that, since $\theta' = 0$ at the extremal values of $\theta$, $K$ may be characterized from (4.25) as

$$K = \cos\theta_m \sin\theta_m = \cos\theta_M \sin\theta_M, \tag{4.29}$$

which explains why $\theta_M = \frac{\pi}{2} - \theta_m$. Also (e) follows immediately from (4.28), since after differentiation we see that

$$2\theta'' = f_K'(\theta),$$

whence the arrows in Figure 1.

To prove (c), we let $\varphi_m$ and $\varphi_M$ be a point of minimum and of maximum, respectively, chosen such that no local extremum exists in between. Then in view of (b)

$$\frac{P}{2} = \int_{\varphi_m}^{\varphi_M} \mathrm{d}\varphi \overset{(4.25)}{=} \int_{\varphi_m}^{\varphi_M} \frac{K\theta'}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\varphi = \int_{\theta_m}^{\theta_M} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta.$$

We now observe that

$$K \overset{(4.29)}{=} \cos\theta_M\sin\theta_M = \cos\theta_M\cos\theta_m \geq \cos\theta_M\cos\theta \quad \text{for all} \quad \theta \in (\theta_m, \theta_M).$$

Therefore

$$\frac{P}{2} \geq \cos\theta_M \int_{\theta_m}^{\theta_M} \frac{\cos\theta}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta,$$

whose primitives may be computed explicitly:

$$\cos\theta_M \int \frac{\cos\theta}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta = \frac{1}{2\sin\theta_M}\arcsin\left(\frac{\sin^2\theta - 2\sin^2\theta_M\cos^2\theta_M}{\sin^2\theta|1 - 2\cos^2\theta_M|}\right).$$

Hence

$$\frac{P}{2} \geq \frac{1}{2\sin\theta_M}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2\sin\theta_M} > \frac{\pi}{2},$$

which proves (c).

To prove (d), let $\varphi_m$ be a minimum point and let $\varphi_*$ be the closest point to $\varphi_m$ such that $\varphi_m \leq \varphi_*$ and $\theta(\varphi_*) = \frac{\pi}{4}$. By (b), the length of the interval within a period where $\theta \leq \frac{\pi}{4}$ is exactly

$$2\int_{\varphi_m}^{\varphi_*} \mathrm{d}\varphi \overset{(4.25)}{=} 2\int_{\theta_m}^{\frac{\pi}{4}} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta$$

Since $\cos\theta \geq \frac{1}{\sqrt{2}}$ if $\theta \in [0, \frac{\pi}{4}]$,

$$2\int_{\varphi_m}^{\varphi_*} \mathrm{d}\varphi \geq \sqrt{2}\int_{\theta_m}^{\frac{\pi}{4}} \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta.$$

The primitives of the right-hand side may computed explicitly (via the substitution $z = \sin^2(2\theta)$):

$$\int \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta = -\arctan\left(\frac{2K\cos(2\theta)}{\sqrt{\sin^2(2\theta) - 4K^2}}\right) + C. \tag{4.30}$$

After a substitution we get (d). □

We will also need the following observation.

**Lemma 4.11.** *Let $\boldsymbol{\gamma}$ be a symmetric shortest path not passing through $(0,0,1)$ and let $\boldsymbol{\gamma} = X(\varphi,\theta)$. If $t_1 \in [0,1]$ is such that $\theta(t_1) < \frac{\pi}{6}$ and $\varphi(t_1) < \frac{\pi}{2}$, then $\theta(t) < \frac{\pi}{6}$ as long as $\varphi(t) < \pi - \varphi(t_1)$.*

*Proof.* Let $w = \sin\theta$ and let $t_2 > t_1$ be the first time in which $\varphi(t_2) = \pi - \varphi(t_1)$. We have

$$J(\boldsymbol{\gamma}\chi_{(t_1,t_2)}) = \int_{t_1}^{t_2} \sqrt{(\dot{w}(t))^2 + (\dot{\varphi}(t))^2 w(t)^2(1 - w(t)^2)}\,\mathrm{d}t.$$

By assumption, $w(t_1) < \frac{1}{2}$. If there is an interval $\tilde{I} \subset [t_1,t_2]$ where $w(t) > \frac{1}{2}$, then a symmetrization of $w$ with respect to $\frac{1}{2}$ would strictly decrease the value of $J$, since

$$(1-w)^2\left(1 - (1-w)^2\right) - w^2(1-w^2) = 2w(1-w)(1-2w) < 0 \quad \text{if} \;\; w \in (1/2, 1).$$

This contradicts that $\boldsymbol{\gamma}$ is a shortest path and thus proves the lemma. □

We are now ready to exclude shortest paths which are contained in $\mathring{\mathcal{P}}_\varepsilon$:

**Lemma 4.12.** *There exists no symmetric shortest path $\boldsymbol{\gamma}$ not passing through $(0,0,1)$ such that $\boldsymbol{\gamma}((0,1)) \subset \mathring{\mathcal{P}}_\varepsilon$.*

*Proof.* Assume by contradiction that such a $\boldsymbol{\gamma}$ exists. We will argue that $J(\boldsymbol{\gamma}) > 2\sin\theta_0$, in contradiction with Lemma 4.2 (note that $\boldsymbol{\gamma}_{min} \subseteq \mathcal{P}_\varepsilon$ for all $\varepsilon$).

Since $\varphi$ has to travel from 0 to $\pi$, it can not be constant in $[0,1]$. Then, it follows from Lemma 4.9 that $\boldsymbol{\gamma}(t) = X(\varphi(t),\theta(t))$, where $\varphi \mapsto \theta(t(\varphi))$ is a smooth solution of (4.24) such that $\theta(0) = \theta(\pi) = \theta_0$. Since $\boldsymbol{\gamma}$ is symmetric we have $\theta'(\pi/2) = 0$. Because of (b) and (c) in Lemma 4.10, $\theta$ is monotone in $(0, \pi/2)$. Hence, letting $\theta_1 = \theta(\pi/2)$, we have

$$K \overset{(4.25)}{=} K(\theta_1) = \cos\theta_1 \sin\theta_1 \quad \text{and} \quad \cos^2\theta \sin^2\theta \geq K^2 \quad \text{for all} \;\; t \in (0,1). \tag{4.31}$$

We claim that $\theta_1 > \pi/4$. If not, it follows from (4.31) that $\theta_1 \leq \theta_0$. Hence $\theta$ is non-increasing in $(0, \pi/2)$, and

$$\begin{aligned} \frac{\pi}{2} &\overset{(4.25)}{=} \int_0^{\frac{\pi}{2}} \frac{K\theta'}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\varphi = \int_{\theta_1}^{\theta_0} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\theta \\ &\leq \cos(\theta_1) \int_{\theta_1}^{\pi/4} \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\theta. \end{aligned} \tag{4.32}$$

By (4.30), we would have $\pi \leq \pi\cos(\theta_1)$, a contradiction. Hence $\theta_1 > \pi/4$.

We note the obvious bound

$$\frac{1}{2}J(\theta) \geq \int_0^{\pi/2} \cos\theta\sin\theta\, \mathrm{d}\varphi \overset{(4.31)}{\geq} \frac{\pi}{2}\sin\theta_1\cos\theta_1.$$

Hence we are done if

$$\frac{\pi}{2}\sin\theta_1\cos\theta_1 > \sin\theta_0,$$

that is, if

$$\theta_0 < \arcsin\left(\frac{\pi}{2}\sin\theta_1\cos\theta_1\right) = \arcsin\left(\frac{\pi}{4}\sin(2\theta_1)\right). \tag{4.33}$$

We claim that (4.33) does hold. If not, recalling also Lemma 4.11, we would have

$$\theta_0 \geq \max\left\{\arcsin\left(\frac{\pi}{4}\sin(2\theta_1)\right), \frac{\pi}{6}\right\} =: f(\theta_1).$$

Then, arguing as in (4.32) we write

$$\begin{aligned} \frac{\pi}{2} &= \int_{\theta_0}^{\theta_1} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\theta \\ &< \cos\left(\frac{\pi}{6}\right) \int_{f(\theta_1)}^{\frac{\pi}{4}} \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\theta \\ &\quad + \cos\left(\frac{\pi}{4}\right) \int_{\frac{\pi}{4}}^{\theta_1} \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\, \mathrm{d}\theta. \\ &= \frac{\sqrt{3}}{2}\arctan\left(\frac{\sin\theta_1\cos\theta_1\cos(2f(\theta_1))}{\sqrt{\sin^2(2f(\theta_1)) - 4\sin^2\theta_1\cos^2\theta_1}}\right) + \frac{\sqrt{2}}{2}\frac{\pi}{2} =: F(\theta_1). \end{aligned}$$

It is now a calculus exercise to check that $F$ is increasing in $\big(\pi/4, \overline{\theta}\big) := \big(0, \frac{1}{2}\left(\pi - \arcsin\frac{2}{\pi}\right)\big)$ and decreasing in $\big(\overline{\theta}, \pi/2\big)$: therefore $F$ has a global maximum at $\overline{\theta}$, with $F(\overline{\theta}) < \pi/2$. Since this is impossible, (4.33) holds and the proof is complete. $\square$

The rest of the section is concerned with estimating the length of candidate symmetric shortest paths which intersect $\partial\mathcal{P}_\varepsilon$ (and do not pass through the north pole). We firstly infer some properties of those candidate shortest paths which reach $\partial\mathcal{P}_\varepsilon$.

**Lemma 4.13.** *Let $\theta_0 < \pi/4$, let $\varepsilon$ be sufficiently small, and let $\boldsymbol{\gamma} = X(\varphi,\theta)$ be a symmetric shortest path not intersecting the north pole. If $t_1 > 0$ exists such that $\boldsymbol{\gamma}(t_1) \in \partial\mathcal{P}_\varepsilon$ and $\boldsymbol{\gamma}(t) \in \mathring{\mathcal{P}}_\varepsilon$ in $[0,t_1)$, then:*

(i) *$\theta(t) \geq \pi/6$ for all $t \in [0,t_1)$;*
(ii) *$\theta$ is increasing in $[0,t_1)$;*
(iii) *$\varphi(t_1) \leq \pi/2 - \varepsilon$.*

*Proof.* (i) follows immediately from Lemma 4.11.

To prove (ii), we note that by Lemma 4.9, (4.22) holds in $[0,t_1)$. Let $\varphi_1 = \varphi(t_1)$. By symmetry, $\varphi_1 \leq \pi/2$. If $K = 0$ we would have $\varphi(t) = \varphi_1$ in $(0,t_1)$: since $\boldsymbol{\gamma}$ does not reach the north pole, this means that $\varphi_1 = 0$ and $\theta$ is increasing from $\theta_0$ up to $\theta(t_1) = \frac{\pi}{2} - \theta_0$. If instead $K > 0$, then (4.23) holds in $(0,\varphi_1)$. We will prove that $\theta' \geq 0$ in $(0,\varphi_1)$, which implies (ii). Assume by contradiction that $\theta' < 0$ somewhere in $(0,\varphi_1)$. Then, by (b) of Lemma 4.10, there is $\varphi_2 \in (0,\varphi_1)$ such that $\theta(\varphi_2) = \theta_m$. By (d) of Lemma 4.10 and since $\theta(\varphi_1) > \pi/4$, we have $\varphi_1 \geq \varphi_1 - \varphi_2 \geq \frac{\pi}{2\sqrt{2}}$. Then, since $\theta^*_\varepsilon$ is increasing in $(0,\pi/2)$ and provided $\varepsilon$ is sufficiently small,

$$\theta_1 := \theta(\varphi_1) = \theta^*_\varepsilon(\varphi_1) \geq \theta^*_\varepsilon\left(\frac{\pi}{2\sqrt{2}}\right) = \theta^*\left(\frac{\pi}{2\sqrt{2}}\right) \overset{(4.6)}{\geq} \arctan\left(\frac{1}{\cos\left(\frac{\pi}{2\sqrt{2}}\right)}\right) > \frac{\pi}{3}.$$

By (4.25), this implies that

$$\sin\theta_m\cos\theta_m \leq \sin\theta_1\cos\theta_1 < \frac{\sqrt{3}}{4}, \quad \text{i.e.} \quad \theta_m < \frac{\pi}{6},$$

which is impossible in view of Lemma 4.11.

To prove (iii), assume by contradiction that $\varphi(t_1) \in (\pi/2 - \varepsilon, \pi/2]$. We have

$$\begin{aligned}
\frac{\pi}{2} - \varepsilon \quad &\leq \quad \int_0^{\varphi(t_1)} \mathrm{d}\varphi \overset{(4.25)}{=} \int_{\theta_0}^{\theta(\varphi(t_1))} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta \\
&\leq \quad \int_{\theta_0}^{\theta_M} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta \qquad (\text{ by (ii)}) \\
&\leq \quad \int_{\pi/6}^{\pi/4} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta \qquad (\text{by (i)}) \\
&\qquad + \frac{\sqrt{2}}{2}\int_{\pi/4}^{\theta_M} \frac{K}{\sin\theta\cos\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta \\
&\overset{(4.30)}{=} \quad \int_{\pi/6}^{\pi/4} \frac{K}{\sin\theta\sqrt{\cos^2\theta\sin^2\theta - K^2}}\,\mathrm{d}\theta + \frac{\pi\sqrt{2}}{4}.
\end{aligned} \tag{4.34}$$

Furthermore, again by (ii), we have

$$K = \sin\theta_M\cos\theta_M \leq \sin\theta(\varphi(t_1))\cos\theta(\varphi(t_1)) \leq \sin\theta^*(\varphi - \varepsilon)\cos\theta^*(\varphi-\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore the integral on the right-hand side of (4.34) vanishes as $\varepsilon \to 0$, yielding a contradiction for $\varepsilon$ sufficiently small. □

We continue with one simple observation.

**Lemma 4.14.** *Let $\varphi \in I = (\varphi_0, \varphi_1) \subseteq [0, \pi/2 - \varepsilon]$. Then*

$$J_I(\theta_\varepsilon^*) = \left[\frac{\sin\theta_0\sin\varphi}{\sqrt{1 + \tan^2\theta_0\cos^2\varphi}}\right]_{\varphi=\varphi_0}^{\varphi=\varphi_1}. \tag{4.35}$$

*Proof.* Since $\theta_\varepsilon^* = \theta^*$ for $\varphi \leq \pi/2 - \varepsilon$, a straightforward computation shows that

$$\cos\theta_\varepsilon^*\sqrt{(\theta_\varepsilon^{*\prime})^2 + \sin^2\theta_\varepsilon^*} = \frac{4\sin\theta_0\cos\varphi}{\sqrt{1+\tan^2\theta_0\cos^2\varphi}(3 + \cos(2\theta_0) + 2\cos(2\varphi)\sin^2\theta_0)}.$$

An integration of this expression yields (4.35). □

We now show that if the graph of a solution to (4.24) emanates from $\partial\mathcal{P} \cap \partial\mathcal{P}_\varepsilon$ into $\mathring{\mathcal{P}}_\varepsilon$, then it does not return to $\partial\mathcal{P} \cap \partial\mathcal{P}_\varepsilon$.

**Lemma 4.15.** *Let $\varphi_1 \in [0, \pi/2 - \varepsilon)$ and let $\theta$ be a solution of (4.24) such that $\theta(\varphi_1) = \theta^*(\varphi_1)$ and $\theta'(\varphi_1) \leq {\theta^*}'(\varphi_1)$. Then $X(\varphi, \theta(\varphi)) \subset \mathring{\mathcal{P}}_\varepsilon$ for all $\varphi \in (\varphi_1, \pi/2 - \varepsilon]$.*

*Proof.* We let $\theta_1 = \theta(\varphi_1)$ and we distinguish two cases.

**Case 1.** $\theta'(\varphi_1) \leq 0$. If $\theta_0 = \pi/4$, $\varphi_1 = 0$, and $\theta'(\varphi_1) = 0$, then $\theta \equiv \pi/4$ and the Lemma is trivially true. Else Lemma 4.10 implies that $\theta$ decreases either until $\varphi = \pi$ or until it reaches its minimum. In the former case the Lemma is proved. In the latter, part (d) of Lemma 4.10 implies that $\theta < \theta^*(\varphi_1)$ at least until $\varphi = \varphi_1 + \pi/\sqrt{2} > \pi/2$.

**Case 2.** $\theta'(\varphi_1) > 0$. It is convenient to let

$$v(\varphi) = \log\left(\tan\left(\tfrac{1}{2}\theta(\varphi)\right)\right).$$

Lengthy but straightforward computations show that

$$v'' = \frac{\cosh(2v) - 3}{\sinh(2v)}(1 + (v')^2).$$

We now observe that

$$\begin{aligned}
\cosh(2v) < 3 \quad &\Longleftrightarrow \quad \frac{1}{2}\log(3 - 2\sqrt{2}) < v < \frac{1}{2}\log(3 + 2\sqrt{2}) \\
&\Longleftrightarrow \quad \log(\tan(\pi/8)) < \log(\tan(\theta/2)) < \log(\tan(3\pi/8) \\
&\Longleftarrow \quad \theta \in (\pi/4, \pi/2), \\
\sinh(2v) > 0 \quad &\Longleftrightarrow \quad v > 0 \;\Longleftarrow\; \theta \in (\pi/8, \pi/2),
\end{aligned}$$

hence $v'' < 0$ if $\theta > \pi/4$. On the other hand, as long as $\varphi \leq \pi/2 - \varepsilon$ we have

$$\theta < \theta_\varepsilon^* = \theta^* \quad \Longleftrightarrow \quad v < v^*(\varphi) := \log\left(\tan\left(\frac{1}{2}\arctan\left(\frac{1}{\tan\theta_0|\cos\varphi|}\right)\right)\right)$$

with

$$v^{*''} = \frac{\sin\theta_0\cos\varphi}{\left(\sin^2\theta_0\cos^2\varphi + \cos^2\theta_0\right)^{3/2}} > 0.$$

Hence $(v - v^*)'' < 0$ as long as $\theta > \pi/4$ and $\varphi \leq \pi/2 - \varepsilon$. Since $v = v^*$ and $v' \leq v^{*'}$ at $\varphi = \varphi_1$, then $v < v^*$ as long as either $\varphi = \pi/2 - \varepsilon$ or $\theta = \pi/4$. In the former case the proof is complete. In the latter case, part (d) of Lemma 4.10 implies that $\theta$ will then remain below $\pi/4$ at least in an interval of length $\pi/\sqrt{2} > \pi/2$, and the proof is complete.

□

We now estimate $J$ over a candidate symmetric shortest path which de-touches from $\partial\mathcal{P} \cap \partial\mathcal{P}_\varepsilon$ and reaches $\varphi = \pi/2 - \varepsilon$:

**Lemma 4.16.** *Let $\boldsymbol{\gamma}$ be a symmetric shortest path not passing through $(0,0,1)$ and let $\boldsymbol{\gamma} = X(\varphi,\theta)$. If $t_1 \geq 0$ exists such that $\varphi_1 = \varphi(t_1) \in [0,\pi/2-\varepsilon)$, $\theta(t_1) = \theta^*(\varphi_1)$, and $\boldsymbol{\gamma} \not\subset \partial\mathcal{P}$ in a right-neighborhood of $t_1$, then*

$$J_{(\varphi_1,\frac{\pi}{2}-\varepsilon)}(\theta) > J_{(\varphi_1,\frac{\pi}{2})}(\theta^*) - \frac{\varepsilon}{2}.$$

*Proof.* By assumption, for all $\sigma > 0$ there exists $t_\sigma \in (t_1, t_1+\sigma)$ such that $\boldsymbol{\gamma}(t_\sigma) \in \mathring{\mathcal{P}}_\varepsilon$. By continuity, there exists $\tilde{t}_\sigma \in [t_1, t_\sigma)$ such that $\boldsymbol{\gamma}(\tilde{t}_\sigma) \in \partial\mathcal{P}_\varepsilon$ and $\boldsymbol{\gamma}(t) \in \mathring{\mathcal{P}}_\varepsilon$ for all $t \in (\tilde{t}_\sigma, t_\sigma]$. Then we may apply Lemma 4.9 in $(\tilde{t}_\sigma, t_\sigma]$.

If $K = 0$ then $\varphi$ is constant, and since the curve is on $\partial\mathcal{P}_\varepsilon$ at $t = \tilde{t}_\sigma$, $\theta$ must decrease. Hence $\boldsymbol{\gamma}$ remains smooth down to $\theta = 0$, the north pole. Therefore this case is excluded.

Then $K > 0$, $\varphi$ is strictly increasing, and $\theta(\varphi)$ solves (4.24) in $(\tilde{t}_\sigma, t_\sigma)$. By Lemma 4.15, we in fact have $X(\varphi,\theta(\varphi)) \subset \mathring{\mathcal{P}}_\varepsilon$ as long as $\varphi \leq \pi/2-\varepsilon$, which in particular implies that $\tilde{t}_\sigma = t_1$ and that $\theta$ solves (4.24) as long as $\varphi \leq \pi/2-\varepsilon$. We let

$$\theta_1 = \theta(\varphi_1) = \theta^*(\varphi_1)$$

and we distinguish two cases.

**Case 1.** $\theta'(\varphi_1) \leq 0$. Lemma 4.10 and the symmetry of the path imply that $\theta$ does not increase until $\pi/2$ and $\theta(\pi/2) = \theta_m > 0$. If $\theta_1 = \theta_0 = \pi/4$ and $\varphi_1 = 0$, then $\boldsymbol{\gamma}((0,1)) \subset \mathring{\mathcal{P}}_\varepsilon$, a case which has already been ruled out in Lemma 4.12. Hence $\theta_1 > \pi/4$.

We claim that

$$\min_{\varphi\in[0,\frac{\pi}{2}]} \sin\theta\cos\theta = \sin\theta_m\cos\theta_m. \tag{4.36}$$

By (4.25),

$$\min_{\varphi\in[\varphi_1,\pi/2]} \sin\theta\cos\theta = \sin\theta_m\cos\theta_m. \tag{4.37}$$

In particular,

$$\sin(\theta_1)\cos(\theta_1) \geq \sin\theta_m\cos\theta_m. \tag{4.38}$$

On the other hand, by Lemma 4.13(ii), $\theta \in [\theta_0,\theta_1]$ for $\varphi \in [0,\varphi_1]$. Hence

$$\min_{\varphi\in[0,\varphi_1]} \sin\theta\cos\theta = \min\{\sin\theta_0\cos\theta_0, \sin\theta_1\cos\theta_1\}. \tag{4.39}$$

Since $\theta^*$ is increasing

$$\sin(\theta_1)\cos(\theta_1) \leq \sin(\theta^*(0))\cos(\theta^*(0)) = \sin\left(\frac{\pi}{2}-\theta_0\right)\cos\left(\frac{\pi}{2}-\theta_0\right) = \sin(\theta_0)\cos(\theta_0),$$

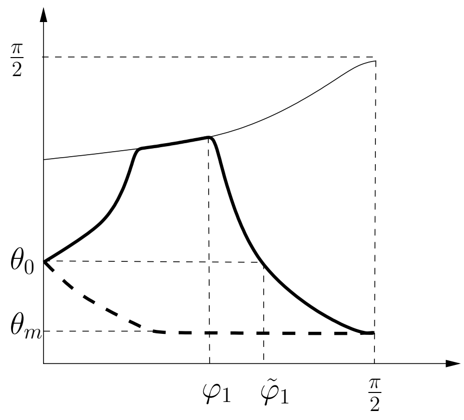


FIGURE 2. Case 1 in the proof of Lemma 4.16. The path $(\varphi, \theta(\varphi))$ (continuous) is beaten by its competitor $(\varphi, \tilde{\theta}(\varphi))$ (dashed).

therefore (4.39) reads as

$$\min_{\varphi\in[0,\varphi_1]} \sin\theta\cos\theta = \sin\theta_1\cos\theta_1 \overset{(4.38)}{\geq} \sin\theta_m\cos\theta_m \tag{4.40}$$

and (4.36) follows from (4.37) and (4.40).

We denote by $\tilde{\varphi}_1 \in (\varphi_1, \pi/2)$ the unique point such that $\theta(\tilde{\varphi}) = \theta_0$ (recall that $\theta_1 > \pi/4$, $\theta(\pi/2) = \theta_m < \pi/4$, and $\theta$ is decreasing in $(\varphi_1, \pi/2)$), and we define (see Figure 2)

$$\tilde{\theta}(\varphi) := \begin{cases} \theta(\varphi + \tilde{\varphi}_1) & \text{if } 0 \leq \varphi \leq \frac{\pi}{2} - \tilde{\varphi}_1 \\ \theta_m & \text{if } \frac{\pi}{2} - \tilde{\varphi}_1 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

We have

$$J_{(0,\tilde{\varphi}_1)}(\theta) > \sin\theta_m\cos\theta_m\tilde{\varphi}_1 = J_{(\pi/2-\tilde{\varphi}_1,\pi/2)}(\tilde{\theta}) \quad \text{and} \quad J_{(0,\frac{\pi}{2}-\tilde{\varphi}_1)}(\tilde{\theta}) = J_{(\tilde{\varphi}_1,\frac{\pi}{2})}(\theta).$$

Therefore $\boldsymbol{\gamma}$ is not a shortest path and this case is excluded.

**Case 2.** $\theta'(\varphi_1) > 0$. Lemma 4.10 and the symmetry of the path imply that $\theta$ increases until $\pi/2 - \varepsilon$. We now estimate its length in $I_\varepsilon = (\varphi_1, \pi/2 - \varepsilon)$. By the assumption of case 2, and since $\boldsymbol{\gamma} \in \mathcal{P}$ in $I_\varepsilon$,

$$0 < \theta'(\varphi_1) \leq (\theta^*)'(\varphi_1). \tag{4.41}$$

By (4.25),

$$\sin^2\theta(\cos^2\theta\sin^2\theta - K^2) = K^2(\theta')^2 \quad \text{for some} \quad K \in (0, 1/2). \tag{4.42}$$

Evaluating this expression at $\varphi_1$, we have

$$K = \frac{\sin^2\theta^*(\varphi_1)\cos\theta^*(\varphi_1)}{\sqrt{(\theta'(\varphi_1))^2+\sin^2\theta^*(\varphi_1)}} \overset{(4.41)}{\geq} \frac{\sin^2\theta^*(\varphi_1)\cos\theta^*(\varphi_1)}{\sqrt{(\theta^{*\prime}(\varphi_1))^2+\sin^2\theta^*(\varphi_1)}}.$$

Of course, we have

$$J_{I_\varepsilon}(\theta) \geq \int_{I_\varepsilon} \cos\theta\sin\theta\,\mathrm{d}\varphi \overset{(4.42)}{\geq} |I_\varepsilon|K.$$

This chain of inequalities implies that

$$J_{I_\varepsilon}(\theta) > |I_\varepsilon|\frac{\sin^2\theta^*(\varphi_1)\cos\theta^*(\varphi_1)}{\sqrt{(\theta^{*\prime}(\varphi_1))^2+\sin^2\theta^*(\varphi_1)}} = \frac{|I_\varepsilon|\sin\theta_0\cos\varphi_1}{\sqrt{1+\tan^2\theta_0\cos^2\varphi_1}}$$

(the latter equality follows from an explicit computation). On the other hand, by Lemma 4.14, the curve which just stays on the obstacle, $\boldsymbol{\gamma}^* = X(\varphi,\theta^*(\varphi))$, $\varphi\in(\varphi_1,\pi/2)$, is such that

$$J_{(\varphi_1,\frac{\pi}{2})}(\theta^*) \quad = \quad \sin\theta_0\left(1-\frac{\sin\varphi_1}{\sqrt{1+\tan^2\theta_0\cos^2\varphi_1}}\right).$$

Hence

$$(J_{I_\varepsilon}(\theta)-J_{(\varphi_1,\frac{\pi}{2})}(\theta^*))\frac{\sqrt{1+\tan^2\theta_0\cos^2\varphi_1}}{\sin\theta_0} > |I_\varepsilon|\cos\varphi_1+\sin\varphi_1-\sqrt{1+\tan^2\theta_0\cos^2\varphi_1}$$
$$> \left(\frac{\pi}{2}-\varepsilon-\varphi_1\right)\cos\varphi_1+\sin\varphi_1-\sqrt{1+\cos^2\varphi_1} =: F(\varphi_1)-\varepsilon\cos\varphi_1.$$

Another calculus exercise shows that $F$ is decreasing $[0,\frac{\pi}{2}]$: since $F(\frac{\pi}{2})=0$, $F$ is positive. Therefore

$$J_{I_\varepsilon}(\theta) > J_{(\varphi_1,\frac{\pi}{2})}(\theta^*)-\varepsilon\frac{\sin\theta_0\cos\varphi_1}{\sqrt{1+\tan^2\theta_0\cos^2\varphi_1}} > J_{(\varphi_1,\frac{\pi}{2})}(\theta^*)-\tfrac{1}{2}\varepsilon$$

and the proof is complete.

□

Next we characterize the candidate shortest paths joining $X(0,\theta^*(0))$ with another point on $\partial\mathcal{P}\cap\partial\mathcal{P}_\varepsilon$ which is on the same side with respect to $\pi/2$.

**Lemma 4.17.** *Let $0<\bar\varphi\leq\pi/2-\varepsilon$. The shortest path which connects $X(0,\frac{\pi}{2}-\theta_0)$ and $X(\bar\varphi,\theta^*(\bar\varphi))$ is (a smooth re-parametrization of) $\boldsymbol{\gamma}^* = X(\varphi,\theta^*(\varphi))$, $\varphi\in[0,\bar\varphi]$.*

*Proof.* Let $I = (0, \bar\varphi)$. We recall by Lemma 4.14 that
$$J_I(\theta^*) = \frac{\sin\theta_0 \sin\bar\varphi}{\sqrt{1+\tan^2\theta_0\cos^2\bar\varphi}} < \sin\theta_0.$$
First of all, we note that $\boldsymbol{\gamma}$ does not reach the north pole. Assume by contradiction that it does at time, say, $\bar t \in (0,1)$. Then
$$J(\boldsymbol{\gamma}) \;\geq\; \int_0^{\bar t} \cos\theta|\dot\theta|\,\mathrm{d}t \geq \sin(\theta(0)) = \sin\left(\frac{\pi}{2} - \theta_0\right) = \cos\theta_0 > \sin\theta_0 > J_I(\theta^*),$$
which is impossible.

Therefore we may use the spherical coordinates (4.4), and arguing as the proof of Lemma 4.8 we see that $\varphi$ in non-decreasing.

Assume by contradiction that $\boldsymbol{\gamma}$ does not coincide (up to a smooth re-parametrization) with $\boldsymbol{\gamma}^*$. Then $t_1 > 0$ and a right-neighborhood $\tilde I$ of $t_1$ exist such that $\varphi_1 := \varphi(t_1) < \bar\varphi$ and $\boldsymbol{\gamma}(\tilde I) \not\subset \partial\mathcal{P}$. Arguing as in the first lines of the proof of Lemma 4.16, one finds that there is $t_2 > t_1$ such that $\boldsymbol{\gamma}(t) \in \mathring{\mathcal{P}}$ for all $t \in (t_1, t_2)$. Then, arguing as in the proof of Lemma 4.9, one finds that (4.22) holds, and $K \geq 0$ since $\varphi$ is non-decreasing. If $K > 0$, then $\theta(\varphi)$ would solve (4.24) in $(t_1, t_2)$; but in view of Lemma 4.15, such solution will not re-hit the constraint until $\varphi = \pi/2 - \varepsilon$, hence $K > 0$ can not occur. If $K = 0$, then $\varphi \equiv \varphi_1$, and since we are on $\partial\mathcal{P}$ at time $t_1$, $\theta$ must move inwards. Hence $\boldsymbol{\gamma}$ remains smooth up to $\theta = 0$, the north pole, a contradiction. □

We are now ready to complete the proof of Theorem 4.1.

*Proof of Theorem 4.1.* First of all, we note that Lemma 4.14 implies that $J(\theta^*) = 2\sin\theta_0$. Hence, in view of Lemma 4.5, it suffices to show that
$$\inf_{\boldsymbol{\gamma}\in\Gamma_\varepsilon} J(\boldsymbol{\gamma}) \geq J(\theta^*) - \omega(\varepsilon) = 2\sin\theta_0 - \omega(\varepsilon), \tag{4.43}$$
where $\omega$ is a universal function which vanishes as $\varepsilon \to 0$. By Lemma 4.6, the inf on the left-hand side of (4.43) is attained. Let $\boldsymbol{\gamma}$ be one such shortest path. If $\boldsymbol{\gamma}$ passes through $(0,0,1)$, then (4.43) follows from Lemma 4.7. If not, we let $\boldsymbol{\gamma} = X(\varphi, \theta)$ and, by Lemma 4.8, we assume w.l.o.g. that $\boldsymbol{\gamma}$ is symmetric. For simplicity, we distinguish between $\theta_0 < \pi/4$ and $\theta_0 = \pi/4$.

**Case 1:** $\theta_0 < \pi/4$. We already know from Lemma 4.12 that $\boldsymbol{\gamma}$ has to intersect $\partial\mathcal{P}_\varepsilon$. Let $t_0$ and $t_1$ be, respectively, the first time in which $\theta(t) = \frac{\pi}{4}$ and the first time in which $\boldsymbol{\gamma}$

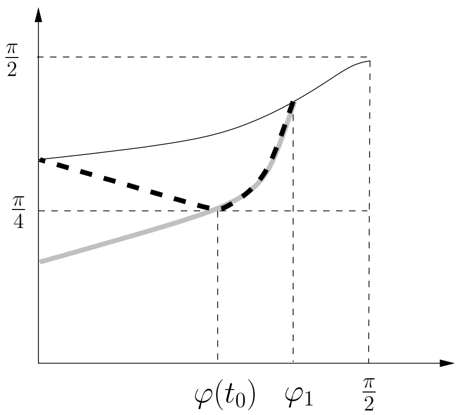


FIGURE 3. Case 1 in the proof of Theorem 4.1. The path $(\varphi(t), \theta(t))$ (gray) is beated by its competitor $(\varphi(t), \tilde{\theta}(t))$ (dashed).

intersects $\partial\mathcal{P}_\varepsilon$:

$$t_1 \quad := \quad \sup\{t > 0 : \boldsymbol{\gamma} \in \mathring{\mathcal{P}}_\varepsilon \ \text{ in } \ [0, t)\} \quad \text{and} \quad \varphi_1 = \varphi(t_1).$$

Provided $\varepsilon$ is sufficiently small, by Lemma 4.13(ii) $\theta$ is increasing in $(0, t_1)$. Hence the curve

$$\tilde{\boldsymbol{\gamma}}(t) := X\left(\varphi(t), \tilde{\theta}(t)\right), \quad \tilde{\theta}(t) := \begin{cases} \frac{\pi}{2} - \theta(t) & t \in [0, t_0] \\ \theta(t) & t \in [t_0, t_1] \end{cases}$$

is contained in $\mathcal{P}_\varepsilon$ (see Figure 3). We claim that

$$J(\tilde{\boldsymbol{\gamma}}\chi_{(0,t_1)}) < J(\boldsymbol{\gamma}\chi_{(0,t_1)}), \tag{4.44}$$

which is equivalent to

$$J(\tilde{\boldsymbol{\gamma}}\chi_{(0,t_0)}) < J(\boldsymbol{\gamma}\chi_{(0,t_0)}). \tag{4.45}$$

By Lemma 4.9, $\boldsymbol{\gamma}$ satisfies (4.22) in $(0, t_0)$. If $K = 0$ then $\varphi \equiv 0$ and (4.45) follows from the expression (4.5) of $J$:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta) < \cos\theta \quad \text{if} \quad \theta \leq \pi/4.$$

Otherwise, by Lemma 4.9 $\varphi \mapsto \theta(\varphi)$ solves (4.24) in $(0, \varphi_0)$, where $\varphi_0 = \varphi(t_0)$. Then, it follows by Lemma 4.13(iii) that $\varphi_0 < \pi/2 - \varepsilon$, and we may use the equivalent expression (4.26) for $J$: since

$$\cos^2\tilde{\theta}((\tilde{\theta}')^2 + \sin^2\tilde{\theta}) = \sin^2\theta((\theta')^2 + \cos^2\theta) < \cos^2\theta((\theta')^2 + \sin^2\theta) \quad \text{in} \quad (0, \varphi_0),$$

(4.45) follows.

Since $\tilde{\boldsymbol{\gamma}}$ is a path connecting $X(0, \frac{\pi}{2} - \theta_0)$ to $X(\varphi_1, \theta^*(\varphi_1))$, Lemma 4.17 implies that $J(\tilde{\boldsymbol{\gamma}}\chi_{(0,t_1)}) \geq J_{(0,\varphi_1)}(\theta^*)$. Together with (4.44), we obtain

$$J_{(0,\varphi_1)}(\theta^*) < J(\boldsymbol{\gamma}\chi_{(0,t_1)}). \tag{4.46}$$

Let now $t_2 \geq t_1$ be defined by

$$t_2 := \max\{t \geq t_1 : \boldsymbol{\gamma} \in \partial\mathcal{P}_\varepsilon \ \text{ in } \ [t_1, t]\} \quad \text{and} \quad \varphi_2 = \varphi(t_2).$$

The estimate in $(t_1, t_2)$ is trivial since $\boldsymbol{\gamma}$ coincides with $\boldsymbol{\gamma}^* := X(\varphi, \theta^*)$:

$$J(\boldsymbol{\gamma}\chi_{(t_1,t_2)}) = J_{(\varphi_1,\varphi_2)}(\theta^*).$$

On $\left(\varphi_2, \frac{\pi}{2} - \varepsilon\right)$, Lemma 4.16 implies that

$$J_{\left(\varphi_2, \frac{\pi}{2} - \varepsilon\right)}(\theta) > J_{\left(\varphi_2, \frac{\pi}{2}\right)}(\theta^*) - \frac{\varepsilon}{2} \quad \text{if } \varphi_2 < \tfrac{\pi}{2} - \varepsilon. \tag{4.47}$$

Finally, we just observe that

$$J_{\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}\right)}(\theta^*) \overset{(4.7)}{\leq} \omega(\varepsilon) \tag{4.48}$$

Collecting (4.46)-(4.48) and recalling the symmetry of $\boldsymbol{\gamma}$ we obtain (4.43).

**Case 2:** $\theta_0 = \pi/4$. This case is simpler. We let

$$t_2 \quad = \quad \max\{t \geq 0 : \boldsymbol{\gamma} \in \partial\mathcal{P}_\varepsilon \ \text{ in } \ [0, t]\} \geq 0 \quad \text{and} \quad \varphi_2 = \varphi(t_2),$$

and we argue exactly as above to obtain $J(\boldsymbol{\gamma}\chi_{(0,t_2)}) = J_{(0,\varphi_2)}(\theta^*)$ and (4.47)-(4.48). $\square$

**Acknowledgements**. The second and third authors have been partially supported by the Spanish MEC project MTM2012–31103. We thank Manuel Ritoré for fruitful discussions on the geometrical problem in Section 4.

## References


[1] S. B. Alexander, I. D. Berg, and R. L. Bishop, *Cut Loci, Minimizers and Wavefronts in Riemannian Manifolds with boundary*, Michigan Math. J. **40 (2)** (1993), 229–237.

[2] R. Alicandro, C. Corbo Esposito and C. Leone, *Relaxation in BV of integral functionals defined on Sobolev functions with values in the unit sphere*, J. Convex Anal. **14** (2007), 69-98.

[3] L. Ambrosio, G. Crippa, S. Maniglia, *Traces and fine properties of a BD class of vector fields and applications*, Ann. Fac. Sci. Toulouse Math. **14** (2005), 527–561.

[4] L. Ambrosio and G. Dal Maso, *A general chain rule for distributional derivatives*, Proc. Amer. Math. Soc. **180** (1990), 691–702.

[5] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000.

[6] Andreu, C. Ballester , F., Caselles, V., Mazón, J.M. (2001). Minimizing Total Variation Flow. *Diff. Int. Eq.* 14(3): 321–360.

[7] F. Andreu, V. Caselles, and J.M. Mazón, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals.* Progress in Mathematics, vol. 223, 2004. Birkhauser.

[8] G. Anzellotti, *Pairings Between Measures and Bounded Functions and Compensated Compactness*, Ann. di Matematica Pura ed Appl. IV **135** (1983), 293-318.

[9] J.W. Barret, X. Feng and A. Prohl, *On p-harmonic map heat flow for* $1 \leq p < \infty$ *and their finite element approximations*, SIAM J. math. Anal. **40** (2008), 1471-1498.

[10] M. Bertsch, R. Dal Passo, A. Pisante, *Point singularities and nonuniqueness for the heat flow for harmonic maps*, Communications in Partial Differential Equations **28(5)** (2003), 1135–1160

[11] M. Bertsch, R. Dal Passo, R. Van der Hout, *Nonuniqueness for the heat flow of harmonic maps on the disk*, Arch. Rat. Mech. Anal. **161(2)**, (2002), 93–112.

[12] M. Bonforte, A. Figalli, *Total variation flow and sign fast diffusion in one dimension*, J. Differential Equations **252** (2012), no. 8, 4455û-4480.

[13] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.

[14] V. Caselles,*On the entropy conditions for some flux limited diffusion*, J. Diff. Equations **250** (2011), 3311-3348.

[15] Y. Chen, *The Weak Solutions to the Evolution Problems of Harmonic Maps*, Math. Z. **201** (1989), 69-74.

[16] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal., **147** (1999) 89-118.

[17] Y. Chen, M-Ch. Hong and N. Hungerbühler, *Heat flow of p-harmonic maps with values into spheres*, Math. Z. **215** (1994), 25-35.

[18] R. Dal Passo, L. Giacomelli and S. Moll, *Rotationally symmetric 1-harmonic maps from* $D^2$ *to* $S^2$, Calc. Var. **32(4)** (2008), 533–554.

[19] R. W. R. Darling, Differential forms and connections, Cambridge University Press, 1994.

[20] A. DeSimone and P. Podio-Guidugli, *On the continuum theory of ferromagnetic solids*, Arch. Rat. Mech. Anal. **136** (1996), 201–233.

[21] J. Diestel and J.J. Uhl, Jr., *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.

[22] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109-160.

[23] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992.

[24] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969.

[25] X. Feng, *Divergence-*$L^q$ *and divergence-measure tensor fields and gradient flows for linear growth functionals of maps into the unit sphere*, Calc. Var. Partial Differential Equations **37** (2010), 111-139.

[26] I. Fonseca and S. Müller, *Relaxation of quasiconvex functionals in* $BV(\Omega, \mathbb{R}^p)$ *for integrands* $f(x, u, \nabla u)$, Arch. Rational Mech. Anal. **123** (1993), 1-49.

[27] I. Fonseca and P. Rybka, *Relaxation of multiple integrals in the space $BV(\Omega, R^p)$*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), 321-348.

[28] L. Giacomelli, J. M. Mazón and S. Moll, *The 1-harmonic flow with values in* $\mathbb{S}^1$, SIAM J. Math. Anal. **45** (2013), 1723-1740.

[29] L. Giacomelli, J. M. Mazón and S. Moll. Solutions to the 1-harmonic flow with values into a hyper-octant of the N-sphere. Appl. Math. Letters 26 (2013), 1061-1064.

[30] L. Giacomelli and S. Moll, *Rotationally Symmetric 1-Harmonic Flows From $D^2$ to $S^2$: Local Well-Posedness and Finite Time Blowup*, SIAM J. Math. Anal., **42**, (2010), 2791–2817.

[31] M. Giaquinta and D. Mucci, *The BV-energy of maps into a manifold: relaxation and density results*, Ann. Scuola Norm. Sup Pisa Cl Sci V, **5 (4)**, (2006), 483–548.

[32] Y. Giga, Y. Kashima and N. Yamazaki, *Local solvability of a constrained gradient system of total variation*, Abstract and Applied Analysis, (2004), 651–682.

[33] Y. Giga and R. Kobayashi, *On constrained equations with singular diffusivity*, Methods Appl. Anal. **10** (2003), 253–277.

[34] Y. Giga and H. Kuroda, *On breakdown of solutions of a constrained gradient system of total variation*, Bol. Soc. Parana. Mat.(3), **22** (2004), 9–20.

[35] Y. Giga, H. Kuroda and N. Yamazaki, *An existence result for a discretized constrained gradient system of total variation flow in color image processing*, Interdisciplinary Information Sciences **11(2)** (2005), 199–204.

[36] Y. Giga, H. Kuroda and N. Yamazaki, *Global solvability of constrained singular diffusion equation associated with essential variation*, International Series of Numerical Mathematics **154**, Free Boundary Problems Theory and Applications, Birkhauser Basel (2007), 209–218.

[37] R. van der Hout, *Flow alignment in nematic liquid crystals in flows with cylindrical symmetry*, Differential and Integral Equations **14** (2001), 189–211.

[38] N. Hungerbühler, *Heat flow into spheres for a class of energies*, Progr. Nonlinear Differential Equations Appl. **59**, (2004), 45û-65.

[39] R. Kobayashi, J.A. Warren and W.C. Carter, *A continuum model for grain boundaries*, Phys. D **140** (2000), 141–150.

[40] M. Misawa, *On the p-harmonic flow into spheres in the singular case*, Nonlinear Anal. **50** (2002), 485–494.

[41] G. Sapiro, *Geometric partial differential equations and image analysis*, Cambridge University Press, 2001.

[42] J. Simon, *Compact Sets in the Space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. **146** (1987), 65-96.

[43] M. Struwe, *The Evolution of Harmonic Maps: Existence, Partial Regularity and Singularities*, Progr. Nonlinear Differential Equations Appl. 7, Birkhauser Boston, Boston, MA, 1992, pp. 485-491.

[44] B. Tang, G. Sapiro and V. Caselles, *Diffusion of general data on non-flat manifolds via harmonic maps theory: The direction diffusion case*, Int. J. Computer Vision, **36(2)** (2000), 149–161.

[45] B. Tang, G. Sapiro and V. Caselles, *Color image enhancement via chromaticity diffusion*, IEEE Transactions on Image Processing, **10** (2001), 701–707.

[46] W.P. Ziemer, *Weakly Differentiable Functions*, GTM 120, Springer Verlag, 1989.

Lorenzo Giacomelli
SBAI Department
Sapienza University of Rome
Via Scarpa, 16
00161 Roma, Italy.

*E-mail address*: lorenzo.giacomelli@sbai.uniroma1.it

José M. Mazón, Salvador Moll
Departament d'Anàlisi Matemàtica, Universitat de València
Dr. Moliner 50
46100 Burjassot, Spain.

*E-mail address*: mazon@uv.es, j.salvador.moll@uv.es