

AN EIGENVALUE PROBLEM WITH VARIABLE EXPONENTS

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ABSTRACT. A highly nonlinear eigenvalue problem is studied in a Sobolev space with variable exponent. The Euler-Lagrange equation for the minimization of a Rayleigh quotient of two Luxemburg norms is derived. The asymptotic case with a “variable infinity” is treated. Local uniqueness is proved for the viscosity solutions.

1. INTRODUCTION

An expedient feature of many eigenvalue problems is that the eigenfunctions may be multiplied by constants. That is the case for our non-linear problem in this note. We will study the eigenvalue problem coming from the minimization of the Rayleigh quotient

$$(1.1) \quad \frac{\|\nabla u\|_{p(x),\Omega}}{\|u\|_{p(x),\Omega}}$$

among all functions belonging to the Sobolev space $W_0^{1,p(x)}(\Omega)$ with variable exponent $p(x)$. Here Ω is a bounded domain in \mathbb{R}^n and the variable exponent $p(x)$ is a smooth function, $1 < p^- \leq p(x) \leq p^+ < \infty$. The norm is the so-called *Luxemburg norm*.

If $p(x) = p$, a constant in the range $1 < p < \infty$, the problem reduces to the minimization of the Rayleigh quotient

$$(1.2) \quad \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}$$

among all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Formally, the Euler-Lagrange equation is

$$(1.3) \quad \operatorname{div} (|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0.$$

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The special case $p = 2$ of this much studied problem yields the celebrated Helmholtz equation

$$\Delta u + \lambda u = 0.$$

It is decisive that homogeneity holds: if u is a minimizer, so is cu for any non-zero constant c . On the contrary, the quotient

$$(1.4) \quad \frac{\int_{\Omega} |\nabla u|^{p(x)} dx}{\int_{\Omega} |u|^{p(x)} dx}$$

with variable exponent does not possess this expedient property, in general. Therefore its infimum over all $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$, is often zero and no minimizer appears in the space $W_0^{1,p(x)}(\Omega)$, except the trivial $\varphi \equiv 0$, which is forbidden. For an example, we refer to [11, pp. 444–445]. A way to avoid this collapse is to impose the constraint

$$\int_{\Omega} |u|^{p(x)} dx = \text{constant}.$$

Unfortunately, in this setting the minimizers obtained for different normalization constants are difficult to compare in any reasonable way, except, of course, when $p(x)$ is constant. For a suitable $p(x)$, it can even happen that any positive λ is an eigenvalue for some choice of the normalizing constant. Thus (1.4) is not a proper generalization of (1.2), which has a well defined spectrum.

A way to avoid this situation is to use the Rayleigh quotient (1.1), where we have used the notation

$$(1.5) \quad \|f\|_{p(x),\Omega} = \inf \left\{ \gamma > 0 : \int_{\Omega} \left| \frac{f(x)}{\gamma} \right|^{p(x)} \frac{dx}{p(x)} \leq 1 \right\}$$

for the Luxemburg norm. This restores the homogeneity. In the integrand, the use of $p(x)^{-1} dx$ (rather than dx) has no bearing, but it simplifies the equations a little.

The existence of a minimizer follows easily by the direct method in the Calculus of Variations, cf. [12]. We will derive the Euler-Lagrange equation

$$(1.6) \quad \operatorname{div} \left(\left| \frac{\nabla u}{K} \right|^{p(x)-2} \frac{\nabla u}{K} \right) + \frac{K}{k} S \left| \frac{u}{k} \right|^{p(x)-2} \frac{u}{k} = 0,$$

where the constants are

$$(1.7) \quad K = \|\nabla u\|_{p(x)}, \quad k = \|u\|_{p(x)}, \quad S = \frac{\int_{\Omega} \left| \frac{\nabla u}{K} \right|^{p(x)} dx}{\int_{\Omega} \left| \frac{u}{k} \right|^{p(x)} dx}.$$

They depend on u . Notice that we are free to fix only one of the norms K and k . The minimum of the Rayleigh quotient (1.1) is $\frac{K}{k}$. Inside the integrals defining the constant S , we now have dx (and not $p(x)^{-1} dx$), indeed. Therefore it is possible that $S \neq 1$. For a constant exponent, $S = 1$ and the Euler-Lagrange equation reduces to (1.3). Sometimes we write K_u, k_u, S_u .

We are interested in replacing $p(x)$ by a “variable infinity” $\infty(x)$. The passage to infinity is accomplished so that $p(x)$ is successively replaced by $jp(x)$, $j = 1, 2, 3 \dots$. In order to identify the limit equation, as $jp(x) \rightarrow \infty$, we use the theory of viscosity solutions. In the case of a constant $p(x)$, the limit equation is

$$(1.8) \quad \max \left\{ \Lambda_\infty - \frac{|\nabla u|}{u}, \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} = 0,$$

where $u \in W_0^{1,\infty}(\Omega)$, $u > 0$ and

$$(1.9) \quad \Lambda_\infty = \frac{1}{\max_{x \in \Omega} \text{dist}(x, \partial\Omega)}.$$

This has been treated in [16] (see also [5, 15, 17]). An interesting interpretation in terms of optimal mass transportation is given in [6]. According to a recent manuscript by Hynd, Smart and Yu, there are domains in which there can exist several linearly independent positive eigenfunctions, see [14]. Thus the eigenvalue Λ_∞ is not always simple.

In our case the limit equation reads

$$(1.10) \quad \max \left\{ \Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)} \left(\frac{u}{K} \right) \right\} = 0,$$

where $K = \|\nabla u\|_{\infty, \Omega}$ and

$$(1.11) \quad \Delta_{\infty(x)} v = \sum_{i,j=1}^n \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} + |\nabla v|^2 \ln(|\nabla v|) \langle \nabla v, \nabla \ln p \rangle.$$

We are able to establish that if Λ_∞ is given the value (1.9), the same as for a constant exponent, then the existence of a non-trivial solution is guaranteed. We also prove a *local uniqueness* result: in very small interior subdomains we cannot “improve” the solution. The technically rather demanding proof is based on the modern theory of viscosity solutions, cf. [7, 18], and we assume that the reader is familiar with this topic.

Needless to say, many open problems remain. To mention one, for a finite variable exponent $p(x)$ we do not know whether the first eigenvalue (the minimum of the Rayleigh quotient) is simple. The methods in [4, 19] do not work well now. There

are also many gaps in the theory available at present: due to the lack of a proper Harnack inequality, we cannot assure that the limit of the $jp(x)$ -eigenfunctions is strictly positive. A discussion about analogous difficulties can be found in [3]. In the present work we restrict ourselves to positive eigenfunctions. We hope to return to this fascinating topic in the future.

2. PRELIMINARIES

We will always assume that Ω is a bounded domain in the n -dimensional space \mathbb{R}^n and that the variable exponent $p(x)$ is in the range

$$(2.12) \quad 1 < p^- \leq p(x) \leq p^+ < \infty,$$

when $x \in \Omega$, and belongs to $C^1(\Omega) \cap W^{1,\infty}(\Omega)$. Thus $\|\nabla p\|_{\infty,\Omega} < \infty$.

Next we define $L^{p(x)}(\Omega)$ and the Sobolev space $W^{1,p(x)}(\Omega)$ with variable exponent $p(x)$. We say that $u \in W^{1,p(x)}(\Omega)$ if u and its distributional gradient ∇u are measurable functions satisfying

$$\int_{\Omega} |u|^{p(x)} dx < \infty, \quad \int_{\Omega} |\nabla u|^{p(x)} dx < \infty.$$

The norm of the space $L^{p(x)}(\Omega)$ is defined by (1.5). This is a Banach space. So is $W^{1,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{p(x),\Omega} + \|\nabla u\|_{p(x),\Omega}.$$

Smooth functions are dense in this space, and so we can define the space $W_0^{1,p(x)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the above norm. We refer to [11] and the monograph [8] about these spaces.

The following properties are used later.

Lemma 2.1 (Sobolev). *The inequality*

$$\|u\|_{p(x),\Omega} \leq C \|\nabla u\|_{p(x),\Omega}$$

holds for all $u \in W_0^{1,p(x)}(\Omega)$; the constant is independent of u .

In fact, even a stronger inequality is valid.

Lemma 2.2 (Rellich-Kondrachev). *Given a sequence $u_j \in W_0^{1,p(x)}(\Omega)$ such that $\|\nabla u_j\|_{p(x),\Omega} \leq M$, $j = 1, 2, 3, \dots$, there exists a $u \in W_0^{1,p(x)}(\Omega)$ such that $u_{j_\nu} \rightarrow u$ strongly in $L^{p(x)}(\Omega)$ and $\nabla u_{j_\nu} \rightharpoonup \nabla u$ weakly in $L^{p(x)}(\Omega)$ for some subsequence.*

Eventually, from now on we shall write $\|\cdot\|_{p(x)}$ rather than $\|\cdot\|_{p(x),\Omega}$, provided this causes no confusion.

We need to identify the space $\bigcap_{j=1}^{\infty} W^{1,jp(x)}(\Omega)$. This limit space is nothing else than the familiar $W^{1,\infty}(\Omega)$. According to the next lemma, it is independent of $p(x)$.

Lemma 2.3. *If u is a measurable function in Ω , then*

$$\lim_{j \rightarrow \infty} \|u\|_{jp(x)} = \|u\|_{\infty}.$$

Proof. The proof is elementary. We use the notation

$$M = \|u\|_{\infty} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|,$$

$$M_j = \|u\|_{jp(x)},$$

and we claim that

$$\lim_{j \rightarrow \infty} M_j = M.$$

To show that $\limsup_{j \rightarrow \infty} M_j \leq M$, we only have to consider those indices j for which $M_j > M$. Then, since $p(x) > 1$,

$$1 = \left(\int_{\Omega} \left| \frac{u(x)}{M_j} \right|^{jp(x)} \frac{dx}{jp(x)} \right)^{\frac{1}{j}} \leq \frac{M}{M_j} \left(\int_{\Omega} \frac{dx}{jp(x)} \right)^{\frac{1}{j}},$$

and the inequality follows.

To show that $\liminf_{j \rightarrow \infty} M_j \geq M$, we may assume that $0 < M < \infty$. Given $\varepsilon > 0$, there is a set $A_{\varepsilon} \subset \Omega$ such that $\operatorname{meas}(A_{\varepsilon}) > 0$ and $|u(x)| > M - \varepsilon$ in A_{ε} . We claim that $\liminf_{j \rightarrow \infty} M_j \geq M - \varepsilon$. Ignoring those indices for which $M_j \geq M - \varepsilon$, we have

$$1 = \left(\int_{\Omega} \left| \frac{u(x)}{M_j} \right|^{jp(x)} \frac{dx}{jp(x)} \right)^{\frac{1}{j}} \geq \left(\int_{A_{\varepsilon}} \left| \frac{u(x)}{M_j} \right|^{jp(x)} \frac{dx}{jp(x)} \right)^{\frac{1}{j}} \geq \frac{M - \varepsilon}{M_j} \left(\int_{A_{\varepsilon}} \frac{dx}{jp^+} \right)^{\frac{1}{j}},$$

and the claim follows. Since ε was arbitrary the Lemma follows. The case $M = \infty$ requires a minor modification in the proof. \square

3. THE EULER LAGRANGE EQUATION

We define

$$(3.13) \quad \Lambda_1 = \inf_v \frac{\|\nabla v\|_{p(x)}}{\|v\|_{p(x)}},$$

where the infimum is taken over all $v \in W_0^{1,p(x)}(\Omega)$, $v \not\equiv 0$. One gets the same infimum by requiring that $v \in C_0^{\infty}(\Omega)$. The Sobolev inequality (Lemma 2.1)

$$\|v\|_{p(x)} \leq C \|\nabla v\|_{p(x)},$$

where C is independent of v , shows that $\Lambda_1 > 0$.

To establish the existence of a non-trivial minimizer, we select a minimizing sequence of admissible functions v_j , normalized so that $\|v_j\|_{p(x)} = 1$. Then

$$\Lambda_1 = \lim_{j \rightarrow \infty} \|\nabla v_j\|_{p(x)}.$$

Recall the Rellich-Kondrachev Theorem for Sobolev spaces with variable exponents (Lemma 2.2). Hence, we can extract a subsequence v_{j_ν} and find a function $u \in W_0^{1,p(x)}(\Omega)$ such that $v_{j_\nu} \rightarrow u$ strongly in $L^{p(x)}(\Omega)$ and $\nabla v_{j_\nu} \rightharpoonup \nabla u$ weakly in $L^{p(x)}(\Omega)$. The norm is weakly sequentially lower semicontinuous. Thus,

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \leq \lim_{\nu \rightarrow \infty} \frac{\|\nabla v_{j_\nu}\|_{p(x)}}{\|v_{j_\nu}\|_{p(x)}} = \Lambda_1.$$

This shows that u is a minimizer. Notice that if u is a minimizer, so is $|u|$. We have proved the following proposition.

Proposition 3.1. *There exists a non-negative minimizer $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, of the Rayleigh quotient (1.1).*

In order to derive the Euler-Lagrange equation for the minimizer(s), we fix an arbitrary test function $\eta \in C_0^\infty(\Omega)$ and consider the competing function

$$v(x) = u(x) + \varepsilon \eta(x),$$

and write

$$k = k(\varepsilon) = \|v\|_{p(x)}, \quad K = K(\varepsilon) = \|\nabla v\|_{p(x)}.$$

A necessary condition for the inequality

$$\Lambda_1 = \frac{K(0)}{k(0)} \leq \frac{K(\varepsilon)}{k(\varepsilon)}$$

is that

$$\frac{d}{d\varepsilon} \left(\frac{K(\varepsilon)}{k(\varepsilon)} \right) = \frac{K'(\varepsilon)k(\varepsilon) - K(\varepsilon)k'(\varepsilon)}{k(\varepsilon)^2} = 0, \quad \text{for } \varepsilon = 0.$$

Thus the necessary condition of minimality reads

$$(3.14) \quad \frac{K'(0)}{K(0)} = \frac{k'(0)}{k(0)}.$$

To find $K'(0)$, differentiate

$$\int_{\Omega} \left| \frac{\nabla u(x) + \varepsilon \nabla \eta(x)}{K(\varepsilon)} \right|^{p(x)} \frac{dx}{p(x)} = 1,$$

with respect to ε . Differentiation under the integral sign is justifiable. We obtain

$$\int_{\Omega} \frac{|\nabla u + \varepsilon \nabla \eta|^{p(x)-2} \langle \nabla u + \varepsilon \nabla \eta, \nabla \eta \rangle}{K(\varepsilon)^{p(x)}} dx = \int_{\Omega} \frac{|\nabla u + \varepsilon \nabla \eta|^{p(x)}}{K(\varepsilon)^{p(x)+1}} K'(\varepsilon) dx.$$

For $\varepsilon = 0$, we conclude

$$\frac{K'(0)}{K(0)} = \frac{\int_{\Omega} K^{-p(x)} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \eta \rangle dx}{\int_{\Omega} \left| \frac{\nabla u}{K} \right|^{p(x)} dx}.$$

A similar calculation yields

$$\frac{k'(0)}{k(0)} = \frac{\int_{\Omega} k^{-p(x)} |u|^{p(x)-2} u \eta dx}{\int_{\Omega} \left| \frac{u}{k} \right|^{p(x)} dx}.$$

Inserting the results into (3.14), we arrive at equation (1.6) in weak form: for all test functions $\eta \in C_0^\infty(\Omega)$, we have

$$\int_{\Omega} \left| \frac{\nabla u}{K} \right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla \eta \right\rangle dx = \Lambda_1 S \int_{\Omega} \left| \frac{u}{k} \right|^{p(x)-2} \frac{u}{k} \eta dx,$$

where $K = \|\nabla u\|_{p(x)}$, $k = \|u\|_{p(x)}$ and S is as in (1.7). Here $\Lambda_1 = K/k$.

The weak solutions with zero boundary values are called *eigenfunctions*, except $u \equiv 0$. We refer to [1, 2, 9, 10, 13] for regularity theory.

Definition 3.1. A function $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, is an *eigenfunction* if the equation

$$(3.15) \quad \int_{\Omega} \left| \frac{\nabla u}{K} \right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla \eta \right\rangle dx = \Lambda S \int_{\Omega} \left| \frac{u}{k} \right|^{p(x)-2} \frac{u}{k} \eta dx$$

holds whenever $\eta \in C_0^\infty(\Omega)$. Here $K = K_u$, $k = k_u$ and $S = S_u$. The corresponding Λ is the *eigenvalue*.

Remark 3.1. According to [2, 10, 9], the weak solutions of equations like (3.15) are continuous if the variable exponent $p(x)$ is Hölder continuous. Thus the eigenfunctions are continuous.

If Λ_1 is the minimum of the Rayleigh quotient in (3.13), we must have

$$\Lambda \geq \Lambda_1,$$

in (3.15), thus Λ_1 is called the *first eigenvalue* and the corresponding eigenfunctions are said to be *first eigenfunctions*. To see this, take $\eta = u$ in the equation, which is possible by approximation. Then we obtain, upon cancellations, that

$$\Lambda = \frac{K}{k} = \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \geq \Lambda_1.$$

We shall restrict ourselves to positive eigenfunctions.

Theorem 3.1. *There exists a continuous strictly positive first eigenfunction. Moreover, any non-negative eigenfunction is strictly positive.*

Proof. The existence of a first eigenfunction was clear, since minimizers of (3.13) are solutions of (3.15). But if u is a minimizer, so is $|u|$, and $|u| \geq 0$. Thus we have a non-negative one. By Remark 3.1 the eigenfunctions are continuous. The strict positivity then follows by the strong minimum principle for weak supersolutions in [13]. \square

We are interested in the asymptotic case when the variable exponent approaches ∞ via the sequence $p(x)$, $2p(x)$, $3p(x)$. . . The procedure requires viscosity solutions. Thus we first verify that the weak solutions of the equation (3.15), formally written as

$$\operatorname{div} \left(\left| \frac{\nabla u}{K} \right|^{p(x)-2} \frac{\nabla u}{K} \right) + \Lambda S \left| \frac{u}{k} \right|^{p-2} \frac{u}{k} = 0,$$

are viscosity solutions. Given $u \in C(\Omega) \cap W_0^{1,p(x)}(\Omega)$, we fix the parameters $k = \|u\|_{p(x)}$, $K = \|\nabla u\|_{p(x)}$ and S . Replacing u by a function $\phi \in C^2(\Omega)$, but keeping k, K, S unchanged, we formally get

$$\Delta_{p(x)}\phi - |\nabla\phi|^2 \log(K) \langle \nabla\phi, \nabla p(x) \rangle + \Lambda^{p(x)} S |\phi|^{p(x)-2} \phi = 0,$$

where

$$\begin{aligned} \Delta_{p(x)}\phi = \operatorname{div} (|\nabla\phi|^{p(x)-2} \nabla\phi) &= |\nabla\phi|^{p(x)-4} \left\{ |\nabla\phi|^2 \Delta\phi + (p(x) - 2) \Delta_\infty\phi \right. \\ &\quad \left. + |\nabla\phi|^2 \ln(|\nabla\phi|) \langle \nabla\phi, \nabla p(x) \rangle \right\}, \end{aligned}$$

and

$$\Delta_\infty\phi = \sum_{i,j=1}^n \frac{\partial\phi}{\partial x_i} \frac{\partial\phi}{\partial x_j} \frac{\partial^2\phi}{\partial x_i \partial x_j}$$

is the ∞ -Laplacian. The relation $\Lambda = K/k$ was used in the simplifications.

Let us abbreviate the expression as

$$\begin{aligned} F(x, \phi, \nabla\phi, D^2\phi) = & |\nabla\phi|^{p(x)-4} \left\{ |\nabla\phi|^2 \Delta\phi + (p(x) - 2) \Delta_\infty\phi + |\nabla\phi|^2 \ln(|\nabla\phi|) \langle \nabla\phi, \nabla p(x) \rangle \right. \\ (3.16) \quad & \left. - |\nabla\phi|^2 \log(K) \langle \nabla\phi, \nabla p(x) \rangle \right\} + \Lambda^{p(x)} S |\phi|^{p(x)-2} \phi = 0. \end{aligned}$$

where we deliberately take $p(x) \geq 2$. Notice that

$$F(x, \phi, \nabla\phi, D^2\phi) < 0$$

exactly when

$$\Delta_{p(x)} \left(\frac{\phi}{K} \right) + \Lambda S \left| \frac{\phi}{k} \right|^{p(x)-2} \frac{\phi}{k} < 0.$$

Recall that k, K, S were dictated by u .

Let $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$. We say that $\phi \in C^2(\Omega)$ touches u from below at the point x_0 , if $\phi(x_0) = u(x_0)$ and $\phi(x) < u(x)$ when $x \neq x_0$.

Definition 3.2. Suppose that $u \in C(\Omega)$. We say that u is a *viscosity supersolution* of the equation

$$F(x, u, \nabla u, D^2 u) = 0$$

if, whenever ϕ touches u from below at a point $x_0 \in \Omega$, we have

$$F(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \leq 0.$$

We say that u is a *viscosity subsolution* if, whenever $\psi \in C^2(\Omega)$ touches u from above at a point $x_0 \in \Omega$, we have

$$F(x_0, \psi(x_0), \nabla \psi(x_0), D^2 \psi(x_0)) \geq 0.$$

Finally, we say that u is a *viscosity solution* if it is both a viscosity super- and subsolution.

Several remarks are appropriate. Notice that the operator F is evaluated for the test function and only at the touching point. If the family of test functions is empty at some point, then there is no requirement on F at that point. The definition makes sense for a merely continuous function u , provided that the parameters k, K, S, Λ have been assigned values. We always have ∇u available for this in our problem.

Theorem 3.2. *The eigenfunctions u are viscosity solutions of the equation*

$$F(x, u, \nabla u, D^2 u) = 0.$$

Proof. This is a standard proof. The equation

$$(3.17) \quad \int_{\Omega} \left| \frac{\nabla u}{K} \right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla \eta \right\rangle dx = \Lambda S \int_{\Omega} \left| \frac{u}{k} \right|^{p(x)-2} \frac{u}{k} \eta dx$$

holds for all $\eta \in W_0^{1,p(x)}(\Omega)$. We first claim that u is a viscosity supersolution. Our proof is indirect. The *antithesis* is that there exist a point $x_0 \in \Omega$ and a test function $\phi \in C^2(\Omega)$, touching u from below at x_0 , such that $F(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) > 0$. By continuity,

$$F(x, \phi(x), \nabla \phi(x), D^2 \phi(x)) > 0$$

holds when $x \in B(x_0, r)$ for some radius r small enough. Then also

$$(3.18) \quad \Delta_{p(x)} \left(\frac{\phi(x)}{K} \right) + \Lambda S \left| \frac{\phi(x)}{k} \right|^{p(x)-2} \frac{\phi(x)}{k} > 0,$$

in $B(x_0, r)$. Denote

$$\varphi = \phi + \frac{m}{2}, \quad m = \min_{\partial B(x_0, r)} (u - \phi).$$

Then $\varphi < u$ on $\partial B(x_0, r)$ but $\varphi(x_0) > u(x_0)$, since $m > 0$. Define

$$\eta = [\varphi - u]_+ \chi_{B(x_0, r)}.$$

Now $\eta \geq 0$. If $\eta \not\equiv 0$, we multiply (3.18) by η and we integrate by parts to obtain the inequality

$$\int_{\Omega} \left| \frac{\nabla \phi}{K} \right|^{p(x)-2} \left\langle \frac{\nabla \phi}{K}, \nabla \eta \right\rangle dx < \Lambda S \int_{\Omega} \left| \frac{\phi}{k} \right|^{p(x)-2} \frac{\phi}{k} \eta dx$$

We have $\nabla \eta = \nabla \phi - \nabla u$ in the subset where $\varphi \geq u$. Subtracting equation (3.17) by the above inequality, we arrive at

$$\begin{aligned} & \int_{\{\varphi > u\}} \left\langle \left| \frac{\nabla \phi}{K} \right|^{p(x)-2} \frac{\nabla \phi}{K} - \left| \frac{\nabla u}{K} \right|^{p(x)-2} \frac{\nabla u}{K}, \frac{\nabla \phi}{K} - \frac{\nabla u}{K} \right\rangle dx \\ & < S \int_{\{\varphi > u\}} \left(\left| \frac{\phi}{k} \right|^{p(x)-2} \frac{\phi}{k} - \left| \frac{u}{k} \right|^{p(x)-2} \frac{u}{k} \right) \left(\frac{\varphi - u}{k} \right) dx, \end{aligned}$$

where the domain of integration is comprised in $B(x_0, r)$. The last integral is negative since $\phi < u$. The first one is non-negative due to the elementary inequality

$$\langle |b|^{p-2}b - |a|^{p-2}a, b - a \rangle \geq 0,$$

which holds for all $p > 1$ because of the convexity of the p -th power. We can take $p = p(x)$. It follows that $\varphi \leq u$ in $B(x_0, r)$. This contradicts $\varphi(x_0) > u(x_0)$. Thus the antithesis was false and u is a viscosity supersolution.

In a similar way we can prove that u is also a viscosity subsolution. \square

4. PASSAGE TO INFINITY

Let us study the procedure when $jp(x) \rightarrow \infty$. The distance function

$$\delta(x) = \text{dist}(x, \partial\Omega)$$

plays a crucial role. We write

$$(4.19) \quad \Lambda_{\infty} = \frac{\|\nabla \delta\|_{\infty}}{\|\delta\|_{\infty}} = \frac{1}{R}$$

where R is the radius of the largest ball inscribed in Ω , the so-called *inradius*. Recall that δ is Lipschitz continuous and $|\nabla \delta| = 1$ a.e. in Ω .

In fact, Λ_∞ is the minimum the Rayleigh quotient in the ∞ -norm:

$$(4.20) \quad \Lambda_\infty = \min_u \frac{\|\nabla u\|_\infty}{\|u\|_\infty},$$

where the minimum is taken among all $u \in W_0^{1,\infty}(\Omega)$. To see this, let $\xi \in \partial\Omega$ be the closest boundary point to $x \in \Omega$. By the mean value theorem

$$|u(x)| = |u(x) - u(\xi)| \leq \|\nabla u\|_\infty |x - \xi| = \|\nabla u\|_\infty \delta(x).$$

It follows that

$$\Lambda_\infty = \frac{1}{\|\delta\|_\infty} \leq \frac{\|\nabla u\|_\infty}{\|u\|_\infty}.$$

Consider

$$(4.21) \quad \Lambda_{jp(x)} = \min_v \frac{\|\nabla v\|_{jp(x)}}{\|v\|_{jp(x)}}, \quad (j = 1, 2, 3 \dots)$$

where the minimum is taken over all v in $C(\bar{\Omega}) \cap W_0^{1,jp(x)}(\Omega)$. When j is large, the minimizer u_j (we do mean $u_{jp(x)}$) is continuous up to the boundary and $u_j|_{\partial\Omega} = 0$. This is a property of the Sobolev space.

Proposition 4.2.

$$(4.22) \quad \lim_{j \rightarrow \infty} \Lambda_{jp(x)} = \Lambda_\infty.$$

Proof. Assume for simplicity that

$$\int_\Omega \frac{dx}{p(x)} = 1.$$

The Hölder inequality implies that

$$\|f\|_{jp(x)} \leq \|f\|_{lp(x)}, \quad l \geq j.$$

Let u_j be the minimizer in the Rayleigh quotient with the $jp(x)$ -norm normalized so that $\|u_j\|_{jp(x)} = 1$. Thus,

$$\Lambda_{jp(x)} = \|\nabla u_j\|_{jp(x)}.$$

Since $\Lambda_{jp(x)}$ is the minimum, we have

$$\Lambda_{jp(x)} \leq \frac{\|\nabla \delta\|_{jp(x)}}{\|\delta\|_{jp(x)}},$$

for all $j = 1, 2, 3 \dots$. Then, by Lemma 2.3,

$$\limsup_{j \rightarrow \infty} \Lambda_{jp(x)} \leq \frac{\|\nabla \delta\|_\infty}{\|\delta\|_\infty} = \Lambda_\infty.$$

It remains to prove that

$$\liminf_{j \rightarrow \infty} \Lambda_{jp(x)} \geq \Lambda_\infty.$$

To this end, observe that the sequence $\|\nabla u_j\|_{jp(x)}$ is bounded. Using a diagonalization procedure we can extract a subsequence u_{j_ν} such that u_{j_ν} converges strongly in each fixed $L^q(\Omega)$ and ∇u_{j_ν} converges weakly in each fixed $L^q(\Omega)$. In other words,

$$u_{j_\nu} \rightarrow u_\infty, \quad \nabla u_{j_\nu} \rightharpoonup \nabla u_\infty, \quad \text{as } \nu \rightarrow \infty,$$

for some $u_\infty \in W_0^{1,\infty}(\Omega)$. By the lower semicontinuity of the norm under weak convergence

$$\|\nabla u_\infty\|_q \leq \liminf_{\nu \rightarrow \infty} \|\nabla u_{j_\nu}\|_q$$

For large indices ν , we have

$$\|\nabla u_{j_\nu}\|_q \leq \|\nabla u_{j_\nu}\|_{j_\nu p(x)} = \Lambda_{j_\nu p(x)}.$$

Therefore,

$$\|\nabla u_\infty\|_q \leq \liminf_{\nu \rightarrow \infty} \Lambda_{j_\nu p(x)}$$

Finally, letting $q \rightarrow \infty$ and taking the normalization into account (by Ascoli's Theorem, $\|u_\infty\|_\infty = 1$) we obtain

$$\frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty} \leq \liminf_{\nu \rightarrow \infty} \Lambda_{j_\nu p(x)},$$

but, since u_∞ is admissible, Λ_∞ is less than or equal to the above ratio. This implies that

$$\lim_{\nu \rightarrow \infty} \Lambda_{j_\nu p(x)} = \Lambda_\infty.$$

By possibly repeating the above, starting with an arbitrary subsequence of variable exponents, it follows that the limit (4.22) holds for the full sequence. This concludes the proof. \square

Using Ascoli's theorem we can assure that the convergence $u_{j_\nu} \rightarrow u_\infty$ is uniform in Ω . Thus the limit of the normalized first eigenfunctions is continuous and we have

$$u_\infty \in C(\bar{\Omega}) \cap W_0^{1,\infty}(\Omega),$$

with $u_\infty|_{\partial\Omega} = 0$, $u_\infty \geq 0$, $u_\infty \not\equiv 0$. However, *the function u_∞ might depend on the particular sequence extracted.*

Theorem 4.3. *The limit of the normalized first eigenfunctions is a viscosity solution of the equation*

$$\max \left\{ \Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)} \left(\frac{u}{K} \right) \right\} = 0,$$

where $K = \|\nabla u\|_\infty$.

Remark 4.2. The limit u of the normalized first eigenfunctions is a non-negative function. At the points where $u > 0$, the equation above means that the largest of the two quantities is zero. At the points¹ where $u = 0$, we agree that first part of the equation is $\Lambda_\infty u \leq |\nabla u|$.

Proof of Theorem 4.3. We begin with the case of viscosity supersolutions. If $\phi \in C^2(\Omega)$ touches u_∞ from below at $x_0 \in \Omega$, we claim that

$$\Lambda_\infty \leq \frac{|\nabla \phi(x_0)|}{\phi(x_0)}, \quad \text{and} \quad \Delta_{\infty(x_0)} \left(\frac{\phi(x_0)}{K} \right) \leq 0,$$

where $K = K_{u_\infty}$. We know that u_j is a viscosity (super)solution of the equation

$$\Delta_{jp(x)} u - |\nabla u|^{jp(x)-2} \ln K_j \langle \nabla u, j \nabla p(x) \rangle + \Lambda_{jp(x)}^{jp(x)} S_{jp(x)} |u|^{jp(x)-2} u = 0$$

where $K_j = \|\nabla u_j\|_{jp(x)}$ and

$$S_{jp(x)} = \frac{\int_{\Omega} \left| \frac{\nabla u_j}{K_j} \right|^{jp(x)} dx}{\int_{\Omega} \left| \frac{u_j}{k_j} \right|^{jp(x)} dx}.$$

We have the trivial estimate

$$\frac{p^-}{p^+} \leq S_{jp(x)} \leq \frac{p^+}{p^-}.$$

We need a test function ψ_j touching u_j from below at a point x_j very near x_0 . To construct it, let $B(x_0, 2R) \subset \Omega$. Obviously,

$$\inf_{B_R \setminus B_r} \{u_\infty - \phi\} > 0,$$

when $0 < r < R$. By the uniform convergence,

$$\inf_{B_R \setminus B_r} \{u_\infty - \phi\} > u_j(x_0) - u_\infty(x_0) = u_j(x_0) - \phi(x_0),$$

provided j is larger than an index large enough, depending on r . For such large indices, $u_j - \phi$ attains its minimum in $B(x_0, R)$ at a point $x_j \in B(x_0, r)$, and letting $j \rightarrow \infty$, we see that $x_j \rightarrow x_0$, as $j \rightarrow \infty$. Actually, $j \rightarrow \infty$ via the subsequence j_ν extracted, but we drop this notation.

Define

$$\psi_j = \phi + (u_j(x_j) - \phi(x_j)).$$

¹When $u < 0$ this is not the right equation, but we keep $u \geq 0$.

This function touches u_j from below at the point x_j . Therefore ψ_j will do as a test function for u_j . We arrive at

$$\begin{aligned}
& |\nabla\phi(x_j)|^{jp(x_j)-4} \left\{ |\nabla\phi(x_j)|^2 \Delta\phi(x_j) \right. \\
& \quad \left. + (jp(x_j) - 2) \Delta_\infty\phi(x_j) + |\nabla\phi(x_j)|^2 \ln(|\nabla\phi(x_j)|) \left\langle \nabla\phi(x_j), j\nabla p(x_j) \right\rangle \right\} \\
& \leq -\Lambda_{jp(x_j)}^{jp(x_j)} S_{jp(x_j)} |\phi(x_j)|^{jp(x_j)-2} \phi(x_j) \\
(4.23) \quad & \quad + |\nabla\phi(x_j)|^{jp(x_j)-2} \ln K_j \left\langle \nabla\phi(x_j), j\nabla p(x_j) \right\rangle.
\end{aligned}$$

First, we consider the case $\nabla\phi(x_0) \neq 0$. Then $\nabla\phi(x_j) \neq 0$ for large indices. Dividing by

$$(jp(x_j) - 2) |\nabla\phi(x_j)|^{jp(x_j)-2}$$

we obtain

$$\begin{aligned}
& \frac{|\nabla\phi(x_j)|^2 \Delta\phi(x_j)}{jp(x_j) - 2} + \Delta_\infty\phi(x_j) + |\nabla\phi(x_j)|^2 \ln|\nabla\phi(x_j)| \left\langle \nabla\phi(x_j), \frac{\nabla p(x_j)}{p(x_j) - 2/j} \right\rangle \\
& \leq \ln K_j \left\langle \nabla\phi(x_j), \frac{\nabla p(x_j)}{p(x_j) - 2/j} \right\rangle - \left(\frac{\Lambda_{jp(x_j)} \phi(x_j)}{|\nabla\phi(x_j)|} \right)^{jp(x_j)-4} \Lambda_{jp(x)}^4 S_{jp(x_j)} \phi(x_j)^3.
\end{aligned}$$

In this inequality, all terms have a limit except possibly the last one. In order to avoid a contradiction, we must have

$$(4.24) \quad \limsup_{j \rightarrow \infty} \frac{\Lambda_{jp(x_j)} \phi(x_j)}{|\nabla\phi(x_j)|} \leq 1.$$

Therefore

$$(4.25) \quad \Lambda_\infty\phi(x_0) - |\nabla\phi(x_0)| \leq 0,$$

as desired. Taking the limit we obtain

$$\Delta_\infty\phi(x_0) + |\nabla\phi(x_0)|^2 \ln \left| \frac{\nabla\phi(x_0)}{K_\infty} \right| \left\langle \nabla\phi(x_0), \nabla \ln p(x_j) \right\rangle \leq 0.$$

Second, consider the case $\nabla\phi(x_0) = 0$. Then the last inequality above is evident. Now the inequality

$$\Lambda_\infty\phi(x_0) - |\nabla\phi(x_0)| \leq 0$$

reduces to $\phi(x_0) \leq 0$. But, if $\phi(x_0) > 0$, then $\phi(x_j) \neq 0$ for large indices. According to inequality (4.23) we must have $|\nabla\phi(x_j)| \neq 0$ and so we can divide by $(jp(x_j) - 2) |\nabla\phi(x_j)|^{jp(x_j)-2}$ and conclude from (4.24) that $\phi(x_0) = 0$, in fact. This shows that we have a viscosity supersolution.

In the case of a subsolution one has to show that for a test function ψ touching u_∞ from above at x_0 at least one of the inequalities

$$\Lambda_\infty \psi_\infty(x_0) - |\nabla \psi(x_0)| \geq 0$$

or

$$\Delta_\infty \psi(x_0) + |\nabla \psi(x_0)|^2 \ln \left| \frac{\nabla \psi(x_0)}{K_\infty} \right| \left\langle \nabla \psi(x_0), \nabla \ln p(x_0) \right\rangle \geq 0$$

is valid. We omit this case, since the proof is pretty similar to the one for supersolutions. \square

5. LOCAL UNIQUENESS

The *existence* of a viscosity solution to the equation

$$\max \left\{ \Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)} \left(\frac{u}{\|\nabla u\|_\infty} \right) \right\} = 0$$

was established in section 4. The question of *uniqueness* is a more delicate one.

In the special case of a constant exponent, say $p(x) = p$, there is a recent counterexample in [14] of a domain (a dumb-bell shaped one) in which there are several linearly independent solutions in $C(\bar{\Omega}) \cap W_0^{1,\infty}(\Omega)$ of the equation

$$\max \left\{ \Lambda - \frac{|\nabla u|}{u}, \Delta_\infty u \right\} = 0, \quad \Lambda = \Lambda_\infty.$$

It is decisive that they have boundary values zero. According to [16, Theorem 2.3], this cannot happen for strictly positive boundary values, which excludes eigenfunctions. This partial uniqueness result implied that there are no positive eigenfunctions for $\Lambda \neq \Lambda_\infty$, cf. [16, Theorem 3.1].

Let us return to the variable exponents. Needless to say, one cannot hope for more than in the case of a constant exponent. Actually, a condition involving the quantities $\min u$, $\max u$, $\max |\nabla \ln p|$ taken over subdomains enters. This complicates the matter and restricts the result.

We start with a normalized positive viscosity solution u of the equation

$$(5.26) \quad \max \left\{ \Lambda_\infty - \frac{|\nabla u|}{u}, \Delta_{\infty(x)} u \right\} = 0.$$

Now $K = \|\nabla u\|_\infty = 1$. The normalization is used in no other way than that the constant K is erased. This equation is not a “proper” one² and the first task is to find the equation for $v = \ln(u)$.

²A term used in the viscosity theory for second order equations

Lemma 5.4. *Let $C > 0$. The function*

$$v = \ln(Cu)$$

is a viscosity solution of the equation

$$(5.27) \quad \max \left\{ \Lambda - |\nabla v|, \Delta_\infty v + |\nabla v|^2 \ln \left(\frac{|\nabla v|}{C} \right) \langle \nabla v, \nabla \ln p \rangle + v |\nabla v|^2 \langle \nabla v, \nabla \ln p \rangle \right\} = 0.$$

We need a *strict* supersolution (this means that the 0 in the right hand side has to be replaced by a negative quantity) which approximates v uniformly. To this end we use the *approximation of unity* introduced in [16]. Let

$$g(t) = \frac{1}{\alpha} \ln(1 + A(e^{\alpha t} - 1)), \quad A > 1, \alpha > 0,$$

and keep $t > 0$. The function

$$w = g(v)$$

will have the desired properties, provided that $v \geq 0$. This requires that

$$Cu(x) \geq 1,$$

which cannot hold globally for an eigenfunction, because $u = 0$ on the boundary. This obstacle restricts the method to local considerations. We are forced to limit our constructions to subdomains.

We use a few elementary results:

$$(5.28) \quad \begin{aligned} 0 < g(t) - t &< \frac{A-1}{\alpha}, \\ A^{-1}(A-1)e^{-\alpha t} &< g'(t) - 1 < (A-1)e^{-\alpha t}, \\ g(t) - t &< \frac{A}{\alpha}(e^{\alpha t} - 1)(g'(t) - 1), \\ g''(t) &= -\alpha(g'(t) - 1)g'(t), \\ 0 < \ln g'(t) &< g'(t) - 1. \end{aligned}$$

In particular, $g'(t) - 1$ will appear as a decisive factor in the calculations. The formula

$$\ln g'(t) = \ln A - \alpha(g(t) - t)$$

is helpful.

We remark that in the next lemma our choice of the parameter α is not optimal, but it is necessary to take $\alpha > 1$, at least. For convenience, we set $\alpha = 2$.

Lemma 5.5. *Take $\alpha = 2$ and assume that $1 < A < 2$. If $v > 0$ is a viscosity supersolution of equation (5.27), then $w = g(v)$ is a viscosity supersolution of the equations*

$$\Lambda - \frac{|\nabla w|}{g'(v)} = 0,$$

and

$$\Delta_\infty w + |\nabla w|^2 \ln \left(\frac{|\nabla w|}{C} \right) \langle \nabla w, \nabla \ln p \rangle + w |\nabla w|^2 \langle \nabla w, \nabla \ln p \rangle + |\nabla w|^4 = -\mu,$$

where

$$\mu = A^{-1}(A-1)|\nabla w|^3 e^{-2v} \left\{ \Lambda - \|e^{2v} \nabla \ln p\|_\infty \right\},$$

provided that

$$\|e^{2v} \nabla \ln p\|_\infty < \Lambda.$$

Remark 5.3. We can further estimate μ and replace it by a constant, viz.

$$A^{-1}\Lambda^3(A-1)e^{-2\|v\|_\infty} \left\{ \Lambda - \|e^{2v} \nabla \ln p\|_\infty \right\},$$

but we prefer not to do so.

Proof. The proof below is only formal and should be rewritten in terms of test functions. One only has to observe that an arbitrary test function φ touching w from below can be represented as $\varphi = g(\phi)$ where ϕ touches v from below.

First we have the expressions

$$\begin{aligned} \nabla w &= g'(v) \nabla v, \\ \Delta_\infty w &= g'(v)^2 g''(v) |\nabla v|^4 + g'(v)^3 \Delta_\infty v, \\ |\nabla w|^2 \ln \left(\frac{|\nabla w|}{C} \right) \langle \nabla w, \nabla \ln p \rangle & \\ &= g'(v)^3 \left\{ |\nabla v|^2 \ln \left(\frac{|\nabla v|}{C} \right) \langle \nabla v, \nabla \ln p \rangle + |\nabla v|^2 \ln(g'(v)) \langle \nabla v, \nabla \ln p \rangle \right\}. \end{aligned}$$

Then, using that v is a supersolution, we get

$$\begin{aligned}
& \Delta_\infty w + |\nabla w|^2 \ln \left(\frac{|\nabla w|}{C} \right) \langle \nabla w, \nabla \ln p \rangle \\
&= g'(v)^2 g''(v) |\nabla v|^4 + g'(v)^3 \left\{ \Delta_\infty v + |\nabla v|^2 \ln \left(\frac{|\nabla v|}{C} \right) \langle \nabla v, \nabla \ln p \rangle \right\} \\
&\quad + g'(v)^3 |\nabla v|^2 \ln(g'(v)) \langle \nabla v, \nabla \ln p \rangle \\
&\leq g'(v)^2 g''(v) |\nabla v|^4 + g'(v)^3 \left\{ -v |\nabla v|^2 \langle \nabla v, \nabla \ln p \rangle - |\nabla v|^4 \right\} \\
&\quad + g'(v)^3 |\nabla v|^2 \ln(g'(v)) \langle \nabla v, \nabla \ln p \rangle.
\end{aligned}$$

Let us collect the terms appearing on the left-hand side of the equation for w . Using the formulas (5.28) for $g'(v)$ and $\ln(g'(v))$ we arrive at

$$\begin{aligned}
& \Delta_\infty w + |\nabla w|^2 \ln \left(\frac{|\nabla w|}{C} \right) \langle \nabla w, \nabla \ln p \rangle + |\nabla w|^4 + w |\nabla w|^2 \langle \nabla w, \nabla \ln p \rangle, \\
&\leq g'(v)^3 |\nabla v|^3 (g'(v) - 1) \left\{ -|\nabla v| + |\nabla \ln p| \right\} + g'(v)^3 |\nabla v|^3 (g(v) - v) |\nabla \ln p|,
\end{aligned}$$

after some arrangements. Using

$$g(t) - t < \frac{A}{2} (e^{2t} - 1) (g'(t) - 1) \leq (e^{2t} - 1) (g'(t) - 1),$$

and collecting all the terms with the factor $|\nabla \ln p|$ separately, observing that $1 + (e^{2t} - 1) = e^{2t}$, we see that the right-hand side is less than

$$g'(v)^3 |\nabla v|^3 (g'(v) - 1) \{-|\nabla v| + |e^{2v} \nabla \ln p|\} \leq |\nabla w|^3 A^{-1} (A - 1) e^{-2v} \{-\Lambda + |e^{2v} \nabla \ln p|\},$$

since the expression in braces is negative. \square

We abandon the requirement of zero boundary values. Thus Ω below can represent a proper subdomain. Eigenfunctions belong to a Sobolev space but we cannot ensure this for an arbitrary viscosity solution. This requirement is therefore included in our next theorem.

Theorem 5.4. *Suppose that $u_1 \in C(\overline{\Omega})$ is a viscosity subsolution and that $u_2 \in C(\overline{\Omega})$ is a viscosity supersolution of equation (5.26). Assume that at least one of them belongs to $W^{1,\infty}(\Omega)$. If $u_1(x) > 0$ and $u_2(x) \geq m_2 > 0$ in Ω , and*

$$(5.29) \quad 3 \left\| \left(\frac{u_2}{m_2} \right)^2 \nabla \ln p \right\|_\infty \leq \Lambda,$$

then the following comparison principle holds:

$$u_1 \leq u_2 \quad \text{on } \partial\Omega \quad \implies \quad u_1 \leq u_2 \quad \text{in } \Omega.$$

Proof. Define

$$v_1 = \ln(Cu_1), \quad v_2 = \ln(Cu_2),$$

with $C = 1/m_2$. Then $v_2 > 0$, but v_1 may take negative values. We define

$$w_2 = g(v_2), \quad \alpha = 2, \quad 1 < A < 2.$$

If $v_2 \geq v_1$, we are done. If not, consider the open subset $\{v_2 < v_1\}$ and denote

$$\sigma = \sup \{v_1 - v_2\} > 0.$$

Note that σ is independent of C . (The antithesis was that $\sigma > 0$.) Then, taking $A = 1 + \sigma$,

$$v_2 < w_2 < v_2 + \frac{A-1}{2} = v_2 + \frac{\sigma}{2}.$$

Note that $v_1 - w_2 = v_1 - v_2 + v_2 - w_2 \geq v_1 - v_2 - \sigma/2$. Taking the supremum on the subdomain $\mathcal{U} = \{w_2 < v_1\}$ we have

$$\sup_{\mathcal{U}} \{v_1 - w_2\} \geq \frac{\sigma}{2} > 0 = \max_{\partial\mathcal{U}} \{v_1 - w_2\}$$

and $\mathcal{U} \Subset \Omega$, i.e. \mathcal{U} is strictly interior. Moreover,

$$(5.30) \quad \sup \{v_1 - w_2\} \leq \frac{3\sigma}{2}.$$

In order to obtain a contradiction, we double the variables and write

$$M_j = \max_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} \left\{ v_1(x) - w_2(y) - \frac{j}{2}|x - y|^2 \right\}.$$

If the index j is large, the maximum is attained at some interior point (x_j, y_j) in $\mathcal{U} \times \mathcal{U}$. The points converge to some interior point, say $x_j \rightarrow \hat{x}$, $y_j \rightarrow \hat{x}$, and

$$\lim_{j \rightarrow \infty} j|x_j - y_j|^2 = 0.$$

This is a standard procedure. According to the ‘‘Theorem of Sums’’, cf. [7] or [18], there exist symmetric $n \times n$ -matrices \mathbb{X}_j and \mathbb{Y}_j such that

$$\begin{aligned} (j(x_j - y_j), \mathbb{X}_j) &\in \overline{J_{\mathcal{U}}^{2,+}} v_1(x_j), \\ (j(x_j - y_j), \mathbb{Y}_j) &\in \overline{J_{\mathcal{U}}^{2,-}} w_2(y_j), \\ \langle \mathbb{X}_j \xi, \xi \rangle &\leq \langle \mathbb{Y}_j \xi, \xi \rangle, \quad \text{when } \xi \in \mathbb{R}^n. \end{aligned}$$

The definition of the semijets and their closures $\overline{J_{\mathcal{U}}^{2,+}}$, $\overline{J_{\mathcal{U}}^{2,-}}$ can be found in the above mentioned references³. The equations have to be written in terms of jets.

³Symbolically the interpretation is: $j(x_j - y_j)$ means $\nabla v_1(x_j)$ and $\nabla w_2(y_j)$, \mathbb{X}_j means $D^2 v_1(x_j)$, and \mathbb{Y}_j means $D^2 w_2(y_j)$.

We exclude one alternative from the equations. In terms of jets

$$\Lambda - \frac{|\nabla w_2|}{g'(v_2)} \leq 0 \quad \text{reads} \quad \Lambda - \frac{j|x_j - y_j|}{g'(v_2(y_j))} \leq 0$$

and, since $v_2 > 0$, $g'(v_2(y_j)) > 1$, and so

$$\Lambda < j|x_j - y_j|.$$

This rules out the alternative $\Lambda - |\nabla v_1(x_j)| \geq 0$ in the equation for v_1 , which reads $\Lambda - j|x_j - y_j| \geq 0$. Therefore we must have that $\Delta_\infty v_1 + \dots + |\nabla v_1|^4 \geq 0$, i.e.

$$\begin{aligned} & \left\langle \mathbb{X}_j j(x_j - y_j), j(x_j - y_j) \right\rangle + j^2|x_j - y_j|^2 \ln \left(\frac{j|x_j - y_j|}{C} \right) \left\langle j(x_j - y_j), \nabla \ln p(x_j) \right\rangle \\ & + v_1(x_j)j^2|x_j - y_j|^2 \ln \left(\frac{j|x_j - y_j|}{C} \right) \left\langle j(x_j - y_j), \nabla \ln p(x_j) \right\rangle + j^4|x_j - y_j|^4 \geq 0. \end{aligned}$$

The equation for w_2 reads

$$\begin{aligned} & \left\langle \mathbb{Y}_j j(x_j - y_j), j(x_j - y_j) \right\rangle + j^2|x_j - y_j|^2 \ln \left(\frac{j|x_j - y_j|}{C} \right) \left\langle j(x_j - y_j), \nabla \ln p(y_j) \right\rangle \\ & + w_2(y_j)j^2|x_j - y_j|^2 \ln \left(\frac{j|x_j - y_j|}{C} \right) \left\langle j(x_j - y_j), \nabla \ln p(y_j) \right\rangle + j^4|x_j - y_j|^4 \\ & \leq -A^{-1}\sigma j^3|x_j - y_j|^3 e^{-2v_2(y_j)} \left\{ \Lambda - \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \right\}. \end{aligned}$$

Subtracting the last two inequalities, we notice that the terms $j^4|x_j - y_j|^4$ cancel. The result is

$$\begin{aligned} & \left\langle (\mathbb{Y}_j - \mathbb{X}_j) j(x_j - y_j), j(x_j - y_j) \right\rangle \\ & + j^2|x_j - y_j|^2 \ln \left(\frac{j|x_j - y_j|}{C} \right) \left\langle j(x_j - y_j), \nabla \ln p(y_j) - \nabla \ln p(x_j) \right\rangle \\ & + j^2|x_j - y_j|^2 \left\langle j(x_j - y_j), w_2(y_j) \nabla \ln p(y_j) - v_1(x_j) \nabla \ln p(x_j) \right\rangle \\ & \leq -A^{-1}\sigma j^3|x_j - y_j|^3 e^{-2v_2(y_j)} \left\{ \Lambda - \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \right\}. \end{aligned}$$

The first term, the one with matrices, is non-negative and can be omitted from the inequality. Then we move the remaining terms and divide by $j^3|x_j - y_j|^3$ to get

$$\begin{aligned} & A^{-1}\sigma e^{-2v_2(y_j)} \left\{ \Lambda - \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \right\} \\ & \leq \left| \ln \frac{j|x_j - y_j|}{C} \right| \left| \nabla \ln p(y_j) - \nabla \ln p(x_j) \right| + |w_2(y_j) \nabla \ln p(y_j) - v_1(x_j) \nabla \ln p(x_j)| \end{aligned}$$

We need the uniform bound

$$\Lambda \leq j|x_j - y_j| \leq L.$$

The inequality with Λ was already clear. We can take $L = 2\|v_1\|_{\infty, \mathcal{U}}$ or $L = \|w_2\|_{\infty, \mathcal{U}} \leq 4\|v_2\|_{\infty, \mathcal{U}}$, using the definition of M_j . Taking the limit as $j \rightarrow \infty$ we use the continuity of $\nabla \ln p$ to arrive at

$$A^{-1} \sigma \left\{ \Lambda - \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \right\} \leq e^{2v_2(\hat{x})} |w_2(\hat{x}) \nabla \ln p(\hat{x}) - v_1(\hat{x}) \nabla \ln p(\hat{x})|.$$

Recall (5.30). Since $A = 1 + \sigma$, the above implies that

$$A^{-1} \sigma \left\{ \Lambda - \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \right\} \leq \|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}} \frac{3\sigma}{2}.$$

Divide out σ . Now $A^{-1} \geq 1/2$. The final inequality is

$$\Lambda \leq 3\|e^{2v_2} \nabla \ln p\|_{\infty, \mathcal{U}}.$$

Thus there is a contradiction, if the opposite inequality is assumed to be valid. Recall that

$$e^{2v_1} = \left(\frac{u_2}{m_2} \right)^2$$

to finish the proof. □

Corollary 5.1. *Local uniqueness holds. In other words, in a sufficiently small interior subdomain we cannot perturb the eigenfunction continuously.*

Proof. We can make

$$\frac{\max_{\mathcal{U}} u}{\min_{\mathcal{U}} u}$$

as small as we please, by shrinking the domain \mathcal{U} . Thus condition (5.29) is valid with the L^∞ norm taken over \mathcal{U} . □

6. DISCUSSION ABOUT THE ONE-DIMENSIONAL CASE

In the one-dimensional case an explicit comparison of the minimization problem for the two Rayleigh quotients (1.1) and (1.2) is possible. Let $\Omega = (0, 1)$ and consider the limits of the problem coming from minimizing either

$$(I) \quad \frac{\|u'\|_{jp(x)}}{\|u\|_{jp(x)}}$$

or

$$(II) \quad \frac{\int_0^1 |v'(x)|^{jp(x)} dx}{\int_0^1 |v(x)|^{jp(x)} dx}, \quad \text{with } \int_0^1 |v(x)|^{jp(x)} dx = C,$$

as $j \rightarrow \infty$. In the second case the equation is

$$\min \left\{ \Lambda - \frac{|v'|}{v}, (v')^2 v'' + (v')^3 \ln(|v'|) \frac{p}{p'} \right\} = 0$$

for $v > 0$ ($v(0) = 0$, $v(1) = 0$, $\|v^p\|_\infty = C$).

The Luxemburg norm leads to the same equation, but with

$$v(x) = \frac{u(x)}{\|u'\|_\infty} = \frac{u(x)}{K}$$

as in equation (1.10). Thus all the solutions violating the condition $\|v'\|_\infty = 1$ are ruled out. This is the difference between the two problems.

Let us return to (II). The equation for v (without any normalization) can be solved. Upon separation of variables, we obtain

$$v(x) = \begin{cases} \int_0^x e^{\frac{A}{p(t)}} dt, & \text{when } 0 \leq x \leq x_0, \\ \int_x^1 e^{\frac{A}{p(t)}} dt, & \text{when } x_0 \leq x \leq 1, \end{cases}$$

where the constant A is at our disposal and the point x_0 is determined by the continuity condition

$$\int_0^{x_0} e^{\frac{A}{p(t)}} dt = \int_{x_0}^1 e^{\frac{A}{p(t)}} dt.$$

Clearly, $0 < x_0 < 1$. Now Λ is determined from

$$\frac{v'(x_0^-)}{v(x_0)} = \Lambda = -\frac{v'(x_0^+)}{v(x_0)}.$$

Provided that the inequality

$$\frac{|v'(x)|}{v(x)} \geq \Lambda \quad (0 < x < 1, x \neq x_0)$$

holds, the number Λ is an eigenvalue for the non-homogeneous problem. What about the value of A ? Given C , we can determine A from

$$\max_{0 < x < 1} v(x)^{p(x)} = C.$$

At least for a suitable $p(x)$, we can this way reach any real number A and therefore Λ can take all positive values, as C varies.

The problem in the Luxemburg norm is different. If u is an eigenfunction and

$$v = \frac{u}{\|u'\|_\infty},$$

then $0 \leq v'(x) \leq 1$ in some interval $(0, x_0)$. But the equation leads to

$$\frac{u'(x)}{\|u'\|_\infty} = e^{-\frac{A_1}{p(x)}}, \quad A_1 \geq 0,$$

in $(0, x_0)$ and

$$-\frac{u'(x)}{\|u'\|_\infty} = e^{-\frac{A_2}{p(x)}}, \quad A_2 \geq 0,$$

in $(x_0, 1)$. (In fact, $A_1 = A_2$). But this is impossible at points where the left-hand side is ± 1 , unless at least one of the constants A_1, A_2 is zero, say that $A_1 = 0$. Then $u(x) = x$ when $0 \leq x \leq x_0$. The determination of Λ from the equation

$$\frac{1}{x_0} = \Lambda = \frac{e^{-A_2/p(x_0)}}{x_0},$$

forces also $A_2 = 0$. It follows that

$$u(x) = \delta(x), \quad \Lambda = \Lambda_\infty = 2$$

is the only positive solution of the equation (1.10). In this problem Λ is unique. Recall that δ is the distance function.

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